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SHORT–TIME EXISTENCE OF THE SECOND ORDER RENORMALIZATION GROUP FLOW IN DIMENSION THREE

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Abstract. Given a compact three–manifold together with a Riemannian metric, we prove the short–time existence of a solution to the renormalization group flow, truncated at the second order term, under a suitable hypothesis on the sectional curvature of the initial metric.

1. **Introduction.** The *renormalization group* (RG) arises in modern theoretical physics as a method to investigate the changes of a system viewed at different distance scales. Since its introduction in the early '50, this set of ideas has given rise to significant developments in *quantum field theory* (QFT) and opened connections between contemporary physics and Riemannian geometry. In spite of this, the RG still lacks of a strong mathematical foundation.

In this paper we deal with a particular example from string theory, the flow equation for the world–sheet nonlinear sigma–models, and we try to analyze the contribution given from its second order truncation. More precisely, let S be the classical (harmonic map) action

$$S(\varphi) = \frac{1}{4\pi a}\int_\Sigma \mathrm{tr}_h(\varphi^* g)\, d\mu_h\,,$$

where $\varphi : \Sigma \to M^n$ is a smooth map between a surface (Σ, h) and a Riemannian manifold (M^n, g) of dimension $n \geq 3$. The quantity $a > 0$ is the so–called *string coupling constant*. Roughly speaking, in order to control the path integral quantization of the action S, one introduces a cut–off momentum Λ which parametrizes the spectrum of fluctuations of the theory as the distance scale is changed according to $1/\Lambda \to 1/\Lambda'$. This formally generates a flow (the renormalization group flow) in the space of actions which is controlled by the induced scale–dependence in (M^n, g). Setting $\tau := -\ln(\Lambda/\Lambda')$, one thus considers the so–called *beta functions* β, associated with the renormalization group of the theory and defined by the formal flow $g(\tau)$ satisfying

$$\frac{\partial g_{ik}(\tau)}{\partial \tau} = -\beta_{ik}\,.$$

In the perturbative regime (that is, when $a|\mathrm{Riem}(g)| \ll 1$) the beta functions β_{ik} can be expanded in powers of a, with coefficients which are polynomial in the curvature tensor of the metric g and its derivatives. As the quantity $a|\mathrm{Riem}(g)|$ is supposed

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to be very small, the first order truncation should provide a good approximation of the full RG–flow

$$\frac{\partial g_{ik}}{\partial \tau} = -a\mathrm{R}_{ik} + o(a)\,,$$

as $a \to 0$.

Hence, the first order truncation (with the substitution $\tau = t/2a$) coincides with the *Ricci flow* $\partial_t g = -2\mathrm{Ric}$, as noted by Friedan [12, 13] and Lott [22], see also [6].

It is a well–known fact that generally the Ricci flow becomes singular in finite time and in [20] Hamilton proved that at a finite singular time $T > 0$, the Riemann curvature blows up. Then, near a singularity, the Ricci flow is no longer a valid approximation of the behavior of the sigma–model. From the physical point of view, it appears then relevant to possibly consider the coupled flow generated by a more general action, as in [5, 25].

Another possibility could be to consider also the second order term in the expansion of the beta functions, whose coefficients are quadratic in the curvature and therefore are (possibly) dominating, even when $a|\mathrm{Riem}(g)| \to 0$. The resulting flow is called *two–loop* RG–flow

$$\frac{\partial g_{ik}}{\partial \tau} = -a\mathrm{R}_{ik} - \frac{a^2}{2}\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,, \tag{1}$$

see [21]. We refer to it as $\mathrm{RG}^{2,a}$–flow.

In [24] Oliynyk investigates the behavior of such flow in dimension two, proving that it can differ substantially from the Ricci flow. In [19] Guenther and Oliynyk prove the existence and the stability of the two–loop RG–flow on the n–dimensional torus, while in negative constant curvature they prove stability for a modified RG–flow by diffeomorphism and scaling. In [17] Gimre, Guenther and Isenberg study the flow on n–dimensional compact manifolds with constant sectional curvature, observing that in negative curvature, the asymptotic behavior of the flow depends on the value of the coupling constant a and of the sectional curvature. In the same paper, the authors also focus on three–dimensional locally homogeneous spaces, where the strong assumptions on the geometry of the initial metric allow to reduce the PDE to a system of ODEs.

The curvature tensor of a Riemannian manifold (M^n, g) is defined, as in [15], by

$$\mathrm{Riem}(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z\,,$$

while the associated $(4,0)$–tensor is defined by $\mathrm{Riem}(X,Y,Z,T) = g(\mathrm{Riem}(X,Y)Z, T)$. In local coordinates, we have

$$\mathrm{R}_{ijkl} = g\Big(\mathrm{Riem}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big)\,,$$

the Ricci tensor is then obtained by tracing $\mathrm{R}_{ik} = g^{jl}\mathrm{R}_{ijkl}$.

