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The Stefan Problem with Mushy Region as a Scaling Limit of Stochastic PDE with Turbulent Transport

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Abstract
This work establishes a scaling limit theorem for the Stefan problem incorporating a mushy region, demonstrating that solutions to stochastic variants with turbulent transport terms converge to the solution to a deterministic partial differential equation. The analysis builds upon recent advances in stochastic phase-change modeling and turbulent flow mathematics in [8]. In the physical interpretation of an ice melting process, our result shows that turbulence accelerates ice melting.

Keywords Stefan problem · Maximal monotone operators · Turbulence · Transport noise · Scaling limit

Mathematics Subject Classification 60H15 · 80A22 · 76D03

1 Introduction

In the present work we are interested in the scaling limit of a Stefan type problem describing the melting/solidification process with mushy region and turbulent transport noise.

First, in Section 1.1 below we describe the physical problem on an heuristic ground. Then in Sections 1.2 and 1.3 we give the precise mathematical formulation.

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1.1 The Physical Model

As said above, this section is written without aiming at mathematical rigour, but with the purpose of explaining the model, and, in particular, the choice of the turbulent transport noise.

We are going to introduce a model for ice and water, where the variables are the velocity field of the water, $u(x,t)$, and the temperature $\theta(x,t)$, both depending on a state variable x and evolving in time t. We assume that $u(x,t)$ is also defined in the icy region, but that, in this case, it is just null. We further assume that water and ice occupy a 2-dimensional torus $\Pi^2 = \mathbb{R}^2/\mathbb{Z}^2$, namely the set $[0,1] \times [0,1]$ with identification of the opposite boundaries.

– To make things clearer, we assume that, at a fixed time $t > 0$, the ice occupies the set $I(t) \subset \Pi^2$. This set evolves in time.

– The water occupies the set $D \diagdown I(t)$.

– As we have already specified, the velocity $u(x,t)$ is equal to zero in the icy zone.

– The specific heat of ice is denoted by C_1, and its thermal conductivity κ_1. The corresponding constants of water are denoted by C_2, and κ_2, respectively. We further assume the presence of a heat source denoted by F which is assumed to be regular enough. This specificity appears into the equations of heat diffusion for the temperature $\theta(x,t)$ in ice

$$C_1 \partial_t \theta(x,t) = \kappa_1 \Delta\theta(x,t) + F(x,t), \qquad \text{for } (x,t) \in I(t) \times [0,T] \qquad \text{(ice)} \tag{1}$$

and of heat diffusion with transport in water:

$$\begin{aligned} C_2 \partial_t \theta(x,t) + u(x,t) \cdot \nabla\theta(x,t) &= \kappa_2 \Delta\theta(x,t) + F(x,t), \\ \text{for } (x,t) \in \big(D \diagdown I(t)\big) \times [0,T] & \qquad \text{(water)} \end{aligned} \tag{2}$$

with an initial condition $\theta(x,0) = \theta_0(x)$ additionally to the boundary condition $\theta|_{\partial I(t)} = 0$ for $t \in [0,T]$.

It is possible to unify these equations in a single model of Stefan type with sharp transition between ice and water, see the detailed model at page 221 from [1]. The model with a mushy region-a semi-solid transition zone between the fully liquid and fully solid phases-is also physically meaningful, and such a model is a better fit for our mathematical analysis requiring regularity. To this purpose, we assume that we have a threshold $\theta_0 > 0$ and three smooth functions γ, K, η such that

$$\gamma(r) = \begin{cases} C_1 r \text{ for } r \le 0, \\ C_2 r \text{ for } r \ge \theta_0, \end{cases}$$

$$K(r) = \begin{cases} \kappa_1 r \text{ for } r \le 0, \\ \kappa_2 r \text{ for } r \ge \theta_0, \end{cases}$$

$$\eta(r) = \begin{cases} 0 \text{ for } r \le 0, \\ r \text{ for } r \ge \theta_0, \end{cases}$$

and such that γ, K, η connect the two pieces in a smooth monotone way. In turn, we replace the equations (1), (2) by the single equation

$$\begin{cases} \partial_t \gamma(\theta(x,t)) + u(x,t) \cdot \nabla\eta(\theta(x,t)) = \Delta K(\theta(x,t)) + F(x,t), \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{3}$$

When $\theta(x,t) \le 0$ we have the equation for the temperature in ice (1); when $\theta(x,t) \ge \theta_0$ we have the equation for the temperature in water (2); and when $\theta(x,t) \in (0,\theta_0)$ we have an hybrid equation, with intermediate variable values of specific heat and thermal conductivity,

and a modified transport effect.
We further emphasize that the concept of icy zone $I(t)$, together with the boundary condition $\theta|_{\partial I(t)} = 0$ are only implicit in this formulation. The region $I(t)$ can be recovered as the level zone where the solution θ of equation (3) satisfies $\theta(x,t) \leq 0$. Likewise, the mushy region corresponds to the constraint zone where $\theta(x,t) \in (0, \theta_0)$.

