Classe di Lettere e Filosofia

Corso di perfezionamento in

Filosofia

XXXV ciclo

Through and beyond classicality: analyticity, embeddings, infinity

Settore scientifico disciplinare: M-FIL/02

Candidato: dr. Matteo TESI

Relatore: Prof. Mario Piazza

Anno accademico: 2022/2023

Contents

Chapter 1

Structure, representation and analyticity

In this chapter we provide a bridge from the first part of the thesis and the second one. The intention is to discuss structural proof theory and its relation to the central topic of analyticity and, in particular, the analytic content of logic.

KEYWORDS. Cut-elimination and analyticity; extensions of sequent calculi; proofs and proof representation.

1.1 Reflecting on proofs

The work of a mathematician centers around the activity of proving theorems. From a pragmatic and descriptive standpoint, a mathematical proof can be conceived of as a procedure which is considered reliable by the community of the mathematicians. The great success of mathematical proofs as a mean to propagate information can be thus assimilated to its claim of universal validity.

Indeed, nowadays mathematical proofs are considered to be a rigorous method in order to communicate and transmit mathematical knowledge. Another relevant aspect of mathematical proofs, which is connected to the claim of universal validity, is the uniformity of the rules employed in formal derivations. The rules are uniform in the sense that they follows the inferential pattern accepted in classical logic and therefore they are common to any part of mathematics.

Hence, by assuming a descriptive approach, we are led to conceive of proofs as an object which has a communicative function and thus a linguistic nature. In fact, proofs are finite strings of symbols which connect certain premises to a conclusion preserving the validity. It is then unsurprising to realize that proofs are a central object of inquiry in logic.

David Hilbert - one of the greatest mathematician of the XX^{th} century - was perfectly aware of the relevance of logic in the study of mathematical proofs. He is responsible for the conceptual shift which initiated the field of studies of *metamathematics*. In particular, metamathematics is devoted to investigate the properties of formal systems and the notion of proof itself. Hilbert's work in logic was motivated by ontological concerns as he identified the existence of a given mathematical object with a proof of consistency, i.e. the unprovability of contradictions.

As it is well-known, Hilbert program - at least in its original and full-fledged form - failed due to the incompleteness phenomena highlighted by Gödel. In particular, the second incompleteness theorem precisely showed that the consistency of arithmetic was among the propositions B such that:

$$\mathsf{PA} \nvdash B \text{ and } \mathsf{PA} \nvdash \neg B$$

However, the investigations promoted by Hilbert were carried on by some of its students. (?) Two main routes were followed. The first one directly stems from Hilbert's original program.

Once one accepts the impossibility of obtaining a purely finitary proof of consistency of a theory with a *modicum* of arithmetic, a proof of consistency can be seen as a way to measure the strength of a certain theory. Roughly speaking, proofs of consistency are obtained via induction principles on specific well-orderings. In this sense one can associate an ordinal to a theory: this line of research is known as *ordinal analysis*.

The second main research direction is referred to as *structural* proof theory. In accordance to the name, structural proof theory investigates the structure of formal proofs and their properties. The investigations on the structure of proofs and on their properties follow various directions. For example:

- The analysis of the constructive content of classical proofs;

- The study of the formal properties of a systems, such as the admissibility of a given rule.

- The development, the analysis and the implementation of calculi for modal and non-classical logics.

- The introduction of new formalisms and suitable generalizations of methods to represent proofs.

These four areas are linked together by a uniform methodology. In particular, the investigations often make use of analytic calculi. Although the notion of analyticity is central in the field of proof theory, the definition of analyticity is neither unique nor unproblematic (87). The concept of analysis is ancient and has a glorious philosophical tradition. Essentially, there are two ways to conceive of analyticity. On the one hand, analyticity can be identified with a lack of information. On the other hand, analyticity could be understood as a method of proof or reasoning.

The first view can be thought as a static or definitional account of analyticity, whereas the second one is dynamic, as it is connected with the way proof are construed and structured.

In this latter sense, a further distinction has to be made, depending on whether the concept applies to a logic, a formal system or a specific proof. A typical criterion to establish the analyticity of a calculus is the so called *subformula property*:

> Every formula occurring in a derivation is a subformula of a formula in the conclusion.

The present chapter, broadly conceived, has three different aims:

1. Analyze and clarify the notion of analyticity in a logical setting.
2. Make explicit the relevance of the notion in the field of logic and, specifically, of proof theory.
3. Understand the interplay between the structure and the representation of proofs and analyticity.

To start with, we shall discuss the first item. In particular, we wish to propose a new definition of analyticity in a logical setting.

In this context, a definition can be thought of as a kind of biconditional. In our case, according to the subformula property criterion, a logic is *analytic* if and only if every derivation has an analytic presentation, i.e. there is a derivation enjoying the subformula property. We claim that this is neither a necessary nor a sufficient criterion.

First, it is not a necessary criterion. In fact, it is reasonable to assume that classical first-order logic is analytic, yet its presentation does not satisfy the subformula property, unless we stipulate the principle:

$A(t)$ is a subformula of $Qx.A$ for every term t of the language.

However, this is rather counterintuitive, as the term t is not a proper part of of the formula $Qx.A$.

Proof-theorists often tend to consider $A(t)$ as a subformula of $Qx.A$ insofar as it still allows for a predicative proof-theoretical treatment of cut-elimination. Indeed, eliminating cuts on universal formulas is possible insofar as the number of connectives does not increase in the normalization procedure. However, we argue that this is true only if we collapse the notion of subformula on the one of logical complexity.

Second, the criterion is not sufficient. In fact, we argue that there are calculi which enjoy a full-fledged subformula property, but they cannot be regarded as fully analytic. Consider the sequent calculus for classical propositional logic by Gentzen, the system **LK** (39). Of course, the law of excluded middle $P \vee \neg P$ has to be derivable in the form of the sequent $\Rightarrow P \vee \neg P$. The naïve attempt at constructing a derivation via a root-first application of logical rules is doomed to fail.

$$\dfrac{\dfrac{P \Rightarrow}{\Rightarrow \neg P}\,\mathrm{R}\neg}{\Rightarrow P \vee \neg P}\,\mathrm{R}\vee$$

The key point is that a derivation can be obtained if we - looking at the proof bottom-up - duplicate the formula $P \vee \neg P$. Indeed, a step of the contraction rule needs to be performed:

$$\dfrac{\dfrac{\dfrac{\dfrac{P \Rightarrow P}{\Rightarrow P, \neg P}\,\mathrm{R}\neg}{\Rightarrow P \vee \neg P, \neg P}\,\mathrm{R}\vee}{\Rightarrow P \vee \neg P, P \vee \neg P}\,\mathrm{R}\vee}{\Rightarrow P \vee \neg P}\,\mathrm{RC}$$

The resulting derivation surely satisfies a subformula property as all the formulas occurring within the derivation are subformulas of a formula in the conclusion. However, it is clear that a new piece information has been introduced looking bottom-up. Indeed, since standard sequent calculi presentations use multisets, the multiplicity of formulas is a relevant parameter.

Also, the step of contraction required in order to properly construct the derivation is not contained by the sequent $\Rightarrow P \vee \neg P$. In particular, the only displayed symbols are P, $\vee$ and $\neg$ which do not indicate the need to duplicate the information of the conclusion in the premise. Therefore we argue that the derivation contains an external piece of information with respect to the conclusion.

We have observed that that the equation:

$$\text{analyticity} = \text{subformula property}$$

is not satisfactory, therefore we would like to propose an alternative account of analyticity. Our proposal stems from the practice of proving theorems and formulas within a formal system. As a provisional attempt we define analyticity as a property of derivations:

A derivation $\mathcal{D}$ is analytic if it can be obtained via the automatic backward application of rules.

We immediately face a difficulty concerning the extension of the predicate of analyticity. In fact, we have just given a definition of an analytic derivation. We deem that it is natural to extend the definition so as to cover also full logical systems. However, the example that we have given above (the case of **LK**) shows that not all sequent-style presentations of classical propositional logic can be regarded as analytic.

This suggests that different calculi for the same logic may or may not be analytic. In particular, the **G3** presentation is analytic, whereas the one by Gentzen is not analytic according to the newly introduced definition, as it fails to eliminate the rule of contraction. We fix the following definition for analyticity for a logic:

A logic is analytic if there is a calculus in which every derivation is analytic.

This shows that our new definition of analyticity strongly depends on fine-grained properties of calculi which are in turn specific properties of a logic. In particular, to assess the analyticity of a certain calculus one needs to take into account the following three parameters:

1. The structure of the objects manipulated in a derivation.
2. The structure or the form of a derivations considered as formal objects.
3. The design of the rules.

We shall see that depending on the variation of these points the analyticity of a calculus changes accordingly.

1.2 Axioms and lines

At the end of the XIXth century the axiomatic method became the prominent approach to formalize mathematics. Its fortune was determined by the request of rigour which seemed to be attainable via the individuation of a minimal set of axioms for a given area of mathematics. An axiomatic system can be described as a finite set of axioms (or axiom schemata) equipped with a set of rules (usually a small one) which allows to combine specific instances of the axioms to obtain proofs.

To give an example, we recall here the axioms and the rule of classical propositional logic.

C

Axioms

1.1 $\vdash A \rightarrow (B \rightarrow A)$

1.2 $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

2.1 $\vdash A \wedge B \rightarrow A$ 2.2 $A \wedge B \rightarrow B$

2.3 $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B \wedge C))$

3.1 $\vdash A \rightarrow A \vee B$ 3.2 $B \rightarrow A \vee B$

3.3 $\vdash (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$

4.1 $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$

4.2 $\vdash A \rightarrow (\neg A \rightarrow B)$

4.3 $\vdash A \vee \neg A$

Inference Rules

$$\frac{\vdash A \qquad \vdash A \rightarrow B}{\vdash B}\,\text{MP}$$

As it is well-known, proving formulas in an axiomatic calculus can be a daunting task. In fact, when faced with the problem:

Is the formula B provable in the axiomatic system $\mathbf{C}$?

there is not a privileged route to follow. Indeed, if B is not a specific instance of an axiom schema, we need to assume, by inspection of the calculus, that it has been obtained by an application of the rule of modus pones of the shape:

$$\frac{\vdash A \to B \qquad \vdash A}{\vdash B}$$

This proof configuration is not revealing with respect to the structure of A which could be extremely different from B and possibly more complex. To give a concrete example of this qualitative argument, let us a consider a proof of the double negation law in the axiomatic system.

1. $\vdash (P \to (\neg\neg P \to P)) \to ((\neg P \to (\neg\neg P \to P)) \to (P \vee \neg P \to (\neg\neg P \to P)))$
2. $\vdash P \to (\neg\neg P \to P)$
3. $\vdash (\neg P \to (\neg\neg P \to P)) \to (P \vee \neg P \to (\neg\neg P \to P))$
4. $\vdash \neg P \to (\neg\neg P \to P)$
5. $\vdash P \vee \neg P \to (\neg\neg P \to P)$
6. $\vdash P \vee \neg P$
7. $\vdash \neg\neg P \to P$

Observe that the formula which occupies the first line of the derivation:

$$\vdash (P \to (\neg\neg P \to P)) \to ((\neg P \to (\neg\neg P \to P)) \to (P \vee \neg P \to (\neg\neg P \to P)))$$

is way more complex than the conclusion.

Trying to frame axiomatic calculi in the categories that we spelled out in the previous section we observe that (i) they manipulate formulas of the language, (ii) they lack structure as they are essentially linear derivations and (iii) there are few rules and prominence is given to axioms and their instances.

As a consequence, the axiomatic calculus is well-suited to characterize the notion of logical theorem, i.e. derivation without assumptions, but fails to give an adequate representation of the notion of derivability. Via the soundness and completeness theorems, theoremhood is seen to correspond faithfully to the semantic notions of logical truth.

It can thus be argued that the axiomatic reasoning does not represent adequately the notion of logical consequence. As pointed out in (113), axiomatic logic fails to capture naturally hypothetical reasoning. Furthermore, proving in an axiomatic (or Hilbert style) system in a sense amounts to studying an algebraic structure and its ordering properties. Although being apparently extremely poor in terms of structure and extremely close to the syntax of the logic, a Hilbert style system can be immediately equipped with a certain algebraic structure.

There is a close interplay between algebraic semantics, metalogical reasoning and axiomatic calculi. In particular, starting from a new logical symbol, one can introduce an axiom governing the behaviour of the new connective and finally consider a corresponding a structure satisfying an inequality.

Axiomatic calculi are thus synthetic systems of proof as derivations cannot be constructed bottom up and they are closely related to an underlying algebraic structure. Such structure directly stems from the intuition of considering n-ary operations in a one-to-one correspondence with the connectives of the logic. In this sense, axiomatic reasoning is close to algebraic reasoning.

1.3 Rules and trees: shaping mathematical reasoning

As hinted at in the introductory section of the present chapter, Gerhard Gentzen irreversibly changed the landscape of proof theory. His principal contribution to the field of structural proof theory was the introduction of two new formalisms for classical and intuitionistic logic. In particular, the two systems were natural deduction and the sequent calculus.Natural deduction falls partially out of the scope of our discussion, therefore we limit ourselves to highlighting some peculiar features of the system.

To start with, natural deduction is a calculus which - contrarily to Hilbert style systems - is mainly based on inference rules. In particular, every connective is equipped with an introduction and an elimination rule: the first kind of rules specifies the meaning of the connective, whereas the second one explains the use of it. The rules manipulate formulas of the language and derivations have a tree-like structure which can be given a kind of normal form.

However, it can be argued that natural deduction exhibits some weaknesses from the standpoint of proof-theoretic reasoning and it fails to offer an analytic presentation of a given logic in the sense specified above. In fact, proofs in natural deduction work with sets of formulas rather than multisets and thus they are not sensible to the multiple application of hypothesis. Also, the rules in natural deduction are not always local, they can be global in the sense that a certain criterion needs to be fulfilled in order to apply them. The rule of implication is a paradigmatic example of this global-like behaviour:

$$\dfrac{\begin{matrix}[A]^1 \\ \vdots\, \mathcal{D} \\ B\end{matrix}}{A \to B}\ {\scriptstyle \mathrm{I}\to,\, 1}$$

This feature of the calculus has two unpleasant consequences. First, it complicates the inductive structure of derivations, thus making the investigation of the metalogical properties of the system less perspicuous. Second, it prevents the possibility to apply a kind of backward reasoning. Indeed, let us consider a derivation of the Frege law in natural deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{3}{A \to (B \to C)} \quad \overset{1}{A}}{B \to C}\ {\scriptstyle \mathrm{E}\to} \qquad \dfrac{\overset{2}{A \to B} \quad \overset{1}{A}}{B}\ {\scriptstyle \mathrm{E}\to}}{C}\ {\scriptstyle \mathrm{E}\to}}{A \to C}\ {\scriptstyle \mathrm{I}\to,\, 1}}{(A \to B) \to (A \to C)}\ {\scriptstyle \mathrm{I}\to,\, 2}}{(A \to (B \to C)) \to ((A \to B) \to (A \to C))}\ {\scriptstyle \mathrm{I}\to,\, 3}$$

Notice that the derivation is composed in two stages. First, we apply the introduction rules root-first, discharging the antecedents of the implications, and then we reason applying the elimination rules to the hypothesis. This latter section of the derivation requires to apply a certain degree of synthetic reasoning.

The sequent calculus overcomes the difficulties of natural deduction in automating the process of the construction of the proof. The crucial point is the choice of the basic syntactic object to manipulate. Indeed, when working with sequent calculi, one does not work with formulas anymore, but with sequents. Now, a sequent is a syntactic object of the shape:

$$\Gamma \Rightarrow \Delta$$

where Γ and Δ are finite multisets (or lists) of formulas. This has the key advantage to allow for reasoning on more complex structures and to manipulate implications rather than single formulas.

One of the immediate upshots is that every rule is now local and every connective is equipped with a pair of symmetric rules: one which acts on the left hand side of the sequent arrow and the other on the right hand side. In sequent calculi derivations are finite trees in which every node is labelled by a sequent and which are built according to the rules of the calculus. Also, the multiplicity of formulas is now controlled as we are now working with multisets of formulas, in which the number of occurrences of a formula counts.

Let us now consider a proof of the same formula - the law of Frege - in a sequent calculus:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \Rightarrow A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \Rightarrow A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$C \Rightarrow C$}
\RightLabel{L$\rightarrow$}
\BinaryInfC{$B \rightarrow C, B \Rightarrow C$}
\RightLabel{L$\rightarrow$}
\BinaryInfC{$A \rightarrow (B \rightarrow C), B, A \Rightarrow C$}
\RightLabel{L$\rightarrow$}
\BinaryInfC{$A \rightarrow (B \rightarrow C), A \rightarrow B, A, A \Rightarrow C$}
\RightLabel{RC}
\UnaryInfC{$A \rightarrow (B \rightarrow C), A \rightarrow B, A \Rightarrow C$}
\RightLabel{R$\rightarrow$}
\UnaryInfC{$A \rightarrow (B \rightarrow C), A \rightarrow B \Rightarrow A \rightarrow C$}
\RightLabel{R$\rightarrow$}
\UnaryInfC{$A \rightarrow (B \rightarrow C) \Rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$}
\RightLabel{R$\rightarrow$}
\UnaryInfC{$\Rightarrow (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$}
\end{prooftree}
$$

The most striking property of Gentzen's calculus is the fact that the rule:

$$
\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \text{Cut}
$$

expressing the transitivity of implication and embodying synthetic reasoning in the form of modus ponens, is redundant. Every derivation containing applications of the cut rule can be effectively transformed (in the sense that a primitive recursive operator can be defined) into a derivation which does not contain applications of cut.

In the context of propositional logic, the elimination of cut stresses the importance of the subformula property, because all the remaining rules enjoy the subformula property. As we have remarked above, a subformula property in the case of first-order classical logic is possible only with the caveat of identifying $A(t)$ as a subformula of $Qx.A$.

1.4 Going beyond Gentzen

So far we have been dealing with sequent calculi for classical logic which can be regarded as a paradigmatic example of analyticity. In particular, the **G3**-style calculus (109) for classical propositional logic fully satisfies the requirements of our alternative definition. It does so by i) working with sequents instead of formulas, ii) absorbing the structural rules within the logical ones and iii) having a tree-like structure. Indeed, the cut rule is admissible and so synthetic reasoning is not required in order to find derivations of sequents. Furthermore, the contraction rules can be dispensed with therefore allowing for a root-first approach.

However, the logical landscape is not limited to classical logic. Indeed, a high number of alternative logics have flourished in the XXth century. The introduction of non-classical logics has essentially been motivated by two different kind of criticisms against classical reasoning. The first is *normative* and a clear example is intuitionistic logic. Intuitionistic logic codifies a kind of mathematical reasoning, just like classical logic does, but a radically alternative one. In this sense, classical logic is simply wrong as it equates existence with consistency of an existential assumption in the eyes of an intuionistic mathematician (by the equivalence $\exists x A \leftrightarrow \neg\neg\exists x A$).

Other logicians deviated from classical reasoning as they did not consider it fit to model certain inferential patterns. In these cases the criticism against classical logic is contextual because it is relative to a specific domain of application. For example, there are circumstances under which the multiple applications of an hypothesis is relevant (as in the case of linear logic (40)).

Non-classical logics have posed a significant problem to researchers in the area of structural proof theory. Indeed, the sequent calculus is surely a flexible tool to accommodate classical, intuitionistic and substructural logics, but fails to encompass well-known logical systems which enjoy a perspicuous semantic or axiomatic presentation (a case is the modal system **S5** which can be proved to be complete with respect to analytic cuts (101)).

The tendency of the researchers in the field was to construct *ad hoc* calculi which could work for a specific logic. However, a paradigm shift changed the perspective on the problem. In particular, instead of adopting an abductive approach which tried to add new rules depending on the shape of the axioms, the base syntactic structure was modified. In the words of Blamey and Humberstone (6):

> This strongly suggests that the move from truth-functional to modal logic is not one best made simply by adding a new primitive connective with new rules governing it, but rather by extending one's conception of the objects to be manipulated by such rules.

This intuition was particularly fruitful and perfectly shows how the syntax and the shape of the derivations is crucial in order to define analytic proof systems.

A first natural generalization of sequent calculi consists in manipulating not a single sequent, but a finite multiset of sequents. A *hypersequent* is an object of the shape:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

Hypersequents have been extensively studied by various authors including (1; 18). They have shown to be particularly suited to model intermediate logics, especially when linearity axioms are added to the base system. Moreover, they proved to be a flexible tool also in the field of many-valued and substructural reasoning.

Hypersequent system usually maintain a tree-like structure, but they manipulate more complex objects. The advantage is the possibility to interpret the structural connective | as a linguistic symbol. Typically, the hypersequent bar admits a disjunctive reading and therefore hypersequents are promising to offer a proof-theoretic treatment of axioms containing disjunctions.

In particular, the right disjunction rule is not invertible in the usual presentations of intuitionistic logic, but it can be regained by the addition of parallel components. For example, the axiom $(P \to Q) \vee (Q \to P)$ can be decomposed into the hypersequent:

$$P \Rightarrow Q \mid Q \Rightarrow P$$

Also, the presence of multiple components naturally suggests the introduction of structural rules which act on them by modifying their structure and rearranging the position of the multisets of formulas occurring therein.

A peculiar feature of the hypersequent calculi is that structural rules now come in pairs. For each structural rule there is an internal and an external variant. While the internal ones are the familiar structural rules, the external ones modify the structure of the hypersequent. In this sense, the rule of external contraction is particularly relevant as it duplicates an entire sequent:

$$\frac{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\text{EC}$$

It is to be remarked that the explicit presence of the rule of external contraction hinders the analyticity of hypersequent calculi, as it could be applied indefinitely many times thus allowing duplications of entire sequents.

Hypersequents are a very natural generalization of standard sequent calculi. One can further enrich the syntactic structure by imposing other properties. For example, by working with lists instead of multisets, one obtains linear nested sequents, which are finite lists of sequents in which the order counts (57). Finally, one can consider nested sequent calculi, in which every node in the derivation is labelled by a tree of sequents (11; 88). Nested sequents have given cut-free sequent systems for several modal logics and more recently also intuitionistic and intermediate logics. We also mention the so-called bunched sequents which add some structural connectives distinguishing between ; and , assigning an additive or a multiplicative reading.

A completely different approach consists in maintaining the familiar structure of sequents without adding extra structure but enriching the base language. The display calculi fit into this category, because they internalize a kind of algebraic semantics by properly extending the language of the calculus (3). The display systems satisfy a very general cut-elimination theorem that holds whenever some requirements are met. The latter can be easily checked by inspection of the rules of the calculus.

Another formalism which builds on a proper extension of the language of a given logic is labelled deduction. Labelled proof systems explicitly internalize semantic elements in the syntax (112; 73). In this way semantic properties are mimicked by the syntax of the calculus. The labelled approach offered a uniform formulation of analytic calculi for modal logics. In particular, every modal and non-classical logic whose semantics can be expressed in a first-order language can be given a labelled sequent calculus which is analytic and preserves the usual structural properties, i.e. admissibility of the rules of weakening, contraction and cut.

The reason behind the success of display and labelled systems lies in the preservation of the relatively simple structure of sequent calculus. This greatly simplifies the investigations of the metalogical properties of the calculi. The main conceptual and techinical consequence is that the target logic is - so to say - embedded in a language which is a conservative extension of it. Indeed, labelled sequent calculi are not *prima facie* a tool to reason on a logic, but rather on its model theory.

The rules directly stem from the truth conditions for a formula, for example the rules for the modal operator $\Box$ immediately follow by the condition:

$$w \Vdash \Box A \Longleftrightarrow \forall u(wRu \Rightarrow u \Vdash A)$$

where the quantifier $\forall$ has a metalogical reading. Whenever the model theory is spelled out in a first-order language, one can exploit the well-known properties of first-order classical logic to study the logic. In particular, labelled sequent calculi enjoy the invertibility of every rule because first-order logic does and the same goes for the structural properties.

However, since the language mirrors the model theory of the logic, asking whether a formula holds in every world of an arbitrary the model amounts to test its validity. We get the following correspondence result: for every logic **X** with frame properties which can be expressed in a first-order language, we have

$$\mathbf{X} \vdash A \Longleftrightarrow \mathbf{G3X} \vdash \Rightarrow w : A$$

where **G3X** is the corresponding labelled sequent system. Therefore, a certain fragment of labelled sequents is enough to characterize the derivability in the given logic.

1.5 The drawbacks of generalizations

We have proposed an overview of the main extensions of the sequent calculus. The key modification to obtain these enhanced systems consists in a generalization of the base syntactic element to be manipulated. On the contrary, the tree-like structure is maintained and the design of the rules varies according to the framework under consideration.

If Hilbert systems work with single formulas and the sequent calculus contains rules which act on implications, the hypersequent calculi deal with disjunctions of implications. The addition of extra-structure allows to model more complex syntactic configurations and increases the number of invertible connectives.

However, a crucial remark is in order. Is the growing complexity of the structure always a welcome addition to the toolbox of structural proof theory? The question is central in order to critically evaluate a large portion of the scientific production in the field of modern structural proof theory. To answer, we need to look at the matter both from a conceptual and a technical point of view.

From a conceptual and philosophical perspective, proof theory has always aimed (at least in purpose) to give a faithful account of logical reasoning (49).

> The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds.

Also, as Gentzen himself speaking about natural deduction notes (39):

> My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return.
>
> In contrast, I intended to set up a system which comes as close as possible to actual reasoning.

The move towards calculus of increased complexity sets a distance between actual reasoning and the syntactic machinery. While manoeuvring formulas or sequents can be conceived as a rather intuitive activity, the same cannot be said with respect to hyper-, nested, display or labelled sequents. These system exhibit a less perspicuous connection with the logic they axiomatize.

There is, however, another worrying aspect which is essentially technical. The more complex a structure gets, the harder it is to analyze the properties of the underlying logic. If the addition of extra-structure often simplifies the analysis of metalogical properties of systems, even proofs of simple such properties can prove to be extremely involved. Also, the elimination of the rule of cut - an essential step towards the analyticity of a system - may be less revealing than in the case of standard Gentzen-style formalisms. The key point is that while the new structure provides the desired analyticity requirement, it needs to be carefully handled and removed in order to unveil the properties of the logic.

To give a concrete example of this qualitative argument, we focus on the case of the disjunction property for intuitionistic propositional logic. It is common knowledge that the cut-elimination theorem for the sequent calculus for intuitionistic logic yields a straightforward proof of the disjunction property, i.e.:

$$\vdash A \vee B \Longrightarrow \quad \vdash A \ or \ \vdash B$$

In fact, once a proof of $\Rightarrow A \vee B$ is obtained, the inspection of the rules implies that the last rule applied has to be the rule R$\vee$ which immediately leads to the desired conclusion.

Working with an hypersequent calculus for intuitionistic logic this is no longer the case. Indeed, the last rule applied could be in principle an instance of external contraction:

$$\dfrac{\begin{matrix}\vdots\\ \Rightarrow A \vee B \mid \ \Rightarrow A \vee B\end{matrix}}{\Rightarrow A \vee B}\ \text{EC}$$

To obtain a proof of the disjunction property it is mandatory to establish the validity of a preliminary lemma which asserts that the derivability of each component of the hypersequent in the calculus is independent, therefore establishing the redundancy of the hypersequential structure for intuitionistic propositional logic.

1.6 Analyticity and explanations

So far, we have described the interplay between the representation of proofs articulated according to their shape, the objects which are manipulated and the design of the rules. We have also explained in which sense analyticity is important for the working proof-theorist explaining the advantages it offers.

However, we still have not discussed the relevance of the notion of analytic proof for logic. Indeed, logic has often been considered analytic. According to our definition of the notion, a logic is analytic if it can be framed into a suitably formulated sequent calculus with certain properties. There is an illustrious philosphical tradition which conceives logic as *tout court* analytic, by equating analyticity with the lack of information.

We would like to maintain the analyticity of logic, especially with respect to classical propositional and first-order logic, but we reject the idea that logical reasoning is uninformative (for an extended discussion on the issue the reader can consult (7)). Indeed, an analytic proof is not - under many respects - the most common presentation of a proof or of an argument. In these contexts the use of synthetic proof methods is often employed for reasons of clarity and conciseness. The use of modus ponens and the argumentation by lemmata is a key ingredient in mathematical proofs. Also, results in the field of structural proof theory and on the length of proofs have showed that analytic proofs can be exponentially longer that proofs containing cuts (97).

Therefore we would like to offer a completely opposite take on the problem. Analyticity is relevant for logic as it precisely shows its informative content. An analytic proof can be seen as the unfolding of a standard mathematical proof in which all the information and the concepts involved in it are explicitly brought to the fore. In this procedure the flow of information in the proof and the interaction between the parts which are independent in a proof containing lemmata becomes evident.

Furthermore, analyticity of the proofs ties to another venerable problem in the field of proof theory, namely simplicity of proofs. The relevance of the theme was stressed by Hilbert who had formulated a 24^{th} problem which was not presented in his lectures in Paris (50):

> The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally,

> if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs.

In a sense, out notion of analyticity is strongly related to a criterion of simplicity of proof. Indeed, the possibility of a bottom-up approach to the search for a proof indicates a simple way to construct a proof which consists in a backward application of the rules within a calculus until all the leaves of the proofs are labelled by initial sequents. Indeed, analyticity entails the fact that each inferential step in the derivation is justified on the base of the shape of the conclusion. Hence analyticity can provide a criterion of simplicity in terms of construction of a proof (although it could be in conflict with the difficulty in reading and interpreting it).

1.7 On the importance of building bridges

We have discussed the relevance of analyticity for proof theory and logic. To conclude the present chapter we would like to stress a further conceptual advantage of developing uniform analytic calculi for non-classical logics. First, presenting various logics as variations of a base common analytic system is particularly desirable as it reduces the fragmentation of the landscape of non-classical logics. Second, the development of uniform analytic calculi is promising in order to build bridges between different proof systems[1].

A key tool in this sense is represented by translations and embeddings. In particular, a logic can be seen as a fragment of another one. A prominent example in this respect is given by the modal embedding for intuitionistic logic which shows that intuitionistic logic can be considereed a fragment of the modal logic **S4**. These connections bring to the fore new interpretations and reading of logical connectives and formulas.

Furthermore, once the embedding is proved to be *sound* - every proof can be translated - and *faithful* - every proof of the translation can be transformed in a proof of the original formula - it can be exploited to obtain proofs of metalogical results.

Embeddings and translations between systems are also a central ingredient in some relatively recent developments in the field of proof-theoretic semantics, such as the project of ecumenical systems (86).

In this thesis we shall focus on the presentation of syntactic enquiries on relations between different non-classical logics. The route we follow is thus

[1] See (**?**) for a discussion of this issue in the context of non-monotonicity and paraconsistency.

described. In Chapter 2 we deal with algebraic and semantical preliminaries concerning the modal embedding of intuitionistic logic. Chapter 3 discusses a nested system for intuitionistic propositional logic and a new system for modal propositional logic which enables to prove a structural refinement of the translation. In Chapter 4 we propose a case study of the embedding of intuitionistic logic in the modal logic of arithmetical provability and we obtain, as a byproduct, terminating sequent calculi for a wide class of intermediate logics. Constructive mathematical theories are the main focus of Chapter 5 which are given a modal interpretation base on the logical shape of their axioms. Chapter 6 introduces the topic of infinitary logics which will be central in the remaining part of the thesis. Infinitary intuitionistic logic is thoroughly investigated both from the syntactic and the semantic viewpoint and the modal embedding is extended to the infinitary setting. The next Chapter 7 is devoted to the study of multiplicative quantifiers and thus, *lato sensu*, of infinitary logics in a substructural setting. The Chapter is connected in a sense to all the previous ones as multiplicative quantifiers can simulate exponentials and thus, by combinig well-known translations, we can represent all the previous systems as fragments of this latter. Chapter 8 discusses a technical point connected to cut-elimination in the presence of infinite sequents, thus introducing a new proof-theoretic techniques. Finally, some brief concluding remarks pave the way for future works.

Chapter 2

The modal interpretation of intuitionistic logic

In this first chapter we reconstruct the proof of the translation as presented by Gödel and McKinsey and Tarski. We start by recalling some basic abstract algebraic notions concerning lattices, Heyting algebras and Boolean algebras with operators. We then give a proof of the algebraic version of Stone's representation theorem for lattices and we use this in order to show faithfulness of the translation along the lines of the work of McKinsey and Tarski.

Keywords: algebraic logic, intuitionistic logic, modal logic

2.1 Algebraic semantics

We shall be mainly concerned with two languages: the one of intuitionistic (and classical) logic and the one of modal logic. We start by presenting the propositional parts of those languages.

Definition 2.1.1. The language FM of intuitionistic propositional logic contains a denumerable set AT of propositional atomic formulas $p_1, p_2, \ldots$ and binary connectives $\wedge, \vee, \rightarrow$ and a zeroary connective $\bot$.

Definition 2.1.2. The language $\mathsf{FM}^{\Box}$ of modal propositional logic contains a denumerable set AT of propositional atomic formulas $p_1, p_2, \ldots$, the binary connectives $\wedge, \vee, \rightarrow$, a unary connective $\Box$ and a zeroary connective $\bot$.

Negation $\neg A$ is defined as $A \rightarrow \bot$ and logical equivalence $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$. $\top$ is defined as $\bot \rightarrow \bot$. $\Diamond A$ abridges $\neg\Box\neg A$.

We also recall the axiomatic presentations of intuitionistic and (some) modal logics. By **I** we denote the axiomatic calculus for intuitionistic propositional logic. An axiomatic calculus **C** (see also the previous chapter) for classical propositional logic is obtained by adding the axiom schema of the double negation law:

$$A \vee \neg A$$

Finally, a calculus for the minimal normal modal logic **K** is obtained by adding the axiom:

$$\textbf{K}. \qquad \Box(A \to B) \to (\Box A \to \Box B)$$

and the inference rule RN: $\dfrac{\vdash A}{\vdash \Box A}\,\mathrm{RN}$, i.e. the rule of necessitation.

I

Axioms

1.1 $A \to (B \to A)$

1.2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

2.1 $A \wedge B \to A$

2.2 $A \wedge B \to B$

2.3 $(A \to B) \to ((A \to C) \to (A \to B \wedge C))$

3.1 $A \to A \vee B$

3.2 $B \to A \vee B$

3.3 $(A \to C) \to ((B \to C) \to (A \vee B \to C))$

4.1 $(A \to B) \to ((A \to \neg B) \to \neg A)$

4.2 $A \to (\neg A \to B)$

Inference Rules

$$\frac{\vdash A \qquad \vdash A \to B}{\vdash B}\,\mathrm{MP}$$

The modal systems which will be studied in the present work are obtained from the base system **K** via the modular addition of suitable axiom schemata:

- **T**. $\Box A \to A$
- **4**. $\Box A \to \Box\Box A$
- **S4**. $\mathbf{T} \oplus \mathbf{4}$
- **S5**. $\mathbf{S4} \oplus A \to \Box\Diamond A$

Definition 2.1.3. Given an axiomatic system **L**, A is derivable in **L**, in symbols $\vdash_{\mathbf{L}} A$, if and only if A is an instance of an axiom schema or is obtained via applications of the rules of the calculus.

Before the development of Kripkean semantics, for a long time the approach to modal and intuitionistic logics was mainly based on algebraic methods. Kripkean semantics interprets logical operators in relational structures and the relations allows for the interpretation of the intensional connectives. Algebraic semantics differs from the kripkean one since it abstracts from the meaning of logical connectives, because it treats them as operations on a given set A.

In this section we will give a presentation of algebraic semantics for intuitionistic logic and modal logic and the representation theorem for Heyting algebras (92). In what follows we will be mainly dealing with axiomatic systems, as it is customary in the field of algebraic logic, hence it is important to specify these points.

Definition 2.1.4. Given an axiomatic system **L**, the logic $\mathbb{L}$ corresponds to the set of theorems of **L**, $\mathbb{L} = \{A | \vdash_{\mathbf{L}} A\}$.

Thus, for example, in the present chapter when we will talk about intuitionistic logic we will be actually referring to the set of theorems of the axiomatic calculus **I** and the same holds for classical logic and normal modal logics.

Definition 2.1.5. Given a non-empty set A and a collection of n-ary maps $o^n : A^n \to A$, with $n \geqslant 0$[1], $\mathfrak{A} = (A, o_1, ..., o_m)$ is an algebra with universe A. An algebra is finite if A is finite. An algebra is degenerate if A is a singleton.

In what follows we will be dealing exclusively with non degenerate algebras.

[1] If $n = 0$, then o is an element in A

Definition 2.1.6. Given two algebras $\mathfrak{A}$, $\mathfrak{B}$ they are similar if they have the same number of operations defined on them and if their arity coincides.

Definition 2.1.7. Algebras of the type $\mathfrak{A} = (A, \wedge^2, \vee^2, \rightarrow^2, \bot^0)$ are called FM-algebras.
Algebras of the type $\mathfrak{A} = (A, \wedge^2, \vee^2, \rightarrow^2, \bot^0, \Box^1)$ are called FM$^\Box$-algebras.

Every formula $F(p_1, ..., p_n)$ gives rise to an n-ary operation in a FM, FM$^\Box$-algebra interpreting F's connectives as the corresponding operations in $\mathfrak{A}$ and the propositional variables $p_1, ..., p_n$ as variables over A. A formula F that defines such an operation is called an FM(FM$^\Box$)-term.
By $F(a_1, ..., a_n)$ we denote the result of applying the operation associated to F to the elements $a_1, ..., a_n$.

Definition 2.1.8. A valuation on a FM$^\Box$-algebra $\mathfrak{A}$ is a map $v : AT \rightarrow A$. Its definition is inductively extended to cover every $F \in$ FM$^\Box$. For every formula $F, G \in$ FM$^\Box$:

- $v(\bot) = \bot$
- $v(F \wedge G) = v(F) \wedge v(G)$
- $v(F \vee G) = v(F) \vee v(G)$
- $v(F \rightarrow G) = v(F) \rightarrow v(G)$
- $v(\Box F) = \Box v(F)$

Clearly, the symbols on the left hand side of the definition are the usual connectives and the modal operator, whereas those on the right hand side represent the corresponding algebraic operations defined on A. The definition of a valuation for the case of FM-algebras is a restriction of the previous one, as we only need to consider the first four items of the definition. In the next pages we will give definitions and results for FM$^\Box$-algebras, since those for FM-algebras are particular cases of the latter.

Definition 2.1.9. Given F, G terms of the FM$^\Box$-algebra $\mathfrak{A}$, $F = G$ is true in $\mathfrak{A}$ if and only if the value of F and G in $\mathfrak{A}$ under v are the same ($v(F) = v(G)$) for every v.

Definition 2.1.10. Given an FM$^\Box$-algebra $\mathfrak{A}$, $Z \subseteq A, Z \neq \emptyset$, we say that $(\mathfrak{A}, Z)$ is a matrix and Z is the set of its distinguished elements. $F \in$ FM$^\Box$ is valid in a matrix if for every valuation v in $\mathfrak{A}$ we have $v(F) \in Z$.

We will deal with algebras where Z contains only an element, i.e. $\top = \bot \to \bot$, so that the previous definition reduces to the following:

Definition 2.1.11. $F \in \mathsf{FM}$ is valid in the matrix $(\mathfrak{A}, \top)$, in symbols $\mathfrak{A} \vDash F$ iff $F = \top$ is true in $\mathfrak{A}$.

Definition 2.1.12. Given two similar algebras $\mathfrak{A}$, $\mathfrak{B}$, a map $\phi : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ if for every operation $o_i \in \mathfrak{A}$ of arity m and every element $a_1, ..., a_m \in \mathfrak{A}$, we have: $\phi(o_i(a_1, ..., a_m))) = o_i(\phi(a_1), ..., \phi(a_m))$.

If ϕ is injective we say that ϕ is an embedding; if ϕ is onto then ϕ is an epimorphism. If ϕ is both injective and onto we say that it is an isomorphism and $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

Given a normal modal logic **L** (i.e. an extension of **K**), its *associated matrix* is the pair $(\mathfrak{A}_L, L)$, such that $\mathfrak{A}_L = (\mathsf{FM}^{\Box}, \wedge, \vee, \to, \bot, \Box)$.[2]

Definition 2.1.13. Given a modal logic **L** we say that **L** is *characterised* by a class of matrices if **L** coincides with the set of formulas that are valid in all matrices in the class.

Theorem 2.1.1. *Given a modal logic L, its associated matrix* $(\mathfrak{A}_L, L)$ *is a characteristic matrix for L.*

Proof. From left to right we proceed as follows. Given a formula $F(p_1, ..., p_n) \in$ **L**, i.e. $\vdash_{\mathbf{L}} F(p_1, ..., p_n)$, for every valuation v we have:

$$v(F) = F(p_1/v(p_1), ..., p_n/v(p_n))$$

but since L is closed under substitution we have $F(p_1/v(p_1), ..., p_n/v(p_n)) \in$ **L**, hence $(\mathfrak{A}, L) \vDash F$.

From right to left we prove the contrapositive. Let us assume $F \notin$ **L**. Hence we consider the valuation v on $\mathfrak{A}_L$ such that $v(p) = p$ for every $p \in AT$, hence via a trivial induction we immediately obtain that $v(F) = F$, but by hypothesis $F \notin$ **L**, hence $(\mathfrak{A}_L, L) \nvDash F$. *qed.*

We will now present an important construction in algebraic logic, that is often exploited in order to prove completeness with respect to algebraic semantics, namely the Lindenbaum-Tarski algebras.

Definition 2.1.14. Given a logic **L**, given the set $\mathsf{FM}^{\Box}$ we define the relation $\sim_{\mathbf{L}}$ such that for every $F, G \in \mathsf{FM}^{\Box}$ we have $F \sim G$ if and only if $\vdash_{\mathbf{L}} F \leftrightarrow G$.

[2]The case for intuitionistic logic is obtained not considering $\Box$.

Lemma 2.1.2. $\sim$ *is an equivalence relation.*

Proof. We have to check that $\sim$ is reflexive, symmetric and transitive.

- We easily obtain $\vdash_{\mathbf{L}} F \leftrightarrow F$, thus $F \sim F$, hence $\sim$ is reflexive.
- Given $F, G \in \mathsf{FM}^{\Box}$, we suppose $F \sim G$, hence $\vdash_{\mathbf{L}} F \leftrightarrow G$, thus by definition of $\leftrightarrow$ and commutativity of $\wedge$ we immediately obtain $\vdash_{\mathbf{L}} G \leftrightarrow F$, that is $G \sim F$.
- Given $F, G, H \in \mathsf{FM}^{\Box}$, we suppose $F \sim G$ and $G \sim H$, hence we have $\vdash_{\mathbf{L}} F \leftrightarrow G$ and $\vdash_{\mathbf{L}} G \leftrightarrow H$, so by definition of $\leftrightarrow$ and transitivity of $\rightarrow$ we obtain $\vdash_{\mathbf{L}} F \leftrightarrow H$.

qed.

Since $\sim$ is an equivalence relation we can now consider the quotient of the set $\mathsf{FM}^{\Box}$, in symbols $\mathsf{FM}^{\Box}_{\sim}$. The elements of the quotient are equivalence classes of the form $[F]_{\sim} = \{G \mid \vdash_{\mathbf{L}} F \leftrightarrow G\}$. As we will see the Lindenbaum algebra is actually the quotient algebra naturally induced by $(\mathfrak{A}, L)$: the advantage is that Lindenbaum algebras have a matrix with a single element.

Theorem 2.1.3 (Lindenbaum Algebra)**.** *Every normal modal logic and intuitionistic logic has a characteristic matrix with a single distinguished element.*

Proof. We deal with the Lindenbaum algebra for modal logics, the case for intuitionistic logic is obtained analogously. For a modal logic L we consider the following algebra $\mathfrak{A}_{/L} = (\mathsf{FM}^{\Box}_{\sim_L}, \wedge, \vee, \rightarrow, \bot, \Box)$, we choose as distinguished element $[\top]$. The operations are so defined:

- $[F] \wedge [G] = [F \wedge G]$
- $[F] \vee [G] = [F \vee G]$
- $[F] \rightarrow [G] = [F \rightarrow G]$
- $\Box[F] = [\Box F]$
- $\bot = [\bot]$

It is routine to check that these operations are well defined in the sense that their definition does not depend on the choice of the representatives of the class.

We now have to prove that for every formula $F \in \mathsf{FM}^{\Box}$: $\vdash_{\mathbf{L}} F$ iff $\mathfrak{A} \vDash F$. In order to do so we state the following proposition (the proof is a trivial induction on the complexity of F and thus we omit it).

Claim For every formula F in which the propositional variables $p_1, ..., p_n$ occur, for every formula $G_1, ..., G_n$, we have: $F([G_1], ..., [G_n]) = [F(p_1/G_1, ..., p_n/G_n)]$.

From left to right we assume $\vdash_{\mathbf{L}} F(p_1, ..., p_n)$. Let v be a valuation on $\mathfrak{A}$ and we consider a substitution p_i/G_i such that $v(p_i) = [G_i]$ for every $p \in AT$. Hence if $\vdash_{\mathbf{L}} F(p_1, ..., p_n)$, then we have $\vdash_{\mathbf{L}} F \leftrightarrow \top$, hence we have $\vdash_{\mathbf{L}} F(G_1, ..., G_n) \leftrightarrow \top$ because **L** is closed under substitution, so by definition $[F(G_1, ..., G_n)] = [\top]$. Combining the chains of equalities above we obtain:

$$v(F) = F(v(p_1), ..., v(p_n)) = F([G_1], ..., [G_n]) = [F(G_1, ..., G_n)] = [\top].$$

where the third equality is justified by the Claim above. So we obtain $\mathfrak{A} \vDash F$.

From right to left we prove the contrapositive, so let us assume $F(p_1, ..., p_n) \notin L$, then $\nvdash_{\mathbf{L}} F \leftrightarrow \top$ (otherwise $F \in \mathbf{L}$). So we have $[F] \neq [\top]$, but we now consider the valuation v such that $v(p) = [p]$ for every $p \in AT$. Clearly we have $v(F) = F(v(p_1), ..., v(p_n)) = F([p_1], ..., [p_n]) = [F(p_1, ..., p_n)] \neq [\top]$, which entails $\mathfrak{A} \nvDash F$. *qed.*

Corollary. *Normal modal logics and intuitionistic logic are characterised by their corresponding Lindenbaum algebra.*

This gives us a completeness theorem for intuitionistic and normal modal logics. We shall see how this construction constitutes a case of a larger class of algebras both for intuitionistic and modal logics.

We would like to characterise a class of algebras in which every intuitionistic formula is valid.

Definition 2.1.15. An FM-algebra $\mathfrak{A} = (A, \wedge, \vee, \to, \bot)$ is a Heyting algebra if for every term of the algebra F, G: $\mathfrak{A} \vDash F = G$ iff $\vdash_{\mathbf{I}} F \leftrightarrow G$. We indicate with AH the class of all Heyting algebras.

We shall at times use 0 to refer to $\bot$ and $\top$ or 1 to refer to $\bot \to \bot$. By the definition it is clear that the Lindenbaum algebra for intuitionistic logic is an Heyting algebra. This gives us the following theorem of completeness for intuitionistic logic with respect to the class of Heyting algebras.

Theorem 2.1.4 (Algebraic completeness)**.** *For every $F \in$ FM and $\mathfrak{A} \in AH$, $\vdash_I F$ iff $\mathfrak{A} \vDash F$.*

Proof. From left to right if $\vdash_{\mathbf{I}} F$, then we have $\vdash_I F \leftrightarrow \top$, then by definition of Heyting algebra we have $\mathfrak{A} \vDash F = \top$, that is $\mathfrak{A} \vDash F$.

From right to left we prove the contrapositive, hence we suppose that $\nvdash_{\mathbf{I}} F$, but then the Lindenbaum-Tarski intuitionistic algebra $\mathfrak{A}_{/I} \nvDash A$ and $\mathfrak{A}_{/I} \in AH$ by definition, hence contradiction. *qed.*

Definition 2.1.16. Given $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot)$, we define over A a relation $\leqslant$ such that $x \leqslant y$ if and only if $x \wedge y = x$.

Next we have an important theorem that gives an alternative characterisation of Heyting algebras; to keep the presentation self contained we do not give the details of the proof, the interested reader may find them in (15).[3]

Theorem 2.1.5. *$\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot) \in AH$ if and only if for every $x, y \in A$:*

1.1	$x \wedge y = y \wedge x$	*1.2*	$x \vee y = y \vee x$
2.1	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	*2.2*	$x \vee (y \vee z) = (x \vee y) \vee z$
3.1	$(x \wedge y) \vee y = y$	*3.2*	$(x \vee y) \wedge y = y$
4.1	$z \wedge x \leqslant y$ *iff* $z \leqslant x \rightarrow y$	*4.2*	$\bot \leqslant x$

The first pair of axioms expresses the commutativity of $\wedge$ and $\vee$, the second one their associativity and the third is the absorption law. An algebra with two operations $\wedge, \vee$ that respect the laws 1.-3. is a lattice. Moreover in a lattice the relation $\leqslant$ introduced above defines a partial order (22).

Theorem 2.1.6. *Given a lattice $A = (A, \wedge, \vee)$ a relation $\leqslant$ such that for every x, y we have $x \leqslant y$ iff $x \wedge y = x$ defines a partial order on it.*

Proof. We have to check that $\leqslant$ is reflexive, transitive and antisymmetric.

- Reflexivity. $x \wedge x = x \wedge (x \vee (x \wedge x)) = x$ (exploiting twice the law of absorption).
- Antisymmetry. Let us suppose $x \leqslant y$ and $y \leqslant x$, hence we have: $x \wedge y = x$ and $y \wedge x = y$, so by transitivity of equalities we have $x = y$.

[3]We recall that this is only one among the various possible presentation of Heyting algebras. For the opposite strategy, i.e. starting with an algebraic characterisation of Heyting algebras see for example (92) (13).

- Transitivity. Let us suppose $x \leqslant y$ and $y \leqslant z$, so we have $x \wedge y = x$ and $y \wedge z = y$, hence we build the following chain of equalities:
  $x = x \wedge y = x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z$.

qed.

Dually, we also have $x \leqslant y$ iff $x \vee y = y$. A lattice in which there are elements 0 and 1 that are the minimum and the maximum with respect to the order is a *bounded lattice*. A lattice that has property 4.1 is a relatively *pseudocomplemented lattice*.
Finally we highlight that from properties 1.-4. it is possible to obtain the following properties:

5.1 $\quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ $\qquad$ 5.2 $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Lattices with this property are called *distributive lattices*. We observe that every Heyting algebra is a bounded pseudocomplemented distributive lattice where the minimum and the maximum are $\bot$ and $\top$, respectively.

Now we will present the theory of filters, a crucial instrument to our development (which will be central in Chapter 6), often employed in model theoretic contruction both in classical and modal logic. In Heyting algebras we use filters in order to represent at a pure set theoretic level propositional theories; i.e. sets of formulas closed under modus ponens and containing all intuitionistic tautologies.

Definition 2.1.17. Given $\mathfrak{A} \in AH$, $\triangledown \subseteq A$ is a filter if for every $x, y \in \mathfrak{A}$:

- $\top \in \triangledown$
- If $x \in \triangledown$ and $x \to y \in \triangledown$, then $y \in \triangledown$ for every $x, y \in A$.

We give an alternative and equivalent characterisation of the concept of filter, that does not contain an explicit reference to $\to$.

Theorem 2.1.7. *Given $\mathfrak{A} \in AH$, $\triangledown \subseteq A$ is a filter if and only if for every $x, y \in \mathfrak{A}$:*

1. $\triangledown \neq \emptyset$
2. *If $x \in \triangledown$ and $y \in \triangledown$, then $x \wedge y \in \triangledown$ for every $x, y \in A$.*
3. *If $x \in \triangledown$ and $x \leqslant y$ then $y \in \triangledown$.*

Proof. From left to right we suppose that $\triangledown$ is a filter, so $\top \in \triangledown$, hence $\triangledown \neq \emptyset$. Moreover, let us suppose $x \in \triangledown$ and $y \in \triangledown$, but since $\mathfrak{A} \vDash x \to (y \to x \wedge y) = \top$, we have $x \to (y \to x \wedge y) \in \triangledown$, so by definition we have $x \wedge y \in \triangledown$. Finally, let us suppose $x \in \triangledown$ and $x \leqslant y$, so we have $x \wedge y = x$ hence $x \wedge y \in \triangledown$, but again $\mathfrak{A} \models x \wedge y \rightarrow y = \top$, so $x \wedge y \rightarrow y \in \triangledown$ and by the separation property $y \in \triangledown$.

From right to left let us suppose that $\triangledown$ satisfies properties 1.-3., so $\triangledown \neq \emptyset$ hence there is $x \in \triangledown$, but $x \leqslant \top$, so $\top \in \triangledown$. Furthermore, let us suppose $x \in \triangledown$ and $x \rightarrow y \in \triangledown$, so $x \wedge (x \rightarrow y) \in \triangledown$. Since $\mathfrak{A} \models x \wedge (x \rightarrow y) = x \wedge y$ we have $x \wedge y \in \triangledown$, but $x \wedge y \leqslant y$ hence $y \in \triangledown$ by 3. *qed.*

Now we will take into account a specific class of filters, the filters generated by a set X.

Definition 2.1.18. Given a lattice A, $X \subset A$,

$$[X\rangle = \{y \in A \mid x_1 \wedge ... \wedge x_n \leqslant y \text{ for some } x_1, ..., x_n \in X\}$$

is the filter generated by X in A.

Lemma 2.1.8. *Given a non-empty set X, $[X\rangle$ is the smallest filter on A that contains X.*

Proof. We first have to check that $[X\rangle$ is a filter.

- $X \neq \emptyset$, so $[X\rangle \neq \emptyset$ as well.
- Given $x, y \in [X\rangle$, then by definition there are $x_1, ..., x_n$ and $y_1, ..., y_m$ such that: $x_1 \wedge ... \wedge x_n \leqslant x$ and $y_1 \wedge ... \wedge y_m \leqslant y$. Hence we build the following chain of equivalences:

$$x_1 \wedge ... \wedge x_n \wedge y_1 \wedge ... \wedge y_m = x_1 \wedge ... \wedge x_n \wedge y_1 \wedge ... \wedge y_m \wedge x \wedge y \text{ iff}$$
$$x_1 \wedge ... \wedge x_n \wedge y_1 \wedge ... \wedge y_m \leqslant x \wedge y \text{ iff}$$
$$x \wedge y \in [X\rangle.$$

- Let us suppose $x \in [X\rangle$ and $x \leqslant y$, then by definition of $[X\rangle$ there are $x_1, ..., x_n$ such that $x_1, ..., x_n \leqslant x$, so by transitivity of $\leqslant$ we obtain $x_1, ..., x_n \leqslant y$, hence $y \in [X\rangle$.

Then we must verify that $[X\rangle$ is the smallest filter that contains X and let $\triangledown$ be a filter on $\mathfrak{A}$ that contains X, let $y \in [X\rangle$. Hence by definition there are $x_1, ..., x_n \in X$ such that $x_1 \wedge ... \wedge x_n \leqslant y$. But by hypothesis $x_1, ..., x_n \in \triangledown$, hence by definition of filter $y \in \triangledown$. Thus $[X\rangle \subseteq \triangledown$ and we have proved that $[X\rangle$ is the minimum element in the set of filters on $\mathfrak{A}$ that contain X. *qed.*

We now discuss a relevant property of a class of filters: *primality*.

Definition 2.1.19. Given a lattice A a filter $\triangledown$ on A is *prime* iff for every $x, y \in A$:

- $\triangledown \neq A$ (i.e. $\triangledown$ is proper)
- If $x \vee y \in \triangledown$, then $x \in \triangledown$ or $y \in \triangledown$

If filters correspond to theories, prime filters intuitively correspond to prime or complete theories. The key point is that theories are, in a sense, closed under implications and conjunctions, whereas prime theories are closed under disjunctions too.

Definition 2.1.20. Given a lattice A, $x \in A$ is prime iff:

- $x \neq \bot$
- If $x = y \vee z$ then $x = y$ or $x = z$

Lemma 2.1.9. *Given a distributive lattice A and $x \in A$ then $[\{x\}\rangle$*[4] *is prime iff x is prime.*

Proof. From left to right we suppose $[x\rangle$ is a prime filter. Hence $x \neq \bot$, for otherwise $[x\rangle = A$. Then let us assume $x = y \vee z$, then we have $y \vee z \in [y \vee z\rangle$, then $y \in [y \vee z\rangle$ or $z \in [y \vee z\rangle$ since $[y \vee z\rangle$ is prime. We consider the first case. By definition of generated filter we have $y \vee z \leqslant y$ and by definition of $\leqslant$ we have $y \leqslant y \vee z$, which yields $y = y \vee z = x$. The other case follows analogously.

From right to left let us suppose that x is a prime element of A and let us assume $y \vee z \in [x\rangle$, then by definition of generated filter we have $x \leqslant y \vee z$, which is equivalent to $x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ by distributivity. Since x is prime we obtain $x = x \wedge y$ or $x = x \wedge z$, that is equivalent by definition to $x \leqslant y$ or $x \leqslant z$. So we have $y \in [x\rangle$ or $z \in [x\rangle$ by definition of generated filter. *qed.*

Corollary. *Given a distributive lattice A, $x \in A$, if x is prime then for every $y, z \in A$ we have: if $x \leqslant y \vee z$, then $x \leqslant y$ or $x \leqslant z$.*

Proof. Let us assume $x \leqslant y \vee z$, with $x \in A$ prime, then by definition of $\leqslant$ we have $x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, so by primality $x = x \wedge y$ or $x = x \wedge z$, therefore $x \leqslant y$ or $x \leqslant z$. *qed.*

Definition 2.1.21. A filter is *maximal* iff every filter that extends it is not proper.

Definition 2.1.22. A filter $\triangledown$ on $\mathfrak{A} \in AH$ is an *ultrafilter* if for every $x \in \mathfrak{A}$ we have: $x \in \triangledown$ or $-x = x \to \bot \in \triangledown$.

[4] From now on simply $[x\rangle$.

A filter $\triangledown$ is *maximal* if, whenever a filter $\triangledown' \supseteq \triangledown$, $\triangledown' = \triangledown$.

Lemma 2.1.10. *Given $\mathfrak{A} \in AH$, a filter $\triangledown$ on $\mathfrak{A}$ is maximal iff it is an ultrafilter.*

In Heyting algebras maximal filters are ultrafilters and they are also prime filters, but the converse does not hold in general. For a proper subclass of Heyting algebras, Boolean algebras, which correspond to **C** as Heyting algebras correspond to **I**, the three properties are equivalent. These algebraic results mirror the differences between classical logic and intuitionistic logic. In view of the representation theorem we limit ourselves to giving a concise presentation of boolean algebras.

Definition 2.1.23. A boolean algebra $\mathfrak{B}$ is a Heyting algebra such that for every $F, G \in \mathsf{FM}$: $\vdash_{\mathbf{C}} F \leftrightarrow G$ iff $\mathfrak{B} \vDash F = G$. We indicate with BA the class of all boolean algebras.

A Boolean algebra may be characterised as a Heyting algebra that for all of its elements x has their complement $-x$ (definable as $x \rightarrow \bot$) that satisfies the following property: $x \vee -x = \top$.We also say that Boolean algebras are Heyting algebras that are closed under the operation of complement. A classical example of boolean algebra is the power set of a given set X, with the operations of intersection, union and complement (corresponding to $X \setminus Y$ for $Y \subseteq X$). In this case we could define the boolean complement of Y relative to X, $Y \rightarrow X$ as $-Y \cup X$,intuitively corresponding to the classical reading of the implication. Hence it is possible to give the following characterisation of Boolean algebras; for the details of the proof see (15).

Lemma 2.1.11. *A Heyting algebra $\mathfrak{B}$ in which for every $x \in \mathfrak{B}$, $\mathfrak{B} \vDash x \vee \neg x$ is a boolean algebra.*

We shall present a theorem that allows us to extend a filter to a prime filter. This is a central result that we will use to prove the Stone representation theorem for Heyting's algebras, but it is non-constructive as it relies essentially on the Zorn lemma which is in turn equivalent to choice axiom.[5]

Lemma 2.1.12 (Zorn)**.** *Given a partial order P, if every totally ordered subset $C \subseteq P$ (i.e. a chain) has an upper bound, then P has a maximal element.*

Theorem 2.1.13 (Prime filter)**.** *Given a filter $\triangledown$ in a distributive lattice A and $x \notin \triangledown$, there is a prime filter $\triangledown'$ such that $\triangledown \subseteq \triangledown'$ and $x \notin \triangledown'$.*

[5]Although we should mention that there have been recent interesting developments in the field of algebraic logic that showed how to avoid the use of the choice axiom to get the prime filter theorem.(5)

Proof. Let a filter $\triangledown$ be given and $x \notin \triangledown$. We consider the set of all proper filters extending $\triangledown$ and not containing x, i.e. $\mathfrak{F}^x_\triangledown = \{H \text{ filter on } A \mid \triangledown \subseteq H \text{ and } x \notin H\}$. Given a chain C in $\mathfrak{F}^x_\triangledown$ it is easy to notice that $\bigcup C$ is an element of $\mathfrak{F}^x_\triangledown$ and is an upper bound of C with respect to the ordering. Hence by Zorn's lemma $\mathfrak{F}^x_\triangledown$ has a maximal element $\triangledown'$. Thus there is a filter $\triangledown'$ such that $\triangledown \subseteq \triangledown'$, $x \notin \triangledown'$ and for every filter H if $\triangledown' \subsetneq H$, then $x \in H$. We claim that $\triangledown'$ is prime.

We argue by contradiction, so let us suppose that $\triangledown'$ is not prime, hence there are $y, z \in \mathfrak{A}$ such that $y \vee z \in \triangledown'$, but $y, z \notin \triangledown'$. So we consider the generated filters $(\triangledown', y) = [\triangledown' \cup \{y\}\rangle, (\triangledown', z) = [\triangledown' \cup \{z\}\rangle$, now $(\triangledown', y), (\triangledown', z) \supset \triangledown'$, so since $\triangledown'$ is the maximal element in the set of filters that do not contain x we have $x \in (\triangledown', y) \cap (\triangledown', z)$. So by definition there are $w_1, w_2 \in \triangledown'$ such that $w_1 \wedge y \leqslant x$ and $w_2 \wedge z \leqslant x$, hence we have $w_1 \wedge w_2 \wedge y \leqslant x$ and $w_1 \wedge w_2 \wedge z \leqslant x$ and so $(w_1 \wedge w_2 \wedge y) \vee (w_1 \wedge w_2 \wedge z) \leqslant x$, which is equivalent, by distributivity, to $(w_1 \wedge w_2) \wedge (y \vee z) \leqslant x$.
But $w_1 \wedge w_2 \in \triangledown'$ and $y \vee z \in \triangledown'$, so by definition of filter we have $(w_1 \wedge w_2) \wedge (y \vee z) \in \triangledown'$, so again via the properties of filters we have $x \in \triangledown'$, which is a contradiction. *qed.*

Now that we have presented the main algebraic instruments we are going to present modal algebras that constitute the analogue of Heyting algebras for modal systems.

Definition 2.1.24. An $\mathsf{FM}^\Box$-algebra $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot, \Box)$ is a $\mathbf{K}^*$ modal algebra if for every $F, G \in \mathsf{FM}^\Box$, $F \leftrightarrow G \in \mathbf{K}^*$ iff $\mathfrak{A} \models F = G$, where $\mathbf{K}^* \in \{\mathbf{K}, \mathbf{T}, \mathbf{4}, \mathbf{S4}, \mathbf{S5}\}$.

Theorem 2.1.14. *Given a modal logic $\mathbf{K}^*$, $F \in \mathsf{FM}^\Box$ we have:*

$$\vdash_{\mathbf{K}^*} F \text{ iff } \mathfrak{A} \models F$$

for every $\mathbf{K}^*$ modal algebra $\mathfrak{A}$.

As in the case of Heyting algebras we give another (more perspicuous) characterisation that will highlight the connection between the algebraic semantics and the system considered.

Theorem 2.1.15. *An $\mathsf{FM}^\Box$-algebra $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot, \Box)$ is a $\mathbf{K}$ modal algebra iff $\mathfrak{A}$ satisfies for every $x, y \in A$:*

1. $\mathfrak{A} \in BA$

2. $\Box(x \wedge y) = \Box x \wedge \Box y$

3. $\Box\top = \top$.

Proof. From left to right let us suppose $\mathfrak{A}$ is a **K** modal algebra, hence 1. holds because **K** includes classical logic and thus we have that $\mathfrak{A}$ is a boolean algebra. Moreover $\vdash_K \Box(F \wedge G) \leftrightarrow \Box F \wedge \Box G$, so by definition $\mathfrak{A} \vDash \Box(F \wedge G) = \Box F \wedge \Box G$, therefore 2. holds. Finally 3. easily follows, because $\vdash_K \Box\top \leftrightarrow \top$ (the implication from left to right is obtained via a fortiori and the one from right to left via the use of the rule RN). Thus we have $\mathfrak{A} \vDash \Box\top = \top$.

From right to left instead we must show that if an algebra $\mathfrak{A}$ satisfies 1., 2., 3. then for every formula $F \in \mathsf{FM}^{\Box}$: if $\vdash_K F$, then $\mathfrak{A} \vDash F$. We proceed by induction on the height of derivations.

- Since $\mathfrak{A}$ is a boolean algebra we only have to show that the axiom **K** is true in $\mathfrak{A}$. So since $\mathfrak{A} \in BA$ we have:

  $$\mathfrak{A} \vDash (x \to y) \wedge x \wedge y = (x \to y) \wedge x$$

  so we apply the operation $\Box$ to the left and to the right, hence we obtain the following chain of equalities:

  $$\Box((x \to y) \wedge x \wedge y) = \Box((x \to y) \wedge x) == \Box(x \to y) \wedge \Box x \wedge \Box y = \Box(x \to y) \wedge \Box x.$$

  Then by definition we obtain $\Box(x \to y) \wedge \Box x \leqslant \Box y$. But a boolean algebra is a Heyting algebra as well, hence $\Box(x \to y) \leqslant \Box x \to \Box y$, so we have $\Box(x \to y) \to (\Box x \to \Box y)$.

- $\mathfrak{A}$ is closed under RN. But this is immediate via property 3. We suppose that $\vdash_K F$, hence we have $\vdash_K F \leftrightarrow \top$, so by induction hypothesis $\mathfrak{A} \vDash F = \top$, so we have $\mathfrak{A} \vDash \Box F = \Box\top$, but by 3. $\mathfrak{A} \vDash \Box F = \top$.

qed.

It is evident that the algebra we have just described mirrors the axioms of the modal system K. In particular 2. corresponds to the distributivity axiom **K**, 3. to the rule of necessitation RN.

2.2 Modal embedding

We are now ready to present the main tool that will allow us to reconstruct a model theoretic proof of the embedding of I into S4 with purely algebraic instruments.

The main ingredient of this procedure is the Stone representation theorem. Stone representation theorem constitutes a turning point in the algebraic approach to philosophical logic. There are two variants of the Stone theorem: a purely algebraic one and a topological one.

As we will see, the algebraic representation theorem for Heyting algebras shows that they are isomorphic to an algebra of the open elements of a topological boolean algebra, i.e. a modal algebra corresponding to the modal logic **S4**, whereas the topological representation theorem for Heyting algebras shows that they are isomorphic to the open sets of a topological space. This duality rests on the fact that the modal operator $\Box$ in **S4** modal algebras has the same properties of an interior operator in a topological space (13) and (65) and therefore it constitutes a conceptualization of the latter. However, in what follows we will focus on the algebraic result, because we will exploit it in order to prove the embedding (98).

Definition 2.2.1. Given a distributive lattice $\mathfrak{A} = (A, \wedge, \vee)$, its associated *Stone space* $W_{\mathfrak{A}}$ is the set of all its prime filters.

Definition 2.2.2. Given $\mathfrak{A}$, $W_{\mathfrak{A}}$ the *Stone function* is an application $\phi_{\mathfrak{A}} : A \to \mathcal{P}(W_{\mathfrak{A}})$ such that for every $x \in A$, $\phi(x) = \{\nabla \in W_{\mathfrak{A}} \,|\, x \in \nabla\}$.

Definition 2.2.3. Given $P_{\mathfrak{A}} = \{\phi_{\mathfrak{A}}(x) | x \in A\}$,we say that $(P_{\mathfrak{A}}, \cap, \cup)$ is the Stone lattice of $\mathfrak{A}$.

The next theorem is crucial from a conceptual point of view and it represents a cornerstone in the field of algebraic approaches to non-classical logics. Its relevance lies in the fact that it gives a - so to say - concrete topological structure which is isomorphic to the purely algebraic one.

Theorem 2.2.1 (Stone representation)**.** *For every distributive lattice $\mathfrak{A}$:*

$$\mathfrak{A} \simeq (P_{\mathfrak{A}}, \cap, \cup)$$

Proof. We claim that the Stone function is an isomorphism from $\mathfrak{A}$ to the Stone lattice of $\mathfrak{A}$. Hence we must prove that $\phi_{\mathfrak{A}}$ is a bijection and a morphism, i.e. a function which commutes with the operations.

- $\phi_{\mathfrak{A}}$ is clearly onto, because by definition $P_{\mathfrak{A}}$ is the image of A through $\phi_{\mathfrak{A}}$.
- $\phi_{\mathfrak{A}}$ is injective. Given $x, y \in A$ let us suppose $x \neq y$, then $x \nleqslant y$ or $y \nleqslant x$. Hence without loss of generality we deal with the first case. So we consider the filter generated by x, $[x\rangle$ that does not contain y (otherwise by definition of generated filters we would obtain $x \leqslant y$) and so we apply

Theorem 2.1.13 and we obtain a prime filter ∇' such that $[x\rangle \subseteq \nabla'$, $x \in \nabla'$ and $y \notin \nabla'$. By definition of $\phi_{\mathfrak{A}}$ we obtain $\nabla' \in \phi_{\mathfrak{A}}(x)$ and $\nabla' \notin \phi_{\mathfrak{A}}(y)$, which entails $\phi_{\mathfrak{A}}(x) \neq \phi_{\mathfrak{A}}(y)$.

- We have to check that $\phi_{\mathfrak{A}}$ preserves the operations. So given $x, y \in A$, we consider $\phi_{\mathfrak{A}}(x \wedge y) = \{\nabla \in W \mid x \wedge y \in \nabla\}$, but since ∇ is a filter we have $\{\nabla \in W \mid x \in \nabla \text{ and } y \in \nabla\} = \phi_{\mathfrak{A}}(x) \cap \phi_{\mathfrak{A}}(y)$.

- Given $x, y \in A$, we consider $\phi_{\mathfrak{A}}(x \vee y) = \{\nabla \in W \mid x \vee y \in \nabla\}$, since ∇ is a prime filter we have $\{\nabla \in W \mid x \in \nabla \text{ or } y \in \nabla\} = \phi_{\mathfrak{A}}(x) \cup \phi_{\mathfrak{A}}(y)$.

Hence the theorem holds, so $\mathfrak{A} \simeq (P_{\mathfrak{A}}, \cap, \cup)$. *qed.*

Before proceeding with the last part of our brief exposition of these algebraic instruments, we will spend a few words on the Stone representation theorem. We observe that if we took as Stone space not the set of all the prime filters on $\mathfrak{A}$, but merely the set of all filters we could not have completed the proof, because we require the primality property when we are dealing with the presevation of $\vee$ under the Stone function. Notice that in contrast this was not necessary with respect to $\wedge$, because filters are closed under intersection. This leads us to consider a crucial point: if surjectivity is easily obtained by construction, injectivity comes at a cost. In fact the proof given above makes an essential use of the prime filter theorem, which in turn relies on the Zorn lemma, a non constructive principle, thus introducing - in the words of Hilbert - an ideal element in the proof.

We are finally in the position to clarify the relation between the modal system **S4** and intuitionistic logic, working at a pure algebraic level. We recall few definitions and lemmata before stating the main results.

Definition 2.2.4. An algebra $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot, \Box)$ is an **S4**-algebra (or a topological boolean algebra) if it satisfies the following conditions for every $x, y \in A$:

- $\mathfrak{A}$ is a K-modal algebra;
- $\Box x \leqslant x$
- $\Box\Box x = \Box x$

Our first step is to show that we can obtain a Heyting algebra from a topological boolean algebra.

Definition 2.2.5. Given an **S4** algebra $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot, \Box)$, an element $x \in \mathfrak{A}$ such that $\Box x = x$ is an open element of $\mathfrak{A}$. We indicate the set of open elements of $\mathfrak{A}$ with $O(A)$.

Lemma 2.2.2. *For every* **S4***-algebra and every open elements* x, y *in it we have* $\Box(x \vee y) = x \vee y$.

Proof. Clearly due to properties of $\Box$ we have $\Box(x \vee y) \leqslant x \vee y$.
Instead in the other direction we have $x \leqslant x \vee y$, so we apply $\Box$ and we have $x = \Box x \leqslant \Box(x \vee y)$. The same can be argued for y, hence $y \leqslant \Box(x \vee y)$. Thus we have $y \wedge \Box(x \vee y) = y$, hence $x \vee y = x \vee (y \wedge \Box(x \vee y)) = (x \vee y) \wedge (x \vee \Box(x \vee y))$, and, since $x \leqslant \Box(x \vee y)$, we have $(x \vee y) \wedge \Box(x \vee y)$, thus $x \vee y \leqslant \Box(x \vee y)$. *qed.*

Definition 2.2.6. Given an **S4** algebra $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot, \Box)$ we say that $\mathfrak{A}^o = (O(A), \wedge, \vee, \rightarrow_\Box, \bot)$, with $x \rightarrow_\Box y = \Box(x \rightarrow y)$ is the algebra of its open elements.

Theorem 2.2.3. *Given an* **S4** *algebra, its algebra of the open elements is a Heyting algebra.*

Proof. First of all we have to check that the algebra of open elements is closed with respect to $\wedge, \vee$, but this is immediate by the properties of $\Box$ and by the previous lemma respectively.
Then we show that it is a Heyting algebra. Due to the characterization of Heyting algebras given by Theorem 2.1.5 we have to check that conditions 1.-4. are respected, but since it is closed under $\wedge, \vee$, we can limit ourselves to check whether 4. is satisfied or not.

From left to right let us suppose $x \wedge y \leqslant z$, then we have $x \leqslant y \rightarrow z$. Applying by both sides the operator $\Box$ we have $\Box x \leqslant \Box(y \rightarrow z)$ and since x is an open element we have $x \leqslant y \rightarrow_\Box z$.

From right to left we suppose $x \leqslant y \rightarrow_\Box z$, hence we have $x \leqslant \Box(y \rightarrow z)$, so since $\Box(y \rightarrow z) \leq \Box y \rightarrow \Box z$ we have $x \leqslant \Box y \rightarrow \Box z$, so we have $x \wedge \Box y \leqslant \Box z$. Since y, z are open elements we obtain $x \wedge y \leqslant z$. *qed.*

It can be easily seen that given a topological boolean algebra $\mathfrak{A}$ and its associated Heyting algebra of the open elements $\mathfrak{A}^o$ we have: if $\mathfrak{A} \vDash F^*$ then $\mathfrak{A}^o \vDash F$, for every formula $F \in \mathsf{FM}$, where $*$ is a modification of Gödel's translation (42). Hence this constitutes a proof of the faithfulness of the embedding result of intuitionistic logic into S4 modulo the theorems of algebraic completeness and the representation theorem for Heyting algebras. As a historical note, we recall that the original Gödel translation was so defined:

- $(p)^{\mathfrak{g}} := p$
- $(\bot)^{\mathfrak{g}} := \bot$
- $(A \wedge B)^{\mathfrak{g}} := A^{\mathfrak{g}} \wedge B^{\mathfrak{g}}$
- $(A \vee B)^{\mathfrak{g}} := \Box A^{\mathfrak{g}} \vee \Box B^{\mathfrak{g}}$
- $(A \to B)^{\mathfrak{g}} := \Box A^{\mathfrak{g}} \to \Box B^{\mathfrak{g}}$

Definition 2.2.7. The function $* : \mathsf{FM} \to \mathsf{FM}^{\Box}$ is so defined:

- $(p)^* = \Box p$
- $(\bot)^* = \bot$
- $(F \wedge G)^* = F^* \wedge G^*$
- $(F \vee G)^* = F^* \vee G^*$
- $(F \to G)^* = \Box(F^* \to G^*)$

is the Gödel translation from **I** to **S4**.