The sectional curvature of a plane $\pi \in T_pM^n$ spanned by a pair of vectors $X, Y \in T_pM^n$ is defined as

$$\mathrm{K}(\pi) = \mathrm{K}(X,Y) = \frac{\mathrm{Riem}(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

After rescaling the flow parameter $\tau \to t/2a$ in equation (1), the $\mathrm{RG}^{2,a}$–flow is given by

$$\partial_t g_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,,$$

which can be seen as a sort of "perturbation" of the Ricci flow $\partial_t g_{ik} = -2\mathrm{R}_{ik}$.

In the paper, we are going to consider the short–time existence of this flow for an initial three–dimensional, smooth, compact Riemannian manifold.

In the special three–dimensional case, thanks to the algebraic decomposition of the Riemann tensor, the evolution equation has the following expression.

$$\partial_t g_{ik} = -2\mathrm{R}_{ik} - a(2\mathrm{R}\mathrm{R}_{ik} - 2\mathrm{R}^2_{ik} + 2|\mathrm{Ric}|^2 g_{ik} - \mathrm{R}^2 g_{ik})\,,$$

where $\mathrm{R}^2_{ik} = \mathrm{R}_{ij}\mathrm{R}_{lk}g^{jl}$.

Theorem 1.1. *Let (M^3, g_0) be a compact, smooth, three–dimensional Riemannian manifold and $a \in \mathbb{R}$. Assume that the sectional curvature K_0 of the initial metric g_0 satisfies*

$$1 + 2a\mathrm{K}_0(X,Y) > 0 \tag{2}$$

for every point $p \in M^3$ and vectors $X, Y \in T_pM^3$. Let

$$Lg_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,,$$

then, there exists some $T > 0$ such that the Cauchy problem

$$\begin{cases} \partial_t g = Lg \\ g(0) = g_0 \end{cases} \tag{3}$$

admits a unique smooth solution $g(t)$ for $t \in [0, T)$.

Notice that, even if not physically relevant, in this theorem we also allow $a < 0$. In such case the condition on the initial metric becomes

$$\mathrm{K}_0(X,Y) < -\frac{1}{2a}$$

which is clearly satisfied by every manifold with negative curvature.

Any manifold with positive curvature satisfies instead condition (2), for every $a > 0$.

2. The principal symbol of the operator L.

The evolution problem involves a fully nonlinear second–order differential operator Lg, which, as for the Ricci flow, can only be weakly elliptic, due to the invariance of the curvature tensors by the action of the group of diffeomorphisms of the manifold M^n. Hamilton in [20] proved the short–time existence of solutions of the Ricci flow using the Nash–Moser implicit function theorem, showing that the flow satisfies a certain first–order integrability condition, namely the contracted second Bianchi identity. In the present paper we establish the short–time existence using the so called *DeTurck's trick* in [9, 10], following the line of Buckland that in [4] showed the short–time existence of solutions of the *cross curvature flow* (see [7]) in dimension three, via the same method.

From the general existence theory of nonlinear parabolic PDEs (see [1, Chapter 4, Section 4], [14] or [23], for instance) it follows that the evolution equation $\frac{\partial}{\partial t}g = Lg$ admits a unique smooth solution for short time if the linearized operator around the initial data $DL_g(h) = \frac{d}{ds}\big|_{s=0} L(g+sh)$ is strongly elliptic, that is, if its principal symbol $\sigma_\xi(DL_g)$ has all the eigenvalues with uniformly positive real parts for any cotangent vector $\xi \neq 0$. We will see that, under the hypotheses of Theorem 1.1, the principal symbol of this linearized operator is nonnegative definite and, even if it always contains some zero eigenvalues, such zero eigenvalues come only from the diffeomorphism invariance. This will allow us to apply DeTurck's trick to the $\mathrm{RG}^{2,a}$–flow.

We start computing the linearized operator DL_g of the operator L at a metric g.

The Riemann and Ricci tensors have the following linearizations, see [2, Theorem 1.174] or [26].

$$D\mathrm{Riem}_g(h)_{ijkm} = \frac{1}{2}\Big(-\nabla_j\nabla_m h_{ik} - \nabla_i\nabla_k h_{jm} + \nabla_i\nabla_m h_{jk} + \nabla_j\nabla_k h_{im}\Big) + \mathrm{LOT}$$
$$D\mathrm{Ric}_g(h)_{ik} = \frac{1}{2}\Big(-\Delta h_{ik} - \nabla_i\nabla_k \operatorname{tr}(h) + \nabla_i\nabla^t h_{tk} + \nabla_k\nabla^t h_{it}\Big) + \mathrm{LOT}$$

where we use the metric g to lower and upper indices and LOT stands for *lower order terms*.

