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Global well-posedness and interior regularity of 2D Navier–Stokes equations with stochastic boundary conditions

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Abstract

The paper is devoted to the analysis of the global well-posedness and the interior regularity of the 2D Navier–Stokes equations with inhomogeneous stochastic boundary conditions. The noise, white in time and coloured in space, can be interpreted as the physical law describing the driving mechanism on the atmosphere–ocean interface, i.e. as a balance of the shear stress of the ocean and the horizontal wind force.

Contents

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1 Introduction

Partial differential equations with boundary noise have been introduced by Da Prato and Zabczyck in the seminal paper [18]. They showed that, also in the one dimensional case, the solutions of the heat equation with white noise Dirichlet or Neumann Boundary conditions have low regularity compared to the case of noise diffused inside the domain. In particular, in the case of Dirichlet boundary conditions the solution is only a distribution. Some improvements in the analysis of the interior regularity of the solutions of these problems and some nonlinear variants have been obtained exploiting specific properties of the heat kernel and of suitable nonlinearities. For some results in this direction we refer to [4, 15, 23, 25, 31]. All these issues make the problem of treating non-linear partial differential equations with boundary noise coming from fluid dynamical models an, almost untouched, field of open problems.

Throughout the manuscript we fix a finite time horizon $T > 0$. Let $a > 0$, $\mathcal{O} = \mathbb{T} \times (0, a)$ and let $\mathbb{T}$ be the one dimensional torus. Finally, we denote by

$$\Gamma_b = \mathbb{T} \times \{0\} \quad \text{and} \quad \Gamma_u = \mathbb{T} \times \{a\}, \tag{1.1}$$

the bottom and the upper part of the boundary of $\mathcal{O}$, respectively.

In this paper we are interested in the global well-posedness and the interior regularity of the 2D Navier–Stokes equations with boundary noise for the unknown velocity field $u(t, \omega, x, z) = (u_1, u_2) : \mathbb{R}_+ \times \Omega \times \mathcal{O} \to \mathbb{R}^2$, formally written as

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u, & \text{on } (0, T) \times \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } (0, T) \times \mathcal{O}, \\ u = 0, & \text{on } (0, T) \times \Gamma_b, \\ \partial_z u_1 = h_b \dot{W}_{\mathcal{H}}, & \text{on } (0, T) \times \Gamma_u, \\ u_2 = 0, & \text{on } (0, T) \times \Gamma_u, \\ u(0) = u_0, & \text{on } \mathcal{O}, \end{cases} \tag{1.2}$$

where $\nabla u = (\partial_j u_i)_{i,j=1}^2$, $W_{\mathcal{H}}(t)$ is a $\mathcal{H}$-cylindrical Brownian motion and $h_b(t, x)$ is a sufficiently regular forcing term; we refer to Sect. 1.1 below for the the relevant assumptions and definitions. To the best of our knowledge this is the first instance of a *global* well-posedness result for a fluid dynamical system driven by stochastic white

in time boundary conditions. We refer to [12, 13] for some homogenization results in the case of Navier–Stokes equations with dynamic boundary conditions driven by a stochastic forcing and to [14] for the local analysis of the three dimensional primitive equations with boundary noise. Finally, we refer to [21, 22] for some limit behaviors of the model (1.2) with $h_b \dot{W}_{\mathcal{H}}$ replaced by a highly oscillating and regular stationary random field.

Following the books by Pedlosky [40, 41] and Gill [30], the model (1.2) is a good idealization of the velocity of the fluid in the ocean. In this scenario, the domain $\mathcal{O} = \mathbb{T} \times (0, a)$ can be considered a vertical slice of the ocean with depth $a > 0$ and we should interpret u_1 (resp. u_2) as the horizontal (resp. vertical) component of the velocity field u. Indeed even if, in principle, one should consider a free surface, instead of $\Gamma_u = \mathbb{T} \times \{a\}$, depending on the time, the approximation of such surface as independent of the time, although highly unrealistic, is justified by the fact that the behavior of the fluid around the surface is in general very turbulent. Hence, as emphasized in [24], only a modelization is tractable and meaningful. The stochastic boundary condition appearing in (1.2) is interpreted as the physical law describing the driving mechanism on the atmosphere-ocean interface, i.e. as a balance of the shear stress of the ocean and the horizontal wind force, see [38] for details.

1.1 Main results

We begin by introducing some notation. Consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$, a separable Hilbert space $\mathcal{H}$ and a cylindrical $\mathcal{F}-$Brownian motion $(W_{\mathcal{H}}(t))_{t \geq 0}$ on $\mathcal{H}$. We say that a process Φ is $\mathcal{F}$-progressive measurable if $\Phi|_{(0,t) \times \Omega}$ is $\mathcal{F}_t \times \mathcal{B}((0, t))$-measurable for all $t > 0$, where $\mathcal{B}$ denotes the Borel σ-algebra. For the relevant notation on function spaces, we refer to Sect. 1.1.1.

