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Epistemic Logics for Relevant Reasoners

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Abstract
We present a neighbourhood-style semantic framework for modal epistemic logic modelling agents who process information using relevant logic. The distinguishing feature of the framework in comparison to relevant modal logic is that the environment the agent is situated in is assumed to be a classical possible world. This framework generates two-layered logics combining classical logic on the propositional level with relevant logic in the scope of modal operators. Our main technical result is a general soundness and completeness theorem.

Keywords Epistemic logic · Relevant logic · Neighbourhood semantics

1 Introduction

The paper [49] introduces a semantic framework for modal logics that combines classical propositional logic with relevant modal logic. The framework generates logics whose propositional fragment is classical and the modal monotonicity rule

$$\varphi \to \psi \Rrightarrow \Box\varphi \to \Box\psi$$

is restricted: it holds only for formulas $\varphi \to \psi$ provable in a given relevant modal logic, not for all $\varphi \to \psi$ provable in the "combined" logic itself. In a sense, relevant logic is confined to the scope of modal operators. The framework models agents *attentive to relevance* when deriving consequences from the information at their disposal without dropping the assumption of classical epistemic logic that the environments these agents

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inhabit are classical possible worlds. This sort of combined epistemic logic goes back to Levesque's [32] combination of classical logic with First Degree Entailment.[1]

Although it is of independent technical interest to study how classical and non-classical logics interact in a uniform framework, it is especially interesting from an epistemological point of view to consider the specific non-classical case of *relevant logic*.

Relevant logics arguably provide a formalisation of epistemic reasoning (cf. [11, 13])[2] and the informational interpretation of Routley-Meyer models aptly describes the complexity of inferential patterns which occur in epistemic contexts (cf. [35, 42, 58]). Moreover, the strengthening of relevant models allows one to characterise, via frame correspondences, increasingly stronger epistemic closure principles (see Fig. 1), thereby obtaining an extremely flexible framework which suitable for many applications.

In a sense, the logics of [49] embody a version of Harman's *clutter avoidance principle* (cf. [29]) according to which agents should not clutter their mind with *irrelevant* consequences of information they posses – if "irrelevant" is construed as "not following by relevant logic".[3] Two characteristic principles of the combined epistemic logics CL based on a relevant logic L of [49] are restricted monotonicity and conjunctive regularity:

$$(\Box\mathsf{C}) \quad \vdash_{\mathsf{CL}} (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$$
$$(\Box\mathsf{M}_{\mathsf{L}}) \quad \vdash_{\mathsf{L}} \varphi \to \psi \implies \vdash_{\mathsf{CL}} \Box\varphi \to \Box\psi$$

Although $(\Box\mathsf{C})$ and $(\Box\mathsf{M}_{\mathsf{L}})$ are not directly at odds with the clutter avoidance principle, there are independent reasons to study relevant epistemic logics without these principles.

Conjunctive regularity $(\Box\mathsf{C})$ is based on the assumption, embodied in the standard Kripke-style relational semantics for $\Box$, that the epistemic state of an agent consists of a single set of states representing all the information available to the agent, *aggregated* into one piece of information. However, there is ample evidence in the philosophy of mind and formal epistemology literature that such a picture of epistemic states is too coarse-grained since, for example, the "information utilizable by cognitive processes is stored in distinct, independently accessible data structures" [7] (p. 80), and since agents reason using distinct "frames of mind" [18].

[1] In what follows, we use the term "epistemic" as broadly encompassing knowledge, belief, evidence etc.

[2] Bílková et al. [11, 13] formulate relevant epistemic logics capturing the idea that the process of forming beliefs based on various pieces of evidence is shaped by relevant logic. They use an existential relational modality, which implies that the associated modality is monotonic. The propositional fragments of their logics are relevant, not classical.

[3] The intuitive link is provided by the variable sharing property of relevant logics, according to which $\varphi \to \psi$ is provable in a relevant logic only if φ and ψ share at least one propositional variable. Some cluttering is present in this framework as well, as witnessed by relevantly provable principles such as $\varphi \to (\varphi \vee \psi)$. It is an interesting topic for future research to develop a version of the framework of [49] where relevant logic is replaced by Parry-style containment logics (cf. [23, 25, 40]) where $\varphi \to \psi$ is provable only if all propositional variables occurring in ψ occur in φ.

Failures of the monotonicity rule, and even its logic-restricted versions such as ($\Box M_L$), are motivated by considerations of justification-based belief (cf. [2]). One may believe φ based on some justification s without having any justification at one's disposal for some ψ entailed by φ. Combinations of relevant and justification logic were explored in e.g. [45, 56].

The above considerations motivate the generalization of the framework of [49] from relational models to neighborhood models of relevant epistemic logic.

The distinctive feature of neighborhood semantics (cf. [14, 39]) for epistemic logics is that an epistemic state is a set of *sets of states*, representing various independent pieces of information available to the agent. The fact that the epistemic state is not necessarily closed under intersections – which leads to failures of ($\Box C$) – means that these pieces of information are not necessarily aggregated by the agent.[4]

The fact that the epistemic state is not necessarily closed under supersets – which leads to the failure of ($\Box M_L$) – implies that epistemic attitudes are not closed under logical entailment.[5]

In the present paper, we develop a framework which combines reasons for studying relevant epistemic logic with reasons for studying modal logics lacking ($\Box C$) and ($\Box M_L$). For the intuitive reasons sketched above, as well as for the sake of generality, we do not assume any specific properties of neighborhoods. The characteristic modal principle of the logics generated by the present framework is the restricted equivalence principle

$$(\Box E_L) \quad \vdash_L \varphi \leftrightarrow \psi \implies \vdash_{CL} \Box\varphi \leftrightarrow \Box\psi.$$

Hence, we obtain *hyperintensional*[6] epistemic logics where objects of epistemic attitudes are *relevant propositions* – sets of states representing pieces of information that are more fine-grained than classical propositions (sets of possible worlds). As usual in neighborhood semantics, assuming specific properties of neighborhoods – such as monotonicity or closure under intersections – leads to stronger logics, and the relational framework of [49] emerges as a special case. Our main technical result is a modular soundness and completeness theorem encompassing a wide family of logics combining relevant and classical modal logic.

[4] A well-known application of neighborhood semantics avoiding closure under intersections is evidence logic (cf. [8, 9]), modelling agents that form their beliefs based on evidence coming from a variety of sources. Neighborhood structures have been fruitfully applied to the logic of linguistic, deontic and metaphysical notions (cf. [14, 20, 34]). For recent applications in formal epistemology, see [28], where a neighborhood semantics is provided for the conditional doxastic logic of [3]. Moreover, topological models (building on the class of S4-neighborhood frames) are used in [4–6] to provide models of several epistemic notions and their dynamics, such as infallible and defeasible knowledge, full and weak belief.

[5] Standefer [54] puts forward a (non-monotonic) relevant modal logic for tracking the reasons supporting the agents' evidence. The logic of "only knowing" of [33] (although in a relational setting) and the conditional logic of [14, ch.10] (although based on classical logic) are other examples of frameworks employing a non-monotonic modality.

[6] We take a logic to be hyperintensional iff ($\Box E$) $\varphi \leftrightarrow \psi \Rrightarrow \Box\varphi \leftrightarrow \Box\psi$ does not preserve theoremhood in the logic. We refer the reader to [48] for a general framework for hyperintensional modal logics and an overview of the topic, and to [38, 55] for recent approaches to hyperintensionality in the context of non-classical logics.

Outline

The rest of the article is structured as follows. Section 2 recalls the neighborhood semantics for relevant modal logics. Section 3 introduces our original extension of relevant neighborhood semantics, based on so-called W-models of [49]. The crucial feature of W-models is that they contain a set of possible worlds – states satisfying specific frame properties that enforce classical behavior of implication and negation – and that validity is defined as satisfaction in all possible worlds. In Section 4 we prove our main technical result, the modular soundness and completeness theorem. The proofs of some of our results are based on the arguments of [49], which we include here for the sake of self-containment.

Related Work

The combination of relevant logic and classical logic has been studied in a number of papers in recent years.

Meyer and Mares [37] add a distinguished set of possible worlds in the context of Routley-Meyer semantics, where De Morgan negation $\sim$ behaves as Boolean negation $\neg$. Their resulting logic is an extension of the classical relevant logic CR and of the classical modal logic S4. Compared to our framework, [37] does not provide a way to reduce relevant implication and negation to material conditional and Boolean negation.

Levesque [32] puts forward an epistemic logic combining classical logic with First Degree Entailment FDE. The propositional fragment of his logic is classical and the characteristic modal principle of the logic is monotonicity restricted to FDE. However, given the restriction to FDE, a sensible relevant conditional connective is lacking. Moreover, nesting of modal operators is not allowed in Levesque's framework (the latter restriction is lifted in [31]).

The framework of [49] can be seen as a generalization of Levesque's framework where FDE can be replaced by any of a wide family of relevant logics with relevant conditional operators, and where nesting of modal operators is allowed. [50] adds to the framework of [49] an important component of Levesque's semantics – the means to model implicit belief seen as closure of explicit belief under classical consequence. The framework of [50] is close to the present semantics in that the set of logical states L is present explicitly in frames and not defined using other means as in [49].

In [58], the framework of [49] is applied to the study of relevant evidence logic, led by considerations of logical omniscience. Compared to the present framework, [58] employs monotonic neighborhood models, it uses Fine's non-standard but equivalent semantics for relevant logic (cf. [24]), and it takes a subsystem of BM with truth constants $\top, \bot$ as the fixed underlying relevant logic. The present framework is more general since it provides a modular characterisation of logics CL which are parametric on a wider set of relevant logics L. The present framework is also more elegant since the completeness proof does not rely on the presence of $\top, \bot$ in the language. Hence, the present paper may be seen as providing a more general framework for logics of relevant evidence that the one in [58].

A generalization of [49] to a first-order setting was provided in [21]. A fully general approach to neighborhood semantics for relevant modal first-order logic is undertaken in [22], where both the semantics for the modal operators and the ternary relational semantics for propositional relevant logic are generalized to neighborhood semantics. This allows one to characterize logics weaker than BM, one of the weakest relevant logic complete with respect to Routley-Meyer frames considered in [49].

In [46, 47], combinations of classical propositional logic with relevant modal logic based on relational semantics are studied. The semantics of [47] is two-sorted and the Hilbert-style axiomatizations provided in the paper use a meta-rule of inference, which is inconvenient if the underlying relevant logic is undecidable, which is often the case. The semantics of [46] is one-sorted, thus more elegant, but the Hilbert-style axiomatizations provided in that paper have two peculiar features: proofs are defined in a non-standard way as *pairs* of finite sequences of formulas, and the completeness proof relies on $\top$, $\bot$.

2 Neighborhood Semantics for Relevant Logic

In this section we introduce frame semantics for relevant modal logic based on neighborhood structures. Our presentation in this section is based on Fuhrmann's [26]. In Section 3.1 we will need to modify Fuhrmann's approach so as to accommodate possible worlds in neighborhood frames. We do so by introducing general bounded structures for modal logic, as done in [51].

Definition 1 (Modal language) Let the *modal language* $\mathcal{L}$ be defined in BNF from a denumerable set of propositional variables At, where $p \in At$, as follows:

$$\varphi \in \mathcal{L} \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Box_L \varphi \mid \Box\varphi$$

As usual, we abbreviate $\varphi \leftrightarrow \psi := \varphi \rightarrow \psi \wedge \psi \rightarrow \varphi$. & , $\forall, \exists, \Rightarrow, \Leftrightarrow$ will denote conjunction, universal quantification, existential quantification, implication and equivalence in the meta-language. The modality $\Box$ denotes the salient epistemic attitude, while the main motivation for $\Box_L$ is technical – it internalises relevant validity in the object language. However, it is possible to give $\Box_L$ a more substantive epistemic reading (see [58] and the discussion in Section 3.1).

Frame semantics for substructural logics is based on partially ordered sets $(S, \leq)$. Before defining more precisely the semantics for the modal language, we introduce some compact notation for operations on partially ordered sets, by which we will be able to encode each frame component's monotonicity conditions with respect to $\leq$.

