



On the existence of isoperimetric regions in manifolds with nonnegative Ricci curvature and Euclidean volume growth

Gioacchino Antonelli¹ · Elia Bruè² · Mattia Fogagnolo³ · Marco Pozzetta⁴

Received: 22 December 2021 / Accepted: 25 January 2022
© The Author(s) 2022, corrected publication 2022

Abstract

In this paper we provide new existence results for isoperimetric sets of large volume in Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth. We find sufficient conditions for their existence in terms of the geometry at infinity of the manifold. As a byproduct we show that isoperimetric sets of big volume always exist on manifolds with nonnegative sectional curvature and Euclidean volume growth. Our method combines an asymptotic mass decomposition result for minimizing sequences, a sharp isoperimetric inequality on nonsmooth spaces, and the concavity property of the isoperimetric profile. The latter is new in the generality of noncollapsing manifolds with Ricci curvature bounded below.

Mathematics Subject Classification 49Q20 · 49J45 · 53A35 · 53C23 · 49J40

Communicated by A. Mondino.

✉ Gioacchino Antonelli
gioacchino.antonelli@sns.it
Elia Bruè
elia.brue@math.ias.edu
Mattia Fogagnolo
mattia.fogagnolo@sns.it
Marco Pozzetta
marco.pozzetta@unina.it

¹ Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italy

² Institute for Advanced Study, 1 Einstein Dr., Princeton, NJ 05840, USA

³ Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore, Piazza dei Cavalieri, 3, 56126 Pisa, Italy

⁴ Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126 Napoli, Italy

Contents

1	Introduction
1.1	Main existence results
1.2	Concavity of the isoperimetric profile
1.3	Strategy of proof of the existence results
1.4	Sharpness and counterexamples
1.5	Plan of the paper
2	Definitions and auxiliary results
2.1	Preliminaries on Riemannian manifolds and metric measure spaces
2.2	RCD spaces
2.3	Finite perimeter sets and minimizing sequences for the isoperimetric problem
2.4	The collapsed case
2.5	Sharp Sobolev inequality on $CD(0, n)$ spaces
3	Concavity properties of the isoperimetric profile
3.1	First and second variations of the penalized perimeter
3.2	Proof of Theorem 1.4
3.3	Consequences of the concavity properties of the isoperimetric profile
4	Main isoperimetric existence results
4.1	Proof of Theorem 1.1
4.2	Proof of Theorem 1.2
4.3	Further existence results
4.4	Proof of Theorem 1.3
	References

1 Introduction

Given a Riemannian manifold (M^n, g) of dimension $n \geq 2$ we denote by d , vol , and Ric the Riemannian distance, the volume measure, and the Ricci tensor of (M^n, g) , respectively.

For any $V \in [0, \text{vol}(M^n))$, the *isoperimetric problem* consists in studying the minimization problem

$$I_{(M^n, g)}(V) := \inf\{P(\Omega) : \Omega \subset M^n \text{ set of finite perimeter with } \text{vol}(\Omega) = V\}, \quad (1.1)$$

where $P(\Omega)$ denotes the perimeter of Ω , see Sect. 2. We shall drop the superscript n in (M^n, g) and the subscript (M^n, g) in $I_{(M^n, g)}$ when there is no risk of confusion. The function $I_{(M^n, g)}$ is the so-called *isoperimetric profile* of (M^n, g) . Any set of finite perimeter $E \subset M^n$ with $\text{vol}(E) = V$ and $P(E) = I_{(M^n, g)}(V)$ is called *isoperimetric set* or *isoperimetric region*.

The problem of the existence of isoperimetric regions in the setting of noncompact manifolds is a hard problem that has seen several important progresses in the last years. Let us just mention some of the most important contributions in the field related to the topics of this work. Major results in the application of a direct method for proving existence of isoperimetric sets in manifolds with lower bounds on the Ricci curvature are contained in [63, 69]. The methods employed there have been generalized in [13]. As we are going to see, such generalization will be fundamental for the present work. In dimension 2, a complete positive answer to the existence issue of isoperimetric sets under nonnegative curvature has been given in [74]. The existence of isoperimetric sets in 3-manifolds with nonnegative scalar curvature and asymptotically flat asymptotics has been established in [25]. Existence results for isoperimetric sets of large volumes in asymptotically flat manifolds were also obtained in [41, 42, 68], in asymptotically hyperbolic spaces in [33], and in the asymptotically conical case in [34]. When the ambient space is a nonnegatively Ricci curved cone, isoperimetric regions exist for any given volume and are actually characterized [66]. In Euclidean solid

cones, the problem has been investigated in [76]. In more general convex bodies, it is treated in [57].

Differential properties of the isoperimetric profile, some of which are generalized in the present paper, have been studied in [18–21,65]. A regularity theory for isoperimetric sets was established for example in [50,64,81,83], and, recently, it has been partly generalized in [14].

1.1 Main existence results

In this paper we provide new existence results for isoperimetric regions of large volume in the setting of manifolds with nonnegative Ricci curvature and *Euclidean volume growth*, i.e., such that

$$\text{AVR}(M^n, g) := \lim_{r \rightarrow +\infty} \frac{\text{vol}(B_r(x))}{\omega_n r^n} \in (0, 1],$$

where ω_n is the volume of the n -dimensional unit ball. The above limit exists as a consequence of the Bishop–Gromov monotonicity (cf. Theorem 2.11). The same notion can be given for $\text{CD}(0, n)$ metric measure spaces (X, d, \mathfrak{m}) (replacing vol with \mathfrak{m}), a class of nonsmooth spaces with a synthetic notion of nonnegative Ricci curvature and dimension bounded above by $n \geq 1$, see Sect. 2.2.

The Euclidean volume growth assumption implies that (M^n, g) is *noncollapsed*, i.e. there exists $v > 0$ such that $\text{vol}(B_1(x)) \geq v$ for every $x \in M$. The latter can be shown to be equivalent to $I_{(M^n, g)}(V) \neq 0$ for some (and in fact for any) $V > 0$ (cf. Proposition 2.18). Hence noncollapsedness is necessary for the existence of isoperimetric regions (see also the counterexamples discussed in [13, Section 4.3]).

Before stating our first result we recall that (X, d, \mathfrak{m}) is said to *split (a line)* if it is isometric as a metric measure space to $(\mathbb{R} \times Y, d_{\mathbb{R}} \otimes d_Y, \mathfrak{m}_{\mathbb{R}} \otimes \mathfrak{m}_Y)$ for some (Y, d_Y, \mathfrak{m}_Y) . Here $d_{\mathbb{R}^k}, \mathfrak{m}_{\mathbb{R}^k}$ are just the Euclidean distance and the Lebesgue measure on \mathbb{R}^k , respectively, for any $k \in \mathbb{N}$.

Theorem 1.1 *Let (M^n, g) be a complete noncompact noncollapsed Riemannian manifold with $\text{Ric} \geq 0$. We write*

$$(M^n, g) = (\mathbb{R}^k \times N^{n-k}, g_{\mathbb{R}^k} + g_N),$$

where (N^{n-k}, g_N) does not split a line, and $0 \leq k \leq n$. Assume that there exists $\varepsilon > 0$ such that every pmGH limit $(X_\infty, d_\infty, \mathfrak{m}_\infty, x_\infty)$ of sequences $\{(M, d, \text{vol}, p_i)\}_{i \in \mathbb{N}_{\geq 0}}$ with $p_i := (0, x_i) \in M$, and $d_N(x_i, x_0) \rightarrow +\infty$ as $i \rightarrow +\infty$, satisfies

$$\text{AVR}(X_\infty, d_\infty, \mathfrak{m}_\infty) \geq \text{AVR}(M^n, g) + \varepsilon.$$

Then there exists $V_0 > 0$ such that for every $V \geq V_0$ there exists an isoperimetric region of volume V in M .

Recall that any sequence $\{(M, d, \text{vol}, p_i)\}_{i \in \mathbb{N}}$ admits limit points in the pmGH topology as a consequence of the Gromov compactness theorem (see Definition 2.4 and Remark 2.7). Such limits could in general be nonsmooth, and just belong to the class of RCD spaces, that are pointed metric measure spaces with a suitable synthetic notion of Ricci curvature bounded below.

The hypothesis in Theorem 1.1 is verified when $\text{AVR}(M^n, g) > 0$ and the manifold N possesses no asymptotic cones that split a line (see Definition 2.14 for the notion of asymptotic cone). Hence we deduce the following.

Theorem 1.2 *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric} \geq 0$ and $\text{AVR}(M^n, g) > 0$. We write*

$$(M^n, g) = (\mathbb{R}^k \times N^{n-k}, g_{\mathbb{R}^k} + g_N),$$

where (N^{n-k}, g_N) does not split a line, and $0 \leq k \leq n$. If no asymptotic cone of N splits a line, then there is $V_0 > 0$ such that for every $V \geq V_0$ there exists an isoperimetric region of volume V in M .

It is well-known that when (M^n, g) has nonnegative sectional curvature the asymptotic cone is unique and it splits if and only if the manifold splits, see [16, pages 58-59], [54, Proposition 4.2]. A clear account on the classical method for proving the latter properties is contained in [51]. Since in this work we are interested in the case of Euclidean volume growth, we provide a proof of the previous facts on asymptotic cones in such special case in Theorem 4.6. The latter phenomenon marks an important difference with the nonnegative Ricci curvature, see [36, Theorem 1.4] and Sect. 1.4 below. In turn, we derive the following existence theorem.

Theorem 1.3 *Let (M^n, g) be a complete Riemannian manifold with nonnegative sectional curvature and Euclidean volume growth. Then there exists $V_0 > 0$ such that for every $V \geq V_0$ there exists an isoperimetric region of volume V .*

Let us briefly discuss the condition in Theorem 1.2. When M does not split any line, it amounts to say that no asymptotic cone splits a line. It is not hard to see that the latter is equivalent to the following spectral gap property: there exists $\varepsilon > 0$ such that

$$\lambda_1(-\Delta_Z) \geq n - 1 + \varepsilon \tag{1.2}$$

for all the cross-sections (Z, d_Z) of asymptotic cones. Here $\lambda_1(-\Delta_Z)$ denotes the first eigenvalue of the Laplacian in Z . The same condition has already appeared in [34, Theorem 1], where its validity was guaranteed by the uniqueness and smoothness of the asymptotic cone.

We point out that (1.2) has a clear variational meaning: $-\Delta_Z - (n - 1)$ coincides with the Jacobi operator associated to the second variation of the perimeter of $B_1(z) \subset C(Z)$, where $z \in C(Z)$ is the tip point. Hence, (1.2) simply says that $B_1(z)$ is a strictly stable isoperimetric region on $C(Z)$. In view of this we can, at least at a heuristic level, rephrase our statement as follows: isoperimetric regions of big volume exist provided the isoperimetric sets of the model spaces at infinity are uniformly stable.

1.2 Concavity of the isoperimetric profile

We now state our last result regarding concavity properties of the isoperimetric profile. This will be an important ingredient in the proof of the previous existence theorems and has an independent interest. Given $I \subset \mathbb{R}$ an interval, and given $f : I \rightarrow \mathbb{R}$, we set, for any $x \in I$,

$$\overline{D}^2 f(x) := \limsup_{h \rightarrow 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}.$$

Theorem 1.4 *Suppose that (M^n, g) is noncollapsed and that $\text{Ric} \geq (n - 1)K$ for $K \leq 0$. Then for any $\tilde{V} \in (0, \text{vol}(M^n))$ there are $\delta > 0$ and $C \geq 0$ such that the function $V \mapsto I_{(M^n, g)}(V) - CV^2$ is concave on $(\tilde{V} - \delta, \tilde{V} + \delta)$. Moreover,*

$$\overline{D}^2 I_{(M^n, g)}(V) \leq \frac{(n - 1)|K|}{I_{(M^n, g)}(V)}, \tag{1.3}$$

holds for every $V \in (0, \text{vol}(M^n))$, and equivalently the inequality

$$I''_{(M^n, g)} \leq \frac{(n - 1)|K|}{I_{(M^n, g)}}, \tag{1.4}$$

holds in the distributional sense on $(0, \text{vol}(M^n))$. As a consequence, $I_{(M^n, g)}$ is twice differentiable almost everywhere and at any value V of twice differentiability it holds the pointwise estimate

$$I''_{(M^n, g)}(V) \leq \frac{(n - 1)|K|}{I_{(M^n, g)}(V)}. \tag{1.5}$$

If $K = 0$ then $I_{(M^n, g)}$ is concave on $(0, \text{vol}(M^n))$.

The previous result is obtained under the natural assumptions for the study of the isoperimetric problem, namely noncollapsedness and Ricci bounded below. In particular, differently from the classical results in [18–21,63,65], the existence of isoperimetric regions is not assumed *a priori*, and, in fact, we will employ Theorem 1.4 as a tool to provide existence. On the other hand, it appears challenging also relying on Nardulli’s generalized existence of isoperimetric sets as done in [63], since in our setting the mass lost at infinity is recovered in RCD spaces where the classical GMT tools needed to compute the second variation of the perimeter are not available at present.¹

To this end, in order to prove Theorem 1.4, we introduce an approximation of the perimeter functional on the manifold M by a sequence of penalized perimeters P_k given by the sum of the usual perimeter and a potential term. The penalization in the definition of P_k , see (3.1), implies the existence of volume constrained minimizers for P_k for any k . Also, the profile I_k corresponding to P_k , see (3.2), converges to the isoperimetric profile $I_{(M^n, g)}$ of the manifold locally uniformly. Hence Theorem 1.4 eventually follows by studying the concavity properties of I_k by means of the existence of minimizers for P_k , and then by passing the estimates to the limit with respect to k .

As a consequence of Theorem 1.4, we get that under the same assumptions the isoperimetric profile is a locally Lipschitz function (Corollary 3.5), which improves the result of local Hölder continuity contained in [67].

As anticipated, Theorem 1.4 has useful consequences for the the existence of isoperimetric regions in manifolds with nonnegative Ricci curvature. This is due to the fact that the concavity of the profile provides information on the asymptotic behavior of $I_{(M^n, g)}$ and $I'_{(M^n, g)}$, see Corollary 3.6. Under the additional assumption of Euclidean volume growth, we can further conclude that $I_{(M^n, g)}$ is strictly increasing (Corollary 3.8). The latter result has to be compared with [75], where the author considers hypotheses on the sectional curvature instead, but without assumptions on the asymptotic volume ratio.

1.3 Strategy of proof of the existence results

Broadly speaking the proof of Theorem 1.1 is based on the direct method of the Calculus of Variations and a concentration-compactness argument, firstly employed in the study of the isoperimetric problem by Nardulli in [69]. Given $V > 0$ we consider a minimizing sequence E_i such that $\text{vol}(E_i) = V$ and $\lim_{i \rightarrow \infty} P(E_i) = I(V)$, and we aim at studying

¹ The first and the fourth author very recently provided, in a joint work with E. Pasqualetto and D. Semola [15], variational tools capable of estimating the second variation of the perimeter in nonsmooth spaces, giving an alternative proof of the concavity properties of the isoperimetric profile in our setting.

limits in a generalized sense. Notice that the existence of a limit in $L^1(M^n, \text{vol})$ would immediately give the existence of isoperimetric regions of volume V , as a consequence of the lower semicontinuity of the perimeter. The main enemy for the existence of the limit is the possibility of having pieces of E_i escaping at infinity.

The asymptotic decomposition result in [13] (see Theorem 2.16 below) tells us that, for noncollapsed Riemannian manifolds with Ricci curvature bounded from below, the mass lost at infinity can be recovered by looking at subsets of metric measure spaces obtained as pmGH limit points of $(M^n, d, \text{vol}, p_i)$, where $p_i \rightarrow \infty$ is a suitable sequence. In the setting of Theorem 1.1, we succeed in controlling the isoperimetric profile of these limit spaces, with the aim of showing that escaping at infinity is not “isoperimetrically convenient”. To this aim we employ the recent result [17, Theorem 1.1], showing that the isoperimetric constant on $\text{CD}(0, n)$ spaces is, roughly speaking, realized by balls of infinite radius, and thus that it is explicitly related to the asymptotic volume ratio. This result generalizes earlier analogous inequalities in the smooth setting [1, 22, 44, 53]. Notice that the pmGH limits at infinity of a noncollapsed nonnegatively Ricci curved manifold are indeed $\text{CD}(0, n)$ spaces. We show, in particular, that in the setting of Theorem 1.1 it is more convenient for minimizing sequences of large volume not to lose mass at infinity along N . On the other hand, if part of the mass of any minimizing sequence is lost at infinity along the factor \mathbb{R}^k , it can be taken back by means of translations along the Euclidean factor.

To perform such a plan we need to know the exact asymptotic behaviour for large volumes of the isoperimetric profile I and its derivative I' , for an arbitrary manifold with nonnegative Ricci curvature. The latter piece of information is used to ensure that every minimizing sequence of sets of sufficiently big volumes can lose at most one piece at infinity along N , see Lemma 4.1.

Loosely speaking, the behaviour at infinity of I is the same as the isoperimetric profile of a cone with opening $\text{AVR}(M^n, g)$. The computation of the asymptotics of I is a direct consequence of Bishop–Gromov comparison theorem and the sharp isoperimetric inequality, while the computation of the asymptotics of I' is a byproduct of the concavity property proven in Theorem 1.4.

We finally stress that the extremal case in the hypothesis in Theorem 1.1, i.e., when $\text{AVR}(X_\infty) = 1$, happens when every limit X_∞ is isometric to \mathbb{R}^n . In this simpler case, since the mass of a minimizing sequence lost along the factor \mathbb{R}^k can be taken back as mentioned above, by a minor variation of the proof of [13, Theorem 5.2] one even gets existence of isoperimetric regions for every volume.

Let us comment on the proof of Theorem 1.2. We show that, under the hypotheses of Theorem 1.2, the hypotheses of Theorem 1.1 are satisfied. This is done in two steps, which correspond to item (i) and (ii) of Lemma 4.2. First, if no asymptotic cone of N splits a line, then the density of every point at distance 1 from the line of the tips of every asymptotic cone of M is uniformly bigger than $\text{AVR}(M^n, g)$. The latter is due to a compactness argument and to the fact that if a point at distance 1 from the line of the tips of an asymptotic cone $\mathbb{R}^k \times C$ of M has density equal to $\text{AVR}(M^n, g)$, then there is a line in $\mathbb{R}^k \times C$ which is not contained in \mathbb{R}^k and then, by the splitting theorem, one has a splitting of the cone C , which is an asymptotic cone of N , resulting in a contradiction. Second, by means of the volume convergence theorem, a lower bound on the density of points at distance 1 from the line of the tips of the asymptotic cones is readily seen to imply a lower bound on the AVR of the pmGH limits at infinity of the manifold along N . It is worth pointing out the reference [32], where this kind of cone splitting argument originated from.

The result in item (i) of Lemma 4.2 discussed above allows to get also other nontrivial existence results for the isoperimetric problem. It can be proved, see Theorem 4.3, that for

manifolds as in Theorem 1.2, if every point at distance 1 from the line of the tips of every asymptotic cone is regular, i.e., has density one, then isoperimetric regions exist for every volume. In particular this implies, see Corollary 4.4, that if a Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth is such that every asymptotic cone has a smooth cross section, then isoperimetric regions exist for every volume. We point out that when the dimension is 2 this partially recovers the aforementioned existence result of isoperimetric sets in nonnegative curvature due to Ritoré [74].

1.4 Sharpness and counterexamples

While the assumptions in Theorem 1.1 turn out to be always satisfied in the setting of manifolds with nonnegative sectional curvature, giving rise to Theorem 1.3, the following example borrowed from [55, pp. 913-914] shows that this is not always the case on manifolds with nonnegative Ricci curvature. Let us consider the metric

$$g := f(r)^2 dt^2 + dr^2 + \eta(r)^2 g_{\mathbb{S}^{k-1}} \quad \text{for } (t, r, p) \in \mathbb{R} \times [0, +\infty) \times \mathbb{S}^{k-1} \simeq \mathbb{R}^{k+1}, k \geq 3,$$

where

$$f(r) = (b + r^2)^{(\beta+2-k)/2} + c \quad \alpha, \beta \in (0, 1), b, c > 1, k - \beta > 2 + \alpha,$$

$$\eta(r) = \frac{1}{2}r + \frac{1}{2a} \int_0^r \int_t^\infty \xi(s) ds,$$

for some smooth $\xi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\xi(t) = t$ on $[0, 1]$, $\xi > 0$ on $[1, 2]$ and $\xi(t) = t^{-1-\alpha}$ on $[2, \infty)$. The parameter $a > 0$ is chosen such that $\eta'(0) = 1$.

