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Research Paper

The field of moduli of a divisor on a rational curve

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ABSTRACT

Let k be a field with algebraic closure $\bar{k}$ and $D \subset \mathbb{P}^1_{\bar{k}}$ a reduced, effective divisor of degree $n \geq 3$, write k_D for the field of moduli of D. A. Marinatto proved that when n is odd, or $n = 4$, D descends to a divisor on $\mathbb{P}^1_{k_D}$.
We analyze completely the problem of when D descends to a divisor on a smooth, projective curve of genus 0 on k_D, possibly with no rational points. In particular, we study the remaining cases $n \geq 6$ even, and we obtain conceptual proofs of Marinatto's results and of a theorem by B. Huggins about the field of moduli of hyperelliptic curves.

1. Introduction

We work over a field k with algebraic closure $\bar{k}$ and separable closure $k^s \subset \bar{k}$. A rational curve is a smooth, projective, geometrically connected curve of genus 0.

The field of moduli. Let $D \subset \mathbb{P}^1_{\bar{k}}$ be an effective reduced divisor. If k is perfect, consider the subgroup $H \subset \mathrm{Gal}(\bar{k}/k)$ of elements σ such that there exists an automorphism of $\mathbb{P}^1_{\bar{k}}/\bar{k}$ mapping D to $\sigma(D)$. The *field of moduli* k_D of D is the subfield of $\bar{k}$ of elements fixed by H. It is possible to generalize this definition to arbitrary fields, see §3.

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If $\bar{k}/k'/k$ is a sub-extension and there exists a rational curve P over k' with a divisor $D_0 \subset P$ such that $(P_{\bar{k}}, D_{0,\bar{k}}) \simeq (\mathbb{P}^1_{\bar{k}}, D)$, we say that k' is a field of definition for $(\mathbb{P}^1_{\bar{k}}, D)$, or that $(\mathbb{P}^1_{\bar{k}}, D)$ descends to k'. Furthermore, we say that D descends to $\mathbb{P}^1_{k'}$ if the above holds with $P \simeq \mathbb{P}^1_{k'}$. The field of moduli is contained in every field of definition.

In our joint paper with A. Vistoli [7] we have given a definition of the field of moduli which works even when the base field k is not perfect. Here, we do not assume that k is perfect as well.

Marinatto's results. Let n be the degree of D, assume $n \geq 3$. Assuming k perfect of characteristic $\neq 2$, A. Marinatto [23] showed that, if n is odd or $n = 4$, then D descends to a divisor of $\mathbb{P}^1_{k_D}$. Hence, if n is odd or $n = 4$, k_D is a field of definition for $(\mathbb{P}^1_{\bar{k}}, D)$.

For every even integer $n \geq 6$, Marinatto constructed examples of divisors $D \subset \mathbb{P}^1_{\bar{k}}$ of degree n which do not descend to $\mathbb{P}^1_{k_D}$. His examples descend to a non-trivial rational curve over k_D, though, hence k_D is a field of definition for $(\mathbb{P}^1_{\bar{k}}, D)$ in these cases.

Our results. Because of this, the problem of fields of definition versus fields of moduli for n even and ≥ 6 is still open. We solve it, and generalize Marinatto's results for n odd and $n = 4$ to an arbitrary base field. The following theorem is a summary of the various results we obtain.

Theorem 1. *Let k be any field, $n \geq 3$ an integer and $D \subset \mathbb{P}^1_{\bar{k}}$ a reduced, effective divisor of degree n. Let $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D) \subset \mathrm{PGL}_2(\bar{k})$ be the subgroup of elements $g \in \mathrm{PGL}_2(\bar{k})$ with $g(D) = D$. Consider the following conditions.*

(1) $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ *is not cyclic of degree even and prime with* $\mathrm{char}\, k$.
(2) The 2-torsion $\mathrm{Br}(k_D)[2]$ *of the Brauer group of k_D is trivial.*
(3) n is odd.
(4) $n = 4$.
(5) $n = 6$.

If at least one of the conditions (1)-(4) holds, then D descends to a divisor of $\mathbb{P}^1_{k_D}$. In particular, if $\mathrm{char}\, k = 2$ *then D always descends to $\mathbb{P}^1_{k_D}$ by condition (1).*

If (5) holds, then D descends to a divisor of some rational curve over k_D.

On the other hand, if k is a field with $\mathrm{char}\, k \neq 2$, $\mathrm{Br}(k)[2] \neq 0$ *and $n \geq 8$ is even, then we may choose D of degree n such that the field of moduli is k and D does not descend to any rational curve over k, i.e. k is not a field of definition for $(\mathbb{P}^1_{\bar{k}}, D)$.*

For $n = 4$ or n odd Theorem 1 is a generalization of Marinatto's theorem to arbitrary fields. For n even and ≥ 6, it is new. In order to prove all the various sub-statements of Theorem 1, we give a characterization of the divisors $D \subset \mathbb{P}^1_{\bar{k}}$ such that the field of moduli is a field of definition for $(\mathbb{P}^1_{\bar{k}}, D)$, see Theorem 31.

We study analogous problems over $\mathbb{P}^2$ in [5], [2] and [6].

Discussion of proofs. Marinatto's proof for n odd is based on a case-by-case analysis of the finite subgroups of PGL_2. Our approach is more conceptual: for n odd, we only use general techniques for problems about fields of moduli (not specific to divisors or rational curves) plus a parity counting. For n even and ≥ 8, we also need to make some elementary geometric arguments. If $\mathrm{Aut}(\mathbb{P}^1, D)$ is not cyclic of degree even and prime with $\operatorname{char} k$, we have a conceptual argument based on the Riemann-Hurwitz formula; from this case we get the cases $n = 4, 6$, too.

Marinatto also uses a theorem of Huggins [20, Theorem 5.3] about fields of moduli of hyperelliptic curves. Huggins' proof is based on a case-by-case analysis of the finite subgroups of PGL_2, too. Our argument with Riemann-Hurwitz directly yields a short, conceptual proof of Huggins' result. Our version is slightly more general than the original one since we do not assume the base field to be perfect.

While we use the general framework constructed in [7], we mention the fact that, in dimension 1 (which is our case), the main ideas are due to P. Dèbes and M. Emsalem [14].

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Conventions. We work over a field k with algebraic closure $\bar{k}$ and separable closure $k^s \subset \bar{k}$. A variety over k is a geometrically integral, separated scheme of finite type over k.

We work with the fppf topology. In particular, when we say "torsor" we implicitly mean "torsor for the fppf topology", and analogously for sheaves and stacks.

Sometimes we write, with an abuse of notation, k instead of $\operatorname{Spec} k$ in order to make the paper easier to read. For instance, if S is a scheme over k, we might write $S \times_k \bar{k}$ for $S \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$.

Given a variety X with a divisor $D \subset X$ over k (or $\bar{k}$), we write $\underline{\mathrm{Aut}}(X, D)$ for the fppf sheaf of automorphisms of X mapping D to itself; we work under the assumption that $\underline{\mathrm{Aut}}(X, D)$ is representable by a finite group scheme over k (or $\bar{k}$); this is true for $X = \mathbb{P}^1_{\bar{k}}$ and D reduced of degree ≥ 3, see Lemma 24. More generally, if (X', D') is another pair, $\underline{\mathrm{Isom}}((X, D), (X', D'))$ is the fppf sheaf of isomorphisms $\phi : X \xrightarrow{\sim} X'$ such that $\phi(D) = D'$.

If k'/k is a field extension and X is a variety over k' with a divisor $D \subset X$, we say that (X, D) descends to k if there exists a pair (Y, E) over k with $(Y, E)_{k'} \simeq (X, D)$.

2. Generalities

In this section, we recall some general facts about torsors, cohomology, gerbes and stacks.

2.1. Torsors and cohomology

Recall that, if G is a group scheme of finite type over k, the *classifying stack* $\mathscr{B}_k G$ [18, Chapitre III, §1.4.4] is defined as follows: if S is a scheme over k, then $\mathscr{B}_k G(S)$ is the groupoid of G-torsors over S; G-torsors are sometimes called principal G-bundles. By definition, the non-abelian cohomology set $\mathrm{H}^1(k, G)$ is the set of fppf G-torsors over k [18, Chapitre III, Définition 2.4.2], i.e. it is $\mathscr{B}_k G(k)/\sim$, and it coincides with classical fppf cohomology in the abelian case [18, Chapitre IV, Théorème 3.4.2].

If G is finite étale, every G-torsor $T \to S$ is trivialized by an étale covering $S' \to S$ (i.e. $S' = T$), hence G-torsors for the fppf topology coincide with G-torsors for the étale topology. Furthermore, if G is abelian, finite and étale, fppf cohomology coincides with étale cohomology [27, Tag 0DDU], and over $\operatorname{Spec} k$ étale cohomology coincides with Galois cohomology because they are both the derived functor of $\Gamma(\operatorname{Spec} k, G) = G(k) = G(k^s)^{\mathrm{Gal}(k^s/k)}$.

In general, we use fppf cohomology. Most of the time, though, we work with finite étale group schemes, so that it coincides with étale/Galois cohomology.

2.2. Regarding stacks

We use stacks, for several reasons. The first is that stacks are a natural language for our problem. Studying the fields of definition of a variety X with some additional structure ξ (such as a divisor) means studying twisted forms of (X, ξ), since a field of definition is just a field where a twisted form is defined, and the object which classifies such twisted forms is usually a stack, more precisely a gerbe. We are going to explain this in detail only in the case in which ξ is a reduced, effective divisor, which is what we need in the present article, see §3.1. We do not want to get into an excessively general discussion to explain why twisted forms are classified by a stack in other cases: the interested reader might want to read [7]. See also [4, §4] for a similar treatment of algebraic cycles.

Another way to see the usefulness of gerbes is the fact that they are the main building block of the theory non-abelian cohomology of Grothendieck–Giraud [18, Chapitre IV, Definition 3.1.1], and in the abelian case the problem of fields of moduli vs. fields of definition has a natural interpretation in terms of second cohomology [11]. The earliest constructions of non-abelian cohomology in classical topology are due to P. Dedecker [15]. The relevance of non-abelian cohomology in the sense of Grothendieck–Giraud to the study of fields of definition was first noticed by M. Fried [17, p. 58] in the case of ramified Galois coverings of $\mathbb{P}^1$, and his observation was developed by P. Dèbes, J.–C. Douai and M. Emsalem [13]. It is actually true in many other cases [7, Examples 5.2].

Without entering into details, let us sketch the reason why non-abelian second cohomology sets, and hence gerbes, are a natural tool for studying fields of moduli. Suppose that k is perfect, and that (X, ξ) is a variety with structure over $\bar{k}$ and with field of moduli k. Suppose that the group of automorphisms G of (X, ξ) is finite. Since k is the field of moduli, for every $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have an isomorphism $(X, \xi) \simeq \sigma^*(X, \xi)$. If G is abelian, this induces an action of $\mathrm{Gal}(\bar{k}/k)$ on G, and two isomorphisms $(X,\xi) \simeq \sigma^*(X,\xi)$ differ by an element of G. This datum can be used to define a 2-cocycle whose associated cohomology class in $\mathrm{H}^2(k,G)$ is the obstruction for (X,ξ) to be defined over k, see e.g. [11].

If G is not abelian, though, the cohomological problem is much more complicated. The main reason is that we do not have a Galois action on G, nor the 2-cohomology set $\mathrm{H}^2(k,G)$. It is still possible to use abelian cohomology by looking at the center of G [11], but it is more conceptually clear to use non-abelian cohomology in the sense of Grothendieck–Giraud [18], see e.g. [12] [13]. By definition [18, Chapitre IV, Definition 3.1.1], the non-abelian H^2 is a set of gerbes. We are not aware of definitions of non-abelian cohomology in degree 2 avoiding the language of gerbes and stacks.

Secondly, if we want to study inseparable field extensions, we have to replace Galois cohomology (as we explained above, this coincides with the étale cohomology of a point) with fppf cohomology: finite, inseparable field extensions are not coverings in the étale topology, but they are coverings in the fppf topology. If we use stacks instead of classical cohomology, this is automatic: the standard formalism of algebraic stacks already uses the fppf topology rather than the étale one, see e.g. [27, Tag 026O].

Finally, even though it is not strictly relevant for this specific article, stacks allow us to generalize to dimension ≥ 2 a foundational result for fields of moduli of curves by P. Dèbes and M. Emsalem [14, Corollary 4.3 (c)], see [7, Theorem 5.4].