Lemma 2.2.4. *Let a topological boolean algebra $\mathfrak{A} = (A, \wedge, \vee, \to, \bot, \Box)$ and its algebra of the open elements $\mathfrak{A}^o = (O(A), \wedge, \vee, \to_{\Box}, \bot)$ be given. Then for every formula $F \in \mathsf{FM}$ if $\mathfrak{A} \vDash F^*$ then $\mathfrak{A}^o \vDash F$.*

Proof. We prove the contrapositive, so let us suppose that $\mathfrak{A}^o \nvDash F$, hence there is a valuation $v : AT \to O(A)$ such that $v(F) \neq \top$. Hence we consider the valuation function $v' : AT \to A$ such that $v'(p) = \Box v(p)$.
Claim For every $F \in \mathsf{FM}$ we have $v'(F) = v(F^*)$.

We proceed by induction on F.

- If $F \equiv p$, then $v'(p) = \Box v(p) = v(\Box p) = v((p)^*)$.
- If $F \equiv C \wedge D$, then $v'(C \wedge D) = v'(C) \wedge v'(D)$, by induction hypothesis we obtain $v(C^*) \wedge v(D^*)$ and so by definition of the valuation function $v((C \wedge D)^*)$. The case for $C \vee D$ is analogous and thus we omit it.
- If $F \equiv C \to D$, then $v'(C \to D) = v'(C) \to_{\Box} v'(D)$, which by induction hypothesis is equal to $v(C^*) \to_{\Box} v(D^*)$, that by definition is equal to $\Box(v(C^*) \to v(D^*)) = v(\Box(C^* \to D^*)) = v((C \to D)^*)$.

Thus since $v(F) \neq \top$, $v'(F^*) \neq \top$, there is a valuation on $\mathfrak{A}$ that falsifies F^*, so $\mathfrak{A} \nvDash F^*$. *qed.*

Hence our last step, following the path crossed by McKinsey and Tarski is to prove that every Heyting algebra is isomorphic to the algebra of open elements of an **S4**-algebra.

Theorem 2.2.5 (Algebraic representation theorem for Heyting algebras)**.** *Given a Heyting algebra* $\mathfrak{A}$*, then* $\mathfrak{A} \simeq (\mathfrak{B})^o$*, for some* **S4***-algebra* $\mathfrak{B}$*.*

Proof. Given $\mathfrak{A}$ we consider its isomorphic Stone lattice $(P_{\mathfrak{A}}, \cap, \cup)$[6] and then its extension $\mathfrak{B} = (\mathcal{P}(W_{\mathfrak{A}}), \cap, \cup) \in BA$, where $\mathcal{P}(X)$ denotes the powerset of X.

Lemma 2.2.6. $(P_{\mathfrak{A}}, \cap, \cup)$ *is a sublattice*[7] *of* $(\mathcal{P}(W_{\mathfrak{A}}), \cap, \cup)$*. Moreover every element* $X \in \mathfrak{B}$ *is of the form* $X = (-Y_1 \cup Z_1) \cap \ldots \cap (-Y_n \cup Z_n)$ *for some* $Y_i, Z_i \in P_{\mathfrak{A}}$*, where* $-$ *indicates the operation of complement in Boolean algebras defined as* $-Y := Y \rightarrow \bot$*.*

Proof. For the details of the proof see (92; 15). *qed.*

We then define the interior operator $\Box$ in $\mathfrak{B}$ as follows: for every $X \in \mathfrak{B}$, if $X = (-Y_1 \cup Z_1) \cap \ldots \cap (-Y_n \cup Z_n)$, then $\Box X = (Y_1 \rightarrow Z_1) \cap \ldots \cap (Y_n \rightarrow Z_n)$, where $\rightarrow$ is the (intuitionistic) relative pseudocomplement.

Hence we have that for every $X, Y \in \mathfrak{B}$:

- $\Box X \in (P_{\mathfrak{A}}, \cap, \cup)$. Immediate since Heyting algebras are closed with respect to the operation of relative pseudocomplement.
- For every $X \in P_{\mathfrak{A}}, \Box X = X$. This easily follows, in fact let us write $X \in P_{\mathfrak{A}}$ as $-W_{\mathfrak{A}} \cup X \in (\mathcal{P}(W_{\mathfrak{A}}), \cap, \cup)$ (notice that $W_{\mathfrak{A}}$ corresponds to $\top$). Hence $\Box X = \top \rightarrow X$. But since $(P_{\mathfrak{A}}, \cap, \cup)$ is a Heyting algebra and moreover $\vdash_I (\top \rightarrow X) \leftrightarrow X$, we can conclude $\Box X = \top \rightarrow X = X$.
- $\Box(X \cap Y) = \Box X \cap \Box Y$, in fact let $X = (-W_1 \cup Z_1) \cap \ldots \cap (-W_n \cup Z_n), Y = (-U_1 \cup V_1) \cap \ldots \cap (-U_m \cup V_m)$, then $\Box(X \cap Y) = (W_1 \rightarrow Z_1) \cap \ldots \cap (W_n \rightarrow Z_n) \cap (U_1 \rightarrow V_1) \cap \ldots \cap (U_m \rightarrow V_m)$, that is equal to $\Box X \cap \Box Y$.
- $\Box X \leqslant X$. Now let $X = (-W_1 \cup Z_1) \cap \ldots \cap (-W_n \cup Z_n)$, then $\Box X = (W_1 \rightarrow Z_1) \cap \ldots \cap (W_n \rightarrow Z_n)$, but as it can be shown $(W_1 \rightarrow Z_1) \cap \ldots \cap (W_n \rightarrow Z_n) \cap (-W_1 \cup Z_1) \cap \ldots \cap (-W_n \cup Z_n)$ is equal to $(W_1 \rightarrow Z_1) \cap \ldots \cap (W_n \rightarrow Z_n)$ (92).
- $\Box\Box X = \Box X$. It is sufficient to observe that since $\Box X \in P_{\mathfrak{A}}$, then $\Box\Box X = \Box X$.

[6]Here the order is simply the set inclusion, the zero element is the empty set and $\rightarrow$ is uniquely determined by the lattice order and $\cap$.

[7]$(A, \wedge', \vee')$ is a sublattice of $(B, \wedge', \vee')$ if $A \subseteq B$ and A is closed under $\wedge$ and $\vee$.

- $\top \in P_{\mathfrak{A}}$, hence $\top = \Box\top$.

Thus $(\mathfrak{B}, \Box)$ is a topological boolean algebra. We consider

$$\mathfrak{B}^o = (\mathcal{O}(\mathcal{P}(W_{\mathfrak{A}})), \cap, \cup, \to_{\Box}, \emptyset)$$

the algebra of the open elements of $\mathfrak{B}$. It clearly coincides with $(P_{\mathfrak{A}}, \cap, \cup)$, hence by transitivity of the isomorphism the Stone function is an isomorphism $\mathfrak{A} \simeq \mathfrak{B}^o$.

This concludes the theorem. *qed.*

Combining Lemma 2.2.4 and Theorem 2.2.5 we obtain the embedding result (actually the faithfulness side that had only been conjectured by Gödel)[8] of intuitionistic logic modulo algebraic completeness, analogously to what Tarski and McKinsey achieved in 1948 (65).

Theorem 2.2.7 (Modal Interpretation)**.** *For every formula $A \in$ FM:*

$$\vdash_I A \textit{ if and only if } \vdash_{S4} A^*.$$

Proof. From left to right the proof follows by an induction on the height of derivations, showing that the translation of every axiom scheme for **I** can be derived in **S4** and proving the closure under modus ponens. For example, we show the admissibility of the translation of modus ponens. The induction hypothesis yields derivations of $\vdash A^*$ and of $\vdash \Box(A^* \to B^*)$ and we construct the following derivation:

$$\dfrac{\dfrac{\vdash A^*}{\vdash \Box A^*}\,\text{RN} \qquad \dfrac{\vdash \Box(A^* \to B^*) \qquad \vdash \Box(A^* \to B^*) \to (\Box A^* \to \Box B^*)}{\vdash \Box A^* \to \Box B^*}\,\text{MP}}{\vdash \Box B^*}\,\text{MP}$$

From right to left we prove the contrapositive. We assume $\nvdash_I A$, hence by algebraic completeness for the intuitionistic propositional calculus there is a Heyting algebra $\mathfrak{A}$ such that $\mathfrak{A} \nvDash A$. By Theorem 2.2.5 $\mathfrak{A}$ is isomorphic to the algebra of open elements of a **S4**-modal algebra $\mathfrak{B}$, we call it $\mathfrak{B}^o$.
So we have $\mathfrak{B}^o \nvDash A$ and by Lemma 2.2.4 we have $\mathfrak{B} \nvDash A^*$, which entails $\nvdash_{S4} A^*$ modulo the validity side of algebraic completeness theorem for **S4**. *qed.*

[8]The validity of the embedding had been already proved by Gödel himself via an induction on the height of derivations in the axiomatic calculus I.

2.3 Kripke semantics

We would like to sketch the proof of the soundness and of the faithfulness of the translation via Kripke semantics. The proof essentially follows the same pattern[9], except that in this case we work with relational structures instead of algebraic ones.

2.3.1 Modal logics

In this section we are going to outline some of the main features of the semantics for modal logic, that will turn out to be essential in the analysis of other calculi. We will deal with kripkean frame-based semantics, that is a framework flexible enough to treat various non classical logics (not only modal logic as we will see). This also gives the opportunity to introduce basic definitions for Kripke-style semantics.

Definition 2.3.1. A frame $\mathcal{F} = \langle W, R \rangle$ is an ordered pair where W is a non empty set and R is a binary relation on W.

Definition 2.3.2. A Kripke model $\mathcal{M} = \langle W, R, v \rangle$ is an ordered triple where $\langle W, R \rangle$ is a frame and v is a function $v : AT \to \mathcal{P}(W)$. We say that $\mathcal{M}$ is based on the frame (W, R).

We call the elements of W possible worlds or states, the relation R is the accessibility relation: given two worlds $x, y \in W$, when xRy we say that x sees y. Intuitively the function v assigns to every propositional variable p the subset $X \subseteq W$ in which p is true. Now we are going to introduce the satisfiability relation for Kripke models.

Definition 2.3.3. Given a Kripke model $\mathcal{M} = \langle W, R, v \rangle$, $x \in W$ and $A \in FM^{\Box}$, the relation $\mathcal{M}, x \Vdash A$, i.e. A is true at world x in the model $\mathcal{M}$, is inductively defined:

- $\mathcal{M}, x \Vdash p$ iff $x \in v(p)$.
- $\mathcal{M}, x \nVdash \bot$;
- $\mathcal{M}, x \Vdash B \wedge C$ iff $\mathcal{M}, x \Vdash B$ and $\mathcal{M}, x \Vdash C$;

[9]This not casual. Indeed, the reason behind this uniformity rests on the so-called duality which brings together four different disciplines: logic, algebra, topology and category theory. The treatment of the topic is beyond the scope of this book and we refer the reader to (30) for an extensive introduction.

- $\mathcal{M}, x \Vdash B \vee C$ iff $\mathcal{M}, x \Vdash B$ or $\mathcal{M}, x \Vdash C$;
- $\mathcal{M}, x \Vdash B \rightarrow C$ iff $\mathcal{M}, x \nVdash B$ or $\mathcal{M}, x \Vdash C$;
- $\mathcal{M}, x \Vdash \Box B$ iff for every $y \in W$: if xRy then $\mathcal{M}, y \Vdash B$;

Definition 2.3.4. Given a Kripke model $\mathcal{M} = \langle W, R, v \rangle$ and $A \in \mathsf{FM}^{\Box}$, A is true in the model $\mathcal{M}$, $\vDash_{\mathcal{M}} A$ iff for every $x \in W$ $\mathcal{M}, x \Vdash A$.

Definition 2.3.5. A is a logical truth of K, in symbols $\vDash_{\mathcal{K}} A$, iff $\vDash_{\mathcal{M}} A$ for every Kripke model $\mathcal{M}$.

Definition 2.3.6. A class of frames C is a set of frames. Classes of frames are classified basing on their accessibility relation:

- $\mathcal{K}$ is the class of all frames;
- $\mathcal{SER}$ is the class of all serial frames, where the relation R is serial ($\forall x \exists y (xRy)$);
- $\mathcal{REF}$ is the class of all the reflexive frames, where the relation R is reflexive ($\forall x (xRx)$);
- $\mathcal{SYM}$ is the class of all the symmetric frames, where the relation R is symmetric ($\forall x \forall y (xRy \rightarrow yRx)$);
- $\mathcal{TRS}$ is the class of all transitive frames, where the relation R is transitive ($\forall x \forall y \forall z (xRy \wedge yRz \rightarrow xRz)$);
- $\mathcal{EUC}$ is the class of all the euclidean frames, where the relation R is euclidean ($\forall x \forall y \forall z (xRy \wedge xRz \rightarrow yRz)$);
- $\mathcal{EQ}$ is the class of all the frames in which R is an equivalence relation (R is symmetric, reflexive and transitive or alternatively reflexive and euclidean);
- $\mathcal{GL}$ is the class of all the frames in which R is transitive and noetherian (i.e. there are not infinite ascending chains).[10]

Definition 2.3.7. $A \in \mathsf{FM}^{\Box}$ is a logical truth with respect to a class of frames C, in symbols $\vDash_{C} A$, iff $\vDash_{\mathcal{M}} A$ for every Kripke model $\mathcal{M}$ whose frame belongs to C.

[10] We underline that in contrast with the classes listed above, $\mathcal{GL}$'s accessibility relation does not admit a first order formulation, since we are quantifying over sets.

Definition 2.3.8. Given $\Gamma \subseteq \mathsf{FM}^{\Box}$, $A \in \mathsf{FM}^{\Box}$, A is a logical consequence of Γ with respect to a class of frames C, in symbols $\Gamma \vDash_C A$, iff for every model $\mathcal{M}$ based on a frame in C, for every $x \in \mathcal{M}$, if $x \Vdash B$ for every $B \in \Gamma$, then $x \Vdash A$.

The table below summarizes the correspondence results between modal axioms and frame properties (9).

Table of correspondences

Name	Axiom	Semantic Frame Property
K	$\Box(A \to B) \to (\Box A \to \Box B)$	none
D	$\Box A \to \Diamond A$	serial
T	$\Box A \to A$	reflexive
4	$\Box A \to \Box\Box A$	transitive
B	$A \to \Box\Diamond A$	symmetric
5	$\Diamond A \to \Box\Diamond A$	euclidean
GL	$\Box(\Box A \to A) \to \Box A$	transitive and noetherian

To conclude our brief overview of the semantics of modal logics[11] we will build a bridge between the syntactic level and the semantic one.

Definition 2.3.9. Given an axiomatic modal calculus **X** we say that **X** is sound with respect to a class of frames C iff for every formula $A \in \mathsf{FM}^{\Box}$: if $\vdash_X A$, then $\vDash_C A$.

Definition 2.3.10. Given an axiomatic modal calculus X we say that X is complete with respect to a class of frames C iff for every formula $A \in \mathsf{FM}^{\Box}$: if $\vDash_C A$, then $\vdash_X A$.

Theorem 2.3.1. *The modal calculi* **K**, **T**, **4**, **B**, **S4**, **S5**, **GL** *are sound and complete with respect to the corresponding classes of frames.*

[11] We will speak again of kripkean semantics when we will be dealing with intuitionistic and intermediate logic.

Proof. The reader is referred to (15) for a proof of these results. *qed.*

2.3.2 Intuitionistic logic

Definition 2.3.11. Given a non empty set P, a relation $\leqslant \subseteq P \times P$ is a partial order relation if:

- $\forall x \in P(x \leqslant x)$ (reflexivity)
- $\forall x \forall y \forall z \in P(x \leqslant y \wedge y \leqslant z \rightarrow x \leqslant z)$ (transitivity)
- $\forall x \forall y \in P(x \leqslant y \wedge y \leqslant x \rightarrow x = y)$ (antisymmetry)

We say that $\langle P, \leqslant \rangle$ is a partial order.

Definition 2.3.12. Given a partial order $\langle P, \leqslant \rangle$, $O(P) \subseteq \mathcal{P}(P)$, where $A \in O(P)$ iff $\forall x \in A \forall y \in P(x \leqslant y \rightarrow y \in A)$, is the set of the open subsets of P.

Now we are in the position to specify the concept of kripkean model for intuitionistic logic:

Definition 2.3.13. A kripkean model for intuitionistic logic $\mathcal{M}$ is an ordered pair $\langle P, v \rangle$, where:

- P is a partial order;
- $v : AT \rightarrow O(P)$ is a function.

We say that the model $\mathcal{M}$ is based on the partial order P.

The elements of a P are often called worlds or states. Intuitively, v assigns to every propositional variable p one of the open subsets of the partial order P. If an atomic formula holds in a world in a model, then it holds in every world accessible from.

From a philosophical viewpoint, kripkean semantics for intuitionistic logic gives us an insight into many of the features of Brouwer's conception of mathematical knowledge. First of all the worlds can be conceptualized as epistemic states, moreover reflexivity and transitivity are coherent with the interpretation of the working mathematician who has constant access to his ideas. The condition imposed by the formulation of the valuation function is the persistence and it represents the fact that once we have obtained a proof of a certain assertion its validity does not cease.

Definition 2.3.14. Given a kripkean model $\mathcal{M} = \langle P, v \rangle$ for intuitionistic logic and a world $x \in P$ and a formula $A \in \mathsf{FM}$ we inductively define the relation $\mathcal{M}, x \Vdash A$ (i.e. A is true at world x):

- $\mathcal{M}, x \Vdash p$ iff $x \in v(p)$
- $\mathcal{M}, x \nVdash \bot$
- $\mathcal{M}, x \Vdash B \wedge C$ iff $\mathcal{M}, x \Vdash B$ and $\mathcal{M}, x \Vdash C$
- $\mathcal{M}, x \Vdash B \vee C$ iff $\mathcal{M}, x \Vdash_{\mathcal{M}} B$ or $\mathcal{M}, x \Vdash C$
- $\mathcal{M}, x \Vdash B \to C$ iff for every y such that $x \leqslant y$, if $\mathcal{M}, y \Vdash B$ then $\mathcal{M}, y \Vdash C$

We prove the lemma of the extension of the persistence property to formulas of whichever form, not only atomic.

Lemma 2.3.2 (Persistence)**.** *Let a formula $A \in \mathsf{FM}$, $\mathcal{M}$ a kripkean intuitionistic model, $x \in \mathcal{M}$, if $\mathcal{M}, x \Vdash A$, then $\mathcal{M}, y \Vdash A$ for every $y \in \mathcal{M}$ such that $x \leqslant y$.*

Proof. The proof is by induction on the complexity of the formula A. *qed.*

Now we have to give the usual definitions of truth in a model, intuitionistic truth and logical consequence.

Definition 2.3.15. A is true in a intuitionistic model $\mathcal{M}$, in symbols $\vDash_{\mathcal{M}} A$, iff for every $x \in \mathcal{M}$, $\mathcal{M}, x \vDash A$.

Definition 2.3.16. A is an intuitionistic truth, in symbols $\vDash_{\mathcal{I}} A$, iff for every intuitionistic model $\mathcal{M}$ we have $\vDash_{\mathcal{M}} A$.

Definition 2.3.17. A is an intutionistic logical consequence from $\Gamma \subseteq \mathsf{FM}$, in symbols $\Gamma \vDash_{\mathcal{I}} A$, iff for every intuitionistic model $\mathcal{M}$, for every $x \in \mathcal{M}$ we have: if $\mathcal{M}, x \Vdash \bigwedge \Gamma$, then $\mathcal{M}, x \Vdash A$.

Theorem 2.3.3 (Completeness)**.** *For every $A \in \mathsf{FM}$, $\vdash_I A$ if and only if $\vDash_{\mathcal{I}} A$.*

The advantage offered by the approach via Kripke models is that the proof of the modal embedding is easily seen to be modularly extendable to intermediate logics, i.e. axiomatic extensions of intuitionistic logic. The strategy is again indirect: we first sketch the general structure of the proof without entering the details. We argue by contraposition, supposing that a formula A is not derivable in the intuitionistic system **I** or an extension thereof. By completeness there is a relational countermodel to such formula. The key point is that the frame on which the model is built is already a frame of the corresponding modal logic. Therefore we leave the valuation unchanged and the resulting structure is the desired modal countermodel. The final step consists in showing by means of an induction on the degree of the formulas that the worlds in the model are equivalent with respect to validity and this suffices to conclude the proof. It is worth noticing that the proof can be easily read in the opposite direction as well, thus leading to a simultaneous proof of soundness and faithfulness of the translation. The main drawback of the present approach (just as in the case of the algebraic proof) is the fact that it is spurious, because it builds on semantic results in order to establish a fact concerning derivability in a logical setting.

Theorem 2.3.4. *For every formula* $A \in \mathsf{FM}$*:* $\vdash_I A \iff \vdash_{S4} A^*$

Proof. We limit ourselves to sketching the right-to-left direction (for the details consult (15)). We argue by contraposition. If $\nvdash_I A$, then by completeness of intuitionistic logic there is a model $\langle P, v\rangle$ a world x which is a countermodel to A, i.e. $x \nVdash A$. We observe $\langle P, v\rangle$ is already an **S4** countermodel. We claim that for every world y in the model we show:

$$y \Vdash B \iff y \Vdash B^*$$

Hence we get $x \nVdash A^*$, therefore we get an **S4** countermodel which gives the desired conclusion by soundness. *qed.*

Chapter 3

A syntactic proof of the embedding

In this chapter we shall be concerned with a presentation of a purely proof-theoretic proof of the soundness and the faithfulness of the translation. We work with standard Gentzen style calculi for intuitionistic and modal logic. We introduce nested sequent calculi for intuitionistic logic and for modal logic, we recall their structural properties and then we prove the soundness and the faithfulness of the translation.

Keywords: proof theory, intuitionistic logic, termination

3.1 Preliminaries

In this section we shall be concerned with a first syntactic proof of the modal embedding of intuitionistic logic and some subintuitionistic logic. Analogous proofs have been obtained using standard sequent-style presentations for intuitionistic logic and **S4** modal logic.

We will offer a new proof which uses nested sequents. The choice is motivated by the three following observations:

- Once the structural properties of the systems are spelled out, the proof is rather elegant and concise.
- The proof transformations are minimal in the sense that the structure of intuitionistic and modal proofs are closely related and they preserve the height of the derivations.
- They can be employed to obtain a rather straightforward generalization to the case of subintuitionistic logics, i.e. weakenings of intuitionistic logic.

A *nested sequent* is a finite tree of multisets of formulas. In ordinary sequents for intuitionistic logic we distinguish between the left and the right hand side of the turnstile. To make this distinction in nested sequents, we use polarities on formulas. There are two polarities, input (intuitively as if on the left of the turnstile in the conventional sequent calculus), denoted by a $\bullet$ superscript and output (intuitively as if on the right of the turnstile), denoted by a $\circ$ superscript. Now, a nested sequent can be written as:

$$\Gamma = A_1^\bullet, ..., A_m^\bullet, B_1^\circ, ..., B_n^\circ, [\Gamma_1], ..., [\Gamma_k] \tag{3.1}$$

where $A_1^\bullet, ..., A_m^\bullet, B_1^\circ, ..., B_n^\circ$ is the multiset of formulas at the root of the sequent tree of Γ, and where $\Gamma_1, \ldots, \Gamma_k$ are its immediate subtrees. We use $\emptyset$ the *empty sequent*, i.e., where $m = n = k = 0$ in (3.1) above. We use capital Greek letters $\Gamma, \Delta, \Sigma, \ldots$, to denote nested sequents, and we assume that the associativity and commutativity of the comma is implicit in our systems, and that acts as its unit. We write $\Gamma^\bullet$ for $A_1^\bullet, ..., A_m^\bullet$ and Γ° for $B_1^\circ, ..., B_n^\circ, [\Gamma_1], ..., [\Gamma_k]$ if Γ is as in (3.1) above. In other words, for every nested sequent Γ we have that $\Gamma = \Gamma^\bullet, \Gamma^\circ$. More generally, we will write $\Gamma^\bullet, \Delta^\bullet, \Sigma^\bullet, \ldots$, for multisets of input formulas (i.e., all formulas have $\bullet$-polarity, and there are no nestings), and we will write Γ°, Δ°, $\Sigma^\circ, \ldots$, for sequents that have only $\circ$-formulas at their root nodes (i.e., there are no $\bullet$-formulas at the root, but there can be nestings with $\bullet$-formulas inside).

The *corresponding formula* of the sequent in (3.1) above is defined as

$$fm(\Gamma) = \bigwedge_{i=1}^{m} A_i \rightarrow \left(\bigvee_{j=1}^{n} B_j \vee \bigvee_{l=1}^{k} fm(\Gamma_l) \right) \tag{3.2}$$

A *(sequent) context* is a nested sequent with a hole $\{\ \}$, taking the place of a formula. Contexts are denoted by $\Gamma\{\ \}$, and $\Gamma\{\Delta\}$ is the sequent obtained from $\Gamma\{\ \}$ by replacing the occurrence of $\{\ \}$ with Δ. We write $\Gamma\{\}$ for the sequent obtained from $\Gamma\{\ \}$ by removing the $\{\ \}$ (i.e., the hole is filled with nothing). The depth of a context $\Gamma\{\ \}$, denoted by $\mathrm{dp}(\Gamma\{\ \})$, is the length of the path in the sequent tree from the root to the hole $\{\ \}$. It is defined inductively as follows: $\mathrm{dp}(\{\ \}) = 0$ and $\mathrm{dp}(\Gamma', \Gamma\{\ \}) = \mathrm{dp}(\Gamma\{\ \})$ and $\mathrm{dp}([\Gamma\{\ \}]) = \mathrm{dp}(\Gamma\{\ \}) + 1$.

We will also use the notation $\Gamma\{\![\Delta]\!\}$ as abbreviation for $\Gamma\{[\Delta]\}$.

Example. Let $\Gamma\{\ \} = A^\bullet, B^\circ, [\{\ \}, [D^\bullet, C^\circ]]$. We have that $\Gamma\{B^\circ\} = A^\bullet, B^\circ, [B^\circ, [D^\bullet, C^\circ]]$ and $\Gamma\{\} = A^\bullet, B^\circ, [[D^\bullet, C^\circ]]$. Let $\Delta = F^\bullet, [G^\circ]$, then $\Gamma\{\Delta\} = A^\bullet, B^\circ, [F^\bullet, [G^\circ], [D^\bullet, C^\circ]]$ and $\Gamma\{\![\Delta]\!\} = A^\bullet, B^\circ, [[F^\bullet, [G^\circ]], [D^\bullet, C^\circ]]$.

Initial Sequents

$$\frac{}{\Gamma\{p^\bullet, \Delta\{p^\circ\}\}} \qquad\qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet$$

Logical Rules

$$\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad\qquad \frac{\Gamma\{A^\circ\} \qquad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma\{A^\bullet\} \qquad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}}\ \vee^\bullet \qquad\qquad \frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}}\ \vee^\circ$$

$$\frac{\Gamma\{A \to B^\bullet, \Delta\{\Sigma, A^\circ\}\} \qquad \Gamma\{A \to B^\bullet, \Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{A \to B^\bullet, \Delta\{\Sigma\}\}}\ \to^\bullet \qquad \frac{\Gamma\{[A^\bullet, B^\circ]\}}{\Gamma\{A \to B^\circ\}}\ \to^\circ$$

Figure 3.1: The calculus **NIPL**

$$\frac{\Gamma\{A^\circ\} \qquad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}\ \mathsf{cut}$$

Figure 3.2: The cut rule

$$\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}}\ \mathsf{w} \quad \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}\ \mathsf{c} \quad \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}\ \mathsf{t} \quad \frac{\Gamma\{\Sigma^\circ\}}{\Gamma\{[\Sigma^\circ]\}}\ \mathsf{4} \quad \frac{\Gamma\{[\Sigma^\bullet, \Delta]\}}{\Gamma\{\Sigma^\bullet, [\Delta]\}}\ \mathsf{l} \quad \frac{\Gamma\{\Sigma^\circ, [\Delta]\}}{\Gamma\{[\Sigma^\circ, \Delta]\}}\ \mathsf{lw}$$

Figure 3.3: Admissible structural rules

3.2 Nested Sequent Calculus for intuitionistic propositional logic

An elegant nested sequent calculus for intuitionistic propositional logic was introduced by Fitting (35), as a notational variant of prefixed tableaux. The lack of a direct cut elimination proof in his calculus has prevented its extension to cover intermediate logics. Indeed, to the best of our knowledge there are no analytic nested calculi for any intermediate logic (other than classical or intuitionistic logic), and for this purpose the nested sequent formalism has been extended in various ways, giving rise to, e.g., linear nested calculi (57), and injective nested calculi (55). In general, proving syntactic cut elimination for nested calculi is harder than for other proof theoretic formalisms, e.g., (hyper)sequent or display calculus. Often this result is obtained by translating the nested calculus at hand to other formalisms, as e.g. in (45; 34).

In this section we present **NIPL**, a variant of Fitting's calculus for designed to have all invertible rules, and to admit a direct cut elimination proof. The system **NIPL**, whose rules are shown in Figure 3.1, is obtained from Fitting's calculus by using multisets instead of sets and by absorbing the rule l into the initial sequents and the rule $\rightarrow^{\bullet}$. Observe indeed that ax and $\rightarrow^{\bullet}$ can be simulated in Fitting's calculus by repeated applications of I. As an immediate consequence follows the soundness of **NIPL** w.r.t. intuitionistic propositional logic.

Terminology: As in standard sequent calculi, we call *context* the part left unchanged from premises to conclusions, we call *principal* the introduced formula in a logical rule, and the rest *active* part/formulas (active formulas in the initial sequents are $p^{\bullet}$, p°, and $\bot^{\bullet}$).

We recall that a rule is *admissible*, whenever the derivability of the premises entails the derivability of the conclusion. A rule is *invertible* if, whenever the conclusion is derivable, so is each of its premises. The *height* of a derivation is the number of nodes minus one in a branch of maximal length. These notions can be strengthened with the property of being *height-preserving*, i.e. the height does not increase in passing from the premises to the conclusion.

As we will show in the next section, **NIPL** satisfies the following properties that guarantee a relatively simple proof of the elimination of the cut rule depicted in Figure 3.2.

(N1) *All rules are height-preserving invertible.*

(N2) *Dedicated structural rules are height-preserving admissible.* These rules, displayed in Figure 3.3, are the usual weakening (w) and contraction (c), the I-rule from (35), variations of the rules for the modal axioms t and 4, from (63), and the new lw-rule which can be seen as the inverse of lift.

(N3) *A cut over formulas that are not principal can be shifted upwards over its premises.* This condition is implied by Belnap's sufficient conditions (C2)–(C7) for cut elimination in display calculi (3).

(N4) *All logical rules are reductive.* This means that they allow the replacement of a cut whose cut formula is principal in the left and right premise of the cut rule by cuts on smaller formulas (possibly using the dedicated structural rules from (N2)). This property is the nested sequent formulation of Belnap's (C8) condition (3).

(N5) *Cuts having an initial sequent as one of their premises can be removed.*

Let us mention two useful features of **NIPL**. The first is standard in well-designed sequent-style calculi: the general form of the ax-rule is derivable.

Lemma 3.2.1 (Axiom expansion)**.** *The sequent* $\Gamma\{A^{\bullet}, \Pi\{A^{\circ}, \Delta\}\}$ *is derivable in* **NIPL** *for every context* Γ, Π, Δ *and every formula* A.

Proof. By induction on the degree of the formula A. We detail the case in which A is of the shape $B \to C$, the other cases being similar.

$$\dfrac{\dfrac{\Gamma\{B \to C^\bullet, \Pi\{[B^\bullet, B^\circ, C^\circ], \Delta\}\} \qquad \Gamma\{B \to C^\bullet, \Pi\{[B^\bullet, C^\bullet, C^\circ], \Delta\}\}}{\Gamma\{B \to C^\bullet, \Pi\{[B^\bullet, C^\circ], \Delta\}\}}\, \to^\bullet}{\Gamma\{B \to C^\bullet, \Pi\{B \to C^\circ, \Delta\}\}}\, \to^\circ$$

The premises are derivable by induction hypothesis.

qed.

The second feature concerns the admissibility of the necessitation rule

$$\frac{\Gamma}{[\Gamma]}$$

Note that unlike all other rules, is shallow, as it cannot be applied inside a context.

Proposition 3.2.2. *If a sequent* Γ *is derivable, then so is* $[\Gamma]$.

3.3 Cut elimination for NIPL

We are going to show that **NIPL** satisfies conditions (N1)–(N5) and the consequent cut elimination theorem.

The preservation of the height of a derivation is crucial for all our arguments. Formally, the *height* of a derivation is the length of the longest path in the tree from its root to one of its leaves. A inference rule with premises $\Gamma_1, \ldots, \Gamma_n$ and conclusion Γ is *height-preserving invertible*, if for every derivation of Γ, there are derivations of $\Gamma_1, \ldots, \Gamma_n$ with at most the same height. The rule is *height-preserving admissible* if, whenever the premises are derivable, the conclusion has a derivation whose height is not bigger than any derivation of a premise.

Lemma 3.3.1. *The weakening rule* w *is height-preserving admissible in* NIPL.

Proof. By induction on the height n of the derivation of $\Gamma\{\emptyset\}$. If $n = 0$, then $\Gamma\{\emptyset\}$ is an initial sequent and so is $\Gamma\{\Delta\}$. If $n > 0$, we apply the induction hypothesis to the premise(s) of the last rule applied and then the rule again. *qed.*

Lemma 3.3.2. *Every rule in* NIPL *is height-preserving invertible.*

Proof. By induction on the height n of the derivation of the conclusion of each rule. The proofs for conjunction and disjunction are standard. The rule $\to^\bullet$ is height-preserving invertible by the height-preserving admissibility of the rule of weakening. We discuss the rule $\to^\circ$. If $\Gamma\{A \to B^\circ\}$ is an initial sequent, then $\Gamma\{[A^\bullet, B^\circ]\}$ is an initial sequent too. If $n > 0$, then we apply the induction hypothesis to each of the premise(s) and then we apply the rule again. *qed.*

Lemma 3.3.3. *The contraction rule* c *is height-preserving admissible in* **NIPL**.

Proof. By induction on the height n of the derivation. If $\Gamma\{\Delta, \Delta\}$ is an initial sequent, the conclusion easily follows. If $n > 0$ and the principal formula is not in Δ, we apply the induction hypothesis to each of the premises and then the rule again. If $n > 0$ and the principal formula is in Δ we exploit the height-preserving invertibility of the logical rules as shown below:

$$\dfrac{\dfrac{\Gamma\{\Delta', \Delta\}}{\Gamma\{\Delta, \Delta\}}\,\rho}{\Gamma\{\Delta\}}\,\mathsf{c} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma\{\Delta', \Delta\}}{\Gamma\{\Delta', \Delta'\}}\,\mathrm{Inv}\rho}{\Gamma\{\Delta'\}}\,\mathsf{c}}{\Gamma\{\Delta\}}\,\rho$$

The application of c is removed invoking the induction hypothesis. The case of a binary rule is analogous and we omit the details. *qed.*

The way we formulated the rules in **NIPL** allows us to establish the admissibility of the lift-rule. A variant of this rule was instead explicitly present in Fitting's system. Its absence (in combination with w and c) permits the use of the additive version of cut, which simplifies the cut elimination argument.

Lemma 3.3.4. *The* lift*-rule is height-preserving admissible in* **NIPL**.

Proof. Proceed by induction on the height n of the derivation of the premise $\Gamma\{[\Sigma^\bullet, \Delta]\}$ of the rule. If $n = 0$ and no formula in $\Sigma^\bullet$ is active, then we can remove it. Otherwise, $\Gamma\{\Sigma^\bullet, [\Delta]\}$ is again an instance of ax. If $n > 0$ and no formula in Σ is principal, we apply the induction hypothesis to the premise(s) of the rule and then the rule again.
If a formula $A^\bullet$ in $\Sigma^\bullet$ is principal in $\wedge^\bullet$ or $\vee^\bullet$, we apply the induction hypothesis (possibly twice). E.g.,

$$\dfrac{\Gamma\{[\Sigma'^\bullet, A^\bullet, B^\bullet, \Delta]\}}{\Gamma\{[\Sigma'^\bullet, A \wedge B^\bullet, \Delta]\}}\,\wedge^\bullet \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma\{[\Sigma'^\bullet, A^\bullet, B^\bullet, \Delta]\}}{\Gamma\{\Sigma'^\bullet, A^\bullet, B^\bullet, [\Delta]\}}\,\mathsf{lift}}{\Gamma\{\Sigma'^\bullet, A \wedge B^\bullet, [\Delta]\}}\,\wedge^\bullet$$

If a formula $A^\bullet$ in $\Sigma^\bullet$ is principal in $\to^\bullet$ as in

$$\dfrac{\Gamma\{[\Sigma'^\bullet, A \to B^\bullet, \Delta\{\Pi, A^\circ\}]\} \qquad \Gamma\{[\Sigma'^\bullet, A \to B^\bullet, \Delta\{\Pi, B^\bullet\}]\}}{\Gamma\{[\Sigma'^\bullet, A \to B^\bullet, \Delta\{\Pi\}]\}}\,\to^\bullet$$

we apply the induction hypothesis and the rule $\to^\bullet$, as in

$$\dfrac{\dfrac{\Gamma\{[\Sigma'^\bullet, A \to B^\bullet, \Delta\{\Pi, A^\circ\}]\}}{\Gamma\{\Sigma'^\bullet, A \to B^\bullet, [\Delta\{\Pi, A^\circ\}]\}}\,\mathsf{lift} \qquad \dfrac{\Gamma\{[\Sigma'^\bullet, A \to B^\bullet, \Delta\{\Pi, B^\bullet\}]\}}{\Gamma\{\Sigma'^\bullet, A \to B^\bullet, [\Delta\{\Pi, B^\bullet\}]\}}\,\mathsf{lift}}{\Gamma\{\Sigma'^\bullet, A \to B^\bullet, [\Delta\{\Pi\}]\}}\,\to^\bullet$$

qed.

Note that with the admissibility of the lift-rule we immediately obtain completeness of **NIPL** with respect to intuitionistic propositional logic via Fitting's system (35).

Lemma 3.3.5. *The* 4*-rule is height-preserving admissible in* **NIPL**.

Proof. By induction on the height n of the derivation of the rule premise. If $\Gamma\{\Sigma^\circ\}$ is an initial sequent, then so is $\Gamma\{[\Sigma^\circ]\}$. If $n > 0$ we assume that a formula in Σ° is principal, otherwise the proof is trivial. We apply the induction hypothesis to the premise(s) of the rule and then the rule again. For example, if the last rule applied is $\to^\circ$, we have:

$$\dfrac{\Gamma\{\Delta^\circ, [A^\bullet, B^\circ]\}}{\Gamma\{\Delta^\circ, A \to B^\circ\}} \to^\circ \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma\{\Delta^\circ, [A^\bullet, B^\circ]\}}{\Gamma\{[\Delta^\circ, [A^\bullet, B^\circ]]\}} \, 4}{\Gamma\{[\Delta^\circ, A \to B^\circ]\}} \to^\circ$$

qed.

Lemma 3.3.6. *The* lw*-rule is height-preserving admissible in* **NIPL**.

Proof. The lw-rule is derivable with the following height-preserving steps:

$$\dfrac{\dfrac{\dfrac{\Gamma\{\Sigma^\circ, [\Delta]\}}{\Gamma\{[\Sigma^\circ], [\Delta]\}} \, 4}{\Gamma\{[\Sigma^\circ, \Delta], [\Sigma^\circ, \Delta]\}} \, \mathsf{w}}{\Gamma\{[\Sigma^\circ, \Delta]\}} \, \mathsf{c}$$

qed.

Lemma 3.3.7. *The* t*-rule is height-preserving admissible in* **NIPL**.

Proof. By induction on the height n of the premise $\Gamma\{[\Delta]\}$. If $n = 0$, then $\Gamma\{[\Delta]\}$ is an initial sequent and so is $\Gamma\{\Delta\}$. If $n > 0$, we apply the induction hypothesis to the premise(s) and then the rule again. As an example, consider the case in which the last rule applied is $\to^\bullet$ and formulas are introduced (bottom-up) in $[\Delta]$. We have:

$$\dfrac{\Gamma\{A \to B^\bullet, [\Delta, A^\circ]\} \qquad \Gamma\{A \to B^\bullet, [\Delta, B^\bullet]\}}{\Gamma\{A \to B^\bullet, [\Delta]\}} \to^\bullet$$

We construct the following derivation:

$$\dfrac{\dfrac{\Gamma\{A \to B^\bullet, [\Delta, A^\circ]\}}{\Gamma\{A \to B^\bullet, \Delta, A^\circ\}} \, \mathsf{t} \qquad \dfrac{\Gamma\{A \to B^\bullet, [\Delta, B^\bullet]\}}{\Gamma\{A \to B^\bullet, \Delta, B^\bullet\}} \, \mathsf{t}}{\Gamma\{A \to B^\bullet, \Delta\}} \to^\bullet$$

where the applications of t are removed by induction hypothesis. *qed.*

This completes the proof of the properties (N1) and (N2). To eliminate cut, we also need (N3)–(N5), which will be shown below.

Theorem 3.3.8 (Cut elimination)**.** *The* cut*-rule is admissible for* **NIPL**.

Proof. We consider a uppermost cut and proceeds by induction on the lexicographically ordered pair (c, n) where c is the degree of its cut formula and n is the height of the derivation of $\Gamma\{A^\bullet\}$.[1]

(N5) If $n = 0$, then $\Gamma\{A^\bullet\}$ is an initial sequent. If $A^\bullet$ is not active, $\Gamma\{\emptyset\}$ is an initial sequent too. If $A^\bullet$ is active in , we have:

$$\dfrac{\Gamma\{p^\circ, \Delta\{p^\circ\}\} \qquad \dfrac{}{\Gamma\{p^\bullet, \Delta\{p^\circ\}\}}}{\Gamma\{\Delta\{p^\circ\}\}}\,\mathsf{cut}$$

The cut is eliminated as follows:

$$\dfrac{\dfrac{\Gamma\{p^\circ, \Delta\{p^\circ\}\}}{\Gamma\{\Delta\{p^\circ, p^\circ\}\}}\,\mathsf{lw}}{\Gamma\{\Delta\{p^\circ\}\}}\,\mathsf{c}$$

The case of axiom $\bot^\bullet$ is handled similarly, noticing that from the derivability in **NIPL** of $\Gamma\{\bot^\circ\}$ follows the derivability of $\Gamma\{\emptyset\}$.

(N3) If $n > 0$ and $A^\bullet$ is not principal, we apply the invertibility of the corresponding rule to $\Gamma\{A^\circ\}$, permute the cut upwards, and remove it by secondary induction hypothesis. E.g., in the case of a binary rule we have:

$$\dfrac{\Gamma\{A^\circ\} \qquad \dfrac{\Gamma'\{A^\bullet\} \qquad \Gamma''\{A^\bullet\}}{\Gamma\{A^\bullet\}}\,\rho}{\Gamma\{\emptyset\}}\,\mathsf{cut}$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma\{A^\circ\}}{\Gamma'\{A^\circ\}}\,\mathrm{Inv}\rho \qquad \Gamma'\{A^\bullet\}}{\Gamma'\{\emptyset\}}\,\mathsf{cut} \qquad \dfrac{\dfrac{\Gamma\{A^\circ\}}{\Gamma''\{A^\circ\}}\,\mathrm{Inv}\rho \qquad \Gamma''\{A^\bullet\}}{\Gamma''\{\emptyset\}}\,\mathsf{cut}}{\Gamma\{\emptyset\}}\,\rho$$

(N4) If $A^\bullet$ is principal in $\wedge$ or $\vee$, the case is handled in the usual way using the rules invertibility. For example

$$\dfrac{\Gamma\{B \vee C^\circ\} \qquad \dfrac{\Gamma\{B^\bullet\} \qquad \Gamma\{C^\bullet\}}{\Gamma\{B \vee C^\bullet\}}\,\vee^\bullet}{\Gamma\{\emptyset\}}\,\mathsf{cut}$$

is eliminated as follows (each cut is on a formula of lesser degree):

$$\dfrac{\dfrac{\dfrac{\Gamma\{B \vee C^\circ\}}{\Gamma\{B^\circ, C^\circ\}}\,\mathrm{Inv} \qquad \dfrac{\Gamma\{B^\bullet\}}{\Gamma\{B^\bullet, C^\circ\}}\,\mathsf{w}}{\Gamma\{C^\circ\}}\,\mathsf{cut} \qquad \Gamma\{C^\bullet\}}{\Gamma\{\emptyset\}}\,\mathsf{cut}$$

The case below in which $A^\bullet$ is principal in $\to^\bullet$

[1] It is enough to consider only the height of the left premise as every right rule is invertible.

$$\dfrac{\Gamma\{B \to C^\circ, \Pi\{\Sigma\}\} \qquad \dfrac{\Gamma\{B \to C^\bullet, \Pi\{B^\circ, \Sigma\}\} \qquad \Gamma\{B \to C^\bullet, \Pi\{C^\bullet, \Sigma\}\}}{\Gamma\{B \to C^\bullet, \Pi\{\Sigma\}\}}\,{\to^\bullet}}{\Gamma\{\Pi\{\Sigma\}\}}\,\mathsf{cut}$$

is handled using some of the structural rules from (N2). We first construct a derivation of $\Gamma\{\Pi\{B^\circ, \Sigma\}\}$:

$$\dfrac{\dfrac{\Gamma\{B \to C^\circ, \Pi\{\Sigma\}\}}{\Gamma\{B \to C^\circ, \Pi\{B^\circ, \Sigma\}\}}\,\mathsf{w} \qquad \Gamma\{B \to C^\bullet, \Pi\{B^\circ, \Sigma\}\}}{\Gamma\{\Pi\{B^\circ, \Sigma\}\}}\,\mathsf{cut}$$

The cut is removed by secondary induction hypothesis. A symmetrical derivation yields $\Gamma\{\Pi\{C^\bullet, \Sigma\}\}$, and the reduction is completed as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma\{\Pi\{B^\circ, \Sigma\}\}}{\Gamma\{\Pi\{B^\circ, C^\circ, \Sigma\}\}}\,\mathsf{w} \qquad \dfrac{\dfrac{\dfrac{\Gamma\{B \to C^\circ, \Pi\{\Sigma\}\}}{\Gamma\{\Pi\{B \to C^\circ, \Sigma\}\}}\,\mathsf{lw}}{\Gamma\{\Pi\{[B^\bullet, C^\circ], \Sigma\}\}}\,\mathsf{Inv}}{\Gamma\{\Pi\{B^\bullet, C^\circ, \Sigma\}\}}\,\mathsf{t}}{\Gamma\{\Pi\{C^\circ, \Sigma\}\}}\,\mathsf{cut} \qquad \Gamma\{\Pi\{C^\bullet, \Sigma\}\}}{\Gamma\{\Pi\{\Sigma\}\}}\,\mathsf{cut}$$

The cuts are removed by the primary induction hypothesis on the degree of the cut formula. *qed.*

We can now also show completeness independently from Fitting's calculus:

Corollary. **NIPL** *is complete with respect to intuitionistic propositional logic.*

Proof. It is easy to check that every axiom of intuitionistic propositional logic can be proved in **NIPL** and modus ponens can be simulated by cut. The claim follows by Theorem 3.3.8. *qed.*

3.4 A new nested sequent system for S4

We now present a new nested sequent system for **S4** which is specifically tailored to prove the soundness and the faithfulness of the modal interpretation of intuitionistic logic. The rules are displayed in Figure 3.4. The modal rules are split according to the shape of the formulas in the scope of the modal operator $\Box$.

We now start the structural analysis of our calculus.

Lemma 3.4.1. *The sequent* $\Gamma\{A^\bullet, A^\circ\}$ *is derivable for every formula* A.

Proof. The proof is by induction on the degree of A. *qed.*

Lemma 3.4.2. *The rule of weakening is height-preserving admissible.*

Initial Sequents

$$\frac{}{\Gamma\{\Box p^\bullet, \Delta\{\Box p^\circ\}\}}\ \mathsf{ax}_1 \qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet$$

$$\frac{}{\Gamma\{p^\bullet, p^\circ\}}\ \mathsf{ax}_2 \qquad \frac{}{\Gamma\{\Box p^\bullet, \Delta\{p^\circ\}\}}\ \mathsf{ax}_3$$

Logical Rules

$$\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}}\ \vee^\bullet \qquad \frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}}\ \vee^\circ$$

$$\frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to B^\bullet\}}\ \to^\bullet \qquad \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \to B^\circ\}}\ \to^\circ$$

$$\frac{\Gamma\{\Box(A \wedge B)^\bullet, \Delta\{\Sigma, A^\bullet, B^\bullet\}\}}{\Gamma\{\Box(A \wedge B)^\bullet, \Delta\{\Sigma\}\}}\ \Box^\bullet_\wedge \qquad \frac{\Gamma\{[A^\circ]\} \quad \Gamma\{[B^\circ]\}}{\Gamma\{\Box(A \wedge B)^\circ\}}\ \Box^\circ_\wedge$$

$$\frac{\Gamma\{\Box(A \vee B)^\bullet, \Delta\{\Sigma, A^\bullet\}\} \quad \Gamma\{\Box(A \vee B)^\bullet, \Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{\Box(A \vee B)^\bullet, \Delta\{\Sigma\}\}}\ \Box^\bullet_\vee \qquad \frac{\Gamma\{[A^\circ, B^\circ]\}}{\Gamma\{\Box(A \vee B)^\circ\}}\ \Box^\circ_\vee$$

$$\frac{\Gamma\{\Box(A \to B)^\bullet, \Delta\{\Sigma, A^\circ\}\} \quad \Gamma\{\Box(A \to B)^\bullet, \Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{\Box(A \to B)^\bullet, \Delta\{\Sigma\}\}}\ \Box^\bullet_\to \qquad \frac{\Gamma\{[A^\bullet, B^\circ]\}}{\Gamma\{\Box(A \to B)^\circ\}}\ \Box^\circ_\to$$

$$\frac{\Gamma\{\Box\Box A^\bullet, \Delta\{\Sigma, A^\bullet\}\}}{\Gamma\{\Box\Box A^\bullet, \Delta\{\Sigma\}\}}\ \Box^\bullet_\Box \qquad \frac{\Gamma\{[[A^\circ]]\}}{\Gamma\{\Box\Box A^\circ\}}\ \Box^\circ_\Box$$

Figure 3.4: The calculus **NS4**

Proof. By straightforward induction on the height of the derivation. *qed.*

An interesting result concerning our calculus is that we do not need explicit rules to handle formulas of the shape $\Box p$.

Theorem 3.4.3. *The rules:*

$$\frac{\Gamma\{\Box p^\bullet, \Delta\{p^\bullet, \Sigma\}\}}{\Gamma\{\Box p^\bullet, \Delta\{\Sigma\}\}}\ \Box^\bullet_{at} \quad \frac{\Gamma\{[p^\circ]\}}{\Gamma\{\Box p^\circ\}}\ \Box^\circ_{at} \quad \frac{\Gamma\{\Box p^\circ\}}{\Gamma\{[p^\circ]\}}\ Inv\Box^\circ_{at}$$

are height-preserving admissible in **NS4**.

Proof. The proof is by induction on the height of the derivation. Since $\Box p$ cannot be principal in an application of a rule, we need to discuss only the cases in which the premises are initial sequents. In all three cases it is enough to observe that if the premise is an initial sequent, so is the conclusion. For example, let us consider the case of the rule $Inv\Box^\circ_{at}$. If it is an initial sequent and $\Box p^\circ$ is not active, then $\Gamma\{[p^\circ]\}$ is an initial sequent too. If it is an initial sequent and $\Box p^\circ$ is active, then the sequent is an instance of ax_1 of the shape $\Gamma\{\Box p^\bullet, \Delta\{\Box p^\circ\}\}$ and so $\Gamma\{\Box p^\bullet, \Delta\{[p^\circ]\}\}$ is an instance of ax_3. *qed.*

Lemma 3.4.4. *Every rule is height-preserving invertible.*

Proof. By induction on the height of the derivation. The modal rules which act on input formulas are height-preserving invertible by height-preserving admissibility of weakening. We consider the case of the modal rule $\Box_\vee^\circ$. If $\Gamma\{\Box(A \vee B)^\circ\}$ is an initial sequent, so is $\Gamma\{[A^\circ, B^\circ]\}$. If it is the conclusion of a rule and $\Box(A \vee B)^\circ$ is principal, we take the premise. Otherwise we apply the induction hypothesis to the premises of the rule applied and then the rule again. *qed.*

Lemma 3.4.5. *The contraction rule is height-preserving admissible.*

Proof. By induction on the height of the derivation exploiting the invertibility of the rules. The proof follows the structure of the one detailed for **NIPL**, so we avoid giving the details. *qed.*

Lemma 3.4.6. *The rules:*

$$\frac{\Gamma\{[\Delta]\}}{\Gamma\{[[\Delta]]\}}\,4 \qquad \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}\,\mathsf{t}$$

are height-preserving admissible.

Proof. The proof follows the structure of the one detailed for **NIPL**, so we avoid giving the details. *qed.*

A similar strategy as the one presented before yields a proof of the cut-elimination theorem.

Theorem 3.4.7. *The* cut *rule is admissible in* **NS4**.

Proof. The proof is by double induction, with main induction hypothesis on the degree of the cut formula and secondary induction hypothesis on the height of the derivation of the right premise of the cut. We distinguish cases according to the shape of the cut formula.

If the cut formula is atomic, then the proof is immediate.

If the cut formula is of the shape $A \wedge B, A \vee B$ or $A \to B$ we appeal to the invertibility of the corresponding connective to replace the cut with cuts on formulas of lesser degree.

If the formula is $\Box A$ we have to distinguish two cases. Either $\Box A^\bullet$ is principal in the right premise of the cut or not. If it is not principal, then we can permute the cut upwards and replace it with cuts of lesser height (since all the right rules are invertible). If it is principal we need to consider five subcases. We limit ourselves to deal with the ones in which $A \equiv \Box p$, $A \equiv \Box(B \to C)$ and $A \equiv \Box B$.

$$\frac{\Gamma\{\Box p^\circ\} \qquad \Gamma\{\Box p^\bullet\}}{\Gamma\{\}}\, Cut$$

We distinguish two subcases. Either $\Gamma\{\Box p^\bullet\}$ is an initial sequent or not. If not, then $\Box p$ is never principal and the cut can be permuted upwards. If it is an initial sequent we assume that $\Box p$ is active (otherwise the reduction is trivial) and we distiguish cases. The premise is of the shape: $\Gamma\{\Box p, \Delta\{\Sigma, p^\circ\}^\bullet\}$ or $\Gamma\{\Box p^\bullet, \Delta\{\Sigma, \Box p^\circ\}\}$. In the second case the cut is eliminated as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma\{\Box p^\circ, \Delta\{\Sigma, \Box p^\circ\}\}}{\Gamma\{[p^\circ], \Delta\{\Sigma, \Box p^\circ\}\}}\, Inv\Box^\circ_{at}}{\Gamma\{\Delta\{\Sigma, [p^\circ], \Box p^\circ\}\}}\, \mathsf{w,4,c}}{\Gamma\{\Delta\{\Sigma, [p^\circ], [p^\circ]\}\}}\, Inv\Box^\circ_{at}}{\Gamma\{\Delta\{\Sigma, [p^\circ]\}\}}\, c}{\Gamma\{\Delta\{\Sigma, \Box p^\circ\}\}}\, \Box^\circ_{at}$$

The other case is analogous and so we omit the details. If the cut formula is $\Box(A \to B)$ we have:

$$\cfrac{\cfrac{\Gamma\{[A^\bullet, B^\circ], \Delta\{\Sigma\}\}}{\Gamma\{\Box(A \to B)^\circ, \Delta\{\Sigma\}\}}\, \Box^\circ_\to \qquad \cfrac{\Gamma\{\Box(A \to B)^\bullet, \Delta\{\Sigma, A^\circ\}\} \qquad \Gamma\{\Box(A \to B)^\bullet, \Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{\Box(A \to B)^\bullet, \Delta\{\Sigma\}\}}\, \Box^\bullet_\to}{\Gamma\{\Delta\{\Sigma\}\}}\, Cut$$

First, we apply height-preserving admissibility of weakening to get $\Gamma\{\Box(A \to B)^\circ, \Delta\{\Sigma, A^\circ\}\}$ and $\Gamma\{\Box(A \to B)^\circ, \Delta\{\Sigma, B^\bullet\}\}$. Hence we perform two cross-cuts to obtain: $\Gamma\{\Delta\{\Sigma, B^\bullet\}\}$ and $\Gamma\{\Delta\{\Sigma, A^\circ\}\}$.

$$\cfrac{\cfrac{\cfrac{\Gamma\{\Delta\{\Sigma, A^\circ\}\}}{\Gamma\{\Delta\{\Sigma, A^\circ, B^\circ\}\}}\, \mathsf{w} \qquad \cfrac{\cfrac{\Gamma\{[A^\bullet, B^\circ], \Delta\{\Sigma\}\}}{\Gamma\{\Delta\{\Sigma, [A^\bullet, B^\circ]\}\}}\, \mathsf{4,c}}{\Gamma\{\Delta\{\Sigma, A^\bullet, B^\circ\}\}}\, \mathsf{t}}{\Gamma\{\Delta\{\Sigma, B^\circ\}\}}\, Cut \qquad \Gamma\{\Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{\Delta\{\Sigma\}\}}\, Cut$$

The corsscuts are removed by secondary induction hypothesis, whereas the displayed cuts are removed by induction on the degree of the cut formula.

If the cut formula is $\Box\Box B$, we have:

$$\cfrac{\cfrac{\Gamma\{[[B^\circ]], \Delta\{\Sigma\}\}}{\Gamma\{\Box\Box B^\circ, \Delta\{\Sigma\}\}}\, \Box^\circ_\Box \qquad \cfrac{\Gamma\{\Box\Box B^\bullet, \Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{\Box\Box B^\bullet, \Delta\{\Sigma\}\}}\, \Box^\bullet_\Box}{\Gamma\{\Delta\{\Sigma\}\}}\, Cut$$

We construct the following derivation:

$$\cfrac{\cfrac{\cfrac{\Gamma\{[[B^\circ]], \Delta\{\Sigma\}\}}{\Gamma\{\Delta\{\Sigma, [[B^\circ]]\}\}}\, \mathsf{4,c}}{\Gamma\{\Delta\{\Sigma, B^\circ\}\}}\, \mathsf{t} \qquad \cfrac{\cfrac{\Gamma\{\Box\Box B^\circ, \Delta\{\Sigma\}\}}{\Gamma\{\Box\Box B^\circ, \Delta\{\Sigma, B^\bullet\}\}}\, \mathsf{w} \qquad \Gamma\{\Box\Box B^\bullet, \Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{\Delta\{\Sigma, B^\bullet\}\}}\, Cut}{\Gamma\{\Delta\{\Sigma\}\}}\, Cut$$

The topmost cut is removed by secondary induction hypothesis, whereas the lowermost is removed by primary induction hypothesis on the degree of the cut formula. *qed.*

Theorem 3.4.8. **NS4** *is sound and complete with respect to Kripke semantics for* **S4**.

Proof. Soundness is easily established by induction on the height of the derivations. Completeness is established by observing that the axioms of **S4** are derivable and that the modus ponens can be simulated via cut. *qed.*

We are now in the position to state and prove the embedding of intuitionistic logic into the modal logic **S4** and to give a syntactic proof of the result. Compared to other proofs, our result has the following advantages:

- The proof follows from a trivial induction.
- The proof is completely syntactic.
- The height of the derivation is preserved in both directions.

In essence, it could be argued that the two calculi are strongly similar in the sense that there is a step to step correspondence in the translation.

Theorem 3.4.9. **NIPL** $\vdash \Gamma$ *if and only if* **NS4** $\vdash \Gamma^*$ *and the height is preserved.*

Proof. The proof is by induction on the height of the derivation in both directions.

$\Rightarrow$ If Γ is an initial sequent in **NIPL**, then Γ^* is an initial sequent in **NS4**. If $n > 0$, the proof follows by applying the induction hypothesis and then the rule. For example, we have:

$$\frac{\Gamma\{[A^\bullet, B^\circ]\}}{\Gamma\{A \to B^\circ\}} \to^\circ$$

We transform the derivation as follows:

$$\frac{\Gamma^*\{[A^{*\bullet}, B^{*\circ}]\}}{\Gamma^*\{\Box(A^* \to B^*)^\circ\}} \Box^\circ_\to$$

$\Leftarrow$ If $n = 0$, then Γ is an initial sequent in **NIPL**. In all the other cases the proof immediately follows by applying the induction hypothesis to the premise(s) of the last rule applied and then the corresponding rule in **NIPL**.

qed.

3.5 Concluding remarks

In this chapter we have proposed a syntactic proof of the modal embedding of intuitionistic logic into the modal logic **S4**. Other proofs in the literature can be found in (109) and, more recently, in (24). A future research direction to be explored might be to exploit these nested sequents in order to study subintuitionistic logics, see (19) and (14) for an introduction. The use of nested calculi seems particularly promising to obtain an extension of the present result to such setting. We would like to point out that the present approach has been extended in order to get analytic and internal (which means without the explicit use of semantic labelling) calculi for the family of intermediate logics of bounded depth (106).

Chapter 4

A formal provability interpretation

In this chapter we deal with the proof of the modal embedding for the logic of provability **GL**. In order to give a syntactic proof of the embedding, we introduce a new semantic characterization of intuitionistic propositional logic. Hence we extract a new labelled sequent calculus which allows us to prove the embedding via a straightfoward induction on the height of the derivations. In doing so, we also analyze the properties of the new calculus which enjoys terminating proof search and can be extended to all intermediate logics with a universal frame condition.

Keywords: proof theory, intuitionistic logic, termination, intermediate logics, provability logic.

4.1 Introduction

The modal embedding of intuitionistic logic gives an interpretation in terms of informal provability. Indeed, interpreting intuitionistic logic in **S4** brings to the fore an epistemic reading of the constructive content of intuitionistic logic. The notion of *informal* proof is crucial, as **S4** cannot be understood as a logic of formal provability, say, in an arithmetic system. This stems from the fact that **S4** validates the reflection schema $\Box A \to A$ and, as a consequence, the schema $\neg\Box\bot$, intuitively asserting the consistency of arithmetic.

However, it is natural to ask whether it is possible to recover an interpretation of intuitionistic propositional logic in terms of provability in a formal system. The answer is positive, because embedding of intuitionistic logic into the modal logics of provability **Grz** and **GL** have been provided. The embeddings have been established first by semantic means, see (31; 56). We have already observed that syntactic proofs of the embedding have been proposed by various authors for the modal logic **S4**. With respect to **Grz**, a syntactic proof was given in (28).

However, a syntactic proof of the embedding for the logic **GL** still was not established. The problem seems to be connected to the difference between the calculi for intuitionistic logic and the modal logic of provability **GL**. In order to propose a solution to this problem we shall introduce a new calculus for intuitionistic propositional logic. As a byproduct, we shall obtain a general methodology to obtain terminating sequent calculi for intuitionistic and intermediate logics. This connects the present work with the extensively discussed topic of terminating sequent calculi for intuitionistic logic and their extensions which we briefly summarize below.

We proceed by considering a slight modification of the usual semantics of intuitionistic logic: we take Kripke frames based on strict orders rather than on partial orders. Then it is easily checked that intuitionistic logic is complete with respect to the class of finite strict orders.

Accordingly, we modify the definition of the truth condition for the implication, by internalizing the finiteness condition of the models (for a similar approach in the context of the modal logic **GL**, see (73)). Thus, by adopting a suitable extension of the basic language of intuitionistic logic, we are in the position to extract a labelled calculus $\mathbf{G3I}_{<}$ from such semantics and we establish its structural properties and then we prove the termination of the proof search.

It is well known that the first proof of decidability of the propositional fragment of intuitionistic logic is due to Gerhard Gentzen. He showed by means of a purely syntactic argument based on the sequent calculus **LJ** that the proof search space could be finitized (39). However, the calculus **LJ** is not totally satisfactory in terms of proof search. This is one of the reasons why modifications of **LJ** have been proposed. In particular, Kleene introduced a single-succedent calculus in which structural rules are absorbed in logical rules (53). Then Maehara devised a multi-succedent calculus **G3i** (59). However, although the contraction rule is not explicitly present in both calculi, it is, so to say, hidden in the left rule for implication which requires the repetition of the principal formula in the antecedent of the premise. Furthermore, although we gain the invertibility of the right rule for disjunction, we lose the invertibility of the right rule for implication.

Hudelmaier (51) and Dyckhoff (24) (independently) elaborating on an idea by Vorob'ev (114), introduced the calculus **G4ip**. In **G4ip** every proof search terminates, because the rule of contraction is eliminated from the calculus by splitting the left implication rule in four different rules according to the shape of the antecedent of the principal formula. The calculus, which has been given also a multisuccedent variant as well as a syntactic proof of cut-elimination (26), eliminates the need of loop-checking, but does not enjoy invertibility of every rule.

More recently, other approaches have been considered to obtain a decision procedure for intuitionistic logic. In particular, Corsi and Tassi (20) presented a calculus which enjoys the subformula property and terminates without the use of *global metarules*. Global metarules are rules which govern the application of the rules of the calculus and whose scope extends to the entire derivation. However, the syntax of the system is rather complex, the rules of the calculi are not invertible and the methodology does not seem to easily extend to other intermediate logics.

Semantic-oriented approaches to the proof theory of intuitionistic logic, such as tableaux calculi and labelled sequent calculi (27) yield systems with good structural properties, but with a more complex decision procedure which requires a loop-checking mechanism in order to obtain termination (74).

As observed by Dyckhoff in (29), there remain some open problems:

1. Find a simple calculus for propositional intuitionistic logic that has the termination property without loop-checking and avoids backtracking via invertibility of the rules. Furthermore, the system should allow for an extraction of a finite countermodel out of a failed proof search.

2. Find a modular approach, i.e. an approach which can be extended so as to cover various superintuitionistic logics.

3. Develop a uniform method for ensuring termination in labelled calculi.

We provide - to the best of our knowledge - the first solution to these problems. In fact, our calculus satisfies the following *desiderata*: invertibility of every rule, extraction of a finite countermodel from failed derivations, an easy termination procedure without loop-checking and the possibility to modularly extend the approach to stronger systems.

The termination of the proof search without loop-checking and backtracking is proved for the system $\mathbf{G3I}_{<}$ exploiting the height-preserving invertibility of every rule as well as the height-preserving admissibility of contraction. Termination is established by showing that the number of variables generated in the proof search process is finite due to the peculiar formulation of the rules. Furthermore, we can directly extract a finite intuitionistic countermodel out of a failed proof search.

We then generalize even further the method to a large class of intermediate logics. In particular, exploiting the modularity of the framework of labelled sequent calculi, we provide a terminating calculus without backtracking and with finite countermodel extraction for every intermediate logic whose frame condition is a universal formula. All the logics discussed, except for Gödel-Dummett logic (25), did not have a terminating calculus.

Finally, we use the new calculi for intuitionistic and universal intermediate logics to obtain a uniform proof of the modal embedding for the logic **GL** and its extensions. The proof is carried out by an induction on the height of the derivation distinguishing cases according to the last rule applied and exploiting the invertibility of the rules of the calculus.

The structure of the chapter is as follows. In the second section we introduce a variation with respect to the usual semantics for intuitionistic logic. In the third section we present the labelled sequent calculus $\mathbf{G3I}_{<}$ and we discuss its structural properties. In the fourth section we prove completeness and termination and we also show how to extract a finite countermodel from a failed proof search. The fifth section is devoted to the study of the extensions of the methodology to intermediate propositional logics. In particular, we show that a large class of intermediate propositional logics can be given a terminating calculus, thus yielding an effective decision procedure.

4.2 An alternative semantics for propositional intuitionistic logic

Intuitionistic logic is complete with respect to the class of *finite* partial orders (15). We refer to these models with the name of standard models.

Definition 4.2.1. A *standard frame* for intuitionistic logic is a pair $\langle P, \leqslant \rangle$ where P is a finite set and $\leqslant$ is a partial order defined on P. A *standard model* is a triple $\mathcal{M} = \langle P, \leqslant, v \rangle$, where $\langle P, \leqslant \rangle$ is a standard frame and $v : AT \to \mathcal{P}(P)$ is a function such that if $x \in v(p)$ and $x \leqslant y$, then $y \in v(p)$. Truth conditions for formulas are inductively defined:

- $\mathcal{M}, x \Vdash p$ iff $x \in v(p)$
- $\mathcal{M}, x \Vdash A \wedge B$ iff $\mathcal{M}, x \Vdash A$ and $\mathcal{M}, x \Vdash B$
- $\mathcal{M}, x \Vdash A \vee B$ iff $\mathcal{M}, x \Vdash A$ or $\mathcal{M}, x \Vdash B$

- $\mathcal{M}, x \Vdash A \to B$ iff $\forall y (if\ x \leqslant y\ and\ \mathcal{M}, y \Vdash A,\ then\ \mathcal{M}, y \Vdash B)$

A formula A is valid in a standard model, $\mathcal{M} \vDash A$, if and only if $\mathcal{M}, x \Vdash A$ for every x in $\mathcal{M}$. A formula A is valid in a standard frame $\mathcal{F}$, $\vDash_{\mathcal{F}} A$, if and only if it is valid in every standard model based on that frame. A formula A is valid in a class C of standard frames, $\vDash_C A$, if and only if it is valid in every frame in the class.

We introduce a variant of such semantic, based on strict orders rather than partial orders, i.e. orders with an irreflexive and transitive relation.

Definition 4.2.2. A *strict frame* for intuitionistic logic is a pair $\langle Q, < \rangle$ where Q is a finite set and $<$ is a strict order defined on Q. A *strict model* is a triple $\mathcal{M} = \langle Q, <, v \rangle$, where $\langle Q, < \rangle$ is a strict frame and $v : AT \to \mathcal{P}(Q)$ is a function such that if $x \in v(p)$ and $x < y$, then $y \in v(p)$.

Truth conditions for formulas are unchanged with respect to standard models, except for the implication:

$$\mathcal{M}, x \Vdash A \to B \text{ iff } \forall y (if\ x < y\ and\ \mathcal{M}, y \Vdash A,\ then\ \mathcal{M}, y \Vdash B) \text{ and (if } \mathcal{M}, x \Vdash A\text{, then } \mathcal{M}, x \Vdash B)$$

The notions of validity in a strict model, in a strict frame and in a class of strict frames are defined as above.

We now show that given a standard model it is always possible to construct a strict model which satisfies the same formulas.

Theorem 4.2.1. *For every standard model $\mathcal{M} = \langle P, \leqslant, v \rangle$, there is a strict model $\mathcal{N}$ and such that every formula $A \in \mathsf{FM}$,*

$$\mathcal{M} \vDash A \text{ if and only if } \mathcal{N} \vDash A$$

Proof. Given the model $\mathcal{M} = \langle P, \leqslant, v \rangle$, we consider the strict model $\mathcal{N} = \langle P, <, v \rangle$, where $x < y$ iff $x \leqslant y$ and $x \neq y$ for every $x, y \in P$. The relation $<$ is clearly irreflexive and transitive.

We check the claim of the theorem by showing that for every x in P, $\mathcal{M}, x \Vdash A$ if and only if $\mathcal{N}, x \Vdash A$. We argue by induction on the complexity of the formula A.

The atomic case follows by definition of v, the cases of conjunction and disjunction are immediate by induction hypothesis.

We discuss the case of implication. We consider the direction from left to right, the other is similar and we omit the details.

Let us assume $\mathcal{M}, x \Vdash A \rightarrow B$ and we suppose $x < y$ and $\mathcal{N}, y \Vdash A$. If $x < y$, then $x \leqslant y$ in $\mathcal{M}$, therefore by induction hypothesis we obtain $\mathcal{M}, y \Vdash A$ and by definition $\mathcal{M}, y \Vdash B$. Again, by induction hypothesis we conclude that $\mathcal{N}, y \Vdash B$.

Furthermore, if $\mathcal{N}, x \Vdash A$, since $x \leqslant x$ in $\mathcal{M}$, then, by using induction hypothesis twice, we obtain $\mathcal{N}, x \Vdash B$. This yields $\mathcal{N}, x \Vdash A \rightarrow B$. *qed.*

We now show that we can present an alternative reformulation of the truth condition for the implication which incorporates information relative to the finiteness of the model. We introduce an abbreviation: for every strict model $\mathcal{M}$ and for every world x in $\mathcal{M}$:

$$x \Vdash A > B \Longleftrightarrow \text{for every } y, \text{ if } x < y \text{ and } y \Vdash A, \text{ then } y \Vdash B$$

Lemma 4.2.2. *For every intuitionistic strict model $\mathcal{M}$, every world x in $\mathcal{M}$, the following are equivalent:*

1. $x \Vdash A > B$

2. *For every y: if $x < y$ and $y \Vdash A > B$ and $y \Vdash A$, then $y \Vdash B$*

Proof. $1 \Rightarrow 2$ is immediate.

We discuss $2 \Rightarrow 1$. Let us assume that 2 holds for x. Let $x < y$ and $y \Vdash A$, we suppose by contradiction that $y \Vdash A > B$ does not hold. Therefore there is z, $y < z$, $z \Vdash A$ and $z \nVdash B$. Now $z \nVdash A > B$, otherwise, since $x < z$ (by transitivity), $z \Vdash B$, which yields a contradiction, therefore there is u such that $z < u$. By iterating this procedure we obtain an infinite chain, against the finiteness condition. Thus $y \Vdash A > B$ and by 2 we obtain $y \Vdash B$, which concludes the proof. *qed.*

This allows us to reformulate the truth condition for implication in strict models.

$(*)$ $x \Vdash A \rightarrow B$ iff the following conditions hold:

1. for every y: if $x < y$ and $y \Vdash A > B$ and $y \Vdash A$, then $y \Vdash B$

2. if $x \Vdash A$, then $x \Vdash B$

A sequent is a syntactic object of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite multisets of formulas in FM. Let **G3i** be the multisuccedent sequent calculus for intuitionistic propositional logic displayed in Figure 4.1. The notion of derivation is defined as follows.

Definition 4.2.3. A derivation in **G3i** of a sequent $\Gamma \Rightarrow \Delta$ is a finite tree in which topmost nodes are instances of an initial sequent and the lower nodes are formed by applications of the rules.

Initial Sequents

$$\frac{}{p, \Gamma \Rightarrow \Delta, p}\ ax \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{A \to B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L\to \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B}\ R\to$$

Figure 4.1: The multisuccedent sequent calculus for intuitionistic propositional logic **G3i**.

Exploiting the completeness of **G3i** with respect to standard Kripke semantics (107), we obtain a result of completeness of intuitionistic logic with respect to finite strict orders in which the truth condition for the implication is rephrased as in $(*)$.