Then, the linearized of L around g, for every $h\in S^2M^n$, is given by

$$\begin{aligned}
& DL_g(h)_{ik}\\
= \;& -2D\mathrm{Ric}_g(h)_{ik} - aD\mathrm{Riem}_g(h)_i{}^{stu}\mathrm{R}_{kstu} - a\mathrm{R}_{istu}D\mathrm{Riem}_g(h)_k{}^{stu} + \mathrm{LOT}\\
= \;& \Delta h_{ik} + \nabla_i\nabla_k\operatorname{tr}(h) - \nabla_i\nabla^t h_{tk} - \nabla_k\nabla^t h_{it}\\
& -\frac{a}{2}\mathrm{R}_{kstu}\big(\nabla^s\nabla^t h_i^u + \nabla_i\nabla^u h^{st} - \nabla^s\nabla^u h_i^t - \nabla_i\nabla^t h^{su}\big)\\
& -\frac{a}{2}\mathrm{R}_{istu}\big(\nabla^s\nabla^t h_k^u + \nabla_k\nabla^u h^{st} - \nabla^s\nabla^u h_k^t - \nabla_k\nabla^t h^{su}\big) + \mathrm{LOT}\\
= \;& \Delta h_{ik} + \nabla_i\nabla_k\operatorname{tr}(h) - \nabla_i\nabla^t h_{kt} - \nabla_k\nabla^t h_{it}\\
& +a\mathrm{R}_{kstu}\big(\nabla_i\nabla^t h^{su} - \nabla^s\nabla^t h_i^u\big) + a\mathrm{R}_{istu}\big(\nabla_k\nabla^t h^{su} - \nabla^s\nabla^t h_k^u\big) + \mathrm{LOT}
\end{aligned}$$

where the last passage follows from the symmetries of the Riemann tensor (interchanging the last two indices makes it change sign).

Now we obtain the principal symbol of the linearized operator in the direction of an arbitrary cotangent vector ξ by replacing each covariant derivative ∇_α with the corresponding component ξ_α,

$$\begin{aligned}
\sigma_\xi(DL_g)(h)_{ik} \;=\;& \xi^t\xi_t h_{ik} + \xi_i\xi_k\operatorname{tr}(h) - \xi_i\xi^t h_{kt} - \xi_k\xi^t h_{it}\\
& +a\mathrm{R}_{kstu}\big(\xi_i\xi^t h^{su} - \xi^s\xi^t h_i^u\big) + a\mathrm{R}_{istu}\big(\xi_k\xi^t h^{su} - \xi^s\xi^t h_k^u\big)\,.
\end{aligned}$$

Since the symbol is homogeneous, we can assume that $|\xi|_g = 1$, furthermore, we can assume to do all the following computations in an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM^n such that $\xi = g(e_1,\cdot)$, hence, $\xi_i = 0$ for $i\neq 1$.

Then, we obtain,

$$\begin{aligned}
\sigma_\xi(DL_g)(h)_{ik} \;=\;& h_{ik} + \delta_{i1}\delta_{k1}\operatorname{tr}(h) - \delta_{i1}\delta^{t1}h_{tk} - \delta_{k1}\delta^{t1}h_{it}\\
& +a\mathrm{R}_{kstu}\big(\delta_{i1}\delta^{t1}h^{su} - \delta^{s1}\delta^{t1}h_i^u\big) + a\mathrm{R}_{istu}\big(\delta_{k1}\delta^{t1}h^{su} - \delta^{s1}\delta^{t1}h_k^u\big)\\
=\;& h_{ik} + \delta_{i1}\delta_{k1}\operatorname{tr}(h) - \delta_{i1}h_{1k} - \delta_{k1}h_{i1}\\
& +a\mathrm{R}_{ks1u}\delta_{i1}h^{su} - a\mathrm{R}_{k11u}h_i^u + a\mathrm{R}_{is1u}\delta_{k1}h^{su} - a\mathrm{R}_{i11u}h_k^u\,.
\end{aligned}$$

So far the dimension n of the manifold was arbitrary, now we carry out the computation in the special case $n=3$ (using again the symmetries of the Riemann tensor),

$$\sigma_\xi(DL_g)\begin{pmatrix} h_{11}\\ h_{12}\\ h_{13}\\ h_{22}\\ h_{33}\\ h_{23}\end{pmatrix} = \begin{pmatrix} h_{22}(1+2a\mathrm{R}_{1212}) + h_{33}(1+2a\mathrm{R}_{1313}) + h_{23}(4a\mathrm{R}_{1213})\\ h_{33}a\mathrm{R}_{1323} + h_{23}a\mathrm{R}_{1223}\\ h_{22}a\mathrm{R}_{1232} + h_{23}a\mathrm{R}_{1332}\\ h_{22}(1+2a\mathrm{R}_{1212}) + h_{23}2a\mathrm{R}_{1213}\\ h_{33}(1+2a\mathrm{R}_{1313}) + h_{23}2a\mathrm{R}_{1213}\\ h_{22}a\mathrm{R}_{1213} + h_{33}a\mathrm{R}_{1213} + h_{23}(1+a\mathrm{R}_{1212}+a\mathrm{R}_{1313})\end{pmatrix}.$$