2.3. Gerbes

Recall that a gerbe $\mathscr{G}$ over k is a non-empty stack over k such that any two objects of $\mathscr{G}$ are fppf-locally isomorphic [18, Chapitre III, §2] [26, Chapter 12]. For us, gerbes are always algebraic stacks of finite type over k. We say that $\mathscr{G}$ is finite (resp. étale, resp. abelian) if the inertia stack $\mathrm{I}_{\mathscr{G}}$ [27, Tag 050P] [26, Definition 8.1.17] is finite (resp. unramified, resp. abelian) over $\mathscr{G}$.

If $\mathscr{G}$ is an abelian gerbe over k, there exists a sheaf of abelian groups A over k such that, for every scheme S over k and every $s \in \mathscr{G}(S)$, the group of automorphisms of s is $A(S)$ [18, Chapitre IV, Proposition 2.2.3.4]. Since we are assuming that $\mathscr{G}$ is an algebraic stack, A is representable by a group scheme. The group scheme A is called the *band* of A, and $\mathrm{H}^2(k,A)$ classifies gerbes with band isomorphic to A [18, Chapitre IV, Théorème 3.4.2] [26, Theorem 12.2.8].

Consider for instance the case of a reduced divisor D on $\mathbb{P}^1_{\bar{k}}$; assume k perfect so that the field of moduli k_D is defined in the usual way. If $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}},D)$ is abelian, then there is a natural action of $\mathrm{Gal}(\bar{k}/k_D)$ on $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}},D)$: this defines a group scheme A over k_D. As we will see in §3.1, twisted forms of $(\mathbb{P}^1_{\bar{k}},D)$ are classified by a gerbe $\mathscr{G}$ over the field of moduli k_D. The group scheme A coincides with the band of $\mathscr{G}$: there is a unique group scheme which describes the automorphism groups of all twisted forms, and it is defined over k_D regardless of the existence of a model of $(\mathbb{P}_{\bar{k}},D)$ on k_D. This is false in the non-abelian case: different twisted forms will define different Galois actions on the automorphism group.

If the gerbe is not abelian, there is still a concept of band [18, Chapitre IV, §2.2], but the definition is more complex than in the abelian case, and it is not a group scheme (abelian bands coincide with sheaves of abelian groups, though [18, Chapitre IV, Proposition 1.2.3]). By definition, non-abelian second cohomology in the sense of Grothendieck–Giraud is the set of gerbes with a given band [18, Chapitre IV, Définition 3.1.1]. We are only going to use bands in the abelian case, so we do not need the general definition. In the non-abelian case, we just use gerbes, regardless of their band.

Classifying stacks are gerbes [18, Chapitre III, Exemple 2.1.2]. A finite gerbe $\mathscr{G}$ over k is isomorphic to the classifying stack of some finite group scheme if and only if $\mathscr{G}(k) \neq \emptyset$ [18, Chapitre III, Corollaire 2.2.6]; gerbes with rational sections are called *neutral* gerbes. As long as gerbes are neutral, working with gerbes is essentially equivalent to working directly with torsors: the main reason why they are useful is precisely the fact that they allow us to work even when there is no rational section.

If $\mathscr{G}$ is a finite étale gerbe over k, then $\mathscr{G}(k^s) \neq \emptyset$, as we will see in Lemma 22. Because of this, $\mathscr{G}_{k^s} \simeq \mathscr{B}_{k^s} G$ for some finite group G, and one should think of a morphism $S \to \mathscr{G}$ as some datum over k whose base change to k^s is the datum of a G-torsor (or G-covering) $T \to S_{k^s}$, even though it might happen that $T \to S_{k^s}$ does *not* descend to a G-torsor of S. For instance, if S is an open subset of a smooth variety S' and $T \to S_{k^s}$ extends to a (possibly ramified) G-covering $T' \to S'_{k^s}$, the fact that $S \to \mathscr{G}$ is defined over k tells us automatically that the ramification locus of $T' \to S'_{k^s}$ descends to a closed subset of S' even if the covering does not descend, see e.g. [3, Corollary A.2].

3. The field of moduli of a divisor

We want now to define the field of moduli of a divisor of $\mathbb{P}^1$ over $\bar{k}$ when the base field k is not perfect. In hindsight it turns out that, if $\operatorname{char} k \neq 0$ and in most cases with $\operatorname{char} k = 0$ (with the notable exception of $k = \mathbb{R}$), our definition coincides with the intersection of the fields of definition [5, Appendix A]. However, we think that using this as a definition is not convenient. We find it more useful to generalize the definition of the field of moduli based on Galois theory. Later, we will give another elementary characterization of the field of moduli, see Proposition 25.

Let X be a proper variety over $\bar{k}$ with an effective, reduced divisor $D \subset X$. Under this hypothesis, the sheaf $\underline{\mathrm{Aut}}(X, D) \subset \underline{\mathrm{Hom}}(X, X)$ of automorphisms of X mapping D to itself is a group algebraic space locally of finite type over $\bar{k}$ [27, Tag 0D1C]. We work under the assumption that $\underline{\mathrm{Aut}}(X, D)$ is finite over $\bar{k}$, so that $\underline{\mathrm{Aut}}(X, D)$ is a finite group scheme [27, Tag 06NH]. Write $G = \underline{\mathrm{Aut}}(X, D)$.

Now consider the tensor product

$$\bar{k} \otimes_k \bar{k} = \bar{k} \otimes_{k^s} (k^s \otimes_k k^s) \otimes_{k^s} \bar{k},$$

it is a product of artinian local algebras of the form $\bar{k} \otimes_{k^s} \bar{k}$, since $k^s \otimes_k k^s$ is a product of copies of k^s naturally indexed by the elements of $\mathrm{Gal}(k^s/k)$.

We have two projections $p_1, p_2 : \operatorname{Spec} \bar{k} \otimes_k \bar{k} \to \operatorname{Spec} \bar{k}$; consider the sheaf of isomorphisms

$$\underline{\mathrm{Isom}}(p_1^*(X,D), p_2^*(X,D)) \to \operatorname{Spec} \bar{k} \otimes_k \bar{k}.$$

By [27, Tag 0D1C], this sheaf is in fact an algebraic space locally of finite presentation over $\operatorname{Spec} \bar{k} \otimes_k \bar{k}$. Denote by $R \subset \operatorname{Spec} \bar{k} \otimes_k \bar{k}$ its scheme theoretic image [27, Tag 01R5].

Lemma 2. *If k is perfect and we identify the $\bar{k}$-points of $\operatorname{Spec} \bar{k} \otimes_k \bar{k}$ with $\mathrm{Gal}(\bar{k}/k)$, then $R \subset \operatorname{Spec} \bar{k} \otimes_k \bar{k} = \mathrm{Gal}(\bar{k}/k) \times \operatorname{Spec} \bar{k}$ is the subgroup of elements $\sigma \in \mathrm{Gal}(\bar{k}/k)$ such that $\sigma^*(X,D) \simeq (X,D)$.*

Proof. An element $\sigma \in \mathrm{Gal}(\bar{k}/k)$ is in R if and only if the fiber over the corresponding point of $\operatorname{Spec} \bar{k} \otimes_k \bar{k}$ is non-empty. The fiber is $\underline{\mathrm{Isom}}(\sigma^*(X,D),(X,D))$, which is non-empty if and only if there exists an isomorphism $\sigma^*(X,D) \simeq (X,D)$ since $\underline{\mathrm{Isom}}(\sigma^*(X,D),(X,D))$ is locally of finite type over $\bar{k}$ [27, Tag 0D1C]. □

Lemma 3. *The scheme $R \subset \operatorname{Spec} \bar{k} \times_k \operatorname{Spec} \bar{k}$ defines an equivalence relation in the sense of [27, Tag 043B] on $\bar{k}$.*

Proof. We have to check that, for every scheme T over k, $R(T) \subset \operatorname{Spec} \bar{k}(T) \times \operatorname{Spec} \bar{k}(T)$ is an equivalence relation on the set $\operatorname{Spec} \bar{k}(T)$.

Let us show that $\underline{\mathrm{Isom}}(p_1^*(X,D), p_2^*(X,D))$ is universally closed over $\operatorname{Spec} \bar{k} \otimes_k \bar{k}$. To check this, we may pass to the reduced structure of $\operatorname{Spec} \bar{k} \otimes_k \bar{k}$, which amounts to replacing k with its perfect closure. Now let k'/k be a finite Galois extension with a model (X', D') of (X,D) over k', it is enough to prove that $\underline{\mathrm{Isom}}(p_1^*(X',D'), p_2^*(X',D'))$ is finite over $\operatorname{Spec} k' \otimes_k k'$. This follows from the fact that $\operatorname{Spec} k' \otimes_k k'$ is a disjoint union of copies of $\operatorname{Spec} k'$ and the fiber over each of them is either empty or an $\underline{\mathrm{Aut}}(X',D')$-torsor, and by assumption $\underline{\mathrm{Aut}}(X',D')$ is finite over k' (since $\underline{\mathrm{Aut}}(X',D')_{\bar{k}} = \underline{\mathrm{Aut}}(X,D)_{\bar{k}}$ is finite over $\bar{k}$).

Hence, $\underline{\mathrm{Isom}}(p_1^*(X,D), p_2^*(X,D))$ maps surjectively on R [27, Tag 0AH6], which in turn implies that $(f,g) \in R(T)$ if and only if $\underline{\mathrm{Isom}}(f^*(X,D), g^*(X,D))$ maps a surjectively on T.

Clearly, $\underline{\mathrm{Isom}}(f^*(X,D), f^*(X,D)) \to T$ is surjective, hence $(f,f) \in R(T)$. Furthermore, $\underline{\mathrm{Isom}}(f^*(X,D), g^*(X,D))$ is isomorphic to $\underline{\mathrm{Isom}}(g^*(X,D), f^*(X,D))$ over T, hence $(f,g) \in R(T)$ if and only if $(g,f) \in R(T)$. Finally, we have a natural morphism

$$\underline{\mathrm{Isom}}(f^*(X,D), g^*(X,D)) \times \underline{\mathrm{Isom}}(g^*(X,D), h^*(X,D)) \to \underline{\mathrm{Isom}}(f^*(X,D), h^*(X,D))$$

over T, hence if $(f,g),(g,h) \in R(T)$ then $(f,h) \in R(T)$. □

Now consider the quotient sheaf $\operatorname{Spec} \bar{k}/R$ [27, Tag 044H], by [27, Tag 04S6] it is an algebraic space over $\operatorname{Spec} k$. The natural morphism $\operatorname{Spec} \bar{k} \to \operatorname{Spec} \bar{k}/R$ is surjective, hence $\operatorname{Spec} \bar{k}/R$ has only one point. By [27, Tag 06NH], this implies that $\operatorname{Spec} \bar{k}/R = \operatorname{Spec} k_D$ is the spectrum of a k-algebra k_D with a unique prime ideal. Since $\operatorname{Spec} \bar{k} \to \operatorname{Spec} \bar{k}/R$ is flat, then $k_D \to \bar{k}$ is injective and hence k_D is reduced, i.e. it is a subfield $k \subset k_D \subset \bar{k}$.

Definition 4. The field k_D/k is the *field of moduli* of (X, D).

Lemma 5. *If k is perfect, our definition of the field of moduli coincides with the usual one given by Galois theory.*

Proof. This is a direct consequence of Lemma 2. $\square$

Lemma 6. *If (X, D) descends to a subextension k'/k, then k' contains the field of moduli k_D.*

Proof. Since (X, D) descends to k', the restriction of $\underline{\mathrm{Isom}}(p_1^*(X, D), p_2^*(X, D))$ to $\operatorname{Spec} \bar{k} \otimes_{k'} \bar{k} \subset \operatorname{Spec} \bar{k} \otimes_k \bar{k}$ is surjective over $R' = \operatorname{Spec} \bar{k} \otimes_{k'} \bar{k}$, hence $R' \subset R$ and we have an induced morphism $\operatorname{Spec} k' = \operatorname{Spec} \bar{k}/R' \to \operatorname{Spec} k_D = \operatorname{Spec} \bar{k}/R$. $\square$

Corollary 7. *The field of moduli k_D is finite over k.*

Proof. Since X is of finite type over $\bar{k}$, then (X, D) descends to a finite extension k' of k. To show this, we can reduce to the case in which X is affine by choosing an affine covering; if X is affine, we can define both X and D with a finite number of polynomials, and choose k' as the extension of k generated by the coefficients of all these polynomials. Since k' contains k_D, then k_D is finite over k as well. $\square$

Lemma 8. *The restrictions of $p_1^*(X, D)$, $p_2^*(X, D)$ to $\operatorname{Spec} \bar{k} \otimes_{k_D} \bar{k} \subset \operatorname{Spec} \bar{k} \otimes_k \bar{k}$ are isomorphic over $\operatorname{Spec} \bar{k} \otimes_{k_D} \bar{k}$.*

Proof. For simplicity, replace k with k_D, so that k is the field of moduli. Assume first that k is separably closed.