Definition 2 (Arrow notation) Let $(S_1, \leq_1)$ and $(S_2, \leq_2)$ be two partially ordered sets and $k_1, \ldots, k_n, k_{n+1} \in \{\downarrow, \uparrow\}$. An n-ary function f from $(S_1, \leq_1)$ to $(S_2, \leq_2)$ is said to be of type $k_1 \ldots k_n \mapsto k_{n+1}$ iff:

$$\bigwedge_{i \leq n} \left(s_i Z_i t_i\right) \implies f(s_1, \ldots, s_n) Z_{n+1} f(t_1, \ldots, t_n)$$

where $Z_i = \leq$ in case $k_i = \uparrow$ and $Z_i = \geq$ in case $k_i = \downarrow$. We denote as $S_1(k_1 \ldots k_n, S_2(k_{n+1}))$ the set of n-ary functions from S_1 to S_2 of type $k_1 \ldots k_n. \mapsto k_{n+1}$.

As a special case, n-ary relations on $(S, \leq)$ are n-ary operations from $(S, \leq)$ to $T = (\{true, false\}, \sqsubseteq)$, where it is assumed that $false \sqsubseteq true$. For example, $S(\uparrow, T(\uparrow))$ denotes the set of all subsets of S that are closed upwards under $\leq$; $S(\downarrow\uparrow, T(\uparrow))$ denotes the set of binary relations on S that are anti-monotonic in the first position and monotonic in the second position; and $S(\uparrow, S(\downarrow))$ denotes the set of anti-monotonic unary functions on S. We will usually omit $T(\uparrow)$; hence $S(\uparrow)$ means $S(\uparrow, T(\uparrow))$. Note that S_1 and S_2 need not be identical: for example $S(\uparrow, \mathcal{P}(\mathcal{P}(S))(\uparrow))$ will denote the set of monotonic functions $f : (S, \leq) \to (\mathcal{P}(\mathcal{P}(S)), \subseteq)$. Finally, if B is a binary relation on S, then $B(s)$ denotes the set $\{t \mid Bst\}$, and if $X \subseteq S$, then $B(X) := \bigcup_{s \in X} B(s)$.

2.1 Semantics of L-models

Definition 3 (Frame) A *frame* is a structure

$$\boldsymbol{F} = (S, L, \leq, R, *, N, N_L)$$

where $(S, \leq)$ is a partially ordered set with S non-empty, $R \in S(\downarrow\downarrow\uparrow)$, $* \in S(\uparrow, S(\downarrow))$, $N \in S(\uparrow, \mathcal{P}(\mathcal{P}(S))(\uparrow))$, $N_L \in S(\downarrow\uparrow)$ and $L \in S(\uparrow)$ is such that:

$$\forall s \exists x (x \in L \ \ \& \ \ Rxss) \tag{1}$$

$$s \in L \ \ \& \ \ Rstu \Rightarrow t \leq u \tag{2}$$

A *model* based on a frame $\boldsymbol{F}$ is a tuple $\boldsymbol{M} = (\boldsymbol{F}, V)$ where $V : At \to S(\uparrow)$.

Definition 4 (Frame operations) For each frame $\boldsymbol{F}$, we define the following operations for all $X, Y \in S(\uparrow)$:

$$X \wedge^{\boldsymbol{F}} Y = X \cap Y$$

$$X \vee^{\boldsymbol{F}} Y = X \cup Y$$

$$X \to^{\boldsymbol{F}} Y = \{s \mid \forall t, u (Rstu \ \ \& \ \ t \in X \Rightarrow u \in Y)\}$$

$$\neg^{\boldsymbol{F}} X = \{s \mid s^* \notin X\}$$

$$\Box^{\boldsymbol{F}} X = \{s \mid NsX\}$$

$$\Box_L^{\boldsymbol{F}} X = \{s \mid \forall t (N_L st \Rightarrow t \in X)\}$$

Definition 5 (Interpretation) For each model $\boldsymbol{M} = (\boldsymbol{F}, V)$, the *interpretation* $[\![\,]\!]_{\boldsymbol{M}} : \mathcal{L} \to S(\uparrow)$ is defined recursively such that $[\![p]\!]_{\boldsymbol{M}} = V(p)$ and:

$$[\![\varphi \wedge \psi]\!]_{\boldsymbol{M}} = [\![\varphi]\!]_{\boldsymbol{M}} \wedge^{\boldsymbol{F}} [\![\psi]\!]_{\boldsymbol{M}}$$

$$[\![\varphi \vee \psi]\!]_{\boldsymbol{M}} = [\![\varphi]\!]_{\boldsymbol{M}} \vee^{\boldsymbol{F}} [\![\psi]\!]_{\boldsymbol{M}}$$

$$[\![\varphi \to \psi]\!]_{\boldsymbol{M}} = [\![\varphi]\!]_{\boldsymbol{M}} \to^{\boldsymbol{F}} [\![\psi]\!]_{\boldsymbol{M}}$$

$$[\![\neg\varphi]\!]_{\boldsymbol{M}} = \neg^{\boldsymbol{F}}[\![\varphi]\!]_{\boldsymbol{M}}$$
$$[\![\Box\varphi]\!]_{\boldsymbol{M}} = \Box^{\boldsymbol{F}}[\![\varphi]\!]_{\boldsymbol{M}}$$
$$[\![\Box_L\varphi]\!]_{\boldsymbol{M}} = \Box^{\boldsymbol{F}}[\![\varphi]\!]_{\boldsymbol{M}}$$

A formula φ is *valid* in a model $\boldsymbol{M}$, written $\boldsymbol{M} \models \varphi$, iff $L \subseteq [\![\varphi]\!]_{\boldsymbol{M}}$; a formula φ is *valid in a class of frames* iff it is valid in each model based on a frame in the class. The set of formulas *valid in all models* is denoted as **BM.E**.

In what follows we often write $\boldsymbol{M}, s \models \varphi$ instead of $s \in [\![\varphi]\!]_{\boldsymbol{M}}$; we omit reference to $\boldsymbol{M}$ when it is clear from context; and we use the relational notation, writing NsX instead of $X \in N(s)$ for brevity.

Frames are partially ordered sets of information states, ordered by information inclusion. That is, $s \leq t$ means that t contains at least as much information as s. Information states, or situations, generalize possible worlds in that they are not closed under the laws of classical logic. Notably, they may support classical contradictions (may not support classical tautologies), i.e. we may have $s \models \varphi \wedge \neg\varphi$ ($s \not\models \varphi \vee \neg\varphi$), as both (none of) φ and its negation holds. However, it does not follow that contradictory information leads to explosion, i.e. $s \not\models \varphi \wedge \neg\varphi \to \psi$. This feature is due to the rich Routley-Meyer semantics for relevant logic, where negation and implication are regarded as intensional modalities and are interpreted via suitable accessibility relations.

The intensional treatment of negation is delivered by the Routley star function $*$, mapping any state s to its maximally compatible state s^*,[7] while the intensional treatment of implication is delivered by the ternary relation R. The introduction of R lies at the heart of the initial motivation for relevant logics, since it allows one to invalidate the so-called "paradoxes of strict implication" such as $\varphi \to (\psi \to \psi)$. In our epistemic setting, the informational interpretation of the semantics lends to a reading of R in terms of information combination (cf. [15, 16, 46]). On this view, $Rstu$ means that "the combination of the pieces of information s and t (not necessarily the union) is a piece of information in u" [16, p. 67] (see Section 3.1 for further discussion).

$N(s)$, the neighborhood of s, contains the propositions in a fixed agent's epistemic state at s. Note that, according to neighborhood semantics, epistemic states consist of distinct sets of information states, which correspond to distinct propositions agents have epistemic access to. On the other hand, according to Kripke semantics epistemic states consist of a single set of information states, which corresponds to the conjunction of the propositions agents have epistemic access to.

To conclude the presentation of frames for relevant logics, L represents the special subset of information states that carry logical information. This reading motivates the plausibility of Conditions (1)–(2): according to (1), there is always a logical information state such that its combination with s does not produce new information, while according to (2) combining a logical information state with s does not result in loss of

[7] Note that $*$ can be defined in terms of a compatibility relation $C \in S(\downarrow, \downarrow)$ by setting $s^* = t \Leftrightarrow Cst \,\&\, \forall t'(Cst' \Rightarrow t' \leq t)$. The existence of s^* in the above definition of $*$ amounts to assuming that C is serial and convergent. A more general semantics without these properties can be provided by using C as negation's accessibility relation.

information. Together, Conditions (1)–(2) are sufficient for the validity of the semantic deduction theorem (see Lemma 2). According to Lemma 2, logical states can be seen as encapsulating informational links from premises to conclusions of relevantly valid implications. Finally, N_L is the accessibility relation associated with the interpretation of $\Box_L$. $\Box_L$ is a conjunctively regular modality which plays a fundamental technical role in the presence of possible worlds (to be defined in Section 3). Since the technical role of $\Box_L$ will become clear after W-models are introduced, we postpone the discussion of N_L to Section 3.1.

We present below some well-known results for relevant modal logic. Lemma 1 holds thanks to the monotonicity properties of each accessibility relation and it generalizes the definition of the valuation V to arbitrary formulas. As a result, propositions expressed by formulas in relevant logic are closed under information containment. Lemma 2 is crucial in proving soundness (see Theorem 2).

Lemma 1 (Heredity) *For all models $\boldsymbol{M}$, $[\![\varphi]\!]_{\boldsymbol{M}} \in S(\uparrow)$.*

Proof The proof, by induction on the complexity of φ, is a standard result of relevant modal logic, and it exploits the interaction between the monotonicity conditions of the frame components (see Definition 3) and the corresponding frame operations (see Definition 4). □

Lemma 2 (Verification) *For all models $\boldsymbol{M}$, $\boldsymbol{M} \models \varphi \to \psi$ iff $[\![\varphi]\!]_{\boldsymbol{M}} \subseteq [\![\psi]\!]_{\boldsymbol{M}}$.*

Proof Assume $s \in [\![\varphi]\!]$ and $s \notin [\![\psi]\!]$ for some $s \in S$. By (1) we have that there is $t \in L$ such that $Rtss$, by which we conclude that $t \not\models \varphi \to \psi$. Conversely, assume for some $s \in L$ that $s \not\models \varphi \to \psi$. Then there are t, u such that $Rstu$, $t \in [\![\varphi]\!]$ and $u \notin [\![\psi]\!]$. By (2) we have that $t \leq u$ and by Lemma 1 we conclude that $t \notin [\![\psi]\!]$. □

Our models are structures with extremely weak constraints, as we do not assume anything besides the monotonicity condition on $*$, R and N.

A number of further assumptions can be imposed on $*$, which result in the validity of stronger principles involving negation. While some conditions are naturally motivated, like $s \leq s^{**}$, which yields the validity of $\varphi \to \neg\neg\varphi$, others are not well suited to a relevant setting, a prominent example being $s = s^*$, which yields the validity of $\varphi \wedge \neg\varphi \to \psi$.[8] Further principles are examined in e.g. [10, 17].

Similarly, we can obtain stronger properties of information combination by imposing further assumptions on R, as illustrated in Fig. 1, where we stipulate $R(st)uv := \exists x(Rstx \wedge Rxuv)$, $Rs(tu)v := \exists x(Rtux \wedge Rsxv)$. Some conditions, like weak commutativity ($Rstu \Rightarrow Rtsu$), idempotence ($Rsss$) and associativity ($R(st)uv \Rightarrow Rs(tu)v$) have been advocated as characteristic of information combination (cf. [57]). However, a precise stance on each frame condition concerning R depends on the specific interpretation of information combination.[9]

8 In semantics using a compatibility relation C, $\varphi \to \neg\neg\varphi$ is valid iff C is symmetric, while $\varphi \wedge \neg\varphi \to \psi$ is valid iff C is reflexive. Symmetric compatibility relations are widely used in frame semantics for relevant logics with epistemic applications (cf. [13, 46]), while proximity (i.e. reflexive and symmetric) relations are used in possibility semantics for orthologic (cf. [30]).

9 Different ways of combining information arise once we make distinctions in the types of information to be combined. For example, we may distinguish between information in implicational form, *programs*, and non-implicational information, *data*. Then, each of weak commutativty, idempotence and associativity can fail depending on whether we combine data, programs or apply programs to data (cf. [52]).