In order to study this equation, we make the change of variable (recall that we are still arguing at an heuristic level, prior to reaching our final model) $X(x,t) = \gamma(\theta(x,t))$. We get

$$\begin{cases} \partial_t X(\xi,t) + u(x,t) \cdot \nabla \eta\left(\gamma^{-1}(X(\xi,t))\right) = \Delta K\left(\gamma^{-1}(X(\xi,t))\right) + F(\xi,t), \\ X(\xi,0) = \gamma(\theta_0(\xi)). \end{cases}$$

Our readers are invited to note that we have renamed the space variable to ξ instead of x, in order to avoid confusion with the notation X. This notation will be kept throughout the paper.
We also introduce the functions

$$\Psi = K \circ \gamma^{-1} \text{ and } \Gamma = \eta \circ \gamma^{-1},$$

and the initial condition $x_0 = \gamma \circ \theta_0$. We write the resulting Cauchy problem in the compact form

$$\begin{cases} \partial_t X - \Delta\Psi(X) + u \cdot \nabla\Gamma(X) = F, \\ X(\xi,0) = x_0(\xi). \end{cases}$$

To enhance readability, we introduce our model of **velocity field**. In order to reach interesting analytical results and formulae, we are not able to directly assume it to be the true solution of the Navier-Stokes equations. On the contrary, we make the assumption that u is a given (generalized) stochastic process, chosen such that its features mimic turbulent fluids. We apply, in a sense, a stochastic parameterization of the small turbulent scales of the fluid. We neglect large scales, which play a smaller part in the melting phenomenon we want to study. We choose

$$u(\xi,t) = \sum_{k=1}^{\infty} \alpha_k \sigma_k(\xi) \frac{d\beta_t^k}{dt},$$

where
– we have denoted by $(\alpha_k)_k$ a sequence of positive constants chosen such that the assumptions below will be satisfied,
– the sequence $(\sigma_k)_k$ is made out of divergence free smooth vector fields, and
– $\left(\beta_t^k\right)_k$ is a sequence of independent standard Brownian motions.
The stochastic product $u \cdot \nabla\Gamma(X)$ will be given a *Stratonovich sense*:

$$dX - \Delta\Psi(X)\,dt + \sum_{k=1}^{\infty} \alpha_k \sigma_k \cdot \nabla\Gamma(X) \circ d\beta_t^k = F dt.$$

This is a crucial element of our modeling, and it is, perhaps, worth explaining it in detail. The simplest motivation is that white noise is the idealization of finite-time-correlation processes, and Wong-Zakai principle prescribes that Stratonovich integration should be used when the idealization to delta correlated noise is made. A concurring explanation is that Stratonovich noise is the one preserving conservation laws. For instance, in the present case of a heat equation, if we assume $\Psi(X) = 0$ and $F = 0$-otherwise conservation is violated independently of the fluid transport-, and $\Gamma_0(X) = X$, the equation with Stratonovich noise

$$dX + \sum_{k=1}^{\infty} \alpha_k \sigma_k \cdot \nabla X \circ d\beta_t^k = 0$$

has the natural conservation property that heat is just transported. Namely, $\int f(X(\xi, t))\, d\xi$ is constant for every smooth function f (property which holds true when u is a smooth process with finite time correlation, and is maintained to the Stratonovich limit). Such properties no longer hold true for the equation driven by Itô noise (we omit the computations).

The choice of Stratonovich noise is not the unique one which preserves the physics of the problem. However, this choice plays a central part in our analysis: owing to the Stratonovich form, the Itô-Stratonovich corrector will arise, corrector responsible for the effect of increased ice melting by turbulence, which constitutes the final purpose of this work.

The noise model we have just described has gained significant attention in contemporary research, particularly regarding its mathematical properties and physical implications. For foundational aspects and recent developments, we direct readers to key works including [11, 13, 14], [15], and [16]. These investigations establish crucial connections between stochastic calculus frameworks and observed turbulent phenomena in multi-phase systems.

1.2 The Regular Problem and the Functional Setting

From now on, we switch to the rigorous study of the following equation

$$\begin{cases} \partial_t X - \Delta\Psi(X) + u \cdot \nabla\Gamma(X) = F \\ X(0, \xi) = x_0(\xi). \end{cases} \tag{4}$$

The reader is invited to note that the equation is set in porous media form with a convection term. For a setting in which the convection term is considered in a non-random framework, we refer our readers to [2]. The case with Robin boundary conditions was studied in [6] and [7]. For stochastic porous media equations in different frameworks, see [3]. For some critical cases see [4] and [9].