Let k'/k be a finite extension with a model (X', D') of (X, D) over k'. Write $\Lambda = k' \otimes_k k'$, since k is separably closed then Λ is an artinian local ring with residue field k'. It is enough to prove that

$$I = \underline{\mathrm{Isom}}(p_1^*(X', D'), p_2^*(X', D')) \to \operatorname{Spec} \Lambda$$

has a global section.

We have that $G' = \underline{\mathrm{Aut}}(X', D')$ is finite, $\operatorname{Spec} \Lambda$ is artinian and the restriction of I to $\operatorname{Spec} k' \subset \operatorname{Spec} \Lambda$ is isomorphic to G': since the topological space of $\operatorname{Spec} \Lambda$ has a unique point, this implies that the underlying topological space of I has a finite number of points, and they are all closed. In particular, by [27, Tag 06NH] I is a scheme locally of finite type over Λ. Since it has a finite number of points and it is locally of finite type over Λ, then it is actually finite over Λ.

There is a preferred point $p \in I(k')$ in the residue fiber of I corresponding to the identity of (X', D'). Consider its local ring $\mathscr{O}_{I,p}$: it is an artinian local ring with a homomorphism $\Lambda \to \mathscr{O}_{I,p}$.

Let $\mathfrak{m} \subset \Lambda$ be the maximal ideal of Λ. Since the restriction of I to the closed fiber is G' and p corresponds to the identity of G', whose local ring in G' is reduced, we get that $\mathscr{O}_{I,p}/\mathfrak{m}\mathscr{O}_{I,p} \simeq k'$ is reduced. By Nakayama's lemma, this implies that $\Lambda \to \mathscr{O}_{I,p}$ is surjective.

Since k is the field of moduli, $I \to \operatorname{Spec} \Lambda = \operatorname{Spec} k' \otimes_k k'$ is scheme–theoretically surjective. We have a transitive action of G' on I by left composition; since the action is transitive the composition $G' \times \operatorname{Spec} \mathscr{O}_{I,p} \to I \to \operatorname{Spec} \Lambda$ is scheme–theoretically surjective as well, i.e. $\Lambda \to k'[G'] \otimes_{k'} \mathscr{O}_{I,p}$ is injective. This implies that $\Lambda \to \mathscr{O}_{I,p}$ is injective: if $N \subset \Lambda$ is the kernel, the action of G' on the restriction $I_{\Lambda/N}$ defines a factorization of $G' \times \operatorname{Spec} \mathscr{O}_{I,p} \to I \to \operatorname{Spec} \Lambda$ through $\operatorname{Spec} \Lambda/N$, hence $N = 0$ since $\Lambda \to k'[G'] \otimes_{k'} \mathscr{O}_{I,p}$ is injective.

As a consequence, $\Lambda = \mathscr{O}_{I,p}$ and we get the desired section $\operatorname{Spec} \Lambda \to I$.

In general, notice that $\bar{k} \otimes_{k^s} \bar{k}$ is the colimit of algebras of the form $h^{\text{perf}} \otimes_h h^{\text{perf}}$ where h/k is a finite Galois extension and h^{perf} is the perfect closure. In fact, the separable closure $k^s = \bigcup_{h/k \text{ Gal.}} h$ is the union (and colimit) of the finite Galois extensions h/k, the algebraic closure $\bar{k} = \bigcup_{h/k \text{ Gal.}} h^{\text{perf}}$ is the union (and colimit) of their perfect closures h^{perf} and tensor product commutes with colimits. The fact that $\bar{k} = \bigcup_{h/k \text{ Gal.}} h^{\text{perf}}$ follows from the fact that every element of $\bar{k}$ is contained in a normal, finite extension of k: if h'/k is finite and normal then the separable elements $h \subset h'$ form a finite Galois extension of k, and h' is purely inseparable over h.

Since we know that the statement holds for separably closed fields and X is of finite type, we get that there exists a finite Galois extension h/k such that (X, D) descends to a pair (Y, E) over h^{perf} and the two restrictions of (Y, E) to $\operatorname{Spec} h^{\text{perf}} \otimes_h h^{\text{perf}}$ are isomorphic.

We may write $\operatorname{Spec} \bar{k} \otimes_k \bar{k} = \operatorname{Spec} \bar{k} \otimes_h (h \otimes_k h) \otimes_h \bar{k}$ as a finite disjoint union $\sqcup_{\sigma \in \mathrm{Gal}(h/k)} \operatorname{Spec} \bar{k} \otimes_h \bar{k}$. By Lemma 2, $\sigma^*(X, D) \simeq (X, D)$ for every $\sigma \in \mathrm{Aut}(\bar{k}/k) = \mathrm{Gal}(\bar{k}/k^{\text{perf}}) = \mathrm{Gal}(k^s/k)$. This, plus the isomorphism between the two restrictions of (Y, E) to $\operatorname{Spec} h^{\text{perf}} \otimes_h h^{\text{perf}}$, induces an isomorphism over each of the finite copies of $\operatorname{Spec} \bar{k} \otimes_h \bar{k}$ contained in $\operatorname{Spec} \bar{k} \otimes_k \bar{k}$, hence we get the statement. $\square$

3.1. The residual gerbe

Let Sch/k_D be the category of schemes over k_D. We define now a stack $\mathscr{G}_D$ which parametrizes twisted forms of (X, D); we will then prove that it is a gerbe over $\operatorname{Spec} k_D$, and we will call it the *residual gerbe* of (X, D).

Definition 9. Define the stack $\mathscr{G}_D$ over Sch/k_D of twisted forms of (X, D) as follows.

If S is a scheme over k_D, then $\mathscr{G}_D(S)$ is the groupoid of morphisms $\mathscr{X} \to S$ where $\mathscr{X}$ is an algebraic space with a reduced, effective Cartier divisor $\mathscr{D} \subset \mathscr{X}$ such that there exists an fppf covering $S' \to S_{\bar{k}}$ with $(\mathscr{X}_{S'}, \mathscr{D}_{S'}) \simeq (X, D) \times_{\bar{k}} S'$ (the fppf-local isomorphism is not part of the datum, we just require existence).

An isomorphism $(\mathscr{X}, \mathscr{D}) \simeq (\mathscr{X}', \mathscr{D}')$ of twisted forms is simply an isomorphism $\sigma : \mathscr{X} \simeq \mathscr{X}'$ over S with $\sigma(\mathscr{D}) = \mathscr{D}'$.

Lemma 10. *If k'/k_D is an extension, the base change $\mathscr{G}_{D,k'}$ is the stack of twisted forms of (X, D) with respect to the base field k'.*

Proof. We may replace k with k_D and assume that k is the field of moduli. Let us write $\mathscr{G}'_D$ for the stack of twisted forms with respect to k', we want to give an isomorphism $\mathscr{G}'_D \simeq \mathscr{G}_{D,k'}$.

Construct a morphism $\mathscr{G}_{D,k'} \to \mathscr{G}'_D$ as follows. Let S be a scheme over k' with a morphism $S \to \mathscr{G}_D$, it corresponds to morphisms $\mathscr{X} \to S$, $\mathscr{D} \subset \mathscr{X}$ such that $(\mathscr{X}, \mathscr{D}) \times_k \bar{k}$ is fppf locally isomorphic to $(X, D) \times_{\bar{k}} (S \times_k \bar{k})$; in particular, $(\mathscr{X}, \mathscr{D}) \times_{k'} \bar{k} \subset (\mathscr{X}, \mathscr{D}) \times_k \bar{k}$ is fppf locally isomorphic to $(X, D) \times_{\bar{k}} (S \times_{k'} \bar{k})$. This defines a morphism $\mathscr{G}_{D,k'} \to \mathscr{G}'_D$.

On the other hand, let S be a k'-scheme and $(\mathscr{X}, \mathscr{D})$ a twisted form over S with respect to k', i.e. $(\mathscr{X}, \mathscr{D}) \times_{k'} \bar{k}$ is fppf locally isomorphic to $(X, D) \times_{\bar{k}} (S \times_{k'} \bar{k})$ over $S \times_{k'} \bar{k}$. Consider now the fibered product $(\mathscr{X}, \mathscr{D}) \times_k \bar{k}$ over k, we want to show that it is fppf locally isomorphic to $(X, D) \times_{\bar{k}} (S \times_k \bar{k})$ over $S \times_k \bar{k}$.

Notice that we may write

$$(\mathscr{X}, \mathscr{D}) \times_k \bar{k} = (\mathscr{X}, \mathscr{D}) \times_{k'} (k' \times_k \bar{k}).$$

The pullback of this pair along the extension $\bar{k}/k'$ is, by assumption, fppf locally isomorphic over $(S \times_{k'} \bar{k}) \times_k \bar{k}$ to the pullback of (X, D) along the first projection to $\bar{k}$. By Lemma 8, this is the same as pulling back along the second projection. Because of this, we get the desired local isomorphism up to replacing S with the flat cover $S \times_{k'} \bar{k} \to S$.

Since everything is of finite type, there exists a finite subextension $k'' \subset \bar{k}$ such that the local isomorphism is defined over $(S \times_{k'} k'') \times_k \bar{k}$ (rather than $(S \times_{k'} \bar{k}) \times_k \bar{k}$). Since $(S \times_{k'} k'') \times_k \bar{k}$ is an fppf covering of $S \times_k \bar{k}$, we finally get the desired fppf local isomorphism over $S \times_k \bar{k}$. Because of this, $(\mathscr{X}, \mathscr{D})$ defines a section $S \to \mathscr{G}_D$, hence we get a morphism $\mathscr{G}'_D \to \mathscr{G}_{D,k'}$. It is straightforward to check that the two morphisms are inverses. □

Lemma 11. *Let $k' \subset \bar{k}$ be a subextension of $\bar{k}$ and (X', D') a pair over k' whose base change to $\bar{k}$ is isomorphic to (X, D). Then (X', D') defines a twisted form of (X, D) in the sense of Definition 9. In particular, $\mathscr{G}_D(k') \neq \emptyset$.*

Proof. By Lemma 6, k' contains the field of moduli. Thanks to Lemma 10, we may assume that $k' = k_D = k$. Under this assumption, the statement is obvious. □

Lemma 12. *The base change $\mathscr{G}_{D,\bar{k}}$ is isomorphic to $\mathscr{B}_{\bar{k}}G$.*

Proof. Thanks to Lemma 10, it is enough to prove this in the case $k = \bar{k}$. A G-torsor $T \to S$ defines a twisted form $((T \times X)/G, (D \times X)/G)$; on the other hand, a twisted form $(\mathscr{X}, \mathscr{D})$ over S defines a G-torsor $\underline{\mathrm{Isom}}((\mathscr{X}, \mathscr{D}), (X, D) \times S)$, and the two constructions are easily checked to be inverses. $\square$

By Lemma 12, we get that $\mathscr{G}_D$ is a finite gerbe over k_D since $\mathscr{B}_{\bar{k}}G$ is a finite gerbe over k_D. The gerbe $\mathscr{G}_D$ is called the *residual gerbe* of (X, D).

It is a well-known consequence of a theorem of S. Lang that a gerbe of finite type over a finite field has a rational section. However, the only reference we have found [16, Theorem 8.1] treats the general case without assuming that the gerbe is of finite type; this general case is much more complex. Since we cannot find a published reference in the simpler case, let us give a proof. See also [24].

Lemma 13. *Let k be a finite field and $\mathscr{G} \to \operatorname{Spec} k$ an algebraic stack which is a gerbe of finite type over k. Then $\mathscr{G}(k) \neq \emptyset$.*

Proof. Let k'/k be a finite field extension such that $\mathscr{G}(k') \neq \emptyset$. Recall that, if X is an algebraic stack over k', the Weil restriction $\mathrm{R}_{k'/k}X$ is defined by $\mathrm{R}_{k'/k}X(S) = X(S_{k'})$ (see [8, after Proposition 6.1] for a discussion of the Weil restriction for algebraic stacks). The fact that $\mathscr{G}_{k'}$ is a gerbe of finite type over k' easily implies that $\mathrm{R}_{k'/k}\mathscr{G}_{k'}$ is a gerbe of finite type over k, and since $\mathrm{R}_{k'/k}\mathscr{G}_{k'}(k) = \mathscr{G}_{k'}(k') \neq \emptyset$ we may identify $\mathrm{R}_{k'/k}\mathscr{G}_{k'}$ with the classifying stack $\mathscr{B}_k G$ of some group scheme G of finite type over k.