Hypothesis 1.1 Let $q > 2$, $p > 2q$, $\alpha \in [0, \frac{1}{q} - \frac{2}{p})$ be such that there exists $\theta \in [0, \frac{1}{2})$ satisfying:

$$\theta + \frac{1}{4} - \frac{1}{p} \geq 0, \qquad -\frac{1}{q} - \alpha - 2\theta + \frac{1}{2} > 0.$$

Assume that $h_b : (0, T) \times \Omega \rightarrow W^{-\alpha,q}(\Gamma_u; \mathcal{H})$ is a $\mathcal{F}$-progressively measurable process with $\mathbf{P} - a.s.$ paths in $L^p(0, T; W^{-\alpha,q}(\Gamma_u; \mathcal{H}))$.

Remark 1.2 Hypothesis 1.1 is for instance satisfied if $q > 2$, $p > 2q > 4$ and $\theta = \alpha = 0$. Note that the case $q = 2$ considered in [18] is not allowed in Hypothesis 1.1.

Following the idea of [17] we split the analysis of (1.2) in two parts. First we consider the stochastic linear problem with non-homogeneous boundary conditions

$$\begin{cases} \partial_t w + \nabla \rho = \Delta w, & \text{on } (0, T) \times \mathcal{O}, \\ \text{div } w = 0, & \text{on } (0, T) \times \mathcal{O}, \\ w = 0, & \text{on } (0, T) \times \Gamma_b, \\ \partial_z w_1 = h_b \dot{W}_{\mathcal{H}}, & \text{on } (0, T) \times \Gamma_u, \\ w_2 = 0, & \text{on } (0, T) \times \Gamma_u, \\ w(0) = 0, & \text{on } \mathcal{O}, \end{cases} \quad (1.3)$$

The solution to the above linear equation (1.3) can be treated in mild form as in [18, 19]. Secondly, denoting by $v = u - w$ we will consider the Navier–Stokes equations with random coefficients

$$\begin{cases} \partial_t v + (v + w) \cdot \nabla(v + w) + \nabla(P - \rho) = \Delta v, & \text{on } (0, T) \times \mathcal{O}, \\ \text{div } v = 0, & \text{on } (0, T) \times \mathcal{O}, \\ v = 0, & \text{on } (0, T) \times \Gamma_b, \\ \partial_z v_1 = 0, & \text{on } (0, T) \times \Gamma_u, \\ v_2 = 0, & \text{on } (0, T) \times \Gamma_u, \\ v(0) = u_0, & \text{on } \mathcal{O}. \end{cases} \quad (1.4)$$

As discussed in [19, Chapter 13], if h_b, u_0, $W_{\mathcal{H}}(t)$ would be regular enough, then $u = v + w$ will be a classical solution of the Navier–Stokes equations with inhomogeneous boundary conditions (1.2).

To state our first result, we introduce some more notation. Here and below, we denote by H (resp. V, $\mathbb{L}^4$) the space $L^2(\mathcal{O}; \mathbb{R}^2)$ (resp. $H^1(\mathcal{O}; \mathbb{R}^2)$, $L^4(\mathcal{O}; \mathbb{R}^2)$) of divergence free vector fields adapted to our framework, introduced rigorously in Sect. 2.1.

Definition 1.3 A process u with paths $\mathbf{P} - a.s.$ in $C([0, T]; H) \cap L^4(0, T; \mathbb{L}^4)$ and progressively measurable with respect to these topologies, is a pathwise weak solution of (1.2) if $u = v + w$, where w has paths in $C(0, T; H) \cap L^4(0, T; \mathbb{L}^4(\mathcal{O}))$, it is progressively measurable with respect to these topologies and is a mild solution of (1.3) while v has paths in $C(0, T; H) \cap L^2(0, T; V)$, it is progressively measurable with respect to these topologies and is a weak solution of (1.4).

The first main result of this paper reads as follows.

Theorem 1.4 (Global well-posedness) *Let Hypothesis 1.1 be satisfied. Then for all $u_0 \in H$ there exists a unique weak solution u to (1.2) in the sense of Definition 1.3.*

According to Remark 1.2, the introduction of the non-Hilbertian setting is necessary in order to prove Theorem 1.4 above, at least with the tools introduced in this article.

Remark 1.5 (*Additional bulk forces*) Without additional difficulties we could also consider in Eq. (1.2) an additive noise diffused inside the domain of the form $h_d(t) \, \mathrm{d}\widetilde{W}_{\mathcal{H}}(t)$, where $\widetilde{W}_{\mathcal{H}}$ is a cylindrical Brownian motion on $\mathcal{H}$ independent of $W_{\mathcal{H}}$ and $h_d : (0, T) \times \Omega \to \gamma(\mathcal{H}, X_{-\lambda, A_q})$ is a progressively measurable process

with paths $\mathbf{P} - a.s.$ in $L^p\left(0, T; \gamma\left(\mathcal{H}, X_{-\lambda, A_q}\right)\right)$, with $p > 2$, $q \geq 2$, $\lambda \in [0, \frac{1}{2})$ such that $1 - \frac{2}{p} - 2\lambda > 0$ and there exists $\theta \in [0, \frac{1}{2})$ satisfying

$$\theta \leq \frac{3}{4} - \lambda - \frac{1}{q}, \qquad \theta \geq \frac{1}{p} - \frac{1}{4}.$$

