Degradation of entanglement in Markovian noise

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The entanglement survival time is defined as the maximum time a system which is evolving under the action of local Markovian, homogeneous-in-time noise is capable of preserving the entanglement it had at the beginning of the temporal evolution. In this paper we study how this quantity is affected by the interplay between the coherent preserving and dissipative contributions of the corresponding dynamical generator. We report the presence of a counterintuitive, nonmonotonic behavior in such a functional, capable of inducing sudden death of entanglement in models which, in the absence of unitary driving, are capable of sustaining entanglement for arbitrarily long times.

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I. INTRODUCTION

Entanglement is a fundamental yet extremely fragile resource of quantum information processing [1]. Preventing its degradation is a fundamental step in the development of quantum technology. Starting from the seminal work on quantum error correction [2], decoherence-free subspaces [3], and dynamical decoupling [4], a number of methods have been proposed to provide partial protection against such a detrimental effect. Most of these approaches typically work under the paradigm of mitigating the environmental noise by properly intertwining the dynamics it induces with external controls. Moreover, such controls usually correspond to Hamiltonian corrections. The basic idea is to fight dissipative and decoherence mechanisms through the action of driving forces that drag the system in regions of the Hilbert space where the former are not so effective. Interestingly enough such external forces do not necessarily need to be coherence preserving: indeed, while typically summing noise sources tends to increase the speed at which entanglement gets lost [5], it may occur that by properly alternating their actions the entanglement survival time can be increased [6]. Similarly, it is clear that coherence-preserving controls do not always help in contrasting the noise: a not carefully designed Hamiltonian driving might amplify the dissipation induced by the environment. Motivated by these observations, in the present paper we study the maximum entanglement survival time $\tau_{\rm ent}$ for a system evolving under the action of a local Markovian, time-homogeneous noise [7]. In the general formalism established by Gorini, Kossakowski, Sudarshan, and Lindblad [8,9], these models are fully described by assigning a dynamical generator $\mathcal{L}$ which includes two distinct contributions: a coherence-preserving term associated with a Hamiltonian operator, and a purely dissipative one associated with a Lindblad superoperator term. For assigned intensity of the latter our goal is to determine how $\tau_{\rm ent}$ varies when increasing the intensity of the former, in order to understand whether Hamiltonian corrections always help in preserving entanglement, and more generally to unveil the interplay between purely dissipative and coherence-preserving contributions in the Lindblad generator. Naively one would expect that a predominance of the Hamiltonian term would tend to increase the survival time of the entanglement. However, for the schemes we have considered this is not the case: the minimal value of $\tau_{\rm ent}$ is reached for a non zero value of the Hamiltonian intensity.

In our analysis we formally identify $\tau_{\rm ent}$ with the smallest time interval after which the dynamics associated with the selected $\mathcal{L}$ becomes an entanglement-breaking (EB) quantum channel [10,11]. This choice makes sure that, irrespective of the initial conditions, no entanglement between the system of interest and any possible ancillary system will survive after $\tau_{\rm ent}$. Conclusive results are presented for the case of qubit systems and for continuous-variable systems evolving under the action of Gaussian noise.

The presented material is organized as follows: We start in Sec. II introducing the formal definition of entanglement survival time for generic open quantum system dynamics and review some basic properties of dynamical semigroups. After presenting a detailed analysis of the general properties of the entanglement survival time in Sec. III A, we focus on some models. Notably, in Sec. IV we present some results dealing with qubit systems while in Sec. IV D we extend the analysis to the case of Gaussian Bosonic channels. Conclusions and final remarks are presented in Sec. V, while technical derivations are presented in the Appendixes.

II. MAXIMUM ENTANGLEMENT SURVIVAL TIME

Consider a quantum system A that is evolving under the noisy influence of an external environment E, whose action we represent by means of a continuous, one-parameter family $\{\Phi_{t,0}\}_{t\geqslant 0}$ of completely positive, trace-preserving (CPt) linear superoperators [12–14]. Assume next that at $t = 0$, A is initialized into a (possibly entangled) joint state $\rho_{AB}(0)$ with an ancillary system B which, without loss of generality, we assume to be isomorphic with A, and which does not couple with E. In this setting we define $t^*(\rho_{AB}(0))$ as the minimum temporal evolution time t at which no entanglement can be found in the associated evolved density matrix

$$\rho_{AB}(t) = (\Phi_{t,0} \otimes \mathrm{id}_B)[\rho_{AB}(0)], \tag{1}$$

(id_B being the identity superoperator on B), i.e., the quantity

$$t^*(\rho_{AB}(0)) := \min\{t \geqslant 0 \text{ s.t. } \rho_{AB}(t) \in \mathfrak{S}_{\mathrm{sep}}(\mathcal{H}_{AB})\}, \tag{2}$$

with $\mathfrak{S}_{\mathrm{sep}}(\mathcal{H}_{AB})$ the subset of separable states of AB. As explicitly indicated by the notation the expression in Eq. (2) is a function of the chosen initial state $\rho_{AB}(0)$: it runs from the minimum value zero (attained when $\rho_{AB}(0)$ is an element of $\mathfrak{S}_{\mathrm{sep}}(\mathcal{H}_{AB})$) to a maximum value

$$\tau_{\mathrm{ent}} := \max_{\rho_{AB}(0)\in\mathfrak{S}(\mathcal{H}_{AB})} t^*(\rho_{AB}(0)), \tag{3}$$

which only depends upon the properties of the maps $\{\Phi_{t,0}\}_{t\geqslant 0}$ and which can be equivalently expressed as the smallest time t for which $\Phi_{t,0}$ becomes EB, i.e.,

$$\tau_{\mathrm{ent}} = \min\{t \geqslant 0 \text{ such that } \Phi_{t,0} \in \mathrm{EB}\}. \tag{4}$$

Since it defines the maximum time interval on which we are guaranteed to have some entanglement between A and B under the evolution (1), we refer to τ_{ent} as the "entanglement survival time" (EST) of the selected dynamical process. Notice, however, that if the maps $\{\Phi_{t,0}\}_{t\geqslant 0}$ exhibit a strong non-Markovian character inducing a significant backflow of information into the system temporal evolution [15–19], nothing prevents the possibility that entanglement between A and B will reemerge at some time t greater than τ_{ent}. The same effect, however, cannot occur in the case of Markovian or weakly non-Markovian models for which instead one has

$$\Phi_{t,0} \in \mathrm{EB} \quad \text{for all } t \geqslant \tau_{\mathrm{ent}}, \tag{5}$$

meaning that the AB entanglement is lost forever at time τ_{ent}. Following the approach of Ref. [20] these two special classes of processes are characterized by families $\{\Phi_{t,0}\}_{t\geqslant 0}$ whose elements fulfill the CP-divisibility or P-divisibility condition respectively, i.e.,

$$\Phi_{t,0} = \Lambda_{t,t'} \circ \Phi_{t',0}\ , \quad \forall t \geqslant t' \geqslant 0\ , \tag{6}$$

where "$\circ$" indicates the composition of superoperators and where the connecting element $\Lambda_{t,t'}$ are CP (Markovian processes) or simply positive transformations (weakly non-Markovian processes). Equation (5) can then be derived by setting $t' = \tau_{\mathrm{ent}}$ in Eq. (6) and exploiting the fact that the composition of an EB channel with a CP, or just positive, map is still EB.

An important subclass of Markovian (CP-divisible) processes is provided by the so-called dynamical semigroups, characterized by channels $\{\Phi_{t,0}\}_{t\geqslant 0}$ which are invariant under translations of the time coordinates or, equivalently, by connecting maps which are time homogeneous, i.e.,

$$\Lambda_{t,t'} = \Lambda_{t-t',0} = \Phi_{t-t',0}\ , \quad \forall t \geqslant t'. \tag{7}$$

Accordingly defining $\Phi_t := \Phi_{t,0}$, Eq. (7) allows us to recast Eq. (6) in terms of the following semigroup identity:

$$\Phi_t \circ \Phi_{\Delta t} = \Phi_{\Delta t} \circ \Phi_t = \Phi_{t+\Delta t}, \quad \forall t, \Delta t \geqslant 0, \tag{8}$$

which ultimately yields a first-order differential equation

$$\dot{\Phi}_t = \mathcal{L} \circ \Phi_t, \quad \Phi_0 = \mathrm{id}, \tag{9}$$

driven by a Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) generator $\mathcal{L}$ [8,9]. The latter admits a standard decomposition in terms of two competing terms: a coherence-preserving contribution gauged by a Hamiltonian term governed by a self-adjoint operator H and by a purely dissipative term $\mathcal{D}$ governing the irreversible process. Specifically, we have

$$\mathcal{L}[\ \cdot\ ] = \gamma\ \mathcal{D}[\ \cdot\ ] - i\omega\ [H,\ \cdot\ ]_-\ , \tag{10}$$

with

$$\mathcal{D}[\ \cdot\ ] = \sum_{j=1}^{d^2-1} \left(L_j[\ \cdot\ ]L_j^\dagger - \frac{1}{2}[L_j^\dagger L_j,\ \cdot\ ]_+ \right), \tag{11}$$

the sum running over a set of no better specified (Lindblad) operators $\{L_j\}_j$, and the symbols $[\ ,\ ]_\pm$ indicating the commutator ($-$) and anticommutator ($+$) brackets, respectively (d being the dimension of A). In Eq. (10) the quantities $\omega, \gamma \geqslant 0$ have dimension of a frequency and gauge of the time scale and the relative strengths of the two competing dynamical mechanisms that act on A: accordingly we refer ω as the (unitary) driving parameter and to γ as the damping parameter (herewith and in the following we set $\hbar = 1$ for the sake of convenience).

As Eq. (9) admits a formal integration

$$\Phi_t = e^{t\mathcal{L}}, \tag{12}$$

it is clear that the EST of a dynamical semigroup must be a functional of its generator, i.e.,

$$\tau_{\mathrm{ent}} = \tau_{\mathrm{ent}}(\mathcal{L}). \tag{13}$$

Analyzing such dependence is the aim of the present work. More precisely, for fixed H and $\mathcal{D}$ we are interested in studying in which way the parameters ω and γ that measure the relative "strengths" of the Hamiltonian and the dissipative contributions of $\mathcal{L}$ affect the value of τ_{ent}. Intuitively one would expect that larger incidence of the first mechanism with respect to the second one would yield longer values of the corresponding EST. Interestingly enough it turns out that this is not always the case: as we explicitly see, in some circumstances the presence of a nonzero value of the Hamiltonian parameter ω induces a drastic reduction of the EST of the model.