It is possible to prove that g has nonnegative Ricci curvature provided b and c are chosen big enough. It is easy to see that (\mathbb{R}^{k+1}, g) has Euclidean volume growth and admits a unique asymptotic cone isometric to $\mathbb{R} \times C(\mathbb{S}_\rho^{k-1})$, where $\rho = 1/2$ is the radius of \mathbb{S}_ρ^{k-1} , which is the cross section of the cone $C(\mathbb{S}_\rho^{k-1})$. Observe that the asymptotic cone splits a line while (\mathbb{R}^{k+1}, g) does not. Moreover, g is invariant under the translation along the direction ∂_t . In particular, if $P_i = (t_i, r_i, p_i)$ satisfies $\sup_i r_i < +\infty$ and $|t_i| \rightarrow \infty$, we deduce that $(\mathbb{R}^{k+1}, d_g, \text{vol}_g, P_i) \rightarrow (\mathbb{R}^{k+1}, d_g, \text{vol}_g, 0)$ in the pmGH topology. Hence the assumptions of Theorem 1.1 are *not* satisfied.

On the other hand, along sequences $Q_i = (t_i, r_i, p_i)$ with $r_i \rightarrow \infty$, one checks that $(\mathbb{R}^{k+1}, d_g, \text{vol}_g, Q_i)$ converges to the Euclidean space $(\mathbb{R}^{k+1}, d_{\text{eu}}, \mathcal{L}^{k+1}, 0)$ in the pmGH topology. Therefore, by the asymptotic mass decomposition Theorem 2.16, one has that the volume of a minimizing sequence lost at infinity ends up in isoperimetric regions in $(\mathbb{R}^{k+1}, g_{\text{eu}})$ or in (\mathbb{R}^{k+1}, g) . However, the presence of such limit regions in $(\mathbb{R}^{k+1}, g_{\text{eu}})$ easily contradicts the minimizing property of the initial sequence (cf. [13, Theorem 5.1 & 5.2] and [65, Theorem 3.5]), while limit isoperimetric regions in (\mathbb{R}^{k+1}, g) can be obviously brought back into the original (\mathbb{R}^{k+1}, g) , recovering the missing volume. Therefore we obtain that (\mathbb{R}^{k+1}, g) has isoperimetric regions for *any* volume.

All in all, we conclude that our assumptions in Theorem 1.1 are not necessary for the existence of isoperimetric sets in the setting of manifolds with nonnegative Ricci curvature.

To date we do not know whether manifolds with nonnegative Ricci curvature and Euclidean volume growth always admit isoperimetric regions for big volumes, and we plan to investigate further this problem in the future.

1.5 Plan of the paper

In Sect. 2 we introduce the preliminary material. We briefly describe the class of CD and RCD spaces, we recall the notion of pmGH convergence and the mass decomposition theorem (Theorem 2.16), and we prove the mentioned sharp Sobolev inequality (Theorem 2.21). Section 3 is devoted to the proof of Theorem 1.4. In Sect. 4 we prove the existence results Theorem 1.1, Theorem 1.2, Theorem 1.3, and further existence results (Sect. 4.3).

2 Definitions and auxiliary results

2.1 Preliminaries on Riemannian manifolds and metric measure spaces

In this preliminary section we introduce basic facts and notations about Riemannian manifolds and metric measure spaces.

Given two Riemannian manifolds (M^m, g) , (N^n, h) , we denote with $(M^m \times N^n, g + h)$ the product of the two Riemannian structure. Given a set $E \subset M$, we denote with $\text{vol}(E)$ its n -dimensional Hausdorff measure $\mathcal{H}^n(E)$ with respect to the distance induced by g . This notion coincides with its classical Riemannian volume measure induced by g . For the notions of BV and Sobolev spaces on Riemannian manifolds we refer the reader to [61, Section 1]. For every finite perimeter set E in Ω we denote with $P(E, \Omega)$ the perimeter of E inside Ω . When $\Omega = M^n$ we simply write $P(E)$. We denote with \mathcal{H}^{n-1} the $(n-1)$ -dimensional Hausdorff measure on M^n relative to the distance induced by g . If E is a C^1 -hypersurface, $P(E)$ coincides with its classical area measure induced by g . We recall that for every finite perimeter set E one has $P(E) = \mathcal{H}^{n-1}(\partial^*E)$ and the characteristic function χ_E belongs to $BV_{\text{loc}}(M^n, \text{vol})$ with generalized gradient $D\chi_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^*E$ for a vector field $\nu_E : M \rightarrow TM^n$ defined $|D\chi_E|$ -a.e. with $|\nu_E| = 1$ at $|D\chi_E|$ -a.e. point, where ∂^*E is the essential boundary of E and $|D\chi_E|$ is the total variation of the measure $D\chi_E$.

We recall the following terminology.

Definition 2.1 (*Convergence of finite perimeter sets*) Let (M^n, g) be a Riemannian manifold. We say that a sequence of measurable (with respect to the volume measure) sets E_i locally converges to a measurable set E if the characteristic functions χ_{E_i} converge to χ_E in $L^1_{\text{loc}}(M^n, g)$. In such a case we simply write that $E_i \rightarrow E$ locally on M^n .

If the sets E_i have also locally finite perimeter, that is, $P(E_i, \Omega) < +\infty$ for any i and any bounded open set Ω , we say that $E_i \rightarrow E$ in the sense of finite perimeter sets if $E_i \rightarrow E$ locally on M^n and the sequence of measures $D\chi_{E_i}$ locally weakly* converges as measures, that is, with respect to the duality with compactly supported continuous functions. In such a case, E has locally finite perimeter and the weak* limit of $D\chi_{E_i}$ is $D\chi_E$.

Remark 2.2 (*Approximation of finite perimeter sets with smooth sets*) It can be proved, see [67, Lemma 2.3], that when M^n is a complete Riemannian manifold every finite perimeter set Ω with $0 < \text{vol}(\Omega) < +\infty$ and $\text{vol}(\Omega^c) > 0$ is approximated by relatively compact sets Ω_i in M^n with smooth boundary such that $\text{vol}(\Omega_i) = \text{vol}(\Omega)$ for every $i \in \mathbb{N}$, $\text{vol}(\Omega_i \Delta \Omega) \rightarrow 0$ when $i \rightarrow +\infty$, and $P(\Omega_i) \rightarrow P(\Omega)$ when $i \rightarrow +\infty$. Thus, by approximation, one can deduce that

$$I(V) = \inf\{\mathcal{H}^{n-1}(\partial\Omega) : \Omega \Subset M^n \text{ has smooth boundary, } \text{vol}(\Omega) = V\},$$

see [67, Theorem 1.1].

We also need to recall the definition of the simply connected radial models with constant sectional curvature.

Definition 2.3 (*Models of constant sectional curvature, cf. [72, Example 1.4.6]*) Let us define

$$\operatorname{sn}_K(r) := \begin{cases} (-K)^{-\frac{1}{2}} \sinh((-K)^{\frac{1}{2}}r) & K < 0, \\ r & K = 0, \\ K^{-\frac{1}{2}} \sin(K^{\frac{1}{2}}r) & K > 0. \end{cases}$$

If $K > 0$, then $((0, \pi/\sqrt{K}] \times \mathbb{S}^{n-1}, dr^2 + \operatorname{sn}_K^2(r)g_1)$, where g_1 is the canonical metric on \mathbb{S}^{n-1} , is the radial model of dimension n and constant sectional curvature K . The metric can be smoothly extended at $r = 0$, and thus we shall write that the metric is defined on the ball $\mathbb{B}_{\pi/\sqrt{K}}^n \subset \mathbb{R}^n$. The Riemannian manifold $(\mathbb{B}_{\pi/\sqrt{K}}^n, g_K := dr^2 + \operatorname{sn}_K^2(r)g_1)$ is the unique (up to isometry) simply connected Riemannian manifold of dimension n and constant sectional curvature $K > 0$.

If instead $K \leq 0$, then $((0, +\infty) \times \mathbb{S}^{n-1}, dr^2 + \operatorname{sn}_K^2(r)g_1)$ is the radial model of dimension n and constant sectional curvature K . Extending the metric at $r = 0$ analogously yields the unique (up to isometry) simply connected Riemannian manifold of dimension n and constant sectional curvature $K \leq 0$, in this case denoted by (\mathbb{R}^n, g_K) .

We denote by $v(n, K, r)$ the volume of the ball of radius r in the (unique) simply connected Riemannian manifold of sectional curvature K of dimension n , and by $s(n, K, r)$ the volume of the boundary of such a ball. In particular $s(n, K, r) = n\omega_n \operatorname{sn}_K^{n-1}(r)$ and $v(n, K, r) = \int_0^r n\omega_n \operatorname{sn}_K^{n-1}(r) dr$, where ω_n is the Euclidean volume of the Euclidean unit ball in \mathbb{R}^n .

On \mathbb{R}^n we denote with $d_{\mathbb{R}^n}$, $g_{\mathbb{R}^n}$, and $m_{\mathbb{R}^n}$, respectively, the Euclidean distance, the Euclidean metric, and the Lebesgue measure.

Let us now briefly recall the main concepts we will need from the theory of metric measure spaces. We recall that a *metric measure space*, m.m.s. for short, (X, d_X, m_X) is a triple where (X, d_X) is a locally compact separable metric space and m_X is a Borel measure bounded on bounded sets. A *pointed metric measure space* is a quadruple (X, d_X, m_X, x) where (X, d_X, m_X) is a metric measure space and $x \in X$ is a point.

For simplicity, and since it will always be our case, we will always assume that given (X, d_X, m_X) a m.m.s. the support $\operatorname{spt} m_X$ of the measure m_X is the whole X .

Given two m.m.s. (X, d_X, m_X) and (Y, d_Y, m_Y) , we denote by $(X \times Y, d_X \otimes d_Y, m_X \otimes m_Y)$ the product m.m.s., where

$$d_X \otimes d_Y((x, y), (x', y')) := \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}, \quad \forall x, x' \in X, \quad \forall y, y' \in Y,$$

and $m_X \otimes m_Y$ is the usual product of measures.

We assume the reader to be familiar with the notion of pointed measured Gromov–Hausdorff convergence, referring to [82, Chapter 27] and to [24, Chapter 7 and 8] for an overview on the subject. In the following treatment we introduce the pmGH-convergence already in a proper realization even if this is not the general definition. Nevertheless, the (simplified) definition of Gromov–Hausdorff convergence via a realization is equivalent to the standard definition of pmGH convergence in our setting, because in the applications we will always deal with locally uniformly doubling measures, see [49, Theorem 3.15 and Section 3.5]. The following definition is taken from the introductory exposition of [4].

Definition 2.4 (*pGH and pmGH convergence*) A sequence $\{(X_i, d_i, x_i)\}_{i \in \mathbb{N}}$ of pointed metric spaces is said to converge in the *pointed Gromov–Hausdorff topology*, in the *pGH sense for short*, to a pointed metric space (Y, d_Y, y) if there exist a complete separable metric space (Z, d_Z) and isometric embeddings

$$\begin{aligned} \Psi_i &: (X_i, d_i) \rightarrow (Z, d_Z), \quad \forall i \in \mathbb{N}, \\ \Psi &: (Y, d_Y) \rightarrow (Z, d_Z), \end{aligned}$$

such that for any $\varepsilon, R > 0$ there is $i_0(\varepsilon, R) \in \mathbb{N}$ such that

$$\Psi_i(B_R^{X_i}(x_i)) \subset \left[\Psi(B_R^Y(y)) \right]_\varepsilon, \quad \Psi(B_R^Y(y)) \subset \left[\Psi_i(B_R^{X_i}(x_i)) \right]_\varepsilon,$$

for any $i \geq i_0$, where $[A]_\varepsilon := \{z \in Z : d_Z(z, A) \leq \varepsilon\}$ for any $A \subset Z$.

Let m_i and μ be given in such a way (X_i, d_i, m_i, x_i) and (Y, d_Y, μ, y) are m.m.s. If in addition to the previous requirements we also have $(\Psi_i)_\# m_i \rightarrow \Psi_\# \mu$ with respect to duality with continuous bounded functions on Z with bounded support, then the convergence is said to hold in the *pointed measured Gromov–Hausdorff topology*, or in the *pmGH sense for short*.

We remark that in the setting we will deal with, product structures are stable under pmGH convergence, i.e., if $(X_n, d_{X_n}, m_{X_n}, x_n) \rightarrow (X, d_X, m_X, x)$ and $(Y_n, d_{Y_n}, m_{Y_n}, y_n) \rightarrow (Y, d_Y, m_Y, y)$ in the pmGH sense then

$$(X_n \times Y_n, d_{X_n} \otimes d_{Y_n}, m_{X_n} \otimes m_{Y_n}, (x_n, y_n)) \rightarrow (X \times Y, d_X \otimes d_Y, m_X \otimes m_Y, (x, y)),$$

in pmGH.

2.2 RCD spaces

Since we will use part of the RCD theory just as an instrument for our purposes and since we will never use in the paper the specific definition of RCD space, we just outline the main references on the subject and we refer the interested reader to the survey of Ambrosio [3] and the references therein.

After the introduction, in the independent works [79,80] and [58], of the curvature dimension condition $CD(K, n)$ encoding in a synthetic way the notion of Ricci curvature bounded from below by K and dimension bounded above by n , the definition of $RCD(K, n)$ m.m.s. was first proposed in [48] and then studied in [11,43,47], see also [27] for the equivalence between the $RCD^*(K, n)$ and the $RCD(K, n)$ condition. The infinite dimensional counterpart of this notion had been previously investigated in [9], see also [8] for the case of σ -finite reference measures.

Remark 2.5 (*pmGH limit of RCD spaces*) We recall that, whenever it exists, a pmGH limit of a sequence $\{(X_i, d_i, m_i, x_i)\}_{i \in \mathbb{N}}$ of (pointed) $RCD(K, n)$ spaces is still an $RCD(K, n)$ metric measure space.

Due to the compatibility of the RCD condition with the smooth case of Riemannian manifolds with Ricci curvature bounded from below and to its stability with respect to pointed measured Gromov–Hausdorff convergence, limits of smooth Riemannian manifolds with Ricci curvature uniformly bounded from below by K and dimension uniformly bounded from above by n are $RCD(K, n)$ spaces. Then the class of RCD spaces includes the class of Ricci limit spaces, i.e., limits of sequences of Riemannian manifolds with the same dimension and with Ricci curvature uniformly bounded from below [28–31]. An extension of

noncollapsed Ricci limit spaces is the class of $RCD(K, n)$ space where the reference measure is the n -dimensional Hausdorff measure relative to the distance, introduced and studied in [12,39,56]. As a consequence of the rectifiability of RCD spaces [62] (see also [23] for an alternative proof), we remark that if (X, d, \mathcal{H}^n) is an $RCD(K, n)$ space then n is an integer.

We state the volume convergence theorems obtained by Gigli and De Philippis in [39, Theorem 1.2 and Theorem 1.3], which are the synthetic version of the celebrated volume convergence of Colding [35] (see also [28]).

Theorem 2.6 *Let $\{(X_i, d_i, \mathcal{H}^n, x_i)\}_{i \in \mathbb{N}}$ be a sequence of pointed $RCD(K, n)$ m.m.s. with $K \in \mathbb{R}$ and $n \in [1, +\infty)$. Assume that (X_i, d_i, x_i) converges in the $pmGH$ topology to (X, d, x) . Then precisely one of the following happens*

- (a) $\limsup_{i \rightarrow \infty} \mathcal{H}^n(B_1(x_i)) > 0$. Then the \limsup is a limit, it coincides with the volume of the limit unitary ball and $(X_i, d_i, \mathcal{H}^n, x_i)$ converges in the $pmGH$ topology to (X, d, \mathcal{H}^n, x) . Hence (X, d, \mathcal{H}^n) is an $RCD(K, n)$ m.m.s. endowed with the n -dimensional Hausdorff measure;
- (b) $\lim_{i \rightarrow \infty} \mathcal{H}^n(B_1(x_i)) = 0$. In this case we have $\dim_H(X, d) \leq n - 1$, where we denoted by $\dim_H(X, d)$ the Hausdorff dimension of (X, d) .

Moreover, for $K \in \mathbb{R}$ and $n \in [1, +\infty)$, let $\mathbb{B}_{K,n,R}$ be the collection of all equivalence classes up to isometry of closed balls of radius R in $RCD(K, n)$ spaces, equipped with the Gromov-Hausdorff distance. Then the map $\mathbb{B}_{K,n,R} \ni Z \rightarrow \mathcal{H}^n(Z)$ is real-valued and continuous.

Remark 2.7 (*Gromov precompactness theorem for RCD spaces*) Here we recall the synthetic variant of Gromov’s precompactness theorem for RCD spaces, see [39, Equation (2.1)]. Let $\{(X_i, d_i, m_i, x_i)\}_{i \in \mathbb{N}}$ be a sequence of $RCD(K_i, n)$ spaces with $n \in [1, +\infty)$, $\text{spt}(m_i) = X_i$ for every $i \in \mathbb{N}$, $m_i(B_1(x_i)) \in [v, v^{-1}]$ for some $v \in (0, 1)$ and for every $i \in \mathbb{N}$, and $K_i \rightarrow K \in \mathbb{R}$. Then there exists a subsequence $pmGH$ -converging to some $RCD(K, n)$ space (X, d, m, x) with $\text{spt}(m) = X$.

We conclude this part by recalling a few basic definitions and results concerning the perimeter functional in the setting of metric measure spaces (see [2,6,60]).

Definition 2.8 (*BV functions and perimeter on m.m.s.*) Let (X, d, m) be a metric measure space. A function $f \in L^1(X, m)$ is said to belong to the space of *bounded variation functions* $BV(X, d, m)$ if there is a sequence $f_i \in \text{Lip}_{\text{loc}}(X)$ such that $f_i \rightarrow f$ in $L^1(X, m)$ and $\limsup_i \int_X \text{lip } f_i \, dm < +\infty$, where $\text{lip } u(x) := \limsup_{y \rightarrow x} \frac{|u(y) - u(x)|}{d(x,y)}$ is the *slope* of u at x , for any accumulation point $x \in X$, and $\text{lip } u(x) := 0$ if $x \in X$ is isolated. In such a case we define

$$|Df|(A) := \inf \left\{ \liminf_i \int_A \text{lip } f_i \, dm : f_i \in \text{Lip}_{\text{loc}}(A), f_i \rightarrow f \text{ in } L^1(A, m) \right\},$$

for any open set $A \subset X$.

If $E \subset X$ is a Borel set and $A \subset X$ is open, we define the *perimeter* $P(E, A)$ of E in A by

$$P(E, A) := \inf \left\{ \liminf_i \int_A \text{lip } u_i \, dm : u_i \in \text{Lip}_{\text{loc}}(A), u_i \rightarrow \chi_E \text{ in } L^1_{\text{loc}}(A, m) \right\},$$

We say that E has *finite perimeter* if $P(E, X) < +\infty$, and we denote by $P(E) := P(E, X)$. Let us remark that the set functions $|Df|, P(E, \cdot)$ above are restrictions to open sets of Borel measures that we denote by $|Df|, |D\chi_E|$ respectively, see [6], and [60].

The *isoperimetric profile* of (X, d, m) is

$$I_X(V) := \inf \{P(E) : E \subset X \text{ Borel, } m(E) = V\},$$

for any $V \in [0, m(X))$. If $E \subset X$ is Borel with $m(E) = V$ and $P(E) = I_X(V)$, then we say that E is an *isoperimetric region*.

It follows from classical approximation results (cf. Remark 2.2) that the above definition yields the usual notion of perimeter on any Riemannian manifold (M^n, g) recalled at the beginning of this section.

We will need the following standard approximation result, that can be obtained by combining [6, Theorem 1.1] with [40, Theorem 4.5.3 and Proposition 4.5.6].

Lemma 2.9 *Let (X, d) be a complete and separable metric space, and let m be a nonnegative measure, finite on bounded sets. Then, for any $f \in BV(X, d, m)$ there exists a sequence $(f_k) \subset \text{Lip}(X, d)$, where f_k has bounded support for any k , such that $f_k \rightarrow f$ pointwise m -a.e. and in $L^1(X, m)$, and $\int_X \text{lip}(f_k) dm \rightarrow \int_X |Df| dm$ as $k \rightarrow \infty$.*

The following general coarea formula will be employed in obtaining the sharp Sobolev inequality on $CD(0, n)$ spaces below.