Theorem 4.2.3 (Completeness)**. G3i** $\vdash\Rightarrow A$ *if and only if* A *is valid in every strict frame.*

Proof. From left to right the proof is straightforward by induction on the height of derivation in the calculus **G3i**.
From right to left we argue by contraposition. If $\Rightarrow A$ is not derivable in **G3i**, then there is a standard intuitionistic model $\mathcal{M}$ based on a finite partial order and a world in $\mathcal{M}$ such that $x \nVdash A$. By Theorem 4.2.1 and Lemma 4.2.2 we obtain the desired conclusion. *qed.*

4.3 The labelled sequent calculus $\mathbf{G3I}_{<}$

We are now in the position to define a labelled sequent calculus from the semantics we introduced in the previous section. Let FM^{+} be the set of formulas built from propositional atoms, the zeroary connective $\bot$ and the binary connectives $\wedge$, $\vee$,

Initial Sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ ax_1 \qquad \frac{}{x < y, x : p, \Gamma \Rightarrow \Delta, y : p}\ ax_2$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Logical Rules

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ R\wedge$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ R\vee$$

$$\frac{x : A > B, \Gamma \Rightarrow \Delta, x : A \qquad x : B, x : A > B, \Gamma \Rightarrow \Delta}{x : A \rightarrow B, \Gamma \Rightarrow \Delta}\ L\rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, x : A > B \qquad x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \rightarrow B}\ R\rightarrow$$

$$\frac{x < y, x : A > B, \Gamma \Rightarrow \Delta, y : A \qquad y : B, x < y, x : A > B, \Gamma \Rightarrow \Delta}{x < y, x : A > B, \Gamma \Rightarrow \Delta}\ L >$$

$$\frac{x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A > B}\ R >, y \text{ fresh}$$

Relational Rules

$$\frac{x < y, y < z, x < z, \Gamma \Rightarrow \Delta}{x < y, y < z, \Gamma \Rightarrow \Delta}\ Trs \qquad \frac{}{x < x, \Gamma \Rightarrow \Delta}\ Irref$$

Figure 4.2: The labelled sequent calculus $\mathbf{G3I}_<$.

$\rightarrow$ and $>$. The language of the calculus $\mathbf{G3I}_<$ contains a countable set of labels $x_0, x_1, x_2, \ldots$

The expressions in the calculus are either labelled formulas or relational atoms. A labelled formula is a syntactic object of the form $x : A$, where $A \in \mathsf{FM}^+$. A relational atom is a syntactic object of the form $x < y$, where x, y are labels and $<$ is a binary relation symbol. Labelled sequents are objects of the form $\Gamma \Rightarrow \Delta$, where Γ is a finite multiset of labelled formulas and relational atoms and Δ is a finite multiset of labelled formulas.

The rules are directly obtained by the truth conditions for the logical operators, for the details concerning the general procedure the reader is referred to (73). The initial sequent ax_2 is added in order to express the monotonicity of the valuation function of intuitionistic models. The rule R$>$ directly stems from the right to left side of the truth definition for $>$. The freshness condition imposed on the rule R$>$ implies that the variable y does not occur in the conclusion of the rule: we say that y is the *eigenvariable* of the inference.

We aim at establishing the structural properties of the calculus $\mathbf{G3I}_<$ by syntactic means. The rules of the system $\mathbf{G3I}_<$ are height-preserving invertible and the structural rules of weakening and contraction are height-preserving admissible. We shall then establish cut-free completeness of the calculus $\mathbf{G3I}_<$ by showing that every derivation in the calculus for intuitionistic propositional logic **G3i** can be transformed into a derivation in $\mathbf{G3I}_<$.

In order to proceed with the structural analysis of the calculus we need to fix some notation and parameters. The principal formula of a rule is the labelled formula displayed in the conclusion. The formulas displayed in the initial sequents are said to be active. The *height* of a derivation is again defined as the length of a maximal branch in the derivation.

Definition 4.3.1. Given a formula A in FM^+, its degree $dg(A)$ is thus defined:

- $dg(p) = dg(\bot) = 0$
- $dg(A \wedge B) = dg(A \vee B) = dg(A > B) = max\{dg(A), dg(B)\} + 1$
- $dg(A \to B) = dg(A > B) + 1$

The degree of a labelled formula $x : A$ coincides with the degree of the formula A.

Lemma 4.3.1. *For every* Γ, Δ *and* A *in* FM^+, *the sequent* $x : A, \Gamma \Rightarrow \Delta, x : A$ *is derivable in* $\mathbf{G3I}_<$.

Proof. The proof is by induction on the degree of the labelled formula $x : A$, we discuss the case in which the labelled formula is of the form $x : B > C$.

$$\dfrac{\dfrac{x : B > C, x < y, y : B > C, y : B, \Gamma \Rightarrow \Delta, y : C, y : B \qquad x : B > C, x < y, y : B > C, y : C, y : B, \Gamma \Rightarrow \Delta, y : C}{x : B > C, x < y, y : B > C, y : B, \Gamma \Rightarrow \Delta, y : C}\text{L>}}{x : B > C, \Gamma \Rightarrow \Delta, x : B > C}\text{R>}$$

The topmost sequents are derivable by induction hypothesis. *qed.*

A rule is *height-preserving admissible* whenever there are derivations of each of its premises, there is a derivation of the conclusion of the same height or of less height. Given a sequent $\Gamma \Rightarrow \Delta$, we denote by $\Gamma[x/y] \Rightarrow \Delta[x/y]$ the sequent obtained by replacing all the occurrences of x in $\Gamma \Rightarrow \Delta$ with y.

Lemma 4.3.2. *The rule:*

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma[x/y] \Rightarrow \Delta[x/y]}\, Sub[x/y]$$

is height-preserving admissible in $\mathbf{G3I}_<$.

Proof. The proof runs by induction on the height n of the derivation of the sequent $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_<$. If $n = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent and so is $\Gamma[x/y] \Rightarrow \Delta[x/y]$. If $n > 0$ and the last rule applied is different from R>, we apply the induction hypothesis and then the rule again. If $n > 0$ and the last rule applied is R>, we distinguish two subcases: y coincides with the eigenvariable of the rule or not. If not, we apply the induction hypothesis and then the rule again. If it does, then we apply the induction hypothesis to replace the eigenvariable of the rule with another fresh variable not occurring in Γ and Δ, then we apply again the induction hypothesis to substitute x with y and then we apply the rule R> to obtain the desired conclusion. *qed.*

Lemma 4.3.3. *The rule:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'} \, Weak$$

is height-preserving admissible in $\mathbf{G3I}_<$.

Proof. The proof is straightforward by induction on the height n of derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_<$. *qed.*

We now show that a generalized version of the initial sequent ax_2 is derivable.

Lemma 4.3.4. *For every formula A and every multiset Γ, Δ, the sequent $x < u_1, ..., u_n < y, x : A, \Gamma \Rightarrow y : A, \Delta$ is derivable in* $\mathbf{G3I}_<$.

Proof. By induction on the degree of the formula A. If A is atomic, then we apply the rule Trs to the sequent $x < u_1, ..., u_n < y, x : A, \Gamma \Rightarrow y : A, \Delta$ until we reach an instance of ax_2. If A is a compound formula not of the form $B > C$, the conclusion easily follows by the induction hypothesis. If it is of form $B > C$, we proceed as follows (where [...] abridges $x < u_1, ..., u_n < y, x < y, y < z, z : B > C$):

$$\cfrac{\cfrac{\cfrac{[...], x < z, z : B, x : B > C, \Gamma \Rightarrow \Delta, z : C, z : B \qquad [...], x < z, z : B, x : B > C, z : C, \Gamma \Rightarrow \Delta, z : C}{x < u_1, ..., u_n < y, y < z, x < z, z : B > C, z : B, x : B > C, \Gamma \Rightarrow \Delta, z : C}\,\text{L>}}{x < u_1, ..., u_n < y, y < z, z : B > C, z : B, x : B > C, \Gamma \Rightarrow \Delta, z : C}\,\text{several Trs}}{x < u_1, ..., u_n < y, x : B > C, \Gamma \Rightarrow \Delta, y : A > B}\,\text{R>}$$

where the topmost sequents are derivable by Lemma 4.3.1. *qed.*

Example. We now give an example of a derivation in the system $\mathbf{G3I}_<$ of the labelled sequent $\Rightarrow x : A \rightarrow (B \rightarrow A)$. First we construct a derivation $\mathcal{D}$ of the sequent $\Rightarrow x : A > (B \rightarrow A)$:

$$
\dfrac{\dfrac{\dfrac{x < y, y : A > (B \to A), y : A, z : B > A, z : B \Rightarrow z : A}{x < y, y : A > (B \to A), y : A \Rightarrow y : B > A}\,\text{R}{>} \qquad x < y, y : A > (B \to A), y : A, y : B \Rightarrow y : A}{x < y, y : A > (B \to A), y : A \Rightarrow y : B \to A}\,\text{R}{\to}}{\Rightarrow x : A > (B \to A)}\,\text{R}{>}
$$

Then we construct the following derivation:

$$
\dfrac{\begin{array}{c}\vdots\,\mathcal{D}\\ \Rightarrow x : A > (B \to A)\end{array} \qquad \dfrac{\dfrac{x < u, u : A > B, u : B, x : A \Rightarrow u : A}{x : A \Rightarrow x : B > A}\,\text{R}{\to} \qquad x : A, x : B \Rightarrow x : A}{x : A \Rightarrow x : B \to A}\,\text{R}{\to}}{\Rightarrow x : A \to (B \to A)}\,\text{R}{\to}
$$

A rule is *invertible* if, whenever the conclusion is derivable, so is each of its premises. The calculus $\mathbf{G3I}_<$ is shown to enjoy height-preserving invertibility of every rule. Invertibility is a desirable property, because it allows to avoid backtracking when searching for a derivation.

Lemma 4.3.5. *Every rule is height-preserving invertible in* $\mathbf{G3I}_<$.

Proof. The rule L> is height-preserving invertible by height-preserving admissibility of weakening. The proof of the invertibility of the rules different from R> follows the usual pattern as detailed in (109). With respect to the rule R>, we need to prove that whenever $\Gamma \Rightarrow \Delta, x : A > B$ is derivable, so is $x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B$ for every label y and the height is preserved. We argue by induction on the height n of the derivation of the sequent $\Gamma \Rightarrow \Delta, x : A > B$ in $\mathbf{G3I}_<$. If $n = 0$, then $\Gamma \Rightarrow \Delta, x : A > B$ is an initial sequent and so is $x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B$. If $n > 0$, we distinguish cases according to the last rule applied. If $x : A > B$ is principal in the last rule applied, then the derivation of the premise yields the desired conclusion applying height-preserving substitution if needed. If $x : A > B$ is not principal, we apply the induction hypothesis to the premise(s) and then we apply again the rule. For example, if the last rule applied is $R >$ and $x : A > B$ is not principal we have:

$$
\dfrac{w < u, u : C > D, u : C, \Gamma \Rightarrow \Delta, x : A > B, u : D}{\Gamma \Rightarrow \Delta, x : A > B, w : C > D}\,\text{R}{>}
$$

We can assume that u does not coincide with y, otherwise we apply height-preserving substitution to replace u with a fresh variable o. Hence we apply the induction hypothesis to obtain a derivation of the sequent $w < u, x < y, u : C > D, y : A > B, u : C, y : A, \Gamma \Rightarrow \Delta, u : D, y : B$ and then we apply again the rule to get $x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B, w : C > D$ which is the desired conclusion. *qed.*

We are in the position to state and prove the height-preserving admissibility of the rules of contraction in $\mathbf{G3I}_<$.

Lemma 4.3.6. *The rule:*

$$\frac{\Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ Ctr$$

is height-preserving admissible in $\mathbf{G3I}_<$.

Proof. We argue by induction on the height n of the derivation. If $n = 0$, then $\Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Sigma$ is an initial sequent and so is $\Gamma, \Pi \Rightarrow \Delta, \Sigma$. If $n > 0$ and no formula in Γ or Δ is principal we apply the induction hypothesis to the premise(s) of the rule and then we apply the rule again. If $n > 0$ and a formula is principal in a unary rule ρ, we have:

$$\frac{\Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Sigma}{\Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Sigma}\ \rho$$

We proceed as follows:

$$\cfrac{\cfrac{\cfrac{\Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Sigma}{\Gamma', \Gamma', \Pi \Rightarrow \Delta', \Delta', \Sigma}\ Inv\rho}{\Gamma', \Pi \Rightarrow \Delta', \Sigma}\ Ctr}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \rho$$

where $Inv\rho$ denotes the application of the invertibility lemma with respect to the rule ρ. The applications of Ctr are removed by invoking the induction on the height of the derivation. The case of binary rules is analgous and we omit the details. *qed.*

In order to show the cut-free completeness of the system $\mathbf{G3I}_<$, we need the proof of the admissibility of the following rule.

Lemma 4.3.7. *The rule:*

$$\frac{x < y, x : A, y : A, \Gamma \Rightarrow \Delta}{x < y, x : A, \Gamma \Rightarrow \Delta}\ Mon$$

is admissible in $\mathbf{G3I}_<$.

Proof. We argue by induction on lexicographically ordered pairs where the first component is $dg(A)$ and the second component is the height n of the derivation of $x < y, x : A, y : A, \Gamma \Rightarrow \Delta$.

If $x < y, x : A, y : A, \Gamma \Rightarrow \Delta$ is an initial sequent, we distinguish subcases. If $y : A$ is not active, then $x < y, x : A, \Gamma \Rightarrow \Delta$ is an initial sequent as well. If $y : A$ is active in ax_1, then $x < y, x : A, \Gamma \Rightarrow \Delta$ is an instance of ax_2. If $y : A$ is active in ax_2, then $x < y, x : A, y : A, \Gamma \Rightarrow \Delta$ is of the form $x < y, y < z, x : p, y : p, \Gamma \Rightarrow \Delta', z : p$. In this case we proceed as follows:

$$\cfrac{\cfrac{}{x < y, y < z, x < z, x : p, \Gamma \Rightarrow \Delta', z : p}\ \mathrm{ax}_2}{x < y, y < z, x : p, \Gamma \Rightarrow \Delta', z : p}\ \mathrm{Trs}$$

If $n > 0$, then we need to distinguish cases according to the last rule applied. If neither $x : A$ nor $y : A$ are principal, then we apply the secondary induction hypothesis and then the rule again. If $x : A$ or $y : A$ are principal, then we can assume that A is not an atomic formula by inspection of the rules. Hence, if A is not of the form $B > C$, we exploit height-preserving invertibility of the rules and the induction hypothesis. For example, if the conclusion is $x < y, x : B \rightarrow C, y : B \rightarrow C, \Gamma \Rightarrow \Delta$, we proceed as follows. First, we construct a derivation of $x < y, x : B > C, \Gamma \Rightarrow \Delta, x : B$:

$$\dfrac{\dfrac{\dfrac{x < y, x : B \rightarrow C, y : B \rightarrow C, \Gamma \Rightarrow \Delta}{x < y, x : B > C, y : B > C, y : C, \Gamma \Rightarrow \Delta, x : B}\,\text{Inv}}{x < y, x : B > C, y : C, \Gamma \Rightarrow \Delta, x : B}\,\text{Mon} \qquad \dfrac{\dfrac{x < y, x : B \rightarrow C, y : B \rightarrow C, \Gamma \Rightarrow \Delta}{x < y, x : B > C, y : B > C, \Gamma \Rightarrow \Delta, x : B, y : B}\,\text{Inv}}{x < y, x : B > C, \Gamma \Rightarrow \Delta, x : B, y : B}\,\text{Mon}}{x < y, x : B > C, \Gamma \Rightarrow \Delta, x : B}\,\text{L>}$$

and we obtain the desired conclusion as follows:

$$\dfrac{x < y, x : B > C, \Gamma \Rightarrow \Delta, x : B \qquad \dfrac{\dfrac{x < y, x : B \rightarrow C, y : B \rightarrow C, \Gamma \Rightarrow \Delta}{x < y, x : C, y : C, y : B > C, x : B > C, \Gamma \Rightarrow \Delta}\,\text{Inv}}{x < y, x : C, x : B > C, \Gamma \Rightarrow \Delta}\,\text{Mon}}{x < y, x : B \rightarrow C, \Gamma \Rightarrow \Delta}\,\text{L}\rightarrow$$

the applications of Mon are removed by the induction hypothesis on the degree of A. The cases in which $y : A$ is $y : B \wedge C$ or $y : B \vee C$ are similar. If A is of the form $B > C$, we need to consider two subcases. If $x : B > C$ is principal, then we apply the induction hypothesis on the height of the derivation to the premises and then the rule again. If $y : B > C$ is principal, then we have:

$$\dfrac{x < y, y < z, x : B > C, y : B > C, \Gamma \Rightarrow \Delta, z : B \qquad x < y, y < z, z : C, x : B > C, y : B > C, \Gamma \Rightarrow \Delta}{x < y, y < z, x : B > C, y : B > C, \Gamma \Rightarrow \Delta}\,\text{L>}$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{x < y, y < z, x : B > C, y : B > C, \Gamma \Rightarrow \Delta, z : B}{x < y, y < z, x < z, x : B > C, y : B > C, \Gamma \Rightarrow \Delta, z : B}\,\text{Weak}}{x < y, y < z, x < z, x : B > C, \Gamma \Rightarrow \Delta, z : B}\,\text{Mon} \qquad \dfrac{\dfrac{x < y, y < z, z : C, x : B > C, y : B > C, \Gamma \Rightarrow \Delta}{x < y, y < z, x < z, z : C, x : B > C, y : B > C, \Gamma \Rightarrow \Delta}\,\text{Weak}}{x < y, y < z, x < z, z : C, x : B > C, \Gamma \Rightarrow \Delta}\,\text{Mon}}{x < y, y < z, x < z, x : B > C, \Gamma \Rightarrow \Delta}\,\text{L>}}{x < y, y < z, x : B > C, \Gamma \Rightarrow \Delta}\,\text{Trs}$$

The applications of Mon are removed by invoking the induction hypothesis on the height of the derivation which is preserved by the application of weakening. *qed.*

We now show that every derivation in the system **G3i** can be transformed into a derivation in the calculus $\mathbf{G3I}_<$. Given a multiset of formulas, we denote by $x : \Gamma$ the multiset of labelled formulas in which every formula is labelled by x.

Theorem 4.3.8. *For every multiset of formulas in* FM*, if* $\Gamma \Rightarrow \Delta$ *is derivable in* **G3i***, then* $x : \Gamma \Rightarrow x : \Delta$ *is derivable in* $\mathbf{G3I}_<$ *for every label* x.

Proof. The proof is by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$ in **G3i**. If $n = 0$, then $x : \Gamma \Rightarrow x : \Delta$ is an initial sequent in $\mathbf{G3I}_<$. If $n > 0$, then we distinguish cases according to the last rule applied. If the last rule applied is different from $L \rightarrow$ and $R \rightarrow$, then we apply the induction hypothesis and the rule again.

If the last rule applied is R→, we have:

$$\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\, R \rightarrow$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \text{IH}}{y : A, y : \Gamma \Rightarrow y : B}}{x < y, y : A > B, y : A, y : \Gamma, x : \Gamma \Rightarrow x : \Delta, y : B}\,\text{Weak}}{\dfrac{x < y, y : A > B, y : A, x : \Gamma \Rightarrow x : \Delta, y : B}{x : \Gamma \Rightarrow x : \Delta, x : A > B}\,\text{R>}}\,\text{Mon} \qquad \dfrac{\dfrac{\vdots\ \text{IH}}{x : A, x : \Gamma \Rightarrow x : B}}{x : A, x : \Gamma \Rightarrow x : \Delta, x : B}\,\text{Weak}}{x : \Gamma \Rightarrow x : \Delta, x : A \rightarrow B}\,\text{R}\rightarrow$$

where IH denotes an application of the inductive hypothesis.

If the last rule applied is L→, we have:

$$\dfrac{A \rightarrow B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\, L \rightarrow$$

We construct the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \text{IH}}{x : A \rightarrow B, x : \Gamma \Rightarrow x : \Delta, x : A}}{x : A > B, x : \Gamma \Rightarrow x : \Delta, x : A, x : A}\,\text{Inv}}{x : A > B, x : \Gamma \Rightarrow x : \Delta, x : A}\,\text{Ctr} \qquad \dfrac{\dfrac{\vdots\ \text{IH}}{x : B, x : \Gamma \Rightarrow x : \Delta}}{x : B, x : A > B, x : \Gamma \Rightarrow x : \Delta}\,\text{Weak}}{x : A \rightarrow B, x : \Gamma \Rightarrow x : \Delta}\, L \rightarrow$$

where IH indicates an application of the induction hypothesis. *qed.*

Notice that every in order to prove the embedding of the system **G3i** into the system $\mathbf{G3I}_<$ we used implicitly or explicitly all the previous lemmata.

Theorem 4.3.9. *For every formula A in* FM*:*

If A is intuitionistically valid, then $\mathbf{G3I}_< \vdash \Rightarrow x : A$.

Proof. If A is intuitionistically valid, then the sequent $\Rightarrow A$ is derivable in **G3i**. Furthermore, every derivation of a sequent $\Rightarrow A$ in **G3i** can be transformed into a derivation of the sequent $\Rightarrow x : A$ in $\mathbf{G3I}_<$ by Theorem 4.3.8. As a consequence, we get the desired conclusion. *qed.*

Remark. The above theorem entails the cut-free derivability of every intuitionistically valid formula in the system $\mathbf{G3I}_<$. Another strategy to obtain this result consists in proving cut-elimination for the system $\mathbf{G3I}_<$. However, the cut-elimination theorem seems difficult to obtain directly for $\mathbf{G3I}_<$.

We have concluded that the calculus $\mathbf{G3I}_<$ enjoys cut-free completeness with respect to intuitionistically valid formulas. Contrarily to the standard sequent calculi for intuitionistic logic (109), the system $\mathbf{G3I}_<$ does not enjoy the subformula property. The violation of the subformula property is double. First, as in the tradition of labelled calculi, there are relational symbols and labels, which do not belong to the language of intuitionistic logic. Second, in $\mathbf{G3I}_<$ the language of intuitionistic logic is extended by formulas of the shape $A > B$ which are not subformulas of $A \to B$. However, the calculus satisfies a relaxed version of the subformula property.

Definition 4.3.2. Given a formula A the set of its generalized subformulas, $\mathcal{G}(A)$ is defined as follows:

- $A \in \mathcal{G}(A)$
- If $B \wedge C, B \vee C, B > C \in \mathcal{G}(A)$, then $B, C \in \mathcal{G}(A)$
- If $B \to C \in \mathcal{G}(A)$, then $B, C, B > C \in \mathcal{G}(A)$

The generalized subformulas of a sequent $\Gamma \Rightarrow \Delta$ are the union of the sets of generalized subformulas of the formulas in Γ and Δ.

This yields the following result which is crucial to establish the termination of our system.

Corollary (Generalized subformula property)**.** *The calculus* $\mathbf{G3I}_<$ *enjoys the generalized subformula property, i.e. every derivation of a sequent* $\Gamma \Rightarrow \Delta$ *contains only labelled formulas in which labels are either eigenvariables or variables in the conclusion and formulas are generalized subformulas of the formulas in the conclusion.*

Proof. Immediate by inspection of the rules. *qed.*

As we shall see, the partial loss of analyticity will allow us to get an easy termination procedure.

4.4 Completeness and termination of $\mathbf{G3I}_<$

We aim at proving termination of the proof search in the labelled sequent calculus $\mathbf{G3I}_<$. We start recalling the notion of validity of a labelled sequent.

Definition 4.4.1. Let $\mathcal{M} = \langle W, <, v \rangle$ be a strict model for intuitionistic propositional logic. An interpretation is a function $|| \; || : Lab \to W$. A labelled sequent is valid in $\mathcal{M}$ with respect to an interpretation $|| \; ||$ if for every labelled formula $x : A$, for every relational atom $y < z$ in Γ, whenever $||x|| \Vdash A$ and $||y|| < ||z||$, then for some $u : B$ in Δ, $||u|| \Vdash B$.

A labelled sequent $\Gamma \Rightarrow \Delta$ is valid in a model $\mathcal{M}$ if it is valid under every interpretation. A labelled sequent is valid in a class of frame C if for every model $\mathcal{M}$ based on a frame in C, $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$.

Theorem 4.4.1 (Soundness)**.** *If* $\mathbf{G3I}_< \vdash \Gamma \Rightarrow \Delta$*, then* $\Gamma \Rightarrow \Delta$ *is valid in every strict model based on a strict frame.*

Proof. The proof is by induction on the height of derivation. *qed.*

In the previous section we have established that if we restrict ourselves to consider formulas of the language of intuitionistic logic, i.e. FM, $\mathbf{G3I}_<$ and the axiomatic intuitionistic calculus prove the same theorems. Thus we have obtained that the calculus $\mathbf{G3I}_<$ is a conservative extension of the axiomatic calculus for intuitionistic logic. We will show how to obtain a more direct completeness result which also enables us to establish the termination of the proof search and to define a procedure to extract a finite countermodel. We first prove some auxiliary lemmata which enable us to avoid redundant applications of rules.

Lemma 4.4.2. *Given a derivation of a sequent* $\Gamma \Rightarrow \Delta$ *in* $\mathbf{G3I}_<$*, there is a derivation of* $\Gamma \Rightarrow \Delta$ *in* $\mathbf{G3I}_<$ *in which rule* $R >$ *has been applied at most once to the same labelled formula* $x : A > B$ *in each branch and the height is preserved.*

Proof. The proof is by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_<$. If $n = 0$ there is nothing to prove. If $n > 0$, but the last rule is not $R >$ the proof follows by applying the induction hypothesis to the premise(s) and then the rule again. If $n > 0$ and the last rule is $R >$ we have:

$$\frac{x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A > B} \; R >$$

By induction hypothesis there is a derivation of $x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B$ which contains at most one application of $R >$ to every formula of the form $x : C > D$ in every branch. If it does not contain an application of R> to $x : A > B$, then we apply $R >$ and we obtain the desired conclusion.

If it contains an application of the rule R> to $x : A > B$, we have:

$$\dfrac{\vdots \mathcal{D} \atop x < z, x < y, y : A > B, z : A > B, z : A, \Gamma'' \Rightarrow \Delta'', z : B}{x < y, y : A > B, \Gamma'' \Rightarrow \Delta'', x : A > B}\, R >$$

$$\vdots \mathcal{E}$$

$$x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots \mathcal{D} \atop x < z, x < y, y : A > B, z : A > B, z : A, \Gamma'' \Rightarrow \Delta'', z : B}{x < y, x < y, y : A > B, y : A > B, y : A, \Gamma'' \Rightarrow \Delta'', y : B}\, \mathrm{Sub}[z/y]}{x < y, y : A > B, y : A, \Gamma'' \Rightarrow \Delta'', y : B}\, Ctr}{x < y, y : A > B, y : A, \Gamma'' \Rightarrow \Delta'', y : B, x : A > B}\, \mathrm{Weak}$$

$$\dfrac{\dfrac{\dfrac{\vdots \mathcal{E} \atop x < y, y : A > B, y : A, y : A, \Gamma \Rightarrow \Delta, y : B, y : B, x : A > B}{x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B, x : A > B}\, Ctr}{\Gamma \Rightarrow \Delta, x : A > B, x : A > B}\, R >}{\Gamma \Rightarrow \Delta, x : A > B}\, Ctr$$

The steps of substitution, contraction and weakening are height-preserving admissible and it can be easily checked (by looking at the corresponding lemmata) that they do not introduce new applications of R>, so we have obtained the desired conclusion.

qed.

Lemma 4.4.3. *Given a derivation of a sequent* $\Gamma \Rightarrow \Delta$ *in* $\mathbf{G3I}_<$*, there is a derivation of the same sequent in* $\mathbf{G3I}_<$ *in which the rule L> has been applied at most once to the same pair of formulas* $x : A > B$ *and* $x > y$ *in every branch of the derivation and the height is preserved.*

Proof. The proof is by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_<$. If $n = 0$, there is nothing to prove. If $n > 0$ and the last rule is different from L>, then we apply the induction hypothesis to the premise and then we apply the rule again. If the last rule applied is L>, then we have the following situation:

$$\dfrac{x < y, x : A > B, \Gamma \Rightarrow \Delta, y : A \qquad x < y, x : A > B, y : B, \Gamma \Rightarrow \Delta}{x < y, x : A > B, \Gamma \Rightarrow \Delta}\, \mathrm{L>}$$

We apply the induction hypothesis to the premises of the rule. Therefore we obtain that there are derivations $\mathcal{D}''$ and $\mathcal{D}'''$ of the premises $x < y, x : A > B, \Gamma \Rightarrow \Delta, y : A$ and $x < y, x : A > B, y : B, \Gamma \Rightarrow \Delta$, respectively, in which the rule L> has not been applied twice to the same pair of formulas $o : C > D$ and $o < w$. The only problematic case is that in which the rule L> has been applied to the formulas $x : A > B$ and $x < y$ in $\mathcal{D}''$ or $\mathcal{D}'''$ or both. We deal with the case in which it has been applied in $\mathcal{D}''$ as an example (the other cases are similar). We have:

$$\dfrac{\dfrac{x < y, x : A > B, \Gamma' \Rightarrow \Delta', y : A \qquad x < y, x : A > B, y : B, \Gamma' \Rightarrow \Delta'}{x < y, x : A > B, \Gamma' \Rightarrow \Delta'}\ \text{L>} \\ \vdots \mathcal{E} \\ x < y, x : A > B, \Gamma \Rightarrow \Delta, y : A \qquad x < y, x : A > B, y : B, \Gamma \Rightarrow \Delta}{x < y, x : A > B, \Gamma \Rightarrow \Delta}\ \text{L>}$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{x < y, x : A > B, \Gamma' \Rightarrow \Delta', y : A}{\vdots \mathcal{E}} \\ x < y, x : A > B, \Gamma \Rightarrow \Delta, y : A, y : A}{x < y, x : A > B, \Gamma \Rightarrow \Delta, y : A}\ \text{RC} \qquad x < y, x : A > B, y : B, \Gamma \Rightarrow \Delta}{x < y, x : A > B, \Gamma \Rightarrow \Delta}\ \text{L>}$$

Since the steps of left and right contraction are height-preserving (and they do not introduce new applications of the rule L>) and since we have removed an application of the rule L>, we have obtained the desired conclusion. *qed.*

We observe that it can be easily proved that an analogous result holds with respect to the rule Trs which needs not be applied twice to the same principal relational atoms in the same branch.
We are now ready to present the main result of the section, namely the termination of the proof search in the calculus $\mathbf{G3I}_<$. Given a sequent $\Gamma \Rightarrow \Delta$, the strategy of the proof of termination consists in constructing a reduction tree, i.e. a tree built from bottom-up applications of the rules of the calculus to $\Gamma \Rightarrow \Delta$ in a given order. The construction goes on until we reach derivable leaves or a topmost sequent to which the rules cannot be further applied. The key point is that it can be shown that the reduction tree is always finite and it yields either a derivation or a failed proof search from which we can extract a countermodel.

Theorem 4.4.4. *Given a sequent $\Gamma \Rightarrow \Delta$ it is decidable whether it is derivable in $\mathbf{G3I}_<$. If it is not derivable we can extract a finite strict countermodel.*

Proof. We define a procedure to construct a reduction tree and then we show that the tree is finite and yields either a derivation or a finite countermodel.
Construction of the reduction tree

The reduction tree is defined inductively in stages. Stage 0: The root is the sequent $\Gamma \Rightarrow \Delta$.
Stage n>0: We distinguish two cases. If every topmost sequent is of the form $x_0 < x_1, ..., x_{n-1} < x_n, x_0 : A > B, \Gamma'' \Rightarrow \Delta'', x_n : A > B$ (which is derivable by Lemma 4.3.4) or is an instance of $ax_1, ax_2, L\bot$ or $Irref$ we have obtained a derivation and we stop. Otherwise, we continue the construction of the tree by writing on top of the topsequents which are not of the form $x_0 < x_1, ..., x_{n-1} < x_n, x_0 : A > B, \Gamma'' \Rightarrow \Delta'', x_n : A > B$ nor instances of $ax_1, ax_2, L\bot$ or $Irref$ other sequents obtained by the applications of the rules of the calculus in a given order. There are 9 different stages, because there are 8 logical rules in the system **G3I**$_<$ and the rule Trs. At stage 10 we repeat stage 1, at stage 11 we repeat stage 2 and so on. The order is as follows: $L\wedge, R\wedge, L\vee, R\vee, L\rightarrow, R\rightarrow, L>, R>$ and Trs. We give the details of cases $L>$ and $R>$, the other cases are analogous.

For $n = 7$ we consider each topmost sequent of the form:

$$x_1 < y_1, ..., x_m < y_m, x_1 : A_1 > B_1, ..., x_m : A_m > B_m, \Gamma \Rightarrow \Delta$$

where $x_1 < y_1, ..., x_m < y_m, x_1 : A_1 > B_1, ..., x_m : A_m > B_m$ and $x_1 < y_1, ..., x_m < y_m$ are all the labelled formulas of the shape $o : C > D$ and oRw, respectively we write on top of it 2^m sequents obtained by the applications of the rule L>:

$$x_1 : A_1 > B_1, ..., x_m : A_n > B_m, y_{i_1} : B_{i_1}, ..., y_{i_k} : B_{i_k}, \Gamma \Rightarrow \Delta, y_{j_{k+1}} : A_{j_{k+1}}, ..., y_{j_m} : A_{j_m}$$

where $\{i_1, ..., i_k\} \subseteq \{1, ..., m\}$ and $j_{k+1}, ..., j_m \in \{1, ..., m\} - \{i_1, ..., i_k\}$.

For $n = 8$ we consider each topmost sequent of the form:

$$\Gamma \Rightarrow \Delta, x_1 : A_1 > B_1, ..., x_m : A_n > B_m$$

where $x_1 : A_1 > B_1, ..., x_m : A_n > B_m$ are all the labelled formulas of the form $o : C > D$ in the succedent and we write on top of it the sequent:

$$x_1 < y_1, ..., x_m < y_m, y_1 : A_1 > B_1, ..., y_m : A_m > B_m, y_1 : A, ..., y_m : A, \Gamma \Rightarrow \Delta, y_1 : B, ..., y_m : B$$

obtained by the applications of the rule R>, where $y_1, ..., y_m$ are variables not occurring in Γ and Δ.

Because of Lemma 4.4.3, we do not need to apply more than once the rule L> to the same principal formulas in the same branch and the same holds for the rule Trs. The reduction tree could be infinite, because the construction might not stop. However, we show that this is not the case. By Corollary 4.3 the formulas occurring in the reduction tree are generalized subformulas of the formulas in $\Gamma \Rightarrow \Delta$ and this set is finite.

As a consequence, if there is an infinite derivation, there must be infinite labels. However, we claim that:

1. Each chain of labels in a branch is finite.

2. Each label in a branch has a finite number of immediate successors.

From the two claims we obtain termination. With respect to item 1., since the set of the generalized subformulas of the formulas in $\Gamma \Rightarrow \Delta$ is finite, it is enough to show that the rule $R >$ cannot be applied twice to the same formula $A > B$ in a chain of labels in a branch, because the rule R> is the only rule which generates new labels in the proof.

Suppose that there is a subchain $x_0 < ... < x_n$ where x_0 and x_n label the same formula $A > B$ in the succedent of two sequents in a branch of a derivation. Therefore we have the following situation:

$$\dfrac{\begin{array}{c}\vdots \\ x_0 < x_1, ..., x_{n-1} < x_n, x_1 : A > B, \Gamma'' \Rightarrow \Delta'', x_n : A > B \\ \vdots \\ x_0 < x_1, x_1 : A > B, x_1 : A, \Gamma' \Rightarrow \Delta', x_1 : B\end{array}}{\Gamma' \Rightarrow \Delta', x_0 : A > B} R >$$

The sequent $x_0 < x_1, ..., x_{n-1} < x_n, x_1 : A > B, \Gamma'' \Rightarrow \Delta'', x_n : A > B$ is derivable in $\mathbf{G3I}_{<}$ by Lemma 4.3.4. Therefore the search can be interrupted, the chain is finite and item 1. is proved.

To show item 2., suppose there is a label which has an infinite number of immediate successors. Due to the generalized subformula property and to the finiteness of the endsequent we can conclude that the rule $R >$ has been applied at least twice to the same labelled formula $x : A > B$ in the branch, but this is prevented by Lemma 4.4.2.

Therefore the construction of the reduction tree in $\mathbf{G3I}_{<}$ is terminating without loop-checking. If all the leaves in the reduction tree are of the form $x_0 < x_1, ..., x_{n-1} < x_n, x_0 : A > B, \Gamma'' \Rightarrow \Delta'', x_n : A > B$ or instances of $ax_1, ax_2, Irref$ or $L\bot$, we have obtained a derivation of $\Gamma \Rightarrow \Delta$. Otherwise $\Gamma \Rightarrow \Delta$ is not derivable and the topsequents which are not of the form $x_0 < x_1, ..., x_{n-1} < x_n, x_0 : A > B, \Gamma'' \Rightarrow \Delta'', x_n : A > B$ nor instances of $ax_1, ax_2, Irref$ or $L\bot$ are called *saturated*.

Construction of the countermodel

Let us suppose $\Gamma \Rightarrow \Delta$ is not derivable in $\mathbf{G3I}_<$, then the proof search terminates and there is a leaf in the reduction tree which is not of the form $x_0 < x_1, ..., x_{n-1} < x_n, x_1 : A > B, \Gamma'' \Rightarrow \Delta'', x_n : A > B$, is not an instance of $ax_1, ax_2, Irref$ or $L\bot$ and is closed under every available rule. Therefore the leaf is a saturated sequent.

Let $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ be the unions of the antecedents and the succedents, respectively, of all the sequents $\Gamma_i \Rightarrow \Delta_i$ of the branch up to the saturated sequent. We define a Kripke model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \Rightarrow \Delta$.

We consider the frame obtained by taking as worlds the labels occurring in $\mathbf{\Gamma}$ with their mutual relations expressed by the relational atoms $x < y$ in $\mathbf{\Gamma}$; we use R to refer to the relation in the model. By the closure under the rules of transitivity we obtain that the frame is a finite strict order.

The valuation is defined as follows: for every atomic labelled formula $x : p$ in $\mathbf{\Gamma}$, we stipulate $x \in v(p)$ and if $x : p$ and $x < y$ are in $\mathbf{\Gamma}$, then $y \in v(p)$: since the sequent is not an instance of ax_1 nor of ax_2 the definition is sound.

It is then easy to show by induction on the complexity of A that $x \Vdash A$ holds in the model if $x : A$ is in $\mathbf{\Gamma}$ and that $x \nVdash A$ if $x : A$ is in $\mathbf{\Delta}$.

We limit ourselves to discussing the case in which the formula is of the form $x : A > B$. If $x : A > B$ is in $\mathbf{\Gamma}$ and xRy, then by the saturation condition we get that either $y : A$ is in $\mathbf{\Delta}$ or $y : B$ is in $\mathbf{\Gamma}$. By induction hypothesis we obtain either $y \nVdash A$ or $y \Vdash B$, which yields $x \Vdash A > B$.

If $x : A > B$ is in $\mathbf{\Delta}$, then by the saturation condition $x < y$, $y : A > B$ and $y : A$ are in $\mathbf{\Gamma}$ and $y : B$ is in $\mathbf{\Delta}$. By induction hypothesis we obtain xRy, $y \Vdash A$ and $y \nVdash B$. Suppose now, towards a contradiction, that $x \Vdash A > B$, then $y \nVdash A > B$, otherwise $y \Vdash B$. Therefore there is z, yRz and $z \Vdash A > B$, $z \Vdash A$ and $z \nVdash B$. Again, $z \nVdash A > B$; iterating this procedure we obtain an infinite chain against the finiteness of the model.

We have obtained the desired countermodel for the sequent $\Gamma \Rightarrow \Delta$. *qed.*

As an immediate corollary, we get the closure under cut of the system $\mathbf{G3I}_<$.

Corollary. *The rule:*

$$\frac{\Gamma \Rightarrow \Delta, x : A \qquad x : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathit{Cut}$$

is admissible in $\mathbf{G3I}_<$.

Proof. Let us assume that we have two derivations of the sequents $\Gamma \Rightarrow \Delta, x : A$ and $x : A, \Gamma' \Rightarrow \Delta'$. By the soundness theorem the sequents are valid and so is $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. Therefore, by the completeness theorem, $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is derivable in $\mathbf{G3I}_<$. *qed.*

Before we proceed, we would like to briefly sum up the results that we have obtained in the following corollary.

Corollary. *The calculus* $\mathbf{G3I}_<$ *enjoys termination without loop-checking and without backtracking due to the height-preserving invertibility of every rule. Furthermore, we can easily extract a finite countermodel out of a failed proof search and there are syntactic proofs of admissibility of the rules of weakening and contraction.*

These results come at the cost of a relaxation of the subformula property. In particular, our system satisfies a generalized subformula property.

Remark. It may be objected that our termination procedure still contains some kind of loop-checking hidden in the Lemmata which ensure the possibility to avoid redundant applications of the rules L> and R>. This is only a *local* form of loop-checking which could be totally dispensed with by resorting to cumulative rules , i.e. for example:

$$\frac{x < y, y : A > B, y : A, \Gamma \Rightarrow \Delta, y : B, x : A > B}{\Gamma \Rightarrow \Delta, x : A > B} \text{ R>}_{c}, y!$$

with the local side condition that either $x < y, y : A > B$ or $y : A$ is not in Γ or $y : B$ is not in Δ. It is immediate to observe that this way we do not require any kind of preliminary lemma and that this formulation of the rules allows for a strong termination property, namely the number of rules applicable to a sequent bottom up are finite. We do not opt for this version of the rules to keep the presentation smooth and easy to follow.

4.5 Extending the methodology

This section is devoted to extend the approach to some families of superintuitionistic (or intermediate) logics which enjoy the finite model property. This can be regarded as a *desideratum*, because the current approaches to termination for intuitionistic logic do not easily generalize to other logical systems. On the contrary, labelled calculi have the advantage of being modular, in the sense that a calculus for an intermediate logic can be obtained by adding some relational rules to the base system.

We start by focussing on Gödel Dummett logic (an interpolable intermediate logic (61)).[1] The systems thus obtained are sound and complete. In particular, completeness, which entails closure under cut, is shown via the extraction of a finite countermodel out of a failed proof search.

Before we proceed, let us recall that the first-order language of standard Kripke models is the language of partial orders $\mathcal{L}_{\leqslant} = \{\leqslant, =\}$, whereas the language of strict orders is $\mathcal{L}_{<} = \{<, =\}$. We also recall that an intermediate logic is said to be characterized by a class of frames C if and only if every formula derivable in the logic is valid in C. We shall also be referring to the method of conversion of frame conditions into relational rules in a labelled sequent calculus, for the details the reader is referred to (73).

4.5.1 Gödel-Dummett logic

We now focus on Gödel-Dummett logic, axiomatized by adding the axiom schema $(A \to B) \vee (B \to A)$ and the cut rule to the calculus **G3i** or, equivalently, by adding the axiom schema to the axiomatic calculus for intuitionistic propositional logic. It is characterized by strongly connected Kripke frames. The condition of strong connectedness, expressed by the first order formula:

$$\forall xyz(x \leqslant y \wedge x \leqslant z \to y \leqslant z \vee z \leqslant y)$$

is easily seen to correspond to the first-order condition

$$\forall xyz(x < y \wedge x < z \to y < z \vee z < y \vee y = z)$$

with respect to strict frames in the language $\mathcal{L}_{<}$. This condition can be converted into a three-premise rule:

$$\frac{y < z, x < y, x < z, \Gamma \Rightarrow \Delta \qquad z < y, x < y, x < z, \Gamma \Rightarrow \Delta \qquad y = z, x < y, x < z, \Gamma \Rightarrow \Delta}{x < y, x < z, \Gamma \Rightarrow \Delta} \, Lin$$

Due to the presence of equality, we also need to add suitable rules to handle the new symbol, see Figure 4.3. The rules of symmetry and transitivity of the equality relation are easily shown to be derivable. The addition of rule *Lin* and the rules of equality to the system $\mathbf{G3I}_{<}$, which yields the system $\mathbf{G3IGD}_{<}$, preserves all the structural properties of the system $\mathbf{G3I}_{<}$. In particular, substitution, weakening and contraction are height-preserving admissible and every rule is

[1]The interpolable intermediate logics are seven and they also include intuitionistic and classical logic.

$$\frac{y = z, x = y, x = z, \Gamma \Rightarrow \Delta}{x = y, x = z, \Gamma \Rightarrow \Delta}\ Euc \qquad \frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref$$

$$\frac{z < y, x < y, x = z, \Gamma \Rightarrow \Delta}{x < y, x = z, \Gamma \Rightarrow \Delta}\ LEq \qquad \frac{x < z, x < y, y = z, \Gamma \Rightarrow \Delta}{x < y, y = z, \Gamma \Rightarrow \Delta}\ REq \qquad \frac{y : p, x : p, x = y, \Gamma \Rightarrow \Delta}{x : p, x = y, \Gamma \Rightarrow \Delta}\ AtRepl$$

Figure 4.3: Rules for equality.

height-preserving invertible. These properties hold for every extension of $\mathbf{G3I}_<$ with equality and relational rules and the proof are essentially the same as in the case of $\mathbf{G3I}_<$, thus we avoid giving the details. The generalization of the rule of replacement, i.e. AtRepl in Figure 4.3, to the case of arbitrary formulas is also admissible.

Lemma 4.5.1. *The rules:*

$$\frac{y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta}{x : A > B, x = y, \Gamma \Rightarrow \Delta}\ Repl_> \qquad \frac{x = y, \Gamma \Rightarrow \Delta, x : p, y : p}{x = y, \Gamma \Rightarrow \Delta, x : p}\ RRepl_{at}$$

are admissible in every extension of $\mathbf{G3I}_<$ *with the rules for equality.*

Proof. We prove the admissibility of Repl$_>$ by induction on the height n of the derivation of the sequent $y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta$. If it is an initial sequent, so is $x : A > B, x = y, \Gamma \Rightarrow \Delta$. If $n > 0$ and $y : A > B$ is not principal, we apply the induction hypothesis to the premise and then the rule again. If $n > 0$ and $y : A > B$ is principal, we have:

$$\frac{y < z, y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta, z : A \qquad y < z, z : B, y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta}{y < z, y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta}\ L>$$

We then proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y < z, y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta, z : A}{y < z, x : A > B, x = y, \Gamma \Rightarrow \Delta, z : A}\ Repl_>}{x < z, y < z, x : A > B, y = x, x = y, \Gamma \Rightarrow \Delta, z : A}\ Weak \qquad \dfrac{\dfrac{y < z, z : B, y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta}{y < z, z : B, x : A > B, x = y, \Gamma \Rightarrow \Delta}\ Repl_>}{x < z, y < z, z : B, x : A > B, y = x, x = y, \Gamma \Rightarrow \Delta}\ Weak}{x < z, y < z, x : A > B, y = x, x = y, \Gamma \Rightarrow \Delta}\ L>}{y < z, x : A > B, y = x, x = y, \Gamma \Rightarrow \Delta}\ LEq}{y < z, x : A > B, x = y, \Gamma \Rightarrow \Delta}\ \text{admissible rule}$$

The applications of $Repl_>$ are removed by the induction hypothesis on the height of the derivation.

We argue by induction on the height n of the derivation of $x = y, \Gamma \Rightarrow \Delta, x : p, y : p$ to prove the admissibility of RRepl$_{at}$. If $n = 0$ and $y : p$ is not principal we remove it. If it is principal it can be principal either in ax_1 or in ax_2. In the first case it is of the form $x = y, y : p, \Gamma' \Rightarrow \Delta, x : p, y : p$ and we construct the following derivation:

$$
\dfrac{\dfrac{\overline{x = y, y = x, x : p, y : p, \Gamma' \Rightarrow \Delta, x : p}\ ax_1}{x = y, y = x, y : p, \Gamma' \Rightarrow \Delta, x : p}\ \text{AtRepl}}{x = y, y : p, \Gamma' \Rightarrow \Delta, x : p}\ \text{Admissible rule}
$$

If it is principal in ax_2, the strategy is similar with the addition of a REq step. If $n > 0$, $y : p$ is never principal, hence we apply the induction hypothesis to the premise(s) of the last rule applied rule and then the rule again. *qed.*

Proposition 4.5.2. *The rules:*

$$
\dfrac{y : A, x : A, x = y, \Gamma \Rightarrow \Delta}{x : A, x = y, \Gamma \Rightarrow \Delta}\ LRepl \qquad \dfrac{x = y, \Gamma \Rightarrow \Delta, x : A, y : A}{x = y, \Gamma \Rightarrow \Delta, x : A}\ RRepl
$$

are admissible in every extension of $\mathbf{G3I}_<$ *with the rules for equality.*

Proof. We prove the admissibility of the rules simultaneously by induction on the degree of the formula $y : A$. We start discussing $LRepl$. If $y : A$ is of the form $y : p$ or $y : B > C$, then the conclusion follows from an application of the rule $AtRepl$ or by the admissibility of $Repl_>$. In the other cases we use the invertibility of the rules and then the primary induction hypothesis followed again by the rule. For example, if $y : A$ is of the form $y : B \rightarrow C$ we proceed as follows:

$$
\dfrac{\dfrac{\dfrac{y : A \rightarrow B, x : A \rightarrow B, x = y, \Gamma \Rightarrow \Delta}{y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta, x : A, y : A}\ \text{Inv L}\rightarrow}{x : A > B, x = y, \Gamma \Rightarrow \Delta, x : A}\ \text{LRepl, RRepl} \quad \dfrac{\dfrac{y : A \rightarrow B, x : A \rightarrow B, x = y, \Gamma \Rightarrow \Delta}{y : B, x : B, y : A > B, x : A > B, x = y, \Gamma \Rightarrow \Delta}\ \text{Inv L}\rightarrow}{x : B, x : A > B, x = y, \Gamma \Rightarrow \Delta}\ \text{LRepl}}{x : A \rightarrow B, x = y, \Gamma \Rightarrow \Delta}\ \text{L}\rightarrow
$$

The applications of LRepl and RRepl are removed by applying the induction hypothesis on the degree of the formula.

We now discuss the rule RRepl. If $y : A$ is of the form $y : p$ we use previous lemma. If $y : A$ is of the form $y : B\#C$, where $\# \in \{\wedge, \vee, \rightarrow\}$, we apply the height-preserving invertibility to $y : A$ and $x : A$ and then the primary induction hypothesis followed again by the rule. If it is of the form $y : B > C$, then we have:

$$
\dfrac{\dfrac{\dfrac{\dfrac{x = y, \Gamma \Rightarrow \Delta, x : B > C, y : B > C}{x = y, x < u, y < u, u : B > C, u : B > C, u : B, u : B, \Gamma \Rightarrow \Delta, u : C, u : C}\ \text{Inv R}>}{x = y, x < u, u : B > C, u : B > C, u : B, u : B, \Gamma \Rightarrow \Delta, u : C, u : C}\ \text{LEq}}{x = y, x < u, u : B > C, u : B, \Gamma \Rightarrow \Delta, u : C}\ \text{LC,RC}}{x = y, \Gamma \Rightarrow \Delta, x : B > C}\ \text{R}>
$$

qed.

As it is well known, Gödel-Dummett logic enjoys the finite model property (15), thus the rule $R >$ is sound with respect the semantics based on linear models.

Theorem 4.5.3 (Soundness)**.** *If* $\mathbf{G3IGD}_< \vdash \Gamma \Rightarrow \Delta$*, then* $\Gamma \Rightarrow \Delta$ *is valid in every model based on a linear strict frame.*

Proof. The proof is by induction on the height of derivation. *qed.*

The proof of termination detailed in Section 4 holds when rule *Lin* is added to the calculus $\mathbf{G3I}_<$.

Theorem 4.5.4 (Completeness)**.** *Given a sequent* $\Gamma \Rightarrow \Delta$ *the proof search terminates without loop-checking and backtracking in* $\mathbf{G3IGD}_<$*. If it is not derivable we can extract a finite linear strict countermodel.*

Proof. We need to introduce some slight modifications to the construction of the reduction tree and of the countermodel.

In particular, we need to add the intuitive steps for equality with the usual proviso of avoiding to apply the same rule to the same active formula(s) (which is justified by height-preserving admissibility of contraction as in 4.4.3). We then need to check that Lemma 4.4.2 still holds, but this is straightforward. The proof search terminates because the rule *Lin* does not introduce new labels in the proof search.

Finally, with respect to the countermodel we do not take the labels as worlds. We consider instead the following relation between labels occurring in the finite reduction tree:

$x \sim y$ if and only if there is an equality atom $x = y$ in the reduction tree.

Due to the steps for equality in the construction of the reduction tree $\sim$ is an equivalence relation and thus we take as worlds of the countermodel the equivalence classes induced by $\sim$. The valuation function is modified accordingly: $[x] \in v(p)$ iff $y : p$ is in $\mathbf{\Gamma}$ for some $y \sim x$. The order of the model is defined as: $[x]R[y]$ iff $z < u$ is in $\mathbf{\Gamma}$ for some $z \sim x$ and $u \sim y$. *qed.*

As a consequence of cut-free completeness we obtain closure under cut.

4.5.2 A general termination result

The termination procedure can be further expanded.

Definition 4.5.1. A universal formula is a first order formula of the form $\forall \overline{x} \varphi(\overline{x})$, where φ is quantifier-free and all its variables are bound. A frame condition is universal if and only if it is a finite conjunction of universal formulas.

Let us now observe that any universal formula is logically equivalent to a formula of the shape $\forall\overline{x}(A \to B)$, where A is a finite (possibly empty) conjunction of atomic formulas and B is a finite disjunction of atomic formulas or $\perp$. Therefore any universal formula in the language $\mathcal{L}_<$ is equivalent to a formula of the shape:

$$\forall\overline{x}(p_1 \wedge ... \wedge p_n \to q_1 \vee ... \vee q_m)$$

where $p_1, ..., p_n$ and $q_1, ..., q_m$ are either relational atoms $x < y$ or equality atoms $x = y$ or $\perp$. Every universal formula $\forall\overline{x}(p_1 \wedge ... \wedge p_n \to q_1 \vee ... \vee q_m)$ can be transformed into a relational rule of the form:

$$\frac{p_1, ..., p_n, q_1, \Gamma \Rightarrow \Delta \qquad ... \qquad p_1, ..., p_n, q_m, \Gamma \Rightarrow \Delta}{p_1, ..., p_n, \Gamma \Rightarrow \Delta}\,\text{R}$$

if $m = 1$ and q_1 is $\perp$ the resulting rule is a zeroary rule; for the details of the procedure in the context of modal logics, see (73). We refer to rules obtained from universal formulas as *universal rules*.

Definition 4.5.2. An application of a universal rule or Ref is *strongly analytic* if all the variables occurring in the premises occur in the conclusion too. A derivation $\mathcal{D}$ is strongly analytic if all the applications of universal rules and of Ref in $\mathcal{D}$ are strongly analytic.

Let $\mathcal{R}$ be a set of universal rules. We denote by $\mathbf{G3I}_< + \mathcal{R}$ the calculus obtained by adding to $\mathbf{G3I}_<$ the rules in $\mathcal{R}$ and, whenever needed, the rules for equality.

Theorem 4.5.5. *For every set $\mathcal{R}$ of universal rules the system $\mathbf{G3I}_< + \mathcal{R}$ enjoys height-preserving admissibility of the rules of substitution, weakening and contraction. Every rule in the system is height-preserving invertible.*

Proof. The proof follows the pattern detailed for the base system $\mathbf{G3I}_<$, because the rules in $\mathcal{R}$ preserve the structural properties of the system. *qed.*

We now show that every derivation in a system extended with universal rules corresponding to a frame condition can be transformed in a strongly analytic derivation (see also (27)).

Lemma 4.5.6. *For every set $\mathcal{R}$ of universal rules and every strongly analytic derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_< + \mathcal{R}$, there is a strongly analytic derivation of $\Gamma[x/y] \Rightarrow \Delta[x/y]$ in $\mathbf{G3I}_< + \mathcal{R}$ provided that x occurs in $\Gamma \Rightarrow \Delta$.*

Proof. Straightforward by induction on the height of the derivation. *qed.*

Lemma 4.5.7. *Let $\mathcal{R}$ be a set of universal rules obtained from a strict frame condition. Any derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_< + \mathcal{R}$ can be transformed in a strongly analytic derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_< + \mathcal{R}$ and the height is preserved.*

Proof. We argue by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_< + \mathcal{R}$. If $n = 0$, there is nothing to prove. If $n > 0$, the critical case is whenever a universal relational rule or Ref introduce in the premises one or more labels not occurring in the conclusion. Then we have:

$$\frac{p_1, ..., p_n, q_1, \Gamma \Rightarrow \Delta \qquad ... \qquad p_1, ..., p_n, q_m, \Gamma \Rightarrow \Delta}{p_1, ..., p_n, \Gamma \Rightarrow \Delta} R$$

where at least one among $q_1, ..., q_m$ contains a variable not occurring in the conclusion. By induction hypothesis we have a strongly analytic derivation of $p_1, ..., p_n, q_i, \Gamma \Rightarrow \Delta$ for every $i \in \{1, ..., m\}$. We apply height-preserving admissibility of substitution to substitute every label not occurring in the conclusion with labels already occurring in the conclusion.By Lemma 4.5.6 the resulting derivations are strongly analytic and then we apply again the rule R to get the desired conclusion. *qed.*

We now show that every universal relational rule need not be instantiated more than once on the same principal formulas in the same branch.

Lemma 4.5.8. *Let $\mathcal{R}$ be a set of universal relational rules. Given a derivation of a sequent $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_< + \mathcal{R}$, there is a derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_< + \mathcal{R}$ in which every universal rule has been applied only once to the same principal formulas in a branch and the height is preserved.*

Proof. The proof is by induction on the height n of the derivation of the sequent $\Gamma \Rightarrow \Delta$ in $\mathbf{G3I}_< + \mathcal{R}$. The interesting case is that in which the last rule applied is a universal rule.

$$\frac{q_1, \overline{p}, \Gamma \Rightarrow \Delta \qquad ... \qquad q_m, \overline{p}, \Gamma \Rightarrow \Delta}{\overline{p}, \Gamma \Rightarrow \Delta} R$$

where $\overline{p}$ is an abbreviation for $p_1, ..., p_n$. We apply the induction hypothesis to every premise of the rule. We distinguish cases: either every derivation of the premises does not contain any application of the rule R to the formulas $\overline{p}$ or it does. In the first case we can apply the rule R. Otherwise let us suppose there is one premise $q_i, \overline{p}, \Gamma \Rightarrow \Delta$ in which there is an application of the rule R to the formulas $\overline{p}$ (the argument can be easily generalized if there are more premises with this property). We have the following situation:

$$\dfrac{q_1, q_i, \overline{p}, \Gamma' \Rightarrow \Delta' \quad \dots \quad q_i, q_i, \overline{p}, \Gamma' \Rightarrow \Delta' \quad \dots \quad q_m, q_i, \overline{p}, \Gamma' \Rightarrow \Delta'}{q_i, \overline{p}, \Gamma' \Rightarrow \Delta'}\, R$$

$$\vdots\, \mathcal{D}$$

$$q_i, \overline{p}, \Gamma \Rightarrow \Delta$$

We construct the following derivation:

$$\dfrac{q_1, \overline{p}, \Gamma \Rightarrow \Delta \quad \dots \quad \dfrac{\begin{matrix} q_i, q_i, \overline{p}, \Gamma' \Rightarrow \Delta' \\ \vdots\, \mathcal{D} \\ q_i, q_i, \overline{p}, \Gamma \Rightarrow \Delta \end{matrix}}{q_i, \overline{p}, \Gamma \Rightarrow \Delta}\, \text{Ctr} \quad \dots \quad q_m, \overline{p}, \Gamma \Rightarrow \Delta}{\overline{p}, \Gamma \Rightarrow \Delta}\, R$$

which yields the desired conclusion. *qed.*

Since relational rules corresponding to universal formulas do not introduce new variables in the derivation (looking bottom-up) by Lemma 4.5.7, we can obtain a new decidability criterion via terminating proof search and easy counter-model extraction for all the intermediate logics with the finite model property and characterized by a universal frame condition. First, we show that every formula A is valid in a class of finite standard frames with a universal condition if and only if is valid in a certain class of finite strict frames with a universal condition. Given a formula φ in the language $\mathcal{L}_{\leqslant}$, we denote by φ^* the $\mathcal{L}_{<}$ formula obtained by replacing every occurrence of $x \leqslant y$ by $x < y \vee x = y$ (notice that if φ is universal, so is φ^*).

Proposition 4.5.9. *Let* $\mathbf{L}(C)$ *be the set of formulas valid in the class of finite standard frames* C *with the universal frame condition* φ *in the language* $\mathcal{L}_{\leqslant}$*, then* $\mathbf{L}(C) = \mathbf{L}(C^*)$*, where* C^* *is the class of finite strict frames with the universal frame condition* φ^**.*

Proof. We prove the right to left inclusion (the other direction is easier). We observe that φ is of the form:

$$\forall \overline{x}(x_1 \leqslant y_1 \wedge \dots \wedge x_n \leqslant y_n \wedge EQ_1 \rightarrow u_1 \leqslant z_1 \vee \dots \vee u_k \leqslant z_k \vee EQ_2)$$

where EQ_1 (EQ_2) is a finite conjunction (disjunction) of equality atoms. We assume that A is valid in the class C^* of strict frames. Let $\mathcal{F} = \langle P, \leqslant \rangle$ be a standard frame in C, $\mathcal{M}$ a model based on $\mathcal{F}$ and x a world in $\mathcal{M}$. We consider the strict frame $\mathcal{F}' = \langle P, < \rangle$ where $x < y$ if and only if $x \leqslant y$ and $x \neq y$ in $\mathcal{F}$. We claim that $\mathcal{F}' \in C^*$, i.e. that φ^* holds in $\mathcal{F}'$. In fact, φ^* is:

$$\forall \overline{x}((x_1 < y_1 \vee x_1 = y_1) \wedge \dots \wedge (x_n < y_n \vee x_n = y_n) \wedge EQ_1 \rightarrow (u_1 < z_1 \vee u_1 = z_1) \vee \dots \vee (u_k < z_k \vee u_k = z_k) \vee EQ_2)$$

Let us assume that $(x_1 < y_1 \vee x_1 = y_1) \wedge ... \wedge (x_n < y_n \vee x_n = y_n) \wedge EQ_1$ holds in $\mathcal{F}'$, by definition of $<$ we get $x_1 \leqslant y_1 \wedge ... \wedge x_n \leqslant y_n \wedge EQ_1$ in $\mathcal{F}$ since $(x_i \leqslant y_i \wedge \neg(x_i = y_i)) \vee x_i = y_i$ is equivalent to $x_i \leqslant y_i$ in every partial order. Since φ holds in $\mathcal{F}$, we get $u_1 \leqslant z_1 \vee ... \vee u_k \leqslant z_k \vee EQ_2$ in $\mathcal{F}$. This holds if and only if:

$$((u_1 \leqslant z_1 \wedge \neg(u_1 = z_1)) \vee u_1 = z_1) \vee ... \vee ((u_k \leqslant z_k \wedge \neg(u_k = z_k)) \vee u_k = z_k) \wedge EQ_2$$

holds in $\mathcal{F}$. By definition of $<$ we get $(u_1 < z_1 \vee u_1 = z_1) \vee ... \vee (u_k < z_k \vee u_k = z_k) \vee EQ_2$ in $\mathcal{F}'$ and so $\mathcal{F}'$ satisfies φ^* and thus is in C^*. We consider the strict model $\mathcal{N} = \langle \mathcal{F}', v \rangle$, where v coincides with the valuation function of $\mathcal{M}$. We can check by induction on the complexity of the formulas that $\mathcal{N} \vDash B$ if and only if $\mathcal{M} \vDash B$ for every formula B as in Theorem 4.2.1. Since by hypothesis A is valid in every strict frame with the property φ^*, we get $\mathcal{N}, x \Vdash A$ and so $\mathcal{M}, x \Vdash A$. Therefore $A \in \mathbf{L}(C)$. *qed.*

Theorem 4.5.10. *Let* **L** *be an intermediate logic with the finite model property and characterized by a class of strict frames with a universal frame condition.* **L** *has a sound and complete labelled sequent calculus with termination without loop-checking and backtracking.*

Proof. To every intermediate logic **L** with the finite model property and characterized by a class of strict frames with a universal frame condition we can associate a labelled calculus $\mathbf{G3IL}_<$. $\mathbf{G3IL}_<$ is obtained by adding to the base calculus $\mathbf{G3I}_<$ the set of relational rules $\mathcal{R}$ corresponding to the universal conditions imposed on the frames and the equality rules whenever needed. Since the logic **L** enjoys the finite model property, the rules of the calculus - in particular the rule R> which internalizes the finiteness of the strict frames - are sound.

Furthermore, universal rules do not introduce new labels in the search of a derivation by Lemma 4.5.7 and therefore Theorem 4.4.4 still holds with obvious modifications such as the ones for equality detailed in Theorem 4.5.4 or adding extra steps corresponding to the relational rules in $\mathcal{R}$ in the construction of the reduction tree. As a consequence, the proof search terminates and it yields either a derivation or a finite countermodel and therefore we have a decision procedure for the logic **L**. *qed.*

The union of Theorem 4.5.10 and Proposition 4.5.9 easily entails that every intermediate logic with the finite model property and a universal frame condition has a terminating sequent calculus.

The class of intermediate logics characterized by a class of frames with a universal frame condition is wide and it includes also logics which we did not analyze, such as the logics with bounded depth, width and the logics with bounded cardinality which are two countable families of intermediate logics (15).

4.6 The provability interpretation of intuitionistic logic

The definition of the calculus **G3I**$_<$ for intuitionistic logic brings it closer to the provability logic **GL**. Indeed, first we overcome the difficulty tied with working with an irreflexive calculus. Second, the finiteness condition is indirectly built in the rule for the implication.

We now recall the rule of the labelled sequent calculus for the modal logic of provability, **GL**, which are displayed in Figure 4.4. The calculus was introduced in (73) and it is obtained from the semantics for the modal logic of provability exploiting the fact that the truth condition for the modal operator can be rewritten as:

$$x \Vdash \Box A \iff \forall y(xRy \wedge y \Vdash \Box A \Rightarrow y \Vdash A)$$

The calculus satisfies the usually desirable structural properties.

Theorem 4.6.1. *The calculus* **G3GL** *enjoys height-preserving invertibility of every rule, height-preserving admissibility of weakening and contraction and admissibility of cut.*

Proof. See (73). *qed.*

We now recall the modal translation from intuitionistic logic into GL provability logic.

Definition 4.6.1. The $*$: $\mathsf{FM} \to \mathsf{FM}^{\Box}$ is inductively defined:

- $(\bot)^* = \bot$
- $(p)^* = \Box p \wedge p$
- $(A \# B)^* = A^* \# B^*$, where $\# \in \{\wedge, \vee\}$
- $(A \to B)^* = \Box(A^* \to B^*) \wedge (A^* \to B^*)$

We extend the $*$ translation to the full language of **G3I**$_<$ by imposing that $(A > B)^* = \Box(A^* \to B^*)$ and that $(x < y)^* = x < y$ and $(x : A) = x : A^*$. We first prove that the soundness lemma holds.

Initial Sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ ax \qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ L\bot$$

$$\frac{}{xRx, \Gamma \Rightarrow \Delta}\ Irref$$

Logical Rules

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ R\wedge$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \rightarrow B, \Gamma \Rightarrow \Delta}\ L\rightarrow \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \rightarrow B}\ R\rightarrow$$

$$\frac{xRy, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{xRy, x : \Box A, \Gamma \Rightarrow \Delta}\ L\Box \qquad \frac{xRy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ R\Box,\ y \text{ fresh}$$

Relational rules

$$\frac{xRy, yRz, xRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\ Trs$$

Figure 4.4: The labelled calculus **G3GL**

Theorem 4.6.2. *If* $\mathbf{G3I}_< \vdash \Gamma \Rightarrow \Delta$*, then* $\mathbf{G3GL} \vdash \Gamma^* \Rightarrow \Delta^*$.

Proof. By induction on the height of derivation in $\mathbf{G3I}_<$. If $n = 0$, then $\Gamma \Rightarrow \Delta$ is either an instance of ax_1, ax_2 or $L\bot$. If it is ax_1 or $L\bot$ then $\Gamma^* \Rightarrow \Delta^*$ is easily seen to be derivable. If it is ax_2, then we have $x < y, x : p, \Gamma \Rightarrow \Delta, y : p$. We show that the sequent $x < y, x : \Box p \wedge p, \Gamma \Rightarrow \Delta, y : \Box p \wedge p$ is derivable in **G3GL**.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x < y, y < z, x < z, x : \Box p, z : \Box p, x : p, z : p, \Gamma^* \Rightarrow \Delta^*, z : p}\,{\scriptstyle ax}}{x < y, y < z, x < z, x : \Box p, z : \Box p, x : p, \Gamma^* \Rightarrow \Delta^*, z : p}\,{\scriptstyle L\Box}}{x < y, y < z, x : \Box p, z : \Box p, x : p, \Gamma^* \Rightarrow \Delta^*, z : p}\,{\scriptstyle Trs}}{x < y, x : \Box p, x : p, \Gamma^* \Rightarrow \Delta^*, y : \Box p}\,{\scriptstyle R\Box} \quad \dfrac{\dfrac{}{x < y, x : \Box p, y : p, x : p, \Gamma^* \Rightarrow \Delta^*, y : p}\,{\scriptstyle Ax}}{x < y, x : \Box p, x : p, \Gamma^* \Rightarrow \Delta^*, y : p}\,{\scriptstyle L\Box}}{\dfrac{x < y, x : \Box p, x : p, \Gamma^* \Rightarrow \Delta^*, y : \Box p \wedge p}{x < y, x : \Box p \wedge p, \Gamma^* \Rightarrow \Delta^*, y : \Box p \wedge p}\,{\scriptstyle L\wedge}}\,{\scriptstyle R\wedge}
$$

If $n > 0$ the cases are routine and we leave them to the reader. *qed.*

We now prove the main result of the section, which is a syntactic proof of the faithfulness of the embedding of intuitionistic logic in provability logic **GL**.

Theorem 4.6.3. *Let* Γ, Δ *be multisets of labelled formulas of the language of* $\mathbf{G3I}_<$*,* Ω *a multiset of relational atoms,* Γ', Δ' *a multiset of labelled atomic formulas and* Θ *a multiset of formulas of the form* $x : \Box p$.

$$\textit{If } \mathbf{G3GL} \vdash \Omega, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', \textit{ then } \mathbf{G3I}_< \vdash \Omega, \Gamma, \Gamma', \Theta' \Rightarrow \Delta, \Delta'$$

where $\Theta' = \{x : p \,|\, x : \Box p \in \Theta\}$.

Proof. We proceed by induction on the height of derivation in the calculus **G3GL**. If $n = 0$, then $\Omega, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta'$ is an initial sequent and so is $\Omega, \Gamma, \Gamma', \Theta' \Rightarrow \Delta, \Delta'$.

If $n > 0$, we distinguish cases according to the last rule applied. Notice that the last rule cannot be $L \rightarrow$ or $R \rightarrow$. If the last rule applied is $L\vee$ or $R\vee$ we simply apply the induction hypothesis to the premises and we apply again the rule in $\mathbf{G3I}_<$.

If the last rule is $R\wedge$ we have to distinguish three subcases.

- If the principal formula is $x : A^* \wedge B^*$, we apply the induction hypothesis to the premises and the we apply again the rule.

- If the principal formula is $x : \Box p \wedge p$, we consider the premise $\Omega, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', x : p$, we apply the induction hypothesis and we obtain the desired conclusion.

- If the principal formula is $x : \Box(A^* \rightarrow B^*) \wedge (A^* \rightarrow B^*)$, then we have the following situation:

$$\dfrac{\Omega, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', x : \Box(A^* \to B^*) \qquad \Omega, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', x : A^* \to B^*}{\Omega, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', x : \Box(A^* \to B^*) \wedge (A^* \to B^*)} \; R\wedge$$

We proceed as follows:

$$\dfrac{\dfrac{\Omega, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', x : \Box(A^* \to B^*)}{\Omega, \Gamma, \Gamma', \Theta' \Rightarrow \Delta, \Delta', x : A > B} \; \text{IH} \qquad \dfrac{\dfrac{\Omega, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', x : A^* \to B^*}{\Omega, \Gamma^*, x : A^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', x : B^*} \; Inv\text{R}\to}{\Omega, \Gamma, x : A, \Gamma', \Theta' \Rightarrow \Delta, \Delta', x : B} \; \text{IH}}{\Omega, \Gamma, \Gamma', \Theta' \Rightarrow \Delta, \Delta', x : A \to B} \; \text{R}\to$$

If the last rule is $L\wedge$ we must once again distinguish three subcases:

- If the principal formula is $x : A^* \wedge B^*$ we apply the induction hypothesis to the premise and then we apply the rule again.
- If the principal formula is $x : \Box p \wedge p$, we consider the premise $\Omega, x : \Box p, x : p, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta'$ and we apply the induction hypothesis to obtain $\Omega, x : p, x : p, \Gamma, \Gamma', \Theta' \Rightarrow \Delta, \Delta'$ and the desired result follows from the admissibility of Ctr in the calculus $\mathbb{G}\nVdash\mathbb{I}_<$.
- If the principal formula is $x : \Box(A^* \to B^*) \wedge (A^* \to B^*)$, the case is similar to the one detailed for the case in which the last rule applied is R$\to$ and thus we omit the details.

If the last rule is $R\Box$, then the principal formulas has to be of the form $x : \Box(A^* \to B^*)$. We have:

$$\dfrac{x < y, \Omega, \Gamma^*, \Gamma', \Theta, y : \Box(A^* \to B^*) \Rightarrow \Delta^*, \Delta', y : A^* \to B^*}{\Omega, \Gamma^*, \Gamma', \Theta, \Rightarrow \Delta^*, \Delta', x : \Box(A^* \to B^*)} \; \text{R}\Box$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{x < y, \Omega, \Gamma^*, \Gamma', \Theta, y : \Box(A^* \to B^*) \Rightarrow \Delta^*, \Delta', y : A^* \to B^*}{x < y, \Omega, \Gamma^*, \Gamma', \Theta, y : \Box(A^* \to B^*), y : A^* \Rightarrow \Delta^*, \Delta', y : B^*} \; Inv\text{R}\to}{x < y, \Omega, \Gamma, y : A, \Gamma', \Theta', y : A > B, \Rightarrow \Delta, \Delta', y : B} \; IH}{\Omega, \Gamma, \Gamma', \Theta, \Rightarrow \Delta, \Delta', x : A > B} \; \text{R}>$$

The last case we must discuss is that in which the last rule applied is $L\Box$. In this case we distinguish two subcases.