The identity $\mathscr{G}_{k'} \to \mathscr{G}_{k'}$ induces a natural representable morphism $\mathscr{G} \to \mathrm{R}_{k'/k}\mathscr{G}_{k'} = \mathscr{B}_k G$ (the fact that it is representable is equivalent to the fact that it is faithful, which follows directly from the definition of Weil restriction). Let H be the fibered product $\mathscr{G} \times_{\mathscr{B}_k G} \operatorname{Spec} k$, where $\operatorname{Spec} k \to \mathscr{B}_k G$ is the tautological section: since $\mathscr{G} \to \mathscr{B}_k G$ is representable, $H \to \operatorname{Spec} k$ is an algebraic space. Recall that $\mathscr{B}_k G$ classifies G-torsors, in particular $H \to \mathscr{G}$ is a G-torsor. This implies that the quotient stack $[H/G]$ is $\mathscr{G}$ and the quotient space H/G is the coarse moduli space of $\mathscr{G}$, i.e. $H/G \simeq \operatorname{Spec} k$ since $\mathscr{G}$ is a gerbe over k.

In particular, this implies that H is a (non-principal) homogeneous space for G. By [21, Theorem 2], $H(k) \neq \emptyset$, hence $\mathscr{G}(k) \neq \emptyset$ too. $\square$

As a consequence, we get the following.

Corollary 14. *If k is finite, (X, D) descends to the field of moduli.* $\square$

3.2. The universal family and the compression

Definition 15. With notation as above, define a category fibered in groupoids $\mathscr{X}_D$ over k_D as follows: if S is a scheme over k_D, then $\mathscr{X}_D(S)$ is the groupoid of triples

$$(\mathscr{X}, \mathscr{D}, s)$$

where $(\mathscr{X}, \mathscr{D})$ is a twist of (X, D) over S as in Definition 9 and $s : S \to \mathscr{X}$ is a section of $\mathscr{X} \to S$. An isomorphism of triples $(\mathscr{X}, \mathscr{D}, s) \simeq (\mathscr{X}', \mathscr{D}', s')$ is an isomorphism $\sigma : \mathscr{X} \to \mathscr{X}'$ over S with $\sigma(D) = D'$ and $s' = \sigma \circ s$. There is an obvious forgetful functor $\mathscr{X}_D \to \mathscr{G}_D$.

We call $\mathscr{X}_D$ the *universal family* of (X, D). The subcategory $\mathscr{D}_D \subset \mathscr{X}_D$ corresponding to sections $S \to \mathscr{X}$ whose image lands in $\mathscr{D} \subset \mathscr{X}$ is called the *universal divisor*.

Lemma 16. *If k'/k_D is an extension, the base change $\mathscr{X}_{D,k'}$ is the universal family of (X, D) with respect to the base field k'.*

Proof. Follows directly from Lemma 10. □

Recall that the quotient stack $[X/G]$ is defined as follows cf. [26, §8.1.12]: if S is a scheme over $\bar{k}$, then $[X/G](S)$ is the groupoid of G-torsors $T \to S$ with a G-equivariant morphism $T \to X$.

Lemma 17. *The base change $\mathscr{X}_{D,\bar{k}}$ is isomorphic to the quotient stack $[X/G]$ over $\bar{k}$, and $\mathscr{D}_{\bar{k}}$ is isomorphic to $[D/G] \subset [X/G]$.*

Proof. We may assume $k = \bar{k}$. Let $(\mathscr{X}, \mathscr{D}) \to S$ be a twisted form corresponding to a G-torsor $T \to S$ and $S \to \mathscr{X} = (X \times T)/G$ a section. The pullback $T' \to S$ of $(X \times T) \to (X \times T)/G$ is a G-torsor with a G-equivariant morphism $T' \to T$, hence $T' \simeq T$ and we get a G equivariant morphism $T \to X$. This defines a morphism $\mathscr{X}_D \to [X/G]$. On the other hand, if $T \to S$ is a G-torsor with a G-equivariant morphism $T \to X$, we get an induced map $S = T/G \to (X \times T)/G$. This defines a morphism $[X/G] \to \mathscr{X}_D$; it is straightforward to check that the two morphisms are inverses. The statement regarding $\mathscr{D}_D$ is analogous. □

By Lemma 17, $\mathscr{D}_D \subset \mathscr{X}_D$ are algebraic stacks with finite inertia cf. [27, Tag 036X] since $[D/G] \subset [X/G]$ are algebraic stacks with finite inertia. For the same reason, if X is smooth, then $\mathscr{X}_D$ is smooth, and if G is reduced then $\mathscr{X}_D$ and $\mathscr{D}_D$ are Deligne–Mumford stacks.

Furthermore, if G is reduced then $\mathscr{D}_D$ is étale over $\operatorname{Spec} k$, since $[D/G]$ is étale over $\operatorname{Spec} \bar{k}$.

Proposition 18. *Let S be an algebraic stack over k, $\mathscr{X} \to S$ a representable morphism, $\mathscr{D} \subset \mathscr{X}$ a divisor finite étale over S. Assume that there exists a finite extension k'/k, an algebraic stack S' over k' and a representable, fppf covering $S' \to S$ such that, if $(\mathscr{X}', \mathscr{D}')$ is the restriction of $(\mathscr{X}, \mathscr{D})$ to S', then $(\mathscr{X}', \mathscr{D}')_{\bar{k}}$ is fppf locally isomorphic to $(X, D) \times S'_{\bar{k}}$ over $S'_{\bar{k}}$.*

There exists a morphism $f : S \to \mathscr{G}_D$ such that $(\mathscr{X}, \mathscr{D})$ is isomorphic to the pullback of $(\mathscr{X}_D, \mathscr{D}_D)$ along f.

Proof. Let us prove this first under the assumption that S is a scheme. Up to enlarging k', we may assume that we have an embedding $k_D \subset k'$. By Lemma 10, the pair $(\mathscr{X}', \mathscr{D}')$ defines a morphism $S' \to \mathscr{G}_D$ and the restriction of $(\mathscr{X}', \mathscr{D}')$ to $S' \times_{k_D} \bar{k}$ is fppf locally isomorphic to $(X, D) \times (S' \times_{k_D} \bar{k})$ (the hypothesis only gives a local isomorphism over the smaller scheme $S' \times_{k'} \bar{k}$). The two compositions $S' \times_S S' \to \mathscr{G}_D$ are isomorphic, since the two restrictions of $(\mathscr{X}', \mathscr{D}')$ are both isomorphic to the restriction of $(\mathscr{X}, \mathscr{D})$.

Since $S' \to S$ is an fppf covering, by descent [28, Theorem 2.55] the composition $S' \to \mathscr{G}_D \to \operatorname{Spec} k_D$ induces a morphism $S \to \operatorname{Spec} k_D$. The fact that the restriction of $(\mathscr{X}', \mathscr{D}')$ to $S' \times_{k_D} \bar{k}$ is fppf locally isomorphic to $(X, D) \times (S' \times_{k_D} \bar{k})$ implies that the restriction of $(\mathscr{X}, \mathscr{D})$ to $S \times_{k_D} \bar{k}$ is fppf locally isomorphic to $(X, D) \times_{k_D} (S \times_{k_D} \bar{k})$, hence we get the desired morphism $S \to \mathscr{G}_D$. The fact that $(\mathscr{X}, \mathscr{D})$ is the pullback of $(\mathscr{X}_D, \mathscr{D}_D)$ follows from the definitions.

Assume now that S is an algebraic stack. For every scheme T with a morphism $t : T \to S$, by the preceding case we obtain a morphism $f_t : T \to \mathscr{G}_D$ associated with $(\mathscr{X}, \mathscr{D})_T$. Notice that, if $f'_t : T \to \mathscr{G}_D$ is another morphism associated with $(\mathscr{X}, \mathscr{D})_T$, by definition of $\mathscr{G}_D$ there exists a *unique* 2-isomorphism $\alpha : f_t \Rightarrow f'_t$ such that the composition

$$(\mathscr{X}, \mathscr{D})_T \simeq f_T^*(\mathscr{X}_D, \mathscr{D}_D) \xrightarrow{\alpha} f_T'^*(\mathscr{X}, \mathscr{D}) \simeq (\mathscr{X}_D, \mathscr{D}_D)_T$$

is the identity. Because of this, the morphisms f_t for varying $t : T \to S$ define a morphism of stacks $S \to \mathscr{G}_D$. □

Lemma 19. *If $(\mathscr{X}, \mathscr{D})$ is a twisted form of (X, D) over a scheme S over k and $S \to \mathscr{G}_D$ is the associated morphism, there is a natural cartesian diagram*

$$\begin{tikzcd} \mathscr{X} \arrow[r] \arrow[d] & \mathscr{X}_D \arrow[d] \\ S \arrow[r] & \mathscr{G}_D. \end{tikzcd}$$

Proof. Fix S' a scheme with a morphism $S' \to S$. Consider a lifting $S' \to \mathscr{X}_D$ of the composition $S' \to S \to \mathscr{G}_D$. By definition, $S' \to \mathscr{X}_D$ corresponds to a triple $(\mathscr{X}', \mathscr{D}', s')$ where $(\mathscr{X}', \mathscr{D}')$ is a twisted form of (X, D) over S' and $s' : S' \to \mathscr{X}'$ is a section.

Asking that $S' \to \mathscr{X}_D$ lifts $S' \to S \to \mathscr{G}_D$ is equivalent to asking that $(\mathscr{X}', \mathscr{D}')$ is isomorphic to $(\mathscr{X}_{S'}, \mathscr{D}_{S'})$. Hence, giving a morphism $S' \to \mathscr{X}_D$ which lifts $S' \to S \to \mathscr{G}_D$ is equivalent to giving a section $S' \to \mathscr{X}$. This means that $\mathscr{X}$ represents the fibered product $S \times_{\mathscr{G}_D} \mathscr{X}_D$. □

Definition 20. The *compression* $\mathbf{X}_D$ of (X, D) is the coarse moduli space of $\mathscr{X}_D$ cf. [10]. The compressed divisor $\mathbf{D}_D$ is the coarse moduli space of $\mathscr{D}_D$.

We remark that the "coarse moduli space" is an algebraic space attached to any algebraic stack with finite inertia (see e.g. [10] for a definition) and it is not necessarily tied to the solution of a moduli problem. Historically, "coarse moduli spaces" were introduced in order to solve moduli problems, but they have since grown to be objects independent from moduli theory.

Lemma 21. *The base change $\mathbf{X}_{D,\bar{k}}$ is isomorphic to the quotient $X/\underline{\mathrm{Aut}}(X,D)$ over $\bar{k}$*

Proof. Formation of coarse moduli space commutes with flat base change [10, Theorem 1.1], hence this is a direct consequence of Lemma 17. $\square$

Since the action of $G = \underline{\mathrm{Aut}}(X,D)$ on X is faithful and G is finite, there exists an open G-invariant subscheme $U \subset X$ where the action is free. It follows that $U/G = [U/G]$ is an algebraic space and hence the morphism $[X/G] \to X/G$ is birational. This implies that the morphism $\mathscr{X}_D \to \mathbf{X}_D$ is birational as well, and we get a birational inverse $\mathbf{X}_D \dashrightarrow \mathscr{X}_D$.

It can be checked that the compression coincides with the *canonical model* defined by Dèbes and Emsalem [14, Remark 3.2] (we do not prove this fact since we do not use it).

3.3. Simplifications if $\underline{\mathrm{Aut}}(X,D)$ is reduced

Recall that we work under the assumption that the fppf sheaf of automorphisms $G = \underline{\mathrm{Aut}}(X,D)$ of (X,D) is representable by a finite group scheme over $\bar{k}$, which we still call G. If G is reduced, then the inertia stack [27, Tag 036X] of $\mathscr{G}_D$ is unramified, and hence $\mathscr{G}_D$ it is a finite étale gerbe over the field of moduli k_D. Under this assumption, in positive characteristic the field of moduli coincides with the intersection of the fields of definition, and with few exceptions (most notably, $k = \mathbb{R}$) this holds in characteristic 0 too, see [5, Corollary 17]. Furthermore, (X,D) always descends to the separable closure of the field of moduli, as we are going to see in Proposition 23.

Lemma 22. *Assume that k is separably closed, and let $\mathscr{G}$ be a finite étale gerbe over k. Then $\mathscr{G}$ has exactly one k-rational section up to isomorphism.*

Proof. Let k'/k be a finite extension with a section $s : \operatorname{Spec} k' \to \mathscr{G}$, the tensor product $k' \otimes_k k'$ is an artinian k algebra with residue field k'. Consider the scheme $I = \underline{\mathrm{Isom}}(p_1^* s, p_2^* s)$ over $\operatorname{Spec} k' \otimes_k k'$, it coincides with the fibered product over $\mathscr{G}$ of the two morphisms $\operatorname{Spec} k' \otimes_k k' \to \mathscr{G}$ and the closed fiber $I_{k'}$ identifies naturally with $\underline{\mathrm{Aut}}(s)$.