Frame condition		Axiom/rule
(DNI)	$s \leq s^{**}$	$\varphi \to \neg\neg\varphi$
(DNE)	$s^{**} \leq s$	$\neg\neg\varphi \to \varphi$
(X)	$s \in L \Rightarrow s^* \leq s$	$\varphi \vee \neg\varphi$
(RD)	Rss^*s	$(\varphi \to \neg\varphi) \to \neg\varphi$
(CP)	$Rstu \Rightarrow Rsu^*t^*$	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
(WB)	$Rstu \Rightarrow Rs(st)u$	$((\varphi \to \psi) \wedge (\psi \to \chi)) \to (\varphi \to \chi)$
(B)	$R(st)uv \Rightarrow Rs(tu)v$	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$
(CB)	$R(st)uv \Rightarrow Rt(su)v$	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(W)	$Rstu \Rightarrow R(st)tu$	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
(C)	$R(st)uv \Rightarrow R(su)tv$	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(M)	$Rstu \Rightarrow (s \leq u \vee t \leq u)$	$\varphi \to (\varphi \to \varphi)$
(ER)	$\exists x(x \in L \ \ \& \ \ Rsxs)$	$\varphi \Rrightarrow (\varphi \to \psi) \to \psi$
(□M)	$NsX \ \ \& \ \ X \subseteq Y \Rightarrow NsY$	$\varphi \to \psi \Rrightarrow \Box\varphi \to \Box\psi$
(□N)	$s \in L \subseteq X \Rightarrow NsX$	$\varphi \Rrightarrow \Box\varphi$
(□C)	$NsX \ \ \& \ \ NsY \Rightarrow Ns(X \cap Y)$	$\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$
(□K)	$Rstu, Ns(X \to Y), NtX \Rightarrow NuY$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(□T)	$NsX \Rightarrow s \in X$	$\Box\varphi \to \varphi$
(□D)	$Ns\neg X \Rightarrow not\ Ns^*X$	$\Box\neg\varphi \to \neg\Box\varphi$
(□4)	$NsX \Rightarrow Ns\{t \mid NtX\}$	$\Box\varphi \to \Box\Box\varphi$
(□5)	$not\ Ns^*X \Rightarrow Ns\{t \mid not\ Nt^*X\}$	$\neg\Box\varphi \to \Box\neg\Box\varphi$

Fig. 1 Some prominent frame conditions with the corresponding axioms and rules

Finally, stronger epistemic properties can be obtained by imposing further constraints on N, two prominent examples being closure under information *aggregation* and *combination*. The difference between the (classically indistinguishable) notions of information aggregation and combination can be explained as follows in a relevant setting. Aggregation is the epistemic action by which φ and ψ are conjoined into $\varphi \wedge \psi$ and, as [53] points out, it requires effort. That is, by the fact alone that an agent bears distinct instances of a given epistemic attitude towards φ and ψ, it does not follow that the agent bears a single instance of the given epistemic attitude towards both of these propositions taken together.[10] On the other hand, combination is an epistemic action operated on the information carried by two propositions via modus ponens, and it differs from aggregation in that the result of combination is greater than the sum of the combined propositions (cf. [53] for a discussion). Formally, aggregation and combination are modeled in neighborhood structures by the closure of N under finite intersection ($NsX \ \ \& \ \ NsY \Rightarrow Ns(X \cap Y)$) and merging ($Rstu, Ns(X \to Y), NtX \Rightarrow NuY$).

[10] An anonymous reviewer has pointed out that there is a tension between non-aggregative modal logics, lacking (□C), and relevant logics, where the agents' reasoning is closed under aggregation by virtue of the adjunction rule $\varphi, \psi \Rrightarrow \varphi \wedge \psi$. The tension is dissipated once it is noted that adjunction, as a rule of inference, poses a stronger requirement than (□C) for aggregation, since φ and ψ must be theorems. Note also that under some important respect relevant logic lacks aggregation in the form of adjunction, as one might conceive of *fusion* $\otimes$ rather than *external conjunction* $\wedge$ as the "right" conjunction for epistemic contexts (cf. [52]).

Figure 1 summarizes the discussion above, providing a table of correspondences between some frame conditions we discussed on information combination, compatibility and the neighborhood function on the one hand, and the axioms/rules characterising the frame conditions on the other (in the precise sense defined by Theorem 1). For the sake of generality, and in view of a flexible framework that could be adapted to the specific modeler's purposes, we do not assume any such property but provide a characterisation result of stronger logics via the method of frame correspondences. However, since equivalent formulas express the same propositions in models, the equivalence rule ($\Box$E) still preserves validity, i.e. $\Box$ is non-hyperintensional.

2.2 Relevant Modal Axiom Systems

We now introduce the relevant axiom systems that will be considered throughout the paper. As for the notation, we will use the terminology X.Y to indicate a system containing X as a propositional subsystem and the modal principles contained in Y.

Definition 6 (Basic relevant modal axiom system) The *axiom system* BM.E consists of the following axioms and rules of inference:

- The following axioms and rules of inference for BM:

$$
\begin{array}{rl}
\text{(A1)} & \varphi \to \varphi \\
\text{(A2)} & \neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi) \\
\text{(A3)} & (\neg\varphi \wedge \neg\psi) \to \neg(\varphi \vee \psi) \\
\text{(A4)} & (\varphi \wedge \psi) \to \varphi \\
\text{(A5)} & (\varphi \wedge \psi) \to \psi \\
\text{(A6)} & \varphi \to (\varphi \vee \psi) \\
\text{(A7)} & \psi \to (\varphi \vee \psi) \\
\text{(A8)} & ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)) \\
\text{(A9)} & ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi) \\
\text{(A10)} & (\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \\
\text{(R1)} & \varphi,\ \varphi \to \psi \Rrightarrow \psi \\
\text{(R2)} & \varphi,\ \psi \Rrightarrow \varphi \wedge \psi \\
\text{(R3)} & \varphi' \to \varphi,\ \psi \to \psi' \Rrightarrow (\varphi \to \psi) \to (\varphi' \to \psi') \\
\text{(R4)} & \varphi \to \psi \Rrightarrow \neg\psi \to \neg\varphi
\end{array}
$$

- The following axioms and rules of inference for the modalities $\Box_L, \Box$:

$$
\begin{array}{rl}
(\Box_L\text{C}) & \Box_L\varphi \wedge \Box_L\psi \to \Box_L(\varphi \wedge \psi) \\
(\Box_L\text{M}) & \varphi \to \psi \Rrightarrow \Box_L\varphi \to \Box_L\psi \\
(\Box\text{E}) & \varphi \leftrightarrow \psi \Rrightarrow \Box\varphi \leftrightarrow \Box\psi
\end{array}
$$

In general, if a relevant modal logic **L** is defined as a set of formulas valid in all frames satisfying a selection of frame conditions from Fig. 1, we will denote the selection of the frame conditions as **L**-*conditions* and any model (frame) which satisfies a given selection of **L**-conditions as an **L**-*model* (**L**-*frame*). For any relevant modal logic **L**, the axiom system L is an extension of BM.E, obtained by adding to BM.E the axioms and rules corresponding to the **L**-conditions according to Fig. 1.

For any axiomatic extension L of BM.E, let derivability in L of a formula φ from a set Γ of formulas, written $\Gamma \vdash_{\mathsf{L}} \varphi$, be defined as usual. Let φ be a theorem of L, written $\vdash_{\mathsf{L}} \varphi$, iff $\emptyset \vdash_{\mathsf{L}} \varphi$. The set of theorems of L is denoted as $Th(\mathsf{L})$. An axiom system L' is an *extension* of an axiom system L iff all axioms of L are axioms of L' and all inference rules of L are inference rules of L'.

Despite the fact that our general level of analysis allows one to consider axiomatic extensions of BM.E in a piecemeal fashion, it will be instructive to present here some notable axiomatic extensions of our minimal system BM.E at the propositional level discussed in the relevant logic literature (cf. [1, 44]). The propositional extensions are obtained by adding to BM.E subsets of the propositional axioms and rules of Fig. 1. In what follows we will adopt the convention of using the same label for axioms/rules and their corresponding frame conditions, when no confusion arises.

$$\begin{aligned}
\mathsf{B.E} &= \mathsf{BM.E} + (\mathsf{DNI}) + (\mathsf{DNE})\\
\mathsf{DW.E} &= \mathsf{B.E} + (\mathsf{CP})\\
\mathsf{TW.E} &= \mathsf{DW.E} + (\mathsf{B}) + (\mathsf{CB})\\
\mathsf{T.E} &= \mathsf{TW.E} + (\mathsf{WB}) + (\mathsf{X}) + (\mathsf{RD}) + (\mathsf{W})\\
\mathsf{E.E} &= \mathsf{T.E} + (\mathsf{ER})\\
\mathsf{R.E} &= \mathsf{E.E} + (\mathsf{C})\\
\mathsf{RM.E} &= \mathsf{R.E} + (\mathsf{M})
\end{aligned}$$

Similarly to the propositional case, several modal extensions of L.E, where L is a propositional extension of BM, have been studied for different epistemic applications.[11] The following are some notable modal extensions of BM.E, obtained by adding to BM.E subsets of the modal axioms and rules of Fig. 1. Note that (extensions of) L.I and L.C are indistinguishable whenever L = CPC (where CPC can be obtained as $\mathsf{R} + \varphi \to (\psi \to \varphi)$), since $\vdash_{\mathsf{CPC}} (\varphi \wedge \psi \to \chi) \leftrightarrow (\varphi \to (\psi \to \chi))$.

$$\begin{aligned}
\mathsf{L.M} &= \mathsf{L.E} + (\Box\mathsf{M})\\
\mathsf{L.I} &= \mathsf{L.M} + (\Box\mathsf{K})\\
\mathsf{L.C} &= \mathsf{L.M} + (\Box\mathsf{C})\\
\mathsf{L.R} &= \mathsf{L.M} + (\Box\mathsf{C}) + (\Box\mathsf{K})
\end{aligned}$$

[11] For example, in [50] L.C is used to model relevant reasoning involving explicit belief, while according to standard models of belief, its logic is in the vicinity of CPC.KD45 (cf. [3]). Moreover, CPC.S4 contains the principles characterising the topological analysis of knowledge as the interior operator (cf. [4–6]), while in most models of knowledge in the computer science and game theory literature, it is assumed that knowledge is a CPC.S5-modality (cf. [36]).

$$\mathsf{L.K} = \mathsf{L.R} + (\Box\mathsf{N})$$
$$\mathsf{L.KD45} = \mathsf{L.K} + (\Box\mathsf{D}) + (\Box 4) + (\Box 5)$$
$$\mathsf{L.S4} = \mathsf{L.K} + (\Box\mathsf{T}) + (\Box 4) + (\Box\mathsf{D})$$
$$\mathsf{L.S5} = \mathsf{L.S4} + (\Box 5)$$

The axiom systems considered contain some redundancies (we refer the reader to [12, 44] for a more extensive discussion of relevant axiom systems). Hence, we will adopt the convention of denoting as L the smallest variant L′ of L such that L′ has the same axioms as L and all inference rules of L are admissible in L′ (note that the rules included in the definition of a system are trivially admissible in the system). We note that (R4) is admissible in any extension L of BM.E containing (CP); (R3) is admissible in any L containing (B) and (CB); and (ER) is admissible in any L containing (C). Hence, in line with our convention, (R4) will not be considered as a basic rule of DW.E (and its extensions) and (R3) will not be considered as a basic rule of TW.E (and its extensions).

Theorem 1 (L-frame charecterisation) *For all* **L***-frames* $\boldsymbol{F}$*,* $\boldsymbol{F}$ *satisfies the* **L***-conditions iff the corresponding axioms and rules are valid in* $\boldsymbol{F}$*.*

Proof See [26]. □

Theorem 2 (L soundness and completeness) *For each* **L**, $\mathbf{L} = Th(\mathsf{L})$.

Proof See [26]. □

3 Relevant Reasoners in a Classical World

We now introduce our semantics, based on so-called W-frames. Doing so requires us to modify frames in three ways: (i) we consider *general frames*, restricting the set of admissible propositions to a distinguished set *Prop*; (ii) we consider *bounded* structures, with two distinguished information states, 1 and 0, representing the full and the empty state respectively; (iii) we add a distinguished set of *possible worlds*.

3.1 W-models

Bounded general frames for modal logic were studied by Seki [51], and their introduction in neighborhood models for relevant modal logic is a necessary requirement for the definition of possible worlds, the main ingredient of W-models.

In W-models, the relevant architecture of models remains unchanged. In particular, the semantic interpretation of formulas is still carried out by the frame operations of Section 2.1. However, at possible worlds the interpretation of such formulas is equivalent to the classical Boolean interpretation. Possible worlds, then, contain complete and consistent information, as opposed to information states containing possibly partial and contradictory information. The crucial feature of our framework is that while propositional formulas are interpreted classically at possible worlds, formulas in the scope of modal operators are interpreted relevantly, since modal accessibility relations are allowed to reach information states other than possible worlds. We then define validity to be truth at all possible worlds, so that classical logic provides a sound and complete axiomatization of the modified semantics with respect to the propositional fragment of $\mathcal{L}$.