Remark 2.10 (*Coarea formula on metric measure spaces*) Let (X, d, m) be a metric measure space. Let us observe that from the definitions given above, a Borel set E with finite measure has finite perimeter if and only if the characteristic function χ_E belongs to $BV(X, d, m)$.

If $f \in BV(X, d, m)$, then $\{f > \alpha\}$ has finite perimeter for a.e. $\alpha \in \mathbb{R}$ and the *coarea formula* holds

$$\int_X u d|Df| = \int_{-\infty}^{+\infty} \left(\int_X u d|D\chi_{\{f>\alpha\}}| \right) d\alpha,$$

for any Borel function $u : X \rightarrow [0, +\infty]$, see [60, Proposition 4.2]. If f is also continuous and nonnegative, then $|Df|(\{f = \alpha\}) = 0$ for every $\alpha \in [0, +\infty)$ and the *localized coarea formula* holds

$$\int_{\{a < f < b\}} u d|Df| = \int_a^b \left(\int_X u d|D\chi_{\{f>\alpha\}}| \right) d\alpha,$$

for every Borel function $u : X \rightarrow [0, +\infty]$ and every $0 \leq a < b < +\infty$, see [4, Corollary 1.9].

We recall a statement for the classical Bishop–Gromov volume and perimeter comparison. The conclusions (2.1), (2.2), and the rigidity part of Theorem 2.11 are consequences, e.g., of [45, Theorem 3.101], [77, Theorem 1.2 and Theorem 1.3], and the arguments within their proofs. The conclusion (2.3) follows from [73, Corollary 2.22, item (i)] and the coarea formula.

Theorem 2.11 (*Bishop–Gromov comparison*) Let (M^n, g) be a complete Riemannian manifold such that $\text{Ric} \geq (n - 1)K$ on M^n for some $K \in \mathbb{R}$. Let us set $T_K := +\infty$ if $K \leq 0$, and $T_K := \pi/\sqrt{K}$ if $K > 0$. Then, for every $p \in M$ and for $r \leq T_K$ the following hold

$$\frac{\text{vol}(B_r(p))}{v(n, K, r)} \rightarrow 1 \text{ as } r \rightarrow 0 \text{ and it is nonincreasing,} \tag{2.1}$$

$$\frac{P(B_r(p))}{s(n, K, r)} \rightarrow 1 \text{ as } r \rightarrow 0 \text{ and it is almost everywhere nonincreasing,} \tag{2.2}$$

$$\frac{P(B_r(p))}{s(n, K, r)} \leq \frac{\text{vol}(B_r(p))}{v(n, K, r)}. \tag{2.3}$$

Remark 2.12 (Bishop–Gromov comparison theorem on m.m.s.) Let us recall that for an arbitrary $\text{CD}((n - 1)K, n)$ space (X, d, m) the classical Bishop–Gromov volume comparison (cf. Theorem 2.11) still holds. More precisely, for a fixed $x \in X$, the function $m(B_r(x))/v(n, K, r)$ is nonincreasing in r and the function $P(B_r(x))/s(n, K, r)$ is essentially nonincreasing in r , i.e., $P(B_R(x))/s(n, K, R) \leq P(B_r(x))/s(n, K, r)$ for almost every radii $R \geq r$, see [82, Theorem 18.8, Equation (18.8), Proof of Theorem 30.11]. Moreover, it holds that $P(B_r(p))/s(n, K, r) \leq \text{vol}(B_r(p))/v(n, K, r)$ for any $r > 0$, indeed the last inequality follows from the monotonicity of the volume and perimeter ratios together with the coarea formula on balls.

Moreover, if (X, d, \mathcal{H}^n) is an $\text{RCD}((n - 1)K, n)$ space, one can conclude that \mathcal{H}^n -almost every point has a unique measure Gromov–Hausdorff tangent isometric to \mathbb{R}^n ([39, Theorem 1.12]), and thus, from the volume convergence in Theorem 2.6, we get

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(B_r(x))}{v(n, K, r)} = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n} = 1, \quad \text{for } \mathcal{H}^n\text{-almost every } x, \tag{2.4}$$

where ω_n is the volume of the unit ball in \mathbb{R}^n . Moreover, since the density function $x \mapsto \lim_{r \rightarrow 0} \mathcal{H}^n(B_r(x))/\omega_n r^n$ is lower semicontinuous ([39, Lemma 2.2]), the latter (2.4) implies that the density is bounded above by the constant 1. Hence, from the monotonicity at the beginning of the remark we deduce that, if (X, d, \mathcal{H}^n) is an $\text{RCD}((n - 1)K, n)$ space, then for every $x \in X$ we have $\mathcal{H}^n(B_r(x)) \leq v(n, K, r)$ for every $r > 0$.

Let us also recall a classical definition for the convenience of the reader.

Definition 2.13 (AVR and Euclidean volume growth) Let (X, d, m) be an arbitrary $\text{CD}(0, n)$ space with $n \in [1, +\infty)$. From Remark 2.12, the Bishop–Gromov monotonicity holds, and thus we can define, for an arbitrary $x \in X$,

$$\text{AVR}(X, d, m) := \lim_{r \rightarrow +\infty} \frac{m(B_r(x))}{\omega_n r^n},$$

the *asymptotic volume ratio* of (X, d, m) , where ω_n is the measure of the unit ball in \mathbb{R}^n . The previous limit is independent on $x \in M$. Notice that $m(B_r(x)) \geq \text{AVR}(X, d, m)\omega_n r^n$ for every $r > 0$, and every $x \in X$. If $\text{AVR}(X, d, m) > 0$ we say that (X, d, m) has *Euclidean volume growth*.

Let us recall the basic definition of asymptotic cone in the setting of RCD spaces with Euclidean volume growth.

Definition 2.14 (Asymptotic cones) Let (X, d, m, x) be a pointed $\text{RCD}(0, n)$ space with $\text{AVR}(X, d, m) > 0$. For every sequence $\{r_i\}_{i \in \mathbb{N}}$ with $r_i \rightarrow +\infty$ the sequence of pointed metric measure spaces $\{(X, r_i^{-1}d, r_i^{-n}m, x)\}_{i \in \mathbb{N}}$ is precompact in the pmGH topology due to Remark 2.7. Every pmGH limit of such a sequence is a metric cone, by a slight modification of the proof of [39, Proposition 2.8] (see also [28, Theorem 7.6]). Any such limit is called an *asymptotic cone of X* and will be denoted by C or by $C(Z)$ when we want to highlight the fact that Z is the metric space that is the basis of the cone. We stress that every such C is an $\text{RCD}(0, N)$ space.

We refer to [39, Definition 2.7] for the precise definition of metric cone. Notice that the class of asymptotic cones of X is independent on the base point $x \in X$. Moreover, it is well known that asymptotic cones may be not unique [28,36,71] and may have a nonsmooth cross-section.

2.3 Finite perimeter sets and minimizing sequences for the isoperimetric problem

In this section we recall the main generalized existence result of [13], which will be crucially used in this paper. We need to recall a generalized notion L^1 -convergence of convergence for sets defined on a sequence of metric measure spaces converging in the pmGH sense. Such a definition is given in [4, Definition 3.1], and it is investigated in [4] capitalizing on the results in [10].

Definition 2.15 (*L^1 -strong and L^1_{loc} convergence*) Let $\{(X_i, d_i, m_i, x_i)\}_{i \in \mathbb{N}}$ be a sequence of pointed metric measure spaces converging in the pmGH sense to a pointed metric measure space (Y, d_Y, μ, y) and let (Z, d_Z) be a realization as in Definition 2.4.

We say that a sequence of Borel sets $E_i \subset X_i$ such that $m_i(E_i) < +\infty$ for any $i \in \mathbb{N}$ converges in the L^1 -strong sense to a Borel set $F \subset Y$ with $\mu(F) < +\infty$ if $m_i(E_i) \rightarrow \mu(F)$ and $\chi_{E_i} m_i \rightarrow \chi_F \mu$ with respect to the duality with continuous bounded functions with bounded support on Z .

We say that a sequence of Borel sets $E_i \subset X_i$ converges in the L^1_{loc} -sense to a Borel set $F \subset Y$ if $E_i \cap B_R(x_i)$ converges to $F \cap B_R(y)$ in L^1 -strong for every $R > 0$.

Observe that in the above definition it makes sense to speak about the convergence $\chi_{E_i} m_i \rightarrow \chi_F \mu$ with respect to the duality with continuous bounded functions with bounded support on Z as $(X_i, d_i), (Y, d_Y)$ can be assumed to be topological subspaces of (Z, d_Z) by means of the isometries Ψ_i, Ψ of Definition 2.4, and the measures m_i, μ can be then identified with the push-forwards $(\Psi_i)_\# m_i, \Psi_\# \mu$ respectively.

Let us recall here for the reader’s convenience the main result of [13], namely [13, Theorem 1.1]. We will crucially need this result for the proof of our main results.

Theorem 2.16 (Asymptotic mass decomposition [13, Theorem 1.1]) *Let (M^n, g) be a non-collapsed noncompact complete manifold with infinite volume, such that $\text{Ric} \geq K$ for some $K \in (-\infty, 0]$, and let $V > 0$. For every minimizing (for the perimeter) sequence of sets $\Omega_i \subset M^n$ of volume V , with Ω_i bounded for any i , up to passing to a subsequence, there exist an increasing sequence $\{N_i\}_{i \in \mathbb{N}} \subseteq \mathbb{N}$, disjoint finite perimeter sets $\Omega_i^c, \Omega_{i,j}^d \subset \Omega_i$, and points $p_{i,j}$, with $1 \leq j \leq N_i$ for any i , such that*

- (i) $\lim_i d(p_{i,j}, p_{i,\ell}) = \lim_i d(p_{i,j}, o) = +\infty$, for any $j \neq \ell < L + 1$ and any $o \in M^n$, where $L := \lim_i N_i \in \mathbb{N} \cup \{+\infty\}$;
- (ii) Ω_i^c converges to $\Omega \subset M^n$ in the sense of finite perimeter sets (Definition 2.1), and we have $\text{vol}(\Omega_i^c) \rightarrow_i \text{vol}(\Omega)$, and $P(\Omega_i^c) \rightarrow_i P(\Omega)$. Moreover Ω is a bounded isoperimetric region on M ;
- (iii) for every $j < L + 1$, $(M^n, d, \text{vol}, p_{i,j})$ converges in the pmGH sense to a pointed RCD(K, n) space (X_j, m_j, p_j) , where m_j is the n -dimensional Hausdorff measure on (X_j, d_j) . Moreover there are isoperimetric regions $Z_j \subset X_j$ such that $\Omega_{i,j}^d \rightarrow_i Z_j$ in L^1 -strong (Definition 2.15) and $P(\Omega_{i,j}^d) \rightarrow_i P_{X_j}(Z_j)$;
- (iv) it holds that

$$I_{(M^n, g)}(V) = P(\Omega) + \sum_{j=1}^L P_{X_j}(Z_j), \quad V = \text{vol}(\Omega) + \sum_{j=1}^L m_j(Z_j). \quad (2.5)$$

We remark that item (ii) in Theorem 2.16 is, in fact, proved in [76, Theorem 2.1], and it consists in the starting point for the rest of the proof of Theorem 2.16.

We will need the following result on the boundedness of isoperimetric regions.

Proposition 2.17 ([13, Corollary 4.2]) *Let (M^n, g) be a complete noncollapsed Riemannian manifold with $\text{Ric} \geq K$ for some $K \in (-\infty, 0]$. Then the isoperimetric regions of (M^n, g) are bounded.*

2.4 The collapsed case

As already observed in [13, Remark 4.7], as a nontrivial consequence of Theorem 2.16 we have that complete noncollapsed manifolds with a lower bound on the Ricci curvature have strictly positive isoperimetric profile for any volume.

With an argument partly inspired by the proof of [57, Proposition 3.13], we are now going to show that, in the nonnegative Ricci case, collapsedness occurs if and only if the isoperimetric profile vanishes for any volume. In particular, no isoperimetric sets exist in this situation.

Proposition 2.18 *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq 0$. Then, $\inf_{x \in M} \text{vol}(B_1(x)) = 0$ if and only if $I(V) = 0$ for any positive volume V .*

Proof As observed above, if M is noncollapsed, then $I(V) > 0$ for any V by [13, Remark 4.7]. Then let us assume that $\inf_{x \in M} \text{vol}(B_1(x)) = 0$. Observe first that $\inf_{x \in M} \text{vol}(B_R(x)) = 0$ for any $R > 0$. Indeed, let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence of points such that $\lim_{j \rightarrow \infty} \text{vol} B_1(x_j) = 0$. Now, for $R \leq 1$ we have $\text{vol} B_R(x_j) \leq \text{vol}(B_1(x_j)) \rightarrow_j 0$, while for $R > 1$, by the Bishop–Gromov volume comparison, we have $\text{vol}(B_R(x_j)) \leq R^n \text{vol}(B_1(x_j)) \rightarrow_j 0$. Fix now a volume $V > 0$, and consider an arbitrary ball $B_R(x)$ such that $\text{vol}(B_R(x)) > V$. This is possible since M^n has infinite volume. Since, as we just showed, $\text{vol}(B_R(x_j)) \rightarrow 0$ along the sequence x_j above we can in particular find a point $\bar{x} \in M$ such that $\text{vol}(B_R(\bar{x})) = V$. By (2.3), we have $P(B_R(\bar{x})) \leq nR^{-1} \text{vol}(B_R(\bar{x}))$, and thus

$$I(V) \leq P(B_R(\bar{x})) \leq \frac{n}{R} V. \tag{2.6}$$

Observing that the argument actually holds true for arbitrarily big radii R , our claim is proved by letting $R \rightarrow \infty$ in (2.6). □

Explicit examples of collapsed manifolds with nonnegative Ricci curvature have been constructed in dimension $n \geq 4$ by Croke and Karcher in [37, Example 1 and Example 2]. Hence for such examples no isoperimetric regions of positive volume can exist.

In the same paper, it is shown that the latter examples do not exist in dimension $n = 2$. As a consequence of Proposition 2.18 and a former result of Ritoré [74, Theorem 1.1], asserting that surfaces with nonnegative curvature admit isoperimetric regions of any given volume, we are able to fully recover such a positive result proved in [37, Theorem A]. We remark that the proof given here is completely different from the one given in [37] and recovers the geometric conclusion of the statement passing through the isoperimetric problem.

Corollary 2.19 *Let (M^2, g) be a noncompact complete Riemannian surface with $\text{Sect} \geq 0$. Hence there exists $C := C(M) > 0$ such that*

$$\text{vol}(B(x, 1)) \geq C, \quad \forall x \in M.$$

Proof If the conclusion does not hold, from Proposition 2.18 we get that $I_{(M^2, g)}(V) = 0$ for every $V > 0$ and hence no isoperimetric regions of positive volume can exist on (M^2, g) . But this is in contradiction with [74, Theorem 1.1 & Theorem 2.1]. □

Remark 2.20 The proof of the previous result Proposition 2.18 builds on the Bishop–Gromov monotonicity results, which hold for $CD(0, n)$ spaces, cf. Remark 2.12. In fact, one can show with the same argument that also on a $CD(0, n)$ space (X, d, m) , if $\inf_{x \in X} m(B_1(x)) = 0$, then the isoperimetric profile identically vanishes.

2.5 Sharp Sobolev inequality on $CD(0, n)$ spaces

Here, we provide a version of the sharp isoperimetric inequality in $CD(0, n)$ spaces recently obtained in [17] (see also the earlier [1,22,44,53]). Namely, an application of [7] allows to obtain it in terms of the perimeter, in place of the Minkowski content. Moreover, we are not requiring the boundedness of the sets involved. We write all the details, for the readers’ sake. We point out also [70, Proposition 2.4] and the proof of [26, Theorem 3.2], that allow to reach for the same conclusion.

Theorem 2.21 (Sharp Sobolev inequality on $CD(0,n)$ spaces) *Let (X, d, m) be a $CD(0, n)$ space. Then for any $f \in BV(X, d, m)$ it holds*

$$n\omega_n^{\frac{1}{n}} \text{AVR}(X, d, m)^{\frac{1}{n}} \left(\int_X |f|^{\frac{n}{n-1}} dm \right)^{\frac{n-1}{n}} \leq |Df|(X). \tag{2.7}$$

In particular, for any set of finite perimeter E , with $m(E) < \infty$, we have

$$P_X(E) \geq n\omega_n^{\frac{1}{n}} \text{AVR}(X, d, m)^{\frac{1}{n}} m(E)^{\frac{n-1}{n}}. \tag{2.8}$$

The isoperimetric inequality [17, Theorem 1.1] reads, for any bounded subset $E \subset X$ of a $CD(0, n)$ space (X, d, m) ,

$$m^+(E) \geq n\omega_n^{\frac{1}{n}} \text{AVR}(X, d, m)^{\frac{1}{n}} m(E)^{\frac{n-1}{n}}, \tag{2.9}$$

where m^+ denotes the (lower) Minkowski content

$$m^+(E) := \liminf_{r \rightarrow 0} \frac{m(B_r(E)) - m(E)}{r}. \tag{2.10}$$

Here we denoted by $B_r(E)$ the r -enlargement of E , i.e. the set of those points $x \in X$ such that $d(x, y) < r$ for some $y \in E$. The proof of Theorem 2.21 substantially follows from (2.9) together with a relaxation argument [7]. Observe that (2.8) also allows for unbounded sets.

We will need the following elementary lemma proven in [5, Lemma A.24].

Lemma 2.22 *Let $G : [0, \infty) \rightarrow [0, \infty)$ a nonincreasing measurable function. Then for any $\alpha \geq 1$ we have*

$$\alpha \int_0^\infty t^{\alpha-1} G(t) dt \leq \left(\int_0^\infty G^{1/\alpha}(t) dt \right)^\alpha. \tag{2.11}$$

Proof of Theorem 2.21 Let $f : X \rightarrow [0, \infty)$ be a Lipschitz function with bounded support. From [7, Proposition 4.2] we know that

$$\int_X \text{lip}(f) dm = \int_0^\infty m^+({f > t}) dt,$$

hence, from (2.9) we deduce

$$\int_X \text{lip}(f) dm = \int_0^\infty m^+({f > t}) dt \geq n\omega_n^{\frac{1}{n}} \text{AVR}(X, d, m)^{\frac{1}{n}} \int_0^\infty m({f > t})^{\frac{n-1}{n}} dt.$$

A simple application of Lemma 2.22 coupled with the coarea formula in Remark 2.10 gives

$$\int_0^\infty m(\{f > t\})^{\frac{n-1}{n}} dt \geq \left(\int_X f^{\frac{n}{n-1}} dm \right)^{\frac{n-1}{n}},$$

for any nonnegative Lipschitz function with bounded support. We can easily drop the assumption on the sign of f by using the decomposition in positive and negative part. Let us finally show (2.7) via approximation argument. Let us fix $f \in BV(X, d, m)$, and approximate it with (f_k) as in Lemma 2.9. For any $k \in \mathbb{N}$ it holds

$$\int_X \text{lip}(f_k) dm \geq n\omega_n^{\frac{1}{n}} \text{AVR}(X, d, m)^{\frac{1}{n}} \left(\int_X |f_k|^{\frac{n}{n-1}} dm \right)^{\frac{n-1}{n}},$$

and passing to the limit as $k \rightarrow \infty$ we deduce

$$\begin{aligned} |Df|(X) &\geq \liminf_{k \rightarrow \infty} n\omega_n^{\frac{1}{n}} \text{AVR}(X, d, m)^{\frac{1}{n}} \left(\int_X |f_k|^{\frac{n}{n-1}} dm \right)^{\frac{n-1}{n}} \\ &\geq \omega_n^{\frac{1}{n}} \text{AVR}(X, d, m)^{\frac{1}{n}} \left(\int_X |f|^{\frac{n}{n-1}} dm \right)^{\frac{n-1}{n}} \end{aligned}$$

where the last inequality follows from Fatou’s lemma.

The last conclusion (2.8) comes by plugging $f = \chi_E$ in (2.7). □

3 Concavity properties of the isoperimetric profile

In this section we study concavity properties of the isoperimetric profile of a noncollapsed Riemannian manifold with Ricci bounded from below. We then apply the obtained results to the case of manifolds with nonnegative Ricci curvature and Euclidean volume growth. In the first part of this section we consider a *penalized* isoperimetric problem and we show that its associated isoperimetric profile converges locally uniformly to the isoperimetric profile of the manifold.