III. EVALUATING EST FOR DYNAMICAL SEMIGROUP

In this section we analyze a few examples of dynamical semigroups and compute their associated EST. We start in Sec. III A by presenting some general properties of the functional (13). In Sec. III B we focus instead on the special cases of qubit systems which allow for an almost complete analytical treatment. Finally in Sec. IV D we discuss the problem in the context of Gaussian Bosonic channels.

A. Preliminary observations

In the study of the functional (13) some structural properties of the GKSL generator should be taken into consideration. First of all, an almost immediate consequence of our definitions is the scaling law

$$\tau_{\rm ent}(q\mathcal{L}) = \tau_{\rm ent}(\mathcal{L})/q \tag{14}$$

that holds for all $q \geqslant 0$ and for all $\mathcal{L}$. Hence for fixed H and $\mathcal{D}$ we can write

$$\tau_{\rm ent}(\mathcal{L}) = \mathcal{T}_{\rm ent}(\kappa)/\gamma, \tag{15}$$

where

$$\kappa := \omega/\gamma \tag{16}$$

is the ratio of the driven and damping constants of the model, and $\mathcal{T}_{\rm ent}(\kappa)$ is a dimensionless quantity associated with the (dimensionless) GKSL generator $\mathcal{D}[\,\cdot\,] - i\kappa\ [H,\ \cdot\,]_-$. Next we remind that the decomposition (10) is not unique because H and the associated Lindblad operators $\{L_j\}_j$ can be freely redefined according to the transformations

$$\begin{aligned} H &\to H' = H + \frac{1}{2i\kappa}\sum_j (c_j^* L_j - c_j L_j^\dagger) + b,\\ L_j &\to L_j' = L_j + c_j, \end{aligned} \tag{17}$$

with c_j being complex numbers and b being an arbitrary real parameter [21], and where the ratio κ on the first term accounts for the strength parameters γ and ω. While the term b plays no role in the derivation (it gets canceled when entering the commutation brackets), the coefficients c_j induce a nontrivial symmetry into the model that we fix by forcing the L_j to be traceless.

A further symmetry of the problem arises from the fact that local unitary transformations can neither create nor destroy entanglement [22]. Accordingly the EST of an arbitrary (not necessarily Markovian) process $\{\Phi_{t,0}\}_{t\geqslant 0}$ is invariant under transformations of the form

$$\Phi'_{t,0} = \mathcal{V}_t \circ \Phi_{t,0} \circ \mathcal{U}_t, \tag{18}$$

where $\mathcal{U}_t[\,\cdot\,] = U_t[\,\cdot\,]U_t^\dagger$ and $\mathcal{V}_t = V_t[\,\cdot\,]V_t^\dagger$ represent unitary conjugations induced by the (possibly time-dependent) operators U_t and V_t, respectively. At the level of the dynamical semigroup this translates into the identity

$$\tau_{\rm ent}(\mathcal{L}) = \tau_{\rm ent}(\mathcal{U}^{-1} \circ \mathcal{L} \circ \mathcal{U}), \tag{19}$$

that holds for a generic (time-independent) unitary conjugation $\mathcal{U}$. Equation (19) can be easily verified by noticing that, given the semigroup Φ_t generated by $\mathcal{L}$ and the semigroup Φ_t' generated by $\mathcal{L}' = \mathcal{U}^{-1} \circ \mathcal{L} \circ \mathcal{U}$, the two are connected as in Eq. (18) by setting $\mathcal{V}_t = \mathcal{U}^{-1}$ and $\mathcal{U}_t = \mathcal{U}$. Notice also that invariance of the EST under Eq. (18) can be used to explicitly verify that in the evaluation of such a parameter it does not matter whether we integrate Eq. (9) directly or by passing through the standard interaction picture. Indeed by setting $\mathcal{U}_t = \mathrm{id}$ and identifying $\mathcal{V}_t^{-1}$ with the evolution induced by the Hamiltonian H of Eq. (10), the integration of Eq. (9) in the standard interaction picture can be seen as a special instance of Eq. (18), with $\Phi'_{t,0}$ being the nonhomogeneous Markovian process characterized by the time-dependent generator $\mathcal{L}_t' = \mathcal{V}_t \circ \mathcal{D} \circ \mathcal{V}_t^{-1}$.

B. Qubit systems

In Ref. [10] it was established that determining whether a given CPt map Φ is EB is equivalent to checking if its associated Choi-Jamiołkowski state $\rho_{AB}^{(\Phi)}$ [23,24] is separable or not. For finite-dimensional systems the latter is defined as the output density matrix generated by Φ when acting locally on a maximally entangled state, i.e.,

$$\rho_{AB}^{(\Phi)} = (\Phi \otimes \mathrm{id}_B)[|\Omega\rangle_{AB}\langle\Omega|], \tag{20}$$

$$|\Omega\rangle_{AB} := \frac{1}{\sqrt{d}}\sum_{k=1}^{d} |k\rangle_A \otimes |k\rangle_B, \tag{21}$$

where d is the dimension of A, and where for $Q = A, B$, $\{|k\rangle_Q\}_{k=1,\dots,d}$ is an orthonormal basis of the system Q. A direct consequence of this fact is that the maximum in Eq. (3) is always attainable on the pure state (21), i.e., that $\tau_{\rm ent}(\mathcal{L})$ of a semigroup $\{\Phi_t\}_{t\geqslant 0}$ can be found as the minimum value of t for which $\rho_{AB}^{(\Phi_t)}$ becomes separable. If A is a qubit, i.e., if $d = 2$, we can address this task by exploiting the positive partial transpose (PPT) criterion [25,26], which states that $\rho_{AB}^{(\Phi_t)}$ is separable if and only if its partial transposes (say, $[\rho_{AB}^{(\Phi_t)}]^{T_B}$) are non-negative, i.e., if and only if all its eigenvalues are greater than or equal to zero. By continuity, $\tau_{\rm ent}$ can then be also identified as the smallest t which nullifies the determinant of $[\rho_{AB}^{(\Phi_t)}]^{T_B}$, i.e.,

$$\tau_{\rm ent} = \min\left\{t \geqslant 0,\ \text{such that}\ \det\left(\left[\rho_{AB}^{(\Phi_t)}\right]^{T_B}\right) = 0\right\}, \tag{22}$$

or, equivalently, as the smallest t which nullifies the corresponding negativity of entanglement [27], i.e.,

$$\tau_{\rm ent} = \min\left\{t \geqslant 0,\ \text{such that}\ \mathcal{N}\left(\rho_{AB}^{(\Phi_t)}\right) = 0\right\}, \tag{23}$$

where given ρ_{AB} is a generic state we have

$$\mathcal{N}(\rho_{AB}) = \frac{1}{2}\sum_\ell (|\lambda_\ell| - \lambda_\ell), \tag{24}$$

with $\{\lambda_\ell\}_\ell$ being the eigenvalues of $\rho_{AB}^{T_B}$. It is worth observing that since the negativity of entanglement is an entanglement monotone [28] (the higher its values the higher is the entanglement present in the system), the function $\mathcal{N}(\rho_{AB}^{(\Phi_t)})$ can also be used to monitor how the entanglement gets degraded before completely disappearing at $\tau_{\rm ent}$. Furthermore, we notice that for $d > 2$ where the PPT criterion provides a sufficient but not necessary condition for separability, the terms on the right-hand side (rhs) of Eqs. (22) and (23) provide an upper bound for $\tau_{\rm ent}$.

Asymptotic analysis. While for $d = 2$ determining the eigenvalues of $[\rho_{AB}^{(\Phi_t)}]^{T_B}$ is always possible in principle, extracting $\tau_{\rm ent}$ from Eqs. (22) or (23) requires in general that one solve a transcendental equation. As a result, only in a few cases it is possible to carry out the entire analysis analytically and one has to resort to numerical methods, for instance, to the Newton-Raphson method, which we employ extensively in the following sections (in particular in the plots shown in Figs. 2–4). Yet by inspecting the asymptotic behavior of the spectrum of $[\rho_{AB}^{(\Phi_t)}]^{T_B}$ (a relatively simple task) it can often be inferred whether the entanglement transmission time of a given dynamical semigroup is finite or infinite. Consider in fact the case where the process admits a single relaxation state, i.e.,

$$\lim_{t\to\infty} \Phi_t(\rho_A(0)) = \bar{\rho}_A, \quad \forall \rho_A(0) \in \mathfrak{S}(\mathcal{H}_A), \tag{25}$$

with $\bar{\rho}_A$ being determined by the identity $\mathcal{L}[\bar{\rho}_A] = 0$. Accordingly the Choi-Jamiołkowski state will converge to the following separable state:

$$\rho_{AB}^{(\Phi_\infty)} := \lim_{t\to\infty} \rho_{AB}^{(\Phi_t)} = \bar{\rho}_A \otimes \mathbb{1}_B/2, \tag{26}$$

which implies

$$\det\left(\left[\rho_{AB}^{(\Phi_\infty)}\right]^{T_B}\right) = \frac{\det(\bar{\rho}_A)}{4} \geqslant 0. \tag{27}$$

Suppose hence that $\det(\bar{\rho}_A) > 0$, which always happens unless the fixed point $\bar{\rho}_A$ is a pure state. Then, considering that for $t = 0$ one has $\det([\rho_{AB}^{(\Phi_0)}]^{T_B}) = \det(|\Omega\rangle_{AB}\langle\Omega|^{T_B}) = -1/16$, by a simple continuity argument it follows that the function $\det([\rho_{AB}^{(\Phi_t)}]^{T_B})$ must cross zero at some finite time t which, via Eq. (22), corresponds to the EST of the problem. If on the contrary we have $\det(\bar{\rho}_A) = 0$, i.e., if $\bar{\rho}_A$ is pure, the continuity argument cannot be applied and the system may exhibit a divergent value of the EST; i.e., the associated dynamical semigroup becomes EB only asymptotically. Borrowing from the terminology introduced in Ref. [1] we can hence conclude that the purity of the relaxation state $\bar{\rho}_A$ provides a sufficient criterion for determining whether the associated dynamical semigroup induces entanglement sudden death (ESD):

ESD criterion. Dynamical semigroups admitting a nonpure density matrix as a relaxation state are characterized by a finite value of the EST.