Then, we conclude that

$$\sigma_\xi(DL_g)=\begin{pmatrix}0&0&0&1+2a\mathrm{R}_{1212}&1+2a\mathrm{R}_{1313}&4a\mathrm{R}_{1213}\\0&0&0&0&a\mathrm{R}_{1323}&a\mathrm{R}_{1223}\\0&0&0&a\mathrm{R}_{1232}&0&a\mathrm{R}_{1332}\\0&0&0&1+2a\mathrm{R}_{1212}&0&2a\mathrm{R}_{1213}\\0&0&0&0&1+2a\mathrm{R}_{1313}&2a\mathrm{R}_{1213}\\0&0&0&a\mathrm{R}_{1213}&a\mathrm{R}_{1213}&1+a\mathrm{R}_{1212}+a\mathrm{R}_{1313}\end{pmatrix}.$$

As expected, in the kernel of the principal symbol there is at least the three–dimensional space of forms $h=\xi\otimes\nu+\nu\otimes\xi\in S^2M^3$ where ν is any cotangent vector, that is, the variations of the metric which are tangent to the orbits of the group of diffeomorphisms (see [8, Chapter 3, Section 2] for more details on this).

Now we use the algebraic decomposition of the Riemann tensor in dimension three in order to simplify the computation of the other eigenvalues.

We recall that

$$\mathrm{R}_{ijkl}=(\mathrm{Ric}\owedge g)_{ijkl}-\frac{\mathrm{R}}{4}(g\owedge g)_{ijkl}$$

where R denotes the scalar curvature, i.e. the trace of the Ricci tensor, and the symbol $\owedge$ denotes the Kulkarni–Nomizu product of two symmetric bilinear forms p and q, defined by

$$\begin{aligned}&(p\owedge q)(X,Y,Z,T)\\=&p(X,Z)q(Y,T)+p(Y,T)q(X,Z)-p(X,T)q(Y,Z)-p(Y,Z)q(X,T)\,,\end{aligned}$$

for every tangent vectors X,Y,Z,T.

By means of the expression of the Riemann tensor in terms of the Ricci tensor and since we are in an orthonormal basis, the principal symbol can be expressed in the simpler form

$$\sigma_\xi(DL_g)=\begin{pmatrix}0&0&0&1+a(\mathrm{R}-2\mathrm{R}_{33})&1+a(\mathrm{R}-2\mathrm{R}_{22})&4a\mathrm{R}_{23}\\0&0&0&0&a\mathrm{R}_{12}&-a\mathrm{R}_{13}\\0&0&0&a\mathrm{R}_{13}&0&-a\mathrm{R}_{12}\\0&0&0&1+a(\mathrm{R}-2\mathrm{R}_{33})&0&2a\mathrm{R}_{23}\\0&0&0&0&1+a(\mathrm{R}-2\mathrm{R}_{22})&2a\mathrm{R}_{23}\\0&0&0&a\mathrm{R}_{23}&a\mathrm{R}_{23}&1+a\mathrm{R}_{11}\end{pmatrix}.$$

In order to apply the argument of DeTurck, we need the weak ellipticity of the linearized operator. To get that we have to compute the eigenvalues of the minor

$$A=\begin{pmatrix}1+a(\mathrm{R}-2\mathrm{R}_{33})&0&2a\mathrm{R}_{23}\\0&1+a(\mathrm{R}-2\mathrm{R}_{22})&2a\mathrm{R}_{23}\\a\mathrm{R}_{23}&a\mathrm{R}_{23}&1+a\mathrm{R}_{11}\end{pmatrix}.$$

We claim that with a suitable orthonormal change of the basis of the plane $span\{e_2,e_2\}=e_1^\perp$ we can always get an orthonormal basis $\{e_1',e_2',e_3'\}$ of T_pM^3 such that $e_1'=e_1$ and $\mathrm{R}_{23}'=\mathrm{Ric}(e_2',e_3')=0$.

Indeed, if $\{e_2',e_3'\}$ is any orthonormal basis of $e_1^\perp$, we can write

$$e_2'=\cos\alpha e_2+\sin\alpha e_3\qquad e_3'=-\sin\alpha e_2+\cos\alpha e_3$$

for some $\alpha\in[0,2\pi)$. Plugging this into the expression of the Ricci tensor, we obtain

$$\begin{aligned}\mathrm{R}_{23}'&=\cos\alpha\sin\alpha(\mathrm{R}_{33}-\mathrm{R}_{22})+(\cos^2\alpha-\sin^2\alpha)\mathrm{R}_{23}\\&=\frac{1}{2}\sin(2\alpha)(\mathrm{R}_{33}-\mathrm{R}_{22})+\cos(2\alpha)\mathrm{R}_{23}\,.\end{aligned}$$