Throughout the remainder of the paper, unless explicitely stated otherwise, we will enforce the following.

Hypotheses

- Ψ and Γ are assumed to be monotone such that $\Psi' \geq \psi_0 > 0$ and $\Gamma' \geq 0$. Note that Ψ and Γ as assumed before are Lipschitz continuos, and to be null in zero.
- $\widetilde{\gamma}^{-1}$ is assumed to satisfy that $\left(\widetilde{\gamma}^{-1}\right)' < 1$. (This assumption is not restrictive and is necessary in order to apply the existence result for the stochastic equation.)
- $F \in C([0, T]; L^2(\Pi^2))$, $x \in L^2(\Pi^2)$.

In order to mathematically deal with the equation, we introduce a suitable theoretical framework.

As we already said, we consider $\Pi^2 = \mathbb{R}^2/\mathbb{Z}^2$ the two dimensional torus and $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$ the nonzero lattice points. Let $\left(H^{s,p}\left(\Pi^2\right), \|\cdot\|_{H^{s,p}}\right)$, $s \in \mathbb{R}$, $p \in (1, \infty)$ be a Bessel space of zero mean periodic functions.

We denote $H^s\left(\Pi^2\right) = H^{s,2}\left(\Pi^2\right)$ and we denote by $\langle \cdot, \cdot \rangle_{H^s}$ the corresponding scalar product. For $s > 0$ we can consider H^{-s} the dual of H^s and denote $\langle \cdot, \cdot \rangle_{H^{-s}, H^s}$ the duality pairing.

The Bessel space of zero mean vector field are defined as

$$\mathbf{H}^{s,p} = \left\{ (u_1, u_2)^t \;\middle|\; u_1, u_2 \in H^{s,p}\left(\Pi^2\right) \right\}$$

endowed with the scalar product

$$\langle u, v \rangle_{\mathbf{H}^s} = \langle u_1, v_1 \rangle_{H^s} + \langle u_2, v_2 \rangle_{H^s}\,, \quad \forall u, v \in \mathbf{H} \text{ and } \forall s \in \mathbb{R}.$$

For $s = 0$ we denote $\mathbf{H}^{0,2} = \mathbf{L}^2$.

If we consider Z a separable Hilbert space endowed with the norm $\|\cdot\|_Z$, we denoted by $C_{\mathcal{F}}^W([0, T]; Z)$ the space of weakly continuous processes $(X_t)_{t\in[0,T]}$ defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, with values in Z and adapted, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_Z^2\right] < \infty$$

and by $L_{\mathcal{F}}^p(0, T; Z)$, $p \in [1, \infty)$ the space of progressively measurable processes $(X_t)_{t\in[0,T]}$ such that

$$\mathbb{E}\left[\int_0^T \|X_t\|_Z^p\, dt\right] < \infty.$$

In several places we will denote by C a constant which is independent and may change from on line to another.

1.3 The Mathematical Model

We can now rigorously define as follows, the mathematical model which allows us to study equation (4).

First, in the turbulence term $u \cdot \nabla\Gamma(X)$ we take the Stratonovich interpretation

$$u(t, \xi) = \sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \circ d\beta_k$$

and we get

$$u \cdot \nabla\Gamma(X) = \sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \cdot \nabla\Gamma(X) \circ d\beta_k \tag{5}$$

where Γ is assumed to be Lipschitz and smooth and such that $\Gamma(r) = 0$ on $(-\infty, \varepsilon)$ for some $\varepsilon > 0$ very small and $\left|\Gamma'(r)\right| \leq C$.

Moreover $(\alpha_k)_{k\geq 1} \in l^2\left(\mathbb{Z}^2\right)$ satisfies

$$\sum_{k\in\mathbb{Z}_0^2} \alpha_k^2 = 1, \text{ and } \alpha_k = \alpha_l \text{ if } |k| = |l|\,. \tag{6}$$

Finally $(\sigma_k)_{k\geq 1}$ is a standard orthonormal basis of divergence free vectors fields in $\mathbf{L}^2$. More precisely, for a two dimensional torus Π^2 which will be seen as $[-1/2, 1/2]$ endowed with periodic boundary conditions, we have the usual trigonometric basis of $L^2\left(\Pi^2\right)$ which is

$$e_k(x) = \sqrt{2}\begin{cases} \cos(2\pi kx)\,, & k \in \mathbb{Z}_+^2 \\ \sin(2\pi kx)\,, & k \in \mathbb{Z}_-^2 \end{cases} \quad \text{for all } x \in \Pi^2$$

for

$$\mathbb{Z}_+^2 = \left\{k \in \mathbb{Z}_0^2 \ |(k_1 > 0) \text{ or } (k_1 = 0, k_2 > 0)\right\}$$

and $\mathbb{Z}_-^2 = -\mathbb{Z}_+^2$.