We conclude by stressing that while explicitly discussed for the qubit case scenario, it is clear that the above argument holds true for system A of arbitrary dimension d, the only difference being associated with the fact that now the rhs terms of Eqs. (26) and (27) get replaced, respectively, by $\bar{\rho}_A \otimes \mathbb{1}_B/d$ and $\det(\bar{\rho}_A)/d^d$.

IV. MODELS

In this section we explicitly explore the behavior of the τ_{ent} for different prototypical models for finite- and infinite-dimensional systems.

A. Phase-flip qubit channels

As a first example of a dynamical semigroup acting on a qubit we consider the case of GKSL generators $\mathcal{L}$ [Eq. (10)] having and arbitrary Hamiltonian term and a unique Lindblad operator L which is Hermitian. For a proper choice of the drift and the damping coefficients ω, γ, and invoking the gauge freedom (17) to make L traceless, the most general example of such processes can be described by setting $L = Z/\sqrt{2}$ and taking $H = \hat{n}\cdot\vec{\sigma}$ with $\hat{n} := (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ being a real unit vector, and with $\vec{\sigma} := (X, Y, Z)$ being the vector of Pauli matrices. Equation (10) hence becomes

$$\mathcal{L}[\,\cdot\,] = \frac{\gamma}{2}\,(Z[\,\cdot\,]Z - \text{id}[\,\cdot\,]) - i\omega\,[\hat{n}\cdot\vec{\sigma},\ \cdot\,]_-\,, \tag{28}$$

which, in the computational basis associated with the eigenvectors of Z, can be interpreted as a phase-flip noise process [22] affecting the qubit A while the latter evolves in the presence of a driving field in the $\hat{n}$ direction. Invoking the equivalence (19) the analysis can be further simplified by observing that a proper unitary rotation along the z axis can be used to bring $\hat{n}$ into the xz plane while keeping the dissipator component invariant. Accordingly, without loss of generality, in our analysis we set equal to zero the azimuthal angle φ, restricting the analysis to Hamiltonian driving of the form

$$\hat{n} = (\sin\theta, 0, \cos\theta). \tag{29}$$

As a preliminary step let us first consider the scenario where no coherent driving is acting on the system ($\kappa = 0$), so that $\mathcal{L} = \gamma\mathcal{D}$. By explicit integration of the system dynamics (see Appendix A) one can easily verify that in this case the negativity of entanglement (24) of the associated Choi-Jamiołkowski state (20) is equal to

$$\mathcal{N}\big(\rho_{AB}^{(\Phi_t)}\big)\Big|_{\kappa=0} = e^{-\gamma t}/2, \tag{30}$$

which shows that the entanglement in the system is degraded exponentially fast, even though it is never completely broken, yielding a divergent value for the associated EST, i.e., using Eqs. (16) and (15),

$$\mathcal{T}_{\text{ent}}(0) = \infty. \tag{31}$$

The same result holds also for arbitrary ω and $\hat{n}$ pointing into the z axis, i.e., $\theta = 0$. In this case in fact passing into the interaction picture representation the driving term can be eliminated without affecting the dissipator, making the former completely irrelevant for the computation of the EST (see comments at the end of Sec. III A).

The problem becomes more interesting when we take $\hat{n}$ as a unit vector that points into the x axis ($\theta = \pi/2$), i.e., $H = X$. Under these assumptions, in the operator basis $\{E^{(00)}, E^{(10)}, E^{(01)}, E^{(11)}\}$ formed by the external products $E^{(ij)} = |i\rangle\langle j|$ of the computational basis, the Lindbladian (28) reads

$$\mathcal{L} = \gamma\begin{pmatrix} 0 & -i\kappa & i\kappa & 0 \\ -i\kappa & -1 & 0 & i\kappa \\ i\kappa & 0 & -1 & -i\kappa \\ 0 & i\kappa & -i\kappa & 0 \end{pmatrix}. \tag{32}$$

By direct evaluation one can verify that for all $\kappa > 0$ it admits as a unique zero eigenvector the completely mixed state $\bar{\rho}_A = \mathbb{1}_A/2$. Hence from the results of the previous section we can conclude that in these cases, at variance with the $\kappa = 0$ scenario (31), the corresponding EST must be finite, yielding ESD [1]. This is a rather remarkable fact as it implies that by adding a unitary (coherence-preserving) contribution to the dissipative dynamics induced by the phase-flip noise generator, we can end up with a "noisier" evolution which becomes EB at a finite time. A more quantitative statement can be obtained by studying the negativity of entanglement, i.e.,

$$\mathcal{N}\big(\rho_{AB}^{(\Phi_t)}\big) = \frac{e^{-\gamma t/2}}{2}\max\{Q_\kappa(\gamma t/2) - \sinh(\gamma t/2), 0\}, \tag{33}$$

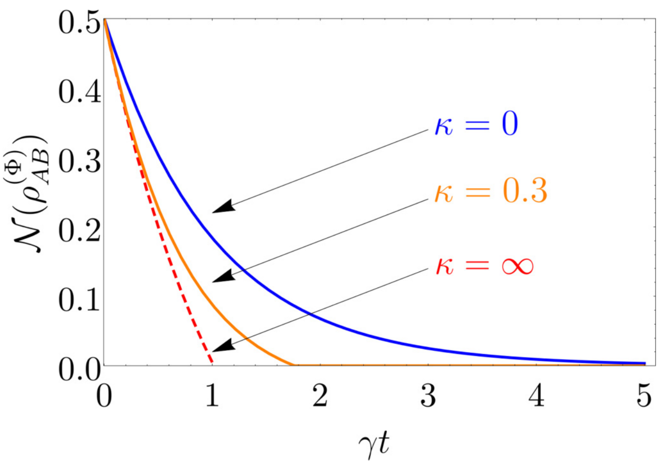


FIG. 1. Temporal evolution of the negativity of entanglement [Eq. (33)] of the phase-flip process [Eq. (28)] for different values of the ratio $\kappa = \omega/\gamma$ and for $\theta = \pi/2$, $\varphi = 0$.

with

$$Q_\kappa(\tau) := \sqrt{\frac{\cosh^2(\tau\sqrt{1-16\kappa^2}) - 16\kappa^2}{1-16\kappa^2}}. \tag{34}$$

The functional dependence of this quantity upon the parameter κ is rather involved, still; as evident from the plots presented in Fig. 1 it clearly emerges that the entanglement present in the model tends to degrade faster as the driving/damping ratio increases. According to Eq. (23) the associated EST can be determined by identifying the zeros of Eq. (33), i.e., by solving the transcendental equation

$$Q_\kappa(\gamma t/2) = \sinh(\gamma t/2)\ , \tag{35}$$

which admits a closed analytical solution for the two extremal cases $\kappa = 0$ and $\kappa \to \infty$. In particular for $\kappa = 0$, since $Q_0(\tau) = \cosh(\tau)$, Eq. (35) allows us to recover the results anticipated in Eqs. (30) and (31). For $\kappa = \infty$ instead one has that $Q_\kappa(\tau)$ converges to 1, allowing us to replace Eq. (33) with

$$\mathcal{N}(\rho_{AB}^{(\Phi_t)})\Big|_{\kappa=\infty} = \frac{e^{-\gamma t/2}}{2}\max\{1-\sinh(\gamma t/2), 0\}, \tag{36}$$

and yielding the following value for the associated rescaled EST functional (15):

$$\mathcal{T}_{\rm ent}(\infty) = \text{arcosh}(3)\ . \tag{37}$$

For the remaining choices of the driving/damping ratio κ an approximate treatment of Eq. (35) allows us to write

$$\mathcal{T}_{\rm ent}(\kappa) \simeq \begin{cases} W(1/4\kappa^2), & \kappa \simeq 0 \\ 2.5 - 3.7\big(\kappa - \frac{1}{4}\big) + 10.6\big(\kappa - \frac{1}{4}\big)^2, & \kappa \simeq \frac{1}{4} \\ \mathcal{T}_{\rm ent}(\infty) + \frac{1-\cos(\mathcal{T}_{\rm ent}(\infty)\sqrt{16\kappa^2-1})}{2\sqrt{2}(16\kappa^2-1)}, & \kappa \simeq \infty, \end{cases} \tag{38}$$

where W is the Lambert function [29] (see Appendix C for details). Furthermore, in the high driving regime $\kappa \geqslant 1/4$, the following inequality can be established:

$$\mathcal{T}_{\rm ent}(\infty) \leqslant \mathcal{T}_{\rm ent}(\kappa) \leqslant \text{arcosh}\left(2 - \frac{1+16\kappa^2}{1-16\kappa^2}\right). \tag{39}$$

In Fig. 2(a) we report a numerical solution of Eq. (35), together with the bounds (39), which confirms the general tendency of the model in translating a high level of unitary driving into a stronger entanglement suppression. A similar behavior is observed for intermediate values of θ in the interval $[0, \pi/2]$ until it eventually diverges everywhere when θ approaches zero: a numerical evaluation of the associated value $\mathcal{T}_{\rm ent}(\kappa)$ is reported in Fig. 2.

B. Generalized amplitude damping process

As our next example we focus on the case where the dissipator $\mathcal{D}$ describes a generalized amplitude damping process (see, e.g., Ref. [30]) inducing bosonic thermalization effects on the qubit dynamics. It can be expressed as in Eqs. (10), and (11) by setting

$$L_1 = \sqrt{N+1}\,\sigma_-\ , \quad L_2 = \sqrt{N}\,\sigma_+\ , \tag{40}$$

with N being a non-negative number that gauges the mean thermal photon number of the system environment and with $\sigma_\pm = \frac{1}{2}(X \pm iY)$ being ladder operators.