Throughout this section we are setting what follows. Let (M^n, g) be a fixed complete Riemannian manifold, and let $o \in M$. Fix a sequence $\varepsilon_k \searrow 0$ and for any $k \in \mathbb{N}$ let U_k be a bounded open smooth set in M containing the ball $B_{1/\varepsilon_k}(o)$ such that $U_k \subset U_{k+1}$; and choose $f_k \in C^\infty(M)$ such that

$$f_k \geq 0, \quad f_k|_{U_k} \equiv 0, \quad |\nabla f_k| \leq \varepsilon_k, \quad f_{k+1} \leq f_k,$$

for any $k \in \mathbb{N}$. The previous choice can be made in such a way that for any k and for any $N > 0$ there is $R > 0$ such that $f_k > N$ on $M \setminus B_R(o)$. Define, for any $k \in \mathbb{N}$,

$$P_k(\Omega) := P(\Omega) + \int_\Omega f_k d \text{vol}, \tag{3.1}$$

for any set of finite perimeter $\Omega \subset M$, and

$$I_k(V) := \inf\{P_k(\Omega) : \Omega \text{ is a finite perimeter set with } \text{vol}(\Omega) = V\}, \tag{3.2}$$

for any $V > 0$. For simplicity, we shall denote by I the isoperimetric profile $I_{(M^n, g)}$.

Lemma 3.1 *Suppose that (M^n, g) is noncollapsed and that $\text{Ric} \geq (n - 1)K$ for some $K \in \mathbb{R}$. Then the following two facts hold.*

- (1) For any $k \in \mathbb{N}$ and any $V > 0$ there exists a set of finite perimeter Ω of volume V such that $I_k(V) = P_k(\Omega)$.
- (2) For any $k \in \mathbb{N}$ the function $I_k : (0, +\infty) \rightarrow \mathbb{R}$ is continuous and the sequence $\{I_k\}_{k \in \mathbb{N}}$ converges from above to the isoperimetric profile $I_{(M^n, g)}$ locally uniformly on $(0, +\infty)$.

Proof For any k and $V > 0$ the existence of a minimizer Ω for P_k at volume V follows by means of the direct method. Indeed, a minimizing sequence $\{\Omega_i\}_{i \in \mathbb{N}}$ for P_k at volume $V = \text{vol}(\Omega_i)$ has uniformly bounded perimeters, and thus it converges up to subsequence to some set Ω in $L^1_{\text{loc}}(M)$ (see e.g. [59, Corollary 12.27], that clearly holds true also in a nonflat setting). The functional P_k is lower semicontinuous with respect to L^1_{loc} -convergence and if by contradiction $\text{vol}(\Omega) < V$, since f_k diverges at infinity, we would get that $\int_{\Omega_i} f_k \rightarrow +\infty$ as $i \rightarrow +\infty$.

For any k and $V > 0$ denote by Ω^k_V a minimizer of P_k at volume V . First observe that I_k is locally bounded; indeed for any $V > 0$ the value of I_k at volumes v in a neighborhood of V is estimated from above by the value of P_k on balls of center o and given volume v .

Let us now prove that I_k is continuous. Indeed, fix $V > 0$ and let $V_i \rightarrow V$ be any sequence. Since $I_k(V_i) = P_k(\Omega^k_{V_i})$, and thus the perimeters of $\Omega^k_{V_i}$ are uniformly bounded in i , we get that, as above, up to a subsequence, $\{\Omega^k_{V_i}\}_{i \in \mathbb{N}}$ converges in L^1_{loc} to some set Ω . As $\{P_k(\Omega^k_{V_i})\}_{i \in \mathbb{N}}$ is uniformly bounded, we deduce that $\text{vol}(\Omega) = V$ and then the convergence $\Omega^k_{V_i} \rightarrow_i \Omega$ holds in L^1 on M . Moreover

$$I_k(V) \leq P_k(\Omega) \leq \liminf_i P_k(\Omega^k_{V_i}) = \liminf_i I_k(V_i).$$

Hence I_k is lower semicontinuous. Let us now finish the proof of the continuity. Indeed, suppose by contradiction that $I_k(V) \leq \limsup_i I_k(V_i) - \delta$ for some $\delta > 0$. There exists a set E of volume V with $P_k(E) \leq I_k(V) + \delta/4$, and then for large i we can find a ball $B_{r_i}(x_i)$ such that $r_i \rightarrow 0^+$ and either $E \cup B_{r_i}(x_i)$ or $E \setminus B_{r_i}(x_i)$ has volume V_i , and, denoting by E_i such a set of volume V_i , we have

$$P_k(E_i) \leq I_k(V) + \delta/2, \tag{3.3}$$

for large i . This inequality follows from $P(E \cup B_{r_i}(x_i)) \leq P(E) + s(n, K, r_i)$, that is a consequence of Bishop-Gromov comparison, together with the fact that $r_i \rightarrow 0$. Finally, $P_k(E_i) \geq I_k(V_i)$, thus inserting the absurd hypothesis in (3.3) and passing to the limsup in i we derive a contradiction.

Therefore $\lim_i I_k(V_i) = I_k(V)$, and the arbitrariness of the sequence implies that I_k is continuous at V , and then I_k is continuous on $(0, +\infty)$.

Since (M^n, g) is noncollapsed with Ricci bounded below, the isoperimetric profile I is continuous and strictly positive by [67, Corollary 2] and [13, Remark 4.7]. Since I_k is continuous and $I \leq I_{k+1} \leq I_k$ for any $k \in \mathbb{N}$, if $I_k \rightarrow_k I$ pointwise, then the same convergence holds locally uniformly by Dini’s Monotone Convergence Theorem. Now for fixed $V > 0$ we show, indeed, that $I_k(V) \rightarrow_k I(V)$. If by contradiction $\limsup_k I_k(V) \geq \delta + I(V)$ for some $\delta > 0$, then one can consider a bounded smooth set Ω of volume V such that $P(\Omega) \leq I(V) + \delta/2$. Since $\Omega \in U_k$ for large k , we get that

$$I(V) + \frac{\delta}{2} \geq P(\Omega) = \limsup_k P_k(\Omega) \geq \delta + I(V),$$

that is impossible. □

3.1 First and second variations of the penalized perimeter

In this section we derive first and second variations for the potential term in P_k , and we collect basic properties of the sets that minimize the penalized perimeter P_k under a volume constraint. We recall that, if $E \subset M$ is a set of finite perimeter, then $D\chi_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$. The vector field ν_E is defined $|D\chi_E|$ -a.e. and it is called *generalized interior unit normal*. In case $\Sigma \subset \partial^* E$ is a smooth hypersurface, then ν_E coincides with the classical interior (with respect to E) unit normal along Σ .

Let us further recall that, if $E \subset M$ is a set of finite perimeter and X is a smooth compactly supported vector field on M , then we can define the *tangential divergence* $\operatorname{div}_E X := \operatorname{div} X - \langle \nabla X(\nu_E), \nu_E \rangle$ at \mathcal{H}^{n-1} -a.e. point on $\partial^* \Omega$. We say that E has *generalized (or distributional) mean curvature of E in direction ν_E* if there is $H_E \in L^1_{\text{loc}}(|D\chi_E|)$ such that

$$\int_{\partial^* E} \operatorname{div}_E X \, d\mathcal{H}^{n-1} = - \int_{\partial^* E} H_E \langle X, \nu_E \rangle \, d\mathcal{H}^{n-1},$$

for every compactly supported vector field X on M . In such a case, the vector $H_E \nu_E$ is called *generalized (or, distributional) mean curvature of E* . Obviously, if $\Sigma \subset \partial^* E$ is a smooth hypersurface, then tangential divergence and generalized mean curvature recover the corresponding classical quantities along Σ .

Lemma 3.2 *Suppose that (M^n, g) is noncollapsed and that $\operatorname{Ric} \geq (n - 1)K$ for $K \in \mathbb{R}$. Let Ω be a minimizer of P_k , see (3.1), at volume V . Then Ω is bounded and there exists a constant $\lambda = \lambda_{k,V} \in \mathbb{R}$ such that*

$$\left. \frac{d}{dt} P_k(\phi_t(\Omega)) \right|_{t=0} = \lambda \int_{\partial^* \Omega} \langle X, -\nu_\Omega \rangle \, d\mathcal{H}^{n-1}, \tag{3.4}$$

for any smooth compactly supported vector field X on M , where ϕ_t is the flow of X .

Moreover, Ω has generalized mean curvature H_Ω in direction ν_Ω given by

$$H_\Omega = \lambda - f_k \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial^* \Omega. \tag{3.5}$$

Furthermore, there is an open representative of Ω and its topological boundary $\partial\Omega$ is given by the disjoint union $\partial\Omega = \partial_r \Omega \sqcup \partial_s \Omega$, where $\partial_r \Omega$ is a smooth hypersurface with topological boundary $\partial_s \Omega$ that is empty if the dimension is $n \leq 7$ and with Hausdorff dimension less than or equal to $n - 8$ when $n \geq 8$.

Proof The first variation $\partial_t P_k(\phi_t(\Omega))|_{t=0}$ vanishes whenever X generates a volume preserving variation, i.e., whenever $\int_{\partial^* \Omega} \langle X, \nu_\Omega \rangle = 0$. Arguing for example as in [59, Theorem 17.20] it is a classical matter to derive the existence of the Lagrange multiplier λ satisfying (3.4).

Being Ω, X, ϕ_t as in the statement, we have that

$$\left. \frac{d}{dt} \left(\int_{\Omega_t} f_k \, d \operatorname{vol} \right) \right|_{t=0} = \int_{\partial^* \Omega} f_k \langle X, -\nu_\Omega \rangle \, d\mathcal{H}^{n-1},$$

where $\Omega_t := \phi_t(\Omega)$. Such variation formula follows by an application of the area formula as in [59, Proposition 17.8], which can be readily reformulated on Riemannian manifolds. The first variation formula for the perimeter then yields that

$$\left. \frac{d}{dt} P_k(\phi_t(\Omega)) \right|_{t=0} = \int_{\partial^* \Omega} (\operatorname{div} X - \langle \nabla X(\nu_\Omega), \nu_\Omega \rangle - f_k \langle X, \nu_\Omega \rangle) \, d\mathcal{H}^{n-1}.$$

We deduce that

$$\int_{\partial^*\Omega} \operatorname{div}_\Omega X \, d\mathcal{H}^{n-1} = \int_{\partial^*\Omega} (f_k - \lambda) \langle X, \nu_\Omega \rangle \, d\mathcal{H}^{n-1},$$

which implies that Ω has generalized mean curvature $H_\Omega \nu_\Omega = -(f_k - \lambda)\nu_\Omega$.

The regularity of $\partial\Omega$ in the statement is classical, and it can be rigorously derived from the recent [14] (see also [50,64,83] also for the Euclidean case). Indeed, its consequences [14, Corollary 1.6 & Remark 1.7]. It implies that Ω has an open representative with topological boundary $\partial\Omega$ given by the disjoint union $\partial\Omega = \partial_r\Omega \sqcup \partial_s\Omega$, where $\partial_r\Omega$ is locally $C^{1,\alpha}$ and it has topological boundary $\partial_s\Omega$ with Hausdorff dimension less than or equal to $n - 8$. Since f_k is smooth, then (3.5) classically implies that the regular part $\partial_r\Omega$ is, in fact, smooth.

Now we can prove that Ω is bounded by a modification of a classical argument appeared in [76, Proposition 3.7] in the Euclidean setting and in [69, Theorem 3] on Riemannian manifolds (see also [13, Appendix B]). Fix $p_0 \in \partial_r\Omega$. Since $\partial_s\Omega$ is closed, there is $R_1 > 0$ such that $\partial\Omega \cap B_{R_1}(p_0)$ is a nonempty smooth hypersurface, and it is well defined the inner normal ν_Ω of Ω on $\partial\Omega \cap B_{R_1}(p_0)$. Let $\varphi \in C_c^\infty(\partial\Omega \cap B_{R_1}(p_0))$ be a nonvanishing nonnegative function and consider the normal vector field $X = -\varphi\nu_\Omega$. Let $\Phi_t(x) := \exp(tX(x))$ and let Ω_t be the varied set whose essential boundary is $\Phi_t(\partial^*\Omega)$, for $|t| < \tau$ and $\tau > 0$. Since $\alpha := \int_{\partial\Omega} \langle X, -\nu_\Omega \rangle > 0$, first variation formulae for perimeter and volume give that

$$\begin{aligned} \operatorname{vol}(\Omega_t) &= \operatorname{vol}(\Omega) + \alpha t + O(t^2), \\ P(\Omega_t, B_{R_1}(p_0)) &= P(\Omega, B_{R_1}(p_0)) + t \int_{\partial\Omega} \varphi H_\Omega \, d\mathcal{H}^{n-1} + O(t^2), \end{aligned} \tag{3.6}$$

for $|t| < \tau$. Moreover, as $\varphi > 0$, we have that

$$\Omega \subset \Omega_t \quad \text{for } 0 \leq t < \tau, \tag{3.7}$$

up to decreasing τ . As $\alpha > 0$ and $|\int \varphi H_\Omega| < +\infty$, (3.6) and (3.7) imply that there exist $\varepsilon_0 > 0$ and $\beta > 0$ such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ there is a finite perimeter set E with

$$\begin{aligned} E \Delta \Omega &\Subset B_{R_1}(p_0), \\ \operatorname{vol}(E) &= \operatorname{vol}(\Omega) + \varepsilon, \\ P(E, B_{R_1}(p_0)) &\leq P(\Omega, B_{R_1}(p_0)) + \beta|\varepsilon|, \\ \varepsilon > 0 &\text{ implies } \Omega \subset E. \end{aligned} \tag{3.8}$$

Now for $r > R_1$ let

$$V(r) := \operatorname{vol}(\Omega \setminus B_r(p_0)), \quad A(r) := P(\Omega, M \setminus B_r(p_0)).$$

Since M is noncollapsed and the Ricci curvature is bounded from below, an isoperimetric inequality holds true for sufficiently small volumes [52, Lemma 3.2], i.e. there is $v_0 > 0$ such that

$$c_0 \operatorname{vol}(\Omega)^{(n-1)/n} \leq P(\Omega),$$

holds true with some $c_0 > 0$ for any finite perimeter set $\Omega \subset M^n$ with $\operatorname{vol}(\Omega) < v_0$. Hence for some $R_2 > R_1$, for $r \geq R_2 > 0$ we can apply such an isoperimetric inequality on $\Omega \setminus B_r(p_0)$, so that

$$\begin{aligned} |V'(r)| + A(r) &= \mathcal{H}^{n-1}(\partial B_r(p_0) \cap \Omega) + P(\Omega, M \setminus B_r(p_0)) \\ &= P(\Omega \setminus B_r(p_0)) \geq c_0 V(r)^{\frac{n-1}{n}}, \end{aligned}$$

for some $c_0 > 0$ for almost every $r \geq R_2$. We want to prove that

$$A(r) \leq |V'(r)| + CV(r), \tag{3.9}$$

for some constant C , and for almost every r sufficiently big. Combining with the previous inequality, in this way we would get

$$c_0 V(r)^{\frac{n-1}{n}} \leq CV(r) + 2|V'(r)| \leq \frac{c_0}{2} V(r)^{\frac{n-1}{n}} - 2V'(r),$$

because $|V'(r)| = -V'(r)$ and $CV(r) \leq \frac{c_0}{2} V(r)^{\frac{n-1}{n}}$ for almost every sufficiently big radius. Hence ODE comparison implies that $V(r)$ vanishes at some $r = \bar{r} < +\infty$, i.e., Ω is bounded as a set of finite perimeter.

So we are left to prove (3.9). For some $R_3 \geq R_2$, for $r \geq R_3$ we can assume that $V(r) < \varepsilon_0$. For fixed $r \geq R_3$ let $\varepsilon := V(r)$ and assume also that $\mathcal{H}^{n-1}(\partial\Omega \cap \partial B_r(p_0)) = 0$ and $V'(r)$ exists, which hold for almost every r . Given such ε , there exists E as in (3.8). Finally let $F = E \cap B_r(p_0)$. It follows that $\text{vol}(F) = \text{vol}(\Omega)$ and (3.8) implies that

$$\Omega \cap B_r(p_0) \subset F, \quad F \setminus \Omega \subset B_{R_1}(p_0), \quad \text{vol}(F \setminus \Omega) = \varepsilon. \tag{3.10}$$

Since Ω is a minimizer for P_k at volume $\text{vol}(\Omega) = \text{vol}(F)$, we deduce

$$\begin{aligned} P_k(\Omega) &\leq P_k(F) = P(E) - P(\Omega, M \setminus B_r(p_0)) + \mathcal{H}^{n-1}(\partial B_r(p_0) \cap \Omega) + \int_F f_k \, d\text{vol} \\ &\leq P(\Omega) + \beta\varepsilon - A(r) + |V'(r)| + \int_F f_k \, d\text{vol}. \end{aligned} \tag{3.11}$$

Up to increase R_3 , we can assume that $\inf_{M \setminus B_r(p_0)} f_k > \sup_{B_{R_1}(p_0)} f_k$. By (3.10) we then get

$$\begin{aligned} \int_F f_k \, d\text{vol} &= \int_{F \cap \Omega} f_k \, d\text{vol} + \int_{F \setminus \Omega} f_k \, d\text{vol} \leq \int_{\Omega \cap B_r(p_0)} f_k \, d\text{vol} + \varepsilon \sup_{B_{R_1}(p_0)} f_k \\ &\leq \int_{\Omega \cap B_r(p_0)} f_k \, d\text{vol} + \int_{\Omega \setminus B_r(p_0)} f_k \, d\text{vol} = \int_{\Omega} f_k \, d\text{vol} \end{aligned}$$

Combining the latter one with (3.11) we obtain (3.9) with $C = \beta$. □

We also need the following second variation formula for the potential term.

Lemma 3.3 *Let M be a complete Riemannian manifold. Let $\Omega \subset M$ be a set of finite perimeter and suppose that there is a smooth hypersurface Σ with boundary contained in the essential boundary $\partial^*\Omega$. Let $X : \Sigma \rightarrow TM$ be a smooth vector field compactly supported in the interior of Σ and denote $\phi_t(x) := \exp(tX(x))$ for $x \in \Sigma$.*

For some $\tau > 0$, if Ω_t is the set of finite perimeter with essential boundary $\phi_t(\partial^\Omega)$ for $|t| < \tau$ and $X_t := \partial_t \phi_t$ is the variation field along $\phi_t(\Sigma)$, then*

$$\frac{d}{dt} \left(\int_{\Omega_t} f_k \, d\text{vol} \right) = \int_{\partial^*\Omega_t} f_k \langle X_t, -\nu_{\Omega_t} \rangle \, d\mathcal{H}^{n-1} \quad \forall |t| < \tau, \tag{3.12}$$

If also X is normal along Σ , then

$$\frac{d^2}{dt^2} \left(\int_{\Omega_t} f_k \, d\text{vol} \right) \Big|_{t=0} = \int_{\Sigma} (f_k H_{\Sigma} \langle X, \nu_{\Omega} \rangle^2 - \langle \nabla f_k, X \rangle \langle X, \nu_{\Omega} \rangle) \, d\mathcal{H}^{n-1}, \tag{3.13}$$

where H_{Σ} is the mean curvature of Σ in direction ν_{Ω} .