Since $\mathscr{G}$ is finite étale over k, then I is finite étale over $\operatorname{Spec} k' \otimes_k k'$, and there is a preferred section $\operatorname{Spec} k' \to \underline{\mathrm{Aut}}(s) \subset I$ corresponding to the identity of s. Since $k' \otimes_k k'$ is artinian and I is étale over $k' \otimes_k k'$, the preferred k'-section extends uniquely to a section $\operatorname{Spec} k' \otimes_k k' \to I$, i.e. we have a global isomorphism $\alpha : p_1^* s \simeq p_2^* s$.

This isomorphism respects the usual cocycle condition on the triple tensor product $k' \otimes_k k' \otimes_k k'$. More precisely, on $k' \otimes_k k' \otimes_k k'$ we have

$$p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha : p_1^* s \to p_3^* s.$$

To prove this equality, notice that the two elements we are comparing are global sections of

$$\underline{\mathrm{Isom}}(p_1^* s, p_3^* s) \to \operatorname{Spec} k' \otimes_k k' \otimes_k k',$$

and that this morphism is finite étale. Since $k' \otimes_k k' \otimes_k k'$ is artinian and the morphism is finite étale, it is enough to prove that the restrictions of the two global sections to the closed point of $\operatorname{Spec} k' \otimes_k k' \otimes_k k'$. Since the restriction of α to the closed point is the identity, this reduces to the fact that $\mathrm{id} \circ \mathrm{id} = \mathrm{id}$.

Since α respects the cocycle condition, (s, α) defines an object with descent data for the fppf covering $\operatorname{Spec} k' \to \operatorname{Spec} k$ [28, Definition 4.2]. Since $\mathscr{G}$ is a stack, we get that s descends to a k-rational section [28, Definition 4.6]. This proves that $\mathscr{G}(k) \neq \emptyset$.

Now assume that $s_1, s_2 \to \mathscr{G}$ are two k-rational sections. Their fibered product over $\mathscr{G}$ is the scheme of isomorphisms $\underline{\mathrm{Isom}}(s_1, s_2)$, which is finite étale over k since $\mathscr{G}$ is finite étale over k. Since k is separably closed, then $\underline{\mathrm{Isom}}(s_1, s_2)(k) \neq \emptyset$, hence $s_1 \simeq s_2$. $\square$

Proposition 23. *Assume that the group scheme representing $\underline{\mathrm{Aut}}(X, D)$ is reduced. Then (X, D) descends uniquely, up to isomorphism, to the separable closure k_D^s of k_D.*

Proof. By Lemma 22, $\mathscr{G}_{D,k_D^s}(k_D^s)$ contains exactly one point up to isomorphism, hence there is exactly one twisted form of (X, D) over k_D^s up to isomorphism. $\square$

The automorphism group scheme is reduced if $X = \mathbb{P}^1$.

Lemma 24. *Let $D \subset \mathbb{P}^1_{\bar{k}}$ a reduced, effective divisor of degree ≥ 3. Then $\underline{\mathrm{Aut}}(\mathbb{P}^1_{\bar{k}}, D)$ is a finite, reduced group scheme over $\bar{k}$.*

Proof. For a start, it is a group scheme of finite type since it is a closed subgroup of PGL_2. Clearly, it is finite since the divisor has degree ≥ 3. We want to show that it is reduced.

This is obvious in characteristic 0, but there is something to say in positive characteristic. It is sufficient to show that if R is an artinian local $\bar{k}$-algebra and $M \in \mathrm{PGL}_2(R)$ is an element fixing $\{0, 1, \infty\} \in \mathbb{P}^1(R)$, then M is the identity. It is a consequence of a general form of Hilbert's 90 theorem [25, Chapter III, Proposition 4.9] that $\mathrm{PGL}_2(R) = \mathrm{GL}_2(R)/R^*$: in fact, Hilbert's 90 theorem implies that $\mathrm{GL}_2 \to \mathrm{PGL}_2$, which is an fppf $\mathbb{G}_m$-bundle and a priori fppf-locally trivial, is actually Zariski-locally trivial, hence $\mathrm{GL}_2(R) \to \mathrm{PGL}_2(R)$ is surjective since R is local. Hence, we may represent M

with a 2×2 square matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with coefficients in R. The assumption that M fixes $\{0, 1, \infty\}$ implies that $b = c = 0$ and $a = d$. $\square$

Using Proposition 23 and Lemma 24, we can get a better understanding of fields of moduli over non-perfect fields. Notice that, up to a change of coordinates, we can always assume that a divisor of degree ≥ 3 contains $0, 1, \infty$.

Proposition 25. *Let $a_1, \dots, a_n \in \bar{k} \smallsetminus \{0, 1\}$ be pairwise different elements and consider the reduced divisor $D = \{0, 1, \infty, a_1, \dots, a_n\} \subset \mathbb{P}^1_{\bar{k}}$. Let $H \subset \mathrm{Aut}(\bar{k}/k) = \mathrm{Gal}(\bar{k}/k^{\mathrm{perf}}) = \mathrm{Gal}(k^s/k)$ be the subgroup of elements σ such that $\sigma^*(\mathbb{P}^1_{\bar{k}}, D) \simeq (\mathbb{P}^1_{\bar{k}}, D)$. The action of H restricts to $k^s(a_1, \dots, a_n)$, and the field of moduli k_D of $(\mathbb{P}^1_{\bar{k}}, D)$ is the fixed field $k^s(a_1, \dots, a_n)^H$.*

Proof. Assume first that k is separably closed. By Lemma 6, $k(a_1, \dots, a_n)$ contains the field of moduli k_D, since D is defined over $k(a_1, \dots, a_n)$. Since k, and thus k_D, is separably closed, by Proposition 23 D descends to a divisor D^s of $\mathbb{P}^1$ over k_D (every rational curve over a separably closed field is isomorphic to $\mathbb{P}^1$).

Since D is étale over $\bar{k}$, then D^s is étale over k_D, it follows that D^s is the union of $n + 3$ rational points $b_1, \dots, b_{n+3} \in \mathbb{P}^1(k_D)$ such that there exists an element $g \in \mathrm{PGL}_2(\bar{k})$ mapping b_i to a_i for $i \leq n$ and $b_{n+1}, b_{n+2}, b_{n+3}$ to $0, 1, \infty$ respectively. Since $b_{n+1}, b_{n+2}, b_{n+3}$ are k_D-rational, then g is defined over k_D, hence $g(b_i) = a_i$ is k_D-rational too for every $i \leq n$ and $k(a_1, \dots, a_n) \subset k_D$. This completes the proof in the case in which k is separably closed.

Assume now that k is an arbitrary field. Let $R = \mathrm{Spec}\, \bar{k} \otimes_{k_D} \bar{k} \subset \mathrm{Spec}\, \bar{k} \otimes_k \bar{k}$ be the equivalence relation defining k_D, i.e. the schematic image of $\underline{\mathrm{Isom}}(p_1^*(X, D), p_2^*(X, D))$. Let k_D^s be the separable closure of k_D, we may write

$$R = \mathrm{Spec}\, \bar{k} \otimes_{k_D^s} (k_D^s \otimes_{k_D} k_D^s) \otimes_{k_D^s} \bar{k}.$$

The field of moduli of D with respect to $\bar{k}/k^s$ is the quotient of $\bar{k}$ by the equivalence relation given by the restriction of R to $\mathrm{Spec}\, \bar{k} \otimes_{k^s} \bar{k}$, which is $\mathrm{Spec}\, \bar{k} \otimes_{k_D^s} \bar{k}$. This implies that k_D^s is the field of moduli of D with respect to $\bar{k}/k^s$. By the previous case, we thus get that $k^s(a_1, \dots, a_n) = k_D^s$ is the separable closure of k_D.

If $\sigma \in \mathrm{Aut}(\bar{k}/k)$, then by construction $\sigma(k_D^s)$ is the separable closure of the field of moduli of $\sigma^*(X, D)$; this implies that $\sigma(k_D^s) = k_D^s$ if $\sigma \in H$. In particular, the action of H restricts to $k^s(a_1, \dots, a_n)$, and we get an embedding $H \subset \mathrm{Gal}(k(a_1, \dots, a_n)/k_D)$. To obtain the statement, it remains to prove that they are equal. This is analogous to the proof of Lemma 2. $\square$

3.4. The case of $\mathbb{P}^1$

From now on, we assume $k = k_D$ and $X = \mathbb{P}^1_{\bar{k}}$. Denote by $\mathscr{P}_D \to \mathscr{G}_D$ and $\mathbf{P}_D$ the universal family and the compression of D respectively, then $\mathbf{P}_D$ is the coarse moduli space of $\mathscr{P}_D$ and there is a birational map $\mathscr{P}_D \to \mathbf{P}_D$ with a generic birational section $\mathbf{P}_D \dashrightarrow \mathscr{P}_D$. The compression $\mathbf{P}_D$ is a rational curve since its base change $\mathbf{P}_{D,\bar{k}}$ to $\bar{k}$ is isomorphic to $\mathbb{P}^1_{\bar{k}}/\underline{\mathrm{Aut}}(\mathbb{P}^1, D) \simeq \mathbb{P}^1_{\bar{k}}$ by Lemma 21.

By definition of $\mathscr{G}_D$, a section $\operatorname{Spec} k \to \mathscr{G}_D$ corresponds to a twisted form of $(\mathbb{P}^1, D)$ over k, i.e. a divisor on some rational curve P. If k is finite, since $\mathbb{P}^1$ is the only rational curve over k, by Corollary 14 we get that D always descends to a divisor of $\mathbb{P}^1$ over the field of moduli. We may thus focus on the case in which k is infinite.

By Lemma 19, the rational curve P is the pullback of the universal family $\mathscr{P}_D \to \mathscr{G}_D$ along the section $\operatorname{Spec} k \to \mathscr{G}_D$. This implies that $\operatorname{Spec} k \to \mathscr{G}_D$ lifts to $\mathscr{P}_D$ if and only if $P(k) \neq \emptyset$, if and only if $P \simeq \mathbb{P}^1_k$. In particular, we get that D descends to $\mathbb{P}^1_k$ if and only if $\mathscr{P}_D(k) \neq \emptyset$.

Clearly, if $\mathscr{P}_D(k) \neq \emptyset$ then $\mathbf{P}_D(k) \neq \emptyset$, hence $\mathbf{P}_D \simeq \mathbb{P}^1_k$. On the other hand, assume that $\mathbf{P}_D(k) \neq \emptyset$, i.e. $\mathbf{P}_D \simeq \mathbb{P}^1_k$, and let $U \subset \mathbf{P}_D$ be an open subset where $\mathbf{P}_D \dashrightarrow \mathscr{P}_D$ restricts to a morphism. Since k is infinite and $\mathbf{P}_D \simeq \mathbb{P}^1_k$, then $U(k) \neq \emptyset$, hence $\mathscr{P}_D(k) \neq \emptyset$. We thus have proved the following.

Lemma 26. *Let k be any field and $D \subset \mathbb{P}^1_{\bar{k}}$ an effective, reduced divisor of degree $n \geq 3$ on $\mathbb{P}^1_{\bar{k}}$ with $n \geq 3$. The following are equivalent.*

- *D descends to a divisor of $\mathbb{P}^1$ over the field of moduli k_D.*
- *The compression $\mathbf{P}_D$ has a k_D-rational point.*

Notice that both conditions of Lemma 26 always hold if k is finite.

If $\operatorname{char} k \neq 2$, Lemma 26 proves that Theorem 1 holds under condition (2), since rational curves are classified by $\mathrm{Br}(k)[2]$ [19, Example 5.2.4]. The case $\operatorname{char} k = 2$ will follow from condition (1).

We thus want to study whether $\mathbf{P}_D$ and $\mathscr{G}_D$ have k_D-rational sections. There is a crucial input for studying this problem: the rational map $\mathbf{P}_D \dashrightarrow \mathscr{G}_D$ obtained by composing $\mathbf{P}_D \dashrightarrow \mathscr{P}_D$ with $\mathscr{P}_D \to \mathscr{G}_D$.