In the absence of the driving term (i.e., $\omega = 0$ or equivalently $\kappa = 0$) the model can be easily integrated, the generator taking the matrix form

$$\mathcal{L} = \begin{pmatrix} -\gamma_2 & 0 & 0 & \gamma_1 \\ 0 & -\frac{1}{2}(\gamma_1+\gamma_2) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\gamma_1+\gamma_2) & 0 \\ \gamma_2 & 0 & 0 & -\gamma_1 \end{pmatrix}, \tag{41}$$

where for ease of notation γ_1 and γ_2 stand for $\gamma_1 = \gamma(N+1)$ and $\gamma_2 = \gamma N$, respectively. In this limit the process admits the density matrix

$$\bar{\rho}_A = \frac{1}{2N+1}\begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix} \tag{42}$$

as a unique stationary solution, which for $N > 0$ is always not pure. For this choice of the parameter we can hence invoke the ESD criterion to establish that the model must exhibit a finite value of the EST parameter. The negativity of entanglement can be computed as well, leading to

$$\mathcal{N}(\rho_{AB}^{(\Phi_t)}) = \frac{e^{-(2N+1)\gamma t/2}}{2}\max\{A_N(\gamma t) - \sinh((2N+1)\gamma t/2), 0\}, \tag{43}$$

where we have introduced the function

$$A_N(\tau) = \sqrt{\frac{1}{2} + \frac{4N(N+1) + \cosh((2N+1)\tau)}{2(2N+1)^2}}. \tag{44}$$

For $N = 0$ (purely lossy dynamics) the above expression reduces to $\mathcal{N}(\rho_{AB}^{(\Phi_t)}) = e^{-\gamma t}/2$ and the process never reaches the EB regime, yielding a divergent value of $\tau_{\rm ent}$, i.e.,

$$\mathcal{T}_{\rm ent}(0)\Big|_{N=0} = \infty\ . \tag{45}$$

For $N > 0$ instead, determining the zero of the r.h.s. term of Eq. (43) shows that the EST is finite and expressed as in

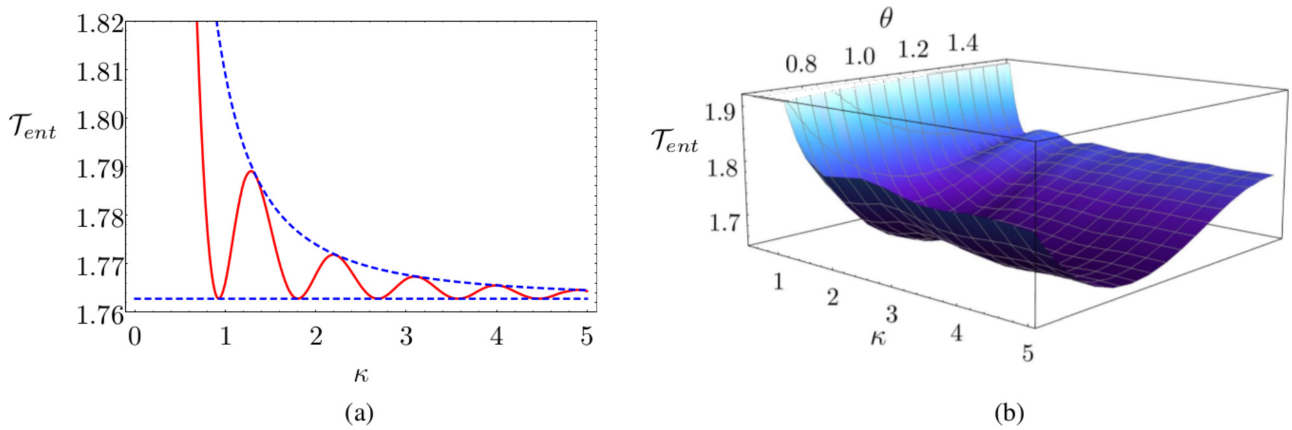


FIG. 2. (a) Plot of the rescaled EST functional $\mathcal{T}_{\rm ent}(\kappa)$ of the phase-flip channel (28) for $\theta = \pi/2$, $\varphi = 0$ as a function of the driving/damping ratio κ (red solid line), together with the bounds (39) (blue dashed lines). (b) Three-dimensional (3D) plot of $\mathcal{T}_{\rm ent}(\kappa)$ as a function of κ and of the rotation parameter θ. Notice that again $\mathcal{T}_{\rm ent}(\kappa)$ diverges for $\kappa \to 0$ and approaches a stationary value for $\kappa \to \infty$. The solid red line in (a) and the 3D plot in (b) have been generated numerically, exploiting the Newton-Raphson method.

Eq. (15) with

$$\mathcal{T}_{\rm ent}(0) = \frac{1}{2N+1}\operatorname{arcosh}\left(1 + \frac{(2N+1)^2}{2N(N+1)}\right). \tag{46}$$

Let us now allow for a nonzero ($\omega > 0$) driving term $H = \hat{n}\cdot\vec{\sigma}$. In analogy with the phase-flip process, if we set $\hat{n} = (0, 0, 1)$ the Hamiltonian part of $\mathcal{L}$ can be eliminated by passing into the interaction picture representation; therefore, the EST does not depend on ω. Also, exploiting the unitary invariance (19), the azimuthal angle φ can be set to zero without loss of generality, leaving us only with the dependence on θ to be resolved. In Fig. 3 we report the entanglement transmission curve for different values of the rotation parameter θ and the mean number of photons,N. We notice how once more the entanglement transmission time decreases with the driving/damping ratio κ. The qualitative behavior of the curves is similar to those observed for the phase-flip model. In particular we notice that at fixed κ, the values of $\mathcal{T}_{\rm ent}(\kappa)$ develop a nontrivial minimum for intermediate values of $\theta \in]0, \pi/2[$, the effect being more evident at large N.

C. The depolarizing process

The last example we consider is the depolarizing process generated by a GKSL generator with the three Lindblad operators

$$L_1 = X/2\,, \quad L_2 = Y/2\,, \quad L_3 = Z/2\,, \tag{47}$$

leading to a dissipator of the form

$$\mathcal{D}[\,\cdot\,] = \tfrac{1}{4}(X[\,\cdot\,]X + Y[\,\cdot\,]Y + Z[\,\cdot\,]Z - 3\,\mathrm{id}[\,\cdot\,]). \tag{48}$$

In this case due to the highly symmetric structure of Eq. (48) any Hamiltonian contribution can be eliminated by passing into the interaction picture without modifying the dissipator. More precisely, one can show that any purely Hamiltonian superoperator (i.e., one with $\gamma = 0$) commutes with Eq. 48). This can be exploited so as to get rid of any functional dependence of EST on ω, and ultimately on k as prescribed by Eq. (14):

$$\mathcal{T}_{\rm ent}(\kappa) = \mathcal{T}_{\rm ent}(0), \tag{49}$$

for all κ. Neglecting hence H, in the basis of the elementary matrices we observe that the generator becomes

$$\mathcal{L} = \frac{\gamma}{2}\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{50}$$

which, by direct exponentiation, leads to

$$\Phi_t = \frac{1}{2}\begin{pmatrix} 1+e^{-\gamma t} & 0 & 0 & 1-e^{-\gamma t} \\ 0 & 2e^{-\gamma t} & 0 & 0 \\ 0 & 0 & 2e^{-\gamma t} & 0 \\ 1-e^{-\gamma t} & 0 & 0 & 1+e^{-\gamma t} \end{pmatrix}. \tag{51}$$

Therefore, via a proper rearrangement of the above matrix elements (divided by 2), the Choi-Jamiołkowski state reads

$$\rho_{AB}^{(\Phi_t)} = \frac{1}{4}\begin{pmatrix} 1+e^{-\gamma t} & 0 & 0 & 2e^{-\gamma t} \\ 0 & 1-e^{-\gamma t} & 0 & 0 \\ 0 & 0 & 1-e^{-\gamma t} & 0 \\ 2e^{-\gamma t} & 0 & 0 & 1+e^{-\gamma t} \end{pmatrix}. \tag{52}$$

The negativity of entanglement can then be computed as

$$\mathcal{N}\big(\rho_{AB}^{(\Phi_t)}\big) = \frac{e^{-\gamma t/2}}{2}\max\{e^{-\gamma t/2} - \sinh(\gamma t/2), 0\}, \tag{53}$$

showing that the entanglement of the system is degraded, again, exponentially fast with rescaled EST value given by

$$\mathcal{T}_{\rm ent}(0) = \ln 3. \tag{54}$$

D. Gaussian Bosonic channels

In this section we address the case of dynamical semigroups acting on infinite-dimensional systems (continuous-variables regime). In particular we focus on the special class