We define

$$\sigma_k(x) = \frac{k^\perp}{|k|} e_k(x), \quad k \in \mathbb{Z}_0^2 \text{ with } k^\perp = (k_2, -k_1).$$

Finally $(\beta_k)_k$ is a sequence of independent standard, one-dimensional Brownian motions supported on the probability space introduced before.

The Stratonovich noise that we introduced previously can be formulated in It ô form by the following transformation

$$\begin{aligned}\sum_{k\in\mathbb{Z}_0^2} \alpha_k \sigma_k \nabla\Gamma(X) \circ d\beta_k = &\sum_{k\in\mathbb{Z}_0^2} \alpha_k\sigma_k \cdot \nabla\Gamma(X)\, d\beta_k \\ &-\frac{1}{2}\sum_{k\in\mathbb{Z}_0^2} \alpha_k^2 div\left[\left(\Gamma'(X)\right)^2 \sigma_k \otimes \sigma_k \nabla X\right] dt.\end{aligned} \tag{7}$$

By applying Lemma 2.6 from [12] we have that

$$\sum_{k\in\mathbb{Z}_0^2} \alpha_k^2 (\sigma_k \otimes \sigma_k) = \frac{1}{2} I_2,$$

where I_2 is a two-dimensional identity matrix.

We denote by

$$g(r) = \frac{1}{4}\int_0^r \left(\Gamma'(x)\right)^2 dx, \quad r \in \mathbb{R}$$

which satisfies $g(0) = 0$ and is Lipschitz from the properties of Γ.

Going back to (7) we get that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}_0^2} \alpha_k\sigma_k \cdot \nabla\Gamma(X) \circ d\beta_k \\ &= \sum_{k\in\mathbb{Z}_0^2} \alpha_k\sigma_k \cdot \nabla\Gamma(X)\, d\beta_k - \frac{1}{4} div\left[\left(\Gamma'(X)\right)^2 \nabla X\right] dt \\ &= \sum_{k\in\mathbb{Z}_0^2} \alpha_k\sigma_k \cdot \nabla\Gamma(X)\, d\beta_k - \Delta g(X)\, dt.\end{aligned} \tag{8}$$

We can rigorously write equation (4) as

$$\begin{cases} dX - \Delta\Psi(X)\, dt - \Delta g(X)\, dt + \sum_{k\in\mathbb{Z}_0^2} \alpha_k\sigma_k \cdot \nabla\Gamma(X)\, d\beta_k = F dt \\ X(0, \xi) = x_0 \end{cases} \tag{9}$$

The existence of solution for the equation above was recently studied in the case of a general domain in [8] in the sense of the definition below. This result obviously applies in the present case by choosing a particular domain and a particular form of the orthonormal basis, which appears in the construction of the noise. Previously, the case of a stochastic porous media equation with divergence Itô noise was first studied in [5].

Definition 1 Let $x \in L^2\left(\Pi^2\right)$. We say that equation (9) has a distributional solution if there exist

- a filtered reference probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$,
- a sequence of independent $\mathcal{F}_t$ Brownian motions,
- an $H^{-1}\left(\Pi^2\right)-$valued continuous $\mathcal{F}_t-$ adapted process $X \in L^2\left(0, T; L^2\left(\Pi^2\right)\right)$

and the following holds true

$$\begin{aligned}\left(X(t), e_j\right)_2 &= \left(x, e_j\right)_2 + \int_0^t \left(F(s), e_j\right)_2 ds\\ &\quad + \int_0^t \left(\Psi(X(s)), \Delta e_j\right)_2 ds + \int_0^t \left(g(X(s)), \Delta e_j\right)_2 ds\\ &\quad + \sum_{k\in\mathbb{Z}_0^2} \int_0^t \alpha_k \left(\sigma_k \Gamma(X(s)), \nabla e_j\right) d\beta_k(s)\end{aligned}$$

$\mathbb{P} - a.s.$, $\forall t \in (0, T)$ and for all test functions e_j.

2 The Scaling Limit and the Main Result

Following the idea introduced in [17] we consider a sequence $\left\{\alpha^N\right\}_{N\in\mathbb{N}} \subseteq l^2\left(\mathbb{Z}_0^2\right)$ where each $\alpha^N = (\alpha_k^N)_{k\geq 1}$ satisfies (6) and

$$\lim_{N\to\infty} \left\|\alpha^N\right\|_{l^\infty} = 0, \tag{10}$$

and we denote X^N the corresponding solutions of equation (9) with $\left\{\alpha_k^N\right\}$ instead of $\{\alpha_k\}$ in the construction of the noise.