- The principal formula is $x : \Box p$, so the premise is $\Omega, x < y, x : \Box p, y : p, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta'$. We perform the following transformation:

$$\dfrac{\dfrac{\Omega, x < y, x : \Box p, y : p, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta'}{\Omega, x < y, x : p, y : p, \Gamma, \Gamma', \Theta \Rightarrow \Delta, \Delta'} \; \text{IH}}{\Omega, x < y, x : p, \Gamma, \Gamma', \Theta \Rightarrow \Delta, \Delta'} \; \text{Mon}$$

- The principal formula is $x : \Box(A^* \to B^*)$, so the premise is $\Omega, x < y, x : \Box(A^* \to B^*), y : A^* \to B^*, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta'$. We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\Omega, x < y, x : \Box(A^* \to B^*), y : A^* \to B^*, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta'}{\Omega, x < y, x : \Box(A^* \to B^*), \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta', y : A^*}\,{\scriptstyle Inv\mathrm{L}\to}}{\Omega, x < y, x : A > B, \Gamma, \Gamma', \Theta \Rightarrow \Delta, \Delta', y : A}\,{\scriptstyle \mathrm{IH}} \quad \dfrac{\dfrac{\Omega, x < y, x : \Box(A^* \to B^*), y : A^* \to B^*, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta'}{\Omega, x < y, x : \Box(A^* \to B^*), y : B^*, \Gamma^*, \Gamma', \Theta \Rightarrow \Delta^*, \Delta'}\,{\scriptstyle Inv\mathrm{L}\to}}{\Omega, x < y, x : A > B, \Gamma, y : B, \Gamma', \Theta \Rightarrow \Delta, \Delta'}\,{\scriptstyle \mathrm{IH}}}{\Omega, x < y, x : A > B, \Gamma, \Gamma', \Theta \Rightarrow \Delta, \Delta'}\,{\scriptstyle \mathrm{L}>}$$

This concludes the proof. *qed.*

As a corollary we obtain:

Corollary. *For every formula* $A \in \mathsf{FM}$*:*

$$\mathbf{G3GL} \vdash \Rightarrow x : A^* \text{ iff } \mathbf{G3I}_< \vdash \Rightarrow x : A$$

Proof. Immediate by the soundness and the faithfulness theorem. *qed.*

The result is preserved by adding relational rules corresponding to the frame conditions of intermediate logics and extensions of **GL**. In particular, the result can be straightforwardly extended so as to encompass all the logics characterized by a universal condition. The proof is unchanged, as relational rules are preserved by the translation.

Corollary. *Let* **L** *be an intermediate logic characterized by a universal frame condition and* $\mathbf{G3I}^*_<$ *its corresponding labelled calculus. We have:*

$$\mathbf{G3I}^*_< \vdash \Rightarrow x : A \text{ if and only if } \mathbf{G3GL}^* \vdash \Rightarrow x : A^*$$

Remark. This result is interesting for three different reasons. First, it constitutes the first proof-theoretic version of the embedding of intuitionistic logic into the provability logic **GL**. The only available proofs thus far appealed to transformations of Kripke models. Our result is instead direct and constructive and it shows precisely how to transform an intuitionistic derivation in a modal one and vice versa. The second reason is that **GL** - contrarily to **S4** - offers a full-fledged formal provability interpretation of intuitionistic logic, as due to Solovay theorem (96) **GL** is the logic capturing the notion of provability in Peano arithmetic. A third and final advantage of the present approach is that it is modular and it can be also seen as a way to import structural results from modal logics to intuitionistic and intermediate ones.

4.7 Conclusion

We have introduced a new terminating calculus for propositional intuitionistic logic. $\mathbf{G3I}_{<}$ is - to the best of our knowledge - the first system which satisfies all the *desiderata* that we mentioned in the introduction. In particular, every rule is invertible and the structural properties are established by syntactic arguments. The proof search is terminating without loop-checking and it allows the extraction of a countermodel out of a failed proof search. Furthermore, the method is extended modularly to every superintuitionistic logic with the finite model property and a universal frame condition.

The novelty of the approach consists in the definition of a calculus in which the rules directly internalize the finite model property of the logics. In our opinion, although the finite model property implies the decidability of intuitionistic logic, this move is not circular. In fact, we aimed at obtaining a terminating sequent calculus rather than at proving a decidability result. In fact, the decidability of a logic (which is well known for intuitionistic propositional logic) does not imply the existence of a terminating sequent calculus for it, whereas the converse clearly holds.

Finally, the newly defined calculus has been exploited to present the first syntactic proof of the modal embedding of intuitionistic logic in the logic of provability **GL**. This shows how the Gödel-McKinsey-Tarski embedding proves to be a flexible tool which inspires also new methodology to solve old open problems. In fact, a syntactic version of the proof of the modal embedding motivated the search for an alternative calculus which exhibited strong structural properties.

There are various future research directions which might be further explored. First of all, it would be tempting to extend the methodology to first-order intermediate logics. In particular, we deem that this method allows the construction of labelled sequent calculi for the first-order intermediate logics characterized by finite Kripke frames. Second, it could be interesting to devise a direct and syntactic cut-elimination theorem for the system $\mathbf{G3I}_{<}$. Finally, it would be interesting to see whether it is possible to extend the methodology to other non-classical logics characterized by relational semantics or neighborhood semantics which enjoy the finite model property.

Chapter 5

Modalizing mathematical theories

We present a uniform proof-theoretic proof of the Gödel-McKinsey-Tarski embedding for a class of first-order theories. This is achieved by adapting to the case of modal logic the methods of proof analysis in order to convert axioms into rules of inference of a suitable sequent calculus. The soundness and the faithfulness of the embedding are proved by induction on the height of the derivations in the augmented calculi. Finally, we apply the result in order to obtain alternative proofs of some metalogical results and we point out new possible lines of research.

Keywords: modal embedding, proof theory, constructive theories, cut-elimination

5.1 Introduction

It was shown how to extend the translation to the setting of first-order intuitionistic logic (92) and to various intermediate logics (23; 15). In this chapter we take a different route and we study the soundness and the faithfulness of the embedding with respect to first-order theories. Previous works in this area focused on specific theories, specifically on the interpretation of Heyting arithmetic in Peano arithmetic extended with modal operators (38; 44; 66). However, a general and uniform approach to the problem has not been developed yet. We identify a class of theories, determined by the shape of their axioms, for which the soundness and the faithfulness of the translation holds.

Proof analysis of first-order theories has obtained considerable results in the last twenty years. For example, the program *axioms as rules* has shown how to convert mathematical axioms into sequent rules while preserving cut-elimination. The resulting system does not enjoy a full subformula property, but a weaker version thereof, which often allows a good structural analysis of the theory.

In particular, a procedure was found in order to transform geometric axioms into rules (72) and to obtain a proof of the proof-theoretic content of first-order Barr's theorem in the form of a conservativity result for geometric formulas.

We show how to use methods of proof analysis to present a uniform proof of the soundness and the faithfulness of the Gödel-McKinsey-Tarski embedding for first-order Horn theories. Furthermore, the proof that we offer is both constructive, in the sense that it avoids appeal to Zorn's lemma or variants thereof and it is also direct. In fact, the methods that we use are purely proof-theoretic and we explicitly define a proof transformation procedure which enables to obtain a modal proof from an intuitionistic one and vice versa.

This is interesting because it yields a modal interpretation of many constructive mathematical theories in terms of (informal) provability and furthermore it allows to exploit modal systems in order to obtain metalogical properties. In particular, we exploit the embedding result to obtain a syntactic proof of the disjunction property and of the witness property for first-order Horn theories which would be harder to obtain working in a multisuccedent intuitionistic sequent calculus. We opted for a sequent calculus style presentation instead of one based on nested sequents as in Chapter 3 mainly because nested sequents are not suited to handle the rules for the intuitionistic universal quantifier as they encode the constant domain condition on the models.

The first section is devoted to the presentation of the sequent calculus for first order **S4** and to the extension of the methods of proof analysis to such system, establishing the usual desired structural properties, especially cut admissibility. The second section discusses Horn theories, which are a subclass of universal theories and we describe some mathematical examples of theories which are axiomatized by Horn sentences. In sections 3. and 4. we present the extension of Gödel-McKinsey-Tarski embedding to Horn theories. Such result is obtained by two separate (non trivial) lemmas of soundness and faithfulness of the translation. We exploit the translation in order to give an alternative proof of the disjunction property and of the witness property for Horn theories. We conclude the chapter by sketching some possible future lines of research.

5.2 Theories based on S4

The language of first-order modal logic is the extension of the language of propositional modal logic $\mathsf{FM}^{\Box}$ with the universal and the existential quantifiers

Initial Sequents

$$\frac{}{\Gamma, p \Rightarrow p, \Delta}\ Ax \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ L\bot$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L\to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ R\to$$

$$\frac{\Box A, A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ L\Box \qquad \frac{\Box\Gamma, \Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Delta, \Box A}\ R\Box$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/y]}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall,\ y \text{ fresh}$$

$$\frac{A[x/y], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists,\ y \text{ fresh} \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

Figure 5.1: The **G3s4** sequent calculus.

$\forall$ and $\exists$. Sequents are syntactic objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. $\Box\Gamma$ is the multisets which contains the formulas $\Box A$ for every A in Γ.

The degree of a formula is here defined as the number of logical symbols occurring in it. The symbol $\equiv$ expresses syntactic equivalence. We consider the sequent calculus **G3s4** for the modal logic **S4** in Figure 5.1.

We show that the calculus **G3s4** can be extended with rules corresponding to certain axioms while preserving the structural properties of the original system.

Definition 5.2.1. A *geometric* formula is a sentence of the form: $\forall\overline{x}(A \to B)$, where A and B do not contain $\to$ and $\forall$.

Any geometric formula can be equivalently reformulated as a sentence of the shape:

$$\forall\overline{x}(p_1 \wedge ... \wedge p_m \rightarrow \exists\overline{y}_1\mathbf{M}_1 \vee ... \vee \exists\overline{y}_n\mathbf{M}_n)$$

where $\mathbf{M}_j$ is a finite conjunction of atomic formulas and y_j are not free in p_i for every $i \in \{1, ..., m\}$ (72). A geometric theory is a theory whose axioms are all geometric sentences.

Definition 5.2.2. For every geometric axiom:

$$\forall\overline{x}(p_1 \wedge ... \wedge p_m \rightarrow \exists\overline{y}_1\mathbf{M}_1 \vee ... \vee \exists\overline{y}_n\mathbf{M}_n)$$

the geometric rule scheme is:

$$\frac{\overline{q}_1[z_1/y_1], \overline{p}, \Gamma \Rightarrow \Delta \qquad ... \qquad \overline{q}_n[z_n/y_n], \overline{p}, \Gamma \Rightarrow \Delta}{\overline{p}, \Gamma \Rightarrow \Delta} \text{Geom}$$

where $\overline{p} \equiv p_1, ..., p_n$ and, for every k, $\overline{q}_k \equiv q_{k1}, ..., q_{kn_k}$, with $\mathbf{M}_k \equiv q_{k1} \wedge ... \wedge q_{kn_k}$. $\overline{q}_k[z_k/y_k]$ denotes the substitution of z_k with y_k in each q_{kj} and y_k do not occur in the conclusion.

If needed, in order to ensure admissibility of contraction it is necessary to add to the system the closure condition.
Closure condition. Given a system of geometric rules, for every instance of the form:

$$\frac{\overline{q}_1[z_1/y_1], p_1, ..., p_{m-2}, p, p, \Gamma \Rightarrow \Delta \qquad ... \qquad \overline{q}_n[z_n/y_n], p_1, ..., p_{m-2}, p, p, \Gamma \Rightarrow \Delta}{p_1, ..., p_{m-2}, p, p, \Gamma \Rightarrow \Delta} \text{Geom}$$

We need to add its closure under contraction:

$$\frac{\overline{q}_1[z_1/y_1], p_1, ..., p_{m-2}, p, \Gamma \Rightarrow \Delta \qquad ... \qquad \overline{q}_n[z_n/y_n], p_1, ..., p_{m-2}, p, \Gamma \Rightarrow \Delta}{p_1, ..., p_{m-2}, p, \Gamma \Rightarrow \Delta}$$

To give a concrete example, consider the case of a theory $\mathcal{L} = \{R\}$, where R is euclidean, i.e. $\forall x\forall y\forall z(xRy \wedge xRz \rightarrow yRz)$, and consider the following instance:

$$\frac{xRy, xRy, yRy, \Gamma \Rightarrow \Delta}{xRy, xRy, \Gamma \Rightarrow \Delta} \text{Euc}$$

In this case the closure condition is:

$$\frac{xRy, yRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta} \text{Euc c.c.}$$

For further discussion on geometric theories and examples thereof, the interested reader is referred to (72).

Given a set of geometric axioms **T**, we denote with **G3s4T** the sequent calculus obtained by adding to **G3s4** the corresponding geometric rules. We proceed with the structural analysis of the calculus.

Lemma 5.2.1. *For every variable x and every term t, the rule:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma[x/t] \Rightarrow \Delta[x/t]}\ Sub[x/t]$$

is height-preserving admissible in **G3s4T**.

Proof. The proof follows the pattern of (72). *qed.*

Lemma 5.2.2. *The rules:*

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LW \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ RW$$

are height-preserving admissible in **G3s4T**.

Proof. By induction on the height of the derivations in **G3s4T**, exploiting Lemma 5.2.1 in order to avoid possible clashes of variables with respect to the rules $L\exists$, $R\forall$ and $Geom$. *qed.*

A rule is *invertible* if, whenever the conclusion is derivable so is (are) the premise(s).

Lemma 5.2.3. *Every rule except for $R\Box$ is height-preserving invertible in* **G3s4T**.

Proof. The rules $L\Box$ and $Geom$ are invertible by Lemma 5.2.2. We limit ourselves to discuss the case of $R\forall$ as an example. If $n = 0$, then $\Gamma \Rightarrow \Delta, \forall x A$ is an initial sequent and so is $\Gamma \Rightarrow \Delta, A[x/t]$. If $n > 0$, we distinguish cases according to the last rule applied. If the last rule is any rule different from $R\Box$, apply the induction hypothesis to the premise(s) (together with height-preserving substitution to avoid clashes of variables) and then apply the rule again. If the last rule is $R\Box$, we have:

$$\frac{\Box\Gamma, \Gamma \Rightarrow B}{\Box\Gamma, \Gamma' \Rightarrow \Box B, \Delta, \forall x A}\ \mathrm{R}\Box$$

In this case we simply apply again the rule $R\Box$ to obtain $\Box\Gamma, \Gamma' \Rightarrow \Box B, \Delta, A[x/t]$. *qed.*

Lemma 5.2.4. *The rules:*

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LC \quad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ RC$$

are height-preserving admissible in **G3s4T**.

Proof. By simultaneous induction on the height of the derivations.

We discuss the left rule of contraction. If $n = 0$, then $A, A, \Gamma \Rightarrow \Delta$ is an initial sequent and so is $A, \Gamma \Rightarrow \Delta$. If $n > 0$, then we distinguish cases according to the last rule applied. If A is not principal, or if it is principal in $L\forall$ or $L\Box$ or is an active formula in the antecedent of rule $R\Box$, apply the induction hypothesis to the premise(s) and the apply the rule again. If it is principal in a propositional rule or in $L\exists$ we apply invertibility of the corresponding rule by Lemma 5.2.3 and then we apply induction hypothesis. If A is principal in a geometric rule we distinguish two subcases. If only one A is principal, we apply the induction hypothesis to the premise and then we apply the rule again. If both A's are principal, we exploit the closure condition.

The case of the right rule of contraction is similar, the most significant case to discuss is that in which the last rule applied is $R\Box$:

$$\frac{\Box\Gamma, \Gamma \Rightarrow A}{\Box\Gamma, \Gamma' \Rightarrow \Box A, \Box A, \Delta}\text{ R}\Box$$

In this case the conclusion follows by applying again the rule to the premise. *qed.*

Theorem 5.2.5 (Cut-elimination)**.** *The rule:*

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

is admissible in **G3s4T**.

Proof. The proof runs by double induction, with main induction hypothesis on the degree of the cut formula and secondary induction hypothesis on the sum of the height of the derivations. We distinguish cases.

1. If the left premise is the conclusion of an application of a zeroary geometric rule, then the conclusion of the cut is an instance of the rule again. If it is the conclusion of an n-ary geometric rule, we have:

$$\cfrac{\cfrac{\Gamma, \overline{p}, \overline{q}_1[z_1/y_1] \Rightarrow \Delta, A \qquad \dots \qquad \Gamma, \overline{p}, \overline{q}_n[z_n/y_n] \Rightarrow \Delta, A}{\Gamma, \overline{p} \Rightarrow \Delta, A}\text{ Geom} \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \overline{p} \Rightarrow \Delta, \Delta'}\text{ Cut}$$

In this case the cuts are replaced by n-cuts of lesser height and the conclusion is obtained by applying the rule again:

$$\cfrac{\cfrac{\Gamma, \overline{p}, \overline{q}_1[z_1/y_1] \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \overline{p}, \overline{q}_1[z_1/y_1] \Rightarrow \Delta, \Delta'}\text{ Cut} \qquad \dots \qquad \cfrac{\Gamma, \overline{p}, \overline{q}_n[z_n/y_n] \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \overline{p}, \overline{q}_n[z_n/y_n] \Rightarrow \Delta, \Delta'}\text{ Cut}}{\Gamma, \Gamma', \overline{p} \Rightarrow \Delta, \Delta'}\text{ Geom}$$

We can assume that no clashes of variables occur by height-preserving substitution. The n-cuts are removed by the secondary induction hypothesis.

2. If a geometric rule is applied in the right premise of the cut, then we distinguish two subcases.

2.1. If the cut formula is not principal, we consider two further subsubcases. If the geometric rule is a zeroary rule, then the conclusion is also an instance of it. If it is an n-ary rule, we reason as in the analogous case in 1.

2.2. If the cut formula is principal we have:

$$\dfrac{\Gamma \Rightarrow \Delta, p_1 \qquad \dfrac{\Gamma', p_1, p_2, .., p_m, \overline{q}_1[z_1/y_1] \Rightarrow \Delta' \quad \dots \quad \Gamma', p_1, p_2, .., p_m, \overline{q}_n[z_n/y_n] \Rightarrow \Delta'}{\Gamma', p_1, p_2, .., p_m \Rightarrow \Delta'}\,\text{Geom}}{\Gamma, \Gamma', p_2, .., p_m \Rightarrow \Delta, \Delta'}\,\text{Cut}$$

In this case we reason by induction on the height of the left premise $\Gamma \Rightarrow \Delta, P_1$ of the cut. If p_1 is principal, since it is an atomic formula in the succedent, it must be principal in an initial sequent and so p_1 occurs in Γ. In this case, the proof follows by applying weakening to $\Gamma', p_1, p_2, .., p_n \Rightarrow \Delta'$.

If p_1 is not principal we distinguish cases according to the last rule applied. The cases in which the last rule is a geometric one have been dealt with in 1. As regards logical rules, if the last rule applied it different from $R\Box$ we permute the cut upwards and we eliminate it by secondary induction hypothesis, applying height-preserving admissibility of substitution in order to avoid clashes of variables. If it is $R\Box$ we have:

$$\dfrac{\dfrac{\Box\Gamma'', \Gamma'' \Rightarrow B}{\Box\Gamma'', \Gamma''' \Rightarrow \Delta'', \Box B, p_1}\,R\Box \qquad \Gamma', p_1, p_2, .., p_n \Rightarrow \Delta'}{\Box\Gamma'', \Gamma''', \Gamma', p_2, .., p_n \Rightarrow \Delta'', \Box B, \Delta'}\,\text{Cut}$$

In this case the desired is obtained by applying rule $R\Box$ to $\Box\Gamma'', \Gamma'' \Rightarrow B$.

3. The last rule applied is not a geometric rule in both premises.

3.1. If the cut formula is not principal in the left premise of the cut in a rule different from $R\Box$ we permute the cut upwards, we eliminate it by secondary induction hypothesis and then we apply the rule again. If it is not principal in $R\Box$ the conclusion follows by applying again the rule $R\Box$ with weakening to the premise. The case in which the cut formula is not principal in the right premise of the cut is analogous.

3.2. If the cut formula is principal in both premises, we discuss only the modal cases (for the other cases the reader is referred to (109)). The possible combinations are $\langle R\Box, R\Box\rangle$ and $\langle R\Box, L\Box\rangle$. In the first case we have:

$$\dfrac{\dfrac{\Box\Gamma, \Gamma \Rightarrow A}{\Box\Gamma, \Gamma'' \Rightarrow \Delta, \Box A}\,\text{R}\Box \qquad \dfrac{\Box A, A, \Box\Gamma', \Gamma' \Rightarrow B}{\Box A, \Box\Gamma', \Gamma''' \Rightarrow \Delta', \Box B}\,\text{R}\Box}{\Box\Gamma, \Box\Gamma', \Gamma'', \Gamma''' \Rightarrow \Delta, \Delta', \Box B}\,\text{Cut}$$

We construct the following derivation:

$$
\dfrac{\dfrac{\Box\Gamma, \Gamma \Rightarrow A \qquad \dfrac{\dfrac{\Box\Gamma, \Gamma \Rightarrow A}{\Box\Gamma, \Gamma \Rightarrow \Box A}\,{\scriptstyle R\Box} \qquad \Box A, A, \Box\Gamma', \Gamma' \Rightarrow B}{A, \Box\Gamma, \Box\Gamma', \Gamma' \Rightarrow B}\,{\scriptstyle Cut}}{\Box\Gamma, \Box\Gamma, \Box\Gamma', \Gamma, \Gamma, \Gamma' \Rightarrow B}\,{\scriptstyle Cut}}{\Box\Gamma, \Box\Gamma', \Gamma'', \Gamma''' \Rightarrow \Delta, \Delta', \Box B}\,{\scriptstyle R\Box}
$$

The cut is removed by main induction hypothesis on the complexity of the cut formula.

Finally, in the second case we have:

$$
\dfrac{\dfrac{\Box\Gamma, \Gamma \Rightarrow A}{\Box\Gamma, \Gamma'' \Rightarrow \Delta, \Box A}\,{\scriptstyle R\Box} \qquad \dfrac{A, \Box A, \Gamma' \Rightarrow \Delta'}{\Box A, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle L\Box}}{\Box\Gamma, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}
$$

The proof is transformed as follows:

$$
\dfrac{\dfrac{\dfrac{\Box\Gamma, \Gamma \Rightarrow A \qquad \dfrac{\Box\Gamma, \Gamma'' \Rightarrow \Delta, \Box A \qquad A, \Box A, \Gamma' \Rightarrow \Delta'}{A, \Box\Gamma, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}}{(\Box\Gamma)^2, \Gamma, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}}{(\Box\Gamma)^3, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\,{\scriptstyle \text{several } L\Box}}{\Box\Gamma, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Ctr}
$$

The topmost cut is removed by secondary induction hypothesis on the sum of the height of the derivations and the lower cut is removed by main induction hypothesis on the degree of the cut formula. *qed.*

Let **G3s4** $\oplus$ **T** denote the sequent calculus obtained by adding every axiom of the theory **T** as an initial sequent.

Corollary. *For every geometric theory* **T***:*

$$\textbf{G3s4} \oplus \textbf{T} \vdash \Rightarrow A \textit{ if and only if } \textbf{G3s4T} \vdash \Rightarrow A$$

Proof. The direction from left to right easily follows by showing that every axiom of **T** is derivable in **G3s4T**. The direction from right to left we exploit the admissibility of cut and contraction. *qed.*

5.3 Horn theories and rules

In the previous section we have shown how to add rules corresponding to geometric axioms while preserving the structural properties of the underlying modal calculus. However, the class of geometric axioms is too large to establish the soundness of the Gödel-McKinsey-Tarski translation. Therefore we focus our attention on a proper subclass of geometric theories.

Definition 5.3.1. A Horn theory is a theory whose axioms are of the form

$$\forall \overline{x}(p_1 \wedge ... \wedge p_n \rightarrow q)$$

where p_i are atomic for every i and q is either an atomic formula or $\perp$.

Roughly speaking, Horn axioms are are universal closure of implications in which the succedent is an atomic formula and the antecedent is a conjunction of atomic formulas.
There are numerous examples of mathematical Horn theories.

- **Groups** Consider the language $\mathcal{L} = \{\cdot, 1, ^{-1}, =\}$. The axioms are:

  1. $\forall xyz(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$ associativity
  2. $\forall x(x \cdot 1 = x)$ right unit
  3. $\forall x(1 \cdot x = x)$ left unit
  4. $\forall x(x \cdot x^{-1} = 1)$ right inverse
  5. $\forall x(x^{-1} \cdot x = 1)$ left inverse

  In order to avoid the presence of existential quantifiers we have considered an equivalent formulation of the theory obtained by expanding the language, adding the inverse and the unit as a unary and a zeroary operation symbol, respectively (108). To obtain commutative groups we add the axiom $\forall x \forall y(x = y \rightarrow y = x)$.

- **Rings** Consider the language $\mathcal{L} = \{\cdot, +, -, \cdot, =, 0, 1\}$. The axioms are:

  Addition

  1. $\forall xyz(x + (y + z) = (x + y) + z)$ associativity
  2. $\forall xy(x + y = y + x)$ commutativity
  3. $\forall x(x + 0 = x)$ unit
  4. $\forall x(x + (-x) = 0)$ inverse

  Multiplication

  1. $\forall xyz(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$ associativity
  2. $\forall x(x \cdot 1 = x)$ right unit
  3. $\forall x(1 \cdot x = x)$ left unit

  Distributivity

1. $\forall xyz(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$ left distributivity
2. $\forall xyz((y + z) \cdot x = (y \cdot x) + (y \cdot z))$ right distributivity

A commutative ring is obtained by adding the axiom $\forall xy(x \cdot y = y \cdot x)$. Once again we considered a suitable formulation of ring theory, by adding a specific function symbol for the inverse of the sum $+$.

- **Irreflexive graphs**. Consider the language $\{R\}$, where R is a binary relation symbol. The axioms are:

  1. $\forall x \neg R(x, x)$ irreflexivity
  2. $\forall x \forall y(R(x, y) \to R(y, x))$ symmetry

- **Partial orders**. Consider the language $\{\leqslant, =\}$. The axioms are:

  1. $\forall x(x \leqslant x)$ Reflexivity
  2. $\forall xyz(x \leqslant y \land y \leqslant z \to x \leqslant z)$ Transitivity
  3. $\forall xy(x \leqslant y \land y \leqslant x \to x = y)$ Antisymmetry

  Clearly, also strict orders, i.e. irreflexive and transitive orders can be treated.

- **Equivalence relations**. Consider the language $\{\sim\}$. The axioms are:

  1. $\forall x(x \sim x)$ reflexivity
  2. $\forall xyz(x \sim y \land y \sim z \to x \sim z)$ transitivity
  3. $\forall xy(x \sim y \to y \sim x)$ symmetry

- **Lattices**. Consider the language $\{\sqcap, \sqcup, =\}$. The axioms are dual for $\sqcap$ and $\sqcup$:

  1. $\forall xyz(x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z)$ associativity
  2. $\forall xy(x \sqcap y = y \sqcap x)$ commutativity
  3. $\forall xy(x \sqcap (x \sqcup y) = x)$ absorption

As examples of theories which are not Horn theories we indicate linear orders and Robinson's arithmetic. In particular, the first is a regular theory (72), whereas Robinson's arithmetic is a geometric theory (72), due to the presence of the axiom $\forall x(x = 0 \lor \exists y(x = s(y)))$.

The rules obtained from Horn axioms are a particular case of geometric rules: in particular, they have a single premise and they do not contain variable restrictions.

Quantifier rules

$$\frac{A[x/y], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta} L\exists, y \text{ fresh} \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A} R\exists$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} L\forall \qquad \frac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \Delta, \forall x A} R\forall, y \text{ fresh}$$

Figure 5.2: The quantifier rulesfor intuitionistic logic.

Definition 5.3.2. For every Horn axiom $\forall\overline{x}(p_1 \wedge ... \wedge p_n \rightarrow q)$, the Horn rule scheme is as follows:

$$\frac{p_1, ..., p_n, q, \Gamma \Rightarrow \Delta}{p_1, ..., p_n, \Gamma \Rightarrow \Delta} \text{Horn}$$

Since Horn rules are a subclass of geometric rules, the results of the previous section hold with respect to these rules as well.

Theorem 5.3.1. *For every Horn theory* **T***, the calculus* **G3s4T** *enjoys admissibility of weakening, contraction and cut.*

Proof. See the previous section. *qed.*

5.4 Soundness of the translation

The sequent calculus for first-order intuitionistic logic is obtained from **G3i** (see Chapter 4, Figure 4.1) by adding the rules for quantifiers displayed in Figure 5.2, where sequents are built from multisets of formulas. A few comments to the formulation of **G3i** are in order. First, we opted for a multi-succedent version of the system as it is closer to the modal system **G3s4** and this is important in order to establish the faithfulness of the translation. Second, the principal formula of rule $L \rightarrow$ is repeated in the left premise and the rules $R \rightarrow$ and $R\forall$ have a context restriction on the premise (otherwise the rules would be unsound).

From now on we denote by **G3iT** and **G3s4T** the extensions of **G3i** and of **G3s4** by rules corresponding to Horn theories, respectively. We summarize the results of proof analysis for the calculus **G3iT**.

Theorem 5.4.1. *The rules of substitution, weakening and contraction are height preserving admissible in* **G3iT**. *Every rule except for $R \rightarrow$ and $R\forall$ is height-preserving invertible. The cut rule is admissible.*

Proof. See (72). *qed.*

We recall the formulation of the modal translation. The present formulation can be found in (15) and differs from the original from Gödel (see (42)).

Definition 5.4.1. The Gödel-McKinsey-Tarski translation is a map from the language of intuitionistic logic to that of modal logic. It is extended to first-order logic as follows:

- $(\exists x A)^* = \exists x A^*$
- $(\forall x A)^* = \Box \forall x A^*$

In this section we will show that every intuitionistic derivation can be transformed into a derivation in the modal calculus of the translation of the endsequent. We first prove an auxiliary lemma, see also (109).

Lemma 5.4.2. *The sequent* $\Rightarrow A^* \leftrightarrow \Box A^*$ *is provable in* **G3s4**.

Proof. One direction, namely $\Box A^* \Rightarrow A^*$ immediately follows by an application of rule $L\Box$. The other direction is proved by induction on the degree of A. If A is atomic, then $\Box p \Rightarrow \Box\Box p$ is easily seen derivable by two applications of $R\Box$. If A is $B \to C$, then $\Box(B^* \to C^*) \Rightarrow \Box\Box(B^* \to C^*)$ is easily seen to be provable; the same argument applies to the universal quantifier.
If A is of the form $B \wedge C$, then we proceed as follows:

$$\dfrac{\dfrac{\overset{\vdots\,\mathrm{IH}}{B^* \Rightarrow \Box B^*} \qquad \dfrac{\overset{\vdots\,\mathrm{IH}}{C^* \Rightarrow \Box C^*} \qquad \Box B^*, \Box C^* \Rightarrow \Box(B^* \wedge C^*)}{\Box B^*, C^* \Rightarrow \Box(B^* \wedge C^*)}\,\mathrm{Cut}}{B^*, C^* \Rightarrow \Box(B^* \wedge C^*)}\,\mathrm{Cut}}{B^* \wedge C^* \Rightarrow \Box(B^* \wedge C^*)}\,\mathrm{L}\wedge$$

The topsequent on the right is easily derivable by applying rule $R\Box$ and $R\wedge$.
If A is of the form $B \vee C$, we have:

$$\dfrac{\dfrac{\overset{\vdots\,\mathrm{IH}}{B^* \Rightarrow \Box B^*} \qquad \Box B^* \Rightarrow \Box(B^* \vee C^*)}{B^* \Rightarrow \Box(B^* \vee C^*)}\,\mathrm{Cut} \qquad \dfrac{\overset{\vdots\,\mathrm{IH}}{C^* \Rightarrow \Box C^*} \qquad \Box C^* \Rightarrow \Box(B^* \vee C^*)}{C^* \Rightarrow \Box(B^* \vee C^*)}\,\mathrm{Cut}}{B^* \vee C^* \Rightarrow \Box(B^* \vee C^*)}\,L\vee$$

The sequents $\Box B^* \Rightarrow \Box(B^* \vee C^*)$ and $\Box C^* \Rightarrow \Box(B^* \vee C^*)$ are derivable by applying rule $R\Box$ followed by $R\vee$.
If A is of the form $\exists x B$, then we proceed as follows.

$$\dfrac{\dfrac{\overset{\vdots\,\mathrm{IH}}{B^*[x/y] \Rightarrow \Box B^*[x/y]} \qquad \Box B^*[x/y] \Rightarrow \Box\exists x B^*}{B^*[x/y] \Rightarrow \Box\exists x B^*}\,\mathrm{Cut}}{\exists x B^* \Rightarrow \Box\exists x B^*}\,\mathrm{L}\exists$$

Once again the topsequent on the right is easily derivable by applying rule $R\Box$ and then rule $R\exists$. *qed.*

We finally prove the soundness of the translation by a proof-theoretic argument based on induction on the height of the derivations.

Theorem 5.4.3 (Soundness)**.** *If* **G3iT** $\vdash \Gamma \Rightarrow \Delta$, *then* **G3s4T** $\vdash \Gamma^* \Rightarrow \Delta^*$.

Proof. The proof is by induction on the height of the derivations in **G3iT**. If $n = 0$, the proof is immediate. If $n > 0$ we distinguish cases according to the rule applied. If the last rule applied is a rule whose principal formula is a finite conjunction, a disjunction or an existential quantifier, then apply the induction hypothesis to the premises and then apply the rule again.

If the last rule applied is $L\to$ or $L\forall$, then apply the induction hypothesis to the premise, apply the corresponding rule again (an extra weakening step is required only in the case of $L\to$ due to the repetition of the principal formula in the left premise of the rule), then the desired conclusion follows by an application of rule $L\Box$. We give an example of this qualitative analysis:

$$\dfrac{A \to B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L\to$$

We transform the proof as follows:

$$\dfrac{\dfrac{\Box(A^* \to B^*), \Gamma^* \Rightarrow \Delta^*, A^* \qquad \dfrac{B^*, \Gamma^* \Rightarrow \Delta^*}{\Box(A^* \to B^*), B^*, \Gamma^* \Rightarrow \Delta^*}\ \text{Weak}}{\Box(A^* \to B^*), A^* \to B^*, \Gamma^* \Rightarrow \Delta^*}\ L\to}{\Box(A^* \to B^*), \Gamma^* \Rightarrow \Delta^*}\ L\Box$$

If the last rule applied is a Horn rule we have:

$$\dfrac{\Gamma, p_1, p_2, .., p_n, q \Rightarrow \Delta}{\Gamma, p_1, .., p_n \Rightarrow \Delta}\ \text{Horn}$$

By applying the induction hypothesis to the premise we obtain a derivation of the sequent $\Gamma^*, \Box p_1, .., \Box p_n, \Box q \Rightarrow \Delta^*$. We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box p_1, ..., \Box p_n, p_1, ..., p_n, q \Rightarrow q}\ \text{ax}}{\Box p_1, ..., \Box p_n, p_1, ..., p_n \Rightarrow q}\ \text{Horn}}{\Box p_1, ..., \Box p_n \Rightarrow \Box q}\ \text{R}\Box \qquad \Gamma^*, \Box p_1, .., \Box p_n, \Box q \Rightarrow \Delta^*}{\dfrac{\Gamma^*, (\Box p_1, .., \Box p_n)^2 \Rightarrow \Delta^*}{\Gamma^*, \Box p_1, .., \Box p_n \Rightarrow \Delta^*}\ \text{Ctr}}\ \text{Cut}$$

The other cases are rather routine. In particular, consider the case of $R\forall$, we have:

$$\dfrac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x A(x), \Delta}\ R\forall$$

By applying the induction hypothesis we get a derivation of $\Gamma^* \Rightarrow A^*[x/y]$. We exploit the fact that, for every A, the sequent $A^* \Rightarrow \Box A^*$ is provable in **G3s4T** by Lemma 5.4.2 by invertibility of the rules $R\wedge$ and $R \rightarrow$. Then we complete the transformation as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma^* \Rightarrow A^*[x/y]}{\Gamma^* \Rightarrow \forall x A^*(x)}\ R\forall}{\Gamma^*, \Box\Gamma^* \Rightarrow \forall x A^*(x)}\ \text{Weak}}{\Box\Gamma^* \Rightarrow \Box\forall x A^*(x), \Delta^*}\ R\Box}{\Gamma^* \Rightarrow \Box\forall x A^*(x), \Delta^*}\ \text{admissible rule}$$

The rule is admissible via cuts with $A^* \Rightarrow \Box A^*$ for every formula A in the multiset Γ. *qed.*

Notice that soundness is a delicate passage, which requires the restriction of the class of geometric theories to the smaller class of Horn theories. In particular, it is necessary to exclude the presence of disjunctions and existential quantifiers in the succedent.

In fact, try to consider the case of the axiom of trichotomy in linear orders on the language $\{<, =\}$:

$$\forall x \forall y (x < y \vee y < x \vee x = y)$$

It is easy to observe that its $*$-translation is not provable in **G3s4T**, where **T** is the theory of linear orders.

Proposition 5.4.4. *The sequent* $\Rightarrow \Box\forall x \Box\forall y(\Box(x < y) \vee \Box(y < x) \vee \Box(x = y))$ *is not derivable in* **G3s4LO***, i.e. the sequent calculus obtained by adding the rule corresponding to the linearity axiom.*

Proof. If the sequent $\Rightarrow \Box\forall x \Box\forall y(\Box(x < y) \vee \Box(y < x) \vee \Box(x = y))$, via cuts we can easily infer the derivability of the sequent $\Rightarrow \Box(x < y), \Box(y < x), \Box(x = y)$. It is easy to observe that this sequent is derivable if and only if one among $\Rightarrow x < y$, $\Rightarrow y < x$ or $\Rightarrow x = y$, which is not the case. *qed.*

5.5 Faithfulness of the translation

A proof of the faithfulness of the embedding for pure logic was presented in (109). Furthermore, embedding results of intuitionistic logic into modal logics have been obtained by exploiting the methodology of labelled sequent calculi (27; 29). By adopting labelled system the faithfulness proof follows from a straightforward induction on the height of derivations in the modal calculus. Our proof extends these results to first-order Horn theories: we reason by induction on the height of the derivations and we use the standard cut-free sequent calculus **G3s4T**.

In order to prove the faithfulness lemma directly, i.e. by induction on the height of the derivations in the modal calculus, we need to devise a suitable strengthening of the induction hypothesis which takes into account the built-in contraction contained in the left rule for the universal quantifier.

Lemma 5.5.1 (Faithfulness)**.** *Let* Π *and* Σ *be multisets of atomic formulas,* $\Gamma^{\forall}$ *a multiset of formulas* $\forall x A^*$*,* Λ *and* Δ *multisets of formulas. Then:*

$$\textit{If } \mathbf{G3s4T} \vdash \Pi, \Gamma^{\forall}, \Lambda^* \Rightarrow \Delta^*, \Sigma, \textit{ then } \mathbf{G3iT} \vdash \Pi, \Gamma^{\forall -}, \Lambda \Rightarrow \Delta, \Sigma$$

where $\Gamma^{\forall -}$ *contains formulas* $\forall x A$ *for every* $\forall x A^*$ *in* $\Gamma^{\forall}$*.*

Proof. The proof is by induction on the height of the derivations in **G3s4T**. If $n = 0$, then the proof is immediate. If $n > 0$, we distinguish cases according to the last rule applied. If the last rule is different from $L\Box$ or $R\Box$, we can simply apply the induction hypothesis and then the rule again (if necessary, as in the case of $L\forall$, we add an extra step of contraction). In particular, if the last rule applied is a Horn rule, we apply the induction hypothesis and then the rule again, because the active formulas of the rule are all atomic.

If the last rule is $L\Box$, we have:

$$\frac{\Pi, \Gamma^{\forall}, \Lambda^*, \Box A^*, A^* \Rightarrow \Delta^*, \Sigma}{\Pi, \Gamma^{\forall}, \Lambda^*, \Box A^* \Rightarrow \Delta^*, \Sigma} \text{ L}\Box$$

where $\Box A^* \equiv B^*$ for some formula B. If $B \equiv \forall x C$ or P, with P atomic, we apply the induction hypothesis to the premise and then we apply height-preserving admissibility of contraction to obtain the desired conclusion. If $B \equiv C \to D$ we have:

$$\frac{\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*), C^* \to D^* \Rightarrow \Delta^*, \Sigma}{\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*) \Rightarrow \Delta^*, \Sigma} \text{ L}\Box$$

In this case we apply height-preserving invertibility of rule $L \to$ to the premise, to obtain two derivations of:

1. $\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*) \Rightarrow \Delta^*, \Sigma, C^*$

2. $\Pi, \Gamma^{\forall}, D^* \Lambda^*, \Box(C^* \to D^*) \Rightarrow \Delta^*, \Sigma$

We proceed as follows:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\Pi, \Gamma^{\forall}, \Lambda^*, \Box(C^* \to D^*) \Rightarrow \Delta^*, \Sigma, C^*}{\Pi, \Gamma^{\forall}, \Lambda, C \to D \Rightarrow \Delta, \Sigma, C}\;\text{IH}
}{\Pi, \Gamma^{\forall}, \Lambda, C \to D, C \to D \Rightarrow \Delta, \Sigma, C}\;\text{Weak}
\qquad
\dfrac{\Pi, \Gamma^{\forall}, D^*, \Lambda^*, \Box(C^* \to D^*) \Rightarrow \Delta^*, \Sigma}{\Pi, \Gamma^{\forall}, D, \Lambda, C \to D \Rightarrow \Delta, \Sigma}\;\text{IH}
}{\Pi, \Gamma^{\forall}, \Lambda, C \to D, C \to D \Rightarrow \Delta, \Sigma}\;\text{L}\to
}{\Pi, \Gamma^{\forall}, \Lambda, C \to D \Rightarrow \Delta, \Sigma}\;\text{Ctr}
$$

where IH denotes the application of the inductive hypothesis.

If the last rule is $R\Box$ we distinguish subcases according to the shape of the principal formula in Δ^*. If it is of the form $\Box P$, we have:

$$
\dfrac{\Box\Lambda''', \Lambda''' \Rightarrow P}{\Pi, \Gamma^{\forall}, (\Lambda'')^*, (\Lambda')^* \Rightarrow \Delta'^*, \Box P, \Sigma}\;R\Box
$$

with $\Lambda^* \equiv (\Lambda'')^*, (\Lambda')^*$ and $(\Lambda'')^* \equiv \Box\Lambda'''$. Now, formulas in Λ''' can be of three types: atomic formulas, implications and universal quantifiers. Namely,

$$
\Lambda''' \equiv q_1, \ldots, q_n, \forall x D_1^*(x), \ldots, \forall x D_l^*(x), B_1^* \to C_1^*, \ldots, B_j^* \to C_j^*
$$

We apply height-preserving invertibility of $L \to$ to reduce the complexity of the implication formulas. Then we apply the induction hypothesis to the 2^j derivations thus obtained. To simplify the explanation and the notation, we assume that there is a single occurrence for each of the three types of formulas, i.e. $n = l = j = 1$, the generalization is straightforward. So we have $\Lambda''' \equiv q, \forall x D^*(x), B^* \to C^*$.
By height-preserving invertibility of $L \to$ we obtain two derivations of the form:

1. $\Box\Lambda''', q, \forall x D^*(x) \Rightarrow p, B^*$

2. $\Box\Lambda''', q, \forall x D^*(x), C^* \Rightarrow p$

Applying the induction hypothesis gives derivations of:

1. $\Lambda'', q, \forall x D(x) \Rightarrow p, B$

2. $\Lambda'', q, \forall x D(x), C \Rightarrow p$

We apply weakening to 1. to add $B \to C$ to the antecedent, then we apply $L \to$, which yields

$$
\Lambda'', B \to C, q, \forall x D(x) \Rightarrow P \;\equiv\; \Lambda'', \Lambda'' \Rightarrow p
$$

Finally admissibility of contraction gives:

$$
\Lambda'' \Rightarrow p
$$

and the conclusion follows by admissibility of weakening.

The cases in which the formula is of the shape $\Box(A^* \to B^*)$ or $\Box\forall x A^*(x)$ are actually similar. In particular, we apply the invertibility of the right rule for $\to$ and $\forall$ in order to be able to apply the induction hypothesis and we repeat the procedure described for the atomic case. We sketch the case of the universal quantifier:

$$\frac{\Box\Lambda''', \Lambda''' \Rightarrow \forall x B^*}{\Pi, \Gamma^{\forall}, (\Lambda'')^*, (\Lambda')^* \Rightarrow \Delta'^*, \Box\forall x B^*, \Sigma}\ R\Box$$

We apply height-preserving invertibility of $R\forall$ and we obtain $\Box\Lambda''', \Lambda''' \Rightarrow B^*[x/y]$ where y is a fresh variable. We then apply height-preserving invertibility of $L\to$ to implicative formulas in Λ''' and thus we can apply the induction hypothesis. Finally, we conclude the proof by an application of $R\forall$ and weakening admissibility to add the missing contexts. *qed.*

By combining the faithfulness lemma with the results presented in the previous section, we obtain the embedding result.

Theorem 5.5.2 (Embedding)**.** *Let* **T** *be a Horn theory, then:*

$$\textbf{G3iT} \vdash \Rightarrow A \textit{ if and only if } \textbf{G3s4T} \vdash \Rightarrow A^*$$

Proof. From left to right we exploit the soundness theorem and from right to left we exploit the faithfulness lemma. *qed.*

We can exploit the soundness and faithfulness result in order to obtain an alternative proof of the disjunction property and of the witness property for Horn theories in a multisuccedent intuitionistic calculus. Namely, instead of searching a proof in the multisuccedent intuitionistic system we can solve the problem by working in the modal calculus.

Theorem 5.5.3 (Disjunction property)**.** *For every Horn theory* **T***, if* **G3iT** $\vdash\Rightarrow A \vee B$*, then* **G3iT** $\vdash\Rightarrow A$ *or* **G3iT** $\vdash\Rightarrow B$*.*

Proof. If **G3iT** $\vdash\Rightarrow A \vee B$, then by soundness we obtain **G3s4T** $\vdash\Rightarrow A^* \vee B^*$. By invertibility of rule R$\vee$ and cuts with $A^* \Rightarrow \Box A^*$ and $B^* \Rightarrow \Box B^*$, we get **G3s4T** $\vdash\Rightarrow \Box A^*, \Box B^*$. The derivation must have the following form:

$$\begin{array}{c} \vdots \\ \dfrac{\Rightarrow C}{\Gamma \Rightarrow \Box A^*, \Box B^*}\ \text{R}\Box \\ \vdots \mathcal{D} \\ \Rightarrow \Box A^*, \Box B^* \end{array}$$

where $\mathcal{D}$ contains only applications of Horn rules, Γ is a multiset of atomic formulas and C is either A^* or B^*, depending on the principal formula of R$\Box$. This yields **G3s4T** $\vdash\Rightarrow A^*$ or **G3s4T** $\vdash\Rightarrow B^*$. By faithfulness of the translation we get the desired conclusion. *qed.*

Theorem 5.5.4 (Witness property)**.** *Every Horn intuitionistic theory enjoys the witness property, i.e. if* **G3iT** $\vdash\Rightarrow \exists x A(x)$*, then* **G3iT** $\vdash\Rightarrow A[x/t]$ *for some term* t*.*

Proof. Assume **G3iT** $\vdash\Rightarrow \exists x A(x)$, by soundness of the translation we get **G3s4T** $\vdash\Rightarrow \exists x A^*(x)$. By a cut with $\exists x A^*(x) \Rightarrow \exists x \Box A^*(x)$, we get **G3s4T** $\vdash\Rightarrow \exists x \Box A^*(x)$. By analyzing the derivation of such sequent, we observe that the derivation must have the following structure:

$$\dfrac{\begin{array}{c}\vdots\\ \Rightarrow A^*[x/t_i]\end{array}}{\Gamma \Rightarrow \exists x \Box A^*(x), \Box A^*[x/t_1], ..., \Box A^*[x/t_n]}\ \text{R}\Box$$
$$\vdots\mathcal{D}$$
$$\Rightarrow \exists x \Box A^*(x)$$

where $\mathcal{D}$ contains only applications of Horn rules and of the rule R$\exists$, $i \in \{1, ..., n\}$ and Γ contains only atomic formulas. By the faithfulness of the translation we get that $\Rightarrow A[x/t_i]$ is derivable in **G3iT**, then take $t_i \equiv t$.

qed.

5.6 Geometric logic and the modal embedding

As we have already observed, the soundness of the modal translation breaks down in the presence of geometric axioms or, more in general, of axioms containing disjunctions or existential quantifiers in the succedent.

Indeed, the modal intepretation still holds for pure logic, in the sense that given an axiom A in first-order geometric logic, we have:

$$\textbf{G3i} \oplus A \vdash \Gamma \Rightarrow \Delta \text{ if and only if } \textbf{G3s4} \oplus A^* \vdash \Gamma^* \Rightarrow \Delta^*$$

However, this solution cannot be regarded as satisfactory. In general, the axiom A is not equivalent over **S4** to its $*$-translation. Therefore we are actually considering a different theory and not the same theory over a modal base.

A very natural question consists in asking which kind of modal system is suitable to reach the following result:

$$\textbf{G3i} \oplus A \vdash \Gamma \Rightarrow \Delta \text{ if and only if } \textbf{G3}? \oplus A \vdash \Gamma^* \Rightarrow \Delta^*$$

To obtain such system we need to properly extend **S4** with an infinite set of axioms. In particular, we require:

$$p \to \Box p \text{ for every atomic first-order formula } p$$

To obtain an analytic system for this logic, we need to slightly modify the rule governing the modal operator.

$$\frac{\Gamma^{at}, \Box\Pi, \Pi \Rightarrow A}{\Gamma^{at}, \Box\Pi, \Pi' \Rightarrow \Delta, \Box A}\text{R}\Box^{+}$$

In other words we require that the atomic propositional formulas are not removed by the application of the rule for the modal operator. Let **G3s4T⁺** be the system obtained by replacing the rule R□ with the rule R□⁺.

Lemma 5.6.1. *The rules of substitution, weakening and contraction are height-preserving admissible in the calculus* **G3s4T⁺**. *Every rule except for R□⁺ is height-preserving admissible.*

Proof. The proofs run by induction and they are minor modifications of the ones for **G3s4T**, therefore we omit the details. *qed.*

Theorem 5.6.2. *The cut rule is admissible in* **G3s4T⁺**.

Proof. The proof runs by double induction with main induction hypothesis on the degree of the cut formula and secondary induction hypothesis on the sum of the height of the derivations of the premises of the cut.

The new relevant case is the one in which the cut formula is atomic and principal in an application of the rule R□⁺ in the right premise of the cut.

$$\frac{\Gamma \Rightarrow \Delta, p \qquad \dfrac{p, \Pi^{at}, \Box\Theta, \Theta \Rightarrow A}{p, \Pi^{at}, \Box\Theta, \Theta' \Rightarrow \Box A, \Lambda}\text{R}\Box^{+}}{\Gamma, \Pi^{at}, \Box\Theta, \Theta' \Rightarrow \Box A, \Lambda, \Delta}\text{Cut}$$

The cut cannot be simply permuted upwards as the rule R□⁺ might not be applicable. Hence, we argue by induction on the left premise of the cut. The case in which it is an initial sequent is trivial. If it is the conclusion of a rule, then p is not principal. If the last rule applied is R□⁺ we consider the premise and we apply the rule again to get the desired conclusion. If the last rule applied is any other rule, we permute the cut upwards and we apply the rule again.

qed.

Although this section is devoted to a syntactic approach to the issue, we would like to point out that a very natural semantics for the system **G3s4T⁺** emerges by considering first-order Kripke models for modal logic with increasing domains and by imposing a monotonicity condition on atomic formulas.

Lemma 5.6.3. *Let* **T** *be a geometric theory. The following statements hold:*

1. **G3s4T**$^+$ $\vdash \Rightarrow p \to \Box p$

2. *There is not a collapse of the modality in* **G3s4T**$^+$*, i.e. there is a formula* A *such that* **G3s4T**$^+$ *does not prove* $\Rightarrow A \leftrightarrow \Box A$.

Proof. Item 1. follows from a routing root-first application of the rule R$\to$ and R$\Box^+$. Notice that the sequent is not provable in **G3S4T**.

Item 2. follows by noticing that $(p \to q) \to \Box(p \to q)$ is not derivable. Suppose towards a contradiction that it is derivable, then by invertibility of the rule L$\to$ so is $\Rightarrow p, \Box(p \to q)$. However, the only applicable rule is R$\Box^+$ which gives $\Rightarrow p \to q$, an underivable sequent. *qed.*

The crucial result for **G3s4T**$^+$ is that for every formula A, we have $A^l \leftrightarrow A^*$, where l is a light translation thus defined.

Definition 5.6.1. The light Gödel-McKinsey-Tarski translation is a map from the language of intuitionistic logic to that of modal logic. It is inductively defined as follows:

- $(p)^l = p$, for p atomic.
- $(\bot)^l = \bot$
- $(A\#B)^l = A^l\#B^l$, where $\# \in \{\wedge, \vee\}$
- $(A \to B)^l = \Box(A^l \to B^l)$
- $(\exists xA)^l = \exists xA^l$
- $(\forall xA)^l = \Box\forall xA^l$

Lemma 5.6.4. **G3s4T**$^+$ $\vdash \Rightarrow A^l \leftrightarrow A^*$ *for every formula* A.

Proof. Immediate by induction on the degree of the formula A. *qed.*

To complete our investigations, we show the following.

Theorem 5.6.5. **G3s4T**$^+ \oplus A^l$ *is equivalent to* **G3s4T**$^+ \oplus A$

Proof. It is trivial to observe that $A^l \to A$, so one direction is easily established via suitable cuts. The converse does not hold in general, so we look at the structure of the derivations. Suppose we have a derivation employing an axiom $\Rightarrow A^l$ as initial sequent. It will be of the form:

$$\Rightarrow \Box\forall\overline{x}\Box(p_1 \wedge \ldots \wedge p_m \rightarrow \exists\overline{y}_1\overline{q}_1 \vee \ldots \vee \exists\overline{y}_n\overline{q}_n)$$

This can be simulated as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \forall\overline{x}(p_1 \wedge \ldots \wedge p_m \rightarrow \exists\overline{y}_1\overline{q}_1 \vee \ldots \vee \exists\overline{y}_n\overline{q}_n)}{\Rightarrow p_1 \wedge \ldots \wedge p_m \rightarrow \exists\overline{y}_1\overline{q}_1 \vee \ldots \vee \exists\overline{y}_n\overline{q}_n}\text{Inv R}\forall}{\Rightarrow \Box(p_1 \wedge \ldots \wedge p_m \rightarrow \exists\overline{y}_1\overline{q}_1 \vee \ldots \vee \exists\overline{y}_n\overline{q}_n)}\text{R}\Box^+}{\Rightarrow \forall\overline{x}\Box(p_1 \wedge \ldots \wedge p_m \rightarrow \exists\overline{y}_1\overline{q}_1 \vee \ldots \vee \exists\overline{y}_n\overline{q}_n)}\text{R}\forall}{\Rightarrow \Box\forall\overline{x}\Box(p_1 \wedge \ldots \wedge p_m \rightarrow \exists\overline{y}_1\overline{q}_1 \vee \ldots \vee \exists\overline{y}_n\overline{q}_n)}\text{R}\Box^+$$

qed.

The modal embedding is established for geometric axiomatic extensions by the following theorem.

Theorem 5.6.6. *For every geometric theory* **T**, **G3iT** $\vdash \Gamma \Rightarrow \Delta$ *if and only if* **G3s4T$^+$** $\vdash \Gamma^l \Rightarrow \Delta^l$.

Proof. From left to right we argue by induction on the height of the derivation. The only new case to check is the one of the geometric rules. Due to the definition of the translation, it is enough to apply the induction hypothesis and then the rule again.

From right to left the strategy follows the pattern detailed in the case of **G3s4T**. *qed.*

Theorem 5.6.7. *For every geometric theory* **T**, **G3iT** $\vdash \Gamma \Rightarrow \Delta$ *if and only if* **G3s4T$^+$** $\vdash \Gamma^* \Rightarrow \Delta^*$.

Proof. We consider the following chain of equivalences. Clearly, **G3iT** $\vdash \Gamma \Rightarrow \Delta$ if and only if **G3s4T$^+$** $\vdash \Gamma^l \Rightarrow \Delta^l$. The latter is equivalent to **G3s4T$^+$** $\vdash \Gamma^* \Rightarrow \Delta^*$, which yields the desired conclusion. *qed.*

5.7 Concluding remarks

We have applied the methods of proof analysis to systems of modal logics and we have proved an extension of the Gödel-McKinsey-Tarski embedding to for first-order Horn theories.

There are various points which might be interesting future line of research. First, it would be interesting to study a similar approach in terms of labelled sequent calculi. This would be convenient as it would greatly simplify the proof of soundness and faithfulness of the embedding and furthermore it could be used to explore the possibility of a Tait-Schütte-Takeuti style completeness for first-order theories based on first-order intuitionistic and modal logic.

Second, it is worth investigating the possibility of extending the approach to systems with first-order axioms containing modal formulas. For example, consider the formula: $\forall x(p(x) \to \Diamond \exists y q(x, y))$, which inside a labelled sequent calculus might be converted into the rule:

$$\frac{a \in D(w), w : p(a), wRo, b \in D(o), o : q(a, b), \Gamma \Rightarrow \Delta}{a \in D(w), w : p(a), \Gamma \Rightarrow \Delta} \text{ Geom, } o, b \text{ fresh}$$

with a double variable condition on worlds and elements of the domain. It is worth considering the scope of such an approach, also in connection with the work of Linnebo and Shapiro (58).

Third, in this chapter we have applied the methods of proof analysis to a domain which lies outside of classical and intuitionistic logic. This naturally poses the intriguing question whether the conversion of axioms into rules can be obtained also considering as a base calculus another non-classical system.

Chapter 6

Infinitary logic and the embedding

The Gödel-McKinsey-Tarski embedding allows to view intuitionistic logic through the lenses of modal logic. In this chapter, an extension of the modal embedding to infinitary intuitionistic logic is introduced. First, infinitary intuitionistic logic is thoroughly investigated, both from a semantic and a syntactic point of view. Next, a neighborhood semantics for a family of axiomatically presented infinitary modal logics is given and soundness and completeness are proved via the method of canonical models. The semantics is then exploited to obtain labelled sequent calculi with good structural properties. Hence, soundness and faithfulness of the embedding are established by transfinite induction on the height of derivations: the proof is obtained directly without resorting to non-constructive principles. Finally, the modal embedding is employed in order to relate classical, intuitionistic and modal derivability in infinitary logic extended with axioms.

6.1 Introduction

Infinitary languages have been extensively studied in logic and especially in proof theory. They are obtained by adding to the language expressions of the shape $\bigwedge_{k>0} A_k$ and $\bigvee_{k>0} A_k$ that denote countable conjunctions and disjunctions, respectively. The proof theory for systems with rules with infinitely many premises has been systematically exploited in order to determine the proof-theoretic strength of arithmetical theories such as Peano arithmetic and ramified analysis.

The works in the area have thoroughly investigated the structural and semantic properties of *classical* infinitary logic. However, the study of infinitary languages in *non-classical* contexts is still underdeveloped. The need to extend the proof theoretical analysis to infinitary intuitionistic logic comes from the fact that infinitary intuitionistic logic is the natural ground to formulate geometric theories, i.e. theories axiomatized by universally closed implications whose antecedent is a finite conjunction of atomic formulas and the succedent is a (possibly) infinite disjunction of existentially quantified finite conjunctions of atomic formulas. Geometric theories are ubiquitous in mathematics as they form the basis of large portions of algebraic theories. Furthermore, in this context the interaction between intuitionistic and classical reasoning obtains a well-defined identification: by Barr's theorem, for derivability of geometric formulas in geometric theories classical and intuitionistic provability coincide.

The most famous interpretation of intuitionistic logic is the BHK interpretation of logical connectives (107), which is based on the conception of truth as a constructive mathematical proof. However such interpretation is informal, as it is based on the elusive notion of construction: this led logicians to consider different formal semantics for intuitionistic logic by exploiting the connections between intuitionistic logic and topology.

The main semantics which were developed are Kripkean semantics (54), Beth semantics (4), topological semantics (66) and algebraic semantics. Among these the most popular one is indeed the Kripke style semantics due to its flexibility. It is to be noted however that from the point of view of validity they are all equivalent with respect to propositional intuitionistic logic.

Nevertheless things change when we consider extensions of intuitionistic logic. In particular if we deal with intuitionistic infinitary logic, that is intuitionistic logic augmented with countable disjunctions and conjunctions, we immediately notice that the equivalence between the different semantics is lost. The infinitary version of the distributivity axiom:

$$\bigwedge_{k>0} (p_k \vee q) \rightarrow \bigwedge_{k>0} p_k \vee q$$

is not intuitionistically acceptable, but it holds in the Kripkean models extended with satisfiability conditions for infinitary conjunctions and disjunctions (71). The reason of this difference is that Kripkean frames are partial orders which in turn correspond to Alexandroff topologies, i.e. topologies closed under infinite intersections (79). This is reflected in the definition of the satisfiability condition for the infinitary conjunction $\bigwedge$ in Kripkean semantics:

$$x \Vdash \bigwedge_{k>0} A_k \text{ iff } x \Vdash A_k \text{ for every } k > 0$$

which generalises the finitary case.

Infinitary intutionistic logic was first introduced by Kalicki in (52) in which a semantics and a tableau system are presented. The semantics is that of complete

Heyting algebras, therein called pseudoboolean algebras, which are Heyting algebras in which sups and infs exist for every subset in the algebra.

Nadel (71) studied intuitionistic infinitary logic and the relations between algebraic, kripkean and Beth's semantics. With respect to algebraic semantics he discusses algebraic models for countable fragments of the language in the more general setting of Heyting algebras rather than complete Heyting algebras (requiring the existence of a sufficient quantity of sups and infs). He also introduced a sequent calculus and proved completeness with respect to the algebraic semantics. More recently a multisuccedent sequent calculus $\mathbf{G3i}_{\omega}$ along with a cut-elimination procedure was given in (72) with an application to Barr's theorem (for further discussion on these topics, see also (93)). $\mathbf{G3i}_{\omega}$ enjoys good structural properties, namely height-preserving admissibility of weakening and contraction, but it lacks full invertibility of every rule. In fact, the right rule for the implication has a single-succedent restriction in the premise as in the finitary case and the same holds for the right rule for the infinitary conjunction.

The restriction in the case of finitary intuitionistic logic can be avoided by resorting to a labelled sequent calculus based on kripkean semantics, see (27). However this strategy cannot be directly adopted in the case of infinitary intuitionistic logic, because kripkean semantics is not adequate for infinitary intuitionistic logic. This suggests that we should search for a more fine grained structure.

Propositional intuitionistic logic and first-order intuitionistic logic are complete with respect to topological semantics (92), (66) and (110). However, to our best knowledge there is no proof of completeness for intuitionistic infinitary logic with respect to topological semantics. We shall therefore give a proof of the result for countable fragments of infinitary intuitionistic logic which ensures completeness with respect to $\mathbf{G3i}_{\omega}$. Completeness with respect to topological semantics in turn allows to obtain a neighborhood semantics for infinitary intuitionistic logic.

Neighborhood semantics for intuitionistic logic was introduced in (69). Neighborhood semantics is a generalisation of Kripkean semantics and it has been extensively used in order to study non-normal modal logics, i.e. logics weaker than the minimal normal modal system **K**. Neighborhood semantics models infinitary intuitionistic logic through restrictions on the closure under intersections between neighborhoods.

We will establish completeness for infinitary intuitionistic logic with respect to neighborhood semantics and exploit it in order to obtain a labelled calculus $\mathbf{G3I}_{\omega}$, along the lines of (76). The calculus enjoys good structural properties: every rule is height-preserving invertible and structural rules are admissible and cut elimination is established in a standard fashion via a double transfinite induction.

The identification between intuitionistic and classical reasoning obtained through Barr's theorem is limited to the geometric fragment. Other fragments have been identified by what are known as Givenko sequent classes. However, there is another - more global - way in which one can give a constructive sense to classical reasoning. In fact, intuitionistic logic can be seen through a modal lens, by extending classical propositional logic with a modality that takes to the object language the intuitionistic notion of provability. This was indeed the original motivation of Gödel's 1933 translation of intuitionistic logic into the modal system **S4**, in which the modal operator $\Box$ received an interpretation in terms of an informal notion of provability (42). Gödel also conjectured the faithfulness of the translation, namely that if the translation of an intuitionistic formula is provable in **S4** then the formula is a theorem of intuitionistic logic. A proof of this fact was published only in 1948 by Tarski and McKinsey (65)[1]. The proof was both indirect, since it used semantic methods, and non-constructive, because the proof essentially required Stone representation theorem, which in turn needs Zorn's lemma.

Extensions of the Gödel-McKinsey-Tarski embedding were considered with respect to first-order intuitionistic logic and first-order modal logic **S4** with increasing domains (91). To our knowledge, no translation has yet been considered for infinitary intuitionistic logic. In order to introduce this generalization we first offer a presentation of infinitary modal logic in terms of neighborhood semantics.

We start by introducing an axiomatization of infinitary $\mathbf{S4}_\omega$ modal logic obtained with the addition of lattice-like axioms for the infinitary connectives $\bigwedge$ and $\bigvee$ to **S4**. We prove the deduction theorem for such system and then completeness with respect to a class of neighborhood frames. See also (102) for a study of neighborhood frames and infinitary modal logic from the perspective of duality theory.

The semantic and its labelled calculus shed further light on the relation between intuitionistic and modal logic. In particular, we present an extension of the Gödel-McKinsey-Tarski embedding of intuitionistic logic into the **S4** modal system (see (15) for an extensive treatment) and prove its soundness and faithfulness both by semantic and proof-theoretic methods. The method presented is of independent interest as it is general and can thus be extended so as to cover even subsystems of the infinitary modal logic $\mathbf{S4}_\omega$. Furthermore, working with

[1]An unpublished proof by Gödel himself was found from the transcription of his stenographic notebooks *Resultate Grundlagen* (43)

labelled calculi we can show how to transform the modal derivation of a translated formula into an intuitionistic derivation of the formula.

The natural - so to say - extension of the $*$ translation to infinitary intuitionistic logic is obtained by adding the following conditions: $(\bigwedge_{k>0} A_k)^* = \bigwedge_{k>0} A_k^*$ and $(\bigvee_{k>0} A_k)^* = \bigvee_{k>0} A_k^*$. As we will see this translation, although sound, is not faithful and infinitary conjunction has to be translated in another way. In particular, it is necessary to modify the interpretation as follows:

$$(\bigwedge_{k>0} A_k)^* = \Box \bigwedge_{k>0} A_k^*$$

This interpretation shows that the elements of non-classicality within intuitionistic logic can be precisely isolated using an **S4** modality even in the infinitary setting.

We conclude the chapter by relating derivability in axiomatic extensions of infinitary intuitionistic, classical and modal logic, by identifying a class of sequents in which derivability coincides under the modal interpretation. Interestingly, we show that the class coincides, in a sense, with a modified version of geometric logic once we add to the infinitary modal logic $\mathbf{S4}_\omega$ the axiom $P \to \Box P$ for every atomic formula P.

6.2 Topological completeness theorem for infinitary intuitionistic logic

The language of intuitionistic infinitary propositional logic is uncountable and in what follows we will refer to suitable countable fragments of such language. In this section we show the completeness theorem of countable fragments of infinitary intuitionistic logic with respect to topological semantics. First of all we recall that formulas in intuitionistic infinitary propositional logic are built extending the usual definition with countable conjunctions and disjunctions. We denote the set of propositional intuitionistic infinitary formulas with FM_ω.

Before we proceed with the study of the different semantics, we briefly recall the sequent calculus $\mathbf{G3i}_\omega$ for infinitary propositional intuitionistic logic and its properties. The calculus is obtained by adding to the calculus **G3i** (see Figure 4.1) the rules governing the infinitary connectives. Sequents are finite multisets of formulas in the language of infinitary intuitionistic logic.

G3i$_\omega$

Infinitary rules

$$\frac{\bigwedge_{k>0} A_k, A_k, \Gamma \Rightarrow \Delta}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}\, L\bigwedge \qquad \frac{\{\Gamma \Rightarrow A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{k>0} A_k}\, R\bigwedge$$

$$\frac{\{A_k, \Gamma \Rightarrow \Delta \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}\, L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k}\, R\bigvee$$

Theorem 6.2.1. *The rules of weakening and contraction are height-preserving admissible in* **G3i$_\omega$**. *Cut is admissible. Every rule except $R \to$ and $R \bigwedge$ is height-preserving invertible.*

Proof. (72). *qed.*

We recall the *algebraic* semantics for infinitary intuitionistic logic, which is based on complete Heyting algebras (see Chapter 2 for an introduction to basic algebraic tools).

Definition 6.2.1. A *complete* Heyting algebra is a Heyting algebra in which the infs and the sups exist for every subset and in which the following distributivity law holds:

$$x \wedge \sup B = \sup_{y \in B}(x \wedge y)$$

for every $x \in H$ and every $B \subseteq H$.

Definition 6.2.2. An *algebraic model for infinitary intuitionistic logic* is an ordered pair $\langle H, v \rangle$ where H is a complete Heyting algebra and $v : AT \to H$ is a function such that:

- $v(A \wedge B) = v(A) \wedge v(B)$
- $v(A \vee B) = v(A) \vee v(B)$

- $v(A \to B) = v(A) \to v(B)$
- $v(\bigwedge_{k>0} A_k) = inf\{v(A_k) \mid k > 0\}$
- $v(\bigvee_{k>0} A_k) = sup\{v(A_k) \mid k > 0\}$

A formula A in Γ is true in the algebraic model $\mathcal{H}$, $\mathcal{H} \vDash A$, iff for every valuation function v we have $v(A) = 1$. A formula is true in the intuitionistic algebraic semantics, $\vDash_H A$, iff it is true in every intuitionistic algebraic model.

Another viable approach consists in considering completeness with respect to Heyting algebras. As pointed out by Nadel this approach turns out to be equivalent to the one presented above (71), due to the fact that every Heyting algebra can be isomorphically embedded in a complete Heyting algebra preserving every infinite sups and infs ((92) IV.9).

The language of infinitary intuitionistic logic is uncountably infinite, however it is often convenient to restrict ourselves to countable fragments thereof, in doing so we borrow some terminology from (68).

Definition 6.2.3. Let A be a formula, the set of subformulas of A, $Sub(A)$ is inductively defined:

- $Sub(p) = \{p\}$
- $Sub(A\#B) = Sub(A) \cup Sub(B) \cup \{A\#B\}$, where $\# \in \{\wedge, \vee, \to\}$
- $Sub(\bigwedge_{k>0} A_k) = \bigcup_{k>0} Sub(A_k) \cup \{\bigwedge_{k>0} A_k\}$
- $Sub(\bigvee_{k>0} A_k) = \bigcup_{k>0} Sub(A_k) \cup \{\bigvee_{k>0} A_k\}$

Lemma 6.2.2. *For every $A \in \mathsf{FM}_\omega$, $|Sub(A)| \leqslant \aleph_0$, where $|X|$ denotes the cardinality of the set X.*

Proof. We prove this fact by induction on the complexity of the formula A. If A is an atomic formula, then $|Sub(A)| = 1$.

If $A \equiv B\#C$, where $\# \in \{\wedge, \vee, \to\}$, then $|Sub(B\#C)| = |Sub(B) \cup Sub(C)| + 1 \leqslant \aleph_0$, because by induction hypothesis $|Sub(B)|, |Sub(C)| \leqslant \aleph_0$.

If $A \equiv \bigwedge_{k>0} B_k$, then $|Sub(\bigwedge_{k>0} B_k)| = |\bigcup_{k>0} Sub(B_k)| + 1 \leqslant \aleph_0$, because, by induction hypothesis, for every $k < 0$, $|Sub(B_k)| \leqslant \aleph_0$ and a countable union of countable sets is countable. *qed.*

Definition 6.2.4. Let $\Gamma \subseteq FOR$ be given. The environment of Γ, $\mathcal{E}(\Gamma)$ is the least subset of FM_ω such that:

1. $\bot \in \mathcal{E}(\Gamma), AT \subseteq \mathcal{E}(\Gamma)$.

2. $Sub(\Gamma) \subseteq \mathcal{E}(\Gamma)$ is closed under subformulas.

3. $\mathcal{E}(\Gamma)$ is closed under *finite* conjunctions, disjunctions and implications.

4. The following distributivity condition holds: if $A, \bigvee_{k>0} B_k \in \mathcal{E}(\Gamma)$, then $\bigvee_{k>0} (A \wedge B_k) \in \mathcal{E}(\Gamma)$.

It is easy to observe that if Γ is countable, then $\mathcal{E}(\Gamma)$ is countable too. When we are dealing with a single formula A we write $\mathcal{E}(A)$ instead of $\mathcal{E}(\{A\})$.

We now recall the notion of a topological model for countable fragments of the language.

Definition 6.2.5. A *topological space* is a pair $\langle W, \tau \rangle$ such that $W \neq \emptyset$ and $\tau \subseteq \mathcal{P}(W)$, where τ contains and W, is closed under arbitrary unions and under finite intersections.

The elements in τ are called the *open subsets* of W.

Definition 6.2.6. Given a *topological space* $\langle W, \tau \rangle$, $x \in W$, the set of open subsets of W which contain x is denoted by $\tau_x = \{a \in \tau \mid x \in a\}$.

We now turn to topological models:

Definition 6.2.7. Let $\langle W, \tau \rangle$ be a topological space, a topological model $\mathcal{M}$ for a countable fragment Γ is a triple $\mathcal{M} = \langle W, \tau, v \rangle$, where v maps atomic formulas in τ, $v : AT \rightarrow \tau$. The definition of v is thus extended to arbitrary formulas in $\mathcal{E}(\Gamma)$ as follows:

- $v(\bot) = \emptyset$
- $v(A \wedge B) = v(A) \cap v(B)$
- $v(A \vee B) = v(A) \cup v(B)$
- $v(A \rightarrow B) = \bigcup\{a \in \tau \mid a \subseteq (W \setminus v(A)) \cup v(B)\} = Int(v(A)^c \cup v(B))$
- $v(\bigvee_{k>0} A_k) = \bigcup_{k>0} v(A_k)$
- $v(\bigwedge_{k>0} A_k) = \bigcup\{a \in \tau \mid a \subseteq \bigcap_{k>0} v(A_k)\} = Int(\bigcap_{k>0} v(A_k))$

A formula A is valid in $\mathcal{M}$ iff $v(A) = W$. A formula A is valid in a topological space $\langle W, \tau \rangle$, in symbols $\vDash_{Top} A$, iff for every valuation v, $v(A) = W$.

Definition 6.2.8. A base $\mathcal{B}_\tau$ for a topological space $\langle W, \tau \rangle$ is a subset of τ such that every set in τ can be obtained by the union of some elements of $\mathcal{B}_\tau$.

In what follows we will make use of the fact that when we consider the interpretation of the connectives $\rightarrow$ and $\bigwedge$ we can restrict ourselves to consider the open sets of any base $\mathcal{B}_\tau$ of the topology instead of every open sets in τ (110). Therefore we can conclude that: $v(\bigwedge_{k>0} A_k) = \bigcup\{a \in \mathcal{B}_\tau \,|\, a \subseteq \bigcap_{k>0} v(A_k)\} = Int(\bigcap_{k>0} v(A_k))$ and $v(A \rightarrow B) = \bigcup\{a \in \mathcal{B}_\tau \,|\, a \subseteq (W \setminus v(A)) \cup v(B)\} = Int(v(A)^c \cup v(B))$, where $\mathcal{B}_\tau$ is a base for τ.

The next step is to show that every countable Heyting algebra can be embedded into a topological space and this yields the desired completeness result with respect to topological semantics. In order to achieve this, we refer to a strategy detailed in (110).

Definition 6.2.9. Let $\mathcal{H}$ be a Heyting algebra, $\triangledown$ a filter, B a subset of $\mathcal{H}$ which has a sup in $\mathcal{H}$. $\triangledown$ *respects* B if whenever $sup(B) \in \triangledown$, there is $b \in \triangledown \cap B$.

The crucial lemma is the following, which shows that there are prime filters which respect a countable quantity of suprema. The following lemma was first stated and proved in (110).