Hence, in order to have $\mathrm{R}'_{23} = 0$, it is sufficient to choose

$$\alpha = \begin{cases} \frac{\pi}{4} & \text{if } \mathrm{R}_{22} = \mathrm{R}_{33}, \\ \frac{1}{2}\arctan(\frac{2\mathrm{R}_{23}}{\mathrm{R}_{22}-\mathrm{R}_{33}}) & \text{otherwise.} \end{cases}$$

The matrix written above represents the symbol $\sigma_\xi(DL_g)$ with respect to a generic orthonormal basis where the first vector coincides with $g(\xi, \cdot)$, so with this change of basis we obtain

$$\sigma_\xi(DL_g) = \begin{pmatrix} 0 & 0 & 0 & 1+a(\mathrm{R}-2\mathrm{R}_{33}) & 1+a(\mathrm{R}-2\mathrm{R}_{22}) & 0 \\ 0 & 0 & 0 & 0 & a\mathrm{R}_{12} & -a\mathrm{R}_{13} \\ 0 & 0 & 0 & a\mathrm{R}_{13} & 0 & -a\mathrm{R}_{12} \\ 0 & 0 & 0 & 1+a(\mathrm{R}-2\mathrm{R}_{33}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+a(\mathrm{R}-2\mathrm{R}_{22}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+a\mathrm{R}_{11} \end{pmatrix}.$$

Hence, the other three eigenvalues of the matrix $\sigma_\xi(DL_g)$ are the diagonal elements of the matrix

$$A = \begin{pmatrix} 1+a(\mathrm{R}-2\mathrm{R}_{33}) & 0 & 0 \\ 0 & 1+a(\mathrm{R}-2\mathrm{R}_{22}) & 0 \\ 0 & 0 & 1+a\mathrm{R}_{11} \end{pmatrix},$$

that is,

$$\lambda_1 = 1 + a(\mathrm{R}-2\mathrm{R}_{33}), \quad \lambda_2 = 1 + a(\mathrm{R}-2\mathrm{R}_{22}), \quad \lambda_3 = 1 + a\mathrm{R}_{11}.$$

Now we recall that, if $\{e_j\}_{j=1,\dots,n}$ is an orthonormal basis of the tangent space, the Ricci quadratic form is the sum of the sectional curvatures,

$$\mathrm{R}_{ii} = \sum_{j\neq i} \mathrm{K}(e_i, e_j)$$

and the scalar curvature R is given by

$$\mathrm{R} = \sum_{i=1}^n \mathrm{R}_{ii} = 2\sum_{i<j} \mathrm{K}(e_i, e_j).$$

Then, in dimension three, denoting by $\alpha = \mathrm{K}(e_2, e_3)$, $\beta = \mathrm{K}(e_1, e_3)$ and $\gamma = \mathrm{K}(e_1, e_2)$, we obtain that the above eigenvalues are

$$\lambda_1 = 1 + 2a\gamma, \qquad \lambda_2 = 1 + 2a\beta, \qquad \lambda_3 = 1 + a(\beta+\gamma).$$

It is now easy to see, by the arbitrariness of the cotangent vector ξ, that these three eigenvalues are positive, hence, the operator L is weakly elliptic, if and only if all the sectional curvatures $\mathrm{K}(X, Y)$ of (M^3, g) satisfy $1 + 2a\mathrm{K}(X, Y) > 0$. If this expression is always positive, then there are exactly three zero eigenvalues, due to the diffeomorphism invariance of the operator L.

Following the work of DeTurck [9, 10] (see also [3]), we show that Problem (3) is equivalent to a Cauchy problem for a strictly parabolic operator, modulo the action of the diffeomorphism group of M^n.

Given a vector field $V \in TM^n$, we will denote the Lie derivative along V with $\mathcal{L}_V$.

Proposition 2.1 (DeTurck's Trick – Existence part [9, 10])**.** *Let (M^n, g_0) be a compact Riemannian manifold.*

Let $L : S^2M^n \to S^2M^n$ and $V : S^2M^n \to TM^n$ be differential operators such that L is geometric, that is, for every smooth diffeomorphism $\psi : M^n \to M^n$ satisfying $L(\psi^* g) = \psi^*(Lg)$. If the linearized operator $D(L - \mathcal{L}_V)_{g_0}$ is strongly elliptic, then the problem

$$\begin{cases} \partial_t g = Lg \\ g(0) = g_0 \end{cases}$$

admits a smooth solution on an open interval $[0, T)$, for some $T > 0$.

The uniqueness part of the statement is more delicate. It exploits an argument relying on existence and uniqueness of solutions of the *harmonic map flow* (see [8, Chapter 3, Section 4]).