Proof The first variation (3.12) follows by an application of the area formula as in [59, Proposition 17.8], which can be readily adapted on Riemannian manifolds. The second variation (3.13) follows by differentiating (3.12). Indeed, rewriting the varied boundary $\partial^*\Omega_t$ as an embedding $F_t : \Sigma \rightarrow M$, where $F_t(\Sigma) = \phi_t(\Sigma)$, we have

$$\frac{d}{dt} \left(\int_{\Omega_t} f_k \, d \text{vol} \right) = \int_{\Sigma} [f_k \langle X_t, -\nu_{\Omega_t} \rangle] \circ F_t \, d \text{vol}(t),$$

where $\text{vol}(t)$ is the volume form on Σ induced by F_t . Hence

$$\begin{aligned} \frac{d^2}{dt^2} \left(\int_{\Omega_t} f_k \, d \text{vol} \right) &= \int_{\Sigma} - \langle \nabla f_k, X_t \rangle \langle X_t, \nu_{\Omega_t} \rangle \circ F_t - f_k \langle \nabla_{X_t} X_t, \nu_{\Omega_t} \rangle \circ F_t + \\ &\quad - f_k \langle X_t, \nabla_{X_t} \nu_{\Omega_t} \rangle \, d \text{vol}(t) + \int_{\Sigma} f_k \langle X_t, -\nu_{\Omega_t} \rangle \circ F_t \, \partial_t \, d \text{vol}(t). \end{aligned}$$

Since $\nabla_{X_t} X_t = 0$ and $\nabla_{X_t} \nu_{\Omega_t}$ is tangent along $F_t(\Sigma)$, evaluating at $t = 0$ we get

$$\begin{aligned} \frac{d^2}{dt^2} \left(\int_{\Omega_t} f_k \, d \text{vol} \right) \Big|_{t=0} &= \int_{\Sigma} - \langle \nabla f_k, X \rangle \langle X, \nu_{\Omega} \rangle \, d\mathcal{H}^{n-1} \\ &\quad + \int_{\Sigma} f_k \langle X, -\nu_{\Omega} \rangle \, \partial_t \, d \text{vol}(t) \Big|_{t=0}. \end{aligned} \tag{3.14}$$

Since X is normal along Σ , i.e., its tangent projection X^T onto $T\Sigma$ vanishes, the identity (3.13) follows by the fact that

$$\partial_t \, d \text{vol}(t) \Big|_{t=0} = \left(\text{div}_{\Sigma}(X^T) - H_{\Sigma} \langle X, \nu_{\Omega} \rangle \right) \, d\mathcal{H}^{n-1} \llcorner \Sigma = -H_{\Sigma} \langle X, \nu_{\Omega} \rangle \, d\mathcal{H}^{n-1} \llcorner \Sigma. \tag{3.15}$$

□

In the hypotheses and notation of Lemma 3.3, if also Ω has generalized mean curvature $H_{\Omega} \nu_{\Omega}$, if φ is a smooth function compactly supported in $\Sigma \subset \partial^*\Omega$, we can plug in the normal field $X = -\varphi \nu_{\Omega}$, so that (3.13) reads

$$\frac{d^2}{dt^2} \left(\int_{\Omega_t} f_k \, d \text{vol} \right) \Big|_{t=0} = \int_{\partial^*\Omega} (\varphi^2 f_k H_{\Omega} - \varphi^2 \langle \nabla f_k, \nu_{\Omega} \rangle) \, d\mathcal{H}^{n-1}, \tag{3.16}$$

indeed $H_{\Sigma} = H_{\Omega}$ on Σ .

3.2 Proof of Theorem 1.4

We will need the following well known lemma that, roughly speaking, gives an approximation in H^1 of the function identically equal to 1 on the boundary of a set, provided its singular part is sufficiently small. We will apply such a result to a volume constrained minimizer of P_k .

Lemma 3.4 *Let $\Omega \subset M^n$ be a bounded open set such that its topological boundary $\partial\Omega$ is given by the disjoint union $\partial\Omega = \partial_r\Omega \sqcup \partial_s\Omega$, where $\partial_r\Omega$ is a smooth hypersurface with topological boundary $\partial_s\Omega$ and the Hausdorff dimension of $\partial_s\Omega$ is less than or equal to $n - 3$.*

Then for any $\delta > 0$ there is $\varphi_{\delta} \in C_c^{\infty}(\partial_r\Omega)$ such that

$$0 \leq \varphi_{\delta} \leq 1, \quad \lim_{\delta \rightarrow 0} \int_{\partial\Omega} \varphi_{\delta}^2 \, d\mathcal{H}^{n-1} = P(\Omega), \quad \lim_{\delta \rightarrow 0} \int_{\partial\Omega} |\nabla \varphi_{\delta}|^2 \, d\mathcal{H}^{n-1} = 0,$$

where $\nabla \varphi_{\delta}$ here denotes the gradient of φ_{δ} as a function on the submanifold $\partial_r\Omega$.

The above lemma has been proved in the Euclidean setting in [78, Lemma 2.4] under additional regularity assumptions first, and then extended to Riemannian manifolds in [20, Proposition 2.5, Remark 2.6]. It is observed in [66, Lemma 3.1] that the regularity assumptions on $\partial\Omega$ as in the statement are sufficient (see also [21, Lemma 3.1]).

In the following proof of Theorem 1.4, we are making variations of a volume constrained minimizer Ω for the penalized perimeter along the normal field $X = -\varphi\nu_\Omega$, with φ given by Lemma 3.4. We point out that in dimension $n \leq 7$, by means of the regularity result recalled in Lemma 3.2, one could safely consider $X = -\nu_\Omega$.

Conclusion of the proof of Theorem 1.4 For the ease of notation we do not explicitly write the volume forms under the integral sign, since it will be clear from the context.

Let $\Omega = \Omega(k, \tilde{V})$ be a fixed minimizer of volume \tilde{V} for P_k . Let $\varphi \in C_c^\infty(\partial_r\Omega)$ be a smooth nonvanishing function with $0 \leq \varphi \leq 1$ supported in the regular part of $\partial\Omega$ and define the normal field $X = -\varphi\nu_\Omega$ and $\phi_t(x) := \exp(tX(x))$ for $x \in \partial^*\Omega$. Denote by Ω_t the varied set whose reduced boundary is $\phi_t(\partial^*\Omega)$, for $|t| < \tau$ and $\tau > 0$. Define

$$v(t) := \text{vol}(\Omega_t), \quad J(v) := P_k(\Omega_t) \quad \text{for the unique } t \text{ s.t. } v = v(t).$$

The definitions of $v(t)$ and $J(v)$ are well posed, indeed

$$v'(t) = \int_{\partial^*\Omega_t} \langle -\nu_{\Omega_t}, \partial_t \phi_t \rangle,$$

and then $v'(0) = \int_{\partial\Omega} \varphi > 0$ and $v' > 0$ for every t if τ is small enough. Hence v is locally invertible and $J(v)$ is well defined for $v \in (\tilde{V} - \tilde{\delta}, \tilde{V} + \tilde{\delta})$ for some $\tilde{\delta} > 0$. Let v^{-1} denote the inverse of $v(t)$. We thus have

$$\begin{aligned} \frac{d}{dt} P(\Omega_t) \Big|_{t=0} &= \int_{\partial\Omega} \varphi H_\Omega, \\ (v^{-1})'(v) &= \left(\int_{\partial^*\Omega_t} \langle -\nu_{\Omega_t}, \partial_t \phi_t \rangle \right)^{-1}, \\ (v^{-1})'(\tilde{V}) &= \left(\int_{\partial\Omega} \varphi \right)^{-1}, \\ (v^{-1})''(\tilde{V}) &= -\frac{1}{\left(\int_{\partial\Omega} \varphi\right)^2} (v^{-1})'(\tilde{V}) v''(0) = -\frac{1}{\left(\int_{\partial\Omega} \varphi\right)^3} \int_{\partial\Omega} \varphi^2 H_\Omega, \end{aligned} \tag{3.17}$$

where the last equality follows by computing $v''(0)$ analogously as in (3.14) and (3.15). Observing that $J(v) = P_k(\Omega_{v^{-1}(v)})$, we can compute

$$\begin{aligned} &\left(\int_{\partial\Omega} \varphi\right)^2 \frac{d^2}{dv^2} J(v) \Big|_{v=\tilde{V}} \\ &= \left(\int_{\partial\Omega} \varphi\right)^2 \frac{d}{dv} \left((v^{-1})' \left(\frac{d}{dt} P_k(\Omega_t) \right) \circ v^{-1} \right) \Big|_{v=\tilde{V}} \\ &= \left(\int_{\partial\Omega} \varphi\right)^2 \left((v^{-1})''(\tilde{V}) \frac{d}{dt} P_k(\Omega_t) \Big|_{t=0} + [(v^{-1})'(\tilde{V})]^2 \frac{d^2}{dt^2} P_k(\Omega_t) \Big|_{t=0} \right). \end{aligned}$$

Using (3.12), (3.16), and (3.17) we get

$$\begin{aligned} \left(\int_{\partial\Omega} \varphi\right)^2 \frac{d^2}{dv^2} J(v) \Big|_{v=\tilde{v}} &= -\frac{1}{\int_{\partial\Omega} \varphi} \int_{\partial\Omega} \varphi^2 H_\Omega \left(\int_{\partial\Omega} \varphi H_\Omega + \frac{d}{dt} \int_{\Omega_t} f_k \Big|_{t=0} \right) + \\ &\quad + \frac{d^2}{dt^2} P(\Omega_t) \Big|_{t=0} + \frac{d^2}{dt^2} \int_{\Omega_t} f_k \Big|_{t=0} \\ &= -\frac{1}{\int_{\partial\Omega} \varphi} \int_{\partial\Omega} \varphi^2 H_\Omega \left(\int_{\partial\Omega} \varphi H_\Omega + \int_{\partial\Omega} \varphi f_k \right) + \\ &\quad + \int_{\partial\Omega} \varphi^2 f_k H_\Omega - \varphi^2 \langle \nabla f_k, \nu_\Omega \rangle + \frac{d^2}{dt^2} P(\Omega_t) \Big|_{t=0}. \end{aligned}$$

The second variation formula for the perimeter yields

$$\frac{d^2}{dt^2} P(\Omega_t) \Big|_{t=0} = \int_{\partial\Omega} |\nabla\varphi|^2 + \varphi^2 (H_\Omega^2 - |A_\Omega|^2 - \text{Ric}(\nu_\Omega, \nu_\Omega)),$$

where A_Ω is the second fundamental form of $\partial_r\Omega$ and $\nabla\varphi$ is the gradient of φ as a function on the submanifold $\partial_r\Omega$. Since $\text{Ric} \geq (n-1)K$, $|\nabla f_k| \leq \varepsilon_k$, $\varphi \leq 1$, and $H_\Omega + f_k = \lambda \equiv \lambda_k, \tilde{v} \in \mathbb{R}$ at \mathcal{H}^{n-1} -almost every point on $\partial^*\Omega$, we estimate

$$\begin{aligned} \left(\int_{\partial\Omega} \varphi\right)^2 \frac{d^2}{dv^2} J(v) \Big|_{v=\tilde{v}} &\leq -\frac{1}{\int_{\partial\Omega} \varphi} \int_{\partial\Omega} \varphi^2 H_\Omega \left(\int_{\partial\Omega} \varphi (H_\Omega + f_k) \right) + \int_{\partial\Omega} \varphi^2 f_k H_\Omega + \varepsilon_k P(\Omega) + \\ &\quad + \int_{\partial\Omega} \varphi^2 H_\Omega^2 + (n-1)|K|P(\Omega) - \int_{\partial\Omega} \varphi^2 |A_\Omega|^2 + \int_{\partial\Omega} |\nabla\varphi|^2 \\ &= -\lambda \int_{\partial\Omega} \varphi^2 H_\Omega + \int_{\partial\Omega} (f_k + H_\Omega)\varphi^2 H_\Omega + \varepsilon_k P(\Omega) + \\ &\quad + (n-1)|K|P(\Omega) - \int_{\partial\Omega} \varphi^2 |A_\Omega|^2 + \int_{\partial\Omega} |\nabla\varphi|^2 \\ &= (\varepsilon_k + (n-1)|K|)P(\Omega) - \int_{\partial\Omega} \varphi^2 |A_\Omega|^2 + \int_{\partial\Omega} |\nabla\varphi|^2. \end{aligned}$$

Hence

$$J''(\tilde{V}) \leq (\varepsilon_k + (n-1)|K|) \frac{P(\Omega)}{(\int_{\partial\Omega} \varphi)^2} + \frac{1}{(\int_{\partial\Omega} \varphi)^2} \left(\int_{\partial\Omega} |\nabla\varphi|^2 - \int_{\partial\Omega} \varphi^2 |A_\Omega|^2 \right). \tag{3.18}$$

By definition of I_k we have that $I_k(v) \leq J(v)$ for any $v \in (\tilde{V} - \delta, \tilde{V} + \delta)$ and $I_k(\tilde{V}) = J(\tilde{V})$. Then by the latter observation,

$$\overline{D}^2 I_k(\tilde{V}) := \limsup_{h \rightarrow 0} \frac{I_k(\tilde{V} + h) + I_k(\tilde{V} - h) - 2I_k(\tilde{V})}{h^2} \leq J''(\tilde{V}).$$

Now observe that we could take $\varphi = \varphi_\delta$ for any δ with φ_δ as in Lemma 3.4. Hence letting $\delta \rightarrow 0$, using that $P(\Omega) \geq I(\tilde{V})$, and exploiting (3.18), we conclude that

$$\overline{D}^2 I_k(\tilde{V}) \leq \frac{\varepsilon_k + (n-1)|K|}{P(\Omega)} \leq \frac{\varepsilon_k + (n-1)|K|}{I(\tilde{V})}. \tag{3.19}$$

Recalling that I is continuous and strictly positive ([67, Corollary 2] and [13, Remark 4.7]), up to taking a smaller $\tilde{\delta}$, we have that $I(V)$ is bounded below by a strictly positive constant independent of $V \in (\tilde{V} - \tilde{\delta}, \tilde{V} + \tilde{\delta})$. Since \tilde{V} above is arbitrary, we deduce that $\bar{D}^2 I_k(V) \leq 2C_{\tilde{V}} < +\infty$ for every $V \in (\tilde{V} - \tilde{\delta}, \tilde{V} + \tilde{\delta})$. The latter, together with the fact that I_k is continuous, see Lemma 3.1, implies that $I_k - C_{\tilde{V}}V^2$ is concave on $(\tilde{V} - \tilde{\delta}, \tilde{V} + \tilde{\delta})$.

Passing to the limit as $k \rightarrow +\infty$, as $I_k \xrightarrow{k} I$ locally uniformly by Lemma 3.1, we deduce that $I - C_{\tilde{V}}V^2$ is concave on $(\tilde{V} - \tilde{\delta}, \tilde{V} + \tilde{\delta})$ as well. Moreover passing to the limit as $k \rightarrow +\infty$ in (3.19), taking into account [19, Lemma B.3.11], we conclude (1.3). Finally, the inequality for the second incremental ratios (1.3) is known to be equivalent to the distributional inequality (1.4), see [19]. Moreover since the function I is locally $C_{\tilde{V}}$ -concave around any $\tilde{V} \in (0, \text{vol}(M^n))$, it is twice differentiable almost everywhere and hence from (1.3) the inequality

$$I''(\tilde{V}) \leq \frac{(n-1)|K|}{I(\tilde{V})},$$

follows for almost all $V \in (0, \text{vol}(M^n))$, thus proving (1.5). □

It would be interesting to understand how the above proof could be improved in order to get concavity properties on the function $I_{(M^n, g)}^{\frac{n-1}{n}}$. Under the a priori assumption of existence of isoperimetric sets, it has been studied in [18–21, 63].

3.3 Consequences of the concavity properties of the isoperimetric profile

In this section we collect some consequences of Theorem 1.4. First, since concave functions are locally Lipschitz, we deduce the following useful corollary, which improves the main result in [67, Theorem 2].

Corollary 3.5 *Suppose that (M^n, g) is noncollapsed and that $\text{Ric} \geq (n-1)K$ for some $K \in \mathbb{R}$. Then the isoperimetric profile $I_{(M^n, g)}$ is a locally Lipschitz function away from 0.*

In the following result we exploit the concavity of I under nonnegative Ricci curvature, together with the sharp isoperimetric inequality (2.8), to deduce the precise asymptotic expansion of I and I' at infinity when $\text{Ric} \geq 0$.

Corollary 3.6 *Let (M^n, g) be a noncompact Riemannian manifold with $\text{Ric} \geq 0$, and denote by I its isoperimetric profile. Then*

$$\lim_{v \rightarrow +\infty} \frac{I(v)}{v^{(n-1)/n}} = n(\omega_n \text{AVR}(M^n, g))^{1/n}. \tag{3.20}$$

Moreover the right derivative $I'_+(v) := \lim_{h \rightarrow 0^+} \frac{I(v+h) - I(v)}{h}$ exists at any $v > 0$ and it satisfies

$$\lim_{v \rightarrow +\infty} v^{1/n} I'_+(v) = (n-1)(\omega_n \text{AVR}(M^n, g))^{1/n}. \tag{3.21}$$

Proof For simplicity, throughout the proof we denote $\theta := \text{AVR}(M^n, g)$. Let us first prove (3.20). We first deal with the case $\theta > 0$. Let us fix $p \in M^n$. From Bishop–Gromov comparison theorem, see Theorem 2.11, we know that $P(B_r(p))/(\omega_n r^{n-1})$ is almost everywhere non-increasing on $[0, +\infty)$. Since $\text{vol}(B_r(p))/(\omega_n r^n) \rightarrow \theta$, as $r \rightarrow +\infty$, a standard use of the coarea formula for $\text{vol}(B_r(p))$, together with the monotonicity ensured by the

Bishop–Gromov comparison result, imply that the essential limit of $P(B_r(p))/(\omega_n r^{n-1})$ as $r \rightarrow +\infty$ is θ as well. Hence, for every $\varepsilon > 0$ there exists $r_0 := r_0(\varepsilon)$ such that for almost every $r \geq r_0$ we have

$$P(B_r(p)) \leq (1 + \varepsilon)\theta n\omega_n r^{n-1}.$$

Moreover, by approximating balls from the outside and by using the semicontinuity of the perimeter, the previous inequality holds for *all* $r \geq r_0$. Hence for every V sufficiently large (recall that our manifold has infinite volume), there exists a ball of center p , radius $r := r(V) \geq r_0$ and volume V such that

$$I(V) \leq P(B_r(p)) \leq (1 + \varepsilon)\theta n\omega_n r^{n-1} \leq (1 + \varepsilon)n(\omega_n\theta)^{1/n} V^{(n-1)/n},$$

where in the last inequality we used $\theta\omega_n r^n \leq V$, which comes from Bishop–Gromov comparison. Since $I(V) \geq n(\omega_n\theta)^{1/n} V^{(n-1)/n}$ for every $V > 0$, as an immediate outcome of Theorem 2.21, we get (3.20) in the case $\theta > 0$.

Let us now prove (3.20) when $\theta = 0$. For every $V > 0$ there exists a radius $r := r(V)$ such that $\text{vol}(B_r(p)) = V$. Notice that $r(V) \rightarrow +\infty$ when $V \rightarrow +\infty$. Moreover

$$\begin{aligned} \frac{I(V)}{V^{(n-1)/n}} &\leq \frac{P(B_r(p))}{\text{vol}(B_r(p))^{(n-1)/n}} = \frac{P(B_r(p))}{\text{vol}(B_r(p))} \left(\frac{\text{vol}(B_r(p))}{\omega_n r^n}\right)^{1/n} r\omega_n^{1/n} \\ &\leq n\omega_n^{1/n} \left(\frac{\text{vol}(B_r(p))}{\omega_n r^n}\right)^{1/n}, \end{aligned} \tag{3.22}$$

where in the first inequality we used the definition of isoperimetric profile, and in the last inequality we used the comparison in (2.3). Since $\theta = 0$ we have that for every ε there exists $r_0 := r_0(\varepsilon)$ such that $\text{vol}(B_r(p))/(\omega_n r^n) \leq \varepsilon$ for every $r \geq r_0$. Hence from (3.22) we conclude the sought limit in (3.20) also in the case $\theta = 0$.

Now we can prove (3.21). If M is not noncollapsed, then (3.21) is trivially verified due to the equivalence in Proposition 2.18. So let us prove (3.21) when M^n is noncollapsed. By Theorem 1.4 we know that I is concave, which implies the existence of I'_+ everywhere and that I'_+ is nonincreasing.