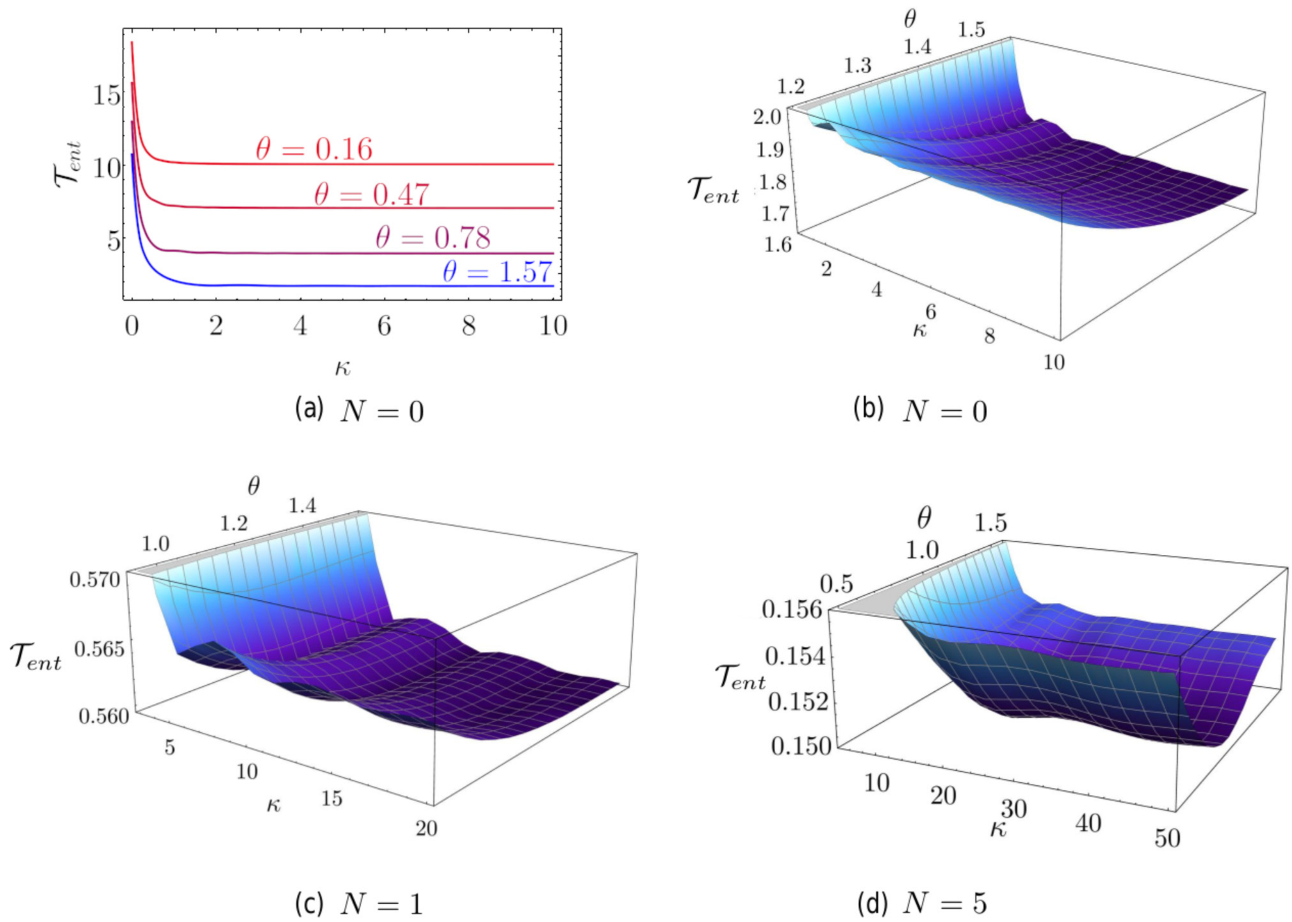


FIG. 3. Panel (a): Plot of the rescaled EST as a function of ratio κ for the amplitude damping process ($N = 0$), for several values of the parameter θ that determines the orientation of the driving Hamiltonian term. Panels (b), (c), and (d): Rescaled EST as a function of κ and of the rotation parameter θ associated to the generalized amplitude damping process, for different values of the environment mean of photons number N. The plots shown in all the panels have been generated numerically employing the Newton-Raphson method.

of CPt maps which belongs to the set of Gaussian Bosonic channels [13,31,32], that we briefly review in Appendix D. Specifically, we consider the continuous-variables analog of the generalized amplitude damping process introduced earlier. This process is described by a GKSL generator (10) with two Lindblad operators

$$L_1 = \sqrt{N+1}\, a\,, \quad L_2 = \sqrt{N}\, a^\dagger, \tag{55}$$

with $N \geqslant 0$ representing the mean photon number of the environment and a and $a^\dagger$ being, respectively, the annihilation and creation bosonic operators, fulfilling the canonical commutation rule $[a, a^\dagger] = 1$. For the Hamiltonian part we take instead the most general quadratic operator which, without loss of generality, we parametrize as

$$H = i\frac{(a^\dagger)^2 - a^2}{2} \sin\theta + a^\dagger a \,\cos\theta, \tag{56}$$

with θ measuring the relative intensity of the squeezing term.

Consider first the case where no driving contribution is present (i.e., $\omega = 0$). By explicit integration the associated CPt transformation Φ_t induced by $\mathcal{L}$ corresponds to a (single-mode) Gaussian Bosonic channel which in the formalism detailed in Appendix D is described by the 2×2 real matrices

$$F_t = e^{-\gamma t/2}\, I, \quad G_t = (2N+1)(1 - e^{-\gamma t})\, I\,. \tag{57}$$

This process belongs in the C class and we can determine its associated EST by finding solutions to the following equation:

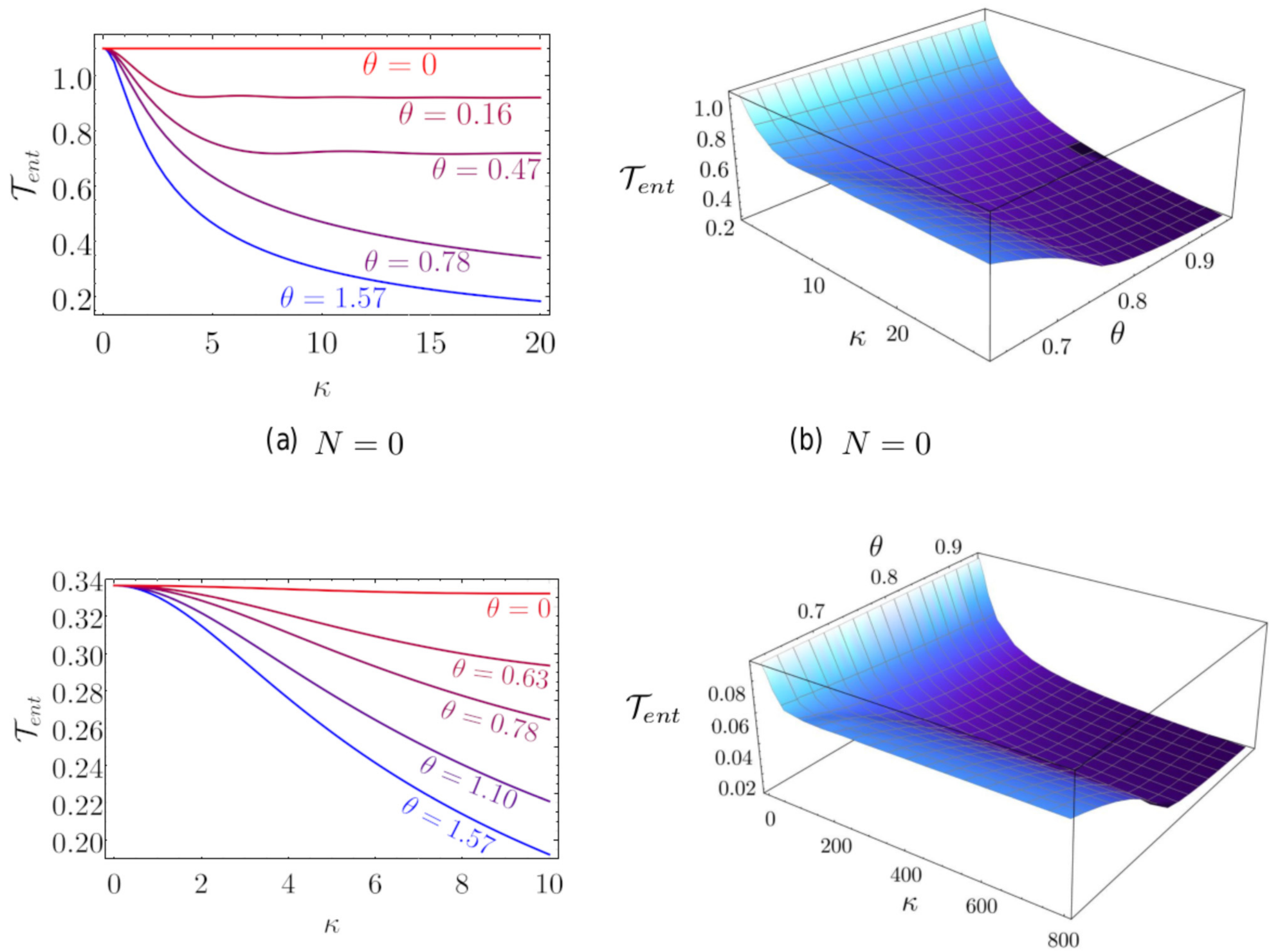
$$\det\left(G_t - \frac{i}{2}(J + F_t^T J F_t)\right) = 0 \tag{58}$$

[see Eq. (D13).] By explicit computation this yields the value for the rescaled functional of Eq. (15), i.e.,

$$\mathcal{T}_{\rm ent}(\kappa = 0) = \ln\frac{4N+3}{4N+1}, \tag{59}$$

which, while being decreasing with N as its qubit counterpart (46), at variance with the latter it does not diverge when N approaches zero [see Eq. (45)].

Consider next the case of a nonzero driving/damping ratio, $\kappa > 0$. For general θ, Eq. (58) yields for the EST an equation analogous to Eq. (35) which we report in Eq. (E11) of Appendix E and whose numerical solution is exhibited in Fig. 4.



(a) $N = 0$ (b) $N = 0$

(c) $N = 1$ (d) $N = 5$

FIG. 4. Panels (a) and (c): Rescaled EST $\mathcal{T}_{\text{ent}}$ as a function of the relative strength κ of dynamics associated to the Gaussian Bosonic Channel model defined by Eqs. (55) and (56), for different values of the parameter θ. Panels (b) and (d): 3D plot of the rescaled EST as a function of κ and of the rotation parameter θ. In all plots the values of $\mathcal{T}_{\text{ent}}$ have been obtained by numerically solving Eq. (E12) of Appendix E, with the Newton-Raphson method.