Lemma 2.2. *Let $V : S^2M^n \to TM^n$ be "DeTurck's vector field",*

$$\begin{aligned} V^j(g) &= -\, g_0^{jk} g^{pq} \nabla_p \Big(\frac{1}{2} \operatorname{tr}_g(g_0) g_{qk} - (g_0)_{qk} \Big) \\ &= -\, \frac{1}{2} g_0^{jk} g^{pq} \big(\nabla_k (g_0)_{pq} - \nabla_p (g_0)_{qk} - \nabla_q (g_0)_{pk} \big)\,, \end{aligned}$$

where g_0 is a Riemannian metric on M^n and g_0^{jk} is the matrix inverse of g_0.

The following facts hold true.

(i) *The linearization in g_0 of the Lie derivative in the direction V is given by*

$$\begin{aligned} (D\mathcal{L}_V)_{g_0}(h)_{ik} = &\frac{1}{2} g_0^{pq} \nabla_i^0 \{ \nabla_k^0 h_{pq} - \nabla_p^0 h_{qk} - \nabla_q^0 h_{pk} \} \\ &+ \frac{1}{2} g_0^{pq} \nabla_k^0 \{ \nabla_i^0 h_{pq} - \nabla_p^0 h_{qi} - \nabla_q^0 h_{pi} \} + \mathrm{LOT}\,, \end{aligned}$$

where ∇^0 is the Levi–Civita connection of the metric g_0. Hence, its principal symbol in the direction ξ, with respect to an orthonormal basis $\{(\xi)^\flat, e_2, \dots, e_n\}$, is

$$\sigma_\xi((D\mathcal{L}_V)_{g_0}) = \left(\begin{array}{c|c|c} -\mathrm{Id}_n & \begin{matrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix} & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right),$$

expressed in the coordinates

$$(h_{11}, h_{12}, \dots, h_{1n}, h_{22}, h_{33}, \dots, h_{nn}, h_{23}, h_{24}, \dots, h_{n-1,n})$$

of any $h \in S^2M^n$.

(ii) *If $\varphi : (M^n, g) \to (M^n, g_0)$ is a diffeomorphism, then $V((\varphi^{-1})^* g) = \Delta_{g,g_0}\varphi$, where the harmonic map Laplacian with respect to g and g_0 is defined by*

$$\Delta_{g,g_0}\varphi = \operatorname{tr}_g(\nabla(\varphi_*))$$

with ∇ the connection defined on $T^*M^n \otimes \varphi^* TM^n$ using the Levi–Civita connections of g and g_0 (see [8, Chapter 3, Section 4] for more details).

Proof of Theorem 1.1. Our operator $Lg_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}$ is clearly invariant under diffeomorphisms, hence, in order to show the smooth existence part in Theorem 1.1 we only need to check that $D(L - \mathcal{L}_V)_{g_0}$ is strongly elliptic, where V is the vector field defined in Lemma 2.2. By the same lemma, with respect to the orthonormal basis $\{e_1, e'_2, e'_3\}$ introduced above, we have

$$\sigma_\xi(D(L-\mathcal{L}_V)_{g_0}) = \begin{pmatrix} 1 & 0 & 0 & a(\mathrm{R}-2\mathrm{R}_{33}) & a(\mathrm{R}-2\mathrm{R}_{22}) & 0 \\ 0 & 1 & 0 & 0 & a\mathrm{R}_{12} & -a\mathrm{R}_{13} \\ 0 & 0 & 1 & a\mathrm{R}_{13} & 0 & -a\mathrm{R}_{12} \\ 0 & 0 & 0 & 1+2a\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+2a\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+a(\beta+\gamma) \end{pmatrix}.$$

Finally, we conclude that a necessary and sufficient condition for the strong ellipticity of the linear operator $D(L - \mathcal{L}_V)_{g_0}$ is then that all the sectional curvatures of (M^3, g_0) satisfy

$$1 + 2a\mathrm{K}_0(X, Y) > 0\,,$$

for every $p \in M^3$ and vectors $X, Y \in T_pM^3$.

The uniqueness of the solution can be proven exactly in the same way as for the Ricci flow. Let $g_1(t)$ and $g_2(t)$ be solutions of the $\mathrm{RG}^{2,a}$–flow with the same initial data g_0. By parabolicity of the harmonic map flow, introduced by Eells and Sampson in [11], there exist $\varphi_1(t)$ and $\varphi_2(t)$ solutions of

$$\begin{cases} \partial_t \varphi_i = \Delta_{g_i, g_0}\varphi_i \\ \varphi(0) = Id_{M^3} \end{cases}$$

Now we define $\widetilde{g}_i = (\varphi_i^{-1})^* g_i$ and, using that $\frac{d}{dt}\varphi^{-1} = -(\varphi^{-1})_*(\frac{d}{dt}\varphi)$, it is easy to show that both $\widetilde{g}_1$ and $\widetilde{g}_2$ are solutions of the Cauchy problem associated to the strong elliptic operator $L - \mathcal{L}_V$ and starting at the same initial metric g_0, hence they must coincide, by uniqueness. By point (ii) of Lemma 2.2, the diffeomorphisms φ_i also coincide because they are both the one–parameter group generated by $-V(\widetilde{g}_1) = -V(\widetilde{g}_2)$. Finally, $g_1 = \varphi_1^*(\widetilde{g}_1) = \varphi_2^*(\widetilde{g}_2) = g_2$ and this concludes the proof of Theorem 1.1. □

We want to spend few words about why the vector field $V(g)$ in Lemma 2.2 is a natural choice.