The case $q = p = 2$ and $\lambda = 0$ is also allowed, see [20, Chapter 5]. Here A_q and γ stands for the Stokes operator on L^q and the class of γ-radonifying operators, see Sect. 2.1 and [32, Chapter 9], respectively. Finally, $X_{-\lambda, A_q}$ is the extrapolated space or order λ w.r.t. A_q as defined in (2.9). To see this, note that, under these assumptions, arguing as in Proposition 3.1 the solution $\widetilde{w}$ to

$$\begin{cases} \partial_t \widetilde{w} + \nabla \widetilde{\rho} = \Delta \widetilde{w} + h_d \dot{\widetilde{W}}_{\mathcal{H}}, & \text{on } (0, T) \times \mathcal{O}, \\ \text{div } \widetilde{w} = 0, & \text{on } (0, T) \times \mathcal{O}, \\ \widetilde{w} = 0, & \text{on } (0, T) \times \Gamma_b, \\ \partial_z \widetilde{w}_1 = 0, & \text{on } (0, T) \times \Gamma_u, \\ \widetilde{w}_2 = 0, & \text{on } (0, T) \times \Gamma_u, \\ \widetilde{w}(0) = 0, & \text{on } \mathcal{O}, \end{cases} \tag{1.5}$$

can be obtained as a stochastic convolution. In particular, the above assumptions on h_d imply that $\widetilde{w}$ is a progressively measurable process with values in $C([0, T]; H) \cap L^4(0, T; \mathbb{L}^4)$. Therefore this term adds no difficulties in order to analyze the well-posedness of Eq. (1.4). For this reason we prefer to not consider this classical source of randomness.

Remark 1.6 (*Comparison with the literature*)

(1) Theorem 1.4 shares strong similarities with [11, Theorem 1.2], which addresses the well-posedness of certain 2D deterministic Navier–Stokes equations with non-homogeneous non-smooth Navier-type boundary conditions. However, it is important to note that our model focuses on a different phenomenon than the one studied in [11]. For this reason, contrary to us, they stress the regularity of the boundary condition of the normal trace of the velocity. From a mathematical viewpoint, the white noise appearing in Eq. (1.2) is rougher both in time and in space compared to the boundary conditions discussed in [11]. However, as discussed in [18], Neumann boundary conditions are more regular than Dirichlet boundary conditions and allow us to treat rougher inputs. Due to these differences, the two results have different ranges of applicability and do not cover each other. Moreover, the tools introduced here differ significantly from the techniques involved in [11].

(2) As discussed in the introduction, the first result in the direction of the analysis of fluid dynamical models with stochastic boundary conditions have been proved in [14, Theorem 5.1], where the authors established local well-posedness of 3D primitive equations with boundary noise modeling wind forces. Both their strategy and ours are based on the splitting technique introduced in [17]. After showing

suitable regularity properties of the stochastic convolution via stochastic maximal L^p-regularity techniques (cf. Proposition 3.1 and [14, Proposition 4.3]), a thorough analysis of certain nonlinear models is required. In contrast, we conduct this analysis within a suitable Hilbertian framework, enabling us to derive energy estimates essential for establishing the global well-posedness of (1.2) (cf. Theorem 3.3 and [14, Section 5.3]). The difference between the global well-posedness result which we are able to obtain and [14, Theorem 5.1] can be seen as consequence of the fact that the 2D Navier–Stokes equations are globally well-posed in the weak setting, while the same cannot be asserted for the primitive equations (cf. [33]). Therefore, in order to prove their local well-posedness result, the authors in [14] need to work with a notion of solution which mixes strong and weak regularity in the space variables. As a byproduct of this fact we are able to consider a noise rougher in space compared to them. Additionally, a minor distinction lies in the boundary conditions applied to the bottom part of the domain Γ_b. We introduce no-slip boundary conditions to accurately model the bottom of the ocean, a choice with theoretical underpinning in works such [21, 22, 30, 40, 41]. In contrast, [14] considered some form of homogeneous Neumann boundary conditions, a choice related to the functional analytic setup of the primitive equations (cf. [14, Remark 3.3]). Beyond the distinct justifications from a modeling perspective, our choice leads to differences in the analysis of the corresponding linear elliptic systems (cf. Section 2.2 and [14, Section 3.5]).

Secondly, we are interested in studying the interior regularity of the solution u provided by Theorem 1.4.

Our second main result reads as follows:

Theorem 1.7 (Interior regularity) *Let Hypothesis 1.1 be satisfied. Let u be the unique weak solution of (1.2) provided by Theorem 1.4. Then for all $t_0 \in (0, T)$ and $\mathcal{O}_0 \subset \mathcal{O}$ such that* $\mathrm{dist}(\mathcal{O}_0, \partial\mathcal{O}) > 0$,

$$u \in C([t_0, T]; C^\infty(\mathcal{O}_0; \mathbb{R}^2)) \quad \mathbf{P} - a.s.$$

According to [47] (see also [36, Section 13.1]), it seems not possible to gain high-order interior time-regularity for the Navier–Stokes problem. This fact is in contrast to the case of the heat equation with white noise boundary conditions, see [16]. The reason behind this is the presence of the unknown pressure P which, due to its non-local nature, provides a connection between the interior and the boundary regularity. Finally, let us mention that other techniques to bootstrap further interior space regularity (e.g. analyticity), such as the 'parameter' trick (see [7, 8] and [43, Subsection 9.4]), seem not to work due to the presence of the noise on Γ_u. Similarly to the proof of Theorem 1.4, we analyze the interior regularity of u combining the interior regularity of w and the interior regularity of v. The interior regularity of w is obtained introducing a proper weak formulation, see Definition 4.1 below. Instead the regularity of v is analyzed via a Serrin's argument exploiting the aforementioned regularity of w.