Definition 7 (W-frames) A *W-frame* is a structure

$$\boldsymbol{F}_W = (S, W, Prop, L, \leq, R, *, N, N_L)$$

where:

- $Prop \subseteq S(\uparrow)$ is a set of admissible propositions such that $X \wedge^{\boldsymbol{F}_W} Y$, $X \vee^{\boldsymbol{F}_W} Y$, $X \rightarrow^{\boldsymbol{F}_W} Y$, $\neg^{\boldsymbol{F}_W} X$, $\Box^{\boldsymbol{F}_W} X$, $\Box_L^{\boldsymbol{F}_W} X \in Prop$ whenever $X, Y \in Prop$, where $\wedge^{\boldsymbol{F}_W}, \vee^{\boldsymbol{F}_W}, \rightarrow^{\boldsymbol{F}_W}, \neg^{\boldsymbol{F}_W}, \Box^{\boldsymbol{F}_W}$ and $\Box_L^{\boldsymbol{F}_W}$ are as in Definition 4;
- $(S, L, \leq, R, *, N, N_L)$ is a frame, with the proviso that $N \in S \rightarrow S(\uparrow, \mathcal{P}(Prop)(\uparrow))$;
- $(S, L, \leq, R, *, N)$ is bounded, i.e. there are elements $0, 1 \in S$ such that, for all $s \in S$, $0 \leq s \leq 1$, and such that, for all $s, t \in S$, $X \in Prop$, the following are satisfied:

$$1^* = 0 \tag{3}$$
$$0^* = 1 \tag{4}$$
$$N1X \tag{5}$$
$$\text{not } N0X \tag{6}$$
$$N_L00 \tag{7}$$
$$N_Ls1 \tag{8}$$
$$N_L1s \Rightarrow s = 1 \tag{9}$$
$$R010 \tag{10}$$
$$R1st \Rightarrow (s = 0 \text{ or } t = 1) \tag{11}$$

- $W \subseteq S$ is a set of possible worlds, i.e. for all $w \in W$ and $s, t \in S$ the following are satisfied:

$$w^* = w \tag{12}$$
$$Rwww \tag{13}$$
$$Rwst \Rightarrow (s = 0 \text{ or } w \leq t) \tag{14}$$
$$Rwst \Rightarrow (t = 1 \text{ or } s \leq w) \tag{15}$$
$$N_L(W) = L \tag{16}$$

A *W-model* based on $\boldsymbol{F}_W$ is a tuple $\boldsymbol{M}_W = (\boldsymbol{F}_W, V)$ where $V : At \rightarrow Prop$ such that $1 \in V(p)$ and $0 \notin V(p)$ for all $p \in Pr$.

Consistently with the notation of Section 2, we will denote any W-model $\boldsymbol{M}_W$ (W-frame $\boldsymbol{F}_W$) which satisfies any selection of frame conditions of Fig. 1 as a **CL**-model $\boldsymbol{M}_{\mathbf{CL}}$ (**CL**-frame $\boldsymbol{F}_{\mathbf{CL}}$), with the proviso that X, Y in the frame conditions are now elements of $Prop$. Given the definition of N in W-frames, this proviso is in fact relevant only in the context of conditions (□M) and (□N).

In our definition of **CL**-models, conditions (3)–(11) on bounded structures ensure that $L, R, *, N, N_L$ continue to satisfy the required monotonicity conditions. Note that, by (1), L is non-empty, which together with (16) implies that W is non-empty.

Possible worlds can be thought of as the context where the agents' reasoning process takes place and where they gather the information forming their epistemic state. Alternatively, possible worlds can be thought of as maximally consistent information states, or worldly-situations, thus obeying to classical logic.

Conditions (12)–(15) above ensure that propositional formulas are interpreted classically at worlds (see Lemma 6). Equation (12) ensures that worlds contain consistent information; (13)–(15) express a maximality requirement for worlds combination, by which no consistent information is lost nor gained by combining a world with itself. Note that (14)–(15) are responsible for the rich semantic structures we need to employ, as they provide a technical motivation for introducing the bounds 0 and 1. To see why we need bounded structures, note that the simplified versions of Conditions (14)–(15), i.e. $Rwst \Rightarrow w \leq t$ and $Rwst \Rightarrow s \leq w$, are not be satisfied by the canonical **CL**-model (to be introduced). As will be clear in Section 4.1, the culprits are to be found in the empty and the full prime theory, respectively, which will be in the canonical model exactly the bounds 0, 1.

Moreover, we restrict the set of admissible propositions to $Prop \subseteq S(\uparrow)$. The restriction is characteristic of *general frames*, which are widely used in modal logics (cf. [51]). The use of general frames simplifies completeness proofs, as they allow us to have a uniform canonical model construction for a wide range of modal logics.[12] It is interesting to note that we resort to general frames in our framework mainly for technical reasons, as they are necessary in order for the completeness construction to yield a **CL**-model. To see why we need general frames, note that the simplified version of Condition (5), i.e. $N1X$ for all $X \in S(\uparrow)$, is sufficient for Lemma 5 to hold, but is not satisfied by the canonical **CL**-model.

Definition 8 (**CL**-Interpretation) For each **CL**-model $\boldsymbol{M}_W = (\boldsymbol{F}_W, V)$, the **CL**-*interpretation* $[\![\,]\!]_{\boldsymbol{M}_W} : \mathcal{L} \to Prop$ is defined recursively as in Definition 5 (using the corresponding W-frames operations). A formula φ is *valid in a* **CL**-*model* $\boldsymbol{M}_W$, written $\boldsymbol{M}_W \models \varphi$, iff $W \subseteq [\![\varphi]\!]_{\boldsymbol{M}_W}$; a formula φ is *valid in a class of* **CL**-*frames* iff it is valid in each **CL**-model based on a **CL**-frame in the class. The set of formulas *valid in all* **CL**-*models* is denoted as **CL**.

Definition 9 ((Classical) Consequence) A formula φ is *a consequence* of a set of formulas Γ in a **CL**-model $\boldsymbol{M}_{\mathbf{CL}}$, written $\Gamma \models_{\boldsymbol{M}_{\mathbf{CL}}} \varphi$, iff for all $s \in S$ and $\psi \in \Gamma$, $\boldsymbol{M}_{\mathbf{CL}}, s \models \psi$ only if $\boldsymbol{M}_{\mathbf{CL}}, s \models \varphi$; a formula φ is a *classical consequence* of a set of

[12] Cf. [14, p.258] and [39, p.66] for a completeness-via-canonicity argument where the choice of a supplemented canonical model is essential to deal with logics containing (□M).

formulas Γ in a **CL**-model $\boldsymbol{M}_{\mathbf{CL}}$, written $\Gamma \models^{c}_{\boldsymbol{M}_{\mathbf{CL}}} \varphi$, iff for all $w \in W$, $\boldsymbol{M}_{\mathbf{CL}}, w \models \varphi$ if $\boldsymbol{M}_{\mathbf{CL}}, w \models \psi$ for all $\psi \in \Gamma$.

To conclude this section, we explain in what sense $\Box_L$ internalises relevant validity. Having defined validity in **CL**-models as truth in all possible worlds, Condition (16) implies that $\Box_L\varphi$ is true at all possible worlds iff φ is true at all the logical states. That is, $\Box_L\varphi$ is valid in all **CL**-models iff φ is valid in all models. This observation, together with Lemma 4 below, lends a more epistemically perspicuous reading of $\Box_L$. According to Lemma 4, the fact that ψ is a relevant consequence of φ means that, according to any possible world the agent is situated in, the logical information at their disposal supports $\varphi \to \psi$. Given our informational interpretation of the ternary relation R, we may analyse this as saying that, when an input φ is processed in a logical context, the conclusion ψ is deducible. Pressing further on this point, we can then analyse $w \models \Box_L(\varphi \to \psi)$ as ψ being deducible in w from φ based solely on logical information.

3.2 Properties of CL-models

CL-models exhibit some interesting features of both classical and relevant logic. For a start, modified versions of Lemmas 1 and 2 (the former restricting the heredity lemma to admissible propositions, the latter exploiting Condition (16)) hold. Lemma 5, which relies on Conditions (3)–(11), shows that 0 and 1 are the full and the empty state. These lemmas are necessary to show Lemma 6, according to which negation and implication at possible worlds are semantically equivalent to Boolean negation and the material conditional. As a consequence, we are able to show that in **CL**-models $\Box$ is a hyperintensional operator, which is nonetheless semantically closed under relevant equivalence (Proposition 1). Lemma 10, prepared by Lemmas 7-9, shows how to construct **CL**-models from **L**-models, a fundamental fact to establish completeness. Finally, Proposition 2 shows that neighborhood semantics constitutes a generalisation of relational semantics for modal logic.

Lemma 3 (**CL**-heredity) *For all* **CL**-*models* $\boldsymbol{M}_W$ *and all* $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\boldsymbol{M}_W} \in Prop$.

Proof By induction on the complexity of φ. The base case follows immediately from the definition of V, while the induction step is established by noting that Definition 7 requires that *Prop* be closed under the **CL**-frame operations. □

Lemma 4 (**CL**-verification) *For all* **CL**-*models* $\boldsymbol{M}_W$ *and all* $\varphi, \psi \in \mathcal{L}$, $\boldsymbol{M}_W \models \Box_L(\varphi \to \psi)$ *iff* $[\![\varphi]\!]_{\boldsymbol{M}} \subseteq [\![\psi]\!]_{\boldsymbol{M}}$.

Proof The following chain of equivalences holds: $W \subseteq [\![\Box_L(\varphi \to \psi)]\!]$ iff $N_L(W) \subseteq [\![\varphi \to \psi]\!]$ iff (by (16)) $L \subseteq [\![\varphi \to \psi]\!]$ iff (by Lemma 1) $[\![\varphi]\!] \subseteq [\![\psi]\!]$. □

Lemma 5 (Full/empty states) *For all* **CL**-*models* $\boldsymbol{M}_W$ *and all* $\varphi \in \mathcal{L}$:

1. $\boldsymbol{M}_W, 1 \models \varphi$;
2. $\boldsymbol{M}_W, 0 \not\models \varphi$.

Proof By simultaneous induction on the complexity of φ. The base case holds by definition of V in $\boldsymbol{M}_{\mathsf{CL}}$. The cases of $\varphi := \psi \wedge \chi$, $\varphi := \psi \vee \chi$ are trivial.

- $\varphi := \neg\psi$. $1 \models \neg\psi$ iff $1^* \not\models \psi$ iff, by (3), $0 \not\models \psi$ which holds by the induction hypothesis; $0 \not\models \neg\psi$ iff $0^* \models \psi$ iff, by (4), $1 \models \psi$ which holds by the induction hypothesis.
- $\varphi := \psi \to \chi$. $1 \models \psi \to \chi$ iff $R1st$ $s \models \psi$ implies $t \models \chi$, for all s, t such that. Hence, assume $s \models \psi$. By (11), either $s = 0$ or $t = 1$. If $s = 0$ then by the induction hypothesis $s \not\models \psi$, which is a contradiction. Therefore $t = 1$, by which we conclude, using the induction hypothesis, that $t \models \chi$. $0 \not\models \psi \to \chi$ iff $s \models \psi$ and $t \not\models \chi$, for some s, t such that $R0st$. This holds by (10) and the induction hypothesis, setting $s = 1$ and $t = 0$.
- $\varphi := \Box_L\psi$. $1 \models \Box_L\psi$ iff $s \models \psi$, for all s such that $N_L 1s$ – which holds since $s = 1$ by (9) and $1 \models \psi$ by the induction hypothesis. $0 \not\models \Box_L\psi$ iff $s \not\models \psi$, for some s such that $N_L 0s$ – which holds by (7) and the induction hypothesis by setting $s = 0$. $\varphi := \Box\psi$. $1 \models \Box\psi$ iff $N1[\![\psi]\!]$, which holds by (5) and Lemma 3. $0 \not\models \Box\psi$ holds since by (6) not $N0[\![\psi]\!]$.