We claim that, for any $\varepsilon > 0$ and $0 < \delta < 2$ there exists $\bar{v} = \bar{v}(\varepsilon, \delta)$ such that

$$\left| \frac{I(v(1 + \delta)) - I(v)}{(v(1 + \delta))^{\frac{n-1}{n}} - v^{\frac{n-1}{n}}} - n(\omega_n\theta)^{1/n} \right| \leq \varepsilon \quad \text{for any } v \geq \bar{v}. \tag{3.23}$$

Indeed the latter follows from (3.20) and the inequality

$$\begin{aligned} &\left| \frac{I(v(1 + \delta)) - I(v)}{(v(1 + \delta))^{\frac{n-1}{n}} - v^{\frac{n-1}{n}}} - n(\omega_n\theta)^{1/n} \right| \\ &\leq \frac{(1 + \delta)^{\frac{n-1}{n}}}{(1 + \delta)^{\frac{n-1}{n}} - 1} \left| \frac{I(v(1 + \delta))}{(v(1 + \delta))^{\frac{n-1}{n}}} - n(\omega_n\theta)^{1/n} \right| + \frac{1}{(1 + \delta)^{\frac{n-1}{n}} - 1} \left| \frac{I(v)}{v^{\frac{n-1}{n}}} - n(\omega_n\theta)^{1/n} \right|. \end{aligned}$$

The monotonicity of I'_+ gives

$$I(v(1 + \delta)) - I(v) = \int_v^{v(1+\delta)} I'_+(t) dt \geq \delta v I'_+(v(1 + \delta)), \tag{3.24}$$

and

$$I(v(1 + \delta)) - I(v) = \int_v^{v(1+\delta)} I'_+(t) dt \leq \delta v I'_+(v), \tag{3.25}$$

for all $v > 0$, where we also employed the fact that I is Lipschitz, as observed in Corollary 3.5, in order to apply the Fundamental Theorem of Calculus. Hence, we get

$$\begin{aligned} \frac{I(v(1 + \delta)) - I(v)}{(v(1 + \delta))^{\frac{n-1}{n}} - v^{\frac{n-1}{n}}} &\geq \frac{\delta v I'_+(v(1 + \delta))}{\frac{n-1}{n} \int_v^{v(1+\delta)} t^{-1/n} dt} \\ &\geq (1 + \delta)^{-1/n} \frac{n}{n-1} (v(1 + \delta))^{1/n} I'_+(v(1 + \delta)), \end{aligned} \tag{3.26}$$

and

$$\frac{I(v(1 + \delta)) - I(v)}{(v(1 + \delta))^{\frac{n-1}{n}} - v^{\frac{n-1}{n}}} \leq \frac{\delta v I'_+(v)}{\frac{n-1}{n} \int_v^{v(1+\delta)} t^{-1/n} dt} \leq (1 + \delta)^{1/n} \frac{n}{n-1} v^{1/n} I'_+(v), \tag{3.27}$$

for all $v > 0$. The latter, together with (3.23), implies

$$\begin{aligned} (1 + \delta)^{-1/n} (n - 1)(\omega_n \theta)^{1/n} - C(n)\varepsilon &\leq \liminf_{v \rightarrow \infty} v^{1/n} I'_+(v) \\ &\leq \limsup_{v \rightarrow \infty} v^{1/n} I'_+(v) \\ &\leq (1 + \delta)^{1/n} (n - 1)(\omega_n \theta)^{1/n} + C(n)\varepsilon. \end{aligned}$$

Letting $\varepsilon, \delta \rightarrow 0$ we get the sought conclusion. □

Remark 3.7 We remark that (3.20) holds for arbitrary $CD(0, n)$ spaces of infinite volume since it only relies on Bishop–Gromov monotonicity, see Remark 2.12, and Theorem 2.21.

In the following corollary we observe that for an arbitrary manifold with nonnegative Ricci curvature and Euclidean volume growth the isoperimetric profile is strictly increasing. This should be compared with [75, Theorem 3.3] where the author shows that on every complete noncompact Riemannian manifold with strictly positive sectional curvature the isoperimetric profile is strictly increasing.

Corollary 3.8 *Let (M^n, g) be a noncompact Riemannian manifold with $\text{Ric} \geq 0$ and $\text{AVR}(M^n, g) > 0$. Then the isoperimetric profile $I_{(M^n, g)}$ is strictly increasing.*

Proof Since $I_{(M^n, g)}$ is concave due to Theorem 1.4, we have that I'_+ is nonincreasing. Moreover, since we have the asymptotic relation in (3.20) and $\text{AVR}(M^n, g) > 0$, we conclude that $I'_+ \geq 0$. Hence, if for some $v_0 > 0$ we have $I'_+(v_0) = 0$, therefore $I'_+(v) = 0$ for every $v \geq v_0$, which is in contradiction with (3.20). In conclusion $I'_+ > 0$ everywhere, and hence I is strictly increasing. □

4 Main isoperimetric existence results

In this section we prove the main existence result Theorem 1.1 and the consequent Theorem 1.2 and Theorem 1.3.

4.1 Proof of Theorem 1.1

Let us denote for simplicity $\theta := \text{AVR}(M^n, g)$. We divide the proof in two steps. The first step is the following Lemma in which we prove that, under the hypotheses of Theorem 1.1,

for large volumes V any minimizing sequence of bounded finite perimeter sets of volume V can lose at most one piece at infinity along N .

Lemma 4.1 *Let $k \geq 0$ be a natural number, and let $(M^n, g) = (\mathbb{R}^k \times N^{n-k}, g_{\mathbb{R}^k} + g_N)$ be a complete noncollapsed Riemannian manifold with $\text{Ric} \geq 0$. Assume that M satisfies the hypothesis of Theorem 1.1. Then there exists $\bar{V} > 0$ such that the following holds.*

For every $V \geq \bar{V}$, let $\{\Omega_i\}_{i \in \mathbb{N}}$ be a minimizing sequence for the volume V of bounded finite perimeter sets. Let $p_{i,j} := (t_{i,j}, x_{i,j}) \in \mathbb{R}^k \times N$ be the points for which Theorem 2.16 applied to $\{\Omega_i\}_{i \in \mathbb{N}}$ holds, using the same notation therein, and for a fixed $x_0 \in N$ let

$$\begin{aligned} J_1 &:= \{j \in [1, L + 1) : \sup_i d_N(x_0, x_{i,j}) < +\infty\}, \\ J_2 &:= \{j \in [1, L + 1) : \limsup_i d_N(x_0, x_{i,j}) = +\infty\}. \end{aligned} \tag{4.1}$$

Then for every $j \in J_2$ we have $\lim_i \text{vol}(\Omega_{i,j}^d) > V/2$. In particular, J_2 consists of at most one element for any $V \geq \bar{V}$.

Proof Let us observe first that (3.21) implies that there exists $\widehat{V} > 0$ such that

$$I'_+(v) < \frac{n}{n-1}(n-1)(\omega_n(\theta + \varepsilon/2))^{1/n} v^{-1/n} \quad \forall v \geq \widehat{V},$$

where the constant ε is the one provided by the hypothesis in Theorem 1.1. In particular

$$\begin{aligned} I(V) - I(V - \delta) &< \int_{V-\delta}^V \frac{n}{n-1}(n-1)(\omega_n(\theta + \varepsilon/2))^{1/n} v^{-1/n} dv \\ &\leq n(\omega_n(\theta + \varepsilon/2))^{1/n} \frac{\delta}{(V - \delta)^{1/n}}, \end{aligned}$$

whenever $V > V - \delta \geq \widehat{V}$. Also, we have that $\delta/(V - \delta)^{1/n} \leq \delta^{1-1/n}$ if $\delta \leq V/2$. Therefore

$$I(V) - I(V - \delta) < n(\omega_n(\theta + \varepsilon/2))^{1/n} \delta^{1-1/n}, \tag{4.2}$$

whenever $V > V - \delta \geq \widehat{V}$ and $\delta \leq V/2$.

So we choose $\bar{V} := 2\widehat{V}$. Let $V \geq \bar{V}$ and $\{\Omega_i\}_{i \in \mathbb{N}}$ be an arbitrary minimizing sequence of bounded finite perimeter sets of volume V . Let $\Omega_{i,j}^d$ be defined as in Theorem 2.16, and we adopt the notation therein. Suppose by contradiction that $m_j(Z_j) = \lim_i \text{vol}(\Omega_{i,j}^d) \leq V/2$ for some $j \in J_2$. From Theorem 2.16 we have

$$I(V) = P_{X_j}(Z_j) + P(\Omega) + \sum_{\ell \neq j} P_{X_\ell}(Z_\ell) = P_{X_j}(Z_j) + I(\text{vol}(\Omega)) + \sum_{\ell \neq j} I_{X_\ell}(m_\ell(Z_\ell)).$$

By Theorem 2.21, by hypothesis, and by [13, Proposition 3.2] we then get

$$\begin{aligned} I(V) &\geq n(\omega_n(\theta + \varepsilon))^{1/n} m_j(Z_j)^{(n-1)/n} + I(\text{vol}(\Omega)) + \sum_{\ell \neq j} I_{X_\ell}(m_\ell(Z_\ell)) \\ &\geq n(\omega_n(\theta + \varepsilon))^{1/n} m_j(Z_j)^{(n-1)/n} + I\left(\text{vol}(\Omega) + \sum_{\ell \neq j} m_\ell(Z_\ell)\right) \\ &= n(\omega_n(\theta + \varepsilon))^{1/n} m_j(Z_j)^{(n-1)/n} + I(V - m_j(Z_j)). \end{aligned}$$

By the absurd hypothesis we can take $\delta = m_j(Z_j)$ in (4.2) contradicting the above estimate, and thus completing the proof of the claim. \square

Let us now conclude the proof of Theorem 1.1 by exploiting the previous Lemma. Let us fix $p := (0, x_0) \in \mathbb{R}^k \times N$. We now show that there exists $V_0 \geq \bar{V}$ such that for every $V \geq V_0$ there exists an isoperimetric region of volume V , thus concluding the proof of the Theorem.

To reach the latter conclusion we start proving that there exists $V_0 \geq \bar{V}$ such that the following improvement of Lemma 4.1 holds. Let $V \geq V_0$, and let $\{\Omega_i\}_{i \in \mathbb{N}}$ be a minimizing sequence for the volume V of bounded finite perimeter sets. Let $p_{i,j} := (t_{i,j}, x_{i,j}) \in \mathbb{R}^k \times N$ be the points for which Theorem 2.16 applied to $\{\Omega_i\}_{i \in \mathbb{N}}$ holds, using the same notation therein, and define

$$J_1 := \{j \in [1, L + 1) : \sup_i d_N(x_0, x_{i,j}) < +\infty\},$$

$$J_2 := \{j \in [1, L + 1) : \limsup_i d_N(x_0, x_{i,j}) = +\infty\},$$

We claim that J_2 is actually empty.

Let us prove the previous claim by contradiction, after having provided the precise value of V_0 . Let ε be the constant given by the hypothesis of Theorem 1.1. Let us first consider the case $\theta > 0$. Let us define

$$\varepsilon^* := (1 + \varepsilon/\theta)^{1/n} - 1.$$

Let $V_0 \geq \bar{V}$ be big enough, where \bar{V} is provided by Lemma 4.1, such that

$$I(V) \leq n(\omega_n \theta)^{1/n} V^{(n-1)/n} \left(1 + 2^{-1} \varepsilon^* (1/3)^{(n-1)/n}\right), \quad \text{for all } V \geq V_0. \tag{4.3}$$

Such a value of V_0 exists thanks to Corollary 3.6. From now on, we fix $V \geq V_0$ and we fix a minimizing sequence $\{\Omega_i\}_{i \in \mathbb{N}}$ of bounded finite perimeter sets of volume V for which, by contradiction, there exists exactly one $j_0 \in J_2$, by Lemma 4.1. From (2.8) and the hypothesis we have

$$\frac{P_{X_{j_0}}(Z_{j_0})}{m_{j_0}(Z_{j_0})^{(n-1)/n}} \geq n\omega_n^{1/n} \text{AVR}(X_{j_0}, d_{j_0}, m_{j_0})^{1/n} \geq n\omega_n^{1/n} (\theta + \varepsilon)^{1/n}. \tag{4.4}$$

From Lemma 4.1 and from item (iii) of Theorem 2.16 we have

$$m_{j_0}(Z_{j_0}) = \lim_i \text{vol}(\Omega_{i,j_0}^d) \geq V/3. \tag{4.5}$$

For every $j \in J_1$, we have that $(M, d, p_{i,j}, \text{vol}) \rightarrow_i (M, d, \bar{p}_j, \text{vol})$ in the pmGH topology, for some $\bar{p}_j \in M$. Indeed, since $j \in J_1$ we have that, up to subsequences, $x_{i,j} \rightarrow_i \bar{x}_j \in N$. Hence it suffices to choose $\bar{p}_j := (0, \bar{x}_j)$ thanks to the homogeneity in the factor \mathbb{R}^k . Hence, by the result in Theorem 2.16, we have that $\Omega_{i,j} \rightarrow_i Z_j$, where Z_j is an isoperimetric region in M . Now, by exploiting item (iv) of Theorem 2.16, taking into account (4.4) and (4.5), exploiting also the isoperimetric inequality in Theorem 2.21, we finally get, recalling that $\varepsilon^* := (1 + \varepsilon/\theta)^{1/n} - 1$, that

$$\begin{aligned}
 I(V) &= P(\Omega) + \sum_{j \in J_1} P(Z_j) + P_{X_{j_0}}(Z_{j_0}) \\
 &\geq n(\omega_n \theta)^{1/n} \left(\text{vol}(\Omega)^{(n-1)/n} + \sum_{j \in J_1} \text{vol}(Z_j)^{(n-1)/n} + (1 + \varepsilon/\theta)^{1/n} m_{j_0}(Z_{j_0})^{(n-1)/n} \right) \\
 &\geq n(\omega_n \theta)^{1/n} \left(\left(\text{vol}(\Omega) + \sum_{j=1}^L m_j(Z_j) \right)^{(n-1)/n} + \varepsilon^* m_{j_0}(Z_{j_0})^{(n-1)/n} \right) \\
 &\geq n(\omega_n \theta)^{1/n} V^{(n-1)/n} \left(1 + \varepsilon^*(1/3)^{(n-1)/n} \right), \tag{4.6}
 \end{aligned}$$

which is a contradiction with (4.3). Hence J_2 is empty.

Let us now consider the case $\theta = 0$. From (3.20) we have that there exists $V_0 \geq \bar{V}$ such that for all $V \geq V_0$ we have

$$I(V) \leq n(\omega_n 2^{-1}(1/3)^{(n-1)/n} \varepsilon)^{1/n} V^{(n-1)/n}. \tag{4.7}$$

Arguing similarly as before we have, for every $V \geq V_0$, and for every minimizing sequence $\{\Omega_i\}_{i \in \mathbb{N}}$ of bounded sets of finite perimeter of volume V , by using the same notation as before,

$$\begin{aligned}
 I(V) &= P(\Omega) + \sum_{j \in J_1} P(Z_j) + P_{X_{j_0}}(Z_{j_0}) \geq P_{X_{j_0}}(Z_{j_0}) \\
 &\geq n(\omega_n \varepsilon)^{1/n} m_{j_0}(Z_{j_0})^{(n-1)/n} \geq n(\omega_n (1/3)^{(n-1)/n} \varepsilon)^{1/n} V^{(n-1)/n}, \tag{4.8}
 \end{aligned}$$

which is a contradiction with (4.7), and thus J_2 is empty also in this case.

Let us now take $V_0 \geq \bar{V}$ such that the previous claim about J_2 to be empty (for every minimizing sequence of bounded sets of finite perimeter of volume greater than V_0) holds. Let us now conclude that for every $V \geq V_0$ there exists an isoperimetric region of volume V in M . Indeed, let us take a minimizing sequence $\{\Omega_i\}_{i \in \mathbb{N}}$ of bounded finite perimeter sets of volume V . We have already shown that for every $j \in J_1$, and thus for all j 's since J_2 is empty as $V \geq V_0$, we have that $(M, d, p_{i,j}, \text{vol}) \rightarrow_i (M, d, \bar{p}_j, \text{vol})$ in the pmGH topology, for some $\bar{p}_j \in M$. From Proposition 2.17 we get that Z_j is bounded for every j since it is an isoperimetric region in M . Hence, by properly translating Z_j along the coordinate \mathbb{R}^k into some Z'_j , we can define $\Omega' := \sqcup_{j=1}^L Z'_j$, where the Z'_j 's are mutually disjoint. Moreover, from (2.5) we get that $\text{vol}(\Omega') = V$. Hence, from (2.5), we conclude that

$$I(V) = P(\Omega) + \sum_{j=1}^L P(Z_j) = P(\Omega') \geq I(V), \tag{4.9}$$

from which Ω' is the sought isoperimetric region of volume V .

4.2 Proof of Theorem 1.2

In order to complete the proof of Theorem 1.2 we will use Theorem 1.1 together with the following Lemma, that we state and prove in the general RCD setting. In item (i) of Lemma 4.2 we show that when the hypothesis of Theorem 1.2 holds, then the density of the points at distance one from the tips of the asymptotic cones of M^n is uniformly greater than $\text{AVR}(M^n, g)$. In item (ii) of Lemma 4.2 we show that when the previous conclusion

on the density holds, then every pmGH limit at infinity has AVR uniformly greater than the AVR(M^n, g). The argument of the proof heavily relies on the cone splitting argument, cf. [32].

Lemma 4.2 *Let $k \geq 0$ be a natural number, let (Y, d_Y, m_Y) be a metric measure space, and assume*

$$(X := \mathbb{R}^k \times Y, d := d_{\mathbb{R}^k} \otimes d_Y, m := m_{\mathbb{R}^k} \otimes m_Y),$$

is an RCD(0, n) space. Assume

$$\text{AVR}(X, d, m) > 0. \tag{4.10}$$

Then the following two statements hold.

(i) *If there exists no asymptotic cone of (Y, d_Y, m_Y) that splits a line, then there exists an $\varepsilon > 0$ such that for every asymptotic cone*

$$(\mathbb{R}^k \times C, d_{\mathbb{R}^k} \otimes d_C, m_{\mathbb{R}^k} \otimes m_C, (0, v_\infty)),$$

of (X, d, m) , and for every $p \in \mathbb{R}^k \times C$ with $d_{\mathbb{R}^k} \otimes d_C(p, \mathbb{R}^k \times \{v_\infty\}) = 1$, we have

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{m_{\mathbb{R}^k} \otimes m_C(B_r(p))}{\omega_n r^n} &=: \vartheta[(\mathbb{R}^k \times C, d_{\mathbb{R}^k} \otimes d_C, m_{\mathbb{R}^k} \otimes m_C, p)] \\ &\geq \text{AVR}(X, d, m) + \varepsilon. \end{aligned}$$

(ii) *Assume that there exists $\alpha > 0$ such that*

$$\lim_{r \rightarrow 0} \frac{m_{\mathbb{R}^k} \otimes m_C(B_r(p))}{\omega_n r^n} \geq \alpha,$$

for every asymptotic cone

$$(\mathbb{R}^k \times C, d_{\mathbb{R}^k} \otimes d_C, m_{\mathbb{R}^k} \otimes m_C, (0, v_\infty)),$$

of (X, d, m) and for every $p \in \mathbb{R}^k \times C$ with $d_{\mathbb{R}^k} \otimes d_C(p, \mathbb{R}^k \times \{v_\infty\}) = 1$. Fix $x_0 := (0, y_0) \in \mathbb{R}^k \times Y$. Fix $\{x_i := (t_i, y_i)\}_{i \geq 1} \subset X$ a sequence with $d_Y(y_i, y_0) \rightarrow_i +\infty$, and let X_∞ be a pmGH limit of a subsequence of (X, d, m, x_i) , namely

$$(X, d, m, x_i) \rightarrow_i (X_\infty, d_\infty, m_\infty, x_\infty),$$

in the pmGH sense up to a subsequence. Hence, $\text{AVR}(X_\infty, d_\infty, m_\infty) \geq \alpha$.

Proof In the proof for simplicity we denote $\theta_X := \text{AVR}(X, d, m)$. Inductively using [47, Theorem 7.4], one deduces that (Y, d_Y, m_Y) is an RCD(0, $n - k$) space. Moreover, one can check that $\text{AVR}(Y, d_Y, m_Y) > 0$ as well.

Let us first prove item (i). Let us assume by contradiction that there exist, for every $i \in \mathbb{N}$, asymptotic cones $(C'_i, d_i, m_i, (0, v_{\infty,i}))$ of (X, d, m) , and points $p_i \in C'_i$ such that $d_i(p_i, \mathbb{R}^k \times \{v_{\infty,i}\}) = 1$ and

$$\vartheta[(C'_i, d_i, m_i, p_i)] \leq \theta_X + 1/i.$$

Recall that $C'_i = \mathbb{R}^k \times C_i$, for some cone C_i with tip $v_{\infty,i}$. Notice that by translating along \mathbb{R}^k we may assume that the \mathbb{R}^k coordinate of p_i is 0. Let us first notice that the sequence $\{(C'_i, d_i, m_i, (0, v_{\infty,i}))\}_{i \in \mathbb{N}}$ is precompact in the pmGH topology. Indeed, since any $(C'_i, d_i, m_i, (0, v_{\infty,i}))$ is the pmGH limit of a rescaling of X , the C'_i 's are RCD(0, n) spaces with $m_i(B_1(0, v_{\infty,i})) = \theta_X$ for every $i \in \mathbb{N}$, due to the volume convergence result

in Theorem 2.6. Hence precompactness follows from Remark 2.7. Moreover, any limit of such a sequence of asymptotic cones must be an asymptotic cone of (X, d, m) , due to the fact that the pmGH-topology, in our case, is metrizable, see [49, Theorem 3.15]. Indeed, by exploiting the latter information, a diagonal argument implies that any pmGH limit point of the sequence $\{(C'_i, d_i, m_i, (0, v_{\infty,i}))\}_{i \in \mathbb{N}}$ is also an asymptotic cone of X .