For $\kappa \simeq 0$ an approximate solution can be obtained in the following form:

$$\mathcal{T}_{\text{ent}}(\kappa) \simeq \ln \frac{4N+3}{4N+1} + \kappa^2(2N+1)(1-\cos 2\theta) \times \left[\frac{4}{(4N+3)(4N+1)} - \mathcal{T}_{\text{ent}}^2(0)\right]. \tag{60}$$

For large values of driving/damping ratio κ instead, Eq. (E11) presents a critical behavior in θ (see Fig. 5). In particular for $\theta \in [0, \pi/4)$ the form of $\mathcal{T}_{\text{ent}}$ is similar to the finite-dimensional case, exhibiting a drop-oscillate-stabilize pattern which can be approximated by the function

$$\mathcal{T}_{\text{ent}}(\kappa) \simeq \mathcal{T}_{\text{ent}}(\infty) + \frac{a\cos[2\kappa\sqrt{-\cos 2\theta}\,\mathcal{T}_{\text{ent}}(\infty)] + b - \cosh(\mathcal{T}_{\text{ent}}(\infty))}{\sinh(\mathcal{T}_{\text{ent}}(\infty))}, \tag{61}$$

with $\mathcal{T}_{\text{ent}}(\infty)$ being the asymptotic value defined as

$$\mathcal{T}_{\text{ent}}(\infty) = \text{arcosh}\frac{2(2N+1)^2 + (8N(N+1)+3)\cos 2\theta}{2(2N+1)^2 + (8N(N+1)+1)\cos 2\theta}, \tag{62}$$

vanishing for $N \to +\infty$. For $\theta \in (\pi/4, \pi/2]$, where Eq. (61) can have a nonzero imaginary part, instead the EST is monotonically decreasing with κ, asymptotically vanishing in the large-κ regime. Since for $\theta \in (\pi/4, \pi/2]$ Eq. (E11) changes form, in this interval the functional dependence is approximated by the function

$$\mathcal{T}_{\text{ent}}(\kappa) \simeq \frac{1}{2\kappa\sqrt{|\cos 2\theta|}}\text{arcosh}\,\frac{2\,\kappa^2\,\alpha(N,\theta)\,\cos 2\theta}{(2N+1)^2(1-\cos 2\theta)}, \tag{63}$$

where the dimensionful quantity α is given by

$$\alpha(N,\theta) = 2\gamma^2(2N+1)^2 + \gamma^2[8N(N+1)+3]\cos 2\theta. \tag{64}$$

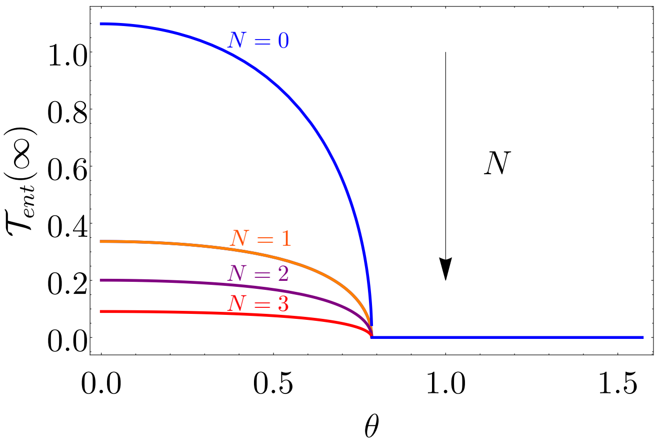


FIG. 5. Asymptotic form of the entanglement survival time [Eq. (62)] of the Gaussian Bosonic channel model defined by Eqs. (55) and (56) as a function of θ for $\kappa \to \infty$.

V. CONCLUSIONS

The present paper focuses on the study of the entanglement transmission time, defined as the time at which a dynamical process induced by the interaction with an external environment becomes entanglement breaking. For the special case of time-homogeneous Markovian systems, we analyze how this quantity is affected by the interplay between the dissipative and the driving contributions of the GKSL generator of the model. We provided both analytical and numerical results for some relevant examples of qubit evolution, described by the bit-flip and the amplitude damping channels. In the simplest cases we evaluate also the negativity of entanglement, which quantifies the entanglement content of the semigroup output state, and therefore provide information also on the rate at which entanglement is being corrupted. We noticed that the dependency of the entanglement transmission time from the damping and driving parameters reflects the form of the eigenvalues of $\mathcal{L}$, the GKSL generator of the quantum dynamical semigroup. The precise form of such dependency can be very complicated even in the simple cases considered, but generally it has been found that oscillations can appear in the entanglement transmission time. This happens in the finite-dimensional case when the eigenvalues of the generator acquire an imaginary part. In the infinite-dimensional case, we observe an oscillatory behavior only for certain values of the rotation parameter.

Somewhat contrary to common intuition, our results clearly show that increasing the driving parameter, by tuning the weight of the unitary dynamics, does not always provide an advantage in the transmission of entanglement. Indeed, in the study cases considered it appears to be detrimental, making the transmission time drop, with the exception of a special driving direction, which makes the driving ineffective. An intuitive explanation of this effect can be attempted by saying that the unitary rotations induced by the presence of coherence-preserving contributions in the GKSL generator could effectively increase the detrimental effects of the dissipative ones, by broadening the range of their action in the phase space of the system. In other words, by exposing the Hilbert space of the latter to attacks that can affect any possible subspaces, these rotations boost the noise level, inducing a "playing both sides of the fence" effect where the system has no hidden places to store the coherence it needs to maintain the entanglement with an eternal ancilla.

ACKNOWLEDGMENTS

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APPENDIX A: FORMAL INTEGRATION OF THE BIT-FLIP CHANNEL MODEL

Setting $\omega = 0$ in the operator basis $\{E^{(00)}, E^{(10)}, E^{(01)}, E^{(11)}\}$ formed by the external products $E^{(ij)} = |i\rangle\langle j|$ of the computational basis, the Lindblad superoperator of the phase=flip channel model takes the matrix form

$$\mathcal{L} = \gamma \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A1}$$

which gives

$$\Phi_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\gamma t} & 0 & 0 \\ 0 & 0 & e^{-\gamma t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A2}$$

as the associated semigroup maps [Eq. (12)]. Adopting hence as the maximally entangled state Eq. (21) the one constructed on the computational basis, i.e., $|\Omega\rangle_{AB}\langle\Omega| = \sum_{j,j'=0,1} E_A^{(jj)} \otimes E_B^{(j'j')}/4$, which for $d = 2$ has the matrix form

$$|\Omega\rangle_{AB}\langle\Omega| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{A3}$$

the partial transpose of the corresponding Choi-Jamiołkowski state can likewise be expressed as

$$\left[\rho_{AB}^{(\Phi_t)}\right]^{T_B} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-\gamma t} & 0 \\ 0 & e^{-\gamma t} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A4}$$

having eigenvalues $1/2$ (twice degenerate) and $\pm e^{-\gamma t}/2$ which leads to Eq. (30) when replaced into Eq. (24).

APPENDIX B: ENTANGLEMENT NEGATIVITY

In this Appendix we provide some details for the derivation of negativity in the models described by the generator (28) with $\hat{n} = (1, 0, 0)$ and the generator (40) with $\omega = 0$.

In the basis $\{E^{(00)}, E^{(10)}, E^{(01)}, E^{(11)}\}$ the Lindbladian (28) is represented by the matrix (32). By means of Eq. (19), we can transform Eq. (28) into the equivalent generator

$$\mathcal{L} = \gamma \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} + 2i\kappa & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} - 2i\kappa & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{B1}$$

which makes our analysis simpler. By taking the exponential, we have

$$\Phi_t = e^{-\frac{1}{2}\gamma t} \begin{pmatrix} \cosh \frac{1}{2}\gamma t & 0 & 0 & \sinh \frac{1}{2}\gamma t \\ 0 & \cosh \frac{1}{2}\gamma t\sqrt{1-16\kappa^2} + \frac{4i\kappa \sinh \frac{1}{2}\gamma t\sqrt{1-16\kappa^2}}{\sqrt{1-16\kappa^2}} & \frac{\sinh \frac{1}{2}\gamma t\sqrt{1-16\kappa^2}}{\sqrt{1-16\kappa^2}} & 0 \\ 0 & \frac{\sinh \frac{1}{2}\gamma t\sqrt{1-16\kappa^2}}{\sqrt{1-16\kappa^2}} & \cosh \frac{1}{2}\gamma t\sqrt{1-16\kappa^2} - \frac{4i\kappa \sinh \frac{1}{2}\gamma t\sqrt{1-16\kappa^2}}{\sqrt{1-16\kappa^2}} & 0 \\ \sinh \frac{1}{2}\gamma t & 0 & 0 & \cosh \frac{1}{2}\gamma t \end{pmatrix}. \tag{B2}$$

Hence, by taking the partial transpose of the associated Choi-Jamiołkowski state, we can find the eigenvalues as functions of time:

$$\lambda(t) = \begin{cases} \frac{1}{2e^{-\gamma t/2}(\sinh(\gamma t/2) \pm Q_\kappa(\gamma t/2))} \\ \frac{1}{2}e^{-\gamma t/2}(\cosh(\gamma t/2) \pm S_\kappa(\gamma t/2)), \end{cases} \tag{B3}$$

where in order to simplify the notation we have defined the functions

$$Q_\kappa(\tau) = \sqrt{\frac{\cosh^2(\tau\sqrt{1-16\kappa^2}) - 16\kappa^2}{1-16\kappa^2}}, \tag{B4}$$

$$S_\kappa(\tau) = \frac{\sinh(\tau\sqrt{1-16\kappa^2}))}{\sqrt{1-16\kappa^2}}. \tag{B5}$$

By summing up the negative part of the eigenvalues, formula (24) for the negativity of the bit-flip channel model follows. In the same way, the Lindbladian (40) is represented by the matrix (41). By taking the exponential of matrix (41), we have

$$(2N+1)\Phi_t = \begin{pmatrix} Ne^{-(2N+1)\gamma t} + N + 1 & 0 & 0 & (N+1)(1 - e^{-(2N+1)\gamma t}) \\ 0 & (2N+1)e^{-\frac{1}{2}(2N+1)\gamma t} & 0 & 0 \\ 0 & 0 & (2N+1)e^{-\frac{1}{2}(2N+1)\gamma t} & 0 \\ N(1 - e^{-(2N+1)\gamma t}) & 0 & 0 & (N+1)e^{-(2N+1)\gamma t} + N \end{pmatrix}, \tag{B6}$$

and therefore the associated Choi-Jamiołkowski state reads

$$(4N+2)\rho_{AB}^{(\Phi_t)} = \begin{pmatrix} Ne^{-(2N+1)\gamma t} + N + 1 & 0 & 0 & (2N+1)e^{-\frac{1}{2}(2N+1)\gamma t} \\ 0 & N(1 - e^{-(2N+1)\gamma t}) & 0 & 0 \\ 0 & 0 & (N+1)(1 - e^{-(2N+1)\gamma t}) & 0 \\ (2N+1)e^{-\frac{1}{2}(2N+1)\gamma t} & 0 & 0 & (N+1)e^{-(2N+1)\gamma t} + N \end{pmatrix}. \tag{B7}$$

The eigenvalue equation for the partial transpose of the Choi-Jamiołkowski state yields

$$\lambda(t) = \begin{cases} ((N+1)e^{-(2N+1)\gamma t} + N)/(2N+1) \\ (N+1+Ne^{-(2N+1)\gamma t})/(2N+1) \\ \frac{1}{2}e^{-(2N+1)\gamma t/2}(\sinh((2N+1)\gamma t/2) \pm A_N(\gamma t)), \end{cases} \tag{B8}$$

where the function $A_N(\tau)$ was defined in Eq. (44). Summing up the negative parts of the eigenvalues, formula (43) for the negativity of the generalized amplitude channel model follows.