□

Lemma 6 (Classical connectives) *For all* **CL***-models* $\boldsymbol{M}_{\mathsf{CL}}$ *and all* $w \in W$*, the following hold:*

1. $\boldsymbol{M}_{\mathsf{CL}}, w \models \neg\varphi$ *iff* $\boldsymbol{M}_{\mathsf{CL}}, w \not\models \varphi$*;*
2. $\boldsymbol{M}_{\mathsf{CL}}, w \models \varphi \to \psi$ *iff* $\boldsymbol{M}_{\mathsf{CL}}, w \not\models \varphi$ *or* $\boldsymbol{M}_{\mathsf{CL}}, w \models \psi$*.*

Proof Item 1 follows from $w = w^*$. The left-to-right implication of Item 2 follows from $Rwww$. The right-to-left implication is established as follows. Assume $Rwst$ and $s \models \varphi$. If $w \models \psi$, then by (14) $s = 0$ or $w \leq t$. The former contradicts $s \models \varphi$ by Lemma 5; hence $w \leq t$, by which we conclude that $t \models \psi$. If $w \not\models \varphi$, then by (15) $t = 1$ or $s \leq w$. If the latter, then by Lemma 3 $s \not\models \varphi$, which is a contradiction; hence $t = 1$, by which we conclude by Lemma 5 that $t \models \psi$. □

Thanks to the above lemma and the definition of validity in **CL**-models, (□E) does not preserve validity in **CL**-models, i.e. □ is a hyperintensional modality. As clarified by the counterexample provided in Proposition 1, this is because two formulas φ and ψ may be true at the same possible worlds but not at the same information states in a **CL**-model. Note that, by Proposition 1, the problem of logical omnisicence (cf. [19] for details) is avoided with respect to classical consequence (Item 1), while □ is still closed under relevant equivalence (Item 2).

Proposition 1 (Closure properties) *For all* **CL***-models* $\boldsymbol{M}_{\mathsf{CL}}$ *and all* $\varphi, \psi \in \mathcal{L}$*, the following hold, where* $\Box\Gamma = \{\Box\gamma \mid \gamma \in \Gamma\}$*:*

1. $\Gamma \models^c_{\boldsymbol{M}_{\mathsf{CL}}} \varphi \not\Rightarrow \Box\Gamma \models^c_{\boldsymbol{M}_{\mathsf{CL}}} \Box\varphi$*;*
2. $\varphi \leftrightarrow \psi \models_{\boldsymbol{M}_{\mathsf{CL}}} \Box\varphi \leftrightarrow \Box\psi$*.*

Proof Item 1 is established by the following counterexample to the stronger claim that (□E) preserves validity in **CL**-models. To this aim, consider the formulas $\neg p \vee q$

and $p \to q$, which are true in the same possible worlds for all **CL**-model $\boldsymbol{M}_{\mathbf{CL}}$. Then, take the **CL**-model $\boldsymbol{M}_{\mathbf{CL}}$ with $S = \{w, s, t\}$, $W = \{w\}$, $s^* = t$, $t^* = s$, $t \notin V(p)$, $s \in V(p)$, $s \notin V(q)$, $t \notin V(q)$, $N(w) = \{\{s\}\}$ and $Rsss$ (the remaining components can be specified so that $\boldsymbol{M}_{\mathbf{CL}}$ is indeed a **CL**-model). In $\boldsymbol{M}_{\mathbf{CL}}$, we have that $w \models \Box(\neg p \vee q)$, since $Nw\{s\} = [\![\neg p \vee q]\!]$, but $w \not\models \Box(p \to q)$, since NwX only if $X = \{s\}$, but $s \not\models p \to q$. Hence, we conclude that $[\![\Box\varphi]\!] \neq [\![\Box\psi]\!]$. Item 2 follows by the fact that, for all **CL**-models $\boldsymbol{M}_{\mathbf{CL}}$ and $s \in S$, $[\![\varphi]\!]_{\boldsymbol{M}_{\mathbf{CL}}} = [\![\psi]\!]_{\boldsymbol{M}_{\mathbf{CL}}}$ implies $Ns[\![\varphi]\!]_{\boldsymbol{M}_{\mathbf{CL}}}$ iff $Ns[\![\psi]\!]_{\boldsymbol{M}_{\mathbf{CL}}}$. □

It is easy to see that for every **L** and every **CL**-model $\boldsymbol{M}_{\mathbf{CL}}$, there is an **L**-model $\boldsymbol{M}'$ such that whenever $\boldsymbol{M}_{\mathbf{CL}} \models \Box_L\varphi$ we have that $\boldsymbol{M}' \models \varphi$. In particular, it suffices to "forget" about bounds and W, and to take *full* general frames, i.e. such that $Prop = S(\uparrow)$.

The converse operation, passing from **L**-models to **CL**-models, however, is less straightforward. To illustrate the procedure, we define, for any **L**-model $\boldsymbol{M}$, a function $+ : \boldsymbol{M} \mapsto \boldsymbol{M}^+$ as follows:

Definition 10 (+ construction) Let $\boldsymbol{M} = (S, L, \leq, R, *, N, N_L, V)$ be a model. Then, let the structure

$$\boldsymbol{M}^+ = (S^+, W, \ Prop, L^+, \leq^+, R^+, *^+, N^+, N_L^+, V^+)$$

be defined as follows for all $p \in At$:

- $S^+ = S \cup \{w, 0, 1\}$;
- $W = \{w\}$;
- $Prop = \{X \mid X \in S^+(\uparrow) \ \ \& \ \ X \neq \emptyset\}$;
- $L^+ = L \cup \{w, 1\}$;
- $\leq^+ = \leq \cup \{(w, w)\} \cup \{(s, 1) \mid s \in S^+\} \cup \{(0, s) \mid s \in S^+\}$;
- $R^+ = R \cup \{(w, w, w)\} \cup \{(0, s, t), (s, 0, t), (s, t, 1) \mid s, t \in S^+\}$;
- $*^+ = * \cup \{(w, w)\} \cup \{(0, 1), (1, 0)\}$;
- $N^+ = \{(s, X) \mid s \in S \ \ \& \ \ X \in Prop \ \ \& \ \ Ns(X \cap S)\} \cup \{(1, X) \mid X \in Prop\}$ $\cup \{(w, X) \mid w \in X \in Prop\}$;
- $N_L^+ = N_L \cup \{(w, w)\} \cup \{(w, s) \mid s \in L\} \cup \{(s, 1) \mid s \in S^+\} \cup \{(0, s) \mid s \in S^+\}$;
- $V^+(p) = V(p) \cup \{1\}$.

Note that $N^+(0) = \emptyset$. We would also like to point out that (i) N^+wX iff $w \in X$ and (ii) $w \in L^+$ are assumed in order to make sure that (□T), (□N) and (ER), respectively, are preserved. Instead of using multiple definitions of $\boldsymbol{M}^+$ depending on the properties satisfied by $\boldsymbol{M}$, we chose to build (i) and (ii) into Definition 10 from the outset even though they are not necessary in some cases.

The $+$ function transforms an **L**-model $\boldsymbol{M}$ into a **CL**-model $\boldsymbol{M}^+$ such that for every invalid formula φ in $\boldsymbol{M}$, $\Box_L\varphi$ is invalid in $\boldsymbol{M}^+$ (Lemma 10). As we will see in Section 4, this semantic result is crucial in proving Lemma 11. The proof requires a number of preliminary results, showing respectively that $\boldsymbol{M}^+$ is a W-model (Lemma 7), and that satisfaction (Lemma 8) and all **CL**-frame conditions (Lemma 9) are preserved by the $+$ function.

Lemma 7 *For all* **BM.E***-models* $\boldsymbol{M}$*,* $\boldsymbol{M}^+$ *is a* W*-model.*

Proof We show (i) that $(S^+, \leq^+)$ is a partially ordered set; (ii) that R^+, L^+, N^+, N_L^+ respect the corresponding monotonicity conditions; (iii) that *Prop* is closed under the frame operations of $\boldsymbol{F}^+$, the frame underlying $\boldsymbol{M}^+$; and (iv) that conditions (3)–(16) are satisfied so that $\boldsymbol{M}^+$ is bounded and w is a possible world. (i) and (ii) can be established by inspection of the definition of $\boldsymbol{M}^+$. We show that N^+ satisfies the monotonicity condition. Assume that $s \leq^+ t$. If $s = 0$ or $t = 1$, then we are done. If $s = 1$, then $t = 1$ and we are done. If $t = 0$, then $s = 0$ and we are done. If $s = w$, then $t \in \{w, 1\}$ and we are done in both cases; if $t = w$, then $s \in \{0, w\}$ and we are also done. The remaining case implies that $s, t \in S$ and $s \leq t$. If $X \in N^+(s)$, then $Ns(X \cap S)$. By monotonicity of N in the original model $\boldsymbol{M}$ we have $Nt(X \cap S)$, and so N^+tX by definition of N^+. Given the definition of *Prop*, (iii) follows from (ii). (iv) can be established by inspection of the definition of $\boldsymbol{M}^+$. □

Lemma 8 *For all* $s \in S$ *and all* $\varphi \in \mathcal{L}$*,* $\boldsymbol{M}, s \models \varphi$ *iff* $\boldsymbol{M}^+, s \models \varphi$*.*

Proof The proof is by induction on the complexity of φ. The base case and the induction steps where the main connective is $\wedge, \vee$ are trivial.

- $\varphi := \neg\psi$. The following chain of equivalences holds: $\boldsymbol{M}, s \models \neg\psi$ iff $\boldsymbol{M}, s^* \not\models \psi$ iff $\boldsymbol{M}^+, s^{*^+} \not\models \psi$ (by the induction hypothesis and the fact that if $s \in S$, then $s^* = s^{*^+}$) iff $\boldsymbol{M}^+, s \models \neg\psi$.
- $\varphi := \psi \to \chi$. If $\boldsymbol{M}, s \not\models \psi \to \chi$, then there are $t, u \in S$ such that $Rstu$, $\boldsymbol{M}, t \models \psi$ and $\boldsymbol{M}, u \not\models \chi$. By $R \subseteq R^+$ and the induction hypothesis we have that R^+stu, $\boldsymbol{M}^+, t \models \psi$ and $\boldsymbol{M}^+, u \not\models \chi$, by which we conclude that $\boldsymbol{M}^+, s \not\models \psi \to \chi$. Conversely, if $\boldsymbol{M}^+, s \not\models \psi \to \chi$, then there are $t, u \in S^+$ such that R^+stu, $\boldsymbol{M}^+, t \models \psi$ and $\boldsymbol{M}^+, u \not\models \chi$. By definition of R^+, either $t = 0$ or $u = 1$ (which are ruled out by Lemma 5), or $t, u \in S$, in which case $Rstu$ and so $\boldsymbol{M}, s \not\models \psi \to \chi$ by the induction hypothesis.
- $\varphi := \Box_L\psi$. If $\boldsymbol{M}, s \not\models \Box_L\psi$, then $\boldsymbol{M}, t \not\models \psi$ for some t such that N_Lst. By $N_L \subseteq N_L^+$ and the induction hypothesis we have that N_L^+st and $\boldsymbol{M}^+, t \not\models \psi$, by which we conclude that $\boldsymbol{M}^+, s \not\models \Box_L\psi$. Conversely, if $\boldsymbol{M}^+, s \not\models \Box_L\psi$, then there is t such that N_L^+st and $\boldsymbol{M}^+, t \not\models \psi$. By definition of N_L^+ we have that either $t = 1$ (which contradicts Lemma 5) or $t \in S$, in which case N_Lst and $\boldsymbol{M}, s \not\models \Box_L\psi$ by the induction hypothesis.
- $\varphi := \Box\psi$. If $\boldsymbol{M}, s \models \Box\psi$, then $Ns[\![\psi]\!]_{\boldsymbol{M}}$. By the induction hypothesis, Lemmas 5 and 7, $[\![\psi]\!]_{\boldsymbol{M}^+} \in \{[\![\psi]\!]_{\boldsymbol{M}} \cup \{1\}, [\![\psi]\!]_{\boldsymbol{M}} \cup \{w, 1\}\}$. In both cases, $N^+s[\![\psi]\!]_{\boldsymbol{M}^+}$ by the definition of N^+, which means that $\boldsymbol{M}^+, s \models \Box\psi$. Conversely, if $\boldsymbol{M}^+, s \models \Box\psi$, then $N^+s[\![\psi]\!]_{\boldsymbol{M}^+}$. By the induction hypothesis, Lemmas 5, and 7, $[\![\psi]\!]_{\boldsymbol{M}^+} \in \{[\![\psi]\!]_{\boldsymbol{M}} \cup \{1\}, [\![\psi]\!]_{\boldsymbol{M}} \cup \{w, 1\}\}$. In both cases, $Ns[\![\psi]\!]_{\boldsymbol{M}}$ by the definition of N^+, which means that $\boldsymbol{M}, s \models \Box\psi$.