Hence, let us take $(C', d_{C'}, m_{C'}, (0, v_{\infty}))$ an asymptotic cone of (X, d, m) , that is also a pmGH limit of the sequence $\{(C'_i, d_i, m_i, (0, v_{\infty,i}))\}_{i \in \mathbb{N}}$. In a proper realization of such a pmGH limit, we can assume that, up to further subsequences, $p_i \rightarrow p \in C'$, and since $d_i(p_i, \mathbb{R}^k \times \{v_{\infty,i}\}) = 1$, we get by continuity of the distance that $d(p, \mathbb{R}^k \times \{v_{\infty}\}) = 1$, and hence $p \notin \mathbb{R}^k \times \{v_{\infty}\}$. Since $(0, v_{\infty})$ is one tip of the asymptotic cone C' , from the volume convergence result in Theorem 2.6, we conclude that

$$m_{C'}(B_r(0, v_{\infty})) = \theta_X \omega_n r^n, \tag{4.11}$$

for every $r > 0$. By an immediate consequence of Bishop–Gromov volume comparison and the volume convergence we also have that

$$\vartheta[(C', d_{C'}, m_{C'}, p)] \geq \theta_X. \tag{4.12}$$

Moreover, since the density ϑ is lower semicontinuous with respect to the pmGH convergence, see [39, Lemma 2.2 (i)], we also conclude that

$$\vartheta[(C', d_{C'}, m_{C'}, p)] \leq \liminf_{i \rightarrow +\infty} \vartheta[(C'_i, d_i, m_i, p_i)] \leq \theta_X. \tag{4.13}$$

Hence, from (4.12) and (4.13), we deduce that

$$\lim_{r \rightarrow 0} \frac{m_{C'}(B_r(p))}{\omega_n r^n} =: \vartheta[(C', d_{C'}, m_{C'}, p)] = \theta_X. \tag{4.14}$$

Since p and $(0, v_{\infty})$ are at finite distance, from (4.11), a simple argument involving the triangle inequality, and the monotonicity of Bishop–Gromov ratios (see Remark 2.12) we also deduce that

$$\lim_{r \rightarrow +\infty} \frac{m_{C'}(B_r(p))}{\omega_n r^n} = \theta_X. \tag{4.15}$$

From (4.14), (4.15), and the monotonicity of the Bishop–Gromov ratios, see Remark 2.12, we deduce that $m_{C'}(B_r(p)) = \theta_X \omega_n r^n$ for every $r > 0$. Hence, from the result in [38, Theorem 1.1], we conclude that C' is a metric cone with tip $p \notin \mathbb{R}^k \times \{v_{\infty}\}$. Hence $C' = \mathbb{R}^k \times C$ is also metric cone over a tip that is not in $\mathbb{R}^k \times \{v_{\infty}\}$, and then from [12, Proposition 1.18] we get that C splits a line. But this is not possible, since C is an asymptotic cone of Y that by hypothesis does not split a line, thus giving the sought contradiction.

Let us now prove item (ii). Since (X, d, m) is an RCD(0, n) space we conclude by stability that also $(X_{\infty}, d_{\infty}, m_{\infty})$ is an RCD(0, n) space. Let us denote $\tilde{x}_i := (0, y_i)$ and observe that (X, d, m, x_i) is isomorphic to (X, d, m, \tilde{x}_i) . Then we may suppose that the pmGH limit $(X_{\infty}, d_{\infty}, m_{\infty}, x_{\infty})$ is obtained through (a sub)sequence of (X, d, m, \tilde{x}_i) .

Since we have that $d_Y(y_i, y_0) \rightarrow_i +\infty$, if we set $\rho_i := d(x_0, \tilde{x}_i)$, we get that, up to subsequences in i , $(X, \rho_i^{-1}d, \rho_i^{-n}m, x_0)$ pmGH-converges to an asymptotic cone $(C', d_{C'}, m_{C'}, (0, v_{\infty}))$. Notice that $C' = \mathbb{R}^k \times C$ for some cone C with tip v_{∞} . Moreover, in a realization of such a convergence, $\tilde{x}_i \rightarrow_i p$ such that $d_{C'}(p, \mathbb{R}^k \times \{z_{\infty}\}) = 1$ from the fact that both x_0 and \tilde{x}_i have zero component along \mathbb{R}^k . From the hypothesis we get that, for some $\alpha > 0$, we have

$$\lim_{r \rightarrow 0} \frac{m_{C'}(B_r(p))}{\omega_n r^n} \geq \alpha.$$

Hence, for every η there exists some $\delta := \delta(\eta) > 0$ for which

$$\frac{m_{C'}(B_\delta(p))}{\omega_n \delta^n} \geq \alpha - \eta. \tag{4.16}$$

Since $(X, \rho_i^{-1}d, \rho_i^{-n}m, \tilde{x}_i)$ pmGH-converges to $(C', d_{C'}, m_{C'}, p)$, the volume convergence result in Theorem 2.6 implies that

$$\lim_{i \rightarrow \infty} \frac{m(B_{\rho_i \delta}(\tilde{x}_i))}{\omega_n (\rho_i \delta)^n} = \frac{m_{C'}(B_\delta(p))}{\omega_n \delta^n}. \tag{4.17}$$

Hence, from (4.16) and (4.17) we get that

$$\frac{m(B_{\rho_i \delta}(\tilde{x}_i))}{\omega_n (\rho_i \delta)^n} \geq \alpha - 2\eta, \quad \text{for all } i \geq i_0(\eta) \text{ large enough.} \tag{4.18}$$

Let us fix $R > 0$. From the volume convergence result in Theorem 2.6 we have that

$$\lim_{i \rightarrow \infty} \frac{m(B_R(\tilde{x}_i))}{\omega_n R^n} = \frac{m_\infty(B_R(x_\infty))}{\omega_n R^n}.$$

By Bishop–Gromov volume comparison on X and (4.18) applied with $i \geq i_1(\eta, R)$ large enough, we have

$$\frac{m_\infty(B_R(x_\infty))}{\omega_n R^n} \geq \alpha - 3\eta.$$

Since $R, \eta > 0$ are arbitrary we get the sought conclusion taking $R \rightarrow +\infty$ and then $\eta \rightarrow 0$. □

Conclusion of the proof of Theorem 1.2 It is a direct consequence of Lemma 4.2, and Theorem 1.1. □

4.3 Further existence results

As a consequence of item (ii) of Lemma 4.2, we can derive further existence results on manifolds with nonnegative Ricci curvature and Euclidean volume growth. The next theorem states that if points located at distance 1 from the tips of every asymptotic cone of a manifold have density 1, then there exist isoperimetric regions of every volume. This is ultimately due to the fact that item (ii) of Lemma 4.2 implies that the essentially relevant pGH limits at infinity are \mathbb{R}^n , allowing us to conclude by the main existence result in [13].

Theorem 4.3 *Let $k \geq 0$ be a natural number, and let $(M^n, g) = (\mathbb{R}^k \times N^{n-k}, g_{\mathbb{R}^k} + g_N)$ be a complete Riemannian manifold such that $\text{Ric} \geq 0$ and $\text{AVR}(M^n, g) > 0$. Assume that for every asymptotic cone $(\mathbb{R}^k \times C, (0, v_\infty))$ of M and every $p \in \mathbb{R}^k \times C$ with $d_{\mathbb{R}^k} \otimes d_C(p, \mathbb{R}^k \times \{z_\infty\}) = 1$, we have*

$$\lim_{r \rightarrow 0} \frac{m_{\mathbb{R}^k} \otimes m_C(B_r(p))}{\omega_n r^n} =: \vartheta[(\mathbb{R}^k \times C, d_{\mathbb{R}^k} \otimes d_C, m_{\mathbb{R}^k} \otimes m_C, p)] = 1. \tag{4.19}$$

Then for every volume $V > 0$ there exists an isoperimetric region of volume V on M^n .

Proof We are in the setting of item (ii) of Lemma 4.2 with $\alpha = 1$. From this we deduce that for any pmGH-limit X_∞ at infinity as in the statement of item (ii) of Lemma 4.2 we have that $\text{AVR}(X_\infty, d_\infty, m_\infty) \geq 1$. But since $\text{AVR}(X_\infty, d_\infty, m_\infty) \leq 1$ due to Bishop–Gromov

comparison and the fact that $m_\infty = \mathcal{H}^n_{d_\infty}$ (cf. Theorem 2.6, and Remark 2.12), we have $\text{AVR}(X_\infty, d_\infty, m_\infty) = 1$. Therefore $(X_\infty, d_\infty, m_\infty) = (\mathbb{R}^n, d_{\mathbb{R}^n}, m_{\mathbb{R}^n})$ as a consequence of [39, Corollary 1.7] and the Bishop–Gromov comparison discussed in Remark 2.12.

Now if $k = 0$, this implies that (M^n, g) is GH-asymptotic to flat \mathbb{R}^n (see [13, Definition 1.2]), and therefore on M^n we have isoperimetric regions for every volume by [13, Theorem 5.2].

More generally, if $k > 0$, we can still conclude that on (M^n, g) we have isoperimetric regions of every volume $V > 0$ arguing like at the end of the proof of Theorem 1.1. Indeed, pGH limits along sequences (x_i, y_i) with $\{y_i\}_i$ bounded in N are isometric to (M^n, g) , and then possible mass of a minimizing sequence lost along these sequence can be obviously recovered as in the proof of Theorem 1.1. On the other hand we obtained that pGH limits obtained along sequences diverging along N^{n-k} are flat \mathbb{R}^n , and then a minor modification in the proof of [13, Theorem 5.2] immediately shows that leak of the mass of a minimizing sequence along such sequences would lead to a contradiction of the minimality assumption on the sequence. □

We record a straightforward consequence of the above statement.

Corollary 4.4 *Let (M^n, g) be a complete Riemannian manifold such that $\text{Ric} \geq 0$ and $\text{AVR}(M^n, g) > 0$.*

Assume that every asymptotic cone of (M^n, g) is smooth outside the tip, i.e., it is of the form $C(Z)$ where Z is a smooth closed manifold. Then for every volume $V > 0$ there exists an isoperimetric region of volume V on M^n .

Proof If every asymptotic cone of M is smooth outside the tip, then (4.19) is clearly satisfied, and thus the existence of isoperimetric regions follows. □

Let us observe that, in the setting of Corollary 4.4, if $n = 2$ then every asymptotic cone is (isometric to) a cone over a circle of some radius, which is obviously smooth outside the tip. Hence the latter statement, when specialized to $n = 2$, recovers a particular case of the existence result of [74] on surfaces with nonnegative curvature.

4.4 Proof of Theorem 1.3

As anticipated in the Introduction, for the proof of Corollary 1.3 we need the fact that manifolds with nonnegative sectional curvature have a unique asymptotic cone that splits if and only if the manifold splits. As explained in the Introduction, this is a standard result in the field, see [16,54]. Nonetheless we give here, for the readers’ convenience, a proof in the only case we need for our aims, i.e., when $\text{AVR}(M^n, g) > 0$. We first prove an auxiliary lemma.

Lemma 4.5 *Let (X, d_X) and (Y, d_Y) be compact metric spaces. If there exist maps $\Phi : X \rightarrow Y$ and $\Psi : Y \rightarrow X$ surjective and 1-Lipschitz, then (X, d_X) and (Y, d_Y) are isometric.*

Proof It is enough to check that $T := \Psi \circ \Phi$ is an isometry, indeed from

$$d_X(x, y) = d_X(T(x), T(y)) \leq d_Y(\Phi(x), \Phi(y)) \leq d_X(x, y)$$

we deduce that Φ is an isometry.

Fix distinct $x, y \in X$ and $\varepsilon \in (0, d_X(x, y))$. Since T is 1-Lipschitz, it is enough to prove that $d_X(T(x), T(y)) \geq d_X(x, y) - \varepsilon$ and then let $\varepsilon \rightarrow 0$. Set $D := d_X(x, y) - \varepsilon/2$.

Given an $\varepsilon/4$ -dense set $S \subset X$, i.e., such that for any $z \in X$ there exists $s \in S$ with $d_X(z, s) \leq \varepsilon/4$, we define $N(S) := \#\{(s_1, s_2) \in S \times S : d_X(s_1, s_2) \geq D\}$. Let S_0 be an $\varepsilon/4$ -dense set that minimizes the function $N(\cdot)$ among $\varepsilon/4$ -dense sets. Since T is 1-Lipschitz, then $d_X(T(s_1), T(s_2)) \leq d_X(s_1, s_2)$ for any $s_1, s_2 \in S_0$, and since T is also surjective, then $T(S_0)$ is $\varepsilon/4$ -dense, therefore $N(T(S_0)) \geq N(S_0)$. This forces $d_X(T(s_1), T(s_2)) \geq D$ for any $s_1, s_2 \in S_0$ with $d_X(s_1, s_2) \geq D$. In particular, if we pick $s_1, s_2 \in S_0$ such that $d_X(x, s_1) \leq \varepsilon/4$ and $d_X(y, s_2) \leq \varepsilon/4$ we have $d_X(s_1, s_2) \geq -\varepsilon/2 + d(x, y) = D$, and therefore, as T is 1-Lipschitz, we conclude

$$d_X(T(x), T(y)) \geq d_X(T(s_1), T(s_2)) - \varepsilon/2 \geq D - \varepsilon/2 = d(x, y) - \varepsilon.$$

□

Theorem 4.6 *If (M^n, g) has nonnegative sectional curvature and Euclidean volume growth, then there exists a unique asymptotic cone (C, d, \mathcal{H}^n) . Moreover C splits if and only if M splits.*

Proof Let us begin by proving the uniqueness of asymptotic cones.

Thanks to Lemma 4.5, it is enough to show that, given two asymptotic cones (C_1, d_1, m_1) and (C_2, d_2, m_2) , there exists a surjective 1-Lipschitz map $\Psi : \overline{B_1}(z_1) \rightarrow \overline{B_1}(z_2)$, where $z_1 \in C_1$ and $z_2 \in C_2$ are tip points.

Fix $p \in M$. Given $1 < R_1 < R_2$ we consider the set $E \subset B_{R_2}(p)$ of those points $x \in B_{R_2}(p)$ such that there exists a unique unit speed geodesic $\gamma_{p,x} : [0, d(x, p)] \rightarrow M$ with $\gamma_{p,x}(0) = p, \gamma_{p,x}(d(x, p)) = x$. It is well known [46, Lemma 3.96] that $\text{vol}(B_{R_2} \setminus E) = 0$, hence $E \subset B_{R_2}(p)$ is dense.

We then define $T : E \subset B_{R_2}(p) \rightarrow B_{R_1}(p)$ as $T(x) := \gamma_{p,x}(d(x, p)R_1/R_2)$. Toponogov’s theorem implies

$$d(T(x), T(y))R_1^{-1} \geq d(x, y)R_2^{-1} \quad \text{for any } x, y \in E, \tag{4.20}$$

in particular $T^{-1} : (T(E) \subset B_{R_1}(p), d/R_1) \rightarrow (B_{R_2}(p), d/R_2)$ is 1-Lipschitz and has dense image, hence it can be extended to a surjective 1-Lipschitz map $\Psi_{R_1, R_2} : (K_{R_1, R_2}, d/R_1) \rightarrow (\overline{B_{R_2}}(p), d/R_2)$, where $K_{R_1, R_2} := \overline{T(E)}$.

Let us now consider two sequences $r_i \rightarrow \infty$ and $s_i \rightarrow \infty$ realizing (C_1, d_1, m_1) and (C_2, d_2, m_2) , respectively. We assume without loss of generality that $s_i < r_i < s_{i+1} < r_{i+1}$ for any $i \geq 1$. By the Gromov–Hausdorff convergence of the sequences of the rescaled space to the asymptotic cones, by a classical Ascoli–Arzelà-type argument we can pass to the limit the maps $\Psi_{s_i, r_i} : ((K_{s_i, r_i}, d/s_i) \rightarrow (\overline{B_{r_i}}(p), d/r_i)$ and we derive the existence of a 1-Lipschitz surjective map

$$\Psi : K \subset \overline{B_1}(z_1) \rightarrow \overline{B_1}(z_2), \tag{4.21}$$

where $z_1 \in C_1$ and $z_2 \in C_2$ are tip points, and K is compact. Since (M^n, g) has Euclidean volume growth, the volume convergence theorem, see Theorem 2.6, guarantees that the reference measures on the asymptotic cones is the Hausdorff measure (with respect to their own distance), and $\mathcal{H}^n((B_1(z_1)) = m_1(B_1(z_1)) = \omega_n \text{AVR}(M^n, g) = m_2(B_1(z_2)) = \mathcal{H}^n(B_1(z_2))$, hence

$$\begin{aligned} \mathcal{H}^n((B_1(z_1)) \setminus K) &= \mathcal{H}^n(B_1(z_1)) - \mathcal{H}^n(K) \\ &= \mathcal{H}^n(B_1(z_2)) - \mathcal{H}^n(K) \\ &\leq \mathcal{H}^n(B_1(z_2)) - \mathcal{H}^n(\Psi(K)) \\ &= 0, \end{aligned}$$

which yields $K = \overline{B}_1(z_1)$ since K is closed. In particular (4.21) provides the sought map.

Let us now prove that the asymptotic cone (C, d, \mathcal{H}^n) splits if and only if M splits. In view of the splitting theorem it is enough to show the following implication

$$C \text{ contains a line} \implies M \text{ contains a line,} \tag{4.22}$$

since the other one is readily verified by stability of the product structure under GH-convergence. Fix $p \in M$. It is enough to show that for any $0 < \varepsilon < 1/9$ there exist $p_1, p_2 \in M$ satisfying $d(p_1, p) = d(p_2, p) = 1/\varepsilon$ and

$$d(p, \gamma_{p,p_1}(s)) + d(p, \gamma_{p,p_2}(s)) \leq d(\gamma_{p,p_1}(s), \gamma_{p,p_2}(s)) + \varepsilon \text{ for any } s \in (0, 1), \tag{4.23}$$

where $\gamma_{p,p_i} : [0, 1] \rightarrow M$ is a minimizing constant speed geodesic such that $\gamma(0) = p$ and $\gamma(1) = p_i$, for $i = 1, 2$.

Indeed the curve $\gamma_\varepsilon : (-1/\varepsilon, 1/\varepsilon) \rightarrow M$, defined as

$$\gamma_\varepsilon : (-1/\varepsilon, 1/\varepsilon) := \begin{cases} \gamma_{p,p_1}(-t\varepsilon) & \text{for } t \in (-1/\varepsilon, 0) \\ \gamma_{p,p_2}(t\varepsilon) & \text{for } t \in (0, 1/\varepsilon), \end{cases} \tag{4.24}$$

is 1-Lipschitz and satisfies $\gamma_\varepsilon(0) = p$. Hence, up to extracting a subsequence, $\gamma_\varepsilon \rightarrow \gamma$ locally uniformly, where $\gamma : \mathbb{R} \rightarrow M$. We claim that γ is a line. To see this, we fix $t > 0$ and we show that $d(\gamma(t), \gamma(-t)) = 2t$. By (4.23) we have

$$\begin{aligned} 2t &= d(p, \gamma_\varepsilon(-t)) + d(p, \gamma_\varepsilon(t)) \leq d(\gamma_\varepsilon(-t), \gamma_\varepsilon(t)) + \varepsilon \leq d(\gamma_\varepsilon(-t), p) + d(p, \gamma_\varepsilon(t)) \\ &+ \varepsilon \leq 2t + \varepsilon. \end{aligned}$$

Hence letting $\varepsilon \rightarrow 0$ implies $2t \leq d(\gamma(-t), \gamma(t)) \leq 2t$ as claimed.