The paper is organized as follows. In Sect. 2 we will introduce the functional setting in order to deal with problem (1.2). In particular, we will introduce the corresponding of the classical spaces and operator needed to deal with Navier–Stokes equations with

no-slip boundary condition to this more involved set of boundary conditions. Indeed, the Stokes operator associated to our problem generates an analytic semigroup which admits an H^∞-calculus of angle strictly less than $\frac{\pi}{2}$ also in the non-Hilbertian setting. This is crucial in order to apply the Stochastic maximal L^p-regularity results of [53], recalled in Sect. 2.4. The proof of Theorem 1.4 is the object of Sect. 3. In particular, in Sect. 3.1 we will consider the linear problem (1.3), while in Sect. 3.2 we will consider the nonlinear problem (1.4). The proof of Theorem 1.7 is the object of Sect. 4. In particular, in Sect. 4.1 we will study the interior regularity of the solution of the linear problem (1.3), while in Sect. 4.2 we will consider the nonlinear problem (1.4). We postpone some technical proofs related to the properties of the Stokes operator in the "Appendix A".

1.1.1 Notation

Here we collect some notation which will be used throughout the paper. Further notation will be introduced where needed. By C we will denote several constants, perhaps changing value line by line. If we want to keep track of the dependence of C from some parameter ξ we will use the symbol $C(\xi)$. Sometimes we will use the notation $a \lesssim b$ (resp. $a \lesssim_\xi b$), if it exists a constant such that $a \leq Cb$ (resp. $a \leq C(\xi)b$).

Fix $q \in (1, \infty)$. For an integer $k \geq 1$, $W^{k,q}$ denotes the usual Sobolev spaces. In the non-positive and non-integer case $s \in (-\infty, \infty) \setminus \mathbb{N}$, we let $W^{s,q} := B^s_{q,q}$ where $B^s_{q,q}$ is the Besov space with smoothness s, and integrability q and microscopic integrability q (in particular $W^{0,q} \neq L^q$). Moreover, $H^{s,q}$ denotes the Bessel potential spaces. Both Besov and Bessel potential spaces can be defined by means of Littlewood-Paley decompositions and restrictions (see e.g. [46], [45, Section 6]) or using the interpolation methods starting with the standard Sobolev spaces $W^{k,q}$ (see e.g. [10, Chapter 6]). Finally, we set $\mathcal{A}^{s,q}(D; \mathbb{R}^d) := (\mathcal{A}^{s,q}(D))^d$ for an integer $d \geq 1$, a domain D and $\mathcal{A} \in \{W, H\}$.

Let $\mathcal{K}$ and Y be a Hilbert and a Banach space, respectively. We denote by $\gamma(\mathcal{K}, Y)$ the set of γ-radonifying operators, see e.g. [32, Chapter 9] for basic definitions and properties. If Y is Hilbert, then $\gamma(\mathcal{K}, Y)$ coincides with the class of Hilbert-Schmidt operator from $\mathcal{K}$ to Y. Below, we need the following Fubini-type result:

$$\mathcal{A}^{s,q}(D; \mathcal{K}) = \gamma(\mathcal{K}, \mathcal{A}^{s,q}(D)) \quad \text{for all } s \in \mathbb{R}, \ q \in (1, \infty), \ \mathcal{A} \in \{W, H\}.$$

The above follows from [32, Theorem 9.3.6] and interpolation.

2 Preliminaries

2.1 The Stokes operator and its spectral properties

In this section we introduce the functional analytic setup in order to define all the object necessarily in the following. In order to improve the readability of the results we will just state the main results on the Stokes operator postponing the proofs to "Appendix A".

Throughout this subsection we let $q \in (1, \infty)$. Recall that $\mathcal{O} = \mathbb{T} \times (0, a)$ where $a > 0$. We begin by introducing the Helmholtz projection on $L^q(\mathcal{O}; \mathbb{R}^2)$, see e.g. [43, Subsection 7.4]. Let $f \in L^q(\mathcal{O}; \mathbb{R}^2)$ and let $\psi_f \in W^{1,q}(\mathcal{O})$ be the unique solution to the following elliptic problem

$$\begin{cases} \Delta \psi_f = \operatorname{div} f & \text{on } \mathcal{O}, \\ \partial_n \psi_f = f \cdot n & \text{on } \Gamma_u \cup \Gamma_b. \end{cases} \tag{2.1}$$

Here n denotes the exterior normal vector field on $\partial \mathcal{O}$. Of course, the above elliptic problem is interpret in its natural weak formulation:

$$\int_{\mathcal{O}} \nabla \psi_f \cdot \nabla \varphi \, dx dz = \int_{\mathcal{O}} f \cdot \nabla \varphi \, dx dz \quad \text{for all } \varphi \in C^\infty(\mathcal{O}). \tag{2.2}$$

By [43, Corollary 7.4.4], we have $\psi_f \in W^{1,q}(\mathcal{O})$ and $\|\nabla \psi_f\|_{L^q(\mathcal{O};\mathbb{R}^2)} \lesssim \|f\|_{L^q(\mathcal{O};\mathbb{R}^2)}$ (the proof of such estimate can also be obtained by the Lax-Milgram theorem in Banach spaces [35, Theorem 1.1], see also the proof of Theorem 2.2 below). Then the Helmholtz projection is given by $\mathbb{P}_q$ is defined as

$$\mathbb{P}_q f = f - \nabla \psi_f, \quad f \in L^q(\mathcal{O}; \mathbb{R}^2).$$