□

Lemma 9 *if* $\boldsymbol{M}$ *is an* **L***-model, then* $\boldsymbol{M}^+$ *is a* **CL***-model.*

Proof We show that the frame conditions of Fig. 1 are preserved through the $()^+$ construction, i.e. each frame condition holds in (the frame underlying) $\boldsymbol{M}^+$ whenever it holds in (the frame underlying) $\boldsymbol{M}$. For all conditions, we distinguish cases depending on whether a state belongs to S or to $\{w, 0, 1\}$. All the cases up until (□M) are established in [49]. We repeat the proofs here for the sake of self-containment.

- (DNI). Assume (DNI) hold in $\boldsymbol{M}$. To show that (DNI) holds in $\boldsymbol{M}^+$, take $s \in S^+$. If $s \in S$ we are done by $R \subseteq R^+$ and $s^* \in S$. If $s \in \{w, 0, 1\}$, we are done by definition of $*^+$. The cases (DNE) and (RD) are established similarly
- (X). Assume that (X) holds in $\boldsymbol{M}$. To show that (X) holds in $\boldsymbol{M}^+$, assume $s \in L^+$. If $s \in L$ then (X) holds by $L \subseteq L^+$ and the fact that (X) holds in $\boldsymbol{M}$. If $s = w$ ($s = 1$) (X) holds since $w^* = w \leq^+ w$ ($1^* = 0 \leq^+ 1$).
- (CP). Assume that (CP) holds in $\boldsymbol{M}$. To show that (CP) holds in $\boldsymbol{M}^+$, assume R^+stu. If $s \in S$, then either $t, u \in S$ and we are done by $R \subseteq R^+$, or $t = 0$ or $u = 1$. In both cases, $R^+su^{*^+}t^{*^+}$ follows from the definition of $*^+$ and R^+. If $s \notin S$, then we reason by cases as follows. If $s = 0$, then we are done since R^+0xy for all x, y. If $s = w$, then either $t = u = w$ and we are done, or $t = 0$ or $u = 1$. In the latter two cases, we can reason as in the case $s \in S$ above. Finally, if $s = 1$, then $t = 0$ or $u = 1$ and we can reason as in the case $s \in S$ again.
- The cases concerning (B), (CB),(W),(C),(WB) follow the same strategy, hence we show the details only for (B). Assume that (B) holds in $\boldsymbol{M}$ and that R^+stx, R^+xuv. To show that (B) holds in $\boldsymbol{M}^+$, we need to show that there is y such that R^+tuy, R^+syv. Let $T = \{s, t, u, v, x\}$. First, if $T \subseteq S$, then $Rstx$ and $Rxuv$ and so we are done, since (B) holds in $\boldsymbol{M}$ and $R \subseteq R^+$. Second, if $1 \in T$ or $0 \in T$, then we distinguish three cases:

  1. If $0 \in \{s, t, u\}$ or $v = 1$, then we are done. (For instance, if $s = 0$, then R^+s1v and R^+tu1; the other cases are similar.)
  2. If $x = 0$, then by (11) either $s = 0$ or $t = 0$ and we are in case 1. If $x = 1$, then by (11) either $u = 0$ or $v = 1$; in both cases we are in case 1. We will use (11) without explicit reference below.
  3. If $s = 1$, then $t = 0$ (case 1) or $x = 1$ (case 2). If $t = 1$, then $s = 0$ (case 1) or $x = 1$ (case 2). If $u = 1$, then $x = 0$ or $v = 1$ (case 2). If $v = 0$, then $x = 0$ (case 2) or $u = 0$ (case 1).

  Third, if $T \subseteq S \cup \{w\}$, then we are either in case 1 or $T = \{w\}$. In the latter case, set $y = w$ and we are done. These three groups of cases exhaust all possibilities.
- (M). Assume that (M) holds in $\boldsymbol{M}$. To show that (M) holds in $\boldsymbol{M}^+$, assume R^+stu. If $s \in S$, then either $t, u \in S$ and we are done by $R \subseteq R^+$ and $\leq \subseteq \leq^+$, or $t = 0$ or $u = 1$. In the former case, $t \leq^+ u$ and in the latter case $s \leq^+ u$. If $s = 0$, then $s \leq^+ u$. If $s = w$, then either $t = u = w$ and we are done, or $t = 0$ or $u = 1$ which is dealt with as above. If $s = 1$, then $t = 0$ or $u = 1$ which is dealt with as above.
- (ER). Assume that (ER) holds in $\boldsymbol{M}$. To show that (ER) holds in $\boldsymbol{M}^+$, assume $s \in S^+$. If $s \in S$, then by (ER) in $\boldsymbol{M}$ there is $x \in L$ such that $Rsxs$, and so R^+sxs by $R \subseteq R^+$. If $s \in \{w, 0, 1\}$, then R^+sws, and by definition of L^+ we have that $w \in L^+$.

We now move to the modal frame conditions.

- ($\Box$M). Assume that ($\Box$M) holds in $\boldsymbol{M}$. To show that ($\Box$M) holds in $\boldsymbol{M}^+$, assume N^+sX and $X \subseteq Y$ for $X, Y \in Prop$. The case $s = 0$ is ruled out by the definition of N^+. If $s = 1$, then trivially N^+sY. If $s = w$ then by N^+wX we have that $w \in X$ and by $X \subseteq Y$ we have that $w \in Y$, hence N^+wY. Finally, if $s \in S$, then we reason as follows. N^+sX means that $Ns(X \cap S)$. Since $X \subseteq Y$ and ($\Box$M) holds in $\boldsymbol{M}$, we have $Ns(Y \cap S)$ and so N^+sY by the definition of N^+ and $Y \in Prop$.
- ($\Box$N). Assume that ($\Box$N) holds in $\boldsymbol{M}$. To show that ($\Box$N) holds in $\boldsymbol{M}^+$, assume $s \in L^+ \subseteq X$ for $X \in Prop$. If $s = 1$ or $s = w$, then trivially N^+sX. If $s \in S$, then $s \in (L^+ \cap S) \subseteq (X \cap S)$. Since ($\Box$N) holds in $\boldsymbol{M}$ and $L^+ \cap S = L$ we have $Ns(X \cap S)$, and so N^+sX.
- ($\Box$C). Assume that ($\Box$C) holds in $\boldsymbol{M}$. To show that ($\Box$C) holds in $\boldsymbol{M}^+$, assume N^+sX, N^+sY for $X, Y \in Prop$. If $s = 1$ then trivially $N^+s(X \cap Y)$. If $s = w$, then by definition of N^+ we have that $w \in X$ and $w \in Y$, hence $w \in X \cap Y$, by which $N^+s(X \cap Y)$. Finally, if $s \in S$, then by N^+sX and N^+sY we have $Ns(X \cap S)$ and $Ns(Y \cap S)$. Since ($\Box$C) holds in $\boldsymbol{M}$, we have $Ns(X \cap Y \cap S)$. Since $X, Y \in Prop$, we have $X \cap Y \in Prop$ and so $N^+s(X \cap Y)$ by the definition of N^+.
- ($\Box$K). Assume that ($\Box$K) holds in $\boldsymbol{M}$. To show that ($\Box$K) holds in $\boldsymbol{M}^+$, assume R^+stu, $N^+s(X \rightarrow^{\boldsymbol{F}^+} Y)$ and N^+tX for $X, Y \in Prop$. If $s = 0$, $t = 0$ or $u = 1$ we are done by definition of N^+. If $s = 1$ by definition of R^+ either $t = 0$ or $u = 1$, and we are done. If $s = w$ then either $t = 0$, $u = 1$ (and we are done) or $t = u = w$. In the latter case, by N^+wX and $N^+w(X \rightarrow^{\boldsymbol{F}^+} Y)$ we have that $w \in X \rightarrow^{\boldsymbol{F}^+} Y$ and $w \in X$. By R^+www then $w \in Y$, and so N^+wY. If $s, t, u \in S$, then we reason as follows. By $N^+s(X \rightarrow^{\boldsymbol{F}^+} Y)$, N^+tX and the definition of N^+ we have $Ns(X \cap S \rightarrow^{\boldsymbol{F}} Y \cap S)$ and $Nt(X \cap S)$. It is easily checked that $(X \rightarrow^+ Y) \cap S = (X \cap S) \rightarrow (Y \cap S)$ if N^+tX. Since ($\Box$K) holds in $\boldsymbol{M}$, we have $Nu(Y \cap S)$ and so N^+uY by the definition of N^+. The remaining cases reduce trivially to the above.
- ($\Box$T). Assume that ($\Box$T) holds in $\boldsymbol{M}$. To show that ($\Box$T) holds in $\boldsymbol{M}^+$, assume N^+sX for $X \in Prop$. Then, $s = 0$ is ruled out by assumption. If $s = w$, $w \in X$ by definition of N^+. If $s = 1$, then $1 \in X$ by $X \in Prop \subseteq S^+(\uparrow)$ and $X \neq \emptyset$. Finally, if $s \in S$, then by N^+sX we have $Ns(X \cap S)$. Since ($\Box$T) holds in $\boldsymbol{M}$ we have $s \in X \cap S$ and so $s \in X$.
- ($\Box$D). Assume that ($\Box$D) holds in $\boldsymbol{M}$. To show that ($\Box$D) holds in $\boldsymbol{M}^+$, assume and $N^+s\neg^{\boldsymbol{F}^+}X$ for $X \in Prop$. Then $s = 0$ is ruled out by assumption. If $s = 1$ then ($\Box$D) trivially holds. If $s = w$ then by definition of $*^+$ $N^+w^{*^+}\neg^{\boldsymbol{F}^+}X$. By definition of N^+ $w \in \neg^{\boldsymbol{F}^+}X$ and by definition of $*^+$ $w^{*^+} = w \notin X$, so not $N^+w^{*^+}X$. Finally, if $s \in S$ we reason as follows. By definition of N^+ and $*^+$ $N^+s\neg^{\boldsymbol{F}^+}X$ means that $Ns\neg^{\boldsymbol{F}}(X \cap S)$. Since ($\Box$D) holds in $\boldsymbol{M}$ we have not $Ns^*(X \cap S)$, and so not $N^+s^{*^+}X$.
- ($\Box$4). Assume that ($\Box$4) holds in $\boldsymbol{M}$. To show that ($\Box$4) holds in $\boldsymbol{M}^+$, assume N^+sX for $X \in Prop$. Then, $s = 0$ is ruled out by assumption. If $s = 1$ then ($\Box$4) trivially holds. If $s = w$, then the assumption $w \in \{t \mid N^+tX\}$ implies $N^+s\{t \mid N^+tX\}$. If $s \in S$, by N^+sX we have $Ns(X \cap S)$. Since ($\Box$4) holds in $\boldsymbol{M}$ we have that $Ns\{t \in S \mid Nt(X \cap S)\}$, and so $N^+s\{t \mid N^+tX\}$.

- ($\Box$5). Assume that ($\Box$5) holds in $\boldsymbol{M}$. To show that ($\Box$5) holds in $\boldsymbol{M}^+$, assume not $N^+s^{*^+}X$ for $X \in Prop$. If $s \in \{w, 0, 1\}$, we reason as in the case of ($\Box$4). If $s \in S$, then we reason as follows. By definition of $*^+$ and N^+, not $N^+s^{*^+}X$ means that not $Ns^*(X \cap S)$. Since ($\Box$5) holds in $\boldsymbol{M}$ we have that $Ns\{t \in S \mid not\ Nt^*(X \cap S)\}$, and so $N^+s\{t \mid not\ N^+t^{*^+}X\}$.

$\Box$

Lemma 10 (**L** to **CL**) *For each* **L***-model* $\boldsymbol{M}$ *there is a* **CL***-model* $\boldsymbol{M}'$ *such that, for all* $\varphi \in \mathcal{L}$*, if* $\boldsymbol{M}' \models \Box_L\varphi$*, then* $\boldsymbol{M} \models \varphi$*.*

Proof For all **BM.E**-models $\boldsymbol{M}$, if $\boldsymbol{M} \not\models \varphi$, then there is $s \in L$ such that $\boldsymbol{M}, s \not\models \varphi$. Then, (i) by Lemma 7 there is a **CBM.E**-model $\boldsymbol{M}^+$ and (ii) by Lemma 8 $\boldsymbol{M}^+, s \not\models \varphi$. Then, (iii) by construction of $\boldsymbol{M}^+$, $s \in N_L^+(W)$, hence $\boldsymbol{M}^+, w \not\models \Box_L\varphi$, by which we conclude that $\boldsymbol{M}^+ \not\models \Box_L\varphi$. Finally, by Lemma 9 steps (i)-(iii) can be performed for all **L**-models. $\Box$

We conclude this section with an observation making precise in which sense the family of logics considered in [49] constitutes a subclass of that considered in the present framework. We refer the reader to [50] for the definition of relational W-models and denote **BM.C** the class of all relational W-models. We recall that a neighborhood function N is *augmented* iff for all $s \in S$ (i) $X \subseteq Y$, $X \in N(s) \Rightarrow Y \in N(s)$ and (ii) $\bigcap N(s) \in N(s)$. We denote with **BM.E**^{a} the set of formulas valid in all W-models such that N is augmented.