In the present work we prove that the law of X^N converges in the usual weak (or, more precisely weak-*) sense to a Dirac mass concentrated on the unique solution to the following deterministic porous media equation

$$\begin{cases} \partial_t X - \Delta\Psi(X) - \Delta g(X) = F \\ X(0, \xi) = x. \end{cases} \tag{11}$$

The new term $\Delta g(X)$ enhances the diffusion given by the term $\Delta\Psi(X)$. Therefore our result may be interpreted as ice melting enhancement by a turbulent fluid.

Since $\Psi + g$ is a maximal monotone and Lipschitz operator (as real-valued function, hence as L^2-defined functional), the equation (11) has a unique distributional solution owing to the classical PDE theory. Precisely, if $x \in H^{-1} \cap L^1$ the equation has a strong solution by Theorem 5.3 of [1]. In our case, since $x \in L^2$ and we are looking for a solution belonging to L^2, we can apply Theorem 3.4.1 from [3] and have existence and uniqueness of a distributional solution, which is exactly what is meant in the Definition 1.

We can write the solution to equation (11) as a PDE weak one

$$X \in L^2\left((0, T) \times \Pi^2\right) \cap C\left([0, T] \times H^{-1}\left(\Pi^2\right)\right)$$

in the following form

$$\left(X\left(t\right),e_j\right)_2=\left(x,e_j\right)_2+\int_0^t\left(F\left(s\right),e_j\right)_2ds$$
$$+\int_0^t\left(\Psi\left(X\left(s\right)\right),\Delta e_j\right)_2ds+\int_0^t\left(g\left(X\left(s\right)\right),\Delta e_j\right)_2ds,$$

for all test functions e_j provided above.

We are now able to state and prove the main result of the paper.

Theorem 2 *Let $\left\{\alpha^N\right\}_{N\in\mathbb{N}}\subseteq l^2\left(\mathbb{Z}_0^2\right)$ be a sequence satisfying the assumptions (6) and (10). We denote*

- *by X^N a sequence of solutions to equation (9), where α is replaced by (α^N), constructed in [8] and fulfilling inequality (12) below and*
- *by $\nu^N:=\mathbb{P}_{X^N}$ the law of these solutions supported by $L^2\left((0,T)\times\Pi^2\right)\cap C\left([0,T];H^{-1}\left(\Pi^2\right)\right)$.*

Then, the family $\left\{\nu^N\right\}$ is tight on $C\left([0,T];H^{-1}\left(\Pi^2\right)\right)$, and it converges weakly to the Dirac measure δ_X where X is the unique solution of the equation (11).

Proof Before providing the main steps of the proof, let us emphasize that, throughout the argument, we drop the F term, as it does not constitute a specific technical difficulty.

Step I (The $\mathbf{L}^2$ estimate) In [8], Theorem 3, we have proved the existence of solutions to equation (9) by a Galerkin scheme, see equation (13) in [8]. From this, the following inequality follows:

$$\begin{aligned}&\left\|X^N\left(t\right)\right\|_2^2+2\int_0^t\left(\nabla\Psi\left(X^N\left(s\right)\right),\nabla X^N\left(s\right)\right)_2ds\\&+2\int_0^t\left(\nabla g\left(X^N\left(s\right)\right),\nabla X^N\left(s\right)\right)_2ds\\&\leq\|x\|_2^2+\frac{1}{2}\int_0^t\left\|\nabla\Gamma\left(X^N\left(s\right)\right)\right\|_2^2ds\\&+2\sum_{k\in\mathbb{Z}_0^2}\int_0^T\left(\alpha_k^N\sigma_k\Gamma\left(X^N\left(s\right)\right),\nabla X^N\right)_2d\beta_k\left(s\right).\end{aligned}\tag{12}$$

We consider, separately

$$\begin{aligned}2\int_0^t\left(\nabla g\left(X^N\left(s\right)\right),\nabla X^N\left(s\right)\right)_2ds&=\frac{2}{4}\int_0^t\left(\left(\Gamma'\right)^2\left(X^N\left(s\right)\right),\left|\nabla X^N\left(s\right)\right|^2\right)_2ds\\&=\frac{1}{2}\int_0^t\left\|\nabla\Gamma\left(X^N\left(s\right)\right)\right\|_2^2ds\end{aligned}$$

and

$$\begin{aligned}\left(\alpha_k^N\sigma_k\Gamma\left(X^N\left(s\right)\right),\nabla X^N\right)_2&=\left(\alpha_k^N\nabla\widetilde{\Gamma}\left(X^N\left(s\right)\right),\sigma_k\right)_2\\&=-\left(\alpha_k^N\widetilde{\Gamma}\left(X^N\left(s\right)\right),div\ \sigma_k\right)_2=0,\end{aligned}$$

where $\widetilde{\Gamma}$ is a primitive of Γ, and the last equality follows from the divergence condition we have mentioned when introducing the σ_k family.