So we are left to prove (4.23). Given $0 < \varepsilon < 1/9$ we can find $R = R(\varepsilon, n, p) > 5/\varepsilon$ such that

$$d_{GH}(B_R(p), B_R(z)) \leq \varepsilon^2 R, \tag{4.25}$$

where $z \in C(Z)$ is a tip. Since C splits a Euclidean factor, there exist $q_1, q_2 \in B_R(p)$ satisfying

$$d(p, q_1) = R, \quad d(p, q_2) = R, \quad \text{and} \quad d(p, q_1) + d(p, q_2) \leq d(q_1, q_2) + \varepsilon^2 R. \tag{4.26}$$

Let $t \in (0, 1)$ be such that $d(p, \gamma_{p,q_1}(t)) = d(p, \gamma_{p,q_2}(t)) = 1/\varepsilon$, where $\gamma_{p,q_i} : [0, 1] \rightarrow M$ is a minimizing constant speed geodesic such that $\gamma(0) = p$ and $\gamma(1) = q_i$, for $i = 1, 2$. We set $p_i := \gamma_{p,q_i}(t)$ and check that

$$d(p, \gamma_{p,q_1}(s)) + d(p, \gamma_{p,q_2}(s)) \leq d(\gamma_{p,q_1}(s), \gamma_{p,q_2}(s)) + \varepsilon \text{ for any } s \in (0, t), \tag{4.27}$$

which amounts to our conclusion.

Toponogov’s theorem implies that $d(\gamma_{p,q_1}(s), \gamma_{p,q_2}(s)) \geq |\tilde{\gamma}_{\tilde{p},\tilde{q}_1}(s) - \tilde{\gamma}_{\tilde{p},\tilde{q}_2}(s)|$, where $(\tilde{p}, \tilde{q}_1, \tilde{q}_2)$ is a comparison triangle in \mathbb{R}^2 corresponding to the triangle (p, q_1, q_2) , and $\tilde{\gamma}_{\tilde{p},\tilde{q}_i}$ is the constant speed Euclidean segment from \tilde{p} to \tilde{q}_i , for $i = 1, 2$. The similarity of the triangles $(\tilde{p}, \tilde{q}_1, \tilde{q}_2)$ and $(\tilde{p}, \tilde{\gamma}_{\tilde{p},\tilde{q}_1}(s), \tilde{\gamma}_{\tilde{p},\tilde{q}_2}(s))$, together with (4.26), give

$$\begin{aligned} d(\gamma_{p,q_1}(s), \gamma_{p,q_2}(s)) &\geq |\tilde{\gamma}_{\tilde{p},\tilde{q}_1}(s) - \tilde{\gamma}_{\tilde{p},\tilde{q}_2}(s)| = |\tilde{q}_1 - \tilde{q}_2| \frac{|\tilde{p} - \tilde{\gamma}_{\tilde{p},\tilde{q}_1}(s)|}{|\tilde{p} - \tilde{q}_1|} \\ &= d(q_1, q_2) \frac{d(p, \gamma_{p,q_1}(s)) + d(p, \gamma_{p,q_2}(s))}{2R} \\ &\geq (d(p, q_1) + d(p, q_2) - \varepsilon^2 R) \frac{d(p, \gamma_{p,q_1}(s)) + d(p, \gamma_{p,q_2}(s))}{2R} \end{aligned}$$

$$\begin{aligned}
 &= \left(1 - \frac{\varepsilon^2}{2}\right)(d(p, \gamma_{p,q_1}(s)) + d(p, \gamma_{p,q_2}(s))) \\
 &\geq d(p, \gamma_{p,q_1}(s)) + d(p, \gamma_{p,q_2}(s)) - \varepsilon,
 \end{aligned}$$

where in the last step we used $d(p, \gamma_{p,q_1}(s)) = d(p, \gamma_{p,q_2}(s)) \leq 1/\varepsilon$. \square

Remark 4.7 The statement of Theorem 4.6 holds in the class of Alexandrov spaces with nonnegative sectional curvature, as a refinement of the arguments above may show.

Conclusion of the proof of Theorem 1.3 The proof immediately follows from Theorem 4.6 and Theorem 1.2. \square

Acknowledgements The first author is partially supported by the European Research Council (ERC Starting Grant 713998 GeoMeG ‘*Geometry of Metric Groups*’). The second author is supported by the Giorgio and Elena Petronio Fellowship at the Institute for Advanced Study. He is grateful to Stefano Nardulli for having introduced him to the isoperimetric problem on noncompact manifolds. The authors are grateful to the anonymous referee for his very careful reading and for his insightful comments, that helped to improve the quality of the presentation. The authors warmly thank Luis Guijarro for his interest in the work and for having pointed out several useful references. The authors also thank Gian Paolo Leonardi, Luciano Mari, Aaron Naber, and Manuel Ritoré for fruitful discussions and comments.

Author Contributions All authors contributed equally to this work. All authors read and approved the final manuscript.

Funding Open access funding provided by Scuola Normale Superiore within the CRUI-CARE Agreement. The first author is partially supported by the European Research Council (ERC Starting Grant 713998 GeoMeG ‘*Geometry of Metric Groups*’). The second author is supported by the Giorgio and Elena Petronio Fellowship at the Institute for Advanced Study.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration

Conflict of interest There is no conflict of interest to disclose.

Informed consent None.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Agostiniani, V., Fogagnolo, M., Mazzieri, L.: Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature. *Invent. Math.* **222**(3), 1033–1101 (2020)
2. Ambrosio, L.: Fine properties of sets of finite perimeter in doubling metric measure spaces. In: vol. 10. 2-3. *Calculus of variations, nonsmooth analysis and related topics*. 111–128 (2002)
3. Ambrosio, L.: Calculus, heat flow and curvature-dimension bounds in metric measure spaces. In: *Proceedings of the International Congress of Mathematicians-Rio de Janeiro*. Vol. I. Plenary lectures, 301–340 (2018)

4. Ambrosio, L., Brué, E., Semola, D.: Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces. *Geom. Funct. Anal.* **29**(4), 949–1001 (2019)
5. Ambrosio, L., Carlotto, A., Massaccesi, A.: Lectures on elliptic partial differential equations. Vol. 18. *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, , x+227 (2018)
6. Ambrosio, L., Di Marino, S.: Equivalent definitions of BV space and of total variation on metric measure spaces. *J. Funct. Anal.* **266**(7), 4150–4188 (2014)
7. Ambrosio, L., Di Marino, S., Gigli, N.: Perimeter as relaxed Minkowski content in metric measure spaces. *Nonlinear Anal.: Theory, Methods Appl.* **153**, 78–88 (2017)
8. Ambrosio, L., Gigli, N., Mondino, A., Rajala, T.: Riemannian Ricci curvature lower bounds in metric measure spaces with σ -finite measure. *Trans. Amer. Math. Soc.* **367**(7), 4661–4701 (2015)
9. Ambrosio, L., Gigli, N., Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded from below. *Duke Math. J.* **163**(7), 1405–1490 (2014)
10. Ambrosio, L., Honda, S.: New stability results for sequences of metric measure spaces with uniform Ricci bounds from below. In: *Measure theory in non-smooth spaces. Partial Differ. Equ. Meas. Theory*, 1–51
11. Ambrosio, L., Mondino, A., Savaré, G.: Nonlinear diffusion equations and curvature conditions in metric measure spaces. *Mem. Amer. Math. Soc.* **262** (2019)
12. Antonelli, G., Brué, E., Semola, D.: Volume bounds for the quantitative singular strata of non collapsed RCD metric measure spaces. *Anal. Geom. Metr. Spaces* **7**(1), 158–178 (2019)
13. Antonelli, G., Fogagnolo, M., Pozzetta, M.: The isoperimetric problem on Riemannian manifolds via Gromov-Hausdorff asymptotic analysis. (2021). [arXiv:2101.12711](https://arxiv.org/abs/2101.12711)
14. Antonelli, G., Pasqualetto, E., Pozzetta, M.: Isoperimetric sets in spaces with lower bounds on the Ricci curvature. (2021). [arXiv:2107.03124](https://arxiv.org/abs/2107.03124)
15. Antonelli, G., Pasqualetto, E., Pozzetta, M., Semola, D.: Sharp isoperimetric comparison and asymptotic isoperimetry on non collapsed spaces with lower Ricci bounds. (2022). [arXiv:2201.04916](https://arxiv.org/abs/2201.04916)
16. Ballmann, W., Gromov, M., Schroeder, V.: *Manifolds of nonpositive curvature*. Vol. 61. *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA , vi+263. isbn: 0-8176-3181-X (1985)
17. Balogh, Z.M., Kristály, A.: Sharp geometric inequalities in spaces with nonnegative Ricci curvature and Euclidean volume growth. Accepted in *Nonlinear Analysis* (2021). [arXiv:2012.11862](https://arxiv.org/abs/2012.11862)
18. Bavard, C., Pansu, P.: Sur le volume minimal de R^2 . *Annales scientifiques de l'École Normale Supérieure* **19**(4), 479–490 (1986)
19. Bayle, V.: Propriétés de concavité du profil isopérimétrique et applications. Ph.D Thesis. <https://tel.archives-ouvertes.fr/tel-00004317v1/document>. Institut Fourier, (2003)
20. Bayle, V.: A differential inequality for the isoperimetric profile. *Int. Math. Res. Not.* **7**, 311–342 (2004)
21. Bayle, V., Rosales, C.: Some isoperimetric comparison theorems for convex bodies in Riemannian manifolds. *Indiana Univ. Math. J.* **54**(5), 1371–1394 (2005)
22. Brendle, S.: Sobolev inequalities in manifolds with nonnegative curvature. To appear in *Comm. Pure. Appl. Math.* (2021). [arXiv:2009.13717](https://arxiv.org/abs/2009.13717)
23. Brué, E., Pasqualetto, E., Semola, D.: Rectifiability of RCD(K,N) spaces via γ -splitting maps. *Ann. Fenn. Math.* **46**(1), 465–482 (2021)
24. Burago, D., Burago, Y., Ivanov, S.: *A course in metric geometry*. Vol. 33. *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, xiv+415 (2001)
25. Carlotto, A., Chodosh, O., Eichmair, M.: Effective versions of the positive mass theorem. *Inventiones Mathematicae* **206**(3), 975–1016 (2016)
26. Cavalletti, F., Manini, D.: Isoperimetric inequality in noncompact MCP spaces. (2021) [arXiv:2110.07528](https://arxiv.org/abs/2110.07528)
27. Cavalletti, F., Milman, E.: The globalization theorem for the curvature-dimension condition. *Invent. Math.* **226**(1), 1–137 (2021)
28. Cheeger, J., Colding, T.H.: Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math.* **144**(1), 189–237 (1996)
29. Cheeger, J., Colding, T.H.: On the structure of spaces with Ricci curvature bounded below I. *J. Differ. Geom.* **46**(3), 406–480 (1997)
30. Cheeger, J., Colding, T.H.: On the structure of spaces with Ricci curvature bounded below. II. *J. Differ. Geom.* **54**(1), 13–35 (2000)
31. Cheeger, J., Colding, T.H.: On the structure of spaces with Ricci curvature bounded below. III. *J. Differ. Geom.* **54**(1), 37–74 (2000)
32. Cheeger, J., Naber, A.: Lower bounds on Ricci curvature and quantitative behavior of singular sets. *Invent. Math.* **191**(2), 321–339 (2013)
33. Chodosh, O.: Large isoperimetric regions in asymptotically hyperbolic manifolds. *Commun. Math. Phys.* **343**(2), 393–443 (2016)

34. Chodosh, O., Eichmair, M., Volkman, A.: Isoperimetric structure of asymptotically conical manifolds. *J. Differ. Geom.* **105**(1), 1–19 (2017)
35. Colding, T.H.: Ricci curvature and volume convergence. *Ann. Math.* **145**(3), 477–501 (1997)
36. Colding, T.H., Naber, A.: Characterization of tangent cones of noncollapsed limits with lower Ricci bounds and applications. *Geom. Funct. Anal.* **23**(1), 134–148 (2013)
37. Croke, C.B., Karcher, H.: Volumes of small balls on open manifolds: lower bounds and examples. *Trans. Amer. Math. Soc.* **309**(2), 753–762 (1988)
38. DePhilippis, G., Gigli, N.: From volume cone to metric cone in the nonsmooth setting. *Geom. Funct. Anal.* **26**(6), 1526–1587 (2016)
39. DePhilippis, G., Gigli, N.: Non-collapsed spaces with Ricci curvature bounded from below. *J. Éc. Polytech. Math.* **5**, 613–650 (2018)
40. Di Marino, S.: Recent advances on BV and Sobolev Spaces in metric measure spaces. Ph.D Thesis. <https://cvgmt.sns.it/paper/2568/>. Scuola Normale Superiore, Pisa, (2014)
41. Eichmair, M., Metzger, J.: Large isoperimetric surfaces in initial data sets. *J. Differ. Geom.* **94**(1), 159–186 (2013)
42. Eichmair, M., Metzger, J.: Unique isoperimetric foliations of asymptotically flat manifolds in all dimensions. *Invent. Math.* **194**(3), 591–630 (2013)
43. Erbar, M., Kuwada, K., Sturm, K.T.: On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces. *Invent. Math.* **201**(3), 993–1071 (2015)
44. Fogagnolo, M., Mazzieri, L.: Minimising hulls, p-capacity and isoperimetric inequality on complete Riemannian manifolds. [arXiv:2012.09490](https://arxiv.org/abs/2012.09490) (2021)
45. Gallot, S., Hulin, D., Lafontaine, J.: *Riemannian Geometry*. Universitext. Springer Berlin Heidelberg, (2004). isbn: 9783540204930
46. Gallot, S., Hulin, D., Lafontaine, J.: *Riemannian geometry*. Third. Universitext. Springer-Verlag, Berlin, (2004), xvi+322. isbn: 3-540-20493-8
47. Gigli, N.: The splitting theorem in non-smooth context. (2013). [arXiv:1302.5555](https://arxiv.org/abs/1302.5555)
48. Gigli, N.: On the differential structure of metric measure spaces and applications. In: *Mem. Amer. Math. Soc.* 236.1113 (2015), vi+91
49. Gigli, N., Mondino, A., Savaré, G.: Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows. *Proc. Lond. Math. Soc.* **111**(5), 1071–1129 (2015)
50. Gonzalez, E., Massari, U., Tamanini, I.: On the regularity of boundaries of sets minimizing perimeter with a volume constraint. *I. Indiana Univ. Math. J.* **32**(1), 25–37 (1983)
51. Guijarro, L., Kapovitch, V.: Restrictions on the geometry at infinity of nonnegatively curved manifolds. *Duke Math. J.* **78**(2), 257–276 (1995)
52. Hebey, E.: *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*. Vol. 5. Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, (1999), x+309
53. John, F.: Sobolev inequalities on manifolds with nonnegative Bakry-Émery Ricci curvature. (2021). [arXiv:2103.08496](https://arxiv.org/abs/2103.08496)
54. Kasue, A.: A compactification of a manifold with asymptotically nonnegative curvature. *Ann. Sci. École Norm. Sup.* **21**(4), 593–622 (1988)
55. Kasue, A., Washio, T.: Growth of equivariant harmonic maps and harmonic morphisms. *Osaka J. Math.* **27**(4), 899–928 (1990)
56. Kitabeppu, Y.: A Bishop-type inequality on metric measure spaces with Ricci curvature bounded below. *Proc. Amer. Math. Soc.* **145**(7), 3137–3151 (2017)
57. Leonardi, G. P., Ritoré, M., Vernadakis, E.: Isoperimetric inequalities in unbounded convex bodies. To appear in *Memoirs of the AMS*. (2016). [arXiv:1606.03906](https://arxiv.org/abs/1606.03906)
58. Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. *Ann. Math.* **169**(3), 903–991 (2009)
59. Maggi, F.: *Sets of finite perimeter and geometric variational problems*. Vol. 135. Cambridge Studies in Advanced Mathematics. An introduction to geometric measure theory. Cambridge University Press, Cambridge, (2012), xx+454
60. Miranda, M., Jr.: Functions of bounded variation on “good” metric spaces. *J. Math. Pures Appl.* **82**(8), 975–1004 (2003)
61. Miranda, M., Jr., Pallara, D., Paronetto, F., Preunkert, M.: Heat semigroup and functions of bounded variation on Riemannian manifolds. *J. Reine Angew. Math.* **613**, 99–119 (2007)
62. Mondino, A., Naber, A.: Structure theory of metric measure spaces with lower Ricci curvature bounds. *J. Eur. Math. Soc.* **21**, 1809–1854 (2014)
63. Mondino, A., Nardulli, S.: Existence of isoperimetric regions in non-compact Riemannian manifolds under Ricci or scalar curvature conditions. *Comm. Anal. Geom.* **24**(1), 115–138 (2016)

64. Morgan, F.: Regularity of isoperimetric hypersurfaces in Riemannian manifolds. *Trans. Am. Math. Soc.* **355**(12), 5041–5052 (2003)
65. Morgan, F., Johnson, D.L.: Some sharp isoperimetric theorems for Riemannian manifolds. *Indiana Univ. Math. J.* **49**(3), 1017–1041 (2000)
66. Morgan, F., Ritoré, M.: Isoperimetric regions in cones. *Trans. Amer. Math. Soc.* **354**(6), 2327–2339 (2002)
67. MuñozFlores, A.E., Nardulli, S.: Local Hölder continuity of the isoperimetric profile in complete non-compact Riemannian manifolds with bounded geometry. *Geom. Dedicata* **201**, 1–12 (2019)
68. MuñozFlores, A.E., Nardulli, S.: The isoperimetric problem of a complete Riemannian manifold with a finite number of C^0 -asymptotically Schwarzschild ends. *Comm. Anal. Geom.* **28**(7), 1577–1601 (2020)
69. Nardulli, S.: Generalized existence of isoperimetric regions in non-compact Riemannian manifolds and applications to the isoperimetric profile. *Asian J. Math.* **18**(1), 1–28 (2014)
70. Nobili, F., Violo, I. Y.: Rigidity and almost rigidity of Sobolev inequalities on compact spaces with lower Ricci curvature bounds. (2021). [arXiv:2108.02135](https://arxiv.org/abs/2108.02135)
71. Perelman, G.: A complete Riemannian manifold of positive Ricci curvature with Euclidean volume growth and nonunique asymptotic cone". In: *Comparison geometry* (Berkeley, CA, 1993-94). Vol. 30. *Math. Sci. Res. Inst. Publ.* Cambridge Univ. Press, Cambridge, (1997), 165-166
72. Petersen, P.: *Riemannian geometry*. Third edition. Vol. 171. *Graduate Texts in Mathematics*. Springer, Cham, (2016), xviii+499
73. Pigola, S., Rigoli, M., Setti, A. G.: *Vanishing and finiteness results in geometric analysis*. Vol. 266. *Progress in Mathematics*. A generalization of the Bochner technique. Birkhäuser Verlag, Basel, (2008), xiv+282
74. Ritoré, M.: The isoperimetric problem in complete surfaces of nonnegative curvature. *J. Geom. Anal.* **11**(3), 509–517 (2001)
75. Ritoré, M.: Continuity of the isoperimetric profile of a complete Riemannian manifold under sectional curvature conditions. *Rev. Mat. Iberoam.* **33**(1), 239–250 (2017)
76. Ritoré, M., Rosales, C.: Existence and characterization of regions minimizing perimeter under a volume constraint inside Euclidean cones. *Trans. Amer. Math. Soc.* **356**(11), 4601–4622 (2004)
77. Schoen, R., Yau, S.-T.: *Lectures on differential geometry*. *Conference Proceedings and Lecture Notes in Geometry and Topology, I*. International Press, Cambridge, MA, (1994), v+235
78. Sternberg, P., Zumbrun, K.: On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint. *Comm. Anal. Geom.* **7**(1), 199–220 (1999)
79. Sturm, K.T.: On the geometry of metric measure spaces. I. *Acta Math.* **196**(1), 65–131 (2006)
80. Sturm, K.T.: On the geometry of metric measure spaces. II. *Acta Math.* **196**(1), 133–177 (2006)
81. Tamanini, I.: Boundaries of Caccioppoli sets with Hölder-continuous normal vector. *J. Reine Angew. Math.* **334**, 27–39 (1982)
82. Villani, C.: *Optimal transport*. Vol. 338. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Old and new. Springer-Verlag, Berlin, (2009), xxii+973
83. Xia, Q.: Regularity of minimizers of quasi perimeters with a volume constraint. *Interfaces Free Bound.* **7**(3), 339–352 (2005)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.