Next we define the Stokes operator on $L^q(\mathcal{O}; \mathbb{R}^2)$. For convenience of notation, we actually define A_q as minus the Stokes operator so that A_q is a positive operator for $q = 2$ (i.e. $\langle A_2 u, u \rangle \geq 0$ for all $u \in \mathsf{D}(A_2)$). Let $\mathbb{L}^q := \mathbb{P}(L^q(\mathcal{O}; \mathbb{R}^2))$. Then, we define the operator $A_q : \mathsf{D}(A_q) \subseteq \mathbb{L}^q \to \mathbb{L}^q$ where

$$\mathsf{D}(A_q) = \big\{ f = (f_1, f_2) \in W^{2,q}(\mathcal{O}; \mathbb{R}^2) \cap \mathbb{L}^q \ : \ f|_{\Gamma_b} = 0,$$
$$f_2|_{\Gamma_u} = \partial_z f_1|_{\Gamma_u} = 0 \big\},$$

and $A_q u = -\mathbb{P}_q \Delta u$ for $u \in \mathsf{D}(A_q)$.

In the main arguments we need stochastic maximal L^q-regularity estimates for stochastic convolutions. By [53] (see also [2, 52]), it is enough to show the boundedness of the H^∞-calculus for A_q. For the main notation and basic results on the H^∞-calculus we refer to [43, Chapters 3 and 4] and [32, Chapter 10].

In the following, we let

$$\mathbb{H}^{s,q}(\mathcal{O}) := H^{s,q}(\mathcal{O}; \mathbb{R}^2) \cap \mathbb{L}^q, \quad s \in \mathbb{R}.$$

Theorem 2.1 (Boundedness H^∞-calculus) *For all $q \in (1, \infty)$, the operator A_q is invertible and has a bounded H^∞-calculus of angle $< \frac{\pi}{2}$. Moreover the domain of the fractional powers of A_q is characterized as follows:*

(1) $\mathsf{D}(A_q^s) = \mathbb{H}^{2s,q}(\mathcal{O})$ *if* $0 \leq s < \frac{1}{2q}$.

(2) $\mathsf{D}(A_q^s) = \big\{ f \in \mathbb{H}^{2s,q}(\mathcal{O}) \mid f|_{\Gamma_b} = 0, \ f_2|_{\Gamma_u} = 0 \big\}$ *if* $\frac{1}{2q} < s < \frac{1}{2} + \frac{1}{2q}$.

(3) $D(A_q^s) = \left\{ f \in \mathbb{H}^{2s,q}(\mathcal{O}) \mid f|_{\Gamma_b} = 0, \ f_2|_{\Gamma_u} = \partial_z f_1|_{\Gamma_u} = 0 \right\}$ if $\frac{1}{2} + \frac{1}{2q} < s < 1$.

The above implies that $-A_q$ generates an analytic semigroup on $\mathbb{L}^q$.
For convenience of notation, we will simply write A in place of A_2. Moreover we define

$$H := \mathbb{L}^2, \quad V := D(A^{1/2}), \quad \mathcal{D}(\mathcal{O}) := \{ f \in C_c^\infty(\mathcal{O}; \mathbb{R}^2) : \operatorname{div} f = 0 \}.$$

We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and the norm in H respectively. In the sequel we will denote by V^* the dual of V and we will identify H with H^*. Every time X is a reflexive Banach space such that the embedding $X \hookrightarrow H$ is continuous and dense, denoting by X^* the dual of X, the scalar product $\langle \cdot, \cdot \rangle$ in H extends to the dual pairing between X and X^*. We will simplify the notation accordingly.

Theorem 2.1 could be known to experts. For the reader's convenience, we provide in "Appendix A" a complete and relatively short proof based on the recent strategy used in [42] for the H^∞-calculus for the Stokes operator on Lipschitz domains [42, Theorem 16].

2.2 The Neumann map

Now we are interested in L^q-estimates for the Neumann map, i.e. we are interested in studying the weak solutions of the elliptic problem

$$\begin{cases} -\Delta u + \nabla \pi = 0, & \text{on } \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } \mathcal{O}, \\ u(\cdot, 0) = 0, & \text{on } \Gamma_b, \\ \partial_z u_1(\cdot, a) = g, & \text{on } \Gamma_u, \\ u_2 = 0, & \text{on } \Gamma_u. \end{cases} \tag{2.3}$$

To state the main result of this subsection, we need to formulate (2.3) in the weak setting. To this end, we argue formally. Take $\varphi = (\varphi_1, \varphi_2) \in C^\infty(\mathcal{O}; \mathbb{R}^2)$ such that $\operatorname{div} \varphi = 0$,

$$\varphi(\cdot, 0) = 0, \quad \text{and} \quad \varphi_2(\cdot, a) = 0.$$

A formal integration by parts shows that (2.3) implies

$$\int_{\mathcal{O}} \nabla u : \nabla \varphi \, dx dz = -\int_{\mathbb{T}} g(x) \varphi_1(x, a) \, dx. \tag{2.4}$$

In particular, the RHS of (2.4) makes sense even in case g is a distribution if we interpret $\int_{\mathbb{T}} g(x) \varphi_1(x, a) \, dx = \langle \varphi_1(\cdot, a), g \rangle$.