Going back to (12), we get, $\mathbb{P}$-a.s.

$$\left\| X^N(t) \right\|_2^2 + 2\int_0^t \left(\nabla\Psi\left(X^N(s)\right), \nabla X^N(s) \right)_2 ds \leq \|x\|_2^2,$$

for all $t \geq 0$, and, due to the strong monotonicity of Ψ, we obtain that

$$\sup_{t\in[0,T]} \left\| X^N(t) \right\|_2^2 + 2\int_0^t \psi_0 \left\| X^N(s) \right\|_{H^1(\Pi^2)}^2 ds \leq \|x\|_2^2. \tag{13}$$

We obtain that, along some subsequence, still indexed by N to simplify reading,

$$\begin{aligned} X^N &\rightharpoonup X \text{ weakly* in } L^2(\Omega; L^\infty\left(0,T; L^2\left(\Pi^2\right)\right)), \\ X^N &\rightharpoonup X \text{ weakly in } L^2(\Omega; L^2\left(0,T; H^1\left(\Pi^2\right)\right)). \end{aligned}$$

Keeping in mind that Ψ and g are Lipschitz continuous we also have that

$$\begin{aligned} \Psi\left(X^N\right) &\rightharpoonup \chi \text{ weakly in } L^2(\Omega; L^2\left(0,T; H^1\left(\Pi^2\right)\right)), \\ g\left(X^N\right) &\rightharpoonup \rho \text{ weakly in } L^2(\Omega; L^2\left(0,T; H^1\left(\Pi^2\right)\right)). \end{aligned}$$

In order to identify the limits $\Psi(X) = \chi$ and $g(X) = \rho$ we need some strong convergence that will be observed as explained in the next step.

Step II (Tightness)

For each $r \geq 2$ there is a constant C independent of N such that for any t and s such that $0 \leq s \leq t \leq T$ we take the difference

$$\begin{aligned} \left(X^N(t) - X^N(s), e_j \right)_2 &= \int_s^t \left(\Psi\left(X^N(l)\right), \Delta e_j \right)_2 dl + \int_s^t \left(g\left(X^N(l)\right), \Delta e_j \right)_2 dl \\ &\quad + \sum_{k\in\mathbb{Z}_0^2} \int_s^t \alpha_k \left(\sigma_k \Gamma\left(X^N(l)\right), \nabla e_j \right) d\beta_k(l) \\ &= I_{s,t}^1 + I_{s,t}^2 + I_{s,t}^3, \end{aligned} \tag{14}$$

and we compute $\mathbb{E}\left(\left(X^N(t) - X^N(s), e_j\right)_2^r\right)$ as follows

$$\begin{aligned} \mathbb{E}\left[\left|I_{s,t}^1\right|^r\right] &= \mathbb{E}\left[\left|\int_s^t \left(\Psi\left(X^N(l)\right), \Delta e_j \right)_2 dl \right|^r\right] \\ &\leq \mathbb{E}\left[\left|\int_s^t \left\| \Psi\left(X^N(l)\right) \right\|_2 \|\Delta e_j\|_\infty dl \right|^r\right] \\ &\leq C\lambda_j^r (t-s)^r, \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left[\left|I_{s,t}^2\right|^r\right] &= \mathbb{E}\left[\left|\int_s^t \left(g\left(X^N(l)\right), \Delta e_j \right)_2 dl \right|^r\right] \\ &\leq \mathbb{E}\left[\left|\int_s^t \left\| g\left(X^N(l)\right) \right\|_2 \|\Delta e_j\|_\infty dl \right|^r\right] \\ &\leq C\lambda_j^r (t-s)^r, \end{aligned}$$

(using the fact that g is Lipschitz because g' is bounded as Γ')

$$\mathbb{E}\left[\left|I_{s,t}^3\right|^r\right] = \mathbb{E}\left[\left|\sum_{k\in\mathbb{Z}_0^2}\int_s^t \alpha_k\left(\sigma_k\Gamma\left(X^N(l)\right),\nabla e_j\right)d\beta_k(l)\right|^r\right]$$
$$\le C\mathbb{E}\left[\left|\int_s^t\sum_{k\in\mathbb{Z}_0^2}\left(\Gamma\left(X^N(l)\right),\alpha_k\sigma_k\nabla e_j\right)_2^2 dl\right|^{\frac{r}{2}}\right]$$
$$\le C\lambda_j^{\frac{r}{2}}|t-s|^{\frac{r}{2}}.$$

Going back in (14) we get that

$$\mathbb{E}\left(\left(X^N(t)-X^N(s),e_j\right)_2^r\right)\le C|t-s|^{\frac{r}{2}}\left(\lambda_j^{\frac{r}{2}}+\lambda_j^r\right).$$

Using this estimate we can show that for some $\beta>4$ and $r\ge 4$ we have

$$\mathbb{E}\left[\left\|X^N(t)-X^N(s)\right\|_{H^{-\beta}}^r\right]\le C|t-s|^{\frac{r}{2}}$$

for some C independent of N.