Lemma 6.2.3. *Let $\mathcal{H}$ be a Heyting algebra, $x, y \in \mathcal{H}$ such that $x \nleqslant y$ and $B_1, ..., B_n, ...$ a countable quantity of subsets of $\mathcal{H}$ which have a supremum in $\mathcal{H}$. Then there exists a prime filter $\triangledown$ of $\mathcal{H}$ which contains x, does not contain y and respects all subsets B_i.*

Proof. We consider the filter $\triangledown_0 =\uparrow x = \{z \,|\, x \leqslant z\}$. We construct a new list $C_1, C_2, ...$ of subsets of $\mathcal{H}$ such that for every i, B_i appears in the list countable times.[2] Now let $x = w_0$, so $\triangledown_0 =\uparrow w_0$. Let us suppose that we have w_n such that $w_n \nleqslant y$ and $\triangledown_n =\uparrow w_n$, then we define w_{n+1} as follows.

If $sup(C_n) \notin \triangledown_n$, then $w_{n+1} = w_n$. If $sup(C_n) \in \triangledown_n$, then $w_{n+1} = w_n \wedge b_n$, for some $b_n \in C_n$ and $w_n \wedge b_n \nleqslant y$. We claim that such b_n always exists. In fact, by hypothesis $sup(C_n) \in \triangledown_n =\uparrow w_n$, so $w_n \leqslant sup(C_n)$. Now if for every $b \in C_n$ we have $w_n \wedge b \leqslant y$, then $w_n = w_n \wedge sup(C_n) = sup_{b \in C_n}(w_n \wedge b) \leqslant y$, contradicting the fact that $w_n \nleqslant y$.

Thus we set $\triangledown_{n+1} =\uparrow w_{n+1}$. We observe that if $w_{n+1} = w_n \wedge b_n \in \triangledown_{n+1}$, then $w_n, b_n \in \triangledown_{n+1}$ by the properties of filters. Clearly $\triangledown_n \subseteq \triangledown_{n+1}$ and $y \notin \triangledown_{n+1}$, because $w_{n+1} \nleqslant y$. We fix $\triangledown = \bigcup_{n \in \omega} \triangledown_n$, which can be easily seen to be a filter such that $x \in \triangledown$ and $y \notin \triangledown$.

[2] For example the list: $C_1 = B_1, C_2 = B_1, C_3 = B_2, C_4 = B_1, C_5 = B_2, C_6 = B_3....$

Furthermore, for every i, $\triangledown$ respect B_i. In fact, if $sup(B_i) \in \triangledown$, then there is $n \in \omega$ such that $sup(B_i) \in \triangledown_n$. Since every B_i appears in the list $C_1, C_2, ..$ a countable number of times, there is a $j \in \omega$ such that $n \leqslant j$ and $B_i = C_j$, thus $sup(B_i) = sup(C_j) \in \triangledown_n \subseteq \triangledown_j$ and so there is $b_j \in B_i$ such that $b_j \in \triangledown_{j+1} \subseteq \triangledown$. This concludes the proof. *qed.*

Notice that in the proof above we have exploited the validity of the infinitary distributive law $w_n \wedge sup(C_n) = sup_{b \in C_n}(w_n \wedge b)$, which is the reason why we required the environment to be closed under some distributivity properties.

Given an Heyting algebra $\mathcal{H}$ and a countable quantity of subsets $B_1, B_2, ...$ of $\mathcal{H}$, we consider a topological space $\langle Pt(\mathcal{H}), \tau_{\mathcal{H}} \rangle$, where

$$Pt(\mathcal{H}) = \{\triangledown \mid \triangledown \text{ prime filter of } \mathcal{H} \text{ and for every } i, \triangledown \text{ respects } B_i\}$$

We choose the set $\mathcal{B}_{\tau_{\mathcal{H}}}$, which contains all the subsets:

$$ext(x) = \{\triangledown \in Pt(\mathcal{H}) \mid x \in \triangledown\}$$

as the base of $\tau_{\mathcal{H}}$.

Theorem 6.2.4. *Given a countable Heyting algebra $\mathcal{H}$ and the topological space $\langle Pt(\mathcal{H}), \tau_{\mathcal{H}} \rangle$, ext is an injective morphism from $\mathcal{H}$ to $\langle \mathcal{P}(Pt(\mathcal{H})), \cap, \cup, \rightarrow, \bot \rangle$, where $x \rightarrow y = Int((x)^c \cup (y))$, for every $x, y \in \mathcal{P}(Pt(\mathcal{H}))$.*

Proof. We prove that for every $x, y \in \mathcal{H}$, $x \leqslant y$ iff $ext(x) \subseteq ext(y)$. From left to right let us suppose $x \leqslant y$, then for every $\triangledown \in ext(x)$ we have $x \in \triangledown$ and $x \leqslant y$ and since $\triangledown$ is a filter we obtain $y \in \triangledown$, which entails $\triangledown \in ext(y)$. From right to left we assume $ext(x) \subseteq ext(y)$ and we argue by contradiction. So we assume $x \nleqslant y$, so by lemma 2.4 there is a prime filter $\triangledown$ which respects all the suprema and furthermore $x \in \triangledown$ and $y \notin \triangledown$. So by definition we have $\triangledown \in ext(x)$, but $ext(x) \subseteq ext(y)$ and therefore $y \in \triangledown$, contradiction.

Now we prove that ext respects all the operations.

$ext(x \wedge y) = \{\triangledown \in Pt(\mathcal{H}) \mid x \wedge y \in \triangledown\} = \{\triangledown \in Pt(\mathcal{H}) \mid x \in \triangledown\} \cap \{\triangledown \in Pt(\mathcal{H}) \mid y \in \triangledown\} = ext(x) \cap ext(y)$. This is clear, because for every filter $\triangledown$, for every $x, y, x \wedge y \in \triangledown$ iff $x \in F$ and $y \in \triangledown$.

$ext(x \vee y) = \{\triangledown \in Pt(\mathcal{H}) \mid x \vee y \in \triangledown\} = \{\triangledown \in Pt(\mathcal{H}) \mid x \in \triangledown\} \cup \{\triangledown \in Pt(\mathcal{H}) \mid y \in \triangledown\} = ext(x) \cup ext(y)$. We exploit the fact that in prime filters $x \vee y \in \triangledown$ iff $x \in \triangledown$ or $y \in \triangledown$.

$ext(x \rightarrow y) = \{\triangledown \in Pt(\mathcal{H}) \mid x \rightarrow y \in \triangledown\} = \bigcup\{ext(z) \mid z \leqslant x \rightarrow y\} = \bigcup\{ext(z) \mid z \wedge x \leqslant y\} = \bigcup\{ext(z) \mid ext(z \wedge x) \subseteq ext(y)\} = \bigcup\{ext(z) \mid ext(z) \cap$

$(x) \subseteq ext(y)\} = ext(x) \to ext(y)$. This is due to the fact that in every Heyting algebra $z \wedge x \leqslant y$ iff $z \leqslant x \to y$.

We also show that the embedding ext preserves the existing infs and sups.

$ext(\bigwedge_{k>0} x_k) = \{\triangledown \in Pt(\mathcal{H}) \mid \bigwedge_{k>0} x_k \in \triangledown\} = \bigcup\{ext(z) \mid z \leqslant \bigwedge_{k>0} x_k\} = \bigcup\{ext(z) \mid z \leqslant x_k$ for every $k > 0\} = \bigcup\{ext(z) \mid ext(z) \subseteq ext(x_k)$ for every $k > 0\} = \bigcup\{ext(z) \mid ext(z) \subseteq \bigcap_{k>0} ext(x_k)\} = Int(\bigcap_{k>0} ext(x_k))$ and $Int(\bigcap_{k>0} ext(x_k)) = inf(\{ext(x_k) \mid k > 0\})$ with respect to the partial order induced by $\subseteq$.

$ext(\bigvee_{k>0} x_k) = \{\triangledown \in Pt(\mathcal{H}) \mid \bigvee_{k>0} x_k \in \triangledown\} = \bigcup\{P \in Pt(\mathcal{H}) \mid x_k \in \triangledown$ for some $k > 0\} = \bigcup_{k>0} ext(x_k)$, because for every prime filter in $Pt(\mathcal{H})$ we have $\bigvee_{k>0} x_k = sup(\{x_k\}_{k>0}) \in \triangledown$ iff $x_k \in \triangledown$ for some $k > 0$, because $\{x_k\}_{k>0}$ is a countable subset of $\mathcal{H}$ and $\triangledown$ respects $\{x_k\}_{k>0}$. Clearly $\bigcup_{k>0} ext(x_k)$ is equal to $sup(\{ext(x_k) \mid k > 0\}$.

We check that ext is injective: if $x \neq y$, then $x \nleqslant y$ or $y \nleqslant x$, in both cases clearly $ext(x) \neq ext(y)$. *qed.*

By exploiting the above theorem we obtain the desired proof of completeness with respect to topological semantics.

Theorem 6.2.5 (Topological completeness)**.** *For every formula A in* FM_ω*: if* $\vDash_{Top} A$*, then* $\mathbf{G3i}_\omega \vdash\Rightarrow A$.

Proof. We proceed by contradiction. Let A be given and let us suppose $\mathbf{G3i}_\omega \nvdash\Rightarrow A$. We construct the Lindenbaum algebra associated to $\mathcal{E}(A)$:

$$\mathcal{A} = \langle \mathcal{E}(A)_{/\sim}, \wedge, \vee, \to, \bot \rangle$$

where $\mathcal{E}(A)_{/\sim}$ is the quotient of the environment of Γ modulo the equivalence relation $B \sim C$ iff $\mathbf{G3i}_\omega \vdash\Rightarrow B \leftrightarrow C$. We observe that $\mathcal{E}(A)$ is countable and so is $\mathcal{E}(A)_{/\sim}$.

The element 0 corresponds to $[\bot]$, that is the equivalence class associated to $\bot$, 1 is $[\bot \to \bot]$. The operation are thus defined: for every $\circ \in \{\wedge, \vee, \to\}$ and every equivalence classes $[B], [C]$ we have $[B] \circ [C] = [B \circ C]$, where $\circ$ on the left side is an operation in the algebra, whereas on the right side of the equality sign it denotes the corresponding connective.

We define a partial order associated to the Lindenbaum algebra: $[B] \leqslant [C]$ iff $\mathbf{G3i}_\omega \vdash B \Rightarrow C$. Thus we observe that for every $k > 0$ we have $inf\{[B]_k \mid k > 0\} = [\bigwedge_{k>0} B_k]$ and $sup\{[B]_k \mid k > 0\} = [\bigvee_{k>0} B_k]$, provided that $\bigwedge_{k>0} B_k, \bigvee_{k>0} B_k \in \mathcal{E}(A)$. By definition of $\mathcal{E}(A)$ for every $[B], [C] \in \mathcal{E}(A)_{/\sim}$

there is a pseudo complement $[B] \to [C]$ such that: $[B] \wedge [D] \leqslant [C]$ iff $[D] \leqslant [B \to C]$. Thus $\mathcal{A}$ is a Heyting algebra. We observe that $[A] \neq [\bot \to \bot]$, otherwise $\mathbf{G3i}_\omega \vdash \Rightarrow A$ against the hypothesis.

Since $\mathcal{A}$ is countable we embed it in its associated topology $\langle Pt(\mathcal{A}), \tau_{\mathcal{A}} \rangle$ and we equip the latter with the valuation $v(B) = ext([B])$. We show by transfinite induction on the complexity of the formulas $B \in \mathcal{E}(A)$ that v is a valuation function.

If B is a propositional atom the claim holds by definition. If $B \equiv \bot$, $v(\bot) = ext([\bot]) = .$

Cases $\wedge, \vee$ and $\bigvee$ follow by induction hypothesis. If $B \equiv C \to D$, then $v(C \to D) = ext([C \to D]) = Int(ext([C])^c \cup ext([D])) =^{IH} Int(v(C)^c \cup v(D))$.

We deal with the infinitary case. $v(\bigwedge_{k>0} C_k) = ext([\bigwedge_{k>0} C_k]) = Int(\bigcap_{k>0} ext([C_k]))$ $=^{IH} Int(\bigcap_{k>0} v(C_k))$.

So we conclude that $\langle Pt(\mathcal{A}), t_{\mathcal{A}}, v \rangle$ is a topological model. Since $[A] \neq [\bot \to \bot]$, we have that $v(A) = ext([A]) \neq W = ext([\bot \to \bot])$ due to injectivity of ext. We have obtained a topological countermodel and this concludes the proof. *qed.*

The theorem that provides the desired topological countermodel crucially relies on the cardinality of the Lindenbaum algebra associated to the environment of the formula A. Moreover the open sets of the topological model thus obtained actually form a complete Heyting algebra and thus Theorem 6.2.5 can also be seen as a further proof of completeness with respect to algebraic semantics for countable fragments of the language. Furthermore, this is interesting insofar as it introduces a more concrete structure (from the mathematical viewpoint) to interpret infinitary intuitionistic logic.

6.3 Neighborhood semantics for infinitary intuitionistic logic

In this section we present the neighborhood semantics for intuitionistic infinitary logic and we discuss some of its properties.

Definition 6.3.1. A *neighborhood frame* for intuitionistic infinitary logic is an ordered pair $\langle W, N \rangle$ with $W \neq \emptyset$, $N : W \to \mathcal{P}(\mathcal{P}(W))$ such that:

1. $W \in N(x)$
2. If $a, b \in N(x)$ then $a \cap b \in N(x)$

3. If $a \in N(x)$ and $a \subseteq b$ then $b \in N(x)$

4. If $a \in N(x)$ then $x \in a$

5. If $a \in N(x)$ then $\{y \mid a \in N(y)\} \in N(x)$

Condition 1. requires that every set of neighborhoods of a point x contains the unit, condition 3. is closure under supersets and conditions 4.-5. actually correspond to reflexivity and transitivity in the preorder associated to the neighborhood frame. The key point is condition 2. which expresses closure under finite intersections. We now present neighborhood models for intuitionistic infinitary logic.

Definition 6.3.2. A *neighborhood model* $\mathcal{M}$ is a triple $\langle W, N, v \rangle$ such that $\langle W, N \rangle$ is a *neighborhood frame* for intuitionistic infinitary logic and v is a map from propositional variables to $\mathcal{P}(W)$ such that $v(p) \in N(x)$ for any $x \in v(p)$.

In order to give a compact presentation of the satisfiability conditions for formulas in *neighborhood models* of intuitionistic infinitary logic we use the local forcing conditions presented in (75). For every world x and every neighborhood a:

$$a \Vdash^{\forall} A \text{ is true iff } x \Vdash A \text{ for every } x \in a$$

Furthermore we introduce two abbreviations: $x \Vdash A \supset B$ means that *if* $x \Vdash A$ *then* $x \Vdash B$ and $x \Vdash \underset{k>0}{\&} A_k$ means that $x \Vdash A_k$ *for every* $k > 0$. In a sense, $\supset$ and & reflect the classical meaning of implication and infinitary conjunction, respectively, whereas $\rightarrow$ and $\bigwedge$ are the proper intuitionistic connectives for implication and infinitary conjunction.

Definition 6.3.3. Given a *neighborhood model* $\mathcal{M} = \langle W, N, v \rangle$, $x \in W$, a formula $A \in \mathsf{FM}_\omega$, the relation $x \Vdash A$ is inductively defined as follows:

- $x \Vdash p$ iff $x \in v(p)$
- $x \Vdash B \wedge C$ iff $x \Vdash B$ and $x \Vdash C$
- $x \Vdash B \vee C$ iff $x \Vdash B$ or $x \Vdash C$
- $x \Vdash B \rightarrow C$ iff $\exists a \in N(x)(a \Vdash^{\forall} B \supset C)$
- $x \Vdash \bigvee_{k>0} B_k$ iff $x \Vdash B_k$ for some $k > 0$
- $x \Vdash \bigwedge_{k>0} B_k$ iff $\exists a \in N(x)(a \Vdash^{\forall} \underset{k>0}{\&} B_k)$

A formula A is valid in a *neighborhood model* $\mathcal{M} = \langle W, N, v \rangle$, in symbols $\underset{\mathcal{M}}{\vDash} A$, iff $x \Vdash A$ for every x in W. A is an infinitary intuitionistic *logical truth*, in symbols $\vDash_{\mathcal{N}} A$, iff $\underset{\mathcal{M}}{\vDash} A$ for every model $\mathcal{M}$ based on an intuitionistic infinitary *neighborhood frame*, i.e. for every valuation v.

Given $A \in \mathsf{FM}_\omega$, let $v(A) = \{x \in W \,|\, x \Vdash_{\mathcal{M}} A\}$. We prove the following useful lemma.

Lemma 6.3.1 (Persistence)**.** *For every $A \in \mathsf{FM}_\omega$, if $x \in v(A)$, then $v(A) \in N(x)$.*

Proof. The proof is by induction on the complexity of formulas.

The atomic case and cases for the connectives $\{\wedge, \vee\}$ are treated in (69). We notice that we never make use of closure under infinite intersections.

We deal with the case of the implication. If $x \Vdash A \rightarrow B$, then there is $a \in N(x)$ such that $a \Vdash^{\forall} A \supset B$. By condition 5. of neighborhood frame we have $m(a) \in N(x)$ and we easily observe that $m(a) \Vdash^{\forall} A \rightarrow B$. However, since $m(a) \subseteq \{y \,|\, y \Vdash A \rightarrow B\} = v(A \rightarrow B)$, by closure under supersets we conclude that $v(A \rightarrow B) \in N(x)$.

We deal with the infinitary connectives. Let us suppose $x \Vdash \bigvee_{k>0} A_k$, so by definition $x \Vdash A_k$ for some $k > 0$. By induction hypothesis $v(A_k) \in N(x)$, but $v(A_k) \subseteq v(\bigvee_{k>0} A_k)$, hence by closure under supersets $v(\bigvee_{k>0} A_k) \in N(x)$.

If $x \in v(\bigwedge_{k>0} B_k)$ by definition there is $a \in N(x)$ such that $a \Vdash^{\forall} \underset{k>0}{\&} B_k$. By condition 5. we have $\{y \,|\, a \in N(y)\} \in N(x)$ and $\{y \,|\, a \in N(y)\} \subseteq v(\bigwedge_{k>0} B_k)$, hence by closure under supersets we have $v(\bigwedge_{k>0} B_k) \in N(x)$, which is the desired conclusion. *qed.*

We observe that we could have defined the finitary conjunction $\wedge$ in terms of the infinitary one $\bigwedge$ by the following lemma.

Lemma 6.3.2. *$x \Vdash A \wedge B$ iff $\exists a \in N(x)(a \Vdash^{\forall} A$ and $a \Vdash^{\forall} B)$.*

Proof. From left to right we assume $x \Vdash A \wedge B$, so $x \Vdash A$ and $x \Vdash B$. By the persistence lemma there are $a \in N(x)$ and $b \in N(x)$ such that $a \Vdash^{\forall} A$ and $b \Vdash^{\forall} B$. By closure under intersection we obtain $a \cap b \in N(x)$ which clearly yields the desired conclusion.

From right to left we observe that if $a \in N(x)$, then $x \in a$ and thus $x \Vdash A$ and $x \Vdash B$. *qed.*

The above lemma yields the interdefinability. In fact, it is easy to check that, if $A = C_0$ and $C_k = B$ for every $k \geqslant 1$, $x \Vdash A \wedge B$ iff $x \Vdash \bigwedge_{k>0} C_k$.

We now prove soundness of the unlabelled sequent calculus $\mathbf{G3i}_\omega$ (as presented in (72)) for intuitionistic infinitary logic with respect to neighborhood semantics. Given finite multisets Γ, Δ of formulas which consist of the formulas $A_1, ..., A_n$ and $B_1, ..., B_m$ respectively, $x \Vdash \Gamma \rightarrow \Delta$ means $x \Vdash A_1 \wedge ... \wedge A_n \rightarrow B_1 \vee ... \vee B_m$.

Theorem 6.3.3 (Soundness)**.** *If* $\vdash_{\mathbf{G3i}_\omega} \Gamma \Rightarrow \Delta$, *then* $\vDash_{\mathcal{N}} \Gamma \rightarrow \Delta$

Proof. We proceed by induction on the height of derivations. The atomic cases as well as the cases for the connectives $\wedge, \vee$ and $\bigvee$ are straightforward. We discuss the cases of implication and infinitary conjunction.

Let us suppose that the last rule applied is $R \rightarrow$. Let $\mathcal{N} = \langle W, N \rangle$ be a neighborhood frame, $\mathcal{M}$ a model based on $\mathcal{N}$ and x a world in W. Let us suppose $x \Vdash \Gamma$. By persistence, for every $C_i \in \Gamma$ there is $a_i \in N(y)$ such that for every i $a_i \Vdash^{\forall} C_i$. By closure under finite intersections[3], $a_1 \cap ... \cap a_n \in N(x)$. We claim that $a_1 \cap ... \cap a_n \Vdash^{\forall} A \supset B$. In fact, let $y \in a_1 \cap ... \cap a_n$ and let us suppose $y \Vdash A$. Since $y \in a_1 \cap ... \cap a_n$ we have $y \Vdash \Gamma$, so by induction hypothesis $y \Vdash B$. Since $a_1 \cap ... \cap a_n \Vdash^{\forall} A \supset B$ and $a_1 \cap ... \cap a_n \in N(x)$, we have $x \Vdash A \rightarrow B$ and thus $x \Vdash \Delta \vee (A \rightarrow B)$. This yields the desired conclusion, because $W \Vdash^{\forall} \Gamma \supset \Delta \vee (A \rightarrow B)$ and $W \in N(x)$, thus $x \Vdash \Gamma \rightarrow \Delta \vee (A \rightarrow B)$.

If the last rule is $L \rightarrow$, then we assume $x \Vdash \Gamma$ and $x \Vdash A \rightarrow B$. By definition there is $a \in N(x)$ such that for every y in a, $y \nVdash A$ or $y \Vdash B$, but we have (item 4. def. 1.1) $x \in a$, thus $x \nVdash A$ or $x \Vdash B$. In the first case by induction hypothesis we obtain $x \Vdash \bigvee \Delta \vee A$, but since $x \nVdash A$, $x \Vdash \bigvee \Delta$. In the second case the induction hypothesis immediately yields the desired conclusion.

If the last rule is $L \bigwedge$, we assume $x \Vdash \Gamma \wedge \bigwedge\limits_{k>0} A_k$, therefore by definition there is $a \in N(x)$ such that for every $a \Vdash^{\forall} \underset{k>0}{\&} A_k$. Since we have $x \in a$, in particular $x \Vdash A_k$ and by induction hypothesis we conclude.

If the last rule is $R \bigwedge$, we assume $x \Vdash \Gamma$, therefore for every $C_i \in \Gamma$ there is $a_i \in N(x)$ such that $a_i \Vdash^{\forall} C_i$ and $a_1 \cap ... \cap a_n \in N(x)$. We claim $a_1 \cap ... \cap a_n \Vdash^{\forall} \underset{k>0}{\&} A_k$. In fact if $y \in a_1 \cap ... \cap a_n$, $y \Vdash \Gamma$, thus by induction hypothesis $y \Vdash A_k$, so $y \Vdash \underset{k>0}{\&} A_k$. Therefore $x \Vdash \bigwedge\limits_{k>0} A_k$ from which follows $x \Vdash \Delta \vee \bigwedge\limits_{k>0} A_k$. *qed.*

Notice that the above proof would not go through if we had the rule $R \bigwedge'$:

$$\frac{\{\Gamma \Rightarrow A_k, \Delta \mid k > 0\}}{\Gamma \Rightarrow \bigwedge_{k>0} A_k, \Delta} \; R \bigwedge'$$

[3]Recall that Γ is a finite multiset of formulas.

This is due to the fact that such a rule is not sound, in fact $\mathbf{G3i}'_\omega$ obtained by substituting $R\bigwedge$ with $R\bigwedge'$ proves $\bigwedge\limits_{k>0}(p_k \vee q) \rightarrow \bigwedge\limits_{k>0} p_k \vee q$. However, such formula is not provable: in order to establish it we provide the following countermodel (for similar examples see also (111) and (71)).

Let $W = \mathbb{R}$ and $N(x) = \{a \mid \exists r > 0((x-r, x+r) \subseteq a)\}$. We define the valuation $v : AT \rightarrow \mathbb{R}$ such that $v(q) = \mathbb{R} \setminus \{0\}$ and $v(p_k) = (-\frac{1}{k}, \frac{1}{k})$ for every $k > 0$. We leave it to the reader to verify that this is a neighborhood model.

We observe that $\mathbb{R} \Vdash^{\forall} \underset{k>0}{\&}(p_k \vee q)$ and $\mathbb{R} \in N(0)$, thus $0 \Vdash \bigwedge_{k>0}(p_k \vee q)$. However $\bigcap_{k>0} v(p_k) = \{0\} \notin N(0)$, thus we can easily conclude that $0 \nVdash \bigwedge\limits_{k>0} p_k$. Furthermore $0 \notin \mathbb{R} \setminus \{0\}$, hence $0 \nVdash q$. Thus $0 \nVdash \bigwedge\limits_{k>0} p_k \vee q$, hence we have provided a countermodel.

6.4 Completeness of the semantics

In order to conclude that our neighborhood semantics is adequate we have to establish completeness with respect to the sequent calculus $\mathbf{G3i}_\omega$. There are various ways in which we could proceed: the first consists in an adaptation of the Tait-Schutte-Takeuti style completeness, obtained via the construction of a suitable reduction tree. However this is hard to achieve due to the lack of invertibility of rules in $\mathbf{G3i}_\omega$. Another possibility is to exploit the standard method of the canonical model, however since we are dealing with a language with uncountably many formulas, some modifications should be taken in order to prove the usual Lindenbaum lemma. We will instead show the correspondence between our neighborhood models and topological ones, thus obtaining an indirect form of completeness. We first sketch the proof and then provide the details. The first step consists in showing that given a topological model, we can obtain a neighborhood model which satisfies the same formulas. The argument to show completeness then goes as follows: we suppose $\nvdash_{\mathbf{G3i}_\omega} \Gamma \Rightarrow \Delta$, hence by topological completeness we obtain a topological model $\mathcal{M}$ such that $\nvDash_{\mathcal{M}} \Gamma \rightarrow \Delta$, hence by our representation theorem we obtain a neighborhood model $\mathcal{N}$ such that $\nvDash_{\mathcal{N}} \Gamma \rightarrow \Delta$ and thus we obtain completeness.

Definition 6.4.1. Given a topological space $\langle W, \tau \rangle$ its *associated neighborhood system* is $\langle W, N_\tau \rangle$, with $N_\tau : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ such that $N_\tau(x) = \{b \mid \exists a \in \tau_x(a \subseteq b)\}$, i.e. supersets of open sets which contain x.

Lemma 6.4.1. *The neighborhood system $\langle W, N_\tau \rangle$ associated to the topological space $\langle W, \tau \rangle$ has the following properties:*

1. *For every $x \in W$ $N_\tau(x)$ is closed under finite intersections, closed under supersets, contains the unit and does not contain the empty set.*

2. *For every $x \in W$, if $a \in N_\tau(x)$, then $x \in a$.*

3. *For every $x \in W$, if $a \in N_\tau(x)$, then $\{y \mid a \in N_\tau(y)\} \in N_\tau(x)$.*

Proof. Items 1.-2. are easy (see also Exercise 1.16 in (79)). We limit ourselves to proving 3. Let us suppose $a \in N_\tau(x)$, then there is $b \in \tau_x$ such that $b \subseteq a$. We want to prove $\{y \mid a \in N_\tau(y)\} \in N_\tau(x)$ This holds iff there is a $c \in \tau_x$ s.t. $c \subseteq \{y \mid a \in N_\tau(y)\}$ iff there is a $c \in \tau_x$ s.t. $c \subseteq \{y \mid \exists d \in \tau_y(d \subseteq a)\}$.

We have that $b \in \tau_x$, furthermore, let us suppose $y \in b \in \tau$, then $b \in \tau_y$ and by hypothesis $b \subseteq a$, hence $b \subseteq \{y \mid \exists d \in \tau_y(d \subseteq a)\} = \{y \mid a \in N_\tau(y)\}$, hence $\{y \mid a \in N_\tau(y)\} \in N_\tau(x)$. *qed.*

Theorem 6.4.2. *Given a topological space $\langle W, \tau \rangle$, the induced neighborhood system $\mathcal{N}_\tau$ is an intuitionistic infinitary neighborhood frame.*

Proof. Immediate by the previous lemma. *qed.*

Before proceeding we recall that an open set which contains x is called an open neighborhood of x, moreover as it is immediate to notice, if $a \in \tau_x$ then $a \in N_\tau(x)$ (whereas the converse does not always hold).

We observe that the satisfiability conditions for topological models defined in the previous section can be reformulated as follows. A point x satisfies A with respect to v iff $x \in v(A)$. Given a point $x \in W$ it is easy to see (8) that the satisfiability conditions in a topological model are equivalent to the following:

- $x \Vdash p$ iff $x \in v(p)$
- $x \nVdash \bot$
- $x \Vdash A \wedge B$ iff $x \Vdash A$ and $\Vdash B$
- $x \Vdash A \vee B$ iff $x \Vdash A$ or $\Vdash B$
- $x \Vdash A \rightarrow B$ iff $\exists a \in \tau_x \forall y \in a$(if $y \Vdash A$ then $y \Vdash B$)
- $x \Vdash \bigvee_{k>0} A_k$ iff $x \Vdash A_k$ for some $k > 0$
- $x \Vdash \bigwedge_{k>0} A_k$ iff $\exists a \in \tau_x \forall y \in a(y \Vdash A_k$ for every $k > 0)$

Now are now ready to prove the last result.

Theorem 6.4.3 (Representation)**.** *Let $\mathcal{M} = \langle W, \tau, v\rangle$ be a topological model. Let $\langle W, N_\tau\rangle$ be its associated neighborhood frame. We consider the neighborhood model $\mathcal{N} = \langle W, N_\tau, v_\tau\rangle$, with $v_\tau(p) = v(p)$, then:*

$$\text{For every } x \in W, \text{ for every } A,\ x \Vdash_{\mathcal{M}} A \text{ iff } x \Vdash_{\mathcal{N}} A$$

Proof. We proceed by induction on the complexity of A.

- If $A \equiv p$, $p \in AT$, then by definition of v, v_τ we have $x \Vdash_{\mathcal{M}} p$ iff $x \Vdash_{\mathcal{N}} p$.

- If $A \equiv B \wedge C$, $A \equiv B \vee C$ and $A \equiv \bigvee\limits_{k>0} A_k$ the proposition follows by induction hypothesis.

- If $A \equiv B \rightarrow C$ we consider the two directions. From left to right let us suppose $x \Vdash_{\mathcal{M}} B \rightarrow C$, then there is $a \in \tau_x$ such that $\forall y \in a($ if $y \Vdash_{\mathcal{M}} B$ then $y \Vdash_{\mathcal{M}} C)$. Then $a \in N_\tau(x)$, so let us suppose $y \in a$ and $y \Vdash_{\mathcal{N}} B$, hence by induction hypothesis $y \Vdash_{\mathcal{M}} B$, so $y \Vdash_{\mathcal{M}} C$ by the assumption, which by induction hypothesis yields $y \Vdash_{\mathcal{N}} C$. Therefore $x \Vdash_{\mathcal{N}} B \rightarrow C$.

  From right to left we suppose $x \Vdash_{\mathcal{N}} B \rightarrow C$, then there is $a \in N_\tau(x)$ such that $\forall y \in a($if $y \Vdash_{\mathcal{N}} B$ then $y \Vdash_{\mathcal{N}} C)$. By definition of $N_\tau(x)$ there is $b \in \tau_x$ such that $b \subseteq a$. Thus let us suppose $y \in b$ and $y \Vdash_{\mathcal{M}} B$. Since $y \in a$, by induction hypothesis and the assumption we obtain $y \Vdash_{\mathcal{N}} C$, which, again by induction hypothesis, yields $y \Vdash_{\mathcal{M}} C$. Therefore $x \Vdash_{\mathcal{M}} B \rightarrow C$.

- If $A \equiv \bigwedge\limits_{k>0} B_k$ we consider the two directions. From left to right let us suppose $x \Vdash_{\mathcal{M}} \bigwedge\limits_{k>0} B_k$, then there is $a \in \tau_x$ such that $\forall y \in a(y \Vdash_{\mathcal{M}} B_k$ for every $k > 0)$. Hence $a \in N_\tau(x)$, thus we suppose $y \in a$, hence by the assumption $y \Vdash_{\mathcal{M}} B_k$ for every $k > 0$, therefore by induction hypothesis $y \Vdash_{\mathcal{N}} B_k$ for every $k > 0$, so by definition $x \Vdash_{\mathcal{N}} \bigwedge\limits_{k>0} B_k$.

  From right to left we suppose $x \Vdash_{\mathcal{N}} \bigwedge\limits_{k>0} B_k$, so by definition there is $a \in N_\tau(x)$ such that $\forall y \in a(y \Vdash_{\mathcal{N}} B_k$ for every $k > 0)$. By definition of $N_\tau(x)$ there is $b \in \tau_x$ such that $b \subseteq a$. Hence let us suppose $y \in b$, so $y \in a$ and by the assumption $y \Vdash_{\mathcal{N}} B_k$ for every $k > 0$, thus, by induction hypothesis $y \Vdash_{\mathcal{M}} B_k$ for every $k > 0$, so $x \Vdash_{\mathcal{M}} \bigwedge\limits_{k>0} B_k$.

qed.

Corollary (Completeness)**.** $\vdash_{\mathbf{G3i}_\omega} \Rightarrow A$ *iff* $\vDash_{\mathcal{N}} A$.

Proof. We have already proved the direction from left to right. For the direction from right to left we assume $\vDash_{\mathcal{N}} A$, therefore by theorem 4.3 we obtain $\vDash_{Top} A$ which, by topological completeness, yields $\mathbf{G3i}_\omega \vdash \Rightarrow A$. *qed.*

Initial Sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ Ax \qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Logical Rules

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ R\wedge$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\ R\supset$$

$$\frac{x : \underset{k>0}{\&} A_k, x : A_k, \Gamma \Rightarrow \Delta}{x : \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta}\ L\&_k \qquad \frac{\{\Gamma \Rightarrow \Delta, x : A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, x : \underset{k>0}{\&} A_k}\ R\&$$

$$\frac{\{x : A_k, \Gamma \Rightarrow \Delta \mid k > 0\}}{x : \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, x : \bigvee_{k>0} A_k, x : A_k}{\Gamma \Rightarrow \Delta, x : \bigvee_{k>0} A_k}\ R\bigvee_k$$

Figure 6.1: The calculus $\mathbf{G3C}_\omega$

Thus we provided a neighborhood semantics for infinitary intuitionistic logic. Our aim is to exploit this framework to develop an analytic calculus for intuitionistic infinitary logic without context restrictions, as opposed to $\mathbf{G3i}_\omega$.

6.5 The labelled sequent calculus $\mathbf{G3I}_\omega$

We first present a classical base for our labelled calculus which in a sense corresponds to the propositional fragment of infinitary classical logic. The satisfiability conditions for $\rightarrow$ and $\bigwedge$ are factorised through those for $\supset$ and & respectively and thus their rules are given in two steps through the rules for $\supset$ and &. The intuitionistic calculus is thus obtained as an extension of a classical one (for an approach similar in spirit, see (86)).

In detail, the rules for the intuitionistic connectives, namely $\rightarrow$ and $\bigwedge$, are obtained as follows.

We first consider the case of $\bigwedge$ and we recall the definition of the forcing condition:

$$x \Vdash \bigwedge_{k>0} A_k \text{ iff } \exists a \in N(x)(a \Vdash^{\forall} \underset{k>0}{\&} A_k)$$

Therefore the rules for $\bigwedge$ are:

$$\frac{a \in N(x), a \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta}{x : \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}\, L\bigwedge, a \text{ fresh} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k, a \Vdash^{\forall} \underset{k>0}{\&} A_k}{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k}\, R\bigwedge$$

Following the same procedure the forcing condition for $\rightarrow$ yields the rules:

$$\frac{a \in N(x), a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta}{x : A \rightarrow B, \Gamma \Rightarrow \Delta}\, L\rightarrow, a \text{ fresh} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : A \rightarrow B, a \Vdash^{\forall} A \supset B}{a \in N(x), \Gamma \Rightarrow \Delta, x : A \rightarrow B}\, R\rightarrow$$

This concludes the list of the logical rules. We now have to introduce rules for the local forcing conditions and for the properties of our neighborhood frames. We introduce the following abbreviation: $m(a) = \{y \,|\, a \in N(y)\}$, thus the fifth condition imposed on neighborhood frames for intuitionistic infinitary logic becomes:

$$\text{If } a \in N(x), \text{ then } m(a) \in N(x)$$

Clearly, for every x and for every $a \in N(x)$, we have: if $y \in m(a)$, then $a \in N(y)$.

The condition imposed on the valuation function, i.e. if $x \in v(p)$ then $v(p) \in N(x)$ actually corresponds to:

$[Mon']$ *If $x \Vdash p$, there is $a \in N(x)$ s.t. $a \Vdash^{\forall} p$ and for every y, if $y \Vdash p$, $y \in a$*

However the following lemma easily shows that in monotonic neighborhood frames the condition can be simplified:

[Mon] *If $x \Vdash p$, there is $a \in N(x)$ s.t. $a \Vdash^{\forall} p$*

Lemma 6.5.1. *In every infinitary intuitionistic neighborhood model Mon and Mon' are equivalent.*

Proof. Clearly Mon' implies Mon. For the converse we assume Mon and let us suppose $x \Vdash p$. Hence by Mon there is $a \in N(x)$ such that $a \Vdash^{\forall} p$. However $a \subseteq \{y \,|\, y \Vdash p\}$, hence by closure under supersets $\{y \,|\, y \Vdash p\} \in N(x)$, thus we have obtained the desired conclusion. *qed.*

Hence Mon justifies the following rule:

$$\frac{a \Vdash^{\forall} p, a \in N(x), x : p, \Gamma \Rightarrow \Delta}{x : p, \Gamma \Rightarrow \Delta}\ {\scriptstyle Mon,\, a \text{ fresh}}$$

A similar simplification can be obtained with respect to the unit condition, in fact it can be easily shown, by exploiting closure under supersets, that the following lemma holds.

Lemma 6.5.2. *In every neighborhood model for infinitary intuitionistic logic for every $x \in W$: $N(x) \neq \emptyset$ iff $W \in N(x)$.*

Proof. From right to left the proof is immediate. From left to right let us suppose that $N(x) \neq \emptyset$, so there is $a \in N(x)$. Clearly $a \subseteq W$ and by closure under supersets we obtain $W \in N(x)$. *qed.*

This justifies the rule $Nondeg$:

$$\frac{a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Nondeg,\, a \text{ fresh}}$$

Furthermore, the condition of closure under finite under finite intersections can be replaced by a weaker condition, which we call *prebasic*.

Lemma 6.5.3. *For every infinitary intuitionistic frame the following are equivalent:*

1. *If $a \in N(x)$ and $b \in N(x)$, then $a \cap b \in N(x)$*
2. *If $a \in N(x)$ and $b \in N(x)$, then there is $c \in N(x)$ and $c \subseteq a$ and $c \subseteq b$*

Proof. Clearly 1 implies 2. With respect to the other direction let us assume $a \in N(x)$ and $b \in N(x)$. By 2 there is $c \in N(x)$ such that $c \subseteq a$ and $c \subseteq b$, which implies $c \subseteq a \cap b$. By closure under supersets we obtain $a \cap b \in N(x)$. *qed.*

This actually tells us that we can replace the requirement of closure under finite intersections with the condition *prebasic*. We extract the following rule from the condition above:

$$\frac{c \in N(x), c \subseteq a, c \subseteq b, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}\ {\scriptstyle Prebasic,\, c \text{ fresh}}$$

We also add to it the rules for the subset relation:

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ {\scriptstyle L \subseteq} \qquad \frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Ref \subseteq} \qquad \frac{a \subseteq b, b \subseteq c, a \subseteq c, \Gamma \Rightarrow \Delta}{a \subseteq b, b \subseteq c, \Gamma \Rightarrow \Delta}\ {\scriptstyle Trs \subseteq}$$

We can now give a complete formulation of the labelled calculus $\mathbf{G3I}_{\omega}$ in Figure 6.2.

Initial Sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ {\scriptstyle Ax} \qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\bot}$$

Logical Rules

The classical base $G3C_\omega$, plus:

$$\frac{a \in N(x), a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta}{x : A \rightarrow B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L \rightarrow,\, a \text{ fresh}} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : A \rightarrow B, a \Vdash^{\forall} A \supset B}{a \in N(x), \Gamma \Rightarrow \Delta, x : A \rightarrow B}\ {\scriptstyle R \rightarrow}$$

$$\frac{a \in N(x), a \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta}{x : \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}\ {\scriptstyle L \bigwedge,\, a \text{ fresh}} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k, a \Vdash^{\forall} \underset{k>0}{\&} A_k}{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k}\ {\scriptstyle R \bigwedge}$$

Auxiliary rules

$$\frac{x \in a, a \Vdash^{\forall} A, x : A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L \Vdash^{\forall}} \qquad \frac{y \in a, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ {\scriptstyle R \Vdash^{\forall},\, y \text{ fresh}}$$

$$\frac{x : p, a \in N(x), a \Vdash^{\forall} p, \Gamma \Rightarrow \Delta}{x : p, \Gamma \Rightarrow \Delta}\ {\scriptstyle Mon,\, a \text{ fresh}} \qquad \frac{a \in N(x), x \in a, \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ {\scriptstyle Ref}$$

$$\frac{a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Nondeg,\, a \text{ fresh}} \qquad \frac{a \in N(x), m(a) \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ {\scriptstyle Trs}$$

$$\frac{x \in m(a), a \in N(x), \Gamma \Rightarrow \Delta}{x \in m(a), \Gamma \Rightarrow \Delta}\ {\scriptstyle Lm} \qquad \frac{c \in N(x), c \subseteq a, c \subseteq b, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}\ {\scriptstyle Prebasic,\, c \text{ fresh}}$$

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Ref \subseteq} \qquad \frac{a \subseteq b, b \subseteq c, a \subseteq c, \Gamma \Rightarrow \Delta}{a \subseteq b, b \subseteq c, \Gamma \Rightarrow \Delta}\ {\scriptstyle Trs \subseteq}$$

Figure 6.2: The calculus $\mathbf{G3I}_\omega$

6.6 Structural analysis of $\mathbf{G3I}_\omega$

In this section we shall study the structural properties of the calculus $\mathbf{G3I}_\omega$. A derivation in $\mathbf{G3I}_\omega$ is a tree which is possibly infinitely branching and every branch is of finite length. Given a derivation $\mathcal{D}$, its height $h(\mathcal{D})$ is thus defined:

1. If $\mathcal{D}$ is an initial sequent, then $h(\mathcal{D}) = 0$
2. If $\mathcal{D}$ is of the form:

$$\dfrac{\dots \qquad \begin{array}{c} \vdots\, \mathcal{D}_n \\ \Gamma_n \Rightarrow \Delta_n \end{array} \qquad \dots}{\Gamma \Rightarrow \Delta}$$

   with possibly countable premises, then $h(\mathcal{D}) = sup_n(h(\mathcal{D}_n)) + 1$, where the latter is a countable ordinal.

The next step is to introduce an appropriate measure for the complexity of labelled formulas: this is crucial in order to carry out our proofs by induction. We recall that labelled formulas may have the following forms: $x : A, a \Vdash^{\forall} A$, whereas relational formulas are $a \in N(x), x \in a$.

Definition 6.6.1. The *label* of a formula $x : A$ is x, the *label* of a formula $a \Vdash^{\forall} A$ is a. The label of a formula ϕ is denoted by $l(\phi)$. The *pure part* of a labelled formula ϕ is obtained removing from ϕ the *label* and the forcing relation and is denoted by $p(\phi)$. The notion of *weight* is defined for *labels* and *pure parts* of formulas:

- For every x and for every a, $w(x) = 0$ and $w(a) = 1 + n(m)$, where $n(m)$ is the number of the occurrences of m in a.
- The *weight* of a pure formula A, $w(A)$ is defined as follows:
  - $w(p) = w(\bot) = 1$
  - $w(A \circ B) = sup(\{w(A), w(B)\}) + 1$, where $\circ \in \{\wedge, \vee, \supset\}$
  - $w(A \rightarrow B) = sup(\{w(A), w(B)\}) + 2$
  - $w(\bigvee_{k>0} A_k) = sup_k(w(A_k)) + 1$
  - $w(\underset{k>0}{\&} A_k) = sup_k(w(A_k)) + 1$
  - $w(\bigwedge_{k>0} A_k) = sup_k(w(A_k)) + 2$

The *degree* of a labelled formula ϕ is an ordered pair $deg(\phi) = (w(p(\phi)), w(l(\phi)))$. For relational formulas we stipulate $deg(x \in a) = deg(a \in N(x)) = (0, w(a))$. *Degrees* of labelled formulas are ordered lexicographically.

From the above definition it is clear that in general for every x, a and A we have $deg(x : A) < deg(a \Vdash^{\forall} A)$ and $deg(a \Vdash \underset{k>0}{\&} A_k) < deg(\bigwedge_{k>0} A_k)$.

Lemma 6.6.1. *Sequents of the form $\phi, \Gamma \Rightarrow \Delta, \phi$ are derivable in* **G3I**$_\omega$ *for every labelled formulas ϕ and for every Γ, Δ:*

Proof. We proceed by transfinite induction on $deg(\phi)$. We discuss the case of $\phi \equiv x : \bigwedge_{k>0} A_k$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, IH\\ a \in N(x), a \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k, a \Vdash^{\forall} \underset{k>0}{\&} A_k\end{array}}{a \in N(x), a \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k}\, R\bigwedge}{x : \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k}\, L\bigwedge$$

The top-sequent is derivable by induction hypothesis because $deg(a \Vdash \underset{k>0}{\&} A_k) < deg(\bigwedge_{k>0} A_k)$. *qed.*

In the calculus **G3I**$_\omega$ labels are of two different kinds: either world labels $x, y, z, ...$ or neighborhood labels $a, b, c, ...$ In the system there are some rules the sound application of which requires special condition on such labels, such as the condition of being fresh. Therefore we need to prove height preserving admissibility of substitution of labels. For every labelled or relational formula ϕ the operation of substitution can assume two different forms: $\phi[x/y]$, i.e. the substitution of every occurrence of x in ϕ with y, and $\phi[a/b]$ the substitution of every occurrence of the neighborhood label a with b. Given a multiset of labelled formulas Γ we indicate with $\Gamma[x/y]$ ($\Gamma[a/b]$) the multiset obtained by substituting x (a) with y (b) in every labelled or relational formula ϕ which occurs in Γ.

Lemma 6.6.2. *The rules*

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma[x/y] \Rightarrow \Delta[x/y]}\, Sub\,[x/y] \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma[a/b] \Rightarrow \Delta[a/b]}\, Sub\,[a/b]$$

are height-preserving admissible in **G3I**$_\omega$.

Proof. By induction on the height of the derivation $\mathcal{D}$.

qed.

Now we are in the position to prove height-preserving admissibility of weakening.

Lemma 6.6.3. *The rules:*

$$\frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\, LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}\, RW$$

are height-preserving admissible for every multiset Γ, Δ *and every* ϕ *labelled formula* $x : A$, $a \Vdash^{\forall} A$ *or relational formula* $x \in a$, $a \in N(x)$. [4]

Proof. Straightforward by induction on the height of derivation, using height-preserving admissibility of substitution whenever needed in order to avoid clashes of variables. *qed.*

A rule is invertible if whenever the conclusion is derivable so are the premises. We now show invertibility of the rules of $\mathbf{G3I}_{\omega}$. We stress that this is one of the main differences with respect to the unlabelled sequent calculus for infinitary intuitionistic logic. In fact in the latter the right rules for $\bigwedge$ and $\rightarrow$ are not invertible due to the context restriction imposed on the premis(es). This has some desirable consequences: first of all invertibility of the rules avoids backtracking, furthermore it simplifies the structural analysis of the calculus and finally it enables a direct proof of completeness via the construction of a suitable reduction tree.

Lemma 6.6.4. *Every rule of* $\mathbf{G3I}_{\omega}$ *is height-preserving invertible.*

Proof. We deal with the rule L$\rightarrow$ and argue by induction on the height of derivation. If $n = 0$, then $x : A \rightarrow B, \Gamma \Rightarrow \Delta$ is an initial sequent and so is $\vdash_{\mathbf{G3I}_{\omega}} a \in N(x), a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta$. If $n > 0$, then we distinguish cases: either $x : A \rightarrow B$ is principal or not. In the first case we take the premise applying, if necessary, height-preserving substitution. In the second case we have to distinguish cases according to the last rule. If the last rule does not have any restriction on neighborhood labels we apply the induction hypothesis to the (possibly infinite) premise(s) and we conclude by an application of the rule. If the last rule contains a condition on neighborhood labels, say, for example, $L\bigwedge$, the premise is $x : A \rightarrow B, b \in N(y), b \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma' \Rightarrow \Delta$ and we can assume (due to admissibility of height-preserving substitution) that $a \not\equiv b$, hence we apply the induction hypothesis to the premise and we obtain

[4]Notice that relational formulas are not needed for RW, because due to the formulation of the rules they are never active.

$a \in N(x), a \Vdash^{\forall} A \supset B, b \in N(y), b \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma' \Rightarrow \Delta$, then we conclude by an application of $L \bigwedge$.

We deal with the rule R& and we argue by induction on the height of derivation. If $n = 0$, then $\Gamma \Rightarrow \Delta, x : \underset{k>0}{\&} A_k$ is an initial sequent and so is $\Gamma \Rightarrow \Delta, x : A_k$ for every $k > 0$. If $n > 0$, then either $x : \underset{k>0}{\&} A_k$ is principal in the last rule applied or not. In the first case we take the premises, in the second case we distinguish cases according to the last rule applied. The general strategy consists in applying the induction hypothesis and then concluding with an application of the rule. We discuss the case in which the last rule applied is $L \bigvee$ with principal formula $y : \bigvee_{t>0} B_t$. For every $k > 0$ we consider the (countably) infinite premises: $\{y : B_t, \Gamma' \Rightarrow \Delta, x : \underset{k>0}{\&} A_k \mid t > 0\}$, hence we apply the induction hypothesis to every premise to obtain $\{y : B_t, \Gamma' \Rightarrow \Delta, x : A_k \mid t > 0\}$ and then we apply again $L \bigvee$ and we obtain the desired result.

qed.

The following lemma shows that the rules of contraction are height-preserving admissible in $\mathbf{G3I}_\omega$.

Lemma 6.6.5. *The rules:*

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\ RC$$

are height-preserving admissible.

Proof. The proof is by simultaneous induction on the height of derivations.

Base. If $n = 0$, then $\phi, \phi, \Gamma \Rightarrow \Delta$ is an initial sequent and the same holds for $\phi, \Gamma \Rightarrow \Delta$.

Induction step If $n > 0$, then we distinguish cases: either ϕ is principal in the last rule applied or not. In the latter case we apply the induction hypothesis to the (possibly infinite) premise(s) and we apply the rule again.

If ϕ is principal then we distinguish between two kinds of rules. If in the last rule the principal formula ϕ is repeated in the premise[5] we apply the induction hypothesis and then the rule again.[6]

We discuss the other cases, which are $L\wedge, R\wedge, L\vee, R\vee, L \supset, R \supset, L \rightarrow, L \bigwedge, L \bigvee, R\&, R \Vdash^{\forall}$. Propositional cases $L\wedge, R\wedge, L\vee, R\vee, R \supset, L \supset$ are dealt with as usual, making use of height-preserving invertibility of the usual rules. With respect to $L \rightarrow$ we consider the following situation:

[5]Notice that such rules are single premise ones.

[6]The only problematic case could be C if the conclusion is $a \in N(x), a \in N(x), \Gamma \Rightarrow \Delta$, but that case is dealt with the closure condition.

$$\frac{a \in N(x), a \Vdash^{\forall} A \supset B, x : A \rightarrow B, \Gamma \Rightarrow \Delta}{x : A \rightarrow B, x : A \rightarrow B, \Gamma \Rightarrow \Delta} \; L \rightarrow, a \text{ fresh}$$

We apply height-preserving invertibility to the premise to obtain $a \in N(x), a \in N(x), a \Vdash^{\forall} A \supset B, a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta$, then we apply the induction hypothesis and we obtain $a \in N(x), a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta$ and we conclude by $L \rightarrow$ (notice that the freshness condition is respected).

If the last rule is an infinitary one, say $L \bigvee$, we have:

$$\frac{\{x : A_k, x : \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta \,|\, k > 0\}}{x : \bigvee_{k>0} A_k, x : \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta} \; L \bigvee$$

We apply height-preserving invertibility to every premise and we obtain countably infinite derivations $\{x : A_k, x : A_k, \Gamma \Rightarrow \Delta \,|\, k > 0\}$, hence we apply to every such derivation the induction hypothesis which yields $\{x : A_k, \Gamma \Rightarrow \Delta \,|\, k > 0\}$ and we conclude by $L \bigvee$.

Finally we deal with $R \Vdash^{\forall}$:

$$\frac{x \in a, \Gamma, \Rightarrow \Delta, x : A, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A, a \Vdash^{\forall} A} \; R \Vdash^{\forall}, x \text{ fresh}$$

We apply height-preserving invertibility to the premise to obtain a derivation of $x \in a, x \in a, \Gamma, \Rightarrow \Delta, x : A, x : A$ of the same height, then we apply the induction hypothesis which yields $x \in a, \Gamma, \Rightarrow \Delta, x : A$ and then we apply again the rule (notice that the freshness condition is respected). *qed.*

Now we can prove the admissibility of the rule:

$$\frac{x : A, a \in N(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta} \; Mon^{+}$$

which corresponds in a sense to the semantical persistence lemma. In order to do so we have to prove some preliminary results.

Lemma 6.6.6. *The following rules are admissible:*

$$\frac{c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, c \Vdash^{\forall} A \wedge B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}{c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta} \; Mon_{\wedge} \qquad \frac{a \Vdash^{\forall} \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} A_k, \Gamma \Rightarrow \Delta} \; Mon_{\bigvee}$$

$$\frac{a \Vdash^{\forall} A \vee B, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \; Mon_{\vee} \qquad \frac{m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} \&_{k>0} A_k, \Gamma \Rightarrow \Delta} \; Mon_{\bigwedge} \qquad \frac{m(a) \Vdash^{\forall} A \rightarrow B, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta} \; Mon_{\rightarrow}$$

Proof. We proceed by induction on the height of derivations. We discuss the cases $[Mon_{\wedge}]$ and $[Mon_{\bigwedge}]$, the other cases are analogous.

$Mon_{\wedge}$. If $c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, c \Vdash^{\forall} A \wedge B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta$ is an initial sequent, then so is the conclusion. If $c \Vdash^{\forall} A \wedge B$ is not principal we apply the induction hypothesis to the premises and then we the rule again. If the last rule is $L \Vdash$ with $c \Vdash^{\forall} A \wedge B$ we have:

$$c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, y \in c, y : A \wedge B, c \Vdash^{\forall} A \wedge B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta$$

We proceed as follows:

$$
\begin{array}{cl}
c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, y \in c, y : A \wedge B, c \Vdash^{\forall} A \wedge B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta & \\ \hline
c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, y \in c, y \in a, y \in b, y : A \wedge B, c \Vdash^{\forall} A \wedge B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta & Weak \\ \hline
c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, y \in c, y \in a, y \in b, y : A, y : B, c \Vdash^{\forall} A \wedge B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta & InvL\wedge \\ \hline
c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, y \in c, y \in a, y \in b, y : A, y : B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta & IH \\ \hline\hline
c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, y \in c, y \in a, y \in b, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta & L \Vdash^{\forall} \\ \hline\hline
c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, y \in c, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta & L \subseteq
\end{array}
$$

$Mon_{\bigwedge}$. If $m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta$ is an initial sequent, so is $a \Vdash^{\forall} \mathop{\&}_{k>0} A_k, \Gamma \Rightarrow \Delta$. If $m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k$ is not principal the proof is straightforward via induction hypothesis. If $m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k$ is principal we have:

$$
\frac{x \in m(a), m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, x : \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}{x \in m(a), m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta} \; L \Vdash^{\forall}
$$

Thus we construct the following derivation:

$$
\begin{array}{cl}
x \in m(a), m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, x : \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta & \\ \hline
x \in m(a), a \in N(x), m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, a \Vdash^{\forall} \mathop{\&}_{k>0} A_k, \Gamma \Rightarrow \Delta & Inv \bigwedge \\ \hline
x \in m(a), a \in N(x), a \Vdash^{\forall} \mathop{\&}_{k>0} A_k, a \Vdash^{\forall} \mathop{\&}_{k>0} A_k, \Gamma \Rightarrow \Delta & IH \\ \hline
x \in m(a), a \in N(x), a \Vdash^{\forall} \mathop{\&}_{k>0} A_k, \Gamma \Rightarrow \Delta & Ctr \\ \hline
x \in m(a), a \Vdash^{\forall} \mathop{\&}_{k>0} A_k, \Gamma \Rightarrow \Delta & Lm
\end{array}
$$

qed.

Theorem 6.6.7. *For every $A \in \mathsf{FM}_{\omega}$, the rule:*

$$
\frac{x : A, a \in N(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta} \; Mon^{+}, a \, fresh
$$

is admissibile in $\mathbf{G3I}_{\omega}$.

Proof. We proceed by induction on the *degree* of the formula $A \in \mathsf{FM}_\omega$.[7] If $A \equiv p$, then Mon^+ reduces to Mon.

If $A \equiv B \wedge C$ we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, c \Vdash^\forall A \wedge B, a \Vdash^\forall A, b \Vdash^\forall B, x : A, x : B, x : A \wedge B, \Gamma \Rightarrow \Delta}{c \in N(x), a \in N(x), b \in N(x), c \subseteq a, c \subseteq b, a \Vdash^\forall A, b \Vdash^\forall B, x : A, x : B, x : A \wedge B, \Gamma \Rightarrow \Delta}{\scriptstyle Mon_\wedge}}{a \in N(x), b \in N(x), a \Vdash^\forall A, b \Vdash^\forall B, x : A, x : B, x : A \wedge B, \Gamma \Rightarrow \Delta}{\scriptstyle Prebasic}}{x : A, x : B, x : A \wedge B, \Gamma \Rightarrow \Delta}{\scriptstyle IH}}{x : A \wedge B, x : A \wedge B, \Gamma \Rightarrow \Delta}{\scriptstyle L\wedge}}{x : A \wedge B, \Gamma \Rightarrow \Delta}{\scriptstyle Ctr}$$

If $A \equiv \bigwedge_{k>0} B_k$, we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{a \in N(x), m(a) \in N(x), x : \bigwedge_{k>0} B_k, m(a) \Vdash^\forall \bigwedge_{k>0} B_k, \Gamma \Rightarrow \Delta}{a \in N(x), m(a) \in N(x), x : \bigwedge_{k>0} B_k, a \Vdash^\forall \&_{k>0} B_k, \Gamma \Rightarrow \Delta}{\scriptstyle Mon_\bigwedge}}{a \in N(x), x : \bigwedge_{k>0} B_k, a \Vdash^\forall \&_{k>0} B_k, \Gamma \Rightarrow \Delta}{\scriptstyle Lm}}{x : \bigwedge_{k>0} B_k, x : \bigwedge_{k>0} B_k, \Gamma \Rightarrow \Delta}{\scriptstyle L\bigwedge}}{x : \bigwedge_{k>0} B_k, \Gamma \Rightarrow \Delta}{\scriptstyle Ctr}$$

The other cases are similar to the ones discussed. *qed.*

We give an example of a derivation: $\vdash_{\mathbf{G3I}_\omega} \Rightarrow x : A \rightarrow (B \rightarrow A)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \in N(x), y \in a, z \in b, b \in N(y), b \Vdash^\forall A, y : A, z : B, z : A \Rightarrow x : A \rightarrow (B \rightarrow A), y : B \rightarrow A, z : A}{a \in N(x), y \in a, z \in b, b \in N(y), b \Vdash^\forall A, y : A, z : B \Rightarrow x : A \rightarrow (B \rightarrow A), y : B \rightarrow A, z : A}{\scriptstyle L\Vdash^\forall}}{a \in N(x), y \in a, z \in b, b \in N(y), b \Vdash^\forall A, y : A \Rightarrow x : A \rightarrow (B \rightarrow A), y : B \rightarrow A, z : B \supset A}{\scriptstyle R\supset}}{a \in N(x), y \in a, b \in N(y), b \Vdash^\forall A, y : A \Rightarrow x : A \rightarrow (B \rightarrow A), y : B \rightarrow A, b \Vdash^\forall B \supset A}{\scriptstyle R\Vdash^\forall}}{a \in N(x), y \in a, b \in N(y), b \Vdash^\forall A, y : A \Rightarrow x : A \rightarrow (B \rightarrow A), y : B \rightarrow A}{\scriptstyle R\rightarrow}}{a \in N(x), y \in a, y : A \Rightarrow x : A \rightarrow (B \rightarrow A), y : B \rightarrow A}{\scriptstyle Mon^+}}{a \in N(x), y \in a \Rightarrow x : A \rightarrow (B \rightarrow A), y : A \supset (B \rightarrow A)}{\scriptstyle R\supset}}{a \in N(x) \Rightarrow x : A \rightarrow (B \rightarrow A), a \Vdash^\forall A \supset (B \rightarrow A)}{\scriptstyle R\Vdash^\forall}}{a \in N(x) \Rightarrow x : A \rightarrow (B \rightarrow A)}{\scriptstyle R\rightarrow}}{\Rightarrow x : A \rightarrow (B \rightarrow A)}{\scriptstyle Nondeg}$$

The topsequent is derivable.

We are now in the position to state and prove the main structural property of our calculus $\mathbf{G3I}_\omega$, namely cut elimination. However since our calculus is infinitary we have to make some modifications: in particular the notion of sum of the heights of the derivations has to be changed. We use the natural sum of ordinals, denoted by #, which has the following two useful properties: $\alpha \# \beta = \beta \# \alpha$ and if $\alpha < \alpha'$, then $\alpha \# \beta < \alpha' \# \beta$ (100).

[7] FM_ω is the language of intuitionistic infinitary logic, thus A does not contain & or $\supset$: in those cases the lemma would not hold.

Theorem 6.6.8. *The rule:*

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ {\scriptstyle Cut}$$

where ϕ is a labelled formula[8] *is admissible in* $\mathbf{G3I}_\omega$.

Proof. We proceed by double transfinite induction on lexicographically ordered pairs with main induction hypothesis on the weight of the cut formula and secondary induction hypothesis on the natural sum of height of derivations of the premises of the cut.

We distinguish five cases: $\langle In, ?\rangle$ and $\langle ?, In\rangle$, i.e. the cases in which one of the two premises is an initial sequent, $\langle Pr, Pr\rangle$, i.e. the case in which the cut formula is principal in both premises, $\langle nPr, ?\rangle$ and $\langle ?, nPr\rangle$, i.e. cases in which cut formula is not principal in one of the two premises.

The first and the second case are similar. We deal with the first. Then we have the following situation:

$$\frac{\overline{\Gamma \Rightarrow \Delta, \phi}\ {\scriptstyle Ax} \qquad \phi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ {\scriptstyle Cut}$$

Hence we distinguish two subcases: either ϕ is active in $\Gamma, \Rightarrow \Delta, \phi$ or not. In the first case then $\phi \equiv x : p$ and $\Gamma \equiv \Gamma'', x : p$, therefore we apply admissibility of weakening to the second premise of cut to obtain the desired conclusion. In the second case the sequent $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is an initial sequent as well.

We deal with the cases in which the cut formula is principal in both premises.[9] Propositional cases in which the cut formula is of the form $A \vee B$, $A \supset B$ or $A \wedge B$ are dealt with as usual, thus we consider the other cases. Let us first consider a case in which the cut formula is $\bigwedge_{k>0} A_k$.

$$\frac{\dfrac{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k, a \Vdash^{\forall} \&_{k>0} A_k}{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k}\ {\scriptstyle R\wedge} \qquad \dfrac{b \in N(x), b \Vdash^{\forall} \&_{k>0} A_k, \Gamma', \Rightarrow, \Delta'}{x : \bigwedge_{k>0} A_k, \Gamma' \Rightarrow \Delta'}\ {\scriptstyle L\wedge}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ {\scriptstyle Cut}$$

Then we proceed as follows:

[8] We can restrict to such cases, because relational formulas are never principal in the succedent of a sequent due to the formulation of the rules.

[9] Notice that $x : p$ cannot be principal in both premises, in fact although it can be principal in an application of Mon in the right premise of cut, in the left premise it can principal only in an initial sequent and that case has already been detailed above.

$$\dfrac{\dfrac{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k, a \Vdash^{\forall} \&_{k>0} A_k \qquad x : \bigwedge_{k>0} A_k, \Gamma' \Rightarrow \Delta'}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\forall} \&_{k>0} A_k}\,Cut \qquad \dfrac{b \in N(x), b \Vdash^{\forall} \&_{k>0} A_k, \Gamma', \Rightarrow, \Delta'}{a \in N(x), a \Vdash^{\forall} \&_{k>0} A_k, \Gamma', \Rightarrow, \Delta'}\,Sub[a/b]}{\dfrac{a \in N(x), a \in N(x), \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Ctr}\,Cut$$

The first cut is removed by secondary induction hypothesis on the natural sum of the height of derivations, whereas the second one is removed by primary induction hypothesis on the degree of the cut formula. The case for $L \rightarrow$ is analogous.

We deal with the case in which the principal formula is $\&_{k>0} A_k$:

$$\dfrac{\dfrac{\{\Gamma \Rightarrow \Delta, x : A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, x : \&_{k>0} A_k}\,R\& \qquad \dfrac{x : \&_{k>0} A_k, x : A_k, \Gamma', \Rightarrow, \Delta'}{x : \&_{k>0} A_k, \Gamma' \Rightarrow \Delta'}\,L\&}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

We observe that $\Gamma \Rightarrow \Delta, x : A_k$ has to be among the premises of the left premise of the inference. The reduction procedure is as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A_k \qquad \dfrac{\Gamma \Rightarrow \Delta, x : \&_{k>0} A_k \qquad x : \&_{k>0} A_k, x : A_k, \Gamma', \Rightarrow, \Delta'}{x : A_k, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\,Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Ctr$$

As before the left cut is eliminated via secondary induction hypothesis and the second one via primary induction induction hypothesis on the degree of the cut formula. The case of $\bigvee$ is dual.

If the cut formula is not principal in one of the two premises, we adopt the usual strategy: in particular we permute upwards the application of cut and we apply the secondary induction hypothesis on the sum of the heights of the derivations in order to remove the cut formula (if necessary we apply height-preserving substitution to avoid clashes of eigenvariables). *qed.*

In order to establish completeness for our labelled sequent system we first have to prove that there is an embedding from the unlabelled sequent calculus $\mathbf{G3i}_\omega$ into it. Given a finite multiset of formulas Γ, $x : \Gamma$ denotes the multiset obtained labelling each formula in Γ by x.

Lemma 6.6.9. *If* $\mathbf{G3i}_\omega \vdash \Gamma \Rightarrow \Delta$, *then* $\mathbf{G3I}_\omega \vdash x : \Gamma \Rightarrow x : \Delta$

Proof. The proof is by induction on the height of derivations in $\mathbf{G3i}_\omega$.

If $n = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent in $\mathbf{G3i}_\omega$ and $x : \Gamma \Rightarrow x : \Delta$ is an initial sequent in $\mathbf{G3I}_\omega$.

If $n > 0$ we distinguish cases according to the last rule applied. Cases $L\wedge, R\wedge, L\vee, R\vee, L\bigvee$ and $R\bigvee$ are straightforward by induction hypothesis. We discuss the cases for the connectives $\rightarrow$ and $\bigwedge$.

If the last rule applied is $R\rightarrow$ we have:

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ \mathrm{R}\!\rightarrow$$

Let $A_1, ..., A_n$ be the formulas in Γ. We construct the following derivation:

$$\begin{array}{cl}
x : A_1, ..., x : A_n, x : A \Rightarrow x : B & \\ \hline
y : A_1, ..., y : A_n, y : A \Rightarrow y : B & Sub[x/y] \\ \hline\hline
y : A_1 \wedge ... \wedge A_n, y : A \Rightarrow y : B & L\wedge \\ \hline
y \in a, a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, y : A_1 \wedge ... \wedge A_n, y : A, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : A \rightarrow B, y : B & Weak \\ \hline
y \in a, a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, y : A, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : A \rightarrow B, y : B & L\Vdash^{\forall} \\ \hline
y \in a, a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : A \rightarrow B, y : A \supset B & R\supset \\ \hline
a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : A \rightarrow B, a \Vdash^{\forall} A \supset B & R\Vdash^{\forall} \\ \hline
a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : A \rightarrow B & R\rightarrow \\ \hline
x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : A \rightarrow B & Mon^{+} \\ \hline
x : A_1, ..., x : A_n \Rightarrow x : \Delta, x : A \rightarrow B & Inv\wedge
\end{array}$$

We can apply the admissible rule Mon^{+} because every formula A_i is a formula in the language of infinitary intuitionistic logic and therefore the same holds for $A_1 \wedge ... \wedge A_n$.

The case of $L\bigwedge$ is similar to the one of $L\rightarrow$ and we omit the details, instead we focus on $R\bigwedge$. If the last rule applied is $R\bigwedge$ we have:

$$\frac{\{\Gamma \Rightarrow B_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{k>0} B_k}\ \mathrm{R}\bigwedge$$

Let $A_1, ..., A_n$ be the formulas in Γ. We construct the following derivation:

$$\begin{array}{cl}
\{x : A_1, ..., x : A_n \Rightarrow x : B_k \mid k > 0\} & \\ \hline
\{y : A_1, ..., y : A_n \Rightarrow y : B_k \mid k > 0\} & Sub[x/y] \\ \hline\hline
\{y : A_1 \wedge ... \wedge A_n \Rightarrow y : B_k \mid k > 0\} & L\wedge \\ \hline
\{y \in a, a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, y : A_1 \wedge ... \wedge A_n, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : \bigwedge_{k>0} B_k, y : B_k \mid k > 0\} & Weak \\ \hline
\{y \in a, a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : \bigwedge_{k>0} B_k, y : B_k \mid k > 0\} & L\Vdash^{\forall} \\ \hline
y \in a, a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : \bigwedge_{k>0} B_k, y : \&_{k>0} B_k & R\& \\ \hline
a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : \bigwedge_{k>0} B_k, a \Vdash^{\forall} \&_{k>0} B_k & R\Vdash^{\forall} \\ \hline
a \in N(x), a \Vdash^{\forall} A_1 \wedge ... \wedge A_n, x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : \bigwedge_{k>0} B_k & R\bigwedge \\ \hline
x : A_1 \wedge ... \wedge A_n \Rightarrow x : \Delta, x : \bigwedge_{k>0} B_k & Mon^{+} \\ \hline
x : A_1, ..., x : A_n \Rightarrow x : \Delta, x : \bigwedge_{k>0} B_k & Inv\wedge
\end{array}$$

qed.

Corollary (Completeness)**.** $\mathbf{G3I}_{\omega} \vdash \Rightarrow x : A$ *iff* $\vDash A$.

Proof. From left to right (soundness) the proof is straightforward by induction on the height of derivation.

From right to left, if $\vDash_{\mathcal{N}} A$, then by completeness of $\mathbf{G3i}_\omega$ we have $\mathbf{G3i}_\omega \vdash \Rightarrow A$ and thus by the embedding we obtain $\mathbf{G3I}_\omega \vdash \Rightarrow x : A$. *qed.*

We have obtained a cut free sequent calculus for intuitionistic infinitary logic with full invertibility of every rule, thus it is also possible to obtain a more direct proof of completeness via the construction of a suitable reduction tree.

6.7 Neighborhood semantics for infinitary S4 modal logic

6.7.1 Infinitary S4 modal logic

Before proceeding, we have to briefly recall some notions relative to the language and the semantics of modal logic.

Definition 6.7.1. The language of infinitary modal logic contains a countably infinite set of propositional atoms AT and connectives $\bot, \wedge, \vee, \rightarrow, \bigvee, \bigwedge$ and $\Box$. The set of formulas of modal infinitary logic $\mathsf{FM}^{\Box}_{\omega}$ is defined inductively as usual.

The notation $\neg A$ abridges $A \rightarrow \bot$. We first introduce a measure of complexity for formulas which requires to be formulated in terms of ordinals, as formulas now include expressions of infinitary length.

Definition 6.7.2. The *weight* of a formula A, $w(A)$, is defined as follows:

- $w(p) = w(\bot) = 1$
- $w(A \circ B) = sup(\{w(A), w(B)\}) + 1$, where $\circ \in \{\wedge, \vee, \rightarrow\}$
- $w(\Box A) = w(A) + 1$
- $w(\bigvee_{k>0} A_k) = sup_k(w(A_k)) + 1$
- $w(\bigwedge_{k>0} A_k) = sup_k(w(A_k)) + 1$

We recall neighborhood semantics for modal logics: since we will be mainly focused on system **S4**, we will discuss neighborhood frames for system **K** and its extensions, namely for normal modal logics (16).

Definition 6.7.3. A **K** neighborhood frame is a pair $\langle W, N \rangle$ with $W \neq \emptyset$ and $N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ such that:

1. If $a, b \in N(x)$, then $a \cap b \in N(x)$

2. If $a \in N(x)$ and $a \subseteq b$, then $b \in N(x)$

3. $W \in N(x)$ for every $x \in W$

A **T** neighborhood frame is a frame in which for every $x \in W$, if $a \in N(x)$, then $x \in a$. An **S4** neighborhood frame is a **T** frame in which for every $x \in W$, if $a \in N(x)$, then $\{y \mid a \in N(y)\} \in N(x)$.

It is easy to observe that **S4** neighborhood frames actually coincide with infinitary intuitionistic neighborhood frames (69; 103).[10] We now introduce neighborhood models for infinitary modal logics and their satisfiability conditions. In what follows, we introduce the following abbreviations. Given a neighborhood frame $\langle W, N\rangle$ and a neighborhood a in $N(x)$ where x is a world in W, we denote by $m(a)$ the set $\{y \mid a \in N(y)\} \in N(x)$. We now introduce the notion of neighborhood model for infinitary modal logic and the truth conditions for a formula at a world. With a slight abuse of notation, we sometimes use first-order quantifiers in an informal manner as an abbreviation of *for every* and *for some* in the metalanguage.

Definition 6.7.4. An **S4** neighborhood model for infinitary modal logic is a triple $\mathcal{M} = \langle W, N, v\rangle$ where $\langle W, N\rangle$ is a **S4** neighborhood frame and $v : AT \to P(W)$ is a valuation function. For every world $x \in W$ and every formula $A \in \mathsf{FM}^{\Box}_{\omega}$ the satisfiability condition in the model $\mathcal{M}, x \Vdash A$, is defined inductively as follows:

- $x \Vdash p$ if and only if $x \in v(p)$
- $x \nVdash \bot$
- $x \Vdash B \wedge C$ if and only if $x \Vdash B$ and $x \Vdash C$
- $x \Vdash B \vee C$ if and only if $x \Vdash B$ or $x \Vdash C$
- $x \Vdash B \to C$ if and only if $x \nVdash B$ or $x \Vdash C$
- $x \Vdash \bigvee_{k>0} B_k$ if and only if $x \Vdash B_k$ for some $k > 0$
- $x \Vdash \bigwedge_{k>0} B_k$ if and only if $x \Vdash B_k$ for every $k > 0$
- $x \Vdash \Box B$ if and only if $\exists a \in N(x) \forall y \in a\ (y \Vdash B)$

[10] See also (102) for a study of neighborhood frames and infinitary modal logic from the perspective of duality theory.