Theorem 2.2 *Let $q \in (1, \infty)$, for all $g \in W^{-1/q,q}(\Gamma_u)$ there exists a unique $(u, \pi) \in W^{1,q}(\mathcal{O}; \mathbb{R}^2) \times L^q(\mathcal{O})/\mathbb{R}$ weak solution of* (2.3). *Moreover (u, π) satisfy*

$$\|u\|_{W^{1,q}(\mathcal{O};\mathbb{R}^2)} + \|\pi\|_{L^q(\mathcal{O})/\mathbb{R}} \leq C \|g\|_{W^{-1/q,q}(\Gamma_u)}. \tag{2.5}$$

Finally, if $g \in W^{1-1/q,q}(\Gamma_u)$, then $(u, \pi) \in W^{2,q}(\mathcal{O}; \mathbb{R}^2) \times W^{1,q}(\mathcal{O})/\mathbb{R}$ and

$$\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)} + \|\pi\|_{W^{1,q}(\mathcal{O})/\mathbb{R}} \leq C \|g\|_{W^{1-1/q,q}(\Gamma_u)}. \tag{2.6}$$

Proof We divide the proof into three steps.

Step 1: Proof of (2.5). Let A_q be as in Sect. 2. We prove (2.5) by applying the Lax-Milgram theorem of [35, Theorem 1.1] to the form $a : Y_1 \times Y_2 \to \mathbb{R}$ where

$$a(u, \varphi) = \int_{\mathcal{O}} \nabla u : \nabla \varphi \, \mathrm{d}x\mathrm{d}z, \quad Y_1 = \mathsf{D}(A_q^{1/2}), \quad Y_2 = \mathsf{D}(A_{q'}^{1/2}).$$

Recall that, by Theorem 2.1,

$$\mathsf{D}(A_q^{1/2}) = \{v = (v_1, v_2) \in \mathbb{H}^{1,q}(\mathcal{O}) : v|_{\Gamma_b} = 0, \ v_2|_{\Gamma_u} = 0\}.$$

Since $W^{1,q'}(\mathcal{O}) \ni \varphi \mapsto \varphi_1|_{\Gamma_u} \in W^{1-1/q',q'}(\Gamma_u) = W^{1/q,q'}(\Gamma_u)$, we have

$$|\langle \varphi_1(\cdot, a), g \rangle| \leq \|g\|_{W^{-1/q,q}(\Gamma_u)} \|\varphi\|_{W^{1/q,q'}(\Gamma_u)} \lesssim \|g\|_{W^{-1/q,q}(\Gamma_u)} \|\varphi\|_{W^{1,q'}(\mathcal{O})}. \tag{2.7}$$

Hence the Lax–Milgram theorem of of [35, Theorem1.1] implies the existence of u as in (2.5) provided, for all $v \in \mathsf{D}(A_p^{1/2})$,

$$\|\nabla v\|_{L^q(\mathcal{O};\mathbb{R}^2)} \approx \sup \left\{ \int_{\mathcal{O}} \nabla v : \nabla f \, \mathrm{d}x\mathrm{d}z \,\Big|\, f \in \mathsf{D}(A_{q'}^{1/2}) \text{ and } \|f\|_{\mathsf{D}(A_{q'}^{1/2})} \leq 1 \right\}. \tag{2.8}$$

The case $\gtrsim$ of (2.8) follows from the Hölder inequality. To prove the opposite inequality, we argue by duality. We start by discussing some known facts about the "Sobolev tower" of spaces associated the operator A_p:

$$\begin{aligned} X_{\alpha, A_q} &= \mathsf{D}(A_p^\alpha) && \text{for } \alpha \geq 0, \\ X_{\alpha, A_q} &= (\mathbb{L}^q, \|A_q^\alpha \cdot \|_{\mathbb{L}^q})^{\sim} && \text{for } \alpha < 0. \end{aligned}$$

Here $\sim$ denotes the completion (since $0 \in \rho(A_q)$ by Theorem 2.1, we have that $f \mapsto \|A_q^\alpha f\|_{\mathbb{L}^q}$ is a norm for all $\alpha < 0$). Since $(A_q)^* = A_{q'}$, it follows that (see e.g. [5, Chapter 5, Theorem 1.4.9])

$$(X_{\alpha, A_q})^* = X_{-\alpha, A_{q'}}. \tag{2.9}$$

Now we can proceed in the proof of $\lesssim$ in (2.8). Firstly, as $\mathsf{D}(A_q) \hookrightarrow \mathsf{D}(A_q^{1/2})$ is dense for all $q \in (1, \infty)$, we can prove such inequality assuming $v \in \mathsf{D}(A_q)$. In the latter case, the duality (2.9) and the Hahn–Banach theorem imply the existence of $g \in X_{-\alpha, A_{q'}}$ of unit norm such that