For details, see Lemma 5 from [8]. Then, for $\alpha\in\left(\frac{1}{r},\frac{1}{2}\right)$ we have

$$\mathbb{E}\left[\int_0^T\int_0^T\frac{\left\|X^N(t)-X^N(s)\right\|_{H^{-\beta}}^r}{|t-s|^{1+\alpha r}}dtds\right]\le C.$$

Combining the relation above with the estimate (13) from *Step I* we get that

$$\mathbb{E}\sup_{t\in[0,T]}\left\|X^N(t)\right\|_2^2+\mathbb{E}\int_0^t\int_0^t\frac{\left\|X^N(t)-X^N(s)\right\|_{H^{-\beta}}^r}{|t-s|^{1+\alpha r}}dtds\le C. \tag{15}$$

We can use the compactness result in Corollary 5 from [19] and obtain that

$$L^\infty\left(0,T;L^2\left(\Pi^2\right)\right)\cap W^{\alpha,r}\left(0,T;H^{-\beta}\left(\Pi^2\right)\right)\subset C\left([0,T];H^{-1}\left(\Pi^2\right)\right),$$

with compact inclusion.

More precisely, if we take

$$K_R=\left\{f\in C\left([0,T];H^{-1}\left(\Pi^2\right)\left|\|f\|_{L^\infty(0,T;L^2(\Pi^2))}+\|f\|_{W^{\alpha,r}(0,T;H^{-\beta}(\Pi^2))}\le R\right)\right\},$$

we have that K_R is compact in $C\left([0,T];H^{-1}\left(\Pi^2\right)\right)$.

Then, for all $\varepsilon>0$ we have by using Markov's inequality that

$$\nu^N\left(K_R^c\right)\le\frac{C}{R}\le\varepsilon$$

for R sufficiently large, and we can conclude that the family of laws $\left\{\nu^N\right\}_N$ is tight in the space $C\left([0,T];H^{-1}\left(\Pi^2\right)\right)$.

Step III (Passage to the limit)

By using the Skorokhod representation theorem we can find an auxiliary probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ and the processes $\left(\widetilde{X}^N,\widetilde{W}^N=\left\{\widetilde{\beta}^{N,k}\right\}_{k\in\mathbb{Z}_0^2}\right)$ and $\left(\widetilde{X},\widetilde{W}=\left\{\widetilde{\beta}^k\right\}_{k\in\mathbb{Z}_0^2}\right)$ such that

$$\begin{aligned} \widetilde{X}^N &\to \widetilde{X} \text{ in } C\left([0,T]; H^{-1}\left(\Pi^2\right)\right) \ \mathbb{P}\text{-a.s.} \\ \widetilde{\beta}^{N,k} &\to \widetilde{\beta}^k \text{ in } C\left([0,T]; \mathbb{Z}_0^2\right) \ \mathbb{P}\text{ -a.s.} \end{aligned}$$

where the convergence of $\widetilde{W}^N$ to $\widetilde{W}$ can be seen as the uniform convergence of cylindrical Wiener processes $\widetilde{W}^N = \sum_{k\in\mathbb{Z}_0^2} e_k \widetilde{\beta}^{N,k}$ to $\widetilde{W} = \sum_{k\in\mathbb{Z}_0^2} e_k \widetilde{\beta}^k$ on the suitable Hilbert space $L^2\left(\Pi^2\right)$.

On the other hand, we have from *Step I* that

$$\begin{aligned} \Psi\left(\widetilde{X}^N\right) &\rightharpoonup \chi \text{ weakly in } L^2(\Omega; L^2\left(0,T; H^1\left(\Pi^2\right)\right)) \\ g\left(\widetilde{X}^N\right) &\rightharpoonup \rho \text{ weakly in } L^2(\Omega; L^2\left(0,T; H^1\left(\Pi^2\right)\right)). \end{aligned}$$

Since Ψ and g are maximal monotone operators in L^2 we get by combining the strong and weak convergences in a classical way $\Psi\left(\widetilde{X}\right) = \chi$ and $g\left(\widetilde{X}\right) = \rho$. The proof is in the spirit of the argument on page 40 from [3], as follows. The Lipschitz-continuity of Ψ, hence the (strong) monotonicity of Ψ^{-1} yields

$$\begin{aligned} \|\Psi(\widetilde{X}^N) - \Psi(\widetilde{X}^M)\|_2^2 &\le C\left(\Psi(\widetilde{X}^N) - \Psi(\widetilde{X}^M), \widetilde{X}^N - \widetilde{X}^M\right)_2 \\ &= -C\left(\Delta\Psi(\widetilde{X}^N) - \Delta\Psi(\widetilde{X}^M), \widetilde{X}^N - \widetilde{X}^M\right)_{H^{-1}}. \end{aligned}$$