Axioms

The axioms of the classical logic C, plus:

C1. $\bigwedge_{k>0} A_k \to A_k$ (for every k)

C2. $A_k \to \bigvee_{k>0} A_k$ (for every k)

C3. $\bigwedge_{k>0} (A \to B_k) \to (A \to \bigwedge_{k>0} B_k)$

C4. $\bigwedge_{k>0} (A_k \to B) \to (\bigvee_{k>0} A_k \to B)$

4 $\Box A \to \Box\Box A$

T $\Box A \to A$

K $\Box(A \to B) \to (\Box A \to \Box B)$

Inference Rules

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B}\ \mathrm{MP} \qquad\qquad \frac{\vdash A}{\Gamma \vdash \Box A}\ RN$$

$$\frac{\{\Gamma \vdash A_k\}_{k>0}}{\Gamma \vdash \bigwedge_{k>0} A_k}\ Adj$$

Figure 6.3: $\mathbf{S4}_\omega$

A formula A is true in a neighborhood model $\mathcal{M}$ if it is true at every world in the model. A formula A is valid in a class of frames C, in symbols $\vDash_C A$, if it is true in every world in every model based on a frame in the class.

Furthermore, we recall the notion of universal forcing: we write $a \Vdash^{\forall} A$ to express the fact that every world in a satisfies A. The satisfiability condition for the modality operator is usually presented as $x \Vdash \Box B$ if and only if $\exists a \in N(x)(a \Vdash^{\forall} B)$ and for every y, if $y \Vdash B$, then $y \in a$, but in the case of neighborhood models based on monotonic frames the condition can be streamlined as above (79).

We write $\vDash_{S4} A$ to denote that A is valid in the class of infinitary **S4** neighborhood frames. We are now going to prove that the axiomatisation in Figure 6.3 is complete with respect to the semantic of infinitary **S4** neighborhood frames. We define derivations in the axiomatic calculus as infinitary branching well-founded trees.

Definition 6.7.5. A derivation in the axiomatic calculus $\mathbf{S4}_\omega$ is a (possibly) infinitely branching well-founded tree where leaves are labelled by expressions of the form $\Gamma \vdash A$, where Γ is a finite set of formulas and either $A \in \Gamma$ or A is an instance of an axiom of the calculus, the tree is built according to locally correct applications of the rules RN, Adj and MP and the conclusion is the root of the tree.

This definition clearly subsumes the standard one in axiomatic calculi: a derivation of a formula $\vdash A$ is simply a derivation of A from the empty set $\emptyset$. The presence of the set of assumptions Γ models the derivability under assumptions in a Hilbert-style system and - together with the built-in weakening in the rule RN - allows one to establish the deduction theorem. Since derivation are now (possibly) infinitely branching well-founded rooted tree, the *height* of a derivation in $\mathbf{S4}_\omega$ needs to be measured by ordinals.

Definition 6.7.6. The *height* of a derivation in $\mathbf{S4}_\omega$ is the length of its longest branch. Given a derivation $\mathcal{D}$, its height $h(\mathcal{D})$ is thus defined:

1. If $\mathcal{D}$ is axiom, then $h(\mathcal{D}) = 0$

2. If $\mathcal{D}$ is of the form:

$$\dfrac{\ldots \qquad \begin{array}{c} \vdots\, \mathcal{D}_n \\ \Gamma_n \Rightarrow \Delta_n \end{array} \qquad \ldots}{\Gamma \Rightarrow \Delta}$$

   with possibly countable premises, then $h(\mathcal{D}) = sup_n(h(\mathcal{D}_n)) + 1$, where the latter is a countable ordinal.

We first show the following fact.

Lemma 6.7.1. $\vdash_{S4_\omega} \Box \bigwedge_{k>0} A_k \to \bigwedge_{k>0} \Box A_k$.

Proof. We construct the following derivation.

$$\dfrac{\dfrac{\cdots \qquad \dfrac{\dfrac{\dfrac{}{\vdash \bigwedge_{k>0} A_k \to A_k}\,\mathrm{C1}}{\vdash \Box(\bigwedge_{k>0} A_k \to A_k)}\,\mathrm{RN} \qquad \dfrac{}{\vdash \Box(\bigwedge_{k>0} A_k \to A_k) \to (\Box \bigwedge_{k>0} A_k \to \Box A_k)}\,\mathrm{K}}{\vdash \Box \bigwedge_{k>0} A_k \to \Box A_k}\,\mathrm{MP} \qquad \cdots}{\vdash \bigwedge_{k>0}(\Box \bigwedge_{k>0} A_k \to \Box A_k)}\,\mathrm{Adj} \qquad \dfrac{}{\vdash \bigwedge_{k>0}(\Box \bigwedge_{k>0} A_k \to \Box A_k) \to (\Box \bigwedge_{k>0} A_k \to \bigwedge_{k>0} \Box A_k)}\,\mathrm{C3}}{\vdash \Box \bigwedge_{k>0} A_k \to \bigwedge_{k>0} \Box A_k}\,\mathrm{MP}$$

qed.

In other words the infinitary variant of the converse Barcan formula is derivable in $\mathbf{S4}_\omega$ and need not be explicitly added to the list of axioms. However, the same does not hold with respect to the infinitary variant of the Barcan formula which is not provable in $\mathbf{S4}_\omega$, due to the fact that $\mathbf{S4}$ neighborhood frames are closed under finite intersections and not under infinite intersections.

6.7.2 Soundness and completeness

In order to show the underivability of the Barcan formula we shall prove the soundness theorem for our axiomatization and then provide a countermodel to the Barcan formula.

Theorem 6.7.2 (Soundness)**.** *If* $\vdash_{S4_\omega} A$*, then* $\vDash_{S4} A$*.*

Proof. The proof runs by induction on the height of the derivation of the formula A in the calculus $\mathbf{S4}_\omega$. The cases of the infinitary axioms and rules are straightforward by the truth conditions for the infinitary connectives. With respect to the modal axioms the strategy follows the pattern detailed in (79). We limit ourselves to observing that closure under infinite intersections is not required in order to show the soundness of the axioms **K**, **T** and **4**. In other words, closure under finite intersections is sufficient in order to establish the soundness result for the modal logic $\mathbf{S4}_\omega$ (the same holds for the finitary modal system **S4**). *qed.*

We now can show that the (infinitary variant) of the Barcan formula is not provable in $\mathbf{S4}_\omega$.

Lemma 6.7.3. $\nvdash_{\mathbf{S4}_\omega} \bigwedge_{k>0} \Box p_k \to \Box \bigwedge_{k>0} p_k$.

Proof. Consider the following countermodel (see also (111) for a similar example). Let $W = \mathbb{R}$ and $N(x) = \{a \mid \exists r > 0((x-r, x+r) \subseteq a)\}$. We define the valuation $v : AT \to \mathbb{R}$ such that $v(q) = \mathbb{R} \setminus \{0\}$ and $v(p_k) = (-\frac{1}{k}, \frac{1}{k})$ for every $k > 0$. We leave it to the reader to check that this is an **S4** neighborhood model. It is immediate to verify that $0 \Vdash \bigwedge\limits_{k>0} \Box p_k$, but $0 \nVdash \Box \bigwedge\limits_{k>0} p_k$, therefore, by soundness we obtain $\nvdash_{S4_\omega} \bigwedge\limits_{k>0} \Box p_k \to \Box \bigwedge\limits_{k>0} p_k$. Informally speaking, the intuition is that each p_k can be forced by a different neighborhood, but their intersection need not be a neighborhood. *qed.*

We now focus on the proof of completeness, in which we make use once again of the notion of environment introduced in (68) in the context of relational semantics, albeit with some modifications. The notion of environment - in this casa employed in a modal setting - is crucial in order to restrict ourselves to consider countable sets of formulas with certain properties. The key insight is that the environment of a countable multiset of formulas Γ contains enough formulas so as to show useful properties of maximal consistent sets, but is not closed under infinitary conjunctions and disjunctions. The upshot of the definition of environment is that the environment of a countable set of formulas is a countable set as well.

We shall exploit this in order to prove the Lindenbaum lemma. We recall that for every set of formulas Γ, $Sub(\Gamma)$ denotes the set of subformulas of the formulas in Γ.

Definition 6.7.7. Given a set of modal formulas Γ, the environment of Γ, $\mathcal{E}(\Gamma)$ is the least subset of $\mathsf{FM}^{\Box}$ such that:

- $AT \subseteq \mathcal{E}(\Gamma)$ and $\bot \in \mathcal{E}(\Gamma)$.
- $Sub(\Gamma) \subseteq \mathcal{E}(\Gamma)$.
- $\mathcal{E}(\Gamma)$ is closed under subformulas and under $\wedge, \vee, \rightarrow, \Box$.
- For every $A \in \mathcal{E}(\Gamma)$ and every $\bigwedge\limits_{k>0} B_k \in \mathcal{E}(\Gamma)$, $\bigwedge\limits_{k>0} (A \vee B_k) \in \mathcal{E}(\Gamma)$.
- For every $A \in \mathcal{E}(\Gamma)$ and every $\bigwedge\limits_{k>0} B_k \in \mathcal{E}(\Gamma)$, if $\bigwedge\limits_{k>0} (A \rightarrow B_k) \in \mathcal{E}(\Gamma)$, then $A \rightarrow \bigwedge\limits_{k>0} B_k \in \mathcal{E}(\Gamma)$.

As before we observe that if Γ is countable, so is $\mathcal{E}(\Gamma)$. In order to show that the calculus is complete it is convenient to show the admissibility of the so called deduction theorem (see also (47) for an extensive discussion in the context of finitary modal logic). We show that it holds for $\mathbf{S4}_\omega$.

Theorem 6.7.4. *If* $\Gamma, A \vdash_{S4_\omega} B$, *then* $\Gamma \vdash_{S4_\omega} A \rightarrow B$.

Proof. We argue by transfinite induction on the height of derivation in $\mathbf{S4}_\omega$. We limit ourselves to discussing the case of rule Adj.

We have:

$$\frac{\ldots \quad \Gamma, A \vdash B_k \quad \ldots}{\Gamma, A \vdash \bigwedge_{k>0} B_k}\,\mathrm{Adj}$$

We proceed as follows:

$$\cfrac{\cfrac{\ldots \quad \cfrac{\Gamma, A \vdash B_k}{\Gamma \vdash A \rightarrow B_k}\,\mathrm{IH} \quad \ldots}{\Gamma \vdash \bigwedge_{k>0}(A \rightarrow B_k)}\,\mathrm{Adj} \qquad \cfrac{}{\Gamma \vdash \bigwedge_{k>0}(A \rightarrow B_k) \rightarrow (A \rightarrow \bigwedge_{k>0} B_k)}\,\mathrm{C3}}{\Gamma \vdash A \rightarrow \bigwedge_{k>0} B_k}\,\mathrm{MP}$$

qed.

Definition 6.7.8. Let $\Gamma \subseteq \mathsf{FM}^{\Box}_\omega$ be given, then:

1. Γ is consistent if $\Gamma \nvdash_{S4_\omega} \bot$.
2. Γ is saturated if:

(a) Γ is consistent.

(b) $\Gamma \cup \neg\Gamma = \mathcal{E}(\Gamma)$ where $\neg\Gamma = \{\neg A \mid A \in \Gamma\}$.

(c) If $\bigwedge_{k>0} A_k \in \mathcal{E}(\Gamma)$, then: if for every $k > 0$ $A_k \in \Gamma$, then $\bigwedge_{k>0} A_k \in \Gamma$.

We start proving some properties of saturated sets.

Lemma 6.7.5. *Let Γ be a saturated set of formulas, then the following statements hold:*

1. *For every $A \in \mathcal{E}(\Gamma)$, either $A \in \Gamma$ or $\neg A \in \Gamma$ and not both.*

2. *For every $A \in \mathcal{E}(\Gamma)$: if $\Gamma \vdash_{S4_\omega} A$, then $A \in \Gamma$.*

3. *If $\bigwedge_{k>0} A_k \in \mathcal{E}(\Gamma)$, then $A_k \in \Gamma$ for all $k > 0$ if and only if $\bigwedge_{k>0} A_k \in \Gamma$.*

4. *If $\bigvee_{k>0} A_k \in \mathcal{E}(\Gamma)$, then $A_k \in \Gamma$ for some $k > 0$ if and only if $\bigvee_{k>0} A_k \in \Gamma$.*

Proof. 1. Let us suppose that $\neg A \notin \Gamma$. If $\neg A \notin \Gamma$, then, since $\neg A \in \mathcal{E}(\Gamma)$, we have that $\neg A \in \neg\Gamma$. Therefore, by definition of saturated set, we get $A \in \Gamma$. Furthermore, it cannot be the case that $A, \neg A \in \Gamma$, otherwise by modus ponens we would get $\Gamma \vdash_{S4_\omega} \bot$, against the definition of saturated sets.

2. Let us assume that $\Gamma \vdash_{S4_\omega} A$. We observe that by 1. either $A \in \Gamma$ or $\neg A \in \Gamma$. In the first case we obtain the desired conclusion. In the second case we observe that Γ would be inconsistent, against the definition of saturated set.

3. From left to right we observe that the proof easily follows by the definition of saturated set. From right to left we observe that if $\bigwedge_{k>0} A_k \in \Gamma$, then by the axiom $\bigwedge_{k>0} A_k \to A_k$ and modus ponens we obtain $A_k \in \Gamma$ for every $k > 0$.

4. From left to right the claim easily follows. From right to left we assume that $\bigvee_{k>0} A_k \in \Gamma$ and we suppose that for every $k > 0$ $A_k \notin \Gamma$, thus $\neg A_k \in \Gamma$ for every $k > 0$. By Adj we obtain $\Gamma \vdash \bigwedge_{k>0} (A_k \to \bot)$. By applying modus ponens to $\Gamma \vdash \bigwedge_{k>0} (A_k \to \bot)$ and the instance $\bigwedge_{k>0} (A_k \to \bot) \to (\bigvee_{k>0} A_k \to \bot)$ of the axiom C4, we get $\Gamma \vdash \bigvee_{k>0} A_k \to \bot$, therefore Γ is inconsistent, which is against the definition of saturated sets.

qed.

We now proceed in the standard fashion, by proving the Lindenbaum lemma.

Lemma 6.7.6. *Given a countable set of formulas* Γ*, if* Γ *is consistent, then for every* $A \in \mathcal{E}(\Gamma)$*,* $\Gamma \cup \{A\}$ *or* $\Gamma \cup \{\neg A\}$ *is consistent.*

Proof. Let us suppose that Γ is consistent and that $\Gamma \cup \{A\}$ and $\Gamma \cup \{\neg A\}$ are inconsistent, then it is easy to obtain, by the deduction theorem and the modus ponens rule, that Γ is inconsistent. *qed.*

Lemma 6.7.7. *Let* Γ *be a countable consistent set of formulas, then there is a saturated set* Δ *and* $\Delta \supseteq \Gamma$.

Proof. We enumerate $\mathcal{E}(\Gamma)$ and construct the following sequence of sets:

- $\Gamma_0 = \Gamma$
- $\Gamma_{n+1} =$
  - $\Gamma \cup \{A_n\}$ if $\Gamma \cup \{A_n\}$ is consistent.
  - $\Gamma \cup \{\neg A_n\}$ if $\Gamma \cup \{\neg A_n\}$ is consistent and $A_n \not\equiv \bigwedge_{k>0} B_k$.
  - $\Gamma \cup \{\neg \bigwedge_{k>0} B_k, \neg B_k\}$ for some $k > 0$ if $A_n \equiv \bigwedge_{k>0} B_k$ and $\Gamma \cup \{\neg A_n\}$ is consistent, where $\Gamma \cup \{\neg \bigwedge_{k>0} B_k, \neg B_k\}$ is consistent.

We show that such B_k always exists. Suppose otherwise, hence for every $k > 0$ we have $\Gamma, \neg \bigwedge_{k>0} B_k, \neg B_k \vdash_{S4_\omega} \bot$. Thus by the deduction theorem we get $\Gamma, \neg \bigwedge_{k>0} B_k \vdash_{S4_\omega} \neg\neg B_k$ and, by propositional classical reasoning, we get $\Gamma, \neg \bigwedge_{k>0} B_k \vdash_{S4_\omega} B_k$. An application of Adj yields $\Gamma, \neg \bigwedge_{k>0} B_k \vdash_{S4_\omega} \bigwedge_{k>0} B_k$. As a result, we obtain $\Gamma, \neg \bigwedge_{k>0} B_k \vdash_{S4_\omega} \bot$, against the hypothesis.

We now set $\Delta = \bigcup_{n \geqslant 0} \Gamma_n$ and it is immediate to prove that Δ is a saturated set. In fact, Δ is consistent and clearly $\Delta \cup \neg\Delta = \mathcal{E}(\Delta) = \mathcal{E}(\Gamma)$. Let us suppose that $\bigwedge_{k>0} A_k \in \mathcal{E}(\Delta)$ and that $A_k \in \Delta$ for every $k > 0$. If $\bigwedge_{k>0} A_k \notin \Delta$, then $\neg \bigwedge_{k>0} A_k \in \Delta$ by construction, therefore, again by construction, we have $\neg A_k \in \Delta$ for some $k > 0$, against the consistency of Δ. *qed.*

We now proceed with the standard construction of a canonical neighborhood model in order to complete our completeness proof. The proof is mostly standard and follows the strategy detailed in (79). We introduce the following notation: $[\![A]\!] = \{x \in W \,|\, x \Vdash A\}$ where W is the set of worlds of a given neighborhood model and $|A| = \{\Delta \ saturated \,|\, A \in \Delta\}$. As usual, the definitions are calibrated in order to build a bridge between syntax and semantics. We now prove some further properties of saturated sets.

Lemma 6.7.8. *The following statements hold:*

1. $|\bot| = \emptyset$
2. $|A \wedge B| = |A| \cap |B|$.
3. $|A \vee B| \subseteq |A| \cup |B|$. *For every* $\Gamma \in |A| \cup |B|$*, if* $A \vee B \in \mathcal{E}(\Gamma)$*, then* $\Gamma \in |A \vee B|$.
4. $|\bigwedge_{k>0} A_k| \subseteq \bigcap_{k>0} |A_k|$. *For every* $\Gamma \in \bigcap_{k>0} |A_k|$*, if* $\bigwedge_{k>0} A_k \in \mathcal{E}(\Gamma)$*, then* $\Gamma \in |\bigwedge_{k>0} A_k|$
5. $|\bigvee_{k>0} A_k| \subseteq \bigcup_{k>0} |A_k|$. *For every* $\Gamma \in \bigcup_{k>0} |A_k|$*, if* $\bigvee_{k>0} A_k \in \mathcal{E}(\Gamma)$*, then* $\Gamma \in |\bigvee_{k>0} A_k|$
6. *If* $|A| \subseteq |B|$ *then* $\vdash_{S4_\omega} A \rightarrow B$*. If* $\vdash_{S4_\omega} A \rightarrow B$*,* $\Gamma \in |A|$ *and* $A \rightarrow B \in \mathcal{E}(\Gamma)$*, then* $\Gamma \in |B|$.

Proof. Straightforward by the definition of the calculus $\mathbf{S4}_\omega$ and the properties of saturated sets. We prove in detail only items 4., 5. and 6., since $1 - 3$ are immediate.

4. If $\Gamma \in |\bigwedge_{k>0} A_k|$, then Γ is saturated and $\bigwedge_{k>0} A_k \in \Gamma$. Since $\Gamma \vdash \bigwedge_{k>0} A_k \rightarrow A_k$ for every $k > 0$, by properties of saturated sets we have $A_k \in \Gamma$ for every $k > 0$, thus $\Gamma \in \bigcap_{k>0} |A_k|$. The other side of the statement is thus proved: let $\Gamma \in \bigcap_{k>0} |A_k|$ such that $\bigwedge_{k>0} A_k \in \mathcal{E}(\Gamma)$ be given. We have $A_k \in \Gamma$ for every $k > 0$ and thus, again by the properties of saturated sets, we have $\bigwedge_{k>0} A_k \in \Gamma$ and by definition $\Gamma \in |\bigwedge_{k>0} A_k|$.
5. If $\Gamma \in |\bigvee_{k>0} A_k|$, then $\bigvee_{k>0} A_k \in \Gamma$ and $\bigvee_{k>0} A_k \in \mathcal{E}(\Gamma)$. By the properties of saturation of Γ, there is $k > 0$ such that $A_k \in \Gamma$, therefore $\Gamma \in |A_k| \subseteq \bigcup_{k>0} |A_k|$. In the other direction we suppose $\Gamma \in \bigcup_{k>0} |A_k|$, so there is $k > 0$ such that $\Gamma \in |A_k|$. Thus $A_k \in \Gamma$ and $\bigvee_{k>0} A_k \in \mathcal{E}(\Gamma)$, since $\Gamma \vdash_{S4_\omega} A_k \rightarrow \bigvee_{k>0} A_k$, by properties of saturated sets we obtain $\bigvee_{k>0} A_k \in \Gamma$ and $\Gamma \in |\bigvee_{k>0} A_k|$.
6. Let us suppose $|A| \subseteq |B|$ and $\nvdash_{S4_\omega} A \rightarrow B$, then $\{A \wedge \neg B\}$ is consistent. There is a saturated set Γ such that $A \wedge \neg B \in \Gamma$, thus $\Gamma \in |A|$ and $\Gamma \notin |B|$, but by hypothesis $|A| \subseteq |B|$, so $\Gamma \in |B|$, a contradiction. For the second part of the proof, we assume that $\vdash_{S4_\omega} A \rightarrow B$ and $\Gamma \in |A|$, so $A \in \Gamma$ and $\Gamma \vdash_{S4_\omega} B$. Since $B \in \mathcal{E}(\Gamma)$, by the properties of saturated sets, we have $B \in \Gamma$ and $\Gamma \in |B|$.

qed.

Definition 6.7.9. The canonical neighborhood model for $\mathbf{S4}_\omega$ is a triple

$$\mathcal{M}^c = \langle MAX, N, v \rangle$$

where:

- $MAX = \{\Gamma \subseteq \mathsf{FM}^{\Box}_{\omega} \mid |\Gamma| \leqslant \aleph_0,\ saturated\}$
- $N : MAX \longrightarrow \mathcal{P}(\mathcal{P}(MAX))$ such that for every $\Gamma \in MAX$,

$$N(\Gamma) = \{X \subseteq MAX \mid exists\, A\ (\,\Box A \in \Gamma\ and\ |A| \subseteq X)\,\}$$

- $v : AT \longrightarrow \mathcal{P}(MAX)$ such that $v(p) = \{\Gamma \in MAX \mid p \in \Gamma\}$.

We now need to check that the neighborhood function N is well defined and that the canonical model is an **S4** model. We start by showing that the function N is well defined. In particular, we need to show that given two sets $|A|$ and $|B|$, with $|A| = |B|$ and $|A| \in N(\Gamma)$, it is never the case that $\neg\Box B \in \Gamma$ (79). If B is not in $\mathcal{E}(\Gamma)$, then $\neg\Box B \notin \mathcal{E}(\Gamma)$ and thus $\neg\Box B \notin \Gamma$. The other case is discussed in the following lemma.

Lemma 6.7.9. *If* $|A| \in N(\Gamma)$*,* $|A| = |B|$ *and* $B \in \mathcal{E}(\Gamma)$*, then* $\Box B \in \Gamma$*.*

Proof. Let us suppose that $|A| \in N(\Gamma)$, $|A| = |B|$ and $B \in \mathcal{E}(\Gamma)$. By definition of N there is $|C|$ such that $\Box C \in \Gamma$ and $|C| \subseteq |A|$, from which we conclude that $\vdash_{S4_\omega} C \rightarrow A$. By applying the rule of necessitation and the rule of modus ponens with a suitable instance of axiom K we get $\vdash_{S4_\omega} \Box C \rightarrow \Box A$. Furthermore, by $|A| = |B|$ we get $\vdash_{S4_\omega} \Box A \leftrightarrow \Box B$ and by pure implicational reasoning we conclude that $\Gamma \vdash_{S4_\omega} \Box B$, which yields $\Box B \in \Gamma$. *qed.*

Lemma 6.7.10. *For every countable saturated set* Γ *and every formula* $A \in \mathcal{E}(\Gamma)$*:*

$$\Box A \in \Gamma \textit{ if and only if } |A| \in N(\Gamma)$$

Proof. From left to right the proof is immediate. From right to left we suppose $|A| \in N(\Gamma)$, then there is $|B| \in N(\Gamma)$ such that $|B| \subseteq |A|$ and $\Box B \in \Gamma$ by definition of N, therefore $B \in \mathcal{E}(\Gamma)$. Since $\vdash B \rightarrow A$, we get $\vdash \Box B \rightarrow \Box A$. Hence, we obtain $\Gamma \vdash_{S4_\omega} \Box A$ by modus ponens, which yields $\Box A \in \Gamma$. *qed.*

We now show that our canonical model is actually an **S4** neighborhood model.

Theorem 6.7.11. *$\mathcal{M}^c = \langle MAX, N, v\rangle$ is an* **S4** *neighborhood model.*

Proof. We verify the conditions imposed on the neighborhood **S4** modal frames.

1. Let us suppose $X, Y \in N(\Gamma)$, then by definition there are $A, B \in \mathsf{FM}^{\Box}_{\omega}$ such that $|A| \subseteq X$ and $|B| \subseteq Y$ and $\Box A \in \Gamma, \Box B \in \Gamma$. We have $\Box(A \wedge B) \in \mathcal{E}(\Gamma)$ and so since $\Gamma \vdash \Box A \to (\Box B \to \Box(A \wedge B))$, we conclude that $\Box(A \wedge B) \in \Gamma$. Furthermore $|A \wedge B| = |A| \cap |B|$ and $|A| \cap |B| \subseteq X \cap Y$, therefore $X \cap Y \in N(\Gamma)$.

2. If $X \in N(\Gamma)$ and $X \subseteq Y$, then there is A such that $|A| \subseteq X \subseteq Y$ and $\Box A \in \Gamma$, therefore we immediately obtain $Y \in N(\Gamma)$.

3. For every $\Gamma \in MAX$, $\bot \to \bot \in \Gamma$, therefore we also have $\Box(\bot \to \bot) \in \Gamma$. Since $|\bot \to \bot| = MAX$, we have $MAX \in N(\Gamma)$.

4. For every $X \in N(\Gamma)$ and every $\Gamma \in MAX$, there is $|A| \subseteq X$ and $\Box A \in \Gamma$. Since $\vdash_{S4_{\omega}} \Box A \to A$ we have $A \in \Gamma$, therefore $\Gamma \in |A|$, thus $\Gamma \in X$.

5. For every X and every $\Gamma \in MAX$, if $X \in N(\Gamma)$, then there is $|A| \subseteq X$ and $\Box A \in \Gamma$. Since $\vdash_{S4_{\omega}} \Box A \to \Box\Box A$ and $\Box\Box A \in \mathcal{E}(\Gamma)$, we have $\Box\Box A \in \Gamma$. We claim that $|\Box A| \subseteq m(X) = \{\Sigma \in MAX \mid X \in N(\Sigma)\}$, which immediately yields that $m(X)$ is in $N(\Gamma)$. Let $\Sigma \in |\Box A|$, then $\Box A \in \Sigma$, but $|A| \subseteq X$ by hypothesis, hence $X \in N(\Sigma)$ and this proves the claim.

qed.

Lemma 6.7.12. *Given the canonical model $\mathcal{M}^c$, for every $\Gamma \in MAX$, for every $A \in \mathcal{E}(\Gamma)$:*

$$\Gamma \in [\![A]\!] \textit{ if and only if } \Gamma \in |A|$$

Proof. The proof is by transfinite induction on the complexity of A. The case in which A is atomic is immediate by the definition of v. Cases $\wedge, \vee, \supset$ are routine via induction hypothesis.

If A is $\bigwedge_{k>0} B_k$, $\Gamma \in [\![\bigwedge_{k>0} B_k]\!]$, then $\Gamma \Vdash A_k$ for every $k > 0$. By induction hypothesis we have $\Gamma \in |B_k|$ for every $k > 0$, i.e. $\Gamma \in \bigcap_{k>0} |B_k|$ and since $\bigwedge_{k>0} B_k \in \mathcal{E}(\Gamma)$ by hypothesis, we obtain $\Gamma \in |\bigwedge_{k>0} B_k|$. In the other direction we assume that $\Gamma \in |\bigwedge_{k>0} B_k| \subseteq \bigcap_{k>0} |B_k|$, therefore $\Gamma \in |B_k|$ for every $k > 0$. By induction hypothesis we have $\Gamma \in [\![B_k]\!]$ for every $k > 0$ and thus $\Gamma \in [\![\bigwedge_{k>0} B_k]\!]$.

If A is $\bigvee_{k>0} B_k$, observe that $[\![\bigvee_{k>0} B_k]\!] = \bigcup_{k>0} [\![B_k]\!]$, so $\Gamma \in \bigcup_{k>0} [\![B_k]\!]$. This is equivalent to $\Gamma \in \bigcup_{k>0} |B_k|$ if and only if $\Gamma \in |\bigvee_{k>0} B_k|$ (notice that in the direction from left to right we make use of the fact that $\bigvee_{k>0} B_k \in \mathcal{E}(\Gamma)$).

The case in which A is $\Box B$ goes as follows. $\Gamma \in [\![\Box B]\!]$ if and only if $[\![B]\!] \in N(\Gamma)$ if and only if (by induction hypothesis) $|B| \in N(\Gamma)$ if and only if $\Box B \in \Gamma$ if and only if $\Gamma \in |\Box B|$. *qed.*

We are now in the position to state and prove the completeness theorem for $\mathbf{S4}_\omega$.

Theorem 6.7.13. *For every $A \in \mathsf{FM}^{\Box}_{\omega}$ we have:*

$$\vdash_{S4_\omega} A \text{ if and only if } \vDash_{S4_\omega} A.$$

Proof. The direction from left to right is the content of the soundness theorem. For the direction from right to left we proceed by contraposition. We suppose $\nvdash_{S4_\omega} A$, thus $\{\neg A\}$ is a consistent set. Therefore there exists $\Gamma \in MAX$ such that $\neg A \in \Gamma$. We have $\Gamma \in |\neg A|$ if and only if $\Gamma \in [\![\neg A]\!]$ and thus $\Gamma \nVdash A$ in the canonical model. *qed.*

6.8 The labelled sequent calculus G3S4$_\omega$

We now introduce a labelled sequent calculus based on neighborhood semantics for infinitary **S4** modal logic.

The relational rules directly stem from the conditions imposed on **S4** neighborhood frames. However, before we proceed we shall reformulate some semantic conditions in order to extract simpler rules.

The unit condition can be streamlined. In fact it can be easily shown, by exploiting closure under supersets, that the following lemma holds as before.

Lemma 6.8.1. *Let $\mathcal{N} = \langle W, N \rangle$ be a neighborhood frame for infinitary* **S4** *logic. Condition (3) of Definition 6.7.3, i.e. $W \in N(x)$, is equivalent to $N(x) \neq \emptyset$.*

Proof. Immediate. *qed.*

The above lemma justifies the rule *Ndeg*:

$$\frac{\alpha \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{Ndeg},\ \alpha \text{ fresh}$$

Lemma 6.8.2. *For every infinitary* **S4** *frame the following are equivalent:*

1. *If $a \in N(x)$ and $b \in N(x)$, then $a \cap b \in N(x)$*

2. *If* $a \in N(x)$ *and* $b \in N(x)$*, then there is* $c \in N(x)$ *and* $c \subseteq a$ *and* $c \subseteq b$

Proof. Immediate. *qed.*

The calculus $\mathbf{G3S4}_\omega$ is an extension of the labelled sequent calculi for non-normal modal logics based on neighborhood semantics, thus we limit ourselves to listing the usual structural properties that the calculus enjoys. A derivation in $\mathbf{G3S4}_\omega$ is a tree which is possibly infinitely branching and every branch is of finite length. Given a derivation $\mathcal{D}$, its height $h(\mathcal{D})$ is thus defined:

1. If $\mathcal{D}$ is an initial sequent, then $h(\mathcal{D}) = 0$

2. If $\mathcal{D}$ is of the form:

$$\dfrac{\dots \quad \begin{array}{c} \vdots\, \mathcal{D}_n \\ \Gamma_n \Rightarrow \Delta_n \end{array} \quad \dots}{\Gamma \Rightarrow \Delta}$$

   with possibly countable premises, then $h(\mathcal{D}) = sup_n(h(\mathcal{D}_n)) + 1$, where the latter is a countable ordinal.

The measures of complexity for the labelled formulas of infinitary modal logic are essentially the same as those for infinitary intuitinoistic logic and thus we omit the details.

Lemma 6.8.3. *The rules*

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma[x/y] \Rightarrow \Delta[x/y]}\, {\scriptstyle Sub\,[x/y]} \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma[\alpha/\beta] \Rightarrow \Delta[\alpha/\beta]}\, {\scriptstyle Sub\,[\alpha/\beta]}$$

are height-preserving admissible in $\mathbf{G3S4}_\omega$.

Proof. By simultaneous transfinite induction on the height of the derivation in $\mathbf{G3S4}_\omega$. *qed.*

The structural rule of weakening is admissible as well.

Lemma 6.8.4. *The rule:*

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\, {\scriptstyle Weak}$$

is height-preserving admissible for every multiset Γ', Δ'.

Proof. By transfinite induction on the height of the derivation. The proof is straightforward, we apply the substitution lemma in order to avoid clashes of variables whenever the last rule has a freshness condition. *qed.*

Initial Sequents

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p} \; Ax \qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta} \; L\bot$$

Logical Rules

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta} \; L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B} \; R\wedge$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta} \; L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B} \; R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta} \; L\to \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B} \; R\to$$

$$\frac{x : \bigwedge_{k>0} A_k, x : A_k, \Gamma \Rightarrow \Delta}{x : \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta} \; L\bigwedge \qquad \frac{\{\Gamma \Rightarrow \Delta, x : A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k} \; R\bigwedge$$

$$\frac{\{x : A_k, \Gamma \Rightarrow \Delta \mid k > 0\}}{x : \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta} \; L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, x : \bigvee_{k>0} A_k, x : A_k}{\Gamma \Rightarrow \Delta, x : \bigvee_{k>0} A_k} \; R\bigvee$$

$$\frac{a \in N(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta} \; L\Box, a \text{ fresh} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A}{a \in N(x), \Gamma \Rightarrow \Delta, x : \Box A} \; R\Box$$

Relational rules

$$\frac{x \in a, a \Vdash^{\forall} A, x : A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \; L\Vdash^{\forall} \qquad \frac{y \in a, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A} \; R\Vdash^{\forall}, y \text{ fresh}$$

$$\frac{a \in N(x), x \in a, \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta} \; Ref \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; Ndeg, a \text{ fresh}$$

$$\frac{a \in N(x), m(a) \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta} \; Trs \qquad \frac{x \in m(a), a \in N(x), \Gamma \Rightarrow \Delta}{x \in m(a), \Gamma \Rightarrow \Delta} \; Lm$$

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; Ref\subseteq \qquad \frac{a \subseteq b, b \subseteq c, a \subseteq c, \Gamma \Rightarrow \Delta}{a \subseteq b, b \subseteq c, \Gamma \Rightarrow \Delta} \; Trs\subseteq$$

$$\frac{c \in N(x), c \subseteq a, c \subseteq b, a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta} \; Prebasic, c \text{ fresh}$$

Figure 6.4: The labelled sequent calculus $\mathbf{G3S4}_{\omega}$.

A rule is invertible whenever the derivability of the conclusion entails the derivability of the premise(s). As opposed to the unlabelled calculi for modal logics, labelled systems ensure height-preserving invertibility of every rule.

Lemma 6.8.5. *The calculus* **G3S4**$_\omega$ *enjoys height preserving invertibility of every rule.*

Proof. The proof is by transfinite induction on the height of the derivation: the structure is analogous to the one detailed in (75). We limit ourselves to discussing the case of rule $L\Box$. If $n = 0$, then $\Gamma, x : \Box A \Rightarrow \Delta$ is an initial sequent and so is $\Gamma, a \Vdash^{\forall} A, a \in N(x) \Rightarrow \Delta$. If $n > 0$ and $x : \Box A$ is principal, we simply take the premise. If $n > 0$ and $x : \Box A$ is not principal we apply the induction hypothesis to the premise(s) of the rule and then we apply the rule again. *qed.*

The rules of contraction are height-preserving admissible as well.

Lemma 6.8.6. *The rules:*

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\,LC \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\,RC$$

are height-preserving admissible.

Proof. The proof is by simultaneous induction on the height of the derivation. The general structure is as follows. If the derivation is of height 0, then the proof is trivial. If ϕ is not principal in the last rule applied or if ϕ is principal and is repeated in the premise(s), then apply the induction hypothesis and the rule again. In the remaining cases, i.e. ϕ is principal and is not repeated in the premise(s), apply height-preserving invertibility of the rule, then apply the induction hypothesis to every rule and apply the rule again to obtain the desired conclusion. We limit ourselves to discussing the case of infinitary conjunction as an example. Suppose we have a derivation of $\Gamma \Rightarrow \Delta, x : \bigwedge_k A_k, x : \bigwedge_k A_k$. If $h = 0$, then $x : \bigwedge_k A_k$ is not principal and it can be removed. If $h > 0$ and $x : \bigwedge_k A_k$ is not principal, then we apply the induction hypothesis to the premise(s) and then the rule again. If $h > 0$ is principal we have the following situation:

$$\frac{\{\Gamma \Rightarrow \Delta, x : \bigwedge_k A_k, x : A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, x : \bigwedge_k A_k, x : \bigwedge_k A_k}\,\mathrm{R}\bigwedge$$

For every $k > 0$ we apply the invertibility lemma to obtain $\Gamma \Rightarrow \Delta, x : A_k, x : A_k$, then we apply the induction hypothesis to get $\Gamma \Rightarrow \Delta, x : A_k$ and we conclude the proof by applying the rule $\mathrm{R}\bigwedge$. *qed.*

We can now state and prove the main syntactic property of the system **G3S4**$_\omega$, namely cut elimination.

Theorem 6.8.7. *The rule:*

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

is admissible in **G3S4**$_\omega$.

Proof. We proceed by double transfinite induction with primary induction hypothesis on the degree of the cut formula and secondary induction hypothesis on the natural sum, denoted by #, of the height of the premises of cut. The natural ordinal sum has the following two useful properties: $\alpha\#\beta = \beta\#\alpha$ and if $\alpha < \alpha'$, then $\alpha\#\beta < \alpha'\#\beta$ (100; 99; 72). We distinguish cases: if one of the two premise is an initial sequent, if the cut formula is principal in both premises of the cut and if the cut formula is not principal in at least one premise of the cut. The first case is straightforward.

With respect to the case in which the cut formula is principal in both premises of the cut we limit ourselves to discussing the case in which it is of the form $\Box A$. We have:

$$\dfrac{\dfrac{a \in N(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A}{a \in N(x), \Gamma \Rightarrow \Delta, x : \Box A}\ R\Box \qquad \dfrac{b \in N(x), b \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}{x : \Box A, \Gamma' \Rightarrow \Delta'}\ L\Box}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

In this case we construct the following derivation:

$$\dfrac{\dfrac{\dfrac{a \in N(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A \qquad x : \Box A, \Gamma' \Rightarrow \Delta'}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta, a \Vdash^{\forall} A}\ Cut \qquad \dfrac{b \in N(x), b \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}{a \in N(x), a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}\ Sub[b/a]}{a \in N(x), a \in N(x), \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}\ Cut}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Ctr$$

where the topmost cut is removed by secondary induction hypothesis and the lower cut is removed by primary induction hypothesis on the degree of the cut formula.

If the cut formula is not principal, the strategy consists in permuting the cut upwards and then removing it by invoking the secondary induction hypothesis. For example, let us consider the case in which the cut formula is not principal in the left premise and the last rule applied is $R\bigwedge$:

$$\dfrac{\dfrac{\{\Gamma \Rightarrow \Delta, x : A_k, \phi \mid k > 0\}}{\Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k, \phi} \qquad \phi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x : \bigwedge_{k>0} A_k}\ Cut$$

For every $k > 0$, we apply a cut between $\Gamma \Rightarrow \Delta, x : A_k, \phi$ and $\phi, \Gamma' \Rightarrow \Delta'$ which is removed by secondary induction hypothesis and we obtain $\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x : A_k$, we then apply rule $R \bigwedge$ to obtain the desired conclusion. *qed.*

Corollary. *If* $\vdash_{S4_\omega} A$*, then* **G3S4**$_\omega \vdash \Rightarrow x : A$.

Proof. We argue by induction on the height of the derivation in the calculus **S4**$_\omega$. Every axiom is derivable and the rules are admissible. In particular, modus ponens is admissible by invertibility of R$\rightarrow$ and admissibility of Cut. *qed.*

Corollary. *The calculus* **G3S4**$_\omega$ *is sound and complete with respect to the neighborhood semantics for* **S4**$_\omega$.

Proof. Soundness is proved by induction on the height of derivations. With respect to completeness we observe that if $\vDash_{S4_\omega} A$, then $\vdash_{S4_\omega} A$ and thus, by Corollary 6.8, **G3S4**$_\omega \vdash \Rightarrow x : A$. *qed.*

Remark. The above corollary shows that when restricted to formulas and to sequents of the shape $x : \Gamma \Rightarrow x : \Delta$ (where Γ and Δ are multisets of formulas) the derivability in the axiomatic system and in the labelled sequent calculus coincide. The labelled sequent calculus has a richer language which properly extends the language of the logic. This has a number of advantages: every rule is now height-preserving invertible and working in a calculus with good analytic properties makes structural analysis available.

6.9 An infinitary extension of the Gödel-McKinsey-Tarksi embedding

In this section we discuss the extension of the embedding of infinitary intuitionistic logic into infinitary **S4** modal logic. The proof that we present exploits the labelled sequent calculi for both systems and proceeds by transfinite induction on the height of derivations in both directions. The natural (or naive) extension to the full language of infinitary intuitionistic is obtained by adding the two following conditions for the infinitary conjunction and disjunction:

- $(\bigwedge_k F_k)^* = \bigwedge_k F_k^*$
- $(\bigvee_k F_k)^* = \bigvee_k F_k^*$

We now can prove the claim concerning the failure of the faithfulness of the natural extension of the embedding $*$.

Theorem 6.9.1. *There is a formula $A \in \mathsf{FM}_\omega$ such that* $\mathbf{G3S4}_\omega \vdash\Rightarrow x : A^*$ *and* $\mathbf{G3I}_\omega \nvdash\Rightarrow x : A$.

Proof. Let us consider the infinitary distributive law: $\bigwedge\limits_{k>0}(p_k \lor q) \to \bigwedge\limits_{k>0} p_k \lor q$. Its natural translation would be:

$$\Box(\bigwedge_{k>0}(\Box p_k \lor \Box q) \to \bigwedge_{k>0}\Box p_k \lor \Box q)$$

We have that $\mathbf{G3S4} \vdash\Rightarrow x : \Box(\bigwedge\limits_{k>0}(\Box p_k \lor \Box q) \to \bigwedge\limits_{k>0}\Box p_k \lor \Box q)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\{x : \bigwedge\limits_{k>0}(\Box p_k \lor \Box q), x : \Box p_k \lor \Box q \Rightarrow x : \Box p_k, x : \Box q \mid k > 0\}}{\{x : \bigwedge\limits_{k>0}(\Box p_k \lor \Box q) \Rightarrow x : \Box p_k, x : \Box q \mid k > 0\}}\ L\bigwedge}{x : \bigwedge\limits_{k>0}(\Box p_k \lor \Box q) \Rightarrow x : \bigwedge\limits_{k>0}\Box p_k, x : \Box q}\ R\bigwedge}{x : \bigwedge\limits_{k>0}(\Box p_k \lor \Box q) \Rightarrow x : \bigwedge\limits_{k>0}\Box p_k \lor \Box q}\ R\lor}{\Rightarrow x : \Box(\bigwedge\limits_{k>0}(\Box p_k \lor \Box q) \to \bigwedge\limits_{k>0}\Box p_k \lor \Box q)}\ \text{Admissible Rule}$$

The topmost sequent is clearly derivable. Were the natural translation faithful, we would have $\mathbf{G3I}_\omega \vdash\Rightarrow x : \bigwedge\limits_{k>0}(p_k \lor q) \to \bigwedge\limits_{k>0} p_k \lor q$, but the infinitary distributivity law is not valid in intuitionistic logic (103). *qed.*

We propose a modification of the $*$ embedding which changes the condition for the translation of the infinitary conjunction:

$$(\bigwedge_{k>0} B_k)^* = \Box(\bigwedge_{k>0} B_k^*)$$

It is now possible to show that the translation of the infinitary distributive law:

$$\Box(\Box(\bigwedge_{k>0}(\Box p_k \lor \Box q)) \to \Box(\bigwedge_{k>0}\Box p_k) \lor \Box q)$$

is not provable in $\mathbf{S4}_\omega$.

Lemma 6.9.2. *The translation of the formula $\bigwedge_{k>0}(p_k \lor q) \to \bigwedge_{k>0} p_k \lor q$ is not derivable in* $\mathbf{S4}_\omega$.

Proof. To show this impossibility we give a countermodel. Let $W = \mathbb{R}$ and $N(x) = \{a \mid \exists r > 0((x - r, x + r) \subseteq a)\}$. We define the valuation $v : AT \to \mathbb{R}$

such that $v(q) = \mathbb{R} \setminus \{0\}$ and $v(p_k) = (-\frac{1}{k}, \frac{1}{k})$ for every $k > 0$. We leave it to the reader to verify that this is an **S4** neighborhood model.

We observe that $\mathbb{R} \Vdash^{\forall} \Box \bigwedge_{k>0} (\Box p_k \vee \Box q)$ and $\mathbb{R} \in N(0)$, thus $0 \Vdash \Box \bigwedge_{k>0}(\Box p_k \vee \Box q)$. However, $\bigcap_{k>0} v(\Box p_k) = \{0\} \notin N(0)$, thus we can easily conclude that $0 \nVdash \Box(\bigwedge_{k>0} \Box p_k)$. Furthermore $0 \notin \mathbb{R} \setminus \{0\}$, hence $0 \nVdash \Box q$. Thus $0 \nVdash \Box(\bigwedge_{k>0} \Box p_k) \vee \Box q$, hence we have provided a countermodel. *qed.*

Therefore we have obtained a neighborhood countermodel for the $*$ translation of the infinitary distributivity axiom. We observe that the countermodel here discussed is identical to the neighborhood countermodel to the infinitary distributivity law for infinitary intuitionistic logic. Furthermore, it is worth observing that if we added closure under arbitrary intersections, i.e. if we considered neighborhood systems based on Alexandroff topologies, the formula could easily be shown to hold.

Remark. It is now a routine matter to verify that for every formula in the language of intuitionistic infinitary logic A, the following holds: $A^* \leftrightarrow \Box A^*$. This parallels the case of finitary intuitionistic and modal logic. This failed in the case of the naive translation, because: $(\bigwedge_{k>0} p_k)^* = \bigwedge_{k>0} \Box p_k$ is not equivalent to $\Box \bigwedge_{k>0} \Box p_k$. Also the new interpretation isolates the constructive features of infinitary intuitionistic logic through the use of an **S4**-like modality. In this way, the constructive content of the infinitary conjunction is clearly visible.

We now prove the faithfulness theorem with respect to neighborhood semantics.

Lemma 6.9.3. *Given a model of intuitionistic logic $\mathcal{M} = \langle W, N, v \rangle$, for every intuitionistic formula B, for every $x \in W$:*

$$x \Vdash_{Int} A \text{ if and only if } x \Vdash_{S4} A^*$$

Proof. We proceed by induction on the complexity of the formula B. If B is atomic, we have $x \Vdash_{Int} p$ if and only if there is $a \in N(x)$ and $a \Vdash^{\forall} p$ if and only if $x \Vdash_{S4} \Box p$.

If $B \equiv C \wedge D, C \vee D, \bigvee_{k>0} C_k$ the proof is immediate by induction hypothesis. If $B \equiv C \rightarrow D$, then if $x \Vdash_{Int} C \rightarrow D$ if and only if there is $a \in N(x)$ and $a \Vdash^{\forall} C \supset D$. Now it is sufficient to show, via induction hypothesis, that $a \Vdash^{\forall} C \supset D$ if and only if $a \Vdash^{\forall} C^* \rightarrow D^*$, hence we can conclude that $x \Vdash_{S4} \Box(C^* \rightarrow D^*)$ by the definition.

If $B \equiv \bigwedge_{k>0} C_k$, then $x \Vdash_{Int} \bigwedge_{k>0} C_k$ if and only if there is $a \in N(x)$ and $a \Vdash^{\forall} \&_{k>0} C_k$ if and only if $a \Vdash^{\forall} \bigwedge_{k>0} C^*_k$ if and only if $x \Vdash \Box \&_{k>0} C^*_k$. *qed.*

Theorem 6.9.4. *For every formula $A \in \mathsf{FM}_\omega$, if $\vDash_{S4} A^*$, then $\vDash_{Int} A$.*

Proof. We proceed by contraposition. If $\nvDash_{Int} A$, there is a neighborhood model $\mathcal{M} = \langle W, N, v \rangle$ for intuitionistic infinitary logic, a world $x \in W$ such that $x \nVdash A$. By Lemma 6.9.3 we obtain $x \nVdash A^*$ and this concludes the proof. *qed.*

The proof we have provided is semantic and thus, in a sense, indirect. In the remaining of the section we will give a proof-theoretic proof of the faithfulness of the translation by exploiting the structural properties of both the calculi $\mathbf{G3I}_\omega$ and $\mathbf{G3S4}_\omega$.

We first define an extension t of the $*$ translation with respect to the language of our labelled sequent calculus $\mathbf{G3I}_\omega$. In particular we supplement the definition of the following two clauses for the added connectives:

$$(A \supset B)^t = A^t \rightarrow B^t \text{ and } (\underset{k>0}{\&} B_k)^t = \bigwedge_{k>0} B^t_k$$

We then extend the translation to the labelled syntax in a way that acts only on the pure part of the sequents.

- $(a \Vdash^\forall B)^t = a \Vdash^\forall B^t$
- $(x : B)^t = x : B^t$
- $(x \in a)^t = x \in a$
- $(a \in N(x))^t = a \in N(x)$

We first prove prove the soundness lemma.

Theorem 6.9.5. *If $\mathbf{G3I}_\omega \vdash \Gamma \Rightarrow \Delta$, then $\mathbf{G3S4}_\omega \vdash \Gamma^t \Rightarrow \Delta^t$.*

Proof. We proceed by induction on the height of derivation in $\mathbf{G3I}_\omega$. If $n = 0$ this is immediate.

If $n > 0$ and the last rule is a rule for connectives $\wedge, \vee, \supset, \bigvee, \&, R \Vdash^\forall, L \Vdash^\forall$ or a relational rule it is straightforward by induction hypothesis.

If $n > 0$ and the last rule is *Mon* we have the following situation:

$$\frac{x : p, a \in N(x), a \Vdash^\forall p, \Gamma \Rightarrow \Delta}{x : p, \Gamma \Rightarrow \Delta}\ \textit{Mon},\ a \text{ fresh}$$

The induction hypothesis yields a derivation of $x : \Box p, a \in N(x), a \Vdash^\forall \Box p, \Gamma^t \Rightarrow \Delta^t$. We construct the following derivation:

$$\cfrac{x : \Box p \Rightarrow x : \Box\Box p \qquad \cfrac{x : \Box p, a \in N(x), a \Vdash^{\forall} \Box p, \Gamma^t \Rightarrow \Delta^t}{x : \Box p, x : \Box\Box p, \Gamma^t \Rightarrow \Delta^t}\, L\Box}{\cfrac{x : \Box p, x : \Box p, \Gamma^t \Rightarrow \Delta^t}{x : \Box p, \Gamma^t \Rightarrow \Delta^t}\, Ctr}\, Cut$$

Where the topmost sequent $x : \Box p \Rightarrow x : \Box\Box p$ on the left is derivable in $\mathbf{G3S4}_\omega$.

If $n > 0$ and the last rule applied is $L \to, L \bigwedge$ we limit ourselves to considering the second case:

$$\frac{a \in N(x), a \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta}{x : \bigwedge\limits_{k>0} A_k, \Gamma \Rightarrow \Delta}\, L \bigwedge, a \text{ fresh}$$

We apply the induction hypothesis to the premise and obtain $a \in N(x), a \Vdash^{\forall} \bigwedge\limits_{k>0} A^t_k, \Gamma^t \Rightarrow \Delta^t$; an application of $L\Box$ yields the desired conclusion, namely $x : \Box \bigwedge\limits_{k>0} A^t_k, \Gamma^t \Rightarrow \Delta^t$.

If $n > 0$ and the last rule applied is $R \to, R \bigwedge$ we deal with the second case:

$$\frac{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge\limits_{k>0} A_k, a \Vdash^{\forall} \underset{k>0}{\&} A_k}{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge\limits_{k>0} A_k}\, R \bigwedge$$

We apply the induction hypothesis to the premise and obtain $a \in N(x), \Gamma^t \Rightarrow \Delta^t, x : \Box \bigwedge\limits_{k>0} A^t_k, a \Vdash^{\forall} \bigwedge\limits_{k>0} A^t_k$; an application of $R\Box$ gives $a \in N(x), \Gamma^t \Rightarrow \Delta^t, x : \Box \bigwedge\limits_{k>0} A^t_k$. *qed.*

We now are ready to prove the main lemma, that will make essential use of a suitable strengthening of the induction hypothesis.

Lemma 6.9.6. *Let Γ, Δ be multisets of labelled formulas, Γ' a multiset of relational atoms and of labelled formulas of the form $a \Vdash^{\forall} p$, Δ' a multiset of labelled formulas, Ω a multiset of relational atoms:*

$$\textit{if } \mathbf{G3S4}_\omega \vdash \Omega, \Gamma^t, \Gamma' \Rightarrow \Delta^t, \Delta', \textit{ then } \mathbf{G3I}_\omega \vdash \Omega, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$$

Proof. We proceed by induction on the height of derivation in $\mathbf{G3S4}_\omega$. If $n = 0$, then it is trivial.

If $n > 0$ and the last rule applied is a rule for connectives $\wedge, \vee, \to, \bigvee, \bigwedge$ or a relational rule the proof is straightforward by induction hypothesis.

If $n > 0$ and the last rule is $L\Box$ the principal formula can be of the form $x : \Box p, x : \Box(A^t \to B^t)$ or $x : \Box(\bigwedge\limits_{k>0} B^t_k)$. In the first case we have:

$$\frac{a \in N(x), \Omega, \Gamma^t, a \Vdash^{\forall} p, \Gamma' \Rightarrow \Delta^t, \Delta'}{\Omega, \Gamma^t, x : \Box p, \Gamma' \Rightarrow \Delta^t, \Delta'}\, L\Box$$

The induction hypothesis yields a derivation of $a \in N(x), \Omega, \Gamma, a \Vdash^{\forall} p, \Gamma' \Rightarrow \Delta, \Delta'$, we add $x : p$ to the antecedent via admissibility of weakening and we obtain the desired conclusion by Mon.

In the second case we have :

$$\dfrac{a \in N(x), \Omega, \Gamma^t, a \Vdash^{\forall} B^t \to C^t, \Gamma' \Rightarrow \Delta^t, \Delta'}{\Omega, \Gamma^t, x : \Box(B^t \to C^t), \Gamma' \Rightarrow \Delta^t, \Delta'}\ L\Box$$

The induction hypothesis yields a derivation of $a \in N(x), \Omega, \Gamma, a \Vdash^{\forall} B \supset C, \Gamma' \Rightarrow \Delta, \Delta'$ and thus an application of $L \to$ yields $\Omega, \Gamma, x : B \to C, \Gamma' \Rightarrow \Delta, \Delta'$. The case of $x : \Box(\bigwedge_{k>0} B^t_k)$ is analogous, so we omit the details.

If $n > 0$ and the last rule is $R\Box$ we have to discuss the same three subcases. If the principal formula is $x : \Box p$ we have:

$$\dfrac{a \in N(x), \Omega, \Gamma^t, \Gamma' \Rightarrow \Delta^t, x : \Box p, a \Vdash^{\forall} p, \Delta'}{a \in N(x), \Omega, \Gamma^t, \Gamma' \Rightarrow \Delta^t, x : \Box p, \Delta'}\ R\Box$$

By height-preserving invertibility of $R \Vdash^{\forall}$ we obtain a derivation of $a \in N(x), x \in a, \Omega, \Gamma^t, \Gamma' \Rightarrow \Delta^t, x : \Box p, x : p, \Delta'$. By induction hypothesis we obtain a derivation of $a \in N(x), x \in a, \Omega, \Gamma, \Gamma' \Rightarrow \Delta, x : p, x : p, \Delta'$ and by admissibility of contraction we obtain $a \in N(x), x \in a, \Omega, \Gamma, \Gamma' \Rightarrow \Delta, x : p, \Delta'$. An application of Ref finally yields the desired conclusion.

If the principal formula is $x : \Box \bigwedge_{k>0} B^t_k$ we have:

$$\dfrac{a \in N(x), \Omega, \Gamma^t, \Gamma' \Rightarrow \Delta^t, x : \Box \bigwedge_{k>0} B^t_k, a \Vdash^{\forall} \bigwedge_{k>0} B^t_k, \Delta'}{a \in N(x), \Omega, \Gamma^t, \Gamma' \Rightarrow \Delta^t, x : \Box \bigwedge_{k>0} B^t_k, \Delta'}\ R\Box$$

By induction hypothesis we obtain a derivation of $a \in N(x), \Omega, \Gamma, \Gamma' \Rightarrow \Delta, x : \bigwedge_{k>0} B_k, a \Vdash^{\forall} \&_{k>0} B_k, \Delta'$ and we obtain the desired conclusion via an application of $R \bigwedge$. The case for $x : \Box(B^t \to C^t)$ is similar and we omit the details.

If the last rule applied is $L \Vdash^{\forall}$ then the principal formula is either $a \Vdash^{\forall} A^t$ or $a \Vdash^{\forall} p$. In the first case we consider the premise, we apply the induction hypothesis and conclude by $L \Vdash^{\forall}$. In the other case we have:

$$\dfrac{\Omega, x \in a, \Gamma^t, a \Vdash^{\forall} p, x : p, \Gamma' \Rightarrow \Delta^t, \Delta'}{\Omega, x \in a, \Gamma^t, a \Vdash^{\forall} p, \Gamma' \Rightarrow \Delta^t, \Delta'}\ L \Vdash^{\forall}$$

In this case we apply the induction hypothesis to the premise and obtain $\Omega, x \in a, \Gamma, a \Vdash^{\forall} p, x : p, \Gamma' \Rightarrow \Delta, \Delta'$ (this is possible due to the stronger induction hypothesis); then we obtain the desired conclusion by $L \Vdash^{\forall}$.

If the last rule applied is $R \Vdash^{\forall}$ we have:

$$\dfrac{\Omega, y \in a, \Gamma^t, \Gamma' \Rightarrow \Delta^t, \Delta', y : B^t}{\Omega, \Gamma^t, \Gamma' \Rightarrow \Delta^t, \Delta', a \Vdash^{\forall} B^t}\ R \Vdash^{\forall}$$

We apply the induction hypothesis to obtain $\Omega, y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', y : B$ and obtain the desired conclusion by $R \Vdash^{\forall}$. *qed.*

As a corollary we obtain that if $\mathbf{G3S4}_\omega \vdash\Rightarrow x : A^t$, then $\mathbf{G3I}_\omega \vdash\Rightarrow x : A$, namely a proof of faithfulness of the embedding.

Corollary. *Given* $A \in \mathsf{FM}_\omega$, $\mathbf{G3I}_\omega \vdash\Rightarrow x : A$ *if and only if* $\mathbf{G3S4}_\omega \vdash\Rightarrow x : A^*$.

Proof. From left to right (soundness) we exploit Theorem 8.12. From right to left (faithfulness) we use Lemma 8.13 and we obtain $\mathbf{G3S4}_\omega \vdash\Rightarrow x : A^t$. But $t_{|\mathsf{FM}_\omega} = *$, i.e., the t translation restricted to the language of intuitionistic infinitary logic coincides with the $*$ translation, and thus $A^* \equiv A^t$ and this concludes the proof. *qed.*

6.10 Relating classical, intuitionistic and modal infinitary derivability

In this final section we relate derivability in classical, intuitionistic and modal infinitary logic extended with suitably formulated axioms. By $\mathbf{G3C}_\omega$ we denote a labelled infinitary sequent calculus for classical logic obtained from $\mathbf{G3S4}_\omega$ by removing the rules for the modal operator and all the auxiliary rules.

First, we identify a class of infinitary formulas where derivability coincides in $\mathbf{G3C}_\omega$, $\mathbf{G3I}_\omega$ and $\mathbf{G3S4}_\omega$.

Definition 6.10.1. A *simple implication* is a formula of the shape $p_1 \wedge \ldots \wedge p_m \rightarrow q$, where p_i and q are atomic propositional formulas. An *infinitary Horn formula* is an infinitary conjunction of simple implications.

Before proceeding, we consider the extension of the calculi obtained by the methodology of conversion of axioms into rules (**?**)[11]. An axiom of the shape $\bigwedge_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow q_k)$ is now converted into infinitely many rules:

$$\frac{x : p_{k,1}, \ldots, x : p_{k,m(k)}, x : q_k, \Gamma \Rightarrow \Delta}{x : p_{k,1}, \ldots, x : p_{k,m(k)}, \Gamma \Rightarrow \Delta}\ \text{Horn}_k$$

to add to a given infinitary labelled calculus.

It is a routine task to check that the structural properties of the calculi $\mathbf{G3C}_\omega$, $\mathbf{G3I}_\omega$ and $\mathbf{G3S4}_\omega$ are preserved. Indeed, weakening, contraction and cut remain

[11] Recently, the methodology of conversion of axioms into rules has been analyzed and studied through the lens of focussing in (64).

admissible and every rule is height-preserving invertible.Given a set of infinitary Horn formulas **T** denote by $\mathbf{G3XT}_\omega$, where $\mathbf{X} \in \{\mathbf{I}, \mathbf{S4}, \mathbf{C}\}$, the calculus obtained by adding to $\mathbf{G3X}_\omega$ the rules corresponding to the axioms in **T**.

Theorem 6.10.1. *The calculi* $\mathbf{G3CT}_\omega$*,* $\mathbf{G3IT}_\omega$ *and* $\mathbf{G3S4T}_\omega$*, where* **T** *is a set of infinitary Horn formulas, enjoy height-preserving invertibility of every rule and height preserving admissibility of the rules of weakening and contraction. The cut rule is admissible.*

Proof. Routine. *qed.*

Furthermore, it can be shown that given a set **T** of infinitary Horn axioms, the calculus $\mathbf{G3XT}_\omega$, where $\mathbf{X} \in \{\mathbf{S4}, \mathbf{C}, \mathbf{I}\}$, is equivalent to $\mathbf{G3X} \oplus \{\Rightarrow A \mid A \in \mathbf{T}\}$, where $\oplus$ denotes the addition of a set of initial sequents or rules to a base calculus.

Lemma 6.10.2. *Let* **T** *be a set of infinitary Horn formulas, then*

$$\mathbf{G3XT}_\omega \vdash \Gamma \Rightarrow \Delta \textit{ if and only if } \mathbf{G3X}_\omega \oplus \mathbf{T} \oplus \{Cut, Weak, Ctr\} \vdash \Gamma \Rightarrow \Delta$$

Proof. From left to right we show that the infinitary Horn rule can be simulated in the calculus $\mathbf{G3X}_\omega \oplus \mathbf{T} \oplus \{Cut, Weak, Ctr\} \vdash \Gamma \Rightarrow \Delta$. We have:

$$\dfrac{\dfrac{x : p_{k,1}, ..., x : p_{k,m(k)} \Rightarrow x : q_k \qquad x : p_{k,1}, ..., x : p_{k,m(k)}, x : q_k, \Gamma \Rightarrow \Delta}{x : p^2_{k,1}, ..., x : p^2_{k,m(k)}, \Gamma \Rightarrow \Delta}\text{Cut}}{x : p_{k,1}, ..., x : p_{k,m(k)}, \Gamma \Rightarrow \Delta}\text{Ctr}$$

From right to left we exploit the admissibility of the structural rules of weakening, contraction and cut. Hence the equivalence follows by showing that the axioms are provable, which follows from a routine root-first application of the rules. *qed.*

Theorem 6.10.3. *Let* **T** *be a set of infinitary Horn theories and* B *be infinitary Horn formula, then the following are equivalent:*

1. $\mathbf{G3CT}_\omega \vdash \Rightarrow x : B$
2. $\mathbf{G3IT}_\omega \vdash \Rightarrow x : B$
3. $\mathbf{G3S4T}_\omega \vdash \Rightarrow x : B^*$

Proof. The equivalence between 1. and 2. is essentially a restricted form of Barr's theorem (cf. (93; 32), and references therein); a proof using labelled systems is presented in Theorem 6.10.11 below. For the equivalence between 2. and 3. we need to extend the embedding to systems with rules for axioms.

2.⇒ 3. We argue by induction on the height of the derivations in the calculus $\mathbf{G3I}_\omega$. The case to be checked is that of rules for axiomatic extensions. In particular, we have:

$$\dfrac{x : p_{k,1}, ..., x : p_{k,m(k)}, x : q_k, \Gamma \Rightarrow \Delta}{x : p_{k,1}, ..., x : p_{k,m(k)}, \Gamma \Rightarrow \Delta}\ \mathrm{Horn}_k$$

which gets transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{x : p_{k,1}, ..., x : p_{k,m(k)}, x : q_k, \Gamma \Rightarrow \Delta}{a_1 \in N(x), \dots, a_m \in N(x), b \in N(x), x \in a_1, \dots, x \in a_m, x \in b, a_1 \Vdash^{\forall} p_{1k}, \dots, a_m \Vdash^{\forall} p_{mk}, b \Vdash q, x : p_{k,1}, ..., x : p_{k,m(k)}, x : q_k, \Gamma \Rightarrow \Delta}\ \mathrm{Weak}}{a_1 \in N(x), \dots, a_m \in N(x), b \in N(x), x \in a_1, \dots, x \in a_m, x \in b, a_1 \Vdash^{\forall} p_{k,1}, \dots, a_m \Vdash^{\forall} p_{k,m(k)}, b \Vdash q, \Gamma \Rightarrow \Delta}\ \mathrm{L}\Vdash}{a_1 \in N(x), \dots, a_m \in N(x), b \in N(x), a_1 \Vdash^{\forall} p_{k,1}, \dots, a_m \Vdash^{\forall} p_{k,m(k)}, b \Vdash q, \Gamma \Rightarrow \Delta}\ \mathrm{Ref}}{x : \Box p_{k,1}, \dots, x : \Box p_{k,m(k)}, x : \Box q_k, \Gamma \Rightarrow \Delta}\ \mathrm{L}\Box$$

We then conclude the proof with the following application of cut:

$$\dfrac{\dfrac{x : \Box p_{k,1}, \dots, x : \Box p_{k,m(k)} \Rightarrow x : \Box q_k \qquad x : \Box p_{k,1}, \dots, x : \Box p_{k,m(k)}, x : \Box q_k, \Gamma \Rightarrow \Delta}{(x : \Box p_{k,1}, \dots, x : \Box p_{k,m(k)})^2, \Gamma \Rightarrow \Delta}\ \mathrm{Cut}}{x : \Box p_{k,1}, \dots, x : \Box p_{k,m(k)}, \Gamma \Rightarrow \Delta}\ \mathrm{Ctr}$$

where the leftmost sequent is easily seen to be provable by root-first applications of the rules. The direction 3.⇒ 2. is a straightforward extension of the faithfulness theorem, as the rules for theories only work with atomic formulas. *qed.*

Remark. The above result cannot be extended to larger classes of axioms. Indeed, any axiom of the form $p \vee q$ can be added to intuitionistic logic and to its labelled calculus by converting it into a rule R of the shape:

$$\dfrac{x : p, \Gamma \Rightarrow \Delta \qquad x : q, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ R$$

However, its $*$ translation $\Box p \vee \Box q$ is not provable in the modal calculus extended with the same rule. This depends on the fact that $p \vee q$ is not a Horn formula.

We now show that a small axiomatic extension of $\mathbf{S4}_\omega$ can be used to extend the result to a larger class of theories. Let us denote by $\mathbf{S4^+}_\omega$ the axiomatic system obtained by adding to $\mathbf{S4}_\omega$ the set of axioms:

$$\{p \to \Box p \mid \text{ for every atomic formula } p\}$$

First, we observe that this does not lead to the collapse of the modalities. Indeed, we do not have $A \to \Box A$ for every A. To witness the failure of the validity of the schema it is enough to consider the formula $p \to q$, because $(p \to q) \to \Box(p \to q)$ is not derivable.

It is not difficult to observe that $\mathbf{S4^+}_\omega$ is sound and complete with respect to the class of **S4** neighborhood models in which the following holds:

$$x \in v(p) \Longrightarrow v(p) \in N(x) \text{ for every world } x \text{ and atomic formula } p$$

Soundness follows by induction on the height of the derivations in the calculus $\mathbf{S4}^+_\omega$. To prove completeness it is enough to check that the construction of the canonical model can be repeated for $\mathbf{S4}^+_\omega$ too. Hence, completeness follows from the next proposition.

Lemma 6.10.4. *The canonical model for* $\mathbf{S4^+}_\omega$ *satisfies* $x \in v(p) \Longrightarrow v(p) \in N(x)$.

Proof. Suppose $\Gamma \in v(p)$, then $\Gamma \Vdash p$, i.e. $p \in \Gamma$. Therefore, since $\Gamma \vdash p \to \Box p$ we get $\Box p \in \Gamma$. This entails $v(p) \in N(\Gamma)$. *qed.*

A labelled sequent calculus for the logic $\mathbf{S4^+}_\omega$ is obtained by adding rule *Mon* to the calculus $\mathbf{G3S4}_\omega$. It is immediate to observe that the axiom $p \to \Box p$ is derivable.

$$\dfrac{\dfrac{a \in N(x), a \Vdash p, x : p \Rightarrow x : \Box p, a \Vdash p}{a \in N(x), a \Vdash p, x : p \Rightarrow x : \Box p}\,\text{R}\Box}{x : p \Rightarrow x : \Box p}\,\text{Mon}$$

Definition 6.10.2. An *infinitary geometric implication* is an infinitary conjunction of formulas of the shape: $p_1 \wedge \ldots \wedge p_m \to \bigvee_{k>0} \overline{q}_k$, where $\overline{q}_k$ is a finite conjunction of atomic formulas.

Infinitary geometric implications can be transformed into the equivalent rules:

$$\dfrac{\{x : p_{k,1}, \ldots, x : p_{k,m(k)}, x : \vec{q}_{k,j(k)}, \Gamma \Rightarrow \Delta \mid j > 0\}}{x : p_{k,1}, \ldots, x : p_{k,m(k)}, \Gamma \Rightarrow \Delta}\,\text{Geom}_k$$

where $x : \vec{q}_{k,j(k)} \equiv x : q_{k,j(k)_1}, \ldots, x : q_{k,j(k)_{nj(k)}}$. We show that the calculus $\mathbf{G3S4}^+_\omega$ satisfies the usual desirable structural properties.

Lemma 6.10.5. *The rules of substitution of labels and weakening are height-preserving admissible in* $\mathbf{G3S4^+T}_\omega$.

Proof. Routine induction. *qed.*

Lemma 6.10.6. *Every rule is height-preserving invertible in* $\mathbf{G3S4^+T}_\omega$. *The rule of contraction is height-preserving admissible in* $\mathbf{G3S4^+T}_\omega$.

Proof. Invertibility of the rule *Mon* follows from the height-preserving admissibility of the weakening rule. With respect to contraction, we argue by transfinite induction. We only need to check the additional case in which the last rule applied is *Mon*.

$$\frac{a \Vdash P, a \in N(x), x : p, x : p, \Gamma \Rightarrow \Delta}{x : p, x : p, \Gamma \Rightarrow \Delta}\text{ Mon}$$

We proceed as follows:

$$\frac{\dfrac{a \Vdash P, a \in N(x), x : p, x : p, \Gamma \Rightarrow \Delta}{a \Vdash P, a \in N(x), x : p, \Gamma \Rightarrow \Delta}\text{ IH}}{x : P, x : P, \Gamma \Rightarrow \Delta}\text{ Mon}$$

The application of contraction is removed by the induction hypothesis. Notice that the repetition of the atomic formula is crucial to make the proof go through. *qed.*

Theorem 6.10.7. *The cut rule is admissible in* $\mathbf{G3S4^{+}T}_{\omega}$.

Proof. The proof is by double transfinite induction. With respect to the proof of cut elimination for $\mathbf{G3S4}_{\omega}$ we need to check the case in which one of the last rule applied is *Mon*. The interesting case is the one in which the cut formula is principal in an application of *Mon*, i.e.

$$\frac{\Gamma \Rightarrow \Delta, x : p \qquad \dfrac{a \Vdash p, a \in N(x), x : p, \Gamma' \Rightarrow \Delta'}{x : p, \Gamma' \Rightarrow \Delta'}\text{ Mon}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\text{ Cut}$$

The cut cannot be immediately permuted upwards, as this would yield $a \Vdash p, a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. In this case we consider the left premise of the cut. If it is an initial sequent, the proof is trivial. Otherwise, $x : p$ is never principal in the succedent and so the cut can be permuted upwards over the premises of $\Gamma \Rightarrow \Delta, x : p$ and eliminated invoking the secondary induction hypothesis. *qed.*

To establish the claim we made about equivalence concerning derivability of geometric formulas it is enough to show that we can streamline the modal interpretation of intuitionistic logic. Indeed, due to the axiom $p \to \Box p$, we can define a light translation l as follows:

- $p^l = p$
- $(A\#B)^l = A^l\#B^l$, where $\# \in \{\wedge, \vee\}$
- $(A \to B)^l = \Box(A^l \to B^l)$
- $(\bigvee_{k>0} A_k)^l = \bigvee_{k>0} A_k^l$
- $(\bigwedge_{k>0} A_k)^l = \Box \bigwedge_{k>0} A_k^l$

Lemma 6.10.8. *For every formula* A, $\mathbf{S4^{+}}_{\omega} \vdash A^* \leftrightarrow A^l$.

Proof. Straightforward by transfinite induction on the weight of the formula A. The only interesting case is the one in which the formula A is atomic, which follows since $\mathbf{S4}^+_\omega \vdash p \leftrightarrow \Box p$. The remaining cases are dealt with by the induction hypothesis. *qed.*

Theorem 6.10.9. *Let* $\mathbf{T}$ *be a set of infinitary geometric axioms:*

$$\mathbf{G3IT}_\omega \vdash \Gamma \Rightarrow \Delta \textit{ if and only if } \mathbf{G3S4^+T}_\omega \vdash \Gamma^* \Rightarrow \Delta^*.$$

Proof. ($\Rightarrow$). By the previous lemma, the proof is reduced to showing that $\mathbf{G3S4^+T}_\omega \vdash \Gamma^l \Rightarrow \Delta^l$. The proof is by induction on the height of the derivations. If $\Gamma \Rightarrow \Delta$ is an initial sequent, then so is $\Gamma^l \Rightarrow \Delta^l$. Otherwise, we distinguish cases according to the last rule applied. The proof follows the pattern detailed for $\mathbf{G3S4}_\omega$. If the last rule applied is an infinitary geometric rule, the proof is immediate (as the l-translation does not affect atomic formulas).

($\Rightarrow$). The proof is analogous the the faithfulness result for $\mathbf{G3S4}_\omega$ and thus we omit the details. *qed.*

We now prove a result which enables removing redundant applications of the monotonicity rule in intuitionistic logic. This is a crucial move in order to bring closer intuitionistic and classical logic in the setting of labelled calculi.