Let us show that the geometric fibers of $\mathbf{P}_D \dashrightarrow \mathscr{G}_D$ are irreducible curves of genus 0. Write $G = \underline{\mathrm{Aut}}(\mathbb{P}^1_{\bar{k},D})$. When we base change to $\bar{k}$, this rational map becomes the natural rational map $\mathbb{P}^1_{\bar{k}}/G \dashrightarrow \mathscr{B}_{\bar{k}}G$ associated with the ramified G-covering $\mathbb{P}^1_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}/G$. If $U \subset \mathbb{P}^1_{\bar{k}}$ is an open subset where the action is free, the rational map restricts to a morphism $U/G \to \mathscr{B}_{\bar{k}}G$ whose (unique up to isomorphism) geometric fiber is U.

4. Rational maps from rational curves to gerbes

As we have seen above, given a field k we are interested in rational maps $P \dashrightarrow \mathscr{G}$ where P is a rational curve over k, $\mathscr{G}$ is a finite étale gerbe over k and such that the geometric fibers are connected of genus 0. In this section, we study such maps.

The following fact is well known, we give a self-contained proof for the convenience of the reader.

Lemma 27. *Let P be a Brauer-Severi variety of dimension 1 over k with $P(k) = \emptyset$. Every divisor on P has even degree.*

Proof. A canonical divisor K on P has degree -2. This implies that, if there is a divisor of odd degree on P, then there is a divisor D of degree 1. By Riemann-Roch, $h^0(D) = 2$, hence there exists an effective divisor D' linearly equivalent to D. Since $\deg D' = \deg D = 1$, then D' is a rational point, which is absurd. □

Proposition 28. *Let k be a field, P a rational curve with $P(k) = \emptyset$, Φ a finite, étale gerbe over k with a rational map $P \dashrightarrow \Phi$. Assume that the geometric fibers of $P \dashrightarrow \Phi$ are irreducible curves of genus 0.*

The gerbe Φ is abelian with cyclic band of order prime to $\operatorname{char} k$, and there exists a separable extension k'/k of degree 2 and a point $p \in P(k')$ such that $P \dashrightarrow \Phi$ restricts to a morphism $P \setminus \{p\} \to \Phi$.

Let us first sketch the proof in the particular case in which k is perfect. Since Φ is étale over k, i.e. it has unramified inertia stack, we may write $\Phi_{\bar{k}} = \mathscr{B}_{\bar{k}} G$ for some finite, reduced group scheme G over $\bar{k}$, and the base change of $P \dashrightarrow \Phi$ to $\bar{k}$ corresponds to a G-covering $\mathbb{P}^1_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. The branch locus of the G-covering descends to P: if $P(k) = \emptyset$, this forces every branch point p of $\mathbb{P}^1_{\bar{k}}$ to have at least one Galois conjugate $\bar{p}$ such that the ramification data over p and $\bar{p}$ are equal. Using the Riemann-Hurwitz formula, it can be checked that this only happens if there are exactly two ramification points and G is cyclic, since otherwise the "doubling" of the ramification data implies that the degree of the ramification divisor is too large.

Proof. Let $\bar{k}$ be an algebraic closure and $\bar{k} \supseteq k^s \supseteq k$ be the separable closure, we have $P_{k^s} \simeq \mathbb{P}^1_{k^s}$, in particular Φ_{k^s} is neutral since there is a rational map $P \dashrightarrow \Phi$. Choose a section $\operatorname{Spec} k^s \to \Phi$ and let G be its group scheme of automorphisms, it is finite and reduced since Φ has finite and unramified inertia. A finite, reduced group scheme over a separably closed field is constant, hence we might think of G as a classical finite group.

Since $\Phi_{k^s} \simeq \mathscr{B}_{k^s} G$ is the classifying stack of G and G is étale, the rational map $\mathbb{P}^1_{k^s} \simeq P_{k^s} \dashrightarrow \Phi_{k^s}$ corresponds to an étale, Galois G-cover $V \to U$ on an open subset $U \subset \mathbb{P}^1_{k^s}$. By hypothesis, V has genus 0, hence by extending $V \to U$ we get a ramified, Galois G-cover $f : \mathbb{P}^1_{k^s} \to \mathbb{P}^1_{k^s}$. In particular, f is separable, and we may apply the Riemann–Hurwitz formula. Since f is separable, the same is true for the base change $f_{\bar{k}}$, hence we may apply Riemann–Hurwitz to $f_{\bar{k}}$ as well.

Let R be the ramification divisor of f as in [27, Tag 0C1B], we think of R as a non-reduced, finite scheme over k^s. There is an action of G on R, in particular on the underlying set. Given a point $r \in R$ denote by d_r the length of R at r, e_r the ramification index, o_r the cardinality of the set-theoretic G-orbit of r. Since f is Galois of degree n, then $n = o_r e_r [k^s(r) : k^s(f(r))]$. By Riemann-Hurwitz [27, Lemma 0C1F] we have

$$2n - 2 = \sum_{r \in R} d_r [k^s(r) : k^s]$$

with $d_r = e_r - 1$ if f is tamely ramified at r and $d_r \geq e_r$ otherwise.

There exists a largest open subset $U \subset P$ such that $U \to \Phi$ is defined [3, Corollary A.2]. Write $Z = P \setminus U$, then Z_{k^s} is the branch locus of f [3, Lemma A.1, Corollary A.2.iv], in particular $f(R) = Z_{k^s}$ set-theoretically. Since P is a non-trivial Brauer-Severi variety, Z has no rational points. This implies that, if $r \in R$ is a point such that $k^s(f(r)) = k^s$, there exists a point $r' \in R$ such that $f(r') \neq f(r)$ is a Galois conjugate of $f(r)$, and hence $o_r = o_{r'}$, $d_r = d_{r'}$, $e_r = e_{r'}$, $k^s(r) \simeq k^s(r')$.

Let us show first that f is tamely ramified at every point. Assume that f has wild ramification at some point r, in particular $d_r \geq e_r$. If $k^s(f(r)) = k^s$, by what we have said above we have

$$2n - 2 \geq 2 o_r e_r [k^s(r) : k^s] = 2n,$$

which is absurd. If $k^s(f(r)) \neq k^s$, then

$$2n - 2 \geq o_r e_r [k^s(r) : k^s] \geq 2 o_r e_r [k^s(r) : k^s(f(r))] = 2n,$$

hence we get a contradiction in this case, too.

We may thus assume that f has tame ramification: in particular, $k^s(r) = k^s(f(r))$ for every r, $n = o_r e_r$ and $d_r = e_r - 1$ for every $r \in R$. Since f is ramified at every $r \in R$, then $e_r \geq 2$ and $o_r \leq n/2$. For every $z \in Z_{k^s}$, write e_z, o_z for e_r, o_r, where $r \in R$ is some point with $f(r) = z$. We have

$$\begin{aligned} 2n - 2 = \sum_{r \in R} (e_r - 1)[k^s(r) : k^s] = \sum_{z \in Z_{k^s}} o_z (e_z - 1)[k^s(z) : k^s] = \\ = \sum_{z \in Z_{k^s}} (n - o_z)[k^s(z) : k^s] \geq \deg Z \cdot n/2 \end{aligned}$$

which implies $\deg Z \leq 3$. By Lemma 27, we also know that $\deg Z$ is even. Since $\deg Z \leq 3$ is even, we may assume $\deg Z = 2$ (if $\deg Z = 0$, then G is trivial).

Assume first Z_{k^s} contains only one point z with $[k^s(z) : k^s] = 2$, in particular $\operatorname{char} k = 2$. Then

$$2n - 2 = o_z (e_z - 1)[k^s(z) : k^s] = 2n - 2o_z,$$

hence $o_z = 1$ and $n = e_z$ is prime with $\operatorname{char} k = 2$, i.e. it is odd. Since n is odd, the base change $f_{\bar{k}}$ of f to $\bar{k}$ is tamely ramified and since f has only one point of ramification then the same is true for $f_{\bar{k}}$, and the ramification index must be n. By Riemann-Hurwitz applied to $f_{\bar{k}}$, we have $2n - 2 = n - 1$, i.e. $n = 1$ which is in contradiction with $\deg Z = 2$.

If Z_{k^s} contains two k^s-rational points, the Galois action swaps them because Z has no rational points, hence

$$2n - 2 = 2o_z(e_z - 1) = 2n - 2o_z,$$

which implies $o_z = 1$ and $n = e_z$ is prime to $\operatorname{char} k$. If $r \in R$ is one of the ramification points, since $|G| = n = e_z$ then r is fixed by G.

Since $|G| = n$ is prime with $\operatorname{char} k$, then every subgroup $H \subset G$ is linearly reductive by Maschke's theorem [22, Chapter XVIII, Theorem 1.2], hence the functor mapping a linear representation V of H to the H-invariants $V^H \subset V$ is exact. Let $\mathfrak{m} \subset \mathscr{O}_{\mathbb{P}^1_{k^s},r}$ be the maximal ideal of the local ring of r. Since the cotangent space $\mathfrak{m}/\mathfrak{m}^2 \simeq k^s$ is a 1-dimensional linear representation of G, if we show that the action of G on $\mathfrak{m}/\mathfrak{m}^2$ is faithful we get that $G \subset k^{s*}$ is cyclic.

Notice that the action of G on $\mathscr{O}_{\mathbb{P}^1_{k^s},r}/\mathfrak{m} \simeq k^s$ is trivial. Let $H \subset G$ be the subgroup of elements acting trivially on $\mathfrak{m}/\mathfrak{m}^2$, then H acts trivially on $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ for every $n \geq 1$. Using the fact that H is linearly reductive, by induction we get that the action of H on $\mathscr{O}_{\mathbb{P}^1_{k^s},r}/\mathfrak{m}^n$ is trivial for every n. It follows that the action of H on $\mathscr{O}_{\mathbb{P}^1_{k^s},r} \subset \varprojlim_n \mathscr{O}_{\mathbb{P}^1_{k^s},r}/\mathfrak{m}^n$ is trivial, which implies that H is trivial since G acts faithfully on $k^s(\mathbb{P}^1_{k_s}) \subset \mathscr{O}_{\mathbb{P}^1_{k^s},r}$. $\square$

Corollary 29. *Let k be a field and $D \subset \mathbb{P}^1_{\bar{k}}$ an effective, reduced divisor of degree $n \geq 3$ defined over $\bar{k}$. If the compression of $(\mathbb{P}^1_{\bar{k}}, D)$ is not isomorphic to $\mathbb{P}^1$ over the field of moduli, then $\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D) = \underline{\operatorname{Aut}}(\mathbb{P}^1_{\bar{k}}, D)(\bar{k})$ is cyclic of order prime to the characteristic.*

Proof. As we have explained at the end of §3.2, there is a rational map $\mathbf{P}_D \dashrightarrow \mathscr{G}_D$, where $\mathbf{P}_D$ is the compression of $(\mathbb{P}^1, D)$, whose base change to $\bar{k}$ is the rational map $\mathbb{P}^1_{\bar{k}}/\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D) \dashrightarrow \mathscr{B}_{\bar{k}} \operatorname{Aut}(\mathbb{P}^1, D)$ corresponding to the ramified covering $\mathbb{P}^1_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}/\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$. Since $\mathbf{P}_D$ is a twisted form of $\mathbb{P}^1_{\bar{k}}/\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$, it is a rational curve. Since the base change of $\mathbf{P}_D \dashrightarrow \mathscr{G}_D$ to $\bar{k}$ corresponds to the ramified covering $\mathbb{P}^1_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}/\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$, the geometric fibers of $\mathbf{P}_D \dashrightarrow \mathscr{G}_D$ are irreducible of genus 0. By hypothesis, $\mathbf{P}_D(k) = \emptyset$. We may then apply Proposition 28 to conclude that $\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ is cyclic of order prime to the characteristic. $\square$

Lemma 30. *Let $P \dashrightarrow \Phi$ be as in Proposition 28. Then Φ is neutral if and only if it has odd degree.*

Proof. By Proposition 28, Φ is abelian with cyclic band A of order prime to $\operatorname{char} k$ (see §2.3 for the concept of band). Gerbes over k banded by A are in natural bijection with $\mathrm{H}^2(k, A)$ [18, Chapitre IV, Théorème 3.4.2], and $0 \in \mathrm{H}^2(k, A)$ corresponds to the gerbe $\mathscr{B}_k A$ [18, Chapitre IV, Définition 3.1.3, Proposition 3.3.2]. If Ψ is a gerbe banded by A with a section $s \in \Psi(k)$, then by definition of band the automorphism group scheme of s is A, hence $\Psi \simeq \mathscr{B}_k A$. Hence, a gerbe banded by A is neutral if and only if it is isomorphic to $\mathscr{B}_k A$. Let $\phi \in \mathrm{H}^2(k, A)$ be the cohomology class corresponding to Φ, by what we have said Φ is neutral if and only if $\phi = 0$. Let n be the degree of Φ, which corresponds to the degree of A i.e. $|A(\bar{k})|$. The group $\mathrm{H}^2(k, A)$ is abelian and n-torsion.