The linearization of a geometric operator (up to lower order terms) always contains a Lie derivative term of the metric $\mathcal{L}_{W(g,h)}g$, due to the invariance by diffeomorphism, where

$$W^j(g, h) = -g^{pq}\big(D(\Gamma^j_{pq})\big)_g(h)\,.$$

This implies the presence of a nonvoid kernel, hence, a degeneracy of the operator. In order to "kill" such term, one can subtract from the operator a Lie derivative of the metric with respect to some vector field $V(g)$ (also depending on the metric g) such that its linearized (up to lower order terms) is precisely $W(g, h)$. A natural choice is then the vector field

$$V^j(g) = -g^{pq}\big(\Gamma^j_{pq} - \overline{\Gamma}^j_{pq}\big)$$

where $\overline{\Gamma}^j_{pq}$ are the Christoffel symbols of some fixed metric (for instance, the initial one g_0). It is easy to see, after some computation, that this vector field $V(g)$ coincides with "DeTurck's vector field" defined in Lemma 2.2 (for more details see [8, Sections 3.2–3.3]).

Finally, we mention that after solving the modified (made nondegenerate) evolution problem, the solution of the original Cauchy problem is recovered by means of the 1–parameter group of diffeomorphisms generated by the vector field $V(g)$ (this argument is contained in the proof of Proposition 2.1).

3. **Some remarks.** In order to continue the study of this flow, it is natural to ask if there exist some Perelman–type entropy functionals which are monotone along the flow, as proposed by Tseytlin in [27]; another possibility is to investigate the evolution of the curvatures along the flow under the hypothesis of Theorem 1.1 and try to find (if there are) some preserved conditions in order to explore the long–time behavior and the structure of the singularities at the maximal time of existence.

The analysis leading to Theorem 1.1 can be repeated step–by–step for the operator L_0, given by

$$L_0 g = -a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,,$$

with associated $\mathrm{RG}_0^{2,a}$–flow

$$\partial_t g_{ik} = -a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,.$$

In this case, along the same lines, the existence of a unique smooth evolution of an initial metric g_0 is guaranteed as long as

$$a\mathrm{K}_0(X,Y) > 0$$

for every point $p \in M^3$ and vectors $X, Y \in T_pM^3$. That is, if $a > 0$ in case of an initial metric with positive curvature and if $a < 0$ in the negative curvature case.

For geometrical purposes, this flow could be more interesting than the $\mathrm{RG}^{2,a}$–flow, in particular because of its scaling invariance, which is not shared by the latter.

Another possibility in this direction is given by the *squared Ricci flow*, that is, the evolution of an initial metric g_0 according to

$$\partial_t g_{ik} = -a\mathrm{R}_{ij}\mathrm{R}^j_k\,,$$

which is scaling invariant and can be analyzed analogously, or a "mixing" with the Ricci flow (non scaling invariant)

$$\partial_t g_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ij}\mathrm{R}^j_k\,,$$

for any constant $a \in \mathbb{R}$, as before.

Indeed, the principal symbol of the operator $H = \mathrm{R}_{ij}\mathrm{R}^j_k$ can be computed as in Section 2. The linearized of the operator H around a metric g, for every $h \in S^2M^n$, is given by

$$\begin{aligned}
DH_g(h)_{ik} &= \mathrm{R}^j_k D\mathrm{Ric}_g(h)_{ij} + \mathrm{R}^j_i D\mathrm{Ric}_g(h)_{jk} + \mathrm{LOT}\\
&= \frac{1}{2}\mathrm{R}^j_k\Big(-\Delta h_{ij} - \nabla_i\nabla_j\,\mathrm{tr}(h) + \nabla_i\nabla^m h_{mj} + \nabla_j\nabla^m h_{im}\Big)\\
&\quad+\frac{1}{2}\mathrm{R}^j_i\Big(-\Delta h_{jk} - \nabla_j\nabla_k\,\mathrm{tr}(h) + \nabla_j\nabla^m h_{mk} + \nabla_k\nabla^m h_{jm}\Big) + \mathrm{LOT}\,.
\end{aligned}$$