Proposition 2 (Neighborhood-relational) **BM.E**^{a} = **BM.C**.

Proof Virtually as in [39]. For any **BM.C**-model $\boldsymbol{M} = (S, W, L, \leq, R, *, Q, N_L, V)$ (where $Q \subseteq S(\downarrow\uparrow)$ is the $\Box$-accessibility relation) we let $N(s) = \{X \mid Q(s) \subseteq X\}$ and $Prop = S(\uparrow)$ define the **BM.E**^{a}-model such that $\boldsymbol{M}' = (S, W, Prop, L, \leq, R, *, N, N_L, V)$ that $\boldsymbol{M}, s \models \varphi \Leftrightarrow \boldsymbol{M}', s \models \varphi$. Conversely, for any **BM.E**^{a}-model $\boldsymbol{M} = (S, W, Prop, L, \leq, R, *, N, N_L, V)$ we let $Q(s) = \{t \mid t \in \bigcap N(s)\}$ define the **BM.C**-model $\boldsymbol{M}' = (S, W, L, \leq, R, *, Q, N_L, V)$ such that $\boldsymbol{M}, s \models \varphi \Leftrightarrow \boldsymbol{M}', s \models \varphi$ (note that V, Q in $\boldsymbol{M}'$ are well defined since $Prop \subseteq S(\uparrow)$). $\Box$

4 Axiomatisation

In this section, we introduce the axiom systems for CL, the logic of agents reasoning according to the relevant logic L while being situated in a classical world. We show how L and CL interact, in the form of meta-rules, via the $\Box_L$ modal operator. Finally, we prove the main result of the paper, a modular soundness and completeness theorem for CL, for any axiom system L of Section 2.2, with respect to the class of **CL**-models.

4.1 The Axiom System CL

Definition 11 ($\Box_L$ versions) Let the $\Box_L$-version of an axiom φ (rule $\varphi_1, \ldots, \varphi_n \Rrightarrow \psi$) be obtained by prefixing the axiom (each of the premises and conclusion of the rule) with $\Box_L$, i.e. $\Box_L\varphi$ ($\Box_L\varphi_1, \ldots, \Box_L\varphi_n \Rrightarrow \Box_L\psi$).

Definition 12 (Axiom system) Let L be the axiom system for one of the relevant modal logics discussed in Section 2.2. We define CL as the axiom system comprising:

1. An axiom system CPC for classical propositional logic with (R1);
2. The $\Box_L$-version of each axiom and rule of inference of L;
3. The Bridge Rule (BR) $\Box_L(\varphi \to \psi) \Rrightarrow \varphi \to \psi$.

Theorem 3 (Soundness) *For all* L *and all* φ, *if* $\varphi \in Th(\mathsf{CL})$, *then* $\varphi \in \mathbf{CL}$.

Proof By induction on the length of CL-proofs, where we use Lemma 4 without mentioning it explicitly. As in previous proofs, we will write $[\![\varphi]\!]$ instead of $[\![\varphi]\!]_{\boldsymbol{M}_{\mathbf{CL}}}$ if $\boldsymbol{M}_{\mathbf{CL}}$ is clear from the context.

- All axioms (rules) of CPC are valid (preserve validity) in all **CL**-frames thanks to Lemma 6.
- The The fact that $\Box_L$-versions of propositional L-axioms are valid is established as follows. A formula $\Box_L\varphi$ is valid in a **CL**-model $\boldsymbol{M}_{\mathbf{CL}}$ iff $N_L(W) \subseteq [\![\varphi]\!]$, which by (16) means that $L \subseteq [\![\varphi]\!]$. For propositional L-axioms φ, the latter is shown as usual in relevant logic (cf. [44]).
- The fact that $\Box_L$-versions of propositional L-rules of inference preserve validity in **CL**-models is established similarly. As an illustration we show the case for $\Box_L$-(ER). Assume that $\Box_L\varphi$ is valid in $\boldsymbol{M}_{\mathbf{CL}}$; we need to show that $\Box_L((\varphi \to \psi) \to \psi)$ is valid in $\boldsymbol{M}_{\mathbf{CL}}$. To show this, assume that $W \subseteq [\![\Box_L\varphi]\!]_{\boldsymbol{M}_{\mathbf{CL}}}$ and pick $w \in W$, $s, t, u \in S$ such that $N_L ws$, $Rstu$ and $t \in [\![\varphi \to \psi]\!]$. We need to show that $u \in [\![\psi]\!]$. By (16) we have that $s \in L$ and by (2) we have that $t \leq u$, hence by Lemma 3 we have that $u \in [\![\varphi \to \psi]\!]$. By (ER), we have that $Ruxu$ for some $x \in L$. Then, since $l \in [\![\varphi]\!]$ for all $l \in L$ by the first assumption and (16), we conclude that $u \in [\![\psi]\!]$.
- The cases corresponding to the $\Box_L$-versions of the modal axioms and rules are established as usual in modal relevant logic (cf. [26]) thanks to Lemma 4. As an illustration, we show the case of $\Box_L$-($\Box$D).[13] Assume that $s \in [\![\Box\neg\varphi]\!]$ for some arbitrary $s \in S$. We may reason as follows: $Ns[\![\neg\varphi]\!]$ entails $Ns\neg[\![\varphi]\!]$ which entails that it is not the case that $Ns^*[\![\varphi]\!]$ (using ($\Box$D)) which entails $s^* \notin [\![\Box\varphi]\!]$ which entails $s \in [\![\neg\Box\varphi]\!]$. Hence, $[\![\Box\neg\varphi]\!] \subseteq [\![\neg\Box\varphi]\!]$, which entails that $\Box_L(\Box\neg\varphi \to \neg\Box\varphi)$ is valid by Lemma 4.
- Finally, the fact that (BR) preserves validity is established as follows. Assume $w \models \Box_L(\varphi \to \psi)$ for all $w \in W$. By Lemma 4 we have that $[\![\varphi]\!] \subseteq [\![\psi]\!]$. Hence, $w \models \varphi$ only if $w \models \psi$ for all $w \in W$. By (6) we conclude that $w \models \varphi \to \psi$ for all $w \in W$.

$\Box$

Lemma 11 (L-CL meta-rule) *For all* L, *the following hold:*

1. $\vdash_{\mathsf{L}} \varphi \iff \vdash_{\mathsf{CL}} \Box_L\varphi$;
2. $\vdash_{\mathsf{L}} \varphi \leftrightarrow \psi \implies \vdash_{\mathsf{CL}} \Box\varphi \leftrightarrow \Box\psi$.

[13] This case slightly differs from the one discussed in [26] since there a different frame condition is assumed to correspond to the axiom $\Box\neg\varphi \to \neg\Box\varphi$.

Proof The left to right implication in item 1 is established by induction on the length of L-proofs. If φ is an L-axiom, then $\Box_L\varphi$ is a CL-axiom by definition of CL. If φ is obtained by applying an L-inference rule to premises $\varphi_1, \ldots, \varphi_n$, then by the induction hypothesis we have that $\vdash_{\mathsf{CL}} \Box_L\varphi_1, \ldots, \vdash_{\mathsf{CL}} \Box_L\varphi_n$. Hence, using the $\Box_L$-version of the rule, we may infer $\vdash_{\mathsf{CL}} \Box_L\varphi$. Conversely, if $\nvdash_{\mathsf{L}} \varphi$, then $\boldsymbol{M} \not\models \varphi$ for some **L**-model $\boldsymbol{M}$. Then, by Lemma 10 there is a **CL**-model $\boldsymbol{M}'$ such that $\boldsymbol{M}' \not\models \Box_L\varphi$ and by Theorem 3 we conclude that $\nvdash_{\mathsf{CL}} \Box_L\varphi$. Item 2 follows from ($\Box$E), item 1, ($\Box_L$M) and (BR). □

It is crucial to note that the converse direction of the meta-rules from the following proposition does not hold. Note also that an immediate consequence of Lemma 11 is that $\vdash_{\mathsf{L}} \varphi \rightarrow \psi$ implies $\vdash_{\mathsf{CL}} \varphi \rightarrow \psi$, by the presence of (BR) in CL. In fact, we can establish a more general result that illustrates the relationship between L and CL.

Proposition 3 (Bridge) *For all* L *not containing* ($\Box$N), *the following hold:*

1. $\vdash_{\mathsf{L}} \varphi \implies \vdash_{\mathsf{CL}} \varphi$;
2. $\vdash_{\mathsf{CL}} \Box_L\varphi \implies \vdash_{\mathsf{CL}} \varphi$.

Proof Item 1 is established by induction on the length of L-proofs.

- All implicational axioms of L are provable in CL by Lemma 11 and (BR).[14] (X) is provable in any CL since the propositional fragment of each L is included in CPC. The cases of the induction step corresponding to rules with implicational conclusions are established using Lemma 11 and (BR).
- The case corresponding to (R1) is established using the induction hypothesis and the fact that (R1) is also an inference rule of CL.
- The case corresponding to (R2) is established using the induction hypothesis and the fact that $\vdash_{\mathsf{CPC}} \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$.
- The case corresponding to ($\Box$E) is established as follows. We assume that $(\Box\varphi \rightarrow \Box\psi) \wedge (\Box\psi \rightarrow \Box\varphi)$ is the last formula of an L-proof that contains also $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. By Lemma 11 (item 1), $\vdash_{\mathsf{CL}} \Box_L((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$, and so $\vdash_{\mathsf{CL}} \Box_L((\Box\varphi \rightarrow \Box\psi) \wedge (\Box\psi \rightarrow \Box\varphi))$ using $\Box_L$-($\Box$E). Since $\vdash_{\mathsf{CL}} \Box_L(\chi \wedge \chi') \rightarrow \Box_L\chi$ (using ($\Box_L$M), Lemma 11 and (BR)) and CL is closed under conjunction elimination, we infer that $\vdash_{\mathsf{CL}} \Box_L(\Box\varphi \rightarrow \Box\psi)$ and $\vdash_{\mathsf{CL}} \Box_L(\Box\psi \rightarrow \Box\varphi)$. Hence, (BR) allows us to infer that $\Box\varphi \rightarrow \Box\psi$ and $\Box\psi \rightarrow \Box\varphi$ are both provable in CL, which means that $\vdash_{\mathsf{CL}} \Box\varphi \leftrightarrow \Box\psi$.

Item 2 follows from item 1 and Lemma 11. □

We note that ($\Box$N) is problematic: φ is not necessarily an implication, so we can not use Lemma 11 and (BR); using induction hypothesis gives us only $\vdash_{\mathsf{CL}} \varphi$ and using only Lemma 11 gives us $\vdash_{\mathsf{CL}} \Box_L\varphi$, from which we can infer only that $\vdash_{\mathsf{CL}} \Box_L\Box\varphi$ using $\Box_L$-($\Box$N).

[14] The fact that *propositional* L-axioms are provable in CL follows from the fact that they are theorems of CPC. However, L contains also modal axioms which require an argument using Lemma 11 and (BR).

4.2 Completeness

In order to obtain a completeness result for CL with respect to **CL**-models via a canonical model construction, we introduce some standard definitions and preliminary results.

Definition 13 (Theories) Let (C)L $\in$ {L, CL}. A (C)L-*theory* is any set Γ of formulas such that (i) $\varphi \in \Gamma$ and $\psi \in \Gamma$ only if $\varphi \wedge \psi \in \Gamma$; and (ii) if $\varphi \in \Gamma$ and $\vdash_{\mathsf{(C)L}} \varphi \rightarrow \psi$, then $\psi \in \Gamma$. A (C)L-theory Γ is *prime* iff $\varphi \vee \psi \in \Gamma$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$; *proper* iff $\Gamma \neq \mathcal{L}$; *regular* iff $Th(\mathsf{(C)L}) \subseteq \Gamma$. A pair of sets of formulas (Γ, Δ) is (C)L-*independent* iff there are no finite non-empty sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{\mathsf{(C)L}} \bigwedge \Gamma' \rightarrow \bigvee \Delta'$.

Recall that a proper prime CL-theory is non-empty iff it is regular, and that Γ is a regular proper prime CL-theory iff it is a maximally CL-consistent theory (that is, a proper CL-theory such that $\varphi \notin \Gamma$ entails that the theory generated by $\Gamma \cup \{\varphi\}$ is non-proper).