Lemma 6.10.10. *Given a derivation of*

$$\Omega, a_1 \Vdash p_1, \dots, a_n \Vdash p_n, x : p_1, \dots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \dots, x : q_t$$

in $\mathbf{G3S4^+T}_\omega$*, where* Ω *contains relational atoms of the shape* $a_i \in N(x), m^n(a_i) \in N(x), c \subseteq a_i, x \in a_i$*, there is a derivation of*

$$x : p_1, \dots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \dots, x : q_t$$

in $\mathbf{G3S4^+T}_\omega$ *which does not use any relational rule.*

Proof. The proof is by induction on the height of the derivation. If $\Omega, a_1 \Vdash p_1, \dots, a_n \Vdash p_n, x : p_1, \dots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \dots, x : q_t$ is an initial sequent, so is $x : p_1, \dots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \dots, x : q_t$, otherwise we distinguish cases according to the last rule applied. We discuss the case in which the last rule applied is L$\Vdash$. We have:

$$\frac{\Omega', x \in a_1, x : p_1, a_1 \Vdash p_1, \dots, a_n \Vdash p_n, x : p_1, \dots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \dots, x : q_t}{\Omega', x \in a_1, a_1 \Vdash p_1, \dots, a_n \Vdash p_n, x : p_1, \dots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \dots, x : q_t} \text{L}\Vdash$$

We proceed as follows:

$$\dfrac{\dfrac{\Omega', x \in a_1, x : p_1, a_1 \Vdash p_1, \ldots, a_n \Vdash p_n, x : p_1, \ldots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \ldots, x : q_t}{x : p_1, x : p_1, \ldots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \ldots, x : q_t}\ \text{IH}}{x : p_1, \ldots, x : p_n \Rightarrow x : \bigvee_{j>0} q_j, x : q_1, \ldots, x : q_t}\ \text{Ctr}$$

It can be checked (we avoid doing so for reasons of space) that the admissible step of contraction does not add any application of relational rules. *qed.*

We can finally relate the derivability in theories based on infinitary classical, intuitionistic and modal logics.

Theorem 6.10.11. *Let* **T** *be a set of infinitary geometric theories and B be infinitary geometric formula, then the following are equivalent:*

1. $\mathbf{G3CT}_\omega \vdash \Rightarrow x : B$
2. $\mathbf{G3IT}_\omega \vdash \Rightarrow x : B$
3. $\mathbf{G3S4^{+}T}_\omega \vdash \Rightarrow x : B^*$

Proof. The equivalence 2. $\Leftrightarrow$ 3. follows from the extending of the modal embedding to infinitary geometric theories.

Assume that $\mathbf{G3IT}_\omega \vdash \Rightarrow x : B$. Let B be the formula $\bigwedge_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \overline{q}_{k,j(k)})$. We have:

$$\dfrac{\Rightarrow x : B \qquad x : B, x : p_{k,1}, \ldots, x : p_{k,m(k)} \Rightarrow x : \bigvee_{j(k)>0} \vec{q}_j(k)}{x : p_{k,1}, \ldots, x : p_{k,m(k)} \Rightarrow x : \bigvee_{j(k)>0} \vec{q}_{j(k)}}\ \text{Cut}$$

Since by Lemma 6.10.10 we can assume that the derivation of $x : p_{k,1}, \ldots, x : p_{k,m(k)} \Rightarrow x : \bigvee_{j(k)>0} \vec{q}_{j(k)}$ does not contain any application of a relational rule, then $x : p_{k,1}, \ldots, x : p_{k,m(k)} \Rightarrow x : \bigvee_{j(k)>0} \vec{q}_{j(k)}$ is derivable in the classical calculus $\mathbf{G3CT}_\omega$. Hence the conclusion follows by applying the rules $L\wedge$, R$\rightarrow$ and R$\bigwedge$ in $\mathbf{G3CT}_\omega$.

To prove 1. $\Rightarrow$ 2. assume $\mathbf{G3CT}_\omega \vdash \Rightarrow x : B$. We apply the invertibility of the rules to obtain the derivability of the sequents $x : p_{k,1}, \ldots, x : p_{k,m(k)} \Rightarrow x : \bigvee_{j(k)>0} \vec{q}_{j(k)}$ for every $k > 0$. By inspection of the rules, the proof is already an intuitionistic derivation and the conclusion is obtained as follows:

$$
\begin{array}{cl}
\{x : p_{k,1}, \ldots, x : p_{k,m(k)} \Rightarrow x : \bigvee_{j(k)>0} \vec{q}_{k,j(k)} \mid k > 0\} & \\ \hline
\{x : p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \Rightarrow x : \bigvee_{j(k)>0} \vec{q}_{k,j(k)} \mid k > 0\} & \text{L}\wedge \\ \hline
\{a \in N(y), x \in a, x : p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \Rightarrow x : \bigvee_{j(k)>0} \vec{q}_{k,j(k)}, y : p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)} \mid k > 0\} & \text{Weak} \\ \hline
\{a \in N(y), x \in a \Rightarrow x : p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \supset \bigvee_{j(k)>0} \vec{q}_{j(k)}, y : p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)} \mid k > 0\} & \text{R}\supset \\ \hline
\{a \in N(y) \Rightarrow a \Vdash p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \supset \bigvee_{j(k)>0} \vec{q}_{k,j(k)}, y : p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)} \mid k > 0\} & \text{R}\Vdash \\ \hline
\{a \in N(y) \Rightarrow y : p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)} \mid k > 0\} & \text{R}\rightarrow \\ \hline
\{\Rightarrow p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)} \mid k > 0\} & \text{Ndeg} \\ \hline
\Rightarrow y : \&_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)}) & \text{R}\& \\ \hline
a \in N(x), y \in a \Rightarrow y : \&_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)}), x : \bigwedge_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)}) & \text{Weak} \\ \hline
a \in N(x) \Rightarrow a \Vdash \&_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)}), x : \bigwedge_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)}) & \text{R}\Vdash \\ \hline
a \in N(x) \Rightarrow x : \bigwedge_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)}) & \text{R}\bigwedge \\ \hline
\Rightarrow x : \bigwedge_{k>0}(p_{k,1} \wedge \ldots \wedge p_{k,m(k)} \rightarrow \bigvee_{j(k)>0} \vec{q}_{k,j(k)}) & \text{Ndeg}
\end{array}
$$

qed.

6.11 Concluding remarks

We introduced an axiomatic system for an infinitary extension of the modal logic S4 and proved its soundness and completeness with respect to a class of neighborhood frames. The proof was obtained by adapting the standard canonical model construction to the infinitary setting. The method here employed is of independent interest. In fact, it can easily be generalized so as to cover first-order extensions of the logics here considered as well as non-normal modal logics.

The new semantics allowed for the extraction of an infinitary labelled sequent calculus in which every rule is height-preserving invertible and the rules of weakening, contraction and cut are shown admissible by means of classical proof-theoretic arguments. The calculus thus obtained has the desirable feature that each rule is height-preserving invertible.

Hence, we have exploited the new system and its similarity to a labelled system for intuitionistic infinitary logic in order to obtain a proof of the Gödel-McKinsey-Tarski translation for infinitary languages. The proof explicitly shows the steps required in order to transform an intuitionistic derivation into a modal one and vice versa. The proof is syntactic, appealing to semantics only through the labelling. We observe that this result could not be achieved if we worked within Kripke style semantics, because it is inadequate for infinitary intuitionistic logic.

Finally, we have explored the connections between classical, intuitionistic and modal derivability in an infinitary setting. In particular, we have considered axiomatic extensions of the labelled calculi here presented and we have provided a cut-free presentation for them. Next, we have shown that with respect to Horn-like infinitary axioms there is a correspondence between derivability in classical, intuitionistic and $\mathbf{S4}_\omega$ logics. The result is extended to a class analogous to the one of geometric axioms by considering an axiomatic extension of $\mathbf{S4}_\omega$. This presents a uniform logical landscape for the study of geometric logic which can be studied through the lens of classical, intuitionistic and modal logic.

An interesting theme for future work may consist in the extension of the present methodology to some intermediate logics, such as Gödel-Dummett logic. Finally, we believe that the extension of the present result to a first-order setting should pose no significant difficulties.

Chapter 7

Infinitary substructural logics and truth theories

The chapter investigates from a proof-theoretic perspective various non-contractive logical systems circumventing logical and semantic paradoxes. Until recently, such systems only displayed additive quantifiers (Grišin, Cantini). Systems with multiplicative quantifers have also been proposed in the 2010s (Zardini), but they turned out to be inconsistent with the naive rules for truth or comprehension. We start by presenting a first-order system for disquotational truth with additive quantifiers and we compare it with Grišin set theory. We then analyze the reasons behind the inconsistency phenomenon affecting multiplicative quantifers: after interpreting the exponentials in affine logic as vacuous quantifiers, we show how such a logic can be simulated within a truth-free fragment of a system with multiplicative quantifiers. Finally, we prove that the logic of these multiplicative quantifiers (but without disquotational truth) is consistent, by showing that an infinitary version of the cut rule can be eliminated. This paves the way to a syntactic approach to the proof theory of infinitary logic with infinite sequents.

7.1 Introduction

Since (33) it is well-known that the contraction rule plays an essential role in the derivation of logical and semantic paradoxes such as the Liar, Russell's and Curry's. In the last few decades, there has been a renewed interest in non-contractive logical systems, as Fitch called them, that block these paradoxes by dropping contraction from their sequent calculus formulation. (46) established the consistency of unrestricted abstraction based on what is nowadays called *affine logic*, i.e. linear logic equipped with the weakening rule. (81) further elaborates on

Grišin's proposal by giving a proof-theoretic analysis of a system with unrestricted abstraction and some additional axioms. Moreover, Cantini embeds combinatory logic in Grišin set theory, thereby establishing its undecidability (12). It is well-known – and it will be recalled below – that Grišin's set theory gives rise to a consistent theory of disquotational truth.

Nevertheless, it also clear that the solution thus provided cannot be the whole story since it only features *additive* quantifiers, which are in effect *classical* quantifiers in disguise. Indeed, as stressed in (70) and (80), the difference between the additive universal quantifier and the multiplicative one may be roughly understood as the one between *any* and *every*. Given the splitting phenomenon of the connectives into additive and multiplicative ones determined by the absence of contraction, the additive quantifiers generalize additive connectives, but there is no logical device corresponding to the generalization of multiplicative ones (10; 70; 62). One spontaneous way of conceiving of multiplicative quantifiers is to identify the universal and the existential quantifiers with *infinitary* multiplicative conjunctions and disjunctions, respectively

$$\forall x A \equiv A(x/t_1) \otimes A(x/t_2) \otimes \dots$$
$$\exists x A \equiv A(x/t1) \parr A(x/t_2) \parr \dots$$

Following this intuition, (115) presented a theory of disquotational truth based on a purely multiplicative fragment of affine logic featuring infinitary quantifiers. However, this theory has received enough attention to make it clear that: (i) it cannot be extended with suitable primitive recursive functions (21); (ii) the attempted proof of consistency of the system via cut-elimination contains a gap (36); and (iii) the system is outright inconsistent given some plausible principles for vacuous quantification (37)

In this chapter we aim to contribute to the understanding of the non-contractive landscape by addressing a set of interconnected issues. Specifically:

- We simplify the cut-elimination proof for Grišin's set theory presented in (12), while also fixing a problem in Cantini's strategy. Furthermore, we show that the seemingly weaker theory of disquotational truth based on affine logic supports Cantini's derivation of Löb's principle given a **K4** modality.

- We show that the rules for vacuous quantification, which are responsible for the inconsistency of Zardini's system, can actually be employed to recover full classical logic in the context of affine logic. We prove that there exists an exact translation of (predicate, infinitary) classical logic into affine logic with vacuous quantification.

- We show that the rules for vacuous quantification, which are responsible for the inconsistency of Zardini's system, can actually be employed to recover full classical logic in the context of affine logic. We prove that there exists an exact translation of (predicate, infinitary) classical logic into affine logic with vacuous quantification.

- We show that the rules for vacuous quantification, which are responsible for the inconsistency of Zardini's system, can actually be employed to recover full classical logic in the context of affine logic. We prove that there exists an exact translation of (predicate, infinitary) classical logic into affine logic with vacuous quantification.

- In the field of linear logic, the dismissed contraction and weakening can be recovered and controlled using exponentials: !, ?, which essentially behave as **S4** modalities. We provide a new perspective on exponentials by interpreting them as vacuous quantifiers. In particular, we show how to simulate affine linear logic within a proper fragment of the system of multiplicative quantifiers by giving a sound and faithful translation. In doing this, we implement Girard's old (but so far unexplored) idea to interpret exponentials as infinitary operations (40).

- We directly show that Zardini's cut-elimination algorithm is based on a proof-manipulation that does not preserve provability.

- Finally, we show that an infinitary version of the cut rule can be eliminated from the purely logical system featuring infinitary quantifiers.

The last point also answers to a question recently posed by (82). However, the proof-theoretic interest of this result extends beyond a non-contractive approach to paradoxes. The proof theory of well-founded infinitely branching derivations has been extensively studied and has found significant application in the context of ordinal analysis (95) and structural proof theory (104). Well-founded infinitary derivations involving sequents with infinitely many formulas have received less attention. The investigations concerning this kind of calculi have been conducted using semantical methods (see (100)).

A semantic argument can be employed to show the cut-free completeness of a calculus for infinitary classical logic with infinite sequents. However, since the logic of multiplicative quantifiers does not enjoy a semantic presentation, this indirect strategy here is not available. We present a syntactic proof of cut-elimination for the system involving sequents with infinitely many formulas for the logic of multiplicative quantifiers.

The chapter is structured as follows: Section 2 discusses a contraction-free and cut-free system for disquotational truth in relation to Grišin set theory. Section 3 shows how the exponentials ! and ? can be demodalized by conceptualizing them in terms of vacuous quantifiers within (a truth-free fragment of) Zardini's system. Section 4 is divided into two parts, a *pars destruens* that investigates the reasons for the inconsistency of Zardini's system and a *pars construens* that presents the cut-elimination procedure for multiplicative quantifiers. Finally, Section 5 concludes by outlining open problems generated by the results of the chapter.

7.2 Contraction and the paradoxes

Non-contractive approaches to the logical and semantic paradoxes are known to be formally successful. Without contraction, it's possible to extend a standard cut-elimination procedure for first-order affine logic without exponentials (henceforth, affine logic AL) to its extension with naïve rules for truth, (class-)membership, predication.

Definition 7.2.1 (Affine Logic AL)**.** $\Gamma, \Delta, \Theta, \Lambda ...$ range over finite multisets of formulae of a countable, first-order Tait language $\mathsf{FM}^{\mathsf{LL}}$.

$$\frac{}{\Rightarrow \Gamma, P, \overline{P}} \text{(IN)}$$

$$\frac{\Rightarrow \Gamma, A_i}{\Rightarrow \Gamma, A_1 \oplus A_2} (A_{i}, {}_{i=1,2}) \qquad \frac{\Rightarrow \Gamma, A \quad \Rightarrow \Gamma, B}{\Rightarrow \Gamma, A \,\&\, B} (\&)$$

$$\frac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \parr B} (\parr) \qquad \frac{\Rightarrow \Gamma, A \quad \Rightarrow \Delta, B}{\Rightarrow \Gamma, \Delta, A \otimes B} (\otimes)$$

$$\frac{\Rightarrow \Gamma, A(y/x)}{\Rightarrow \Gamma, \forall x A} (\forall, y!) \qquad \frac{\Rightarrow \Gamma, A(t/x)}{\Rightarrow \Gamma, \exists x A} (\exists)$$

where $y!$ expresses the fact that the variable y does not occur in the conclusion.

We follow the linear logic tradition and use $\oplus$ and & for, respectively, additive disjunction and conjunction, and $\parr$ and $\otimes$ for, respectively, multiplicative disjunction and conjunction. Furthermore, we use $\exists$ and $\forall$ for the additive quantifiers. Lastly, we use $\overline{A}$ for the negation of the formula A. In following the Tait convention for negation (see also (107)), however, we take A to be defined as follows:

- If A is the atomic formula P, then $\overline{A} = \overline{P}$.
- If A is the atomic formula $\overline{P}$, then $\overline{A} = P$.
- If A is the formula $B \oplus C$, then $\overline{A} = \overline{B} \mathbin{\&} \overline{C}$.
- If A is the formula $B \mathbin{\&} C$, then $\overline{A} = \overline{B} \oplus \overline{C}$.
- If A is the formula $B \parr C$, then $\overline{A} = \overline{B} \otimes \overline{C}$.
- If A is the formula $B \otimes C$, then $\overline{A} = \overline{B} \parr \overline{C}$.
- If A is the formula $\forall x A$, then $\overline{A} = \exists x \overline{A}$.
- If A is the formula $\exists x A$, then $\overline{A} = \forall x \overline{A}$.

Linear logic without exponentials is obtained from affine logic by restricting the initial sequents to those of the form initial sequents of the form $\Rightarrow P, \overline{P}$. We use a double line to denote a multiple, but finite, application of the rules of the calculi.

Let $\mathsf{FM}^{\mathsf{LL}^+}$ be a language featuring:

- For $n, m \in \mathbb{N}$, n-ary predicates $\mathrm{S}^{n,m}$ and their negated dual $\overline{\mathrm{S}^{n,m}}$;
- The logical symbols of `AL`;
- The λ term forming operator $\lambda \cdot . \cdot$;
- Variables $v_1, v_2, \ldots$ (we employ x, y, z for metavariables).

In the predicate $\mathrm{S}^{n,m}$ the superscript n denotes the arity, whereas m indicates the number of free variables.. For formulae $A \in \mathsf{FM}^{\mathsf{LL}^+}$, $\lambda x A$ is a term whose free variables are the free variables of A minus x. We abbreviate:

$$\lambda x_1 \ldots x_n\, A := \lambda x_1.(\ldots \lambda x_n\, A \ldots).$$

Notice that we allow for "self-referential" names to be built in the system. For instance, we allow for the existence of terms l such that

$$l := \ulcorner \overline{\mathrm{S}^{1,0}}(l) \urcorner.$$

The term l, as we shall see shortly, plays the role of a name for a Liar sentence. Similar terms are available for other paradoxical sentences such as Russell's, Curry's, and so on.

Definition 7.2.2 (Semantic Extensions of AL)**.**

(i) The system $\mathtt{UTS}^{n,m}$ is obtained by formulating AL in $\mathsf{FM}^{\mathsf{LL}^+}$ and by adding the rules

$$\frac{\Rightarrow \Gamma, A(t_1, \ldots, t_n)}{\Rightarrow \Gamma, \mathrm{S}^{n,m}(\lambda x_1 \ldots x_n\, A, t_1 \ldots t_n)}\ (\mathrm{S}^{n,m})$$

$$\frac{\Rightarrow \Gamma, \overline{A}(t_1, \ldots, t_n)}{\Rightarrow \Gamma, \overline{\mathrm{S}^{n,m}}(\lambda x_1 \ldots x_n\, A, t_1 \ldots t_n)}\ (\overline{\mathrm{S}^{n,m}})$$

for all formulae A with exactly m free variables.

(ii) **UTS** comprises rules for $\mathrm{S}^{n,m}$ for all $n, m \in \mathbb{N}$.

Remark. The template provided by the theories $\mathtt{UTS}^{n,m}$ enables us to define several systems that are relevant for the analysis of the paradoxes in a non-contractive setting. As we shall see shortly, the systems $\mathtt{UTS}^{1,m}$, for each $m \in \mathbb{N}$, correspond to Grišin set theory[1]. A non-contractive theory of *disquotational truth* corresponds to $\mathtt{UTS}^{1,0}$.

Derivations in AL and extensions thereof are finite trees that are locally correct with respect to the rules just given. (12) provides a cut-elimination strategy for the system $\mathtt{UTS}^{1,m}$. The strategy relies on a triple induction on, respectively, the number of naïve comprehension rules, the depth of the cut-formula, i.e. the number of logical symbols occurring in it, and the level of the cut. The strategy, as it stands, cannot deal satisfactorily with some of the cases, for instance the one in which the last inference in one of the branches before a cut is an additive conjunction and in which the cut formula is not principal in the last inference.[2] We circumvent the problem by showing that an induction on a *single parameter* suffices. In order to do this, we provide a slightly nonstandard measure of length of the derivation.[3]

Definition 7.2.3. Given a proof $\mathcal{D}$, its height $h(\mathcal{D})$ is given by the following recursion:

- $h(\mathcal{D}) = 1$ for $\mathcal{D}$ an instance of (IN);

[1] Grišin calls his theory a *set theory*, but since it is non-extensional it ought rather to be viewed as a property theory.

[2] The triple induction may be repairable – as suggested by Cantini in personal communication – by redefining what Cantini calls $\in$-complexity for additive rules, by taking in particular the maximum of the $\in$-complexity of the premisses instead of their sum.

[3] After circulating a preprint version of this article, Pierluigi Minari brought to our attention the manuscript (**?**), which contains essentially the same fix to Cantini's strategy.

- $h(\mathcal{D}) = \max(h(\mathcal{D}_0), h(\mathcal{D}_1)) + 1$, with $\mathcal{D}$ ending with an application of (&) to $\mathcal{D}_0$ and $\mathcal{D}_1$;

- $h(\mathcal{D}) = h(\mathcal{D}_0) + h(\mathcal{D}_1)$, with $\mathcal{D}$ ending with an application of ($\otimes$) to $\mathcal{D}_0$ and $\mathcal{D}_1$;

- $h(\mathcal{D}) = h(\mathcal{D}_0) + 1$ in all other cases.

Proposition 7.2.1. *Cut is admissible in* **UTS**. *Therefore,* **UTS** *is consistent.*

Proof. The proof rests on the following reduction lemma:

(R) if $\mathcal{D}_0$ and $\mathcal{D}_1$ are cut-free proofs of Γ, A and $\Delta, \overline{A}$, respectively, then there is a cut-free proof $\mathcal{D}$ of Γ, Δ with $h(\mathcal{D}) \leq h(\mathcal{D}_0) + h(\mathcal{D}_1)$.

(R) is proved by an induction on $h(\mathcal{D}_0) + h(\mathcal{D}_1)$. We consider two cases for illustration. If the "cut formulae" are principal in the last inference, and they are obtained by (for notational simplicity) $\mathrm{S}^{n,n}$ and $\overline{\mathrm{S}^{n,n}}$, respectively, then we have

$$\dfrac{\begin{array}{c}\mathcal{D}_{00}\\ \Rightarrow \Gamma, A(x_1 \ldots x_n)\end{array}}{\Rightarrow \Gamma, \mathrm{S}^{n,n}(\lambda\vec{x}A, \vec{x})} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{10}\\ \Rightarrow \Delta, \overline{A}(x_1 \ldots x_n)\end{array}}{\Rightarrow \Delta, \overline{\mathrm{S}^{n,n}}(\lambda\vec{x}A, \vec{x})}$$

We can then simply apply the induction hypothesis to $\mathcal{D}_{00}$ and $\mathcal{D}_{10}$. If the last rules applied are ($\otimes$) and ($⅋$), respectively, we have:

$$\dfrac{\begin{array}{c}\mathcal{D}_{00}\\ \Rightarrow \Gamma, A\end{array} \qquad \begin{array}{c}\mathcal{D}_{01}\\ \Rightarrow \Delta, B\end{array}}{\Rightarrow \Gamma, \Delta, A \otimes B} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{10}\\ \Rightarrow \Theta, \overline{A}, \overline{B}\end{array}}{\Rightarrow \Theta, \overline{A} ⅋ \overline{B}}$$

Then the desired $\mathcal{D}$ is obtained by applying the induction hypothesis to, e.g., $\mathcal{D}_{00}$ and $\mathcal{D}_{10}$, and then to the resulting derivation and $\mathcal{D}_{01}$.[4] *qed.*

Cantini shows that the addition of a **K4** modality to Grišin set theory – that is, a rule corresponding to the modal principle 4 – and a necessitation rule is strong enough to derive the Löb's principle $\Box(\Box A \rightarrow A) \rightarrow \Box A$. We strengthen Cantini's observation and show that the schema $\mathtt{UTS}^{1,0}$ suffices for the task. In what follows, it will be convenient to refer to the canonical name $\ulcorner A \urcorner$ of a sentence A of $\mathsf{FM}^{\mathsf{LL}}$, and to the corresponding truth-ascription $\mathrm{Tr}\ulcorner A \urcorner$. We let, for A a sentence:

$$\ulcorner A \urcorner := \lambda v_0\, A, \qquad \mathrm{Tr}\ulcorner A \urcorner :\leftrightarrow \mathrm{S}^{1,0}(\ulcorner A \urcorner, v_0).$$

[4]It's here that the definition of $h(\cdot)$ plays a role: if length was defined as the number of nodes in the maximal branch of the proof-tree, then the induction would not go through in this case as, potentially, $h(\mathcal{D}) > h(\mathcal{D}_0) + h(\mathcal{D}_1)$.

Definition 7.2.4. The system $\mathtt{UTS}^{1,0} + \mathbf{K4}$ is obtained by extending $\mathtt{UTS}^{1,0}$ with the rules:

$$\dfrac{\Rightarrow \Diamond\Gamma, \Gamma, A}{\Rightarrow \Delta, \Diamond\Gamma, \Box A}\,(\textsc{nec}) \qquad \dfrac{\Rightarrow \Diamond\Gamma, \Delta, A \qquad \Rightarrow \Diamond\Gamma, \Theta, B}{\Rightarrow \Diamond\Gamma, \Delta, \Theta, A \otimes B}\,(\Box\otimes) \qquad \dfrac{\Rightarrow \Gamma, A \qquad \Rightarrow \Delta, \overline{A}}{\Rightarrow \Gamma, \Delta}\,(\textsc{Cut})$$

Lemma 7.2.2. $\mathtt{UTS}^{1,0} + \mathbf{K4}$ *derives the schema* $\Box(\Box A \to A) \to \Box A$.

Proof. Let $C :\leftrightarrow (\Box\mathrm{Tr}\ulcorner C\urcorner \to A)$, for arbitrary A. We show that if $\Diamond\overline{A}, A$ is derivable, then so is A. We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \Diamond\overline{\mathrm{Tr}}\ulcorner C\urcorner, \Box\mathrm{Tr}\ulcorner C\urcorner \qquad \Rightarrow \overline{A}, A}{\Rightarrow \Diamond\overline{\mathrm{Tr}}\ulcorner C\urcorner, \Box\mathrm{Tr}\ulcorner C\urcorner \otimes \overline{A}, A}\,(\otimes)}{\Rightarrow \Diamond\overline{\mathrm{Tr}}\ulcorner C\urcorner, \overline{\mathrm{Tr}}\ulcorner C\urcorner, A}\,(\overline{\mathrm{Tr}})}{\Rightarrow \Diamond\overline{\mathrm{Tr}}\ulcorner C\urcorner, \Box A}\,(\textsc{nec}) \qquad \Rightarrow \Diamond\overline{A}, A}{\Rightarrow \Diamond\overline{\mathrm{Tr}}\ulcorner C\urcorner, A}\,(\textsc{Cut})}{\Rightarrow \Diamond\overline{\mathrm{Tr}}\ulcorner C\urcorner \parr A}\,(\parr) \qquad \Rightarrow \Box\mathrm{Tr}\ulcorner C\urcorner \otimes \overline{A}, C}{\dfrac{\dfrac{\Rightarrow C}{\Rightarrow \mathrm{Tr}\ulcorner C\urcorner}\,(\mathrm{Tr})}{\Rightarrow \Box\mathrm{Tr}\ulcorner C\urcorner}\,(\textsc{nec})}\,(\textsc{Cut}) \qquad \Rightarrow \Diamond\overline{\mathrm{Tr}}\ulcorner C\urcorner, A}{\Rightarrow A}\,(\textsc{Cut})$$

Since $\Rightarrow \Diamond(\Box(\Diamond\overline{A} \vee A) \wedge \Diamond\overline{A}), \Diamond(\Box P \wedge \overline{P}) \vee \Box P$ is easily seen to be derivable, we immediately get the desired conclusion as reported also in (12, Thm 2.8). *qed.*

By translating the box modality as $P \parr \overline{P}$ for a designated atom P, we immediately obtain the conservativity of $\mathtt{UTS}^{1,0} + \mathbf{K4}$ over $\mathtt{UTS}^{1,0}$ which in turns immediately yields the consistency of the former system.

We would like to conclude this section by observing that the calculus $\mathtt{UTS}^{1,0} + \mathbf{K4}$ provably does not admit cut-elimination. To witness this it is enough to consider the sequent $\Rightarrow \Diamond(\Box P \otimes \overline{P}), \Box P$. The latter is indeed provable via cut as shown by the above derivation, but does not admit a cut-free proof by inspection of the rules.

Open problem 1. *Can we obtain a cut-free system equivalent to* $\mathtt{UTS}^{1,0} + \mathbf{K4}$*? A natural approach would be to substitute the modal rule with:*

$$\dfrac{\Rightarrow \Diamond\Gamma, \Gamma, \Diamond\overline{A}, A}{\Rightarrow \Diamond\Gamma, \Box A}$$

The systems considered so far feature only additive quantifiers, which can be viewed as straightforward generalizations of the additive conjunction and disjunction. However, this straightforward solution to the logical paradoxes may not be completely satisfactory: the system lacks quantifiers that generalize

$$\dfrac{}{\Rightarrow \Gamma, P, \overline{P}}\ (\textsc{in})$$

$$\dfrac{\Rightarrow \Gamma, A}{\Rightarrow \Gamma, \mathrm{Tr}(\ulcorner A \urcorner)}\ (\mathrm{Tr}) \qquad \dfrac{\Rightarrow \Gamma, \overline{A}}{\Rightarrow \Gamma, \overline{\mathrm{Tr}}(\ulcorner A \urcorner)}\ (\overline{\mathrm{Tr}})$$

$$\dfrac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \parr B}\ (\parr) \qquad \dfrac{\Rightarrow \Gamma, A \qquad \Rightarrow \Delta, B}{\Rightarrow \Gamma, \Delta, A \otimes B}\ (\otimes)$$

$$\dfrac{\ldots \qquad \Rightarrow \Gamma_i, A(t_i/x) \qquad \ldots}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \forall x A}\ (\forall) \qquad \dfrac{\Rightarrow \Gamma, A(t_1/x), A(t_2/x), \ldots}{\Rightarrow \Gamma, \exists x A}\ (\exists)$$

Figure 7.1: IKTω

multiplicative connectives. Several logicians and philosophers encouraged such a strengthening of the basic non-contractive theory (10; 70; 62). The challenge was taken up by Zardini in (115).

7.3 Multiplicative quantifiers and inconsistency

(115) attempts to establish a cut elimination theorem for the multiplicative fragment of affine logic extended with a combination of multiplicative quantifiers and naïve truth (IKTω), see Figure 7.3. We opted for a Tait style presentation of the original calculus by Zardini. Indeed, the two calculi are easily seen to be equivalent in terms of provability. By $\biguplus_{i\in I} \Gamma_i$ we denote the infinitary multiset union of the Γ_i. Terms $t_1, t_2, t_3, \ldots$ constitutes an exhaustive enumeration of the terms of the language.

Zardini motivates the theory by emphasizing that additive connectives are not compatible with the solutions to the semantic paradoxes he defends; as a consequence, multiplicative quantifiers become the natural extension of multiplicative conjunction and disjunction. The proposal consists in equating multiplicative universal and existential quantifiers with an infinitary multiplicative conjunction and disjunction, respectively. This move is not without consequences from the point of view of the structural analysis of the system. In particular, the choice of such a reading of quantifiers has the immediate consequence of working with sequents with infinite multisets of formulas.

It is worth noting a nonstandard feature of the multiset notion employed by Zardini, which is specifically required by his formulation of the ($\forall$) rule.[5]

[5]Thanks to Francesco Paoli for highlighting this point in personal communication.

While standard multisets (even infinite ones) allow only for finite multiplicities of formulas, Zardini's multisets permit ω-many repetitions of formulae. However, one of the problems with Zardini's notion is that it does not allow for tracking copies of different infinite multiplicities (which may be made up of infinite repetitions of the same formula), thereby reintroducing a form of contraction into the system. Many of the results that follow are based on the consequences of this choice. Formally, multisets are not, as usual, functions $\Gamma : \mathsf{FM}^{\mathsf{LL}} \to \omega$, but rather functions $\Gamma : \mathsf{FM}^{\mathsf{LL}} \to \omega + 1$. We write $\Gamma(A) > 0$ to denote the fact that A occurs in Γ (possibly infinitely many times).

Several problems have been found with Zardini's proposal, but his work contains insightful ideas that prompted interest in the study of infinitary systems with multiplicative quantifiers and their interaction with paradox-breeding notions. Da Ré and Rosenblatt showed that extending Zardini's system with basic arithmetical axioms leads to inconsistency (21), while Fjellstad identified a gap in the cut-elimination proof (36). In §7.4.1, we directly show that Zardini's cut-elimination algorithm is based on a proof-manipulation that does not preserve provability. In a recent paper, Fjellstad also show that the system $\mathtt{IKT}\omega$ is outright inconsistent, if the rules for the multiplicative quantifiers are used to deal with vacuous quantification in a natural way (37). In this section we show how, even without a truth predicate or similar semantic resources, the implicit rules for vacuous quantification in $\mathtt{IKT}\omega$ are problematic. In particular, we prove that vacuous quantification simulates the role played by exponentials in linear logic. Therefore, vacuous quantification in the setting of Zardini's system allows one to faithfully interpret classical logic as a fragment.

Since the system $\mathbf{IKT}_\omega$ and its fragment obtained from removing the truth predicate are systems in which derivations are infinitely branching well-founded trees, we need to suitably modify the notion of height in order to carry out inductive arguments. To deal with infinitary derivations we assign ordinals to measure the heights of the derivations (as in Section 6 of the present work). The assignment is the standard one as can be found in (95), the key point is that for every rule ρ:

$$\frac{\dots \qquad \Rightarrow \Gamma_i \qquad \dots}{\Rightarrow \Gamma} \ (\rho)$$

the height of the premise Γ_i is strictly less than the height of the conclusion Γ for every i. More generally, the height of a derivation $\mathcal{D}$ is inductively defined as follows: with h_i the heights of the direct sub-derivations of $\mathcal{D}$, $i \in I$, the height of $\mathcal{D}$ is $sup_{i \in I}(h_i) + 1$.

7.3.1 Vacuous quantifiers and classical logic

We recall that a rule is said to be (height-preserving) *invertible* if the derivability of the conclusion entails the derivability of each of its premises (and the height is less or equal). We start by showing that the rule for the existential quantifier is height-preserving invertible.

Lemma 7.3.1. *The rule* $\exists$ *is height-preserving invertible.*

Proof. By induction on the height of the derivation. If the sequent $\Rightarrow \Gamma, \exists x A$ is an initial sequent, then so is $\Rightarrow \Gamma, A(t_0/x), A(t_1/x), ...$ since $\exists x A$ is not an atomic formula and initial sequents are all on the form $\Rightarrow \Gamma, P, \overline{P}$ for atomic P's. If the formula $\exists x A$ is principal, the premise gives the desired conclusion. If the last rule applied is any other rule, we apply the induction hypothesis to each of the premise(s) and then the rule again. For example, if the last rule applied is $\forall$, we have:

$$\dfrac{\ldots \qquad \Rightarrow \Gamma_i, B(t_i/y), \exists x A \qquad \ldots}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \forall y B, \exists x A}\ (\forall)$$

We construct the following derivation:

$$\dfrac{\ldots \qquad \overset{\vdots \text{IH}}{\Rightarrow \Gamma_i, B(t_i/y), A(t_1/x), A(t_2/x), ...} \qquad \ldots}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \forall y B, A(t_1/x), A(t_2/x), ...}\ (\forall)$$

where IH denotes the application of the inductive hypothesis. *qed.*

We also observe that the weakening rule (Weak) is height-preserving admissible in the system $\mathbf{IZ}_\omega$. This fact will shortly be employed in the proof of Proposition 7.3.3.

Lemma 7.3.2. *The weakening rule:*

$$\dfrac{\Rightarrow \Gamma}{\Rightarrow \Gamma, \Delta}\ \textit{Weak}$$

is height-preserving admissible for every multiset Δ.

Proof. Straightforward by induction on the height of the derivation. *qed.*

Definition 7.3.1. The translation from classical logic in a language containing signed propositional atoms, conjunctions and disjunctions (in what follows we assume that the quantifiers are vacuous).

- $(P)^* = P$
- $(\overline{P})^* = \overline{P}$
- $(A \vee B)^* = \exists x A^* \parr \exists y B^*$
- $(A \wedge B)^* = \exists x A^* \otimes \exists y B^*$

The translation extends to multisets: if Γ is a finite multiset of formulae in the classical language, we let $\Gamma^* = \exists x \Gamma^*$, where $\exists x \Gamma^*$ stands for the multiset obtained by prefixing every formula in Γ with a vacuous quantifier. We write A^∞ to denote the multiset of formula containing infinitely many copies of A. The definition naturally extends to multisets of formulas.

Definition 7.3.2 (**ALV**)**.** **ALV** extends **AL** with the following rules for vacuous quantification:

$$\frac{\Rightarrow \Gamma, A^\infty}{\Rightarrow \Gamma, \exists x A}\ (\text{v}\exists) \qquad\qquad \frac{\dots \quad \Rightarrow \Gamma_i, A \quad \dots}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \forall x A}\ (\text{v}\forall)$$

Proposition 7.3.3. *Classical propositional logic is a subsystem of affine propositional logic extended with infinitary rules for vacuous quantification (***ALV***).*

The proof of the proposition rests on the following Lemma which ensures the admissibility of an infinitary form of contraction for vacuously existentially quantified formulas.

Lemma 7.3.4. *The following rule is admissible in* **ALV***:*

$$\frac{\Rightarrow \Gamma, \exists x A^\infty}{\Rightarrow \Gamma, \exists x A}$$

Proof. We argue by induction on the height of the derivation. If $\Rightarrow \Gamma, \exists x A^\infty$ is an initial sequent, so is $\Rightarrow \Gamma, \exists x A$, because only literals can be principal in initial sequents. If one of the existential quantifiers is principal, we have:

$$\frac{\Rightarrow \Gamma, A^\infty, \exists x A^\infty}{\Rightarrow \Gamma, \exists x A^\infty}\ (\text{v}\exists)$$

By applying the invertibility of the rule for the existential quantifier we get a derivation of $\Rightarrow \Gamma, A^\infty$, because the countable union of a countable multiset of formulas is a countable multiset. The desired conclusion follows by an application of the rule $\exists$.

$$\Rightarrow \Gamma, P, \overline{P} \quad (\textsc{cin})$$

$$\frac{\Rightarrow \Gamma, A \qquad \Rightarrow \Gamma, B}{\Rightarrow \Gamma, A \wedge B}\,(\wedge) \qquad \frac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \vee B}\,(\vee)$$

Figure 7.2: **CPL**

If the last rule is a unary rule and $\exists x A$ is not principal, we apply the induction hypothesis to the premise and then the rule again. If the last rule applied is $\otimes$, we have:

$$\frac{\Rightarrow \Gamma, B, \exists x A^{\infty} \qquad \Rightarrow \Delta, C, \exists x A^{\infty}}{\Rightarrow \Gamma, \Delta, B \otimes C, \exists x A^{\infty}}\,(\otimes)$$

In this case we construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma, B, \exists x A^{\infty}}{\Rightarrow \Gamma, B, \exists x A}\,(\textsc{ih})}{\Rightarrow \Gamma, B, A^{\infty}}\,(\textsc{inv}) \qquad \dfrac{\dfrac{\Rightarrow \Delta, C, \exists x A^{\infty}}{\Rightarrow \Gamma, C, \exists x A}\,(\textsc{ih})}{\Rightarrow \Gamma, C, A^{\infty}}\,(\textsc{inv})}{\dfrac{\Rightarrow \Gamma, \Delta, B \otimes C, A^{\infty}}{\Rightarrow \Gamma, \Delta, B \otimes C, \exists x A}\,(\text{v}\exists)}\,(\otimes)$$

qed.

Proof of Proposition. We first prove that, for $\Rightarrow \Gamma$ a finite sequent in the classical logical language,

$$\textbf{CPL} \text{ derives } \Rightarrow \Gamma \text{ only if } \textbf{ALV} \text{ derives } \Rightarrow \exists x \Gamma^{*} \tag{7.1}$$

where **CPL** is a Tait-style formulation of classical logic – cf. Figure 7.2. (7.1) is obtained by induction on the length n of the proof of $\Rightarrow \Gamma$ in **CPL**, where length can be taken to be the number of nodes in the maximal path of the derivation tree. If $n = 1$, we have the following derivation of $\Rightarrow \exists x P, \exists x \overline{P}$ in **ALV**[6]

$$\frac{\Rightarrow \exists x \Gamma, P^{\infty}, \overline{P}^{\infty}}{\Rightarrow \exists x \Gamma, \exists x P, \exists x \overline{P}}\,(\text{v}\exists)$$

For $n > 1$, we consider the two different cases of $(\wedge)$ and $(\vee)$. In the former case, we reason as follows:

[6]For a definition of the convention involving the double line, we refer to p. 184.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \exists x\Gamma^*, \exists xA^* \qquad \Rightarrow \exists x\Gamma^*, \exists xB^*}{\Rightarrow (\exists x\Gamma^*)^2, \exists xA^* \otimes \exists xB^*}(\otimes)}{\Rightarrow \exists x\Gamma^*, \exists xA^* \otimes \exists xB^*}(\text{Lemma 7.3.4})}{\Rightarrow \exists x\Gamma^*, (\exists xA^* \otimes \exists xB^*)^\infty}(\text{Weak})}{\Rightarrow \exists x\Gamma^*, \exists x(\exists xA^* \otimes \exists xB^*)}(\text{v}\exists)$$

In the latter, we consider the following proof in **ALV**:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \exists x\Gamma^*, \exists xA^*, \exists xB^*}{\Rightarrow \exists x\Gamma^*, \exists xA^* \parr \exists xB^*}(\parr)}{\Rightarrow \exists x\Gamma^*, (\exists xA^* \parr \exists xB^*)^\infty}(\text{Weak})}{\Rightarrow \exists x\Gamma^*, \exists x(\exists xA^* \parr \exists xB^*)}(\text{v}\exists)$$

qed.

We observe that Lemma 7.3.4 can be proved also if the premise $\Rightarrow \Gamma, \exists xA^n$ for every $n \geq 1$.

Lemma 7.3.5. *If* $\Rightarrow A_1^{*\infty}, \ldots, A_n^{*\infty}$ *is derivable in* **ALV**, *then* **CPL** *derives* $\Rightarrow A_1, \ldots, A_n$.

Proof. The proof is by induction on the height of the derivation in **ALV**. If $\Rightarrow A_1^{*\infty}, \ldots, A_n^{*\infty}$ is an initial sequent, then $\Rightarrow A_1, \ldots, A_n$ is an initial sequent in **CPL**. If $\Rightarrow A_1^{*\infty}, \ldots, A_n^{*\infty}$ is the conclusion of a logical rule we distinguish cases according to the last rule applied. If the last rule applied is $\otimes$ we have:

$$\dfrac{\Rightarrow \exists xB^*, (\exists xB^* \otimes \exists xC^*)^\infty, \ldots, A_n^{*\infty} \qquad \Rightarrow \exists xC^*, (\exists xB^* \otimes \exists xC^*)^\infty, \ldots, A_n^{*\infty}}{\Rightarrow \exists xB^* \otimes \exists xC^*, (\exists xB^* \otimes \exists xC^*)^\infty, \ldots, A_n^{*\infty}}(\otimes)$$

We proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \exists xB^*, (\exists xB^* \otimes \exists xC^*)^\infty, \ldots, A_n^{*\infty}}{\Rightarrow B^{*\infty}, (\exists xB^* \otimes \exists xC^*)^\infty, \ldots, A_n^{*\infty}}(\text{inv})}{\Rightarrow B, B \wedge C, \ldots, A_n}(\text{IH}) \qquad \dfrac{\dfrac{\Rightarrow \exists xC^*, (\exists xB^* \otimes \exists xC^*)^\infty, \ldots, A_n^{*\infty}}{\Rightarrow C^{*\infty}, (\exists xB^* \otimes \exists xC^*)^\infty, \ldots, A_n^{*\infty}}(\text{inv})}{\Rightarrow C, B \wedge C, \ldots, A_n}(\text{IH})}{\Rightarrow B \wedge C, B \wedge C, \ldots, A_n}(\wedge)}{\Rightarrow B \wedge C, \ldots, A_n}(\text{C})$$

where (C) denotes an application of height-preserving admissibility of the rule of contraction in the calculus for classical logic. If the last rule applied is $\parr$, we have:

$$\dfrac{\Rightarrow \exists xB^*, \exists xC^*, \exists xB^{*\infty} \parr \exists xC^{*\infty}, \ldots, A_n^{*\infty}}{\Rightarrow \exists xB^{*\infty} \parr \exists xC^{*\infty}, \ldots, A_n^{*\infty}}(\parr)$$

$$\frac{\Rightarrow \Gamma, \exists x A, A(t/x)}{\Rightarrow \Gamma, \exists x A}\ (\exists) \qquad \frac{\Rightarrow \Gamma, A(y/x)}{\Rightarrow \Gamma, \forall x A}\ (\forall, \text{Y!})$$

Figure 7.3: Classical rules for quantifiers

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \exists x B^*, \exists x C^*, \exists x B^{*\infty} \parr \exists x C^{*\infty}, \ldots, A_n^{*\infty}}{\Rightarrow B^{*\infty}, C^{*\infty}, \exists x B^{*\infty} \parr \exists x C^{*\infty}, \ldots, A_n^{*\infty}}\ (\text{INV})}{\Rightarrow B, C, B \vee C, \ldots, A_n}\ (\text{IH})}{\Rightarrow B \vee C, B \vee C, \ldots, A_n}\ (\vee)}{\Rightarrow B \vee C, \ldots, A_n}\ (\text{C})$$

qed.

We can now prove the faithfulness of the embedding.

Theorem 7.3.6. $\Rightarrow \Gamma$ *is derivable in* **CPL** *if and only if* $\Rightarrow \exists x \Gamma^*$ *is derivable in* **ALV**.

Proof. From left to right we exploit the soundness of the translation. From right to left we apply invertibility of the rule for the existential quantifier and we get a derivation of $\Rightarrow \Gamma^{*\infty}$. We then apply the faithfulness lemma which yields the desired conclusion. *qed.*

7.3.2 Extension to first-order and infinitary logic

We now extend to first-order logic the soundness of the embedding. To do so, we need to introduce clauses which translate the universal and the existential quantifiers. We propose the following:

- $(\exists x A)^* = \exists x \exists y A^*$, y does not occur in A.
- $(\forall x A)^* = \forall x \exists y A^*$, y does not occur in A.

We recall the rules for the universal and existential quantifiers in classical logic in Figure 7.3.2. The rule $(\exists)$ is formulated in a Kleene-style version in order to eliminate the need for an explicit contraction rule (107).

Proposition 7.3.7. *The embedding extends to first-order classical logic.*

Proof. We only need to check the case of the existential quantifier and the universal one. If the last rule applied is $\exists$, we have:

$$\frac{\Rightarrow \Gamma, \exists x A, A(t/x)}{\Rightarrow \Gamma, \exists x A}\ (\exists)$$

By induction on the height of the derivation we get:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Rightarrow \exists y \Gamma^*, \exists y \exists x \exists y A^*, \exists y A^*(t/x)}{\Rightarrow \exists y \Gamma^*, \exists y \exists x \exists y A^*, (\exists y A^*(t/x))^\infty}\ (\textsc{Weak})}{\Rightarrow \exists y \Gamma^*, \exists y \exists x \exists y A^*, \exists x \exists y A^*(t/x)}\ (\mathrm{v}\exists)}{\Rightarrow \exists y \Gamma^*, \exists y \exists x \exists y A^*, (\exists x \exists y A^*(t/x))^\infty}\ (\textsc{Weak})}{\Rightarrow \exists y \Gamma^*, \exists y \exists x \exists y A^*}\ (\mathrm{v}\exists)$$

In the case of the rule $\forall$, we proceed as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\dots \quad \Rightarrow \exists y \Gamma^*, \exists y A^*(t_i/x) \quad \dots}{\Rightarrow (\exists y \Gamma^*)^\infty, \forall x \exists y A^*}\ (\mathrm{v}\forall)}{\Rightarrow \exists y \Gamma^*, \forall x \exists y A^*}\ (\textsc{Lemma}\ 7.3.4)}{\Rightarrow \exists y \Gamma^*, (\forall x \exists y A^*)^\infty}\ (\textsc{Weak})}{\Rightarrow \exists y \Gamma^*, \exists y \forall x \exists y A^*}\ (\mathrm{v}\exists)$$

qed.

The embedding can be further extended to encompass infinitary classical logic, that is the extension of classical logic with the rule:

$$\frac{\Rightarrow \Gamma, A(t_1/v) \dots \Rightarrow \Gamma, A(t_n/v) \dots}{\Rightarrow \Gamma, \forall v A}\ (\forall^\infty\text{-}\textsc{cl})$$

with Γ a finite multiset. The claim follows immediately from

Lemma 7.3.8. *The rule ($\forall^\infty$-*CL*) is admissible in* **ALV** *via the translation* * *of its formulas.*

Proof. We proceed as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Rightarrow \exists x \Gamma^*, \exists x A^*(t_1/v) \quad \dots \quad \Rightarrow \exists x \Gamma^*, \exists x A^*(t_n/v) \dots}{\Rightarrow (\exists x \Gamma^*)^\infty, \forall y \exists x A^*}\ (\mathrm{v}\forall)}{\Rightarrow \exists x \Gamma^*, \forall y \exists x A^*}\ (\textsc{Lemma}\ 7.3.4)}{\Rightarrow \exists x \Gamma^*, (\forall y \exists x A^*)^\infty}\ (\textsc{Weak})}{\Rightarrow \exists x \Gamma^*, \exists z \forall y \exists x A^*}\ (\mathrm{v}\exists)$$

qed.

In the case of infinitary classical logic, we can show that the embedding is indeed faithful, in the sense that if the translation of a sequent is provable in **ALV**, then the sequent is provable in infinitary classical logic.

Theorem 7.3.9. *For any sequent $\Rightarrow \Gamma$, if $\Rightarrow \Gamma^{*\infty}$ is provable in* **ALV***, then $\Rightarrow \Gamma$ is provable in infinitary classical logic.*

Proof. The proof is by induction on the height of the derivation in **ALV** distinguishing cases according to the last rule applied.

Suppose the last rule applied is $\forall$ with principal formula $\forall x \exists y A^*$, we have:

$$\frac{\Rightarrow \Gamma^{*\infty}, (\forall x \exists y A^*)^\infty, \exists y A^*(t_1/x) \qquad \dots \qquad \Rightarrow \Gamma^{*\infty}, (\forall x \exists y A^*)^\infty, \exists y A^*(t_n/x) \dots}{\Rightarrow \Gamma^{*\infty}, (\forall x \exists y A^*)^\infty, \forall x \exists y A^*} (\forall^\infty\text{-CL})$$

we safely assume that the premises contain infinitely many copies of each of the formulas. We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \Gamma^{*\infty}, (\forall x \exists y A^*)^\infty, \exists y A^*(t_1/x)}{\Rightarrow \Gamma^{*\infty}, (\forall x \exists y A^*)^\infty, (A^*(t_1/x))^\infty} (\text{INV})}{\Rightarrow \Gamma, \forall x A, A(t_1/x)} (\text{IH}) \qquad \dots \qquad \dfrac{\dfrac{\Rightarrow \Gamma^{*\infty}, (\forall x \exists y A^*)^\infty, \exists y A^*(t_n/x) \dots}{\Rightarrow \Gamma^{*\infty}, (\forall x \exists y A^*)^\infty, (A^*(t_n/x))^\infty \dots} (\text{INV})}{\Rightarrow \Gamma, \forall x A, A(t_n/x) \dots} (\text{IH})}{\dfrac{\Rightarrow \Gamma, \forall x A, \forall x A}{\Rightarrow \Gamma, \forall x A} (\text{C})} (\forall^\infty\text{-CL})$$

qed.

7.3.3 Vacuous Quantification and Exponentials

In this section we show that affine logic with exponentials can be embedded via a faithful translation in **ALV**.[7]

First we recall the rules which govern the exponentials in affine logic

$$\frac{\Rightarrow \Gamma, ?A, ?A}{\Rightarrow \Gamma, ?A} (?\text{C}) \qquad \frac{\Rightarrow \Gamma, A}{\Rightarrow \Gamma, ?A} (?) \qquad \frac{\Rightarrow ?\Gamma, A}{\Rightarrow \Delta, ?\Gamma, !A} (!)$$

We call **ALE** the resulting system – Affine Logic with Exponentials.

Consider the translation:

- $(P)^\circ = P$
- $(\overline{P})^\circ = \overline{P}$
- $(A \parr B)^\circ = A^\circ \parr B^\circ$
- $(A \otimes B)^\circ = A^\circ \otimes B^\circ$

[7]It is fairly obvious that **ALV** can be faithfully translated in the extension of **AL** with infinitary rules for quantifiers.

- $(?A)^\circ = \exists x A^\circ$
- $(!A)^\circ = \forall x A^\circ$

where the quantifiers are vacuous.

Proposition 7.3.10. *$\Rightarrow \Gamma$ is provable in* **ALE** *if and only if $\Rightarrow \Gamma^\circ$ is provable in* **ALV**.

The proof of Proposition 7.3.10 follows immediately from the the next lemmata.

Lemma 7.3.11. *The following rule is admissible in* **ALV** *for every finite multiset Γ:*

$$\frac{\Rightarrow \exists y\Gamma, A}{\Rightarrow \exists y\Gamma, \forall x A}$$

Proof. The admissibility is proved with the following steps.

$$\cfrac{\cfrac{\ldots \quad \Rightarrow \exists y\Gamma, A \quad \ldots}{\Rightarrow (\exists y\Gamma)^\infty, \forall x A}\,(\mathrm{v}\forall)}{\Rightarrow \exists y\Gamma, \forall x A}\,(\textsc{Lm}.\ 7.3.4)$$

qed.

Lemma 7.3.12. *If* **ALE** *proves $\Rightarrow \Gamma$, then* **ALV** *proves $\Rightarrow \Gamma^\circ$.*

Proof. We argue by induction on the height of the derivation of $\Rightarrow \Gamma$ in **ALE**. The only cases to check are the ones involving exponentials. If the last rule applied is ?C or ! we exploit Lemma 7.3.4 and Lemma 7.3.11. If the last rule applied is ? we use height-preserving admissibility of weakening and the rule $\exists$.

qed.

Lemma 7.3.13. *Let Γ be a finite multiset of formulas of* **ALE** *and $A_1, ..., A_n$ be formulas of* **ALE***:*

If **ALV** *derives $\Rightarrow \Gamma^\circ, A_1^{\circ\infty}, ..., A_n^{\circ\infty}$, then $\Rightarrow \Gamma, ?A_1, ..., ?A_n$ is derivable in* **ALE**.

Proof. We argue by induction on the height of the derivation of $\Rightarrow \Gamma^\circ, A_1^{\circ\infty}, ..., A_n^{\circ\infty}$ in **ALV** distinguishing cases according to the last rule applied.

Since we are working in a setting with admissible weakening, we can safely assume that in applications of the rule $\otimes$ and $\forall$ for every $i \in \{1, \ldots, n\}$ infinitely many occurrences of $A_i^{\circ\infty}$ are present in each premise. If the last rule applied is $\forall$ and the principal formula is in Γ°, we have:

$$\dfrac{\dots \qquad \Rightarrow \Gamma_i^{\circ'}, B^\circ, A_1^{\circ\infty}, ..., A_n^{\circ\infty} \qquad \dots}{\Rightarrow \Gamma^{\circ'}, \forall x B^\circ, A_1^{\circ\infty}, ..., A_n^{\circ\infty}} \;(\mathrm{v}\forall)$$

Since by assumption $\Gamma^{\circ'}$ is finite, there must be an $i < \omega$ such that $\Gamma_i = \emptyset$. We consider that premise $\Rightarrow B^\circ, A_1^{\circ\infty}, ..., A_n^{\circ\infty}$ and we construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Rightarrow B^\circ, A_1^{\circ\infty}, ..., A_n^{\circ\infty}}{\Rightarrow B, ?A_1, ..., ?A_n}\;(\textsc{ih})}{\Rightarrow !B, ?A_1, ..., ?A_n}\;(!)}{\Rightarrow \Gamma', !B, ?A_1, ..., ?A_n}\;(\textsc{Weak})$$

If $\forall x B$ is a formula among $A_1^{\circ\infty}, ..., A_n^{\circ\infty}$ we proceed analogously with an extra application of the rule ?.

If the last rule applied is $\exists$ and the principal formula is among the formulas in $A_1^{\circ\infty}, ..., A_n^{\circ\infty}$, we have:

$$\dfrac{\Rightarrow \Gamma^{\circ'}, B^{\circ\infty}, A_1^{\circ\infty}, ..., A_n^{\circ\infty}}{\Rightarrow \Gamma^{\circ'}, \exists x B^\circ, A_1^{\circ\infty}, ..., A_n^{\circ\infty}} \;(\mathrm{v}\exists)$$

We construct the following derivation:

$$\dfrac{\Rightarrow \Gamma^{\circ'}, B^{\circ\infty}, A_1^{\circ\infty}, ..., A_n^{\circ\infty}}{\Rightarrow \Gamma', ?B, ?A_1, ..., ?A_n} \;(\textsc{ih})$$

The application of the inductive hypothesis suffices.

The remaining cases are easily provable by applications of the inductive hypothesis followed by applications of the rules of the calculus **ALE**. *qed.*

Lemma 7.3.13 gives a formal representation of the intuitive claim about the infinitary nature of exponentials. Indeed, the context-restriction imposed on the rule for the operator ! is simulated by the fact that the infinitary multiplicative rule for $\forall$ yields a premise in which the context not under the scope of ? is absent.

Remark. We observe that due to the transitivity of faithful translations we obtain an alternative proof of the embedding of classical logic into **ALV** as follows:

$$\textbf{CL} \text{ proves } \Rightarrow \Gamma \Leftrightarrow \textbf{ALE} \text{ proves } \Rightarrow \Gamma^\bullet \Leftrightarrow \textbf{ALV} \text{ proves } \Rightarrow (\Gamma^\bullet)^\circ$$

where $\bullet$ is the translation of affine logic into classical logic.

7.3.4 Exponential Liar

From the previous results linking vacuous quantification and the exponentials, and the inconsistency Zardini's system established by (37), we can restore the propositional structure of the derivation of the Liar paradox in full linear and affine logics extended with rules for full disquotation. By our assumptions on λ-terms, we can assume that there is a term $l := \ulcorner ?\overline{\mathrm{Tr}}(l) \urcorner$. We abbreviate with L the sentence $?\overline{\mathrm{Tr}}(l)$. We are also assuming that $\overline{L}$ abbreviates $!\mathrm{Tr}(l)$. Therefore, the rules

$$\dfrac{\Rightarrow \Gamma, ?\overline{\mathrm{Tr}}(l)}{\Rightarrow \Gamma, L}\,(L) \qquad\qquad \dfrac{\Rightarrow \Gamma, !\mathrm{Tr}(l)}{\Rightarrow \Gamma, \overline{L}}\,(\overline{L})$$

are obviously admissible – in fact, the conclusions are just notational variants of the premisses.

Proposition 7.3.14. *Full, propositional linear and affine logics are inconsistent with the rules*

$$\dfrac{\Rightarrow \Gamma, A}{\Rightarrow \Gamma, \mathrm{Tr}\ulcorner A \urcorner}\,(\mathrm{Tr}) \qquad\qquad \dfrac{\Rightarrow \Gamma, \overline{A}}{\Rightarrow \Gamma, \overline{\mathrm{Tr}}\ulcorner A \urcorner}\,(\overline{\mathrm{Tr}})$$

for A a sentence possibly containing exponentials.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Rightarrow \overline{\mathrm{Tr}}(l), \mathrm{Tr}(l)}\,(\textsc{in})}{\Rightarrow ?\overline{\mathrm{Tr}}(l), \mathrm{Tr}(l)}\,(?)}{\Rightarrow ?\overline{\mathrm{Tr}}(l), !\mathrm{Tr}(l)}\,(!)}{\Rightarrow ?\overline{\mathrm{Tr}}(l), \overline{L}}\,(\overline{L})}{\Rightarrow ?\overline{\mathrm{Tr}}(l), \overline{\mathrm{Tr}}(l)}\,(\overline{\mathrm{Tr}})}{\Rightarrow ?\overline{\mathrm{Tr}}(l), ?\overline{\mathrm{Tr}}(l)}\,(?)}{\Rightarrow ?\overline{\mathrm{Tr}}(l)}\,(?\textsc{c})}{\Rightarrow L}\,(L) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Rightarrow \overline{\mathrm{Tr}}(l), \mathrm{Tr}(l)}\,(\textsc{in})}{\Rightarrow ?\overline{\mathrm{Tr}}(l), \mathrm{Tr}(l)}\,(?)}{\Rightarrow ?\overline{\mathrm{Tr}}(l), !\mathrm{Tr}(l)}\,(!)}{\Rightarrow ?\overline{\mathrm{Tr}}(l), \overline{L}}\,(\overline{L})}{\Rightarrow ?\overline{\mathrm{Tr}}(l), \overline{\mathrm{Tr}}(l)}\,(\overline{\mathrm{Tr}})}{\Rightarrow ?\overline{\mathrm{Tr}}(l), ?\overline{\mathrm{Tr}}(l)}\,(?)}{\Rightarrow ?\overline{\mathrm{Tr}}(l)}\,(?\textsc{c})}{\Rightarrow L}\,(L)}{\Rightarrow \mathrm{Tr}(l)}\,(\mathrm{Tr})}{\Rightarrow !\mathrm{Tr}(l)}\,(!)}{\Rightarrow \overline{L}}\,(\overline{L})}{\Rightarrow}\,(\textsc{cut})$$

qed.

Remark. The content of Proposition 7.3.14 shows that - in general - full linear logic with exponentials is enough to simulate the liar paradox when paired with rules for naïve truth. We would like to point out that in our setting the faithful embedding of the exponentials in **ALV** requires the presence of the structural rule of weakening.

7.4 Cut-elimination for multiplicative quantifiers

7.4.1 Zardini's cut-elimination: another visit

The results in the previous sections tell us that Zardini's cut-elimination argument for the theory of naïve truth based on his multiplicative quantifiers cannot work. This leaves open the question whether Zardini's procedure could work in the absence of the rules for the truth predicate. The answer is still negative: (36) found a gap in Zardini's reduction for the quantifiers. Fjellstad isolates an example of a sequent which is obviously cut-free derivable, but such that the cut involved in its proof cannot be eliminated following Zardini's instructions. Although pointing to a serious gap in Zardini's reduction, Fiellstad's example involves a case that can nonetheless be dealt with by supplementing Zardini's original reduction strategy with extra conditions.[8] By contrast, we directly show that Zardini's cut-elimination algorithm is based on a proof-manipulation that does not preserve provability.

The problem involves the elimination of cuts in which the cut formula is principal in both the premises of the cut and is a universal or existential formula. Consider the cut which needs to be eliminated.

$$\dfrac{\dfrac{\dots \quad \Rightarrow \Gamma_i, A(t_i/x) \quad \dots}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \forall x A}\,\forall \qquad \dfrac{\Rightarrow \overline{A}(t_1/x), \overline{A}(t_2/x), ..., \Delta}{\Rightarrow \exists x \overline{A}, \Delta}\,\exists}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \Delta}\,\text{Cut}$$

The solution proposed by Zardini is to reduce the size of the multiset of cut formulas $\overline{A}(t_1/x), \overline{A}(t_2/x), ...$ introduced by the application of $\exists$. In particular, one should trace up the multiset in the derivation until it becomes finite in a branch. By the design of the system a countably infinite (sub)multiset of $\overline{A}(t_1/x), \overline{A}(t_2/x), ...$ can only be introduced by the rule $\forall$ or by a weakened initial sequent, we detail the first case.

[8] To be sure, we believe that Fjellstad's example points to a fundamental flaw in Zardini's strategy, but the specific example does not amount to a knock-down case.

$$\dfrac{\dfrac{\ldots \quad \Rightarrow \Gamma_i,, A(t_i/x) \quad \ldots}{\Rightarrow \biguplus_{i<\omega}\Gamma_i, \forall x A}\,\forall \qquad \dfrac{\begin{array}{c}\dfrac{\ldots \quad \Rightarrow \overline{A}(t_i/x), \Delta_i' \quad \ldots}{\Rightarrow \overline{A}(t_i/x), \overline{A}(t_{i+1}/x), \Delta'}\,\forall \\ \vdots\, \mathcal{D} \\ \Rightarrow \overline{A}(t_1/x), \overline{A}(t_2/x), ..., \Delta\end{array}}{\Rightarrow \exists x \overline{A}, \Delta}\,\exists}{\Rightarrow \biguplus_{i<\omega}\Gamma_i, \Delta}\,\text{Cut}$$

Notice that the principal formula in $\forall$ is not displayed. According to Zardini, we should pick the premise $\Rightarrow \overline{A}(t_i/x), \Delta_i'$ and construct the following derivation.

$$\begin{array}{c}\Rightarrow \overline{A}(t_i/x), \Delta_i' \\ \vdots\, \mathcal{D} \\ \Rightarrow \overline{A}(t_1/x), \overline{A}(t_2/x), ..., \overline{A}(t_i/x), \Delta'\end{array}$$

The cut is then replaced by i many cuts and the desired conclusion follows from the application of the weakening rule. Now, the gap in Zardini argument is exactly in the passage displayed above. In fact, while the sequent $\Rightarrow \overline{A}(t_i/x), \Delta_i'$ is indeed provable, the same cannot be said of the sequent $\Rightarrow \overline{A}(t_1/x), \overline{A}(t_2/x), ..., \overline{A}(t_i/x), \Delta'$. In other words, Zardini's reduction is based on the idea that the derivation $\mathcal{D}$ could be performed *even if one focused on a single premiss only, instead of infinitely many.* For instance, according to the reduction, one could start with the derivation

$$\begin{array}{c}\dfrac{\ldots \quad \Rightarrow \overline{P}(t_i/x), P(t_i/x) \quad \ldots}{\Rightarrow \overline{P}(t_i/x), \overline{P}(t_{i+1}/x), ..., \forall x P}\,\forall \\ \vdots\, \mathcal{D} \\ \Rightarrow \overline{P}(t_1/x), \overline{P}(t_2/x), ..., \Delta'\end{array}$$

According to the reduction, one could then transform the derivation into:

$$\begin{array}{c}\dfrac{\Rightarrow \overline{P}(t_i/x), P(t_i/x)}{\Rightarrow \overline{P}(t_i/x), \forall x P}\,\forall \\ \vdots\, \mathcal{D} \\ \Rightarrow \overline{P}(t_1/x), \overline{P}(t_2/x), ..., \Delta'\end{array}$$

The sequent $\Rightarrow \overline{P}(t_i/x), \forall x P$, however, is clearly not (cut-free) provable.

7.4.2 Eliminating cuts

Zardini's reduction is flawed even if one considers the system without the truth predicate. However, as we shall now demonstrate, cut is eliminable in Zardini's infinitary logic (without truth), i.e. the system $\mathbf{IK}_\omega$.

$$\dfrac{}{\Rightarrow \Gamma, P, \overline{P}}\ (\textsc{in})$$

$$\dfrac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \parr B}\ (\parr) \qquad \dfrac{\Rightarrow \Gamma, A \qquad \Rightarrow \Delta, B}{\Rightarrow \Gamma, \Delta, A \otimes B}\ (\otimes)$$

$$\dfrac{\ldots \qquad \Rightarrow \Gamma_i, A(t_i/x) \qquad \ldots}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \forall x A}\ (\forall) \qquad \dfrac{\Rightarrow \Gamma, A(t_1/x), A(t_2/x), \ldots}{\Rightarrow \Gamma, \exists x A}\ (\exists)$$

Our strategy is based on a double induction, on the length of the derivation and on a modified notion of the degree of formulas which is extended so as to measure the complexity of (possibly infinite) multisets of formulas: for this reason, the proof cannot be lifted to the system with a fully disquotational truth predicate since, as it is well-known, truth collapses the depth of sentences.

We shall eliminate cuts of the form:

$$\dfrac{\Rightarrow \Gamma, \Phi \qquad \{\Rightarrow \Delta_\varphi, \overline{\varphi} \mid \Phi(\varphi) > 0\}}{\Rightarrow \Gamma, \Delta}\ (\textsc{Cut})$$

Intuitively, the (CUT) rule allows one to cut infinitely many formulas simultaneously. Hence we have one premise $\Rightarrow \Gamma, \Phi$, where Φ is the multiset of formulas to cut and (possibly) infinitely many premises $\Rightarrow \Delta_\varphi, \overline{\varphi}$, one for every formula φ with $\Phi(\varphi) > 0$. Finally, the multiset Δ in the conclusion denotes the infinitary multiset union of all the multisets Δ_φ. To eliminate the cut we need to distinguish between complexity of formulas and complexity of multisets of formulas.

The depth of a formula $dp(\varphi)$ is the number of logical symbols (including quantifiers) occurring in it. We shall reason by double induction, with main induction hypothesis on the degree of the multiset of cut formulas, i.e. $dg(\Phi) = sup_{\Phi(\varphi)>0}(dp(\varphi)) + 1$ (the degree of a multiset will be - in general - an ordinal), and secondary induction hypothesis on the Hessenberg ordinal sum of the height of the derivations (which is commutative, associative, left and right cancellative and strictly monotone in both arguments). The key point of the reduction is the fact that infinite multisets of the form $[A(t_i/x) \mid i \in I]$ have a *finite degree*, because all the formulas occurring inside them have the same depth.

We first prove an auxiliary lemma which enables us to remove cuts on atomic formulas.

Lemma 7.4.1. *For any multiset* Γ, Δ *and any literal* P*, the rule:*

$$\frac{\Rightarrow \Gamma, P \qquad \Rightarrow \Delta, \overline{P}}{\Rightarrow \Gamma, \Delta} \text{(CUTAT)}$$

is admissible.

Proof. The proof is by induction on the height of $\Rightarrow \Gamma, P$. If Γ, P is an initial sequent, the proof follows by admissibility of weakening. If $\Rightarrow \Gamma, P$ is not an initial sequent, then it is the conclusion of a rule and P cannot be the principal formula. In this case, we permute the cut upward and we eliminate it by induction on the height of the derivation. *qed.*

Theorem 7.4.2. *The cut rule is admissible in* $\mathbf{IK}_\omega$.

Proof. By double (transfinite) induction with main induction hypothesis on the degree of the multiset of cut formulas and secondary induction hypothesis on the height of the left premise of the cut, i.e. Γ, Φ.

If $\Rightarrow \Gamma, \Phi$ is an initial sequent, we distinguish cases. If no formula is active in Φ, then $\Rightarrow \Gamma, \Delta$ is an initial sequent too. If one formula is active in Φ, then the proof follows by weakening. If both the atomic formulas are active in Φ, i.e. if $\Phi \equiv \Phi', P, \overline{P}$, then we have two premises $\Rightarrow \Delta_P, P$ and $\Rightarrow \Delta_{\overline{P}}, \overline{P}$ and the desired conclusion follows by an application of the admissible rule CUTAT.

If no formula in Φ is principal, the cut is permuted upwards (possibly replaced by infinitely many cuts) and removed by secondary induction hypothesis.

If a formula is principal in Φ, we distinguish cases according to its shape. We focus on the cases of the quantifiers, as they are the relevant ones. If a formula of the shape $\forall x A$ is principal, we have:

$$\frac{\Rightarrow \Gamma_1, \Phi_1, A(t_1/x) \qquad \dots \qquad \Rightarrow \Gamma_n, \Phi_n, A(t_n/x) \dots}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \biguplus_{i<\omega} \Phi_i, \forall x A} (\forall)$$

The other premises of the cut will be $\Delta, \exists x \overline{A}$ and $\Theta_\varphi, \overline{\varphi}$ for every φ in Φ. First, for every $i < \omega$, we perform the following reduction:

$$\frac{\Rightarrow \Gamma_i, \Phi_i, A(t_i/x) \qquad \{\Rightarrow \Theta_\varphi, \overline{\varphi} \mid \Phi_i(\varphi) > 0\}}{\Rightarrow \Gamma_i, \Theta_i, A(t_i/x)} \text{(CUT)}$$

where Θ_i is the multiset union of all the multisets Θ_φ with $\Phi_i(\varphi) > 0$. The cut is removed by secondary induction hypothesis on the height of the left premise of the cut. We then apply height-preserving invertibility of the rule $\exists$ to $\Rightarrow \Delta, \exists x \overline{A}$

to get $\Rightarrow \Delta, \overline{A}(t_1/x), \overline{A}(t_2/x), \ldots$. Finally we proceed with the following cut:

$$\frac{\Rightarrow \Delta, \overline{A}(t_1/x), \overline{A}(t_2/x), \ldots \qquad \{\Rightarrow \Gamma_i, \Theta_i, A(t_i/x) \mid i < \omega\}}{\Rightarrow \biguplus_{i<\omega} \Gamma_i, \Theta, \Delta} \text{(Cut)}$$

This cut is removed by primary induction hypothesis on the degree of the multiset of cut formulas which is strictly decreased.

If the principal formula is an existential one, we have

$$\frac{\Rightarrow \Gamma, \Phi, A(t_1/x), A(t_2/x), \ldots}{\Rightarrow \Gamma, \Phi, \exists x A} (\exists)$$

In this case we look at the premise of the cut of the shape $\Rightarrow \Delta, \forall x\overline{A}$ and we distinguish two subcases. Either $\forall x\overline{A}$ is principal in an inference rule in the derivation or not. In the latter case, then Δ is already derivable and we obtain the desired conclusion via weakening. In the former case we go upwards to the point in which $\forall x\overline{A}$ is principal (by the design of the rules $\forall x\overline{A}$ will be only in one branch). We have:

$$\begin{array}{c}\dfrac{\Rightarrow \Delta'_1, \overline{A}(t_1/x) \qquad \ldots \qquad \Rightarrow \Delta'_n, \overline{A}(t_n/x)}{\Rightarrow \biguplus_{i<\omega} \Delta'_i, \forall x\overline{A}} (\forall) \\ \vdots \mathcal{D} \\ \Rightarrow \Delta, \forall x\overline{A}\end{array}$$

We perform the following reduction:

$$\begin{array}{c}\dfrac{\dfrac{\Rightarrow \Gamma, \Phi, A(t_1/x), A(t_2/x), \ldots \qquad \{\Rightarrow \Theta_\varphi, \overline{\varphi} \mid \Phi(\varphi) > 0\}}{\Rightarrow \Theta, \Gamma, A(t_1/x), A(t_2/x), \ldots} \text{(Cut)} \qquad \{\Rightarrow \Delta'_i, \overline{A}(t_i/x) \mid i \in I\}}{\Rightarrow \Theta, \Gamma, \biguplus_{i<\omega} \Delta'_i} \text{(Cut)} \\ \vdots \mathcal{D} \\ \Rightarrow \Theta, \Gamma, \Delta\end{array}$$

The topmost cut is removed by secondary induction hypothesis on the height of the left premise of the cut, whereas the lowermost is removed by induction on the degree of the multiset of cut formulas which has - again - strictly decreased. *qed.*

We have introduced an approach to cut-elimination for multiplicative quantifiers. It seems hard to generalize it so as to encompass a theory of truth. Indeed, we use a double induction on two measures, one of which is a kind of measure of complexity of formulas. It is well known that rules for naïve truth collapse the depth of sentences: any attempt to reduce a cut on $\mathrm{Tr}\ulcorner A \urcorner$ to a cut on A need to deal with the fact that the depth of A is arbitrary larger than the minimal depth of $\mathrm{Tr}\ulcorner A \urcorner$. However, this is coherent with what we know about the interaction of truth and Zardini's rules, given that the original system by Zardini is inconsistent. We believe that – as pointed out also in (82) – the explicit presence of a double inductive parameter in the cut-elimination procedure brings to the fore the hidden presence of contraction.