APPENDIX C: PERTURBATIVE EXPANSION OF EST

Let us rewrite Eq. (35) in terms of adimensional variables

$$\cosh(\tau) = 2 + \frac{\cosh(\tau\sqrt{1-16\kappa^2}) - 16\kappa^2}{1-16\kappa^2}, \tag{C1}$$

where $\tau = \gamma t$ and $\kappa = \omega/\gamma$. We now solve this equation in three different regimes.

By expanding the equation in powers of κ and keeping only the first nontrivial term, we have the equation

$$\tau \sinh(\tau) + 2(1 - \cosh(\tau)) \simeq \frac{1}{4\kappa^2}. \tag{C2}$$

Furthermore, we know that for small values of κ, $\tau_{\rm ent}$ diverges because the process is asymptotically EB. Therefore, by expanding the equation for large τ's, we find

$$\tau e^{\tau} \simeq \frac{1}{2\kappa^2}, \ \Rightarrow \ \mathcal{T}_{\rm ent} \simeq W\left(\frac{1}{2\kappa^2}\right). \tag{C3}$$

We can now find corrections perturbatively. Let us introduce the perturbative parameter ϵ and consider the deformed equation

$$\begin{aligned}&\epsilon\cosh(\tau) + (1-\epsilon)\tau e^{\tau}\\ &\quad = 2\epsilon + \epsilon\frac{\cosh(\tau\sqrt{1-16\kappa^2}) - 16\kappa^2}{1-16\kappa^2} + \frac{(1-\epsilon)}{2\kappa^2}.\end{aligned} \tag{C4}$$

This equation interpolates between the known asymptotic value of $\mathcal{T}_{\rm ent}$ at $\epsilon = 0$ and the unknown value of $\mathcal{T}_{\rm ent}$ at $\epsilon = 1$. We therefore look for solutions to the above equation in the form

$$\tau(\epsilon) = \sum_{n=0}^{\infty} \tau_n(\kappa)\epsilon^n. \tag{C5}$$

If the series converges, we have $\mathcal{T}_{\rm ent}(\kappa) = \sum_{n=0}^{\infty} \tau_n(\kappa)$. By expanding Eq. (C4) in powers of ϵ and applying Eq. (C5) to it, we can recursively determine the coefficients $\tau_n(\kappa)$ by imposing equality order-by-order in ϵ. The first correction to Eq. (C3) turns out to be

$$\begin{aligned}\tau_1(\kappa) &= \frac{(1-2\kappa^2 W(1/2\kappa^2))^2 - 4\kappa^2 W(1/2\kappa^2)}{2(1+W(1/2\kappa^2))}\\ &\quad + \frac{1 + (2\kappa^2 W(1/2\kappa^2))^{1-16\kappa^2}}{2\sqrt{1-16\kappa^2}(2\kappa^2 W(1/2\kappa^2))^{\sqrt{1-16\kappa^2}-1}}.\end{aligned} \tag{C6}$$

At the critical ratio $\tau = 1/4$, Eq. (C1) simplifies considerably, leaving us with

$$\cosh(\tau) - 1 - \frac{\tau^2}{2} = 2 \ \Rightarrow \ \mathcal{T}_{\rm ent} \simeq 2.5. \tag{C7}$$

We can expand around this value by looking for solutions of the form

$$\mathcal{T}_{\rm ent}(\kappa) = \sum_{n=0}^{\infty} \tau_n \left(\kappa - \frac{1}{4}\right)^n. \tag{C8}$$

We can determine the coefficients τ_n recursively by plugging the above expansion into Eq. (C1) and expanding it in power of $(\tau - 1/4)$. The first few coefficients of the expansion read

$$\tau_0 \simeq 2.5, \tag{C9}$$

$$\tau_1 = -\frac{\tau_0^4}{3(\sinh(\tau_0) - \tau_0)} \simeq -3.7, \tag{C10}$$

$$\begin{aligned}\tau_2 &= \frac{\tau_0^4\left(4\tau_0^2\cosh^2(\tau_0) - 5\tau_0^4\cosh(\tau_0)\right)}{90(\sinh(\tau_0)-\tau_0)^3}\\ &\quad + \frac{4\tau_0^4\left(\left(\tau_0^2 - 15\right)\sinh^2(\tau_0) + 6\tau_0\left(\tau_0^2+5\right)\sinh(\tau_0)\right)}{90(\sinh(\tau_0)-\tau_0)^3}\\ &\quad + \frac{\tau_0^2\left(27\tau_0^2 + 64\right)}{90(\sinh(\tau_0)-\tau_0)^3} \simeq 10.6\,.\end{aligned} \tag{C11}$$

By taking the limit $\kappa \to +\infty$ of Eq. (C1), we have the simple solution

$$\lim_{\kappa\to\infty} \mathcal{T}_{\rm ent}(\kappa) = \operatorname{arcosh}(3) := \mathcal{T}_{\rm ent}(\infty). \tag{C12}$$

One might therefore think of looking for corrections by expanding $\mathcal{T}_{\rm ent}$ as a Laurent series: $\mathcal{T}_{\rm ent}(\kappa) = \sum_{n=0}^{\infty} \frac{\tau_{2n}}{\kappa^{2n}}$. However, when κ is continued to the complex plane, Eq. (C1) has an essential singularity at $\kappa = \infty$. As a result, an expansion of the form above cannot be found. Instead, we can obtain a perturbative series about infinity by introducing a perturbative parameter ϵ, deforming Eq. (C1) into

$$\cosh(\tau) = 2 + \epsilon\frac{\cosh(\tau\sqrt{1-4\kappa^2}) - 4\kappa^2}{1-4\kappa^2} + 1 - \epsilon. \tag{C13}$$

We can thus perform a perturbative expansion similar to the $\kappa \simeq 0$ regime:

$$\tau(\epsilon) = \sum_{n=0}^{\infty} \tau_n(\kappa)\epsilon^n, \tag{C14}$$

the first few coefficients of the expansion being

$$\tau_0(\kappa) = \mathcal{T}_{\rm ent}(\infty), \tag{C15}$$

$$\tau_1(\kappa) = \frac{\cosh(\mathcal{T}_{\rm ent}(\infty)\sqrt{1-16\kappa^2}) - 1}{2\sqrt{2}(1-16\kappa^2)}. \tag{C16}$$

APPENDIX D: INTRODUCTION TO GAUSSIAN BOSONIC CHANNELS

A formal definition of Gaussian Bosonic channels can be obtained by passing into the Heisenberg representation [14] and assigning their action on the Weyl operators of the system [33]. We remind that assuming the system of interest is composed of n independent modes described by canonical coordinates $\{Q_j, P_j\}_{j=1,\dots,n}$ fulfilling the the canonical commutation relations

$$[Q_i, P_j] = i\,\delta_{ij}, \quad [Q_i, Q_j] = [P_i, P_j] = 0, \tag{D1}$$

a generic Weyl operator is defined as the unitary transformation $W_\xi = e^{i\xi\cdot R}$ with $\xi \in \mathbb{R}^{2n}$ and $R = (Q_1, P_1, \dots, Q_n, P_n)$. A zero-mean Gaussian channel Φ can then be uniquely identified by two $2n \times 2n$ real matrices F and G that, in the Heisenberg representation, define the mapping

$$W_\xi \ \longrightarrow \ W_{F\xi}\, e^{-\frac{i}{2}\xi^T G \xi}, \quad \forall \xi\ . \tag{D2}$$

The CPt condition imposes on F, G the inequality

$$G \geqslant \frac{i}{2}(J - F^T J F), \tag{D3}$$

where J is the standard symplectic metric of the system, i.e.,

$$J = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{D4}$$

It can be proven [34] that a Gaussian channel (F, G) is EB if and only if it admits a decomposition of the form $G = \mu + \nu$, and such that

$$\nu \geqslant \frac{i}{2}J, \quad \mu \geqslant \frac{i}{2}F^T J F. \tag{D5}$$

Therefore, a necessary condition for (F, G) to be EB is

$$G \geqslant \frac{i}{2}(J + F^T JF). \tag{D6}$$

Let us now restrict our attention to one-mode Gaussian channels (i.e., $n = 1$). For such channels, a complete characterization can be given, based upon the Williamson theorem [31,33]. As it turns out, depending on the value of the quantity $F^T JF$, there exists canonical unitary transformations U_1, U_2 such that, via the mapping

$$\Phi(\rho) \longrightarrow U_2\, \Phi(U_1\, \rho\, U_1^\dagger)\, U_2^\dagger, \tag{D7}$$

the Gaussian channel $\Phi \simeq (F, G)$ can be reduced to one of the following normal forms:

(A) $F^T JF = 0$. Then (F, G) can be reduced to the form

$$F = k\, E^{(00)}, \quad G = \left(q + \frac{1}{2}\right) I. \tag{D8}$$

(B_1) $F^T JF = J$. Then (F, G) can be reduced to the form

$$F = I, \quad G = \tfrac{1}{2}\, E^{(11)}. \tag{D9}$$

(B_2) $F^T JF = J$. Then (F, G) can be reduced to the form

$$F = I, \quad G = q\, I. \tag{D10}$$

(C) $F^T JF = k^2 J$, $k > 0$, $k \neq 1$. Then (F, G) can be reduced to the form

$$F = k\, I, \quad G = \left(q + \frac{|1 - k^2|}{2}\right) I. \tag{D11}$$

(D) $F^T JF = -k^2 J$, $k > 0$. Then (F, G) can be reduced to the form

$$F = k\, Z, \quad G = \left(q + \frac{1 + k^2}{2}\right) I, \tag{D12}$$

where k is a real number and $q \geqslant 0$. Combining the above result with Eq. (D5), we have the following EB conditions for one-mode Gaussian channels [34]:

(A) Φ is EB (in fact, it is c-q).
(B_1) Φ is not EB.
(B_2) Φ is EB if and only if $q \geqslant 1$.
(C) Φ is EB if and only if $q \geqslant \min\{1, k^2\}$.
(D) Φ is EB.