Hence, the principal symbol in the direction of the cotangent vector ξ, as before, is

$$\sigma_\xi(DH_g)(h)_{ik} = -\frac{1}{2}\mathrm{R}^j_k\Big(\xi^m\xi_m h_{ij} + \xi_i\xi_j\,\mathrm{tr}(h) - \xi_i\xi^m h_{jm} - \xi_j\xi^m h_{im}\Big)$$

$$
\begin{aligned}
& -\frac{1}{2}\mathrm{R}_i^j\Big(\xi^m\xi_m h_{jk}+\xi_j\xi_k\,\mathrm{tr}(h)-\xi_j\xi^m h_{km}-\xi_k\xi^m h_{jm}\Big)\\
= & -\frac{1}{2}\mathrm{R}_k^j\Big(h_{ij}+\delta_{1i}\delta_{1j}\,\mathrm{tr}(h)-\delta_{1i}h_{1j}-\delta_{1j}h_{1i}\Big)\\
& -\frac{1}{2}\mathrm{R}_i^j\Big(h_{jk}+\delta_{1j}\delta_{1k}\,\mathrm{tr}(h)-\delta_{1j}h_{1k}-\delta_{1k}h_{1j}\Big)\\
= & -\frac{1}{2}\Big(\mathrm{R}_{1k}(\delta_{1i}\,\mathrm{tr}(h)-h_{1i})+\mathrm{R}_{1i}(\delta_{1k}\,\mathrm{tr}(h)-h_{1k})\Big)\\
& -\frac{1}{2}\Big(\mathrm{R}_k^j(h_{ij}-\delta_{1i}h_{1j})+\mathrm{R}_i^j(h_{jk}-\delta_{1k}h_{1j})\Big)\,,
\end{aligned}
$$

where $\xi = g(e_1,\cdot)$ and $\{e_i\}$ is an orthonormal basis of T_pM^n.

Again, by specifying the initial metric to be g_0 and diagonalizing the restriction of the Ricci tensor (which is still symmetric) to the hyperspace $e_1^\perp$, we can find an orthonormal basis $\{e'_2,\dots,e'_n\}$ of $e_1^\perp$ such that $\mathrm{Ric}(e'_i,e'_k)=0$ if $i\neq k$, the principal symbol of the operator H, computed in the basis $\{e_1,e'_2,\dots,e'_n\}$, is described by

$$
\begin{aligned}
\sigma_\xi(DH_{g_0})(h)_{11} &= -\frac{1}{2}\Big(2\mathrm{R}_{11}\sum_{j=2}^n h_{jj}\Big)\\
\sigma_\xi(DH_{g_0})(h)_{1k} &= -\frac{1}{2}\Big(2\mathrm{R}_{1k}h_{kk}+\mathrm{R}_{1k}\sum_{j\neq 1,\,k}h_{jj}+\sum_{j\neq 1,\,k}\mathrm{R}_{1j}h_{jk}\Big)\\
\sigma_\xi(DH_{g_0})(h)_{kk} &= -\frac{1}{2}\Big(2\mathrm{R}_{kk}h_{kk}\Big)\\
\sigma_\xi(DH_{g_0})(h)_{ik} &= -\frac{1}{2}\Big((\mathrm{R}_{kk}+\mathrm{R}_{ii})h_{ik}\Big)
\end{aligned}
$$

for every $i,k\in\{2,\dots,n\}$ with $i\neq k$.

It is easy to see that the matrix associated to $\sigma_\xi(DH_{g_0})$ expressed in the coordinates

$$(h_{11},h_{12},\dots,h_{1n},h_{22},h_{33},\dots,h_{nn},h_{23},h_{24},\dots,h_{n-1,n})$$

of S^2M^n is upper triangular with n zeroes on the first n diagonal elements, then the next $(n-1)$ ones are the values $-\mathrm{R}_{kk}$ for $k\in\{2,\dots,n\}$ and finally, the last $(n-1)(n-2)/2$ ones are given by $-(\mathrm{R}_{ii}+\mathrm{R}_{kk})/2$ for every $i,k\in\{2,\dots,n\}$ with $i\neq k$.

Now, applying Proposition 2.1 with the same vector field V of Lemma 2.2, we obtain that the squared Ricci flow

$$\partial_t g_{ik} = -a\mathrm{R}_{ij}\mathrm{R}_{lk}g^{jl}\,,$$

has a unique smooth solution for short time, when $a>0$ for every initial manifold (M^n,g_0) with positive Ricci curvature and when $a<0$, for every initial manifold (M^n,g_0) with negative Ricci curvature.

We conclude this discussion mentioning the *cross curvature flow*, introduced by Chow and Hamilton in [7], which belongs to this "family" of quadratic flows. The short–time existence and uniqueness of a smooth evolution of every initial metric of a three–dimensional manifold with curvature not changing sign, was established by Buckland in [4].

Note. *Recently, Gimre, Guenther and Isenberg extended the short–time existence of the $RG^{2,a}$–flow, Theorem 1.1, to any dimensions in* [18] *(see also* [16]*).*

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