7.5 Concluding remarks and future work

We conducted an investigation into contraction-free systems and their potential use in solving paradoxes in the context of truth theories Furthermore, we proposed a novel way of understanding exponentials, which offers an alternative interpretation of an inherently modal concept. Our study ultimately led us to develop a new cut-elimination procedure for infinitary sequents, allowing for a proof-theoretical analysis of multiplicative quantifiers.

Moving forward, several open problems warrant further investigation. For example, finding a suitable truth predicate to incorporate into the base theory while maintaining consistency is an intriguing challenge, given that systems based on multiplicative quantifiers are not entirely contraction-free. Moreover, Grišin set theory is inconsistent modulo the addition of extensionality. A natural question arises as to whether there exists a natural corresponding property in the case of truth theories based on contraction-free systems with additive (or classical, one may say) quantifiers.

Furthermore, it is important to determine whether the cut-elimination theorem can be generalized to the case of infinitary logic with infinite sequents, with particular attention to the strength of the resulting system.

Finally, in order to avoid the implicit contraction found in the notion of infinite multiset in Zardini's naive non-contractive system, it would be beneficial to investigate multiplicative, infinitary rules developed using a notion of multiset that can account for copies of different infinite multiplicities.

Chapter 8

Proof theory for infinite sequents

We deal with with a purely syntactic analysis of infinitary logic with infinite sequents. In particular, we discuss sequent calculi calculi for classical and intuitionistic infinitary logic with good structural properties based on sequents possibly containing infinitely many formulas. A cut-elimination proof is proposed which employs a new strategy and is based on the new inductive parameter introduced in Chapter 7. We conclude the chapter by discussing related issues and possible themes for future research.

8.1 Introduction

Infinitary logics are described by languages including expressions of infinite length. In particular, we shall be concerned with languages augmented with countable conjunctions and disjunctions (99). From a proof-theoretic point of view, infinitary logics have been investigated with various different approaches. We recall some of the most common methods employed in Gentzen style proof theory:

- Derivations are well-founded trees possibly infinitely branching.
- Derivations are well-founded trees possibly infinitely branching in which every node is occupied by a possibly infinite sequent.
- Derivations are non well-founded trees.

The first approach is the one which has proved to be the most flexible in the context of predicative proof theory and ordinal analysis (see (95; 89)). The structural analysis of such systems allowed to establish cut-elimination and therefore analyticity (for another, more recent approach to the issue, see (76) and (93)).

Derivations as non well-founded trees are the central ingredient of cyclic proof theory in which derivations are allowed to contain branches of infinite length. This method has proved to be particularly promising to accommodate the proof theory of modal fixpoint logics (2).

Finally, derivations with infinite sequents have been considered in the literature in the work of Takeuti (100) and Lopez-Escobar (60). However, the structural properties of the systems were established by means of semantic approaches, such as by showing that every sequent admits either a proof or a countermodel. This approach cannot be regarded as completely satisfactory for two distinct reasons.

First, cut-elimination is a syntactic property of a system and a semantic proof thereof is not conceptually pure as it uses tools which are external to the system itself. Second, the use of semantics to establish cut-free completeness is a move available only in the presence of a suitable structure to interpret the logic, whereas syntactic cut-elimination does not require it.

There is an evident difficulty concerning the cut-elimination theorem in the context of infinitary logic with infinite sequents. To witness this, consider the reduction in which the cut formula formula is an infinitary conjunction principal in both the premises of the cut.

$$\dfrac{\dfrac{\{\Gamma \Rightarrow \Delta, A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{k>0} A_k}\,\text{R}\wedge \qquad \dfrac{\{A_k\}_{k>0}, \Pi \Rightarrow \Sigma}{\bigwedge_{k>0} A_k, \Pi \Rightarrow \Sigma}\,\text{L}\wedge}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,\text{Cut}$$

A naive attempt at eliminating the cut could consist in replacing it with infinitely many cuts on the formulas A_k. However, this strategy is not available insofar as derivations have to respect the well-foundedness conditions and this would lead to non-terminating cut reductions.

A syntactic proof of the elimination of these cuts requires to eliminate a infinitary generalization of the cut and to devise a new reduction strategy, together with a new inductive parameter.

We shall focus on intuitionistic and classical infinitary logic. We start discussing classical infinitary logic in a **G3**-style sequent setting with infinite sequents. In classical infinitary logic every rule is height-preserving invertible. Next, we consider intuitionistic infinitary logic.

Infinitary intuitionistic logic has been investigated by Kalicki (52) and Nadel (71) who provided an interpretation in terms of complete Heyting algebras and showed that Kripke semantics turned out to be inadequate. The closure under infinite intersections of opens in the underlying topology forces the intuitionistically unacceptable distributivity principle:

$$\bigwedge_{k>0}(p_k \vee q) \rightarrow \bigwedge_{k>0} p_k \vee q$$

In the work (103) a new semantics for intuitionistic infinitary logic was provided. In particular, starting from the topological interpretation of intuitionistic logic - see (66) for a topological semantics of intuitionistic propositional logic - and building on the work of Moniri and Maleki (69) a topological and a neighborhood semantics (see (79) for an introduction) were introduced and studied.

The results contained in the chapter show that systems with infinite sequents are conservative with respect to those with finite ones (at least considering rules in which there is a single principal formula). Furthermore, soundness and completeness are preserved.

This chapter can be conceived as an extension of the strategy employed for multiplicative quantifiers to intuitionistic and classical logic with infinite sequents.

The plan of the chapter is as follows. In Section 2 we provide a gentle introduction to the notion of infinite sequents and we introduce the classical calculus. The usual structural properties are established and cut is eliminated for the propositional fragment. Section 3 is devoted to the analysis of the intuitionistic calculus: a full cut elimination theorem is proved. Next, Section 4 comes full circle by inducing a syntactic cut-elimination in the full system for classical logic *modulo* an extension of the negative translation of classical logic into intuitionistic logic in the infinitary setting. Finally, Section 5 discusses some themes which may be object of future research.

8.2 Infinite sequents

The language of infinitary logic is built from predicate letters, connectives and quantifiers as usual and two infinitary connectives $\bigwedge$ and $\bigvee$ which denote the countable infinitary conjunction and disjunction, respectively.

As in the previous chapter, a multiset of formulas is here defined as a function $f : A \rightarrow \omega + 1$. It is immediate from this stipulation that we accept multisets containing countably many formulas. As a consequence, sequents are now conceivable as syntactic objects of the shape:

$$\Gamma \Rightarrow \Delta$$

where Γ and Δ are multisets of formulas.

G3C$_\omega^\infty$

Initial Sequents

$$\frac{}{p, \Gamma \Rightarrow \Delta, p}\ Ax \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\ L\rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ R\rightarrow$$

$$\frac{\{A_k\}_{k>0}, \Gamma \Rightarrow \Delta}{\bigwedge\limits_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigwedge \qquad \frac{\{\Gamma \Rightarrow \Delta, A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge\limits_{k>0} A_k}\ R\bigwedge$$

$$\frac{\{A_k, \Gamma \Rightarrow \Delta \mid k > 0\}}{\bigvee\limits_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, \{A_k\}_{k>0}}{\Gamma \Rightarrow \Delta, \bigvee\limits_{k>0} A_k}\ R\bigvee$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/y]}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall, y!$$

$$\frac{A[x/y], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists, y! \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

The most striking difference between the above defined system and the usual presentations of infinitary logic lies in the fact that we opt for a *multiplicative* formulation of the unary rules for the infinitary connectives. In infinitary logic the multiplicative formulation of such rules imposes a change in the structures which are manipulated.

Namely, sequents are now built from infinite multisets of formulas. This requires a new approach to the structural analysis of the calculus. The principal tool in our investigations will be transfinite induction which we shall use to study the properties of derivations.

Definition 8.2.1. A derivation is a (possibly infinitely branching) rooted tree, where the leaves are initial sequents and the other nodes are constructed according to the rules.

Notice that this definition rules out the possibility of proofs with branches of infinite length. To measure the length of the derivations we assign once again countable ordinals as in the previous sections.

We shall also use measures to assess the complexity of formulas. Once again, these will be countable ordinals.

Definition 8.2.2. The *weight* of a formula A is inductively defined:

- $w(P) = 0$ if P atomic.
- $w(A\#B) = sup(w(A), w(B)) + 1$, where $\# \in \{\wedge, \vee, \rightarrow\}$.
- $w(\bigwedge_{k>0} A_k) = w(\bigvee_{k>0} A_k) = (sup_{k>0} w(A_k)) + 1$.
- $w(QxA) = w(A) + 1$.

Lemma 8.2.1. *The sequent* $\Gamma, A \Rightarrow \Delta, A$ *is derivable for every formula* A *and multiset* Γ *and* Δ.

Proof. The proof is by transfinite induction on the weight of the formula A. We detail the case in which A is an infinite disjunction.

$$\dfrac{\dfrac{\{\Gamma, A_k \Rightarrow \Delta, \{A_k\}_{k>0} \mid k > 0\}}{\Gamma, \bigvee_{k>0} A_k \Rightarrow \Delta, \{A_k\}_{k>0}}\, L\bigvee}{\Gamma, \bigvee_{k>0} A_k \Rightarrow \Delta, \bigvee_{k>0} A_k}\, R\bigvee$$

The topmost sequent is provable by induction hypothesis. *qed.*

The substitution of a variable x with a term t is defined as usual.

Lemma 8.2.2. *For every variable x and term t for the language, the rule:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma[x/t] \Rightarrow \Delta[x/t]}\ Sub[x/t]$$

is height-preserving admissible in $\mathbf{G3C}^{\infty}_{\omega}$.

Proof. The proof runs by transfinite induction on the height of the derivation of the sequent $\Gamma \Rightarrow \Delta$. *qed.*

Next, we need to establish the admissibility of the rule of weakening. In this case, we need to be able to add infinite multisets of formulas.

Lemma 8.2.3. *The rule:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ Weak$$

is height-preserving admissible in $\mathbf{G3C}^{\infty}_{\omega}$.

Proof. We argue by transfinite induction, possibly using the admissibility of the rule of substitution in order to avoid clashes of variables whenever the last rule applied is L$\exists$ or R$\forall$. *qed.*

We now need to establish the invertibility of the rules of the calculus with preservation of the height. This is a relevant point, as usual formulations of the unary rules for infinitary connectives are invertible due to the repetition of the principal formula in the premise of the rule (76).

Lemma 8.2.4. *Every rule is height-preserving invertible in* $\mathbf{G3C}^{\infty}_{\omega}$.

Proof. The proof is by transfinite induction on the height of the derivation. We detail the case of the rule R$\bigvee$. If $\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k$ is an initial sequent, then so is $\Gamma \Rightarrow \Delta, \{A_k\}_{k>0}$. If it is the conclusion of a rule, we apply the induction hypothesis and then the rule again. For example, if the last rule applied is R$\bigwedge$, we have:

$$\frac{\{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k, B_i \mid i > 0\}}{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k, \bigwedge_{i>0} B_i}\ \mathrm{R}\bigwedge$$

We proceed as follows:

$$\frac{\dfrac{\{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k, B_i \mid i > 0\}}{\{\Gamma \Rightarrow \Delta, \{A_k\}_{k>0}, B_i \mid i > 0\}}\ \mathrm{IH}}{\Gamma \Rightarrow \Delta, \{A_k\}_{k>0}, \bigwedge_{i>0} B_i}\ \mathrm{R}\bigwedge$$

The other cases are dealt with analogously. *qed.*

Next key step is the admissibility of the rule of contraction. In this case we need to contract infinitely many formulas.

Lemma 8.2.5. *The rule:*

$$\frac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\, Ctr$$

is height-preserving admissible in $\mathbf{G3C}^{\infty}_{\omega}$.

Proof. We argue by induction on the height of the derivation of the sequent $\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma$. If it is an initial sequent, then so is $\Gamma, \Pi \Rightarrow \Delta, \Sigma$. If it is the conclusion of a rule, we need to distinguish two subcases. If the principal formula is in Γ or Δ, then we apply the induction hypothesis to each of the premises of the rule and then the rule again. The general structure of the argument is as follows:

$$\frac{\ldots \quad \Gamma', \Pi, \Pi \Rightarrow \Delta', \Sigma, \Sigma \quad \ldots}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma}\, \rho$$

We proceed as follows:

$$\frac{\ldots \quad \dfrac{\Gamma', \Pi, \Pi \Rightarrow \Delta', \Sigma, \Sigma}{\Gamma', \Pi \Rightarrow \Delta', \Sigma}\, \text{IH} \quad \ldots}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\, \rho$$

If one of the formulas in Π or Σ is principal, then we need to distinguish cases according to the shape of the formulas. The strategy consists in applying the invertibility of the corresponding rule and then the induction hypothesis. Let us consider the case in which the principal formula is in Σ and is $\bigvee_{k>0} A_k$.

$$\frac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma', \Sigma', \bigvee_{k>0} A_k, \{A_k\}_{k>0}}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma', \Sigma', \bigvee_{k>0} A_k, \bigvee_{k>0} A_k}\, \text{R}\bigvee$$

We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma', \Sigma', \bigvee_{k>0} A_k, \{A_k\}_{k>0}}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma', \Sigma', \{A_k\}_{k>0}, \{A_k\}_{k>0}}\, \text{InvR}\bigvee}{\Gamma, \Pi \Rightarrow \Delta, \Sigma', \{A_k\}_{k>0}}\, \text{IH}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma', \bigvee_{k>0} A_k}\, \text{R}\bigvee$$

This is the critical case which requires to prove a stronger version of the admissibility of contraction, namely the contraction of infinite multisets of formulas. *qed.*

Finally, the last part of the section is devoted to the structural analysis of the classical calculus for infinitary classical logic. We shall eliminate a generalized form of the cut rule. In essence, the crucial move is the shift from a cut between formulas to a more complex cut between multisets of formulas. This rule can be found in the literature, see (60), (100) and (68).[1] Recently, a variant of such rule

[1]However, a syntactic proof of its admissibility is - to the best of our knowledge - not present in the literature.

was shown to be admissible in a system for multiplicative quantifiers through a purely proof-theoretic argument (77).

The rule is:

$$\frac{\{\Pi_\varphi \Rightarrow \Sigma_\varphi, \varphi \mid \varphi \in \Phi\} \qquad \Phi, \Gamma \Rightarrow \Delta, \Psi \qquad \{\psi, \Theta_\psi \Rightarrow \Lambda_\psi \mid \psi \in \Psi\}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\text{Cut}$$

where $\Pi = \bigcup_{\varphi\in\Phi} \Pi_\varphi$, $\Sigma = \bigcup_{\varphi\in\Phi} \Sigma_\varphi$, $\Theta = \bigcup_{\varphi\in\Phi} \Theta_\varphi$ and $\Lambda = \bigcup_{\varphi\in\Phi} \Lambda_\varphi$. We need to introduce an ordinal measure for multisets of formulas. Essentially we define the degree of a multiset of formulas to be the supremum of the weight of the formulas in it plus one. Formally,

$$deg(\Xi) = sup_{\xi\in\Xi} w(\xi) + 1$$

for any multiset Ξ. The proof will run by double transfinite induction, with main induction on $deg(\Phi, \Psi)$ and a secondary induction hypothesis on the height of the derivation of the sequent $\Phi, \Gamma \Rightarrow \Delta, \Psi$.

We first have to prove a preliminary theorem which involves the elimination of atomic cuts.

Theorem 8.2.6. *The rule:*

$$\frac{\Gamma \Rightarrow \Delta, P \qquad P, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\mathit{Cutat}$$

is admissible in $\mathbf{G3C}^{\infty}_{\omega}$.

Proof. The proof is by induction on the height of $P, \Pi \Rightarrow \Sigma$. If it is an initial sequent and P is not active, we remove it. Otherwise, we take the premise $\Gamma \Rightarrow \Delta, P$ and we obtain the desired conclusion via height-preserving admissibility of weakening. If $P, \Pi \Rightarrow \Sigma$ is the conclusion of a rule, then we notice that P is never principal by the design of the rules, so the cut can be permuted upwards and removed by the induction hypothesis (if necessary making use of the substitution lemma in order to avoid clashes of variables). *qed.*

We are now ready to prove the cut-elimination theorem. We shall establish the result for the propositional fragment of $\mathbf{G3C}^{\infty}_{\omega}$, i.e. the calculus $\mathbf{G3C}^{\infty}_{\omega}$, obtained from the previous one by dropping the rules for the quantifiers.[2]

Theorem 8.2.7. *The cut rule is admissible in* $\mathbf{G3C}^{\infty}_{\omega}$.

[2]The full cut-elimination theorem will be proved in Section 4 exploiting the result obtained for the intuitionistic calculus.

Proof. The proof is by double transfinite induction as explained above. If $\Phi, \Gamma \Rightarrow \Delta, \Psi$ is an initial sequent, four subcases have to be distinguished.

- The active formula P is neither in Φ, nor in Ψ. In this case the conclusion of the cut is an initial sequent too.
- The active formulas are in Φ and in Δ. In this case, we consider the premise $\Pi_P \Rightarrow \Sigma_P, P$ and we apply weakening in order to obtain the desired conclusion.
- The active formulas are in Γ and in Ψ. Symmetric to the previous case.
- The active formulas are in Φ and in Ψ. We consider the premises $\Pi_P \Rightarrow \Sigma_P, P$ and $P, \Theta_P \Rightarrow \Lambda_P$ and we perform the following transformation:

$$\dfrac{\dfrac{\Pi_P \Rightarrow \Sigma_P, P \qquad P, \Theta_P \Rightarrow \Lambda_P}{\Pi_P, \Theta_P \Rightarrow \Lambda_P, \Sigma_P}\,\text{Cutat}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Lambda, \Sigma}\,\text{Weak}$$

If $\Phi, \Gamma \Rightarrow \Delta, \Psi$ is the conclusion of a rule and the principal formula is neither in Φ or in Ψ, then the cut is permuted upwards and replaced by possibly infinite cuts of lesser height. The general structure of the reduction is:

$$\dfrac{\{\Pi_\varphi \Rightarrow \Sigma_\varphi, \varphi \mid \varphi \in \Phi\} \qquad \dfrac{\{\Phi, \Gamma_i \Rightarrow \Delta_i, \Psi \mid i \in I\}}{\Phi, \Gamma \Rightarrow \Delta, \Psi}\,\rho \qquad \{\psi, \Theta_\psi \Rightarrow \Lambda_\psi \mid \psi \in \Psi\}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Cut}$$

where $|I|$ is the cardinality of the set of premises of the rule ρ. We construct the following derivation:

$$\dfrac{\ldots \qquad \dfrac{\{\Pi_\varphi \Rightarrow \Sigma_\varphi, \varphi \mid \varphi \in \Phi\} \qquad \Phi, \Gamma_i \Rightarrow \Delta_i, \Psi \qquad \{\psi, \Theta_\psi \Rightarrow \Lambda_\psi \mid \psi \in \Psi\}}{\Gamma_i, \Pi, \Theta \Rightarrow \Delta_i, \Sigma, \Lambda}\,\text{Cut} \qquad \ldots}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\rho$$

The cuts are removed invoking the secondary induction hypothesis on the height of the premise $\Phi, \Gamma_i \Rightarrow \Delta_i, \Psi$ which has strictly decreased.

The final and crucial case is the one in which one of the formulas in Φ or Ψ is principal in the last rule applied. The general strategy consists in removing all the non principal formulas via cuts which are permuted upwards and then finishing with cuts on multisets of lesser degree. Of course, we need to distinguish cases according to the shape of the principal formula and to its position in the sequent.

We consider the case in which the principal formula is $\bigwedge_{k>0} A_k$ in Φ.

$$\dfrac{\{\Pi_\varphi \Rightarrow \Sigma_\varphi, \varphi \mid \varphi \in \Phi\} \qquad \dfrac{\Phi', \{A_k\}_{k>0}, \Gamma \Rightarrow \Delta, \Psi}{\Phi', \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta, \Psi}\,\text{L}\bigwedge \qquad \{\psi, \Theta_\psi \Rightarrow \Lambda_\psi \mid \psi \in \Psi\}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Cut}$$

In this case we first permute the cut upwards as follows:

$$\dfrac{\{\Pi_\varphi \Rightarrow \Sigma_\varphi, \varphi \mid \varphi \in \Phi'\} \qquad \Phi', \{A_k\}_{k>0}, \Gamma \Rightarrow \Delta, \Psi \qquad \{\psi, \Theta_\psi \Rightarrow \Lambda_\psi \mid \psi \in \Psi\}}{\{A_k\}_{k>0}, \Gamma, \Pi', \Theta \Rightarrow \Delta, \Sigma', \Lambda}\,\text{Cut}$$

where $\Pi' = \bigcup_{\varphi\in\Phi'} \Pi_\varphi$ and $\Sigma' = \bigcup_{\varphi\in\Phi'} \Sigma_\varphi$. This cut is removed by the secondary induction hypothesis on the height of the derivation. We then consider the premise $\Pi_{\bigwedge_{k>0} A_k} \Rightarrow \Sigma_{\bigwedge_{k>0} A_k}, \bigwedge_{k>0} A_k$ and we apply height-preserving invertibility of the rule R$\bigwedge$ to get a derivation of $\Pi_{\bigwedge_{k>0} A_k} \Rightarrow \Sigma_{\bigwedge_{k>0} A_k}, A_k$ for every $k > 0$. Hence we conclude the reduction as follows:

$$\dfrac{\{\Pi_{\bigwedge_{k>0} A_k} \Rightarrow \Sigma_{\bigwedge_{k>0} A_k}, A_k \mid k > 0\} \qquad \{A_k\}_{k>0}, \Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Cut and Ctr}$$

Since, by definition of degree,

$$deg([A_k]_{k>0}) = sup_k w(A_k){+}1 < sup_k w(A_k){+}2 = sup w(\bigwedge_{k>0} A_k){+}1 = deg([\bigwedge_{k>0} A_k]) \leq deg(\Phi, \Psi)$$

the application of cut can be removed invoking the primary induction hypothesis on the degree of the multiset of cut formulas.

We discuss the case in which $\bigwedge_{k>0} A_k$ is principal in Ψ. In this case, the procedure is similar.

$$\dfrac{\{\Pi_\varphi \Rightarrow \Sigma_\varphi, \varphi \mid \varphi \in \Phi\} \qquad \dfrac{\{\Phi, \Gamma \Rightarrow \Delta, \Psi', A_k \mid k > 0\}}{\Phi, \Gamma \Rightarrow \Delta, \Psi', \bigwedge_{k>0} A_k}\,\text{R}\bigwedge \qquad \{\psi, \Theta_\psi \Rightarrow \Lambda_\psi \mid \psi \in \Psi\}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\,\text{Cut and Ctr}$$

We start by permuting the cut upwards as before. First, for every $k > 0$ we construct the following derivation:

$$\dfrac{\{\Pi_\varphi \Rightarrow \Sigma_\varphi, \varphi \mid \varphi \in \Phi\} \qquad \Phi, \Gamma \Rightarrow \Delta, \Psi', A_k \qquad \{\psi, \Theta_\psi \Rightarrow \Lambda_\psi \mid \psi \in \Psi'\}}{\Gamma, \Pi, \Theta' \Rightarrow \Delta, \Sigma, \Lambda', A_k}\,\text{Cut}$$

where $\Theta' = \bigcup_{\psi\in\Psi'} \Theta_\psi$ and $\Lambda' = \bigcup_{\psi\in\Psi'} \Lambda_\psi$. For every $k > 0$ the cut is removed by secondary induction hypothesis on the height of the premise $\Phi, \Gamma \Rightarrow \Delta, \Psi', A_k$. We thus get a set of derivations of sequents

$$\{\Gamma, \Pi, \Theta' \Rightarrow \Delta, \Sigma, \Lambda', A_k \mid k > 0\}$$

Next, we complete the reduction as follows:

$$\dfrac{\{\Gamma, \Pi, \Theta' \Rightarrow \Delta, \Sigma, \Lambda', A_k \mid k > 0\} \qquad \dfrac{\bigwedge_{k>0} A_k, \Theta_{\bigwedge_{k>0} A_k} \Rightarrow \Lambda_{\bigwedge_{k>0} A_k}}{\{A_k\}_{k>0}, \Theta_{\bigwedge_{k>0} A_k} \Rightarrow \Lambda_{\bigwedge_{k>0} A_k}}\text{Inv}}{\Gamma, \Pi, \Theta \Rightarrow \Delta, \Sigma, \Lambda}\text{Cut and Ctr}$$

The cut can be removed by primary induction hypothesis. *qed*.

8.3 Infinitary intuitionistic logic

We introduce a single succedent sequent calculus for intuitionistic logic with infinite sequents. In intuitionistic logic we achieve a syntactic cut-elimination theorem for the full calculus with infinite sequents. Indeed, the previous strategy is fully extendable due to the lack of a built-in contraction rule in the right rule for the existential quantifier.

$\mathbf{G3I}^{\infty}_{\omega}$

Initial Sequents

$$\frac{}{p, \Gamma \Rightarrow p}\, Ax \qquad \frac{}{\bot, \Gamma \Rightarrow C}\, L\bot$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\, L\wedge \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\, R\wedge$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\, L\vee \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\, R\vee_i$$

$$\frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\, L\to \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\, R\to$$

$$\frac{\{A_k\}_{k>0}, \Gamma \Rightarrow C}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow C}\, L\bigwedge \qquad \frac{\{\Gamma \Rightarrow A_k \mid k > 0\}}{\Gamma \Rightarrow \bigwedge_{k>0} A_k}\, R\bigwedge$$

$$\frac{\{A_k, \Gamma \Rightarrow C \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow C}\, L\bigvee \qquad \frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow \bigvee_{k>0} A_k}\, R\bigvee_k$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\, L\forall \qquad \frac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x A}\, R\forall, y!$$

$$\frac{A[x/y], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\, L\exists, y! \qquad \frac{\Gamma \Rightarrow A[x/t]}{\Gamma \Rightarrow \exists x A}\, R\exists$$

Preliminary results still need to be established for the system $\mathbf{G3I}^{\infty}_{\omega}$. The notions of height, admissibility and weight are left unchanged.

Lemma 8.3.1. *The sequent* $\Gamma, A \Rightarrow A$ *is provable for every formula* A *in* $\mathbf{G3I}^{\infty}_{\omega}$.

Proof. By transfinite induction on the weight of A. We discuss the case of the infinitary conjunction:

$$\dfrac{\dfrac{\{\{A_k\}_k > 0, \Gamma \Rightarrow A_k \mid k > 0\}}{\{A_k\}_k > 0, \Gamma \Rightarrow \bigwedge_{k>0} A_k}\ \mathrm{R}\bigwedge}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow \bigwedge_{k>0} A_k}\ \mathrm{L}\bigwedge$$

qed.

The rule of substitution is height-preserving admissible too.

Lemma 8.3.2. *The rule:*

$$\frac{\Gamma \Rightarrow C}{\Gamma[x/t] \Rightarrow C[x/t]}\ Sub[x/t]$$

is height-preserving admissible in $\mathbf{G3I}^{\infty}_{\omega}$.

Proof. The proof is by transfinite induction on the height of the derivation of the premise $\Gamma \Rightarrow C$. *qed.*

The weakening rule is admissible too.

Lemma 8.3.3. *The rules of weakening:*

$$\frac{\Gamma \Rightarrow C}{\Pi, \Gamma \Rightarrow C}\ LW \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow C}\ RW$$

are height-preserving admissible in $\mathbf{G3I}^{\infty}_{\omega}$.

Proof. The proof is by transfinite induction on the height of the derivation of the premise of the rule. *qed.*

We now have to discuss the invertibility of the rules of the calculus. As it well known, in intuitionistic logic we lose invertibility of some rules. This actually ensures the constructive reading of the connectives.

Lemma 8.3.4. *The rule L→ is height-preserving invertible with respect to its right premise. Every other rule except for* $R\vee_i$, $R\bigvee_k$ *and* $R\exists$ *is height-preserving invertible in* $\mathbf{G3I}^{\infty}_{\omega}$.

Proof. The rule L∀ is invertible by height-preserving admissibility of weakening. In the remaining cases the proof is by induction on the height of the derivation. *qed.*

Contraction is here restricted to the antecedent. Indeed, the rule is of the shape:

$$\frac{\Pi, \Pi, \Gamma \Rightarrow C}{\Pi, \Gamma \Rightarrow C}\ \mathrm{Ctr}$$

Theorem 8.3.5. *The rule Ctr is height-preserving admissible in* $\mathbf{G3I}^{\infty}_{\omega}$.

Proof. The proof runs by induction on the height of the derivation of $\Gamma, \Pi, \Pi \Rightarrow C$. If it is an initial sequent, then so is $\Pi, \Gamma \Rightarrow C$. If it is the conclusion of a rule, we distinguish cases. If no formula in Π is principal, then we apply the induction hypothesis to each of the premises of the rule and then the rule again. If a formula in Π is principal, one needs to distinguish cases according to the shape of the principal formula.

We consider the case in which the formula is $\bigwedge_{k>0} A_k$. We have:

$$\frac{\{A_k\}_{k>0}, \bigwedge_{k>0} A_k, \Pi', \Pi', \Gamma \Rightarrow C}{\bigwedge_{k>0} A_k, \bigwedge_{k>0} A_k, \Pi', \Pi', \Gamma \Rightarrow C}\ \mathrm{L}\bigwedge$$

We construct the following derivation:

$$\cfrac{\cfrac{\cfrac{\{A_k\}_{k>0}, \bigwedge_{k>0} A_k, \Pi', \Pi', \Gamma \Rightarrow C}{\{A_k\}_{k>0}, \{A_k\}_{k>0}, \Pi', \Pi', \Gamma \Rightarrow C}\ \mathrm{InvL}\bigwedge}{\{A_k\}_{k>0}, \Pi', \Gamma \Rightarrow C}\ \mathrm{Ctr}}{\bigwedge_{k>0} A_k, \Pi', \Gamma \Rightarrow C}\ \mathrm{L}\bigwedge$$

The application of the rule Ctr is removed by induction on the height of the premise. *qed.*

We then have to prove the crucial result, cut-elimination. In this case, the cut to be eliminated is of the following shape:

$$\frac{\{\Pi_\varphi \Rightarrow \varphi \mid \varphi \in \Phi\} \qquad \Phi, \Gamma \Rightarrow C}{\Pi, \Gamma \Rightarrow C}\ \mathrm{Cut}$$

As we shall see, the proof is easier then the one detailed for classical logic.

Theorem 8.3.6. *The cut rule is admissible in* $\mathbf{G3I}^{\infty}_{\omega}$.

Proof. The proof is by double induction, with main induction on the degree of the multiset of cut formulas Φ and secondary induction hypothesis on the height of the derivation of $\Phi, \Gamma \Rightarrow C$. If $\Phi, \Gamma \Rightarrow C$ is an initial sequent, then the proof is immediate. If $\Phi, \Gamma \Rightarrow C$ is the conclusion of a rule, but no formula in Φ is principal, then the conclusion follows by permuting the cut upwards. Hence the cut is replaced by (possibly infinitely many) cuts which are removed by secondary induction hypothesis. The last case to consider is the one in which a formula in Φ is principal. In this case we need to distinguish cases according to the shape of the formula.

- The principal formula is $A \to B$, then we have:

$$\frac{\Phi', A \to B, \Gamma \Rightarrow A \qquad \Phi', B, \Gamma \Rightarrow C}{\Phi', A \to B, \Gamma \Rightarrow C}$$

In this case, we first construct the following derivation $\mathcal{D}'$:

$$\frac{\{\Pi_\varphi \Rightarrow \varphi \mid \varphi \in \Phi\} \qquad \Phi', A \to B, \Gamma \Rightarrow A}{\Gamma, \Pi \Rightarrow A}\ \text{Cut}$$

The cut is removed by induction on the height of the premise of the cut. Analogously, we construct a derivation $\mathcal{D}''$ of $\Pi', B, \Gamma \Rightarrow C$ where $\Pi' = \bigcup_{\varphi \in \Phi'} \Pi_\varphi$. Next, we perform the following reduction:

$$\dfrac{\dfrac{\overset{\mathcal{D}'}{\Gamma, \Pi \Rightarrow A} \qquad \dfrac{\Pi_{A\to B} \Rightarrow A \to B}{\Pi_{A\to B}, A \Rightarrow B}\ \text{Inv}}{\Gamma, \Pi_{A\to B}, \Pi \Rightarrow B}\ \text{Cut} \qquad \overset{\mathcal{D}''}{\Pi', B, \Gamma \Rightarrow C}}{\dfrac{\Gamma, \Pi, \Pi \Rightarrow C}{\Gamma, \Pi \Rightarrow C}\ \text{Ctr}}\ \text{Cut}$$

The cuts are removed by induction on the degree of the multiset of cut formulas. Indeed, we have:

$$deg([A]), \deg([B]) < deg([A \to B]) \leq deg([\Phi', A \to B])$$

The cases in which the principal formula is $A \wedge B$ is dealt with analogously.

- The cases in which the cut formula is $\bigvee_{k>0} A_k$ is as follows:

$$\frac{\{\Phi', A_k, \Gamma \Rightarrow C \mid k > 0\}}{\Phi', \bigvee_{k>0} A_k, \Gamma \Rightarrow C}$$

For every $k > 0$ we construct the derivation $\mathcal{D}_k$:

$$\frac{\{\Pi_\varphi \Rightarrow \varphi \mid \varphi \in \Phi'\} \qquad \Phi', A_k, \Gamma \Rightarrow C}{A_k, \Pi', \Gamma \Rightarrow C}\ \text{Cut}$$

We then consider the premise $\Pi_{\bigvee A_k} \Rightarrow \bigvee_{k>0} A_k$. We distinguish two cases. Either the formula $\bigvee_{k>0} A_k$ is never principal in the subderivation above $\Pi_{\bigvee A_k} \Rightarrow \bigvee_{k>0} A_k$ or it is principal (in possibly infinitely many inferences). In the first case, then $\Pi_{\bigvee A_{k>0}} \Rightarrow$ is derivable and the desired conclusion follows by weakening. For any of the countable branches in which $\bigvee_{k>0} A_k$ is principal, we trace this application and we have:

$$\begin{array}{c} \dfrac{\Pi''_{\bigvee_{k>0} A_k} \Rightarrow A_k}{\Pi''_{\bigvee_{k>0} A_k} \Rightarrow \bigvee_{k>0} A_k}\ \text{R}\bigvee \\ \vdots_{\rho} \\ \Pi_{\bigvee_{k>0} A_k} \Rightarrow \bigvee_{k>0} A_k \end{array}$$

Notice that by the design of the rules we can assume that in the branch in ρ from $\Pi_{\bigvee_{k>0} A_k} \Rightarrow \bigvee_{k>0} A_k$ to $\Pi''_{\bigvee_{k>0} A_k} \Rightarrow A_k$ the formula $\bigvee_{k>0} A_k$ was never principal. We construct the following derivation:

$$\dfrac{\Pi''_{\bigvee_{k>0} A_k} \Rightarrow A_k \qquad A_k, \Pi', \Gamma \Rightarrow C}{\Pi''_{\bigvee_{k>0} A_k}, \Pi', \Gamma \Rightarrow C}\,\text{Cut}$$
$$\vdots_{\rho'}$$
$$\Pi \Rightarrow C$$

where ρ' is obtained from ρ by adding whenever needed the weakened context Π'. The cuts can be removed by invoking the primary induction hypothesis on the degree of the multiset of cut formulas. The cases of R$\vee$ and R$\exists$ are dealt with analogously (in the latter case one only needs to apply height-preserving substitution in order to perform the reduction, we leave the details to the reader)[3].

- If the principal formula is $\bigwedge_{k>0} A_k$, we have:

$$\dfrac{\{A_k\}_{k>0}, \Phi', \Gamma \Rightarrow C}{\bigwedge_{k>0} A_k, \Phi', \Gamma \Rightarrow C}\,\text{L}\wedge$$

We perform the following reduction:

$$\dfrac{\dfrac{\Pi_{\bigwedge_{k>0} A_k} \Rightarrow \bigwedge_{k>0} A_k}{\{\Pi_{\bigwedge_{k>0} A_k} \Rightarrow A_k \mid k > 0\}}\,\text{Inv} \qquad \dfrac{\{\Pi_\varphi \Rightarrow \varphi \mid \varphi \in \Phi'\} \qquad \{A_k\}_{k>0}, \Phi', \Gamma \Rightarrow C}{\{A_k\}_{k>0}, \Pi', \Gamma \Rightarrow C}\,\text{Cut}}{\Pi, \Gamma \Rightarrow C}\,\text{Cut and Ctr}$$

The topmost cut is removed by secondary induction hypothesis, whereas the lowermost one is removed by primary induction hypothesis since:

$$deg([A_k]_{k>0}) < deg([\bigwedge_{k>0} A_k]) \leq deg([\Phi', \bigwedge_{k>0} A_k])$$

- The only remaining case is the one in which the principal formula is $\forall x A$.

$$\dfrac{A[x/t], \forall x A, \Phi', \Gamma \Rightarrow C}{\forall x A, \Phi', \Gamma \Rightarrow C}\,\text{L}\forall$$

We construct the following derivation:

[3]Notice that this is the only troublesome case in infinitary classical logic

$$\dfrac{\dfrac{\Pi_{\forall xA} \Rightarrow \forall xA}{\Pi_{\forall xA} \Rightarrow A[x/t]}\,\text{Inv} \qquad \dfrac{\{\Pi_\varphi \Rightarrow \varphi \mid \varphi \in \Phi\} \qquad A[x/t], \forall xA, \Phi', \Gamma \Rightarrow C}{A[x/t], \Pi, \Gamma \Rightarrow C}\,\text{Cut}}{\Pi_{\forall xA}, \Pi, \Gamma \Rightarrow C}\,\text{Cut}$$

The topmost cut is removed by secondary induction hypothesis, whereas the lowermost one is removed by primary induction hypothesis since:

$$deg([A[x/t]]) < deg([\forall xA]) \leq deg([\Phi', \forall xA])$$

qed.

As a corollary to cut-elimination we get the subformula property and the disjunction property.

Corollary. *If* $\mathbf{G3I}^{\infty}_{\omega} \vdash \Rightarrow \bigvee_{k>0} A_k$*, then* $\mathbf{G3I}^{\infty}_{\omega} \vdash \Rightarrow A_k$ *for some* $k > 0$.

Proof. Immediate by inspection of the rules. *qed.*

As shown in (71), a distinctive feature of infinitary intuitionistic logic lies in the refutation of an infinitary distributivity principle, namely:

$$\bigwedge_{k>0}(P_k \vee Q) \rightarrow \bigwedge_{k>0} P_k \vee Q$$

one may wonder whether the present version of intuitionistic infinitary logic is still sound. Indeed, the calculus $\mathbf{G3I}^{\infty}_{\omega}$ is sound with respect to the semantics of complete Heyting algebras and this can be easily shown via a routine induction on the height of the derivation (we will prove soundness and completeness in the final section of the chapter, by showing the equivalence with the systems with finite sequents). Furthermore, exploiting the analyticity of the system resulting from the cut admissibility theorem, we can also show that the sequent $\Rightarrow \bigwedge_{k>0}(P_k \vee Q) \rightarrow \bigwedge_{k>0} P_k \vee Q$ is not derivable.

Lemma 8.3.7. *The sequent* $\Rightarrow \bigwedge_{k>0}(P_k \vee Q) \rightarrow \bigwedge_{k>0} P_k \vee Q$ *is not derivable in* $\mathbf{G3I}^{\infty}_{\omega}$.

Proof. By invertibility of R$\rightarrow$ and L$\bigwedge$, it is equivalent to consider the derivability of the sequent $\{P_k \vee Q\}_{k>0} \Rightarrow \bigwedge_{k>0} P_k \vee Q$.

By the design of the sequent rules, the only way to reach an initial sequent is to apply the rule R$\vee_i$. In this case, we shall encounter two kind of sequents:

- $\{P_k \vee Q\}_{k>0}, \Pi \Rightarrow \bigwedge_{k>0} P_k$

- $\{P_k \vee Q\}_{k>0}, \Pi \Rightarrow Q$

where Π contains $P_1, \ldots, P_n$ and Q for some n. It is now a trivial task to check that both the sequents are not derivable. *qed.*

8.4 Syntactic cut-elimination modulo negative translation

We have obtained a direct cut-elimination for intuitionistic logic, but not for classical infinitary logic. Therefore, we shall prove our result by embedding classical infinitary logic into intuitionistic infinitary logic via a natural extension of Gödel-Gentzen's negative translation.

Definition 8.4.1. The infinitary Gödel-Gentzen translation $g : FOR \to FOR$ is inductively defined:

- $g(\bot) = \bot$
- $g(p) = \neg\neg p$
- $g(A\#B) = g(A)\#g(B)$, where $\# \in \{\wedge, \to\}$
- $g(A \vee B) = \neg(\neg g(A) \wedge \neg g(B))$
- $g(\forall x A) = \forall x g(A)$
- $g(\exists x A) = \neg\forall x \neg g(A)$
- $g(\bigwedge_{k>0} A_k) = \bigwedge_{k>0} g(A_k)$
- $g(\bigvee_{k>0} A_k) = \neg \bigwedge_{k>0} \neg g(A_k)$

We prove the following lemma.

Lemma 8.4.1. *For every formula A, we have:* $\neg\neg A^g \Rightarrow A^g$ *in* $\mathbf{G3I}_\omega^\infty$.

Proof. The proof is by transfinite induction on the degree of the formula A. *qed.*

Next, we establish the embedding.

Theorem 8.4.2. *If* $\mathbf{G3C}_\omega^\infty \vdash \Gamma \Rightarrow \Delta$, *then* $\mathbf{G3I}_\omega^\infty \vdash \Gamma^g, \neg\Delta^g \Rightarrow$.

Proof. The proof is by transfinite induction on the height of the derivation of the sequent $\Gamma \Rightarrow \Delta$ in the calculus $\mathbf{G3C}^{\infty}_{\omega}$. If the sequent is an initial sequent the proof is immediate. We focus on the cases involving the infinitary connectives.

If the last rule is $R\bigvee$ we have:

$$\dfrac{\neg\neg\bigwedge_{k>0}\neg g(A_k) \Rightarrow \bigwedge_{k>0}\neg g(A_k) \qquad \dfrac{\begin{array}{c}\vdots\,\text{IH}\\ g(\Gamma),\neg g(\Delta),\{\neg g(A_k)\}_{k>0} \Rightarrow\end{array}}{g(\Gamma),\neg g(\Delta),\bigwedge_{k>0}\neg g(A_k) \Rightarrow}\,\text{L}\bigwedge}{g(\Gamma),\neg g(\Delta),\neg\neg\bigwedge_{k>0}\neg g(A_k) \Rightarrow}\,\text{Cut}$$

where the leftmost sequent is provable by root-first applications of the rules.

If the last rule applied is $L\bigvee$ we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\text{IH}\\ \{g(A_k),g(\Gamma),\neg g(\Delta) \Rightarrow \mid k>0\}\end{array}}{\{g(\Gamma),\neg g(\Delta) \Rightarrow \neg g(A_k) \mid k>0\}}\,R\to}{g(\Gamma),\neg g(\Delta) \Rightarrow \bigwedge_{k>0}\neg g(A_k)}\,R\bigwedge}{g(\Gamma),\neg g(\Delta),\neg\bigwedge_{k>0}\neg g(A_k) \Rightarrow \bigwedge_{k>0}\neg g(A_k)}\,LW}{g(\Gamma),\neg g(\Delta),\neg\bigwedge_{k>0}\neg g(A_k) \Rightarrow}\,L\to$$

qed.

In order to obtain full cut-elimination for classical infinitary logic with infinite sequent a last move has to be made.

Lemma 8.4.3. *For every sequent* $\Pi,\Gamma \Rightarrow \Delta,\Lambda$*, if* $\Pi,\Gamma^g \Rightarrow \Delta^g,\Lambda$ *is derivable in* $\mathbf{G3C}^{\infty}_{\omega}$*, then so is* $\Pi,\Gamma \Rightarrow \Delta,\Lambda$*, where* Π *and* Λ *only contain atomic formulas.*

Proof. This result is indeed trivial in the presence of the cut rule, but we need to establish it without resorting to it. We argue by induction on the height of the derivation. If $\Pi,\Gamma^g \Rightarrow \Delta^g,\Lambda$ is an initial sequent, so is $\Pi,\Gamma \Rightarrow \Delta,\Lambda$.

If the last rule applied is any rule different from L$\neg$ or R$\neg$, then the proof follows by applying the induction hypothesis and then the rule again. For example, if the last rule applied is R$\to$, we have:

$$\dfrac{\Pi,g(A),\Gamma^g \Rightarrow (\Delta^g)',g(B),\Lambda}{\Pi,\Gamma^g \Rightarrow (\Delta^g)',g(A)\to g(B),\Lambda}\,\text{R}\to$$

We apply the induction hypothesis to get a derivation of $\Pi,A,\Gamma \Rightarrow \Delta,B,\Lambda$ and the desired conclusion follows from an application of the rule R$\to$.

If the last rule applied is L$\neg$ or R$\neg$, then we need to distinguish cases according to the shape of the principal formula. We discuss the case of L$\neg$, the case of R$\neg$ is symmetric. If the principal formula is $\neg\neg P$, we have:

$$\frac{\Pi, \Gamma^g \Rightarrow \Delta^g, \Lambda, \neg P}{\Pi, \neg\neg P, \Gamma^g \Rightarrow \Delta^g, \Lambda}\ \mathrm{L}\neg$$

We proceed as follows:

$$\cfrac{\cfrac{\Pi, , \Gamma^g \Rightarrow \Delta^g, \Lambda, \neg P}{\Pi, P, \Gamma^g \Rightarrow \Delta^g, \Lambda}\ \mathrm{Inv}}{\Pi, P, \Gamma \Rightarrow \Delta, \Lambda}\ \mathrm{IH}$$

The application of the induction hypothesis is justified, because the invertibility of the rule preserves the height of the derivation. If the principal formula is $\neg \bigwedge_{k>0} \neg g(A_k)$, we have:

$$\frac{\Pi, , \Gamma^g \Rightarrow \Delta^g, \Lambda, \bigwedge_{k>0} \neg g(A_k)}{\Pi, \neg \bigwedge_{k>0} \neg g(A_k), \Gamma^g \Rightarrow \Delta^g, \Lambda}\ \mathrm{L}\neg$$

We construct the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Pi, , \Gamma^g \Rightarrow \Delta^g, \Lambda, \bigwedge_{k>0} \neg g(A_k)}{\{\Pi, \Gamma^g \Rightarrow \Delta^g, \Lambda, \neg g(A_k) \mid k > 0\}}\ \mathrm{InvR}\bigwedge}{\{\Pi, g(A_k), \Gamma^g \Rightarrow \Delta^g, \Lambda \mid k > 0\}}\ \mathrm{InvR}\neg}{\{\Pi, A_k, \Gamma \Rightarrow \Delta, \Lambda \mid k > 0\}}\ \mathrm{IH}}{\Pi, \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta, \Lambda}\ \mathrm{L}\bigvee$$

The cases of in which the principal formulas is of the shape $\neg(\neg g(A) \wedge \neg g(B))$ and $\neg\forall x \neg g(A)$ are analogously dealt with and we omit the details. *qed.*

We can now obtain a purely syntactic proof of the cut-elimination theorem by exploiting the negative translation of classical into intuitionistic logic.

Theorem 8.4.4. *The cut rule can be eliminated in* $\mathbf{G3C}^{\infty}_{\omega}$.

Proof. Suppose we have derivations of $\Gamma \Rightarrow \Delta, A$ and $A, \Pi \Rightarrow \Sigma$. By the embedding we get derivations of $\Gamma^g, \neg\Delta, \neg A^g \Rightarrow$ and $A^g, \Pi^g, \neg\Sigma^g \Rightarrow$ in $\mathbf{G3I}^{\infty}_{\omega}$. Exploiting cut we get a derivation of:

$$\Gamma^g, \Pi^g, \neg\Delta^g, \neg\Sigma^g \Rightarrow$$

in $\mathbf{G3I}^{\infty}_{\omega}$. Clearly, $\mathbf{G3C}^{\infty}_{\omega} \vdash \Gamma^g, \Pi^g, \neg\Delta^g, \neg\Sigma^g \Rightarrow$, so we argue as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma^g, \Pi^g, \neg\Delta^g, \neg\Sigma^g \Rightarrow}{\Gamma^g, \Pi^g, \bigwedge \neg\Delta^g, \bigwedge \neg\Sigma^g \Rightarrow}\ \mathrm{L}\bigwedge}{\Gamma^g, \Pi^g \Rightarrow \neg \bigwedge \neg\Delta^g, \neg \bigwedge \neg\Sigma^g}\ \mathrm{R}\neg}{\Gamma^g, \Pi^g \Rightarrow (\bigvee \Delta)^g, (\bigvee \Sigma)^g}\ \text{rewriting}}{\Gamma, \Pi \Rightarrow \bigvee \Delta, \bigvee \Sigma}\ \text{Lemma 8.4.3}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathrm{InvR}\bigvee$$

qed.

8.4.1 Equivalence with the finite sequents calculi

In this section we show the equivalence of the calculi for infinitary logic with finite and infinite sequents. We focus on the case of the calculus for intuitionistic logic (the classical case is similar and thus we omit the details). We recall the single succedent calculus $\mathbf{G3i}^s_\omega$ for intuitionistic infinitary logic (which is a variant of $\mathbf{G3i}_\omega$, see Chapter 6).

$\mathbf{G3i}^s_\omega$

Initial Sequents

$$\frac{}{p, \Gamma \Rightarrow p}\,{\scriptstyle Ax} \qquad \frac{}{\bot, \Gamma \Rightarrow C}\,{\scriptstyle L\bot}$$

Logical Rules

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\,{\scriptstyle L\wedge} \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,{\scriptstyle R\wedge}$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\,{\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\,{\scriptstyle R\vee_i}$$

$$\frac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\,{\scriptstyle L\to} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,{\scriptstyle R\to}$$

$$\frac{A_k, \bigwedge_{k>0} A_k, \Gamma \Rightarrow C}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow C}\,{\scriptstyle L\bigwedge_k} \qquad \frac{\{\Gamma \Rightarrow A_k \mid k > 0\}}{\Gamma \Rightarrow \bigwedge_{k>0} A_k}\,{\scriptstyle R\bigwedge}$$

$$\frac{\{A_k, \Gamma \Rightarrow C \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow C}\,{\scriptstyle L\bigvee} \qquad \frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow \bigvee_{k>0} A_k}\,{\scriptstyle R\bigvee_k}$$

$$\frac{\forall x A, A[x/t], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\,{\scriptstyle L\forall} \qquad \frac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x A}\,{\scriptstyle R\forall,\, y!}$$

$$\frac{A[x/y], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\,{\scriptstyle L\exists,\, y!} \qquad \frac{\Gamma \Rightarrow A[x/t]}{\Gamma \Rightarrow \exists x A}\,{\scriptstyle R\exists}$$

Essentially, $\mathbf{G3i}^s_\omega$ is obtained from the calculus $\mathbf{G3I}^\infty_\omega$ by replacing the rule L$\bigwedge$ with the infinitely many rules L$\bigwedge_k$.

Theorem 8.4.5. *The calculus* $\mathbf{G3i}^s_\omega$ *satisfies the admissibility of the structural rules of weakening, contraction and cut.*

One direction of the embedding is easier. Indeed, if a sequent is derivable in the calculus with finite sequents, it is derivable also in the one based on infinite ones.

Lemma 8.4.6. *If* $\Gamma \Rightarrow C$ *is derivable in* $\mathbf{G3i}^s_\omega$*, then so is in* $\mathbf{G3I}^\infty_\omega$*.*

Proof. The proof is by induction on the height of the derivations. The only new cases to detail are the ones in which the last rule applied is L$\bigwedge$. We have:

$$\dfrac{\dfrac{\dfrac{\bigwedge_{k>0} A_k, A_k, \Gamma \Rightarrow C}{\{A_k\}_{k>0}, A_k, \Gamma \Rightarrow C}\,\text{InvL}\bigwedge}{\{A_k\}_{k>0}, \Gamma \Rightarrow C}\,\text{Ctr}}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow C}\,\text{L}\bigwedge$$

qed.

The other direction is slightly more complex. We have:

Lemma 8.4.7. *If* $\mathbf{G3I}^\infty_\omega$ *proves* $\Gamma \Rightarrow C$*, then* $\mathbf{G3i}^s_\omega$ *proves* $\bigwedge \Gamma \Rightarrow C$*.*

Proof. The proof is by induction on the height of the derivation. If it is an initial sequent, the proof is trivial. Otherwise, we distinguish cases according to the last rule applied. If the last rule is R→, we have:

$$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\text{R}\to$$

The induction hypothesis yields a derivation of:

$$\bigwedge \Gamma \wedge A \Rightarrow B$$

in $\mathbf{G3i}^s_\omega$. We apply a cut with the derivable sequent $\bigwedge \Gamma, A \Rightarrow \bigwedge \Gamma \wedge A$ and we conclude the transformation via an application of the rule R→. The other cases involving right rules are immediate (apply the induction hypothesis and then the rule again). If the last rule applied is L$\bigwedge$, the proof follows immediately by an application of the induction hypothesis. The case in which the last rule applied is L$\bigvee$, we have:

$$\dfrac{\{A_k, \Gamma \Rightarrow C \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow C}\,\text{L}\bigvee$$

For every $k > 0$, we get: $A_k \wedge \bigwedge \Gamma \Rightarrow C$. So we proceed as follows:

$$\dfrac{\bigvee_{k>0} A_k \wedge \bigwedge \Gamma \Rightarrow \bigvee_{k>0}(A_k \wedge \bigwedge \Gamma) \qquad \dfrac{\{A_k \wedge \bigwedge \Gamma \Rightarrow C \mid k > 0\}}{\bigvee_{k>0}(A_k \wedge \bigwedge \Gamma) \Rightarrow C}\,\text{L}\bigvee}{\bigvee_{k>0} A_k \wedge \bigwedge \Gamma \Rightarrow C}\,\text{Cut}$$

where the leftmost sequent is derivable in $\mathbf{G3i}^s_\omega$ via straightforward root-first applications of the rules. *qed.*

The above theorem immediately yields a completeness result in the following form, where $\Vdash$ is the validity relation in the semantics based on complete Heyting algebras.

Theorem 8.4.8. $\mathbf{G3I}^\infty_\omega \vdash \Gamma \Rightarrow A$ *if and only if* $\vDash \bigwedge \Gamma \rightarrow A$.

Proof. From left to right we argue by induction on the height of the derivation in the calculus $\mathbf{G3I}_\omega$. From right to left, if $\vDash \bigwedge \Gamma \rightarrow A$ we get $\mathbf{G3i}_\omega \vdash \Rightarrow \bigwedge \Gamma \rightarrow A$ and via the embedding $\mathbf{G3I}_\omega \vdash \Rightarrow \bigwedge \Gamma \rightarrow A$. The desired conclusion follows from invertibility of the rules R$\rightarrow$ and L$\bigwedge$. *qed.*

Notice that the formulation of the completeness theorem is crucial. Indeed, the sequent $\{P_k\}_{k>0} \Rightarrow \bigwedge_{k>0} P_k$ would not be valid if interpreted as expressing logical consequence. In fact, there are various counterexamples to $\{P_k\}_{k>0} \vDash \bigwedge_{k>0} P_k$ which rest on the fact that infinite intersections of open sets need not be open.

8.5 Concluding remarks and future works

We have discussed and analyzed the proof theory of infinitary logic with infinite sequents. We have provided a structural analysis of the calculi for classical and intuitionistic logic. The calculi enjoy admissibility of the structural rules of weakening and contraction. Furthermore, cut is eliminated employing a new strategy which runs by a double transfinite induction with a new parameter, the degree of a multiset of cut formulas.

The cut-elimination for full classical infinitary logic (including quantifiers) is obtained via the negative translation into full infinitary intuitionistic logic which enjoys a full and direct cut-elimination theorem. The results presented in the chapter show that the extension of an infinitary calculus for classical and intuitionistic logic with infinite sequents is, in a sense, inessential as it can be interpreted with finite sequents.

However, this is not always the case. As shown by Minari in (68), working with infinite sequents in the context of infinitary modal logic marks a difference as it enables the derivability of the (infinitary variant of the) Barcan formula (see (104) for a cut-free sequent calculus for infinitary modal logic and (105) for an application to the modal interpretation of intuitionistic logic). Furthermore, in substructural logics the presence of infinitary conjunctions and disjunctions can be used to simulate contraction and exponential modalities, see (40).

Therefore we deem that the techniques developed in the present chapter can be interesting as they might be employed to investigate other areas of infinitary logic. An interesting point to be addressed is the possibility of a full fledged syntactic approach to the proof theory of infinitary logic with infinite sequents and with rules which can act on infinite multisets of formulas. A similar approach was pursued in (100), but only with semantic methods.

Conclusion and future work

In this thesis we have presented various results in the field of structural proof theory. These are unified from a unitary conceptual perspective which is aimed to bring together different logical systems (84). We conclude the work by sketching some themes of possible future research:

(i) It could be interesting to explore the possibility of a direct cut-elimination result for the system $\mathbf{G3I}_{<}$ in Chapter 4.

(ii) It would be natural to consider extensions of the investigations to systems with rules acting on infinitely many formulas simultaneously.

(iii) Methods here employed could be explored in the context of non-monotonic logics too.

(iv) Use nested calculi to obtain well-behaved and uniform proof-theory for other families of non-classical logics.

Bibliography

[1] Avron A., *Hypersequents, logical consequence and intermediate logics for concurrency*, Annals of Mathematics and Artificial Intelligence, 4: 225-248, 1991.

[2] Afshari B., Leigh G., *Cut-free completeness for modal mu-calculus*, Mathematics 2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), 2017.

[3] Nuel D. Belnap Jr., *Display logic*, Journal of Philosophical Logic volume 11, pp. 375–417, 1982.

[4] Beth E. W., *Semantic construction of intuitionistic logic*, Koninklijke Nederlandse Akademie van Wetenschappen, Mededelingen, NieuweReeks 19, 357-388, 1956.

[5] Bezhanishvili N ., Holliday W., *Choice-free Stone duality*, The Journal of Symbolic Logic vol. 85:1,pp. 109 - 148 , 2020.

[6] Blamey S., Humberstone L., *A perspective on modal sequent logic*, in Publications of the Research Institute for Mathematical Sciences, vol. 27, pp. 763-82, 1991

[7] D'Agostino M., Floridi L., *The enduring scadanl of deduction*, Synthese 167, pp. 271-315, 2009

[8] Bezhanishvili G., Holliday W., *A semantic hierarchy for intuitionistic logic*, Indagationes Mathematicae, 2019

[9] Blackburn P., de Rijke M., Venema Y., *Modal Logic*, Cambridge University Press, 2001

[10] Blass A., *A game semantics for linear logic*, Annals of Pure and Applied logic, vol. 56, n. 1-3, pp. 183–220, 1992, Elsevier

[11] Brunnler K., *Deep sequent systems for modal logics*, Archive for Mathematical Logic 48 (6):551-577 , 2009

[12] Cantini A., *The undecidability of Grisin's set theory*, Studia Logica, Vol. 74, No. 3 (Aug., 2003), pp. 345-368

[13] Casari E., *La matematica della verità. Strumenti matematici della semantica logica*, Bollati Boringhieri, 2006.

[14] Celani S., Jansana R., *A closer look at some subintuitionistic logics*, Notre Dame Journal of Formal Logic 42, 225-255.

[15] Chagrov A., Zakharyaschev, M., *Modal Logic*, Oxford University Press, Clarendon Press, 1997.

[16] Chellas B. F., *Modal Logic - An Introduction*, Cambridge University Press, 1998.

[17] Ciabattoni A., Ferrari M., *Hypertableau and Path-Hypertableau Calculi for some Families of Intermediate Logics* in : Dyckhoff, R. (eds) Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2000. Lecture Notes in Computer Science(), vol 1847. Springer, Berlin, Heidelberg, 2000.

[18] A. Ciabattoni, N. Galatos, and K. Terui, *From axioms to analytic rules in nonclassical logics*, in Proceedings of LICS 08, IEEE, 2008, pp. 229–240.

[19] Corsi G., *Weak Logics with strict implication*, Zeitschrift fur Mathematische Logik und Grundlagen der Mathematic, vol. 33, 1987.

[20] Corsi, G., Tassi, G., *Intuitionistic logic freed of all metarules*, Journal of Symbolic Logic, vol. 72, 4, pp. 1204-1218, 2007.

[21] Da Re B., Rosenblatt L., *Contraction, Infinitary Quantifiers, and Omega Paradoxes* Journal of Philosophical Logic, n. 4, vol. 47, pp. 611–629, doi:10.1007/s10992-017-9441-2, Springer Verlag, 2018

[22] Davey B. A., Priestley A. H., *Introduction to Lattices and Order, 2^{nd} edition*, Cambridge University Press, 2002.

[23] M. A. E. Dummet, E. J. Lemmon, *Modal logics between S4 and S5*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 5, 250-264 1959

[24] Dyckhoff, R., *Contraction-free sequent calculi for intuitionistic logic*, Journal of Symbolic Logic, vol. 75, pp. 795-807, 1992.

[25] Dyckhoff R., *A deterministic terminating sequent calculus for Gödel-Dummett logic*, Logic Journal of the IGPL, Volume 7, Issue 3, May 1999, Pages 319–326, 1999.

[26] Dyckhoff, R., Negri, S. *Admissibility of structural rules for contraction-free systems of intuitionistic logic*, Journal of Symbolic Logic, vol 65, pp. 1499-1518, 2000.

[27] Dyckhoff, R., Negri, S. *Proof analysis in intermediate logics*, Archive for Mathematical Logic, vol. 51, pp. 71-92, 2012.

[28] Dyckhoff, R., Negri, S., *A cut-free sequent system for Grzegorczyk logic with an application to the Gödel-McKinsey-Tarski embedding*, Journal of Logic and Computation, vol. 26, pp. 169-187, 2016.

[29] Dyckhoff, R., *Intuitionistic decision procedures since Gentzen*, in Kahle R., Strahm T., Studer T. (eds) Advances in Proof Theory. Progress in Computer Science and Applied Logic, vol 28. Birkhäuser, Cham., 2016.

[30] Esakia L., *Leo Esakia on Duality in Modal and Intuitionistic Logics*, ed. Guram Bezhanishvili, Springer, 2016.

[31] Esakia L., *On modal counterparts of superintuitionistic logics*, In The Seventh All-Union Symposium on Logic and Methodology of Science, Abstracts (in Russian), Kiev, pp. 135–136, 1976.

[32] Fellin, G., S. Negri and E. Orlandelli. Glivenko sequent classes and constructive cut elimination in geometric logic, Archive for Mathematical Logic, 2022, https://doi.org/10.1007/s00153-022-00857-z.

[33] Fitch F. B., *A system of formal logic without an analogue to the Curry W operator*, The journal of symbolic logic, vol. 1 (1936), pp. 92–100

[34] Fitting M., *Prefixed tableaux and nested sequents*, Annals of Pure and Applied Logic, vol. 163:3, pp. 291-313, 2012.

[35] Fitting M., *Nested sequents for intuitionistic logics*, Notre Dame Journal of Formal Logic vol. 55:1, pp. 41-61, 2014.

[36] Fjellstad A., *A Note on the Cut-Elimination Proof in "Truth Without Contraction"*, Review of Symbolic Logic, pp. 1-5, Cambridge University Press, year = 2020, doi:10.1017/s1755020319000571.

[37] Fjellstad A., Olsen J.-F. *IKTω and Łukasiewicz-Models*, Notre Dame Journal of Formal Logic, vol. 62, n. 2, pp. 247–256, 2021, Duke University Press

[38] R. C. Flagg, H. Friedman, *Epistemic and intuitionistic formal systems*, Annals of Pure and Applied Logic 32, 53- 60, 1986

[39] Gentzen, G., *Untersuchungen über das logische Schliessen*, Math. Zeitschrift 39, 176-210, 1935

[40] Girard J.-Y., *Linear logic*, Theoretical Computer Science 50, l-102 North-Holland, 1987

[41] Girard J.-Y., *Locus solum: from the rules of logic to the logic of rules*, Mathematical Structures in Computer Science 11, 301-506, 2001.

[42] K. Gödel, *Eine interpretation der intuitionistischen aussagenkalüls*, Ergebnisse eines mathematischen Kolloquiums, vol. 4, pp. 39-40, 1933

[43] Gödel, K. *Results on Foundations*. Edited by M. Hämeen-Anttila and J. von Plato. Sources and Studies in History of Mathematics and Physical Science, Springer, 2023.

[44] N.D. Goodman, *Epistemic arithmetic is a conservative extension of intuitionistic arithmetic*, Journal of Symbolic Logic, 49 (1984), pp. 192-203

[45] Goré R., Postniece L., Tiu A., *Cut-Elimination and Proof Search for Bi-Intuitionistic Tense Logic* , Advances in Modal Logic, ed. Beklemishev L., College Publications, 2010.

[46] Predicate and set-theoretic calculi based on logic without contractions, Math. USSR Izvestija (English translation), vol. 18 (1982), No.1 , 41-59.

[47] Hakli R., Negri S., *Does the deduction theorem fail for modal logic?*, Synthese 187, 849-867, 2012

[48] Harrop R., *Some Structure Results for Propositional Calculi*, Journal of Symbolic Logic vol. 30:3, pp. 271-292, 1965.

[49] Hilbert D., *Die Grundlagen der Mathematik*, Vortrag gehalten auf Einladung des Mathematischen Seminars im Juli 1927 in Hamburg, published in Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, 6(1/2): 65–85; translated in van Heijenoort 1967: 464–479. doi:10.1007/BF02940602, 1927 [1967]

[50] Hilbert D., *Mathematische Notizbiicher, 3 notebooks*, Niedersachsische Staats- und Universititsbibliothek Gottingen, Handschriftenabteilung, Cod. Ms. D. Hilbert 600:1-3.

[51] Hudelmaier, J., *Bounds for cut-elimination in intuitionistic propositional logic*, Archive for Mathematical Logic, vol 31, pp. 331-353, 1992.

[52] Kalicki C., *Infinitary propositional intuitionistic logic*, Notre Dame Journal of Formal Logic, vol. XXI n. 2, 1980

[53] Kleene, S. C., *An Introduction to Metamathematics*, North-Holland, 1952.

[54] Kripke S., *Semantical analysis of intuitionistic logic I*, Studies in Logica and the Foundations of Mathematics 40, 92-130, 1965.

[55] Kuznets R., Lellmann B., *Interpolation for intermediate logics via injective nested sequents*, Journal of Logic and Computation, Volume 31, Issue 3, April 2021, Pages 797–831.

[56] A. V. Kuznetsov and A. Yu. Muravitsky. *On superintuitionistic logics as fragments of proof logic extensions*, Studia Logica, 45, 77-99, 1986.

[57] Lellmann B., *Linear Nested Sequents, 2-Sequents and Hypersequents* in Automated Reasoning with Analytic Tableaux and Related Methods, ed. De-Nivelle H., Springer-Verlag, 2015.

[58] Linnebo O., Shapiro S., it Predicativism as a form of potentialism, The Review of Symbolic Logic, 16:1, 2021.

[59] Maehara, S., *Eine Darstellung der Intuitionistichen Logik in der Klassischen*, Nagoya Math. Journal, vol 7, pp. 45-64, 1954.

[60] Lopez-Escobar E. G. K., *An interpolation theorem for denumerably long formulas*, Fundamenta Mathematicae, 57, 253–272, 1965.

[61] Maksimova, L. *Interpolation properties of superintuitionistic logics*, Studia Logica 38, pp. 419 - 428, 1979.

[62] Mares E., Paoli F., *Logical consequence and the paradoxes*, Journal of Philosophical Logic, vol. 43, n. 2, pp. 439–469, 2014, Springer

[63] Marin S., Strassburger L., *Label-free Modular Systems for Classical and Intuitionistic Modal Logics*, in Advances in Modal Logic, Volume 10, ed. Rajeev Goré, Barteld Kooi, and Agi Kurucz, 2014.

[64] Marin S., Miller D., Pimentel E., Volpe M. From axioms to synthetic inference rules via focusing. *Annals of Pure and Applied Logic* 173(5): 103091, 2022.

[65] J.C. McKinsey, A. Tarski, *Some theorems about the sentential calculi of Lewis and Heyting*, Journal of Symbolic Logic 13, 1-15, 1948

[66] G. Mints, *On Novikov hypothesis*, in Selected papers in Proof Theory, Bibliopolis, 1992

[67] Mints G., *A short introduction to intuitionistic logic*, Springer Verlag, 2000

[68] Minari P., *Some remarks on the proof-theory and the semantics of infinitary modal logic*, in R. Kahle, T. Strahm, T. Studer (eds.). Advances in Proof Theory, pp. 291-318, 2016

[69] Moniri M., Maleki F. S., *Neighborhood semantics for basic and intuitionistic logic*, Logic and Logical Philosophy 24, 339-355, 2015

[70] Montagna F., *Storage operators and multiplicative quantifiers in many-valued logics*, Journal of Logic and Computation, vol. 14, 2, 299–322, 2004, OUP

[71] Nadel M. E., *Infinitary intuitionistic logic from a classical point of view*, Annals of Mathematical Logic 14, 159 - 191, 1978.

[72] S. Negri, *Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem*, Archive for Mathematical Logic, 2003

[73] Negri S., *Proof Analysis in Modal Logic*, in Journal of Philosophical Logic, vol. 34. n. 4, pp. 507-544, 2005.

[74] Negri S., *On the duality between proofs and countermodels in labelled sequent calculi*, in TABLEAUX 2013, LNCS 8123, pp. 5-9, 2013.

[75] Negri, S.Proof theory for non-normal modal logics: The neighbourhood formalism and basic results. *IfCoLog Journal of Logics and their Applications* 4, 1241–1286, 2017

[76] Negri S. Geometric rules in infinitary logic. In *Arnon Avron on Semantics and Proof Theory of Non-Classical Logics*, Outstanding Contributions to Logic. Springer, 2021.

[77] Nicolai C., Piazza M., Tesi M., *Non-contractive logics, Paradoxes, and Cut-elimination*, 2022, submitted.

[78] Orevkov, V. P. On Glivenko sequent classes. *Proc. Steklov Inst. Math.*, vol. 98, pp. 147–173 (translated from the Russian original V. P. Orevkov, Glivenko's sequence classes, Logical and logical-mathematical calculus. Part I, *Tru dy Mat. Inst. Steklov*, vol. 98, 1968, pp. 131–154), 1968.

[79] Pacuit E., *Neighborhood semantics for modal logic*, Springer International Publishing, 2017

[80] Paoli F., *The ambiguity of quantifiers*, Philosophical Studies, vol. 124, n 3, pp. 313–330, 2005

[81] Petersen U., *On Zardini's rules for multiplicative quantification as the source of contra (di) ctions*, The Review of Symbolic Logic, pp. 1–10, 2022

[82] Petersen U., *On Zardini's rules for multiplicative quantification as the source of contra(di)ctions*, The Review of Symbolic Logic, 2022.

[83] Piazza M., Castellan M., *Quantales and Structural Rules*, Journal of Logic and Computation 5:6, pp. 709-724, 1996.

[84] Piazza M., *Regole logiche e razionalità umana* in M. Dell'Utri, A. Rainone (eds.), *Modelli filosofici della razionalità*, Mimesis, pp. 11-28, 2016.

[85] Piazza M., Pulcini G., *Unifying logics through context-sensitiveness*, Journal of Logic and Computation, 2017.

[86] Pimentel E., Pereira L. C., de Paiva V., *An Ecumenical Notion of Entailment*, Synthese, 2019.

[87] Poggiolesi F., *Are the Validities of Modal Logic Analytic? Or Analyticity Again, through Information, Proof, Modal Logic and Hintikka*, Philosophia Scientiae 19-2, 2005.

[88] Poggiolesi F., *Gentzen Calculi for Modal Propositional Logic*, trends in Logic, vol. 32, 2010.

[89] Pohlers W., *Proof Theory: The First Step into Impredicativity*, Springer-Verlag Berlin Heidelberg, 2009.

[90] D. Prawitz, P.E. Malmnas, *A survey of some connections between classical, intuitionistic and minimal logic*, in: H. Arnold Schmidt et al., eds., Contributions to Mathematical Logic, North-Holland, Amsterdam, 215-229, 1966

[91] Priest, G. *An Introduction to Non-Classical Logic. From If to Is*, Cambridge University Press

[92] H. Rasiowa, R. Sikorski, *The mathematics of metamathematics*, Polish Scientific Publishers, Warsaw, 1963

[93] Rathjen M., *Remarks on Barr's theorem: Proofs in geometric theories*, In D. Probst and P. Schuster (eds) *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, Series: Ontos Mathematical Logic 6, De Gruyter, 2016.

[94] Restall G., *Subintuitionistic logics*, Notre Dame Journal of Formal Logic 35, 116-129, 1994

[95] Schütte K., *Proof Theory*, Springer-Verlag Berlin Heidelberg, 1977

[96] Solovay, R. M., *Provability interpretations of modal logic*, Israel Journal of Mathematics, vol. 25 (1976), pp. 287-304.

[97] R. Statman, *Bounds for proof-search and speed-up in the predicate calculus*, Annals of Pure and Applied Logic, 15, 225-287, 1987.

[98] M. Stone, *Topological representations of distributive lattices and Brouwerian logics*, Casopis pro Pestovan Matematiky a Fysiky 67, 1-25, 1937

[99] Sundholm B. G., *Proof Theory: a Survey of the Omega-Rule*, University of Oxford (Phd thesis), 1983

[100] Takeuti G., *Proof Theory*, Dover Publications, 2013.

[101] Takano M., *Subformula property as a substitute for cut-elimination in modal propositional logics*, Mathematica japonica, 37:1129-1145, 1992.

[102] Tanaka Y. Duality for κ-additive complete atomic modal algebras. *Algebra universalis* 82: 31, 2021.

[103] Tesi M., Negri S., *Neighbourhood semantics and labelled calculus for intuitionistic infinitary logic*, Journal of Logic and Computation, 2021.

[104] Tesi M., *On the proof theory of infinitary modal logic*, Studia Logica, 110, pp. 1349–1380, Springer, 2022.

[105] Tesi M., Negri S., *The Gödel-McKinsey-Tarski embedding for infinitary intuitionistic logic and its extensions*, Annals of Pure and Applied Logic, Elsevier, 2023.

[106] Ciabattoni A., Strassburger L., Tesi M., *Taming bounded depth with nested sequents*, Proceedings of AIML, College Publications, 2022.

[107] Troelstra A. S., van Dalen D., *Constructivism in Mathematics Vol. 1*, Elsevier, 1988

[108] A. S. Troelstra, D. van Dalen, *Constructivism in Mathematics, vol. 2*, North Holland, 1998

[109] Troelstra A., Schwichtenberg H., *Basic Proof Theory*, Cambridge University Press, 2000.

[110] Valentini S., *A simple proof of the completeness theorem of the intuitionistic predicate calculus with respect to the topological semantics*, unpublished manuscript, 1994

[111] Vickers S., *Topology via logic*, Cambridge University Press, 2008.

[112] Viganò L., *Labelled Non-Classical Logics*, Springer, 2000

[113] von Plato J., *From axiomatic logic to natural deduction*, Studia Logica 102, pp. 1167-1184, 2014.

[114] Vorob'ev, N. N., *The derivability problem in the constructive propositional calculus with strong negation*, Doklady Akademii Nauk SSSR, vol. 85, 689-692, 1952.

[115] Zardini E., *Truth Without Contra(di)ction*, Review of Symbolic Logic, pp. 498–535, 2011