$$
\begin{aligned}
\|A_q^{1/2} v\|_{L^q(\mathcal{O};\mathbb{R}^2)} &= \int_{\mathcal{O}} A_q^{1/2} v \cdot A_{q'}^{-1/2} g \, \mathrm{d}x\mathrm{d}z \\
&\overset{(i)}{=} \int_{\mathcal{O}} A_q v \cdot A_{q'}^{-1} g \, \mathrm{d}x\mathrm{d}z \\
&\overset{(ii)}{=} -\int_{\mathcal{O}} \Delta v \cdot A_{q'}^{-1} g \, \mathrm{d}x\mathrm{d}z \\
&\overset{(iii)}{=} -\int_{\mathcal{O}} \nabla v : \nabla(A_{q'}^{-1} g) \, \mathrm{d}x\mathrm{d}z
\end{aligned}
$$

where in (i) we used that $A_q^{1/2} v = A_q^{-1/2}(A_q v)$ and $(A_q^{-1/2})^* = A_{q'}^{-1/2}$, in (ii) that $A_q = -\mathbb{P}_q \Delta_q$ and therefore $\mathbb{P}_{q'} A_{q'}^{-1} g = A_{q'}^{-1} g$ as $A_{q'}^{-1} g \in \mathsf{D}(A_{q'}^{1/2}) \subseteq \mathbb{L}^{q'}(\mathcal{O})$. Finally, in (iii) we used that no boundary terms appear due to the boundary conditions and $\partial_z v_1(\cdot, a) = 0$ as $v \in \mathsf{D}(A_q)$.

Hence the case $\lesssim$ of (2.8) follows from the above chain of equality, the fact that $\mathsf{D}(A_q^{1/2}) \hookrightarrow W^{1,q}(\mathcal{O};\mathbb{R}^2)$ and $A_{q'}^{-1} : X_{-1/2, A_{q'}} \to X_{1/2, A_{q'}}$ is an isomorphism.

Now, the existence of the pressure π satisfying the estimate (2.5) is standard and follows from the De Rham theorem, see e.g. [29, Corollary III.5.1, Lemma IV.1.1].

Step 2: Proof of (2.6). By Step 1, it suffices to prove the existence of a solution $(u, \pi) \in W^{2,q}(\mathcal{O}) \times W^{1,q}(\mathcal{O})/\mathbb{R}$ for which (2.3) holds. In case of $g \in C^\infty(\Gamma_u)$, the conclusion follows from standard L^2-theory and we will present the argument in this case at the end of the proof. In the remaining case we argue by density. Note that, arguing as in the proof of Proposition A.4, a localization argument and [43, Theorem 7.2.1] (applied with time as a dummy variable) yield the following a-priori estimates for solutions $(u, \pi) \in W^{2,q}(\mathcal{O}; \mathbb{R}^2) \times W^{1,q}(\mathcal{O})/\mathbb{R}$ to (2.3):

$$
\begin{aligned}
\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)} + \|\nabla \pi\|_{W^{1,q}(\mathcal{O};\mathbb{R}^2)} &\leq C\|u\|_{W^{2-2/q,q}(\mathcal{O};\mathbb{R}^2)} + C\|g\|_{W^{1-1/q,q}(\Gamma_u)} \\
&\leq \varepsilon\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)} + C_\varepsilon\|u\|_{W^{1,q}(\mathcal{O};\mathbb{R}^2)} + C\|g\|_{W^{1-1/q,q}(\Gamma_u)} \\
&\leq \varepsilon\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)} + C_\varepsilon\|g\|_{W^{1-1/q,q}(\Gamma_u)},
\end{aligned}
$$

where $\varepsilon > 0$ is arbitrary and in the last step we applied Step 1.

The above shows $\|u\|_{W^{2,q}(\mathcal{O};\mathbb{R}^2)} + \|\nabla \pi\|_{W^{1,q}(\mathcal{O})} \lesssim \|g\|_{W^{1-1/q,q}(\mathbb{T})}$ for all solutions $(u, \pi) \in W^{2,q}(\mathcal{O}; \mathbb{R}^2) \times W^{1,q}(\mathcal{O})/\mathbb{R}$ to (2.3). Combining this, the density of $C^\infty(\Gamma_u)$ in $W^{1-1/q,q}(\Gamma_u)$, and the above mentioned solvability for $g \in C^\infty(\Gamma_u)$; one readily obtains the existence of solutions to (2.3) in the class $W^{2,q}(\mathcal{O}) \times W^{1,q}(\mathcal{O})/\mathbb{R}$.

Step 3: Proof of the regularity of (u, π) *in case of* $g \in C^\infty(\mathbb{T})$. The proof of this fact follows the lines of Proposition A.2. First, by Lax-Milgram Lemma and [51, Proposition 1.1, Proposition 1.2], there exists a unique couple, $(u, \pi) \in V \times L^2(\mathcal{O})$ such that

$$\int_{\mathcal{O}} \nabla u : \nabla \phi \, dx dz = - \int_{\mathbb{T}} g(x) \phi_1(x, a) \, dx \quad \forall \phi \in V \tag{2.10}$$

$$\langle -\Delta u + \nabla \pi, \varphi \rangle = 0 \quad \forall \varphi \in C_c^\infty(\mathcal{O}; \mathbb{R}^2) \tag{2.11}$$

$$\|u\|_V + \|\pi\|_{L^2/\mathbb{R}} \lesssim \|g\|_{H^{-1/2}(\Gamma_u)}. \tag{2.12}$$