Notice that the only channels which could either be EB or non-EB, depending on k and q, are the ones in classes B_2 and C, and that for them Eq. (D6) is in fact equivalent to the EB conditions (D5). Furthermore, by continuity, for these maps the entanglement transmission time can be determined studying the zeros of the following equation:

$$\det\left(G - \frac{i}{2}(J + F^T JF)\right) = 0. \tag{D13}$$

APPENDIX E: EST FOR THE GAUSSIAN AMPLITUDE DAMPING CHANNEL

For the model described by the GKSL generator in Eqs. (55) and (56), the matrices F_t and G_t can be expressed as

$$F_t = \exp\left[\gamma t \begin{pmatrix} -\frac{1}{2}N + \kappa \sin\theta & -\kappa\cos\theta \\ \kappa\cos\theta & -\frac{1}{2}N - \kappa\sin\theta \end{pmatrix}\right], \tag{E1}$$

$$G_t = (2N + 1)\gamma \int_0^t \mathrm{ds}\, \mathrm{F_s^T F_s}. \tag{E2}$$

More explicitly, the matrix elements of F_t are

$$F_t|_{11} = \frac{i}{2} e^{-\frac{1}{2}N\tau}\left[\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}} - i\right) e^{-i\kappa\tau\sqrt{\cos 2\theta}} - \left(\frac{\sin\theta}{\sqrt{\cos 2\theta}} + i\right) e^{i\kappa\tau\sqrt{\cos 2\theta}}\right], \tag{E3}$$

$$F_t|_{21} = \frac{\cos\theta}{\sqrt{\cos 2\theta}} e^{-\frac{1}{2}N\tau} \sin(\kappa\tau\sqrt{\cos 2\theta}), \tag{E4}$$

$$F_t|_{12} = -F_t|_{21}, \tag{E5}$$

$$F_t|_{22} = -\frac{i}{2} e^{-\frac{1}{2}N\tau}\left[\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}} + i\right) e^{-i\kappa\tau\sqrt{\cos 2\theta}} - \left(\frac{\sin\theta}{\sqrt{\cos 2\theta}} - i\right) e^{i\kappa\tau\sqrt{\cos 2\theta}}\right], \tag{E6}$$

where $\tau = \gamma t$, while for the matrix G_t we have

$$\begin{aligned} G_t|_{11} = &-\frac{(2N+1)\sec 2\theta}{1 + 4\kappa^2\cos 2\theta}\Big[e^{-\tau}\kappa^2 - \cos 2\theta(1 + 2\kappa^2 \\ &+ 2\kappa(\kappa\cos 2\theta + \sin\theta)) + \frac{1}{2}e^{-\tau}(2\kappa^2\cos 4\theta \\ &+ \cos 2\theta(1 + 4\kappa^2 + \cos(2\kappa\tau\sqrt{\cos 2\theta}))(1 + 4\kappa\sin\theta) \\ &+ 2\sqrt{\cos 2\theta}\sin\theta(2\kappa\sin\theta + 1)\sin(2\kappa\tau\sqrt{\cos 2\theta})) \\ &+ 2\sin^2(\kappa\tau\sqrt{\cos 2\theta})\Big], \end{aligned} \tag{E7}$$

$$\begin{aligned} G_t|_{21} = &\frac{(2N+1)\tan 2\theta}{1 + 4\kappa^2\cos 2\theta} e^{-\tau}[2(1 - e^\tau)\kappa^2\cos 2\theta \\ &+ \kappa\sqrt{\cos 2\theta}\sin(\kappa\tau\sqrt{\cos 2\theta}) + \sin^2(\kappa\tau\sqrt{\cos 2\theta})], \end{aligned} \tag{E8}$$

$$G_t|_{12} = G_t|_{21}, \tag{E9}$$

$$\begin{aligned} G_t|_{22} = &-\frac{(2N+1)\sec 2\theta}{1 + 4\kappa^2\cos 2\theta}\Big[e^{-\tau}\kappa^2 - \cos 2\theta(1 + 2\kappa^2 \\ &+ 2\kappa(\kappa\cos 2\theta - \sin\theta)) + \frac{1}{2}e^{-\tau}(2\kappa^2\cos 4\theta \\ &+ \cos 2\theta(1 + 4\kappa^2 + \cos(2\kappa\tau\sqrt{\cos 2\theta}))(1 - 4\kappa\sin\theta) \\ &+ 2\sqrt{\cos 2\theta}\sin\theta(2\kappa\sin\theta - 1)\sin(2\kappa\tau\sqrt{\cos 2\theta})) \\ &+ 2\sin^2(\kappa\tau\sqrt{\cos 2\theta})\Big]. \end{aligned} \tag{E10}$$

Let us now set $\gamma_1 = \gamma(N+1)$ and $\gamma_2 = \gamma N$ for the sake of convenience. Equation (58) gives the following equation for the EST:

$$\cosh(\tau) - a(\gamma_1,\gamma_2,\kappa,\theta)\cos(2\tau\kappa\sqrt{\cos 2\theta}) = b(\gamma_1,\gamma_2,\kappa,\theta), \tag{E11}$$

where

$$a = \frac{2(\gamma_1+\gamma_2)^2(1-\sec 2\theta)}{(3\gamma_1+\gamma_2)(\gamma_1+3\gamma_2)+4\kappa^2\beta}, \tag{E12}$$

$$b = \frac{2(\gamma_1+\gamma_2)^2\sec 2\theta + \left(3\gamma_1^2+2\gamma_1\gamma_2+3\gamma_2^2\right)+4\kappa^2\alpha}{(3\gamma_1+\gamma_2)(\gamma_1+3\gamma_2)+4\kappa^2\beta}, \tag{E13}$$

$$\alpha = \left(\gamma_1^2+\gamma_2^2\right)(2+3\cos 2\theta)+2\gamma_1\gamma_2(2+\cos 2\theta), \tag{E14}$$

$$\beta = \left(\gamma_1^2+\gamma_2^2\right)(2+\cos 2\theta)+2\gamma_1\gamma_2(2+3\cos 2\theta). \tag{E15}$$

The equation presents a critical behavior at $\theta = \pi/4$, and becomes

$$\cosh(\tau) = \frac{(\gamma_1-\gamma_2)^2+4(\gamma_1+\gamma_2)^2+4\kappa^2(\gamma_1+\gamma_2)^2(2+\tau^2)}{(3\gamma_1+\gamma_2)(\gamma_1+3\gamma_2)+8\kappa^2(\gamma_1+\gamma_2)^2}, \tag{E16}$$

while for $\kappa = 0$ it yields the purely dissipative EST value

$$\mathcal{T}_{\text{ent}} = \ln\frac{3\gamma_1+\gamma_1}{\gamma_1+3\gamma_2}, \tag{E17}$$

which we have reported as Eq. (59) of the main text.

For small values of κ we look for solutions of the form

$$\mathcal{T}_{\text{ent}}(\kappa) = \sum_{n=0}^{\infty}\tau_n\kappa^n. \tag{E18}$$

The first corrections are

$$\tau_0 = \ln\frac{3\gamma_1+\gamma_2}{\gamma_1+3\gamma_2}, \quad \tau_1 = 0, \tag{E19}$$

$$\tau_2 = \frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}\left[\frac{4(\gamma_1-\gamma_2)^2}{(3\gamma_1+\gamma_2)(\gamma_1+3\gamma_2)} - \ln^2\frac{3\gamma_1+\gamma_2}{\gamma_1+3\gamma_2}\right] \times (1-\cos 2\theta), \tag{E20}$$

and this is valid for all forms of the equation.

Let us now consider the behavior of the equation for large κ. When $0 \leqslant \theta < \pi/4$, then by taking the limit we have

$$\cosh(\tau) = \lim_{\kappa\to\infty} b(\gamma_1,\gamma_2,\kappa,\theta) \tag{E21}$$

and therefore

$$\lim_{\kappa\to+\infty}\mathcal{T}_{\text{ent}}(\kappa) = \operatorname{arcosh}\frac{\alpha(\gamma_1,\gamma_2,\theta)}{\beta(\gamma_1,\gamma_2,\theta)}. \tag{E22}$$

Using the same deformation procedure employed for Eq. (C13) we can find the first correction

$$\frac{a\cos(2\kappa\sqrt{-\cos 2\theta}\,\tau_0)+b-b_\infty}{\sqrt{b_\infty^2-1}}, \tag{E23}$$

where $b_\infty = \lim_{\kappa\to\infty} b(\kappa)$, yielding Eq. (61) of the main text. Instead for $\theta \in (\pi/4, \pi/2)$, we have asymptotically for $\kappa \to +\infty$

$$\mathcal{T}_{\text{ent}} \simeq \frac{1}{2\kappa\sqrt{-\cos 2\theta}}\operatorname{arcosh}\frac{2\kappa^2\alpha(\gamma_1,\gamma_2,\theta)}{(\gamma_1+\gamma_2)^2(\sec 2\theta-1)}, \tag{E24}$$

which coincides with Eq. (63) of the main text.

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