Owing to the convergence $\widetilde{X}^N \to \widetilde{X}$ in $C\left([0,T]; H^{-1}\left(\Pi^2\right)\right)$ $\mathbb{P}$-a.s., completed with the boundedness of $\Psi(\widetilde{X}^N)$ in $L^2(\Omega; L^2\left(0,T; H^1\left(\Pi^2\right)\right))$, hence $\mathbb{P}$-a.s. in $L^2\left(0,T; H^1\left(\Pi^2\right)\right)$, it follows that

$$\int_0^T \|\Psi(\widetilde{X}^N(t)) - \Psi(\widetilde{X}^M(t))\|_2^2 dt \to 0, \mathbb{P} - a.s., \text{ as } M, N \to \infty.$$

The reader is reminded that Ψ has linear growth and that the bounds in (13) hold uniformly in $\omega \in \Omega$. As a simple consequence, one applies Lebesgue's dominated convergence to get

$$\mathbb{E}\Big[\int_0^T \|\Psi(\widetilde{X}^N(t)) - \Psi(\widetilde{X}^M(t))\|_2^2 dt\Big] \to 0.$$

Combining this with

$$\widetilde{X}^N \rightharpoonup \widetilde{X} \text{ weakly* in } L^2(\Omega \times [0,T] \times \Pi^2),$$

and the fact that

$$\Psi \text{ is maximal monotone on } L^2(\Omega \times [0,T] \times \Pi^2),$$

one is able to identify the limit. The argument for g is identical and will be omitted.

We can now pass to the limit in

$$\begin{aligned} \left(\widetilde{X}^N(t), e_j\right)_2 &= \left(x, e_j\right)_2 \\ &\quad + \int_0^t \left(\Psi\left(\widetilde{X}^N(s)\right), \Delta e_j\right)_2 ds + \int_0^t \left(g\left(\widetilde{X}^N(s)\right), \Delta e_j\right)_2 ds \\ &\quad + \sum_{k\in\mathbb{Z}_0^2} \int_0^t \alpha_k^N \left(\sigma_k \Gamma\left(\widetilde{X}^N(s)\right), \nabla e_j\right) d\widetilde{\beta}_k(s) \end{aligned}$$

as follows.

1. First, let us note that the left-hand term converges $\mathbb{P}$-a.s. to $\left(\widetilde{X}(t), e_j\right)_2$.
2. For the stochastic term, we use the Burkholder-Davis-Gundi inequality and we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\sum_{k\in\mathbb{Z}_0^2}\int_0^t \alpha_k^N\left(\Gamma\left(\widetilde{X}^N(s)\right), \sigma_k\nabla e_j\right)_2 d\widetilde{\beta}_k(s)\right|^2\right] \\
&\leq C\mathbb{E}\left[\sum_{k\in\mathbb{Z}_0^2}\int_0^T\left(\alpha_k^N\right)^2\left(\Gamma\left(\widetilde{X}^N(s)\right), \sigma_k\nabla e_j\right)_2^2 ds\right] \\
&\leq C\left\|\alpha^N\right\|_{l^\infty}^2\mathbb{E}\left[\int_0^T\left\|\Gamma\left(\widetilde{X}^N(s)\right)\nabla e_j\right\|_2^2 ds\right] \\
&\leq C\left\|\alpha^N\right\|_{l^\infty}^2\lambda_j^2\mathbb{E}\left[\int_0^T\left\|\widetilde{X}^N(s)\right\|_2^2 ds\right]\to 0.
\end{aligned}
$$

   for $N\to\infty$.
3. We deduce that both the left-hand member and the stochastic integral converge to $\left(\widetilde{X}(t), e_j\right)_2$ and 0 respectively in $L^2(\Omega; L^2(0,T;\mathbb{R}))$ in the strong, hence the weak topology. Passing to the limit in this space and in the weak sense, we get

$$
\begin{aligned}
\left(\widetilde{X}(t), e_j\right)_2 &= \left(x, e_j\right)_2 \\
&\quad+\int_0^t\left(\Psi\left(\widetilde{X}(s)\right), \Delta e_j\right)_2 ds+\int_0^t\left(g\left(\widetilde{X}(s)\right), \Delta e_j\right)_2 ds.
\end{aligned}
$$

This concludes our proof. □

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Declarations

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