Lemma 12 (Extension Lemma) *For all* L*:*

1. *If* (Γ, Δ) *is* L*-independent, then there is a prime* L*-theory* Σ *such that* $\Gamma \subseteq \Sigma$ *and* $\Delta \cap \Sigma = \emptyset$*;*
2. *If* (Γ, Δ) *is* CL*-independent and both* Γ *and* Δ *are non-empty, then there is a non-empty proper prime* CL*-theory* Σ *such that* $\Gamma \subseteq \Sigma$ *and* $\Delta \cap \Sigma = \emptyset$*.*

Proof Item (1) is established as follows. If $\Gamma = \emptyset$, then let $\Sigma := \emptyset$. If $\Gamma \neq \emptyset$ and $\Delta = \emptyset$, then let $\Sigma := \mathcal{L}$. If both Γ and Δ are non-empty, then use the standard “prime extension” argument (cf. [43, Theorem 5.17]). Item (2) is established similarly (note that Σ needs to be proper if $\Delta \neq \emptyset$). In fact, this is the well-known Lindenbaum Lemma. □

Definition 14 (Canonical model) The *canonical* CL*-frame* is the structure

$$\boldsymbol{F}^c_{\mathsf{CL}} = (S^c, W^c, Prop^c, L^c, \leq^c, R^c, *^c, N^c, N^c_L)$$

where:

- S^c is the set of all prime L-theories, where $0 := \emptyset$ and $1 := \mathcal{L}$;
- W^c is the set of all non-empty proper prime CL-theories;
- $Prop^c = \{[\varphi]_{\mathsf{L}} \mid \varphi \in \mathcal{L}\}$ for $[\varphi]_{\mathsf{L}} = \{s \in S^c \mid \varphi \in s\}$;
- L^c is the set of all regular prime L-theories;
- $\leq^c$ is set inclusion;
- $R^c stu$ iff $\forall \varphi, \psi$, if $\varphi \rightarrow \psi \in s$ and $\varphi \in t$, then $\psi \in u$;
- $s^{*^c} = \{\varphi \mid \neg\varphi \notin s\}$;
- $N^c s[\varphi]_{\mathsf{L}}$ iff $\Box\varphi \in s$;
- $N^c_L st$ iff, $\forall \varphi$, $\Box_L \varphi \in s$ only if $\varphi \in t$.

The *canonical* CL*-model* is $\boldsymbol{M}^c_{\mathsf{CL}} = (F^c, V^c)$ where $V^c(p) = [p]_{\mathsf{L}}$.

The canonical frame is well defined. First, $W^c \subseteq S^c$ since every CL-theory is an L-theory by Lemma 11 (item 1) and (BR). Second, the neighborhood function N^c is well defined since $[\varphi] = [\psi]$ iff $\vdash_{\mathsf{L}} \varphi \leftrightarrow \psi$. This equivalence follows easily from the definition of an L-theory and Lemma 12. In particular, $[\varphi] = [\psi]$ only if $\vdash_{\mathsf{L}} \varphi \leftrightarrow \psi$, which entails $\vdash_{\mathsf{L}} \Box\varphi \leftrightarrow \Box\psi$ by ($\Box$E), which entails $[\Box\varphi] = [\Box\psi]$.

Note the similarities between the definition of $\boldsymbol{M}^c_{\mathsf{CL}}$ and the canonical L-model of e.g. [26, 27]. In what follows we omit the superscript from $\boldsymbol{M}^c_{\mathsf{CL}}$ ($[\varphi]_{\mathsf{L}}$) whenever the context allows us to do that.

Lemma 13 (Canonical *Prop*) *For all* $f \in \{\wedge, \vee, \neg, \rightarrow, \Box, \Box_L\}$,

$$f^{\boldsymbol{F}^c_W}([\varphi_1], \ldots, [\varphi_n]) = [f(\varphi_1, \ldots, \varphi_n)] .$$

Proof The lemma is established by a standard case-by-case argument encountered in the relevant logic literature; cf. [22, 26, 43]. We give the argument for two cases as an illustration. First, $s \in \neg^{\boldsymbol{F}_W}[\varphi]$ iff (by definition of $\neg^{\boldsymbol{F}_W}$) $s^* \notin [\varphi]$ iff $\varphi \notin s^*$ iff (by definition of $*$) $\neg\varphi \in s$ iff $s \in [\neg\varphi]$. Second, $s \in \Box^{\boldsymbol{F}_W}[\varphi]$ iff $Ns[\varphi]$ (by definition of $\Box^{\boldsymbol{F}_W}$) iff $\Box\varphi \in s$ (by definition of N) iff $s \in [\Box\varphi]$. □

We proceed to show that $\boldsymbol{M}^c_{\mathsf{CL}}$ is a **CL**-model whenever the underlying axiom system is L. This requires a number of preliminary lemmas.

Lemma 14 $\boldsymbol{M}^c_{\mathsf{CL}}$ *is based on a bounded general frame.*

Proof It suffices to show the following:

- *Prop* $\subseteq S(\uparrow)$ and *Prop* is closed under $\boldsymbol{F}_W$'s frame operations. This holds by definition of $[\varphi]$ and $\leq^c$, and by Lemma 13.
- $\boldsymbol{F}_W$ is based on a frame. This is established by combining standard arguments pertaining to canonical models from the relevant modal logic literature (cf. [22, 26, 43], for instance). As an illustration, we show that $N \in S(\uparrow, \mathcal{P}(Prop)(\uparrow))$. Assume for some arbitrary $s, t \in S$ and $X \in Prop$ that $s \subseteq t$ and NsX. Then for some φ we have that $X = [\varphi]$. By $Ns[\varphi]$, we have that $\Box\varphi \in s$ and by $s \subseteq t$ we have that $\Box\varphi \in t$, hence $Nt[\varphi]$.
- The conditions on bounds are satisfied, which is established as follows. $0 \leq s \leq 1$ holds since $\emptyset \subseteq s \subseteq \mathcal{L}$. Equations (3)–(4) hold since $1^* = \{\varphi \mid \neg\varphi \notin 1\} = \emptyset = 0$ and $0^* = \{\varphi \mid \neg\varphi \notin 0\} = \mathcal{L} = 1$. To show that (5)–(6) hold, note that, by $X \in Prop$, there is φ such that $X = [\varphi]$. By definition of $\boldsymbol{F}_W$, we have that $\Box\varphi \in 1$ ($\Box\varphi \in 0$) for all (no) φ, hence we conclude using the definition of N that $N1X$ ($N0X$) for all (no) X. Equations (7)–(8) follow easily from the definition of N_L and $0, 1$. To show that (9) holds, assume by contradiction that $N_L 1s$ for some $s \neq 1$. Hence, there is $\varphi \notin s$, by which we conclude by definition of N_L that $\Box_L\varphi \notin 1$, which is a contradiction. Equation (10) follows easily from the definition of R and $0, 1$. Equation (11) holds since, if $R1st$ and, by contradiction, $s \neq 0$ and $t \neq 1$, then there is $\varphi \in s$ and $\psi \notin t$ such that $\varphi \rightarrow \psi \in 1$, which contradicts $R1st$.

□

Lemma 15 *W^c is a set of possible worlds.*

Proof We show that $\boldsymbol{F}^c_W$ satisfies conditions (12)–(16).

- Equation (12) holds since non-empty proper prime CL-theories are maximal CL-consistent theories, and so $\varphi \in s$ iff $\neg\varphi \notin s$.
- Equation (13) follows from $\vdash_{\mathsf{CPC}} (\varphi \wedge (\varphi \to \psi)) \to \psi$.
- Equation (14) is established as follows. Assume for some arbitrary $w \in W$ and $s, t \in S$ that $Rwst$ and $s \neq 0$; we have to prove that $w \subseteq t$. Thus assume for some arbitrary φ, ψ that $\varphi \in w$ and $\psi \in s$. Since $\vdash_{\mathsf{CPC}} \varphi \to (\psi \to \varphi)$, we have that $\psi \to \varphi \in w$, and so $\varphi \in t$ by the definition of R. Since φ was arbitrary, we established that $w \subseteq t$.
- Equation (15) is established as follows. Assume for some arbitrary $w \in W$ and $s, t \in S$ that $Rwst$ and $t \neq 1$; we have to prove that $s \subseteq w$. Hence, assume by contradiction for some arbitrary $\varphi \in s$ that $\varphi \notin w$. Hence, since w is maximal we have that $\neg\varphi \in w$ and by $t \neq 1$ we have that $\psi \notin t$ for some ψ. Since $\vdash_{\mathsf{CPC}} \neg\varphi \to (\varphi \to \psi)$, we have that $\varphi \to \psi \in w$, and so we conclude that $\psi \in w$ using the definition of R. But this is a contradiction.
- Equation (16) is established as follows. First, to show that $N_L(W) \subseteq L$, take some arbitrary $s \notin L$. Hence, $\varphi \notin s$ for some $\varphi \in Th(\mathsf{L})$. By Lemma 11 we have that $\vdash_{\mathsf{CL}} \Box_L\varphi$, hence $\Box_L\varphi \in w$ for all $w \in W$, which implies that there is no $w \in W$ such that $N_L ws$. To prove that $L \subseteq N_L(W)$, assume that $s \in L$. If $s = 1$, then there is $w \in W$ such that $N_L ws$ by (8), the fact that CL is consistent (by Theorem 3), and Lemma 12(item 2). If $s \neq 1$, then we reason as follows. The pair $\big(Th(\mathsf{CL}), \{\Box_L\varphi \mid \varphi \notin s\}\big)$ contains non-empty sets and is CL-independent. If it were not, then

  - $\vdash_{\mathsf{CL}} \bigvee_{i<n} \Box_L\varphi_i$ for some $n > 0$, hence
  - $\vdash_{\mathsf{CL}} \Box_L \bigvee_{i<n} \varphi_i$ by $\vdash_{\mathsf{CL}} \Box_L\varphi \vee \Box_L\psi \to \Box_L(\varphi \vee \psi)$, hence
  - $\vdash_{\mathsf{L}} \bigvee_{i<n} \varphi_i$ by Lemma 11, hence
  - $\bigvee_{i<n} \varphi_i \in s$ since $s \in L$, which entails that
  - $\varphi_i \in s$ for some $i < n$ since s is prime.

  This is a contradiction, so the pair has to be CL-independent. It follows from the Extension Lemma 12 that there is a non-empty proper prime CL-theory w such that $N_L ws$.

□

Lemma 16 *If* L *is obtained by adding a set of axioms and rules X from* Fig. 1 *to* BM.E, *then the frame conditions corresponding to X hold in $\boldsymbol{F}^c_W$.*

Proof This claim is established by combining standard canonicity arguments found in the literature on relevant modal logic; cf. [22, 26, 43]. □

Lemma 17 (Canonical model) *For all* L, *$\boldsymbol{M}^c_{\mathsf{CL}}$ is a* **CL**-*model.*

Proof By Lemma 14, $\boldsymbol{M}_{\mathsf{CL}}$ is based on a bounded general frame. By Lemma15 $W \subseteq S$ is a set of possible worlds. By Lemma 16 that for each frame condition Φ in Fig. 1 corresponding to specific axioms or rules of L, Φ holds in $\boldsymbol{F}_W$. This concludes the proof. □

Lemma 18 (Truth) *For all* L *and all* $\varphi \in \mathcal{L}$, $\boldsymbol{M}^c_{\mathsf{CL}}, s \models \varphi$ *iff* $\varphi \in s$.

Proof By induction on the complexity of φ. The base case follows from the definition of V^c and the induction step follows from Lemma 13. □

Theorem 4 (Soundness and Completeness) *For all* L, $Th(\mathsf{CL}) = \mathbf{CL}$.

Proof One direction is Theorem 3, the other one follows from Lemmas 17 and 18. □

5 Conclusion

In this article we developed a neighborhood-style semantic framework for modal epistemic logics capturing the idea that objects of epistemic attitudes are fine-grained *relevant propositions* that are individuated using relevant logic instead of classical logic. At the same time, the propositional fragment of the logics generated by our framework is classical. Hence, these logics are combinations of classical and relevant logic where the latter is confined to the scope of modal operators. Our main technical result is a modular soundness and completeness theorem covering a wide range of logics. The semantic framework presented here generalizes the relational semantic framework of [49].

Interesting problems for future research include the development of multi-agent extensions of our logics with group operators for common and distributed epistemic modalities and different forms of dynamic updates, in the style of [41].

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