Let k' be the residue field of the point p given by Proposition 28, then k'/k is a separable extension of degree 2 and $\phi_{k'} = 0$ since k' splits P. We have $2\phi = \mathrm{cor}_{k'/k}(\phi_{k'}) = 0$: if n is odd, then $\phi = \frac{n+1}{2} 2\phi = 0$.

If Φ has even degree, by Proposition 28 $\mathrm{char}\, k$ is prime with n and hence $\mathrm{char}\, k \neq 2$. Write $U = P \smallsetminus \{p\}$, there exists a morphism $U \to \Gamma$ where Γ is a non-neutral gerbe banded by μ_2: for a construction, see the proof of [8, Proposition 13.2] using parameters $p = r = 2$ and $X = U$.

Now consider the morphism $U \to \Phi \times \Gamma$. By [8, Remark 5.11, Lemma 5.12], there exists a factorization $U \to \Phi' \to \Phi \times \Gamma$ with Φ' a finite gerbe such that $U \to \Phi'$ induces a surjective map of étale fundamental groups, while $\Phi' \to \Phi \times \Gamma$ is faithful (equivalently, representable); in particular Φ' is finite étale since $\Phi \times \Gamma$ is finite étale. The base change $\Phi'_{\bar{k}}$ is isomorphic to $\mathscr{B}_{\bar{k}} G$ for some finite group G, and we get morphisms $U_{\bar{k}} \to \mathscr{B}_{\bar{k}} G \to \mathscr{B}_{\bar{k}} \mu_n \times \mathscr{B}_{\bar{k}} \mu_2$ such that $\pi_1(U_{\bar{k}}) \to G$ is surjective and $G \to \mu_n \times \mu_2$ is injective, i.e. G is the image of $\pi_1(U_{\bar{k}})$ in $\mu_n \times \mu_2$.

Since $\mathrm{char}\, k \neq 2$, the largest 2-adic quotient of $\pi_1(U_{\bar{k}})$ is isomorphic to $\mathbb{Z}_2$. This, plus the fact that n is even, implies that the image G of $\pi_1(U_{\bar{k}})$ in $\mu_n \times \mu_2$ is cyclic of order n and the composition $G \to \mu_n \times \mu_2 \to \mu_n$ is an isomorphism. This implies that the composition $\mathscr{B}_{\bar{k}} G \to \mathscr{B}_{\bar{k}} \mu_n \times \mathscr{B}_{\bar{k}} \mu_2 \to \mathscr{B}_{\bar{k}} \mu_n$ is an isomorphism, which in turn implies that the composition $\Phi' \to \Phi \times \Gamma \to \Phi$ is an isomorphism as well. Hence, we get a morphism $\Phi = \Phi' \to \Gamma$, and $\Phi(k) = \emptyset$ since $\Gamma(k) = \emptyset$. $\square$

5. Which divisors descend to a rational curve

We characterize which pairs $(\mathbb{P}^1_{\bar{k}}, D)$ descend to the field of moduli. We will prove Theorem 1 using this characterization.

Theorem 31. *Let k be a field and $D \subset \mathbb{P}^1_{\bar{k}}$ an effective, reduced divisor of degree $n \geq 3$ on $\mathbb{P}^1_{\bar{k}}$ with $n \geq 3$. The following are equivalent.*

- *$(\mathbb{P}^1_{\bar{k}}, D)$ is not defined over its field of moduli.*
- *The compression has no rational points and $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ has even degree.*
- *The compression has no rational points and $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ is cyclic of degree even and prime to $\mathrm{char}\, k$.*

Proof. If k is finite, then $(\mathbb{P}^1_{\bar{k}}, D)$ is defined over the field of moduli by Corollary 14 and the compression has rational points by [21, Theorem 3]. Assume that k is infinite.

By base change we may assume that k is the field of moduli. Let $\mathscr{G}, \mathbf{P}$ be the residual gerbe and the compression respectively, there is a rational map $\mathbf{P} \dashrightarrow \mathscr{G}$ whose geometric fibers are irreducible of genus 0.

If $(\mathbb{P}^1_{\bar{k}}, D)$ is not defined over k, i.e. $\mathscr{G}$ is not neutral, then $\mathbf{P}$ is not isomorphic to $\mathbb{P}^1$ and hence $\mathscr{G}$ is abelian with cyclic band of degree even and prime with $\mathrm{char}\, k$ by Proposition 28 and Lemma 30. Since $\mathscr{G}_{\bar{k}} = \mathscr{B}_{\bar{k}} \mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$, we obtain that the first condition implies the third, which in turn clearly implies the second. If $\mathbf{P}(k) = \emptyset$ and $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ has even degree, then by Lemma 30 the residual gerbe $\mathscr{G}$ is not neutral, hence we conclude. □

Corollary 32. *Let k be a field of characteristic 2 and $D \subset \mathbb{P}^1_{\bar{k}}$ an effective, reduced divisor with $n \geq 3$. Then $(\mathbb{P}^1_{\bar{k}}, D)$ is defined over its field of moduli.*

6. Generalizations of results of A. Marinatto and B. Huggins

Thanks to Theorem 31, we can easily reprove and generalize the results of A. Marinatto and B. Huggins.

Theorem 33 *([23, Theorem 1]). Let k be a field and $D \subset \mathbb{P}^1_{\bar{k}}$ a reduced, effective divisor of $\mathbb{P}^1_{\bar{k}}$ of degree $n \geq 3$. If n is odd, then D descends to a divisor of $\mathbb{P}^1$ over the field of moduli.*

Proof. Thanks to Corollary 14 and Corollary 32, we may assume that k is infinite with $\mathrm{char}\, k \neq 2$. By base change, we can also assume that k is the field of moduli of $(\mathbb{P}^1_{\bar{k}}, D)$. Let $\mathbf{P}$ be the compression and $\mathbf{D} \subset \mathbf{P}$ the compressed divisor, i.e. the coarse moduli space of the universal divisor $\mathscr{D} \subset \mathscr{P}$, we have that $\mathbf{D}$ is a finite étale scheme over k.

By Lemma 26 it is enough to show that, if $\mathbf{P}(k) = \emptyset$, then n is even. If $p, q \in \mathbf{D}(k^s) = \mathbf{D}(\bar{k})$ are in the same Galois orbit, then the fibers of $D \to \mathbf{D}_{\bar{k}} = D/\mathrm{Aut}(\mathbb{P}^1, D)$ over p and q have the same cardinality. Since $\mathbf{P}$ is non-trivial and $\mathbf{D}$ is étale over k, the residue fields of the points of $\mathbf{D}$ are separable of even degree by Lemma 27, hence the Galois-orbits of $\mathbf{D}(k^s)$ have even cardinality. It follows that D has even cardinality, too. □

Proposition 34 *([23, Proposition 2.12]). Let k be a field and $D \subset \mathbb{P}^1_{\bar{k}}$ an effective, reduced divisor of degree 4. Then D descends to a divisor of $\mathbb{P}^1$ over the field of moduli.*

Proof. For every such divisor, it is well-known that there is a copy of $\mathbb{Z}/2 \times \mathbb{Z}/2$ inside $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$. In fact, we may assume $D = \{0, 1, \infty, a\}$, and the transformations $z \mapsto \frac{a}{z}$, $z \mapsto \frac{a-z}{1-z}$ generate such a subgroup. It follows that the compression has a rational point by Proposition 28. We conclude by Lemma 26. □

Theorem 35 *([20, Theorem 5.3]). Let k be a field of characteristic $\neq 2$ and H a hyperelliptic curve over $\bar{k}$ with hyperelliptic involution ι. If H is not defined over its field of moduli, then $\underline{\mathrm{Aut}}(H)/ < \iota >$ is cyclic of order prime to $\mathrm{char}\, k$.*

Proof. Let $D \subset \mathbb{P}^1_{\bar{k}} = H/\iota$ be the branch divisor, we have $\mathrm{Aut}(X)/ < \iota >= \mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$. Let $\mathscr{G}_H, \mathscr{H}_H, \mathscr{G}_D, \mathscr{P}_D$ be the residual gerbes and the universal families of $(H, \emptyset)$, (X, D) respectively. It is a direct consequence of the definitions that the field of moduli of $(H, \emptyset)$ contains the field of moduli of $(\mathbb{P}^1_{\bar{k}}, D)$, and Lemma 8 implies that they are actually equal; up to base change, we may then assume that k is the field of moduli of both pairs. There is a natural 2-commutative diagram

$$\begin{tikzcd} \mathscr{H}_H \arrow[r] \arrow[d] & \mathscr{P}_D \arrow[d] \\ \mathscr{G}_H \arrow[r] & \mathscr{G}_D \end{tikzcd}$$

where the morphism $\mathscr{G}_H \to \mathscr{G}_D$ is a relative abelian gerbe banded by μ_2, and $\mathscr{H}_H \to \mathscr{P}_D$ is birational: these two facts can be checked after base changing to $\bar{k}$, where they follow from Lemma 12 and 17. In particular, $(H, \emptyset)$ and $(\mathbb{P}^1_{\bar{k}}, D)$ have equal compression $\mathbf{P}$. Thanks to Lemma 13, we may assume that k is infinite. Since H is not defined over the field of moduli, i.e. $\mathscr{G}_H(k) = \emptyset$, and there is a rational map $\mathbf{P} \dashrightarrow \mathscr{G}_H$ obtained by composing $\mathbf{P} \dashrightarrow \mathscr{H}_H$ with $\mathscr{H}_H \to \mathscr{G}_H$, we get that $\mathbf{P}(k) = \emptyset$. We conclude by applying Proposition 28 to $\mathbf{P} \dashrightarrow \mathscr{G}_D$. □

7. Divisors of degree 6

Proposition 36. *Let k be a field and $D \subset \mathbb{P}^1_{\bar{k}}$ an effective, reduced divisor of degree 6. Then $(\mathbb{P}^1_{\bar{k}}, D)$ is defined over its field of moduli.*

Proof. Thanks to Corollary 32, we may assume $\operatorname{char} k \neq 2$. Thanks to Proposition 23, we may assume that D is defined over k^s. Up to base change, we may also assume that k is the field of moduli. Write $G = \operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$. Let $\mathbf{P}$ be the compression and $\mathbf{D}$ the compressed divisor, i.e. the coarse moduli spaces of the universal family $\mathscr{P}_D$ and of the universal divisor $\mathscr{D}_D \subset \mathscr{P}_D$. By Lemma 17, we have that $\mathscr{P}_{D,\bar{k}} \simeq \mathbb{P}^1_{\bar{k}}/G$ and $\mathbf{D}_{\bar{k}} \simeq D/G$, in particular $\mathbf{D}$ is finite étale over k. Assume by contradiction that $(\mathbb{P}^1_{\bar{k}}, D)$ is not defined over k, then by Theorem 31 $\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ is cyclic of degree even and prime to $\operatorname{char} k$, and $\mathbf{P}$ is a non-trivial rational curve.

Since $\mathbf{P}$ is non-trivial, then $\mathbf{D}$ has even degree, and hence $\mathbf{D}_{\bar{k}} = D/\operatorname{Aut}(\mathbb{P}^1, D)$ has either $2, 4$ or 6 points. Since $\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ is cyclic of degree even and prime with $\operatorname{char} k$, then its action on $\mathbb{P}^1_{\bar{k}}$ has two fixed points, while all the other orbits have even cardinality equal to $|\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)|$. Using the fact that D has degree 6 and $D/\operatorname{Aut}(\mathbb{P}^1, D)$ either $2, 4$ or 6, it is easy to check that the only possibility is that D contains the 2 fixed points, $\mathbf{D}_{\bar{k}}$ has 4 points and $\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ has order 2.

Choose coordinates on $\mathbb{P}^1_{\bar{k}}$ such that $0, \infty \in D$ are the two fixed points, and 1 is another point of D. With respect to these coordinates, we have that the only non-trivial element of $\operatorname{Aut}(\mathbb{P}^1_{\bar{k}}, D)$ is the map $x \mapsto -x$, we may thus write $D = \{0, \infty, 1, -1, \lambda, -\lambda\}$. It is immediate to check that $x \mapsto \lambda/x$ defines another non-trivial automorphism of $(\mathbb{P}^1, D)$, hence we get a contradiction. □

8. Divisors of even degree $n \geq 8$

We want now to construct a divisor D of even degree $n \geq 8$ over $\bar{k}$ which does not descend to any rational curve over its field of moduli.