Now, let us fix $h > 0$, extend periodically either u and g in the x direction and consider $\phi = \tau_h \tau_{-h} u$ as a test function in (2.10), where $\tau_h v = \frac{v(x+h,z)-v(x,z)}{h}$. Then by change of variables, it follows that

$$\begin{aligned}
\|\tau_h \nabla u\|_{L^2(\mathcal{O})}^2 &\le C \|\tau_h g\|_{L^2(\Gamma_u)} \|\tau_h u\|_{L^2(\Gamma_u)} \\
&\le C \|\tau_h g\|_{L^2(\Gamma_u)} \|\tau_h u\|_{L^2(\mathcal{O})} + C \|\tau_h g\|_{L^2(\Gamma_u)} \|\tau_h \nabla u\|_{L^2(\mathcal{O})} \\
&\le C \|g\|_{C^1(\Gamma_u)} \|\tau_h u\|_{L^2(\mathcal{O})} + \frac{\|\tau_h \nabla u\|_{L^2(\mathcal{O})}^2}{2} + C \|g\|_{C^1(\Gamma_u)}^2.
\end{aligned}$$

Therefore

$$\|\tau_h \nabla u\|_{L^2(\mathcal{O})}^2 \le C \|g\|_{C^1(\Gamma_u)} \|\tau_h u\|_{L^2(\mathcal{O})} + C \|g\|_{C^1(\Gamma_u)}^2. \tag{2.13}$$

Since $u \in V$ and (2.12) holds the right hand side of inequality (2.13) is uniformly bounded in $h \to 0$ and this implies

$$\|\partial_x \nabla u\|_{L^2(O)}^2 \le C \|g\|_{C^1(\Gamma_u)}^2. \tag{2.14}$$

Let us now consider $\phi = \partial_x \psi$, $\psi \in \mathcal{D}(\mathcal{O})$ as test function in (2.10). Thanks to [51, Proposition 1.1, Proposition 1.2], $\partial_x \pi \in L^2(\mathcal{O})$ and $\|\partial_x \pi\|_{L^2} \lesssim \|g\|_{C^1(\Gamma_u)}$. Since u is divergence free and (2.11) holds, then

$$\partial_z \pi = \partial_{xx} u_2 - \partial_{xz} u_1 \in L^2(\mathcal{O}).$$

Therefore $\|\nabla \pi\|_{L^2} \lesssim \|g\|_{C^1(\Gamma_u)}$. Lastly, again by relation (2.11)

$$\partial_{zz} u_1 = \partial_x \pi - \partial_{xz} u_1 \in L^2(\mathcal{O})$$

Combining all the information obtained we get

$$\|u\|_{H^2(\mathcal{O}; \mathbb{R}^2)} + \|\pi\|_{H^1(\mathcal{O})/\mathbb{R}} \le C \|g\|_{C^1(\Gamma_u)}^2$$

Iterating the argument one gets that $(u, \pi) \in H^{k+1}(\mathcal{O}; \mathbb{R}^2) \times H^k(\mathcal{O})$ provided $g \in C^k(\Gamma_u)$ for all $k \ge 1$. Now the claim of Step 3 follows from Sobolev embeddings. $\square$

Next we denote by $\mathcal{N}$ the solution map defined by Theorem 2.2 which associate to a boundary datum g the velocity u solution of (2.3), i.e. $\mathcal{N} g := u$. From the above result we obtain

Corollary 2.3 *Let $\mathcal{N}$ and $\mathcal{H}$ be the Neumann map and a Hilbert space, respectively. Then, for all $q \geq 2$ and $\varepsilon > 0$,*

1. $\mathcal{N} \in \mathcal{L}(W^{-\alpha,q}(\Gamma_u; \mathcal{H}); \gamma(\mathcal{H}, D(A_q^{\frac{1-\alpha}{2}+\frac{1}{2q}-\varepsilon})))$ *for* $\alpha \in [0, \frac{1}{q}]$.
2. $\mathcal{N} \in \mathcal{L}(L^q(\Gamma_u; \mathcal{H}); \gamma(\mathcal{H}, D(A_q^{\frac{1}{2}+\frac{1}{2q}-\varepsilon})))$.

Proof To begin, recall that $W^{s,q}(\Gamma_u; \mathcal{H}) = \gamma(\mathcal{H}, W^{s,q}(\Gamma_u))$ for all $s \in \mathbb{R}$ and $q \in (1, \infty)$, see Sect. 1.1.1. Hence, due to the ideal property of γ-radonifying operators [32, Theorems 9.1.10 and 9.1.20], it is enough to consider the scalar case $\mathcal{H} = \mathbb{R}$.

(1): By interpolating with the real method $(\cdot, \cdot)_{\theta,q}$ where $\theta \in (0, 1)$ (see e.g. [10, Theorem 6.4.5]), the estimates in Theorem 2.2 yield

$$\mathcal{N} : W^{\theta-1/q,q}(\Gamma_u) \to W^{\theta+1,q}(\mathcal{O}) \quad \text{for all } \theta \in (0, 1).$$

Moreover, by construction $\mathcal{N}[u]$ satisfies

$$\mathcal{N}[u]|_{\Gamma_b} = 0, \quad \text{and} \quad (\mathcal{N}[u])_2|_{\Gamma_u} = 0,$$

where $(\cdot)_2$ denotes the second component. Hence (1) follows from the description of the fractional power of A_q in Theorem 2.1 and