In order to do this, given an infinite field k of characteristic $\neq 2$ and P a non-trivial rational curve, we construct a divisor $E \subset P$ with certain special properties. We will then construct D so that k is the field of moduli, P is the compression and E is the compressed divisor; the special properties of $E \subset P$ will guarantee that D does not descend to any rational curve over k.

Given a morphism $f : X \to Y$, we write $\mathrm{Aut}(f) \subset \mathrm{Aut}(X)$ for the subgroup of automorphisms $g \in \mathrm{Aut}(X)$ such that $f \circ g = f$.

Lemma 37. *Let k be an infinite field of characteristic $\neq 2$, P a non-trivial rational curve, $n \geq 8$ an even integer.*

There exists an effective, reduced divisor $E \subset P$ étale over k, a quadratic extension k'/k, a point $p \in P(k')$ with Galois conjugate $\bar{p} \in P(k')$ and a cyclic cover $f : \mathbb{P}^1_{\bar{k}} \to P_{\bar{k}}$ of degree 2 ramified over $p, \bar{p}$ such that the divisor $D = f^{-1}(E_{\bar{k}})$ with the reduced structure has degree n and $\mathrm{Aut}(f)$ is its own centralizer in $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$.

Proof. Let ω be the canonical bundle of P. Write either $n = 4m$ with $m \geq 2$ or $n = 4m-2$ with $m \geq 3$. Consider the natural map $\mathbb{P}(\mathrm{H}^0(\omega^{-1})) \times \mathbb{P}(\mathrm{H}^0(\omega^{-m+1})) \to \mathbb{P}(\mathrm{H}^0(\omega^{-m}))$ corresponding to the sum of divisors, it is dominant.

If $n \geq 10$, i.e. $m \geq 3$, a generic rational point of $\mathbb{P}(\mathrm{H}^0(\omega^{-m}))$ corresponds to a divisor étale over k with trivial automorphism group scheme. Moreover, any rational point of $\mathbb{P}(\mathrm{H}^0(\omega^{-1}))$ corresponds to a point of P whose residue field is a quadratic extension of k. Taking a generic rational point of $\mathbb{P}(\mathrm{H}^0(\omega^{-1})) \times \mathbb{P}(\mathrm{H}^0(\omega^{-m+1}))$, we can thus find a divisor $E \subset P$ of degree $2m$ étale over k with trivial automorphism group scheme and a quadratic extension k'/k such that $E(k') \neq \emptyset$. If $n = 8$, with an analogous argument we get a reduced divisor $E \subset P$ of degree 4 such that $\mathrm{Aut}(\mathbb{P}^1, E_{\bar{k}})$ is the Klein group.

If $p \in P(k')$ is a k'-rational point, denote by $\bar{p}$ its Galois conjugate. If $n = 4m - 2$, choose $p \in E(k')$, otherwise $p \in P \smallsetminus E(k')$. If $n = 8$, for every non-trivial $g \in \mathrm{Aut}(P_{\bar{k}}, E_{\bar{k}})$ the set of points $p \in E(k')$ such that either $g(p) = p$ or $g(p) = \bar{p}$ is not Zariski-dense, since k is infinite we may choose $p \in P \smallsetminus E(k')$ so that $g(p) \neq p$, $g(p) \neq \bar{p}$ for every non-trivial $g \in \mathrm{Aut}(P_{\bar{k}}, E_{\bar{k}})$.

Consider a cyclic cover $f : \mathbb{P}^1_{\bar{k}} \to P_{\bar{k}}$ of degree 2 which ramifies at p and $\bar{p}$ and let $D \subset \mathbb{P}^1_{\bar{k}}$ be the inverse image of $E_{\bar{k}}$ with the reduced structure, it is an effective divisor of degree n. The group of automorphisms $\mathrm{Aut}(\mathbb{P}^1, D)$ has a cyclic subgroup $\mathrm{Aut}(f)$ of order 2. Let $C \subset \mathrm{Aut}(\mathbb{P}^1, D)$ be the centralizer of $\mathrm{Aut}(f)$, then $C/\mathrm{Aut}(f)$ acts faithfully on $(P_{\bar{k}}, E_{\bar{k}})$. If $n \geq 10$, $\mathrm{Aut}(P_{\bar{k}}, E_{\bar{k}})$ is trivial and hence $C = \mathrm{Aut}(f)$. If $n = 8$ and $g \in C/\mathrm{Aut}(f) \subset \mathrm{Aut}(P_{\bar{k}}, E_{\bar{k}})$, then $g(p)$ is a branch point for f and hence either $g(p) = p$ or $g(p) = \bar{p}$. By our choice of p, we get that g is trivial, hence $C = \mathrm{Aut}(f)$ in this case, too. $\square$

Lemma 38. *Let G be a finite group and $g \in G$ a non trivial element with $g^2 = \mathrm{id}$ and such that the centralizer of g is $< g >$. Then $|G|/2$ is odd.*

Proof. Let $P \subset G$ be a 2-Sylow subgroup containing g, the center Z of P is non-trivial. Since the centralizer of g is $< g >$, we get that $Z \subset < g >$ and hence $Z =< g >$. It follows that P centralizes g, hence $P = Z =< g >$ and $|G|/2 = [G : P]$ is odd. $\square$

Proposition 39. *Let k be a field of characteristic $\neq 2$, assume that the 2-torsion $\mathrm{Br}(k)[2]$ of the Brauer group is non-trivial. Let $n \geq 8$ be an even integer. There exists an effective, reduced divisor $D \subset \mathbb{P}^1_{\bar{k}}$ of degree n with field of moduli equal to k such that $(\mathbb{P}^1_{\bar{k}}, D)$ is not defined over k.*

Proof. Since the 2-torsion of the Brauer group is non trivial, in particular k is infinite, and there exists a non-trivial rational curve P. Let $E \subset P$, $p \in P(k')$, $f : \mathbb{P}^1_{\bar{k}} \to P_{\bar{k}}$, $D \subset \mathbb{P}^1_{\bar{k}}$ be as in Lemma 37.

Let k_D/k be the field of moduli of $(\mathbb{P}^1_{\bar{k}}, D)$ and $\mathscr{G}$ the residual gerbe. Consider p as a divisor of degree 2 and let $\sqrt[2]{P, p}$ be 2nd root stack [1, Appendix B]. Write $k' = k(p)$. By definition of root stack, the $2 : 1$ map $\mathbb{P}^1_{k'} \to \mathbb{P}^1_{k'}$ ramified over the two points of $p_{k'} \subset \mathbb{P}^1_{k'}$ induces a morphism $\mathbb{P}^1_{k'} \to \sqrt[2]{\mathbb{P}^1_{k'}, p_{k'}} = (\sqrt[2]{P, p})_{k'}$ which is finite étale. This gives an identification of $(\sqrt[2]{P, p})_{k'}$ with the quotient stack $[\mathbb{P}^1_{k'}/\mu_2]$ over k. Because of this, the étale fundamental gerbe Π of $\sqrt[2]{P, p}$ [8, §8] is an abelian gerbe banded by μ_2 and the structure morphism $\sqrt[2]{P, p} \to \Pi$ is representable, since its base change to k' corresponds to the natural morphism $[\mathbb{P}^1_{k'}/\mu_2] \to \mathscr{B}_{k'}\mu_2$.

Let $E' \subset \sqrt[2]{P, p}$ be the inverse image of $E \subset P$ with the reduced structure, the base change $E'_{\bar{k}}$ naturally identifies with $[D/\mu_2] \subset [\mathbb{P}^1_{\bar{k}}/\mu_2] = (\sqrt[2]{P, p})_{\bar{k}}$. By applying Proposition 18 to $S = \Pi$ and $S' = \operatorname{Spec} k'$, we get a morphism $\Pi \to \mathscr{G}$ such that $\sqrt[2]{P, p}$ is the pullback of the universal family $\mathscr{P} \to \mathscr{G}$. Since $\operatorname{Spec} k$ is the coarse moduli space of Π, we obtain a factorization $\Pi \to \operatorname{Spec} k \to \operatorname{Spec} k_D$, i.e. $k_D = k$.

By Lemma 38, $\mathrm{Aut}(f)$ has odd index in $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$. Let $2a$ with a odd be the order of $\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)$. Let $\mathbf{P}$ be the compression of $(\mathbb{P}^1_{\bar{k}}, D)$. The morphism $\sqrt[2]{P, p} \to \mathscr{P}$ induces a morphism of coarse moduli spaces $P \to \mathbf{P}$ of degree a. Since a is odd and P is a non-trivial rational curve, it follows that $\mathbf{P}$ is a non-trivial rational curve as well, since otherwise P would have a divisor of odd degree. Since $|\mathrm{Aut}(\mathbb{P}^1_{\bar{k}}, D)|$ is even and $\mathbf{P}(k) = \emptyset$, by Theorem 31 $(\mathbb{P}^1_{\bar{k}}, D)$ is not defined over k. $\square$

Example 40. Let us construct some explicit examples of divisors not defined over the field of moduli. The underlying idea is the same of Proposition 39, but its realization will be much more elementary thanks to the fact that we work over the base field $k = \mathbb{R}$ of real numbers. Assume that $n = 4m \geq 12$. The example can be adapted to every $n \geq 8$ even, but the assumption $n = 4m \geq 12$ makes the argument a bit easier: since Proposition 39 is already very general, here we strive for simplicity rather than generality.

Choose m algebraically independent complex numbers $a_1, \ldots, a_m \in \mathbb{C}$ of norm 1, i.e. such that $\mathbb{Q}(a_1, \ldots, a_m)$ has transcendence degree m and $a_i\bar{a}_i = 1$, and define

$$D = \{\pm 2, \pm i/2, \pm a_1, \ldots, \pm a_m, \pm ia_1, \ldots, \pm ia_m\}.$$

The holomorphic transformation $z \mapsto iz^{-1}$ maps D to $\bar{D}$, so the field of moduli is $\mathbb{R}$. I claim that there are no twisted forms of $(\mathbb{P}^1_{\mathbb{C}}, D)$ over $\mathbb{R}$.

If there is such a twisted form (P, D') over $\mathbb{R}$, then the natural action of $\mathrm{Gal}(\mathbb{C}, \mathbb{R})$ on $(P(\mathbb{C}), D'(\mathbb{C})) \sim (\mathbb{P}^1_{\mathbb{C}}, D)$ defines an anti-holomorphic involution g of $(\mathbb{P}^1_{\mathbb{C}}, D)$, i.e. $g(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, $g^2 = 1$ and $g(D) = D$.

Recall that a generalized circle in $\mathbb{P}^1_{\mathbb{C}}$ is either a straight line or a circle in the usual sense. It is well known that the elements of $\mathrm{PGL}_2(\mathbb{C})$ map generalized circles into generalized circles; since this is also true for conjugation, we get that g maps generalized circles into generalized circles as well.

Write B for the circle of radius one centered in 0. Notice that B is the only generalized circle containing $4m$ points of D: in fact, since $n \geq 12$, a generalized circle containing $4m$ points of D contains at least $4m - 4 \geq 12 - 8 = 4$ points of $\{\pm a_1, \ldots, \pm a_m, \pm ia_1, \ldots, \pm ia_m\}$, and B is the only generalized circle containing four such points (given three different points, there is exactly one generalized circle passing through them). It follows that $g(B) = B$, which in turn implies that $g(\{\pm 2, \pm i/2\}) = \{\pm 2, \pm i/2\}$.

Since $g(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ is determined by its value on three points, then a, b, c, d are rational functions of $g(2), g(-2), g(i/2) \in \bar{\mathbb{Q}}$ (up to a scalar), hence we can assume that $a, b, c, d \in \bar{\mathbb{Q}}$ as well. Since $a_1, \ldots, a_m$ are algebraically independent over $\mathbb{Q}$, this implies that $g(a_1) \in \{\pm a_1, \pm ia_1\}$. In particular, we get the equality

$$\frac{a\bar{a}_1 + b}{c\bar{a}_1 + d} = i^r a_1$$

for some $0 \leq r \leq 3$. Since $\bar{a}_1 a_1 = 1$, rearranging the terms we obtain the equation

$$i^r d a_1^2 + (i^r c - b)a_1 - a = 0.$$

Since a_1 is transcendental, this equation implies $d = a = 0$ and $b = i^r c$, i.e. $g(z) = i^r \bar{z}^{-1}$. The identity $g^2 = 1$ implies that r is either 0 or 2, i.e. $g(z) = \pm\bar{z}^{-1}$. This is absurd, since $g(2) = \pm 1/2 \notin D$.

Data availability

No data was used for the research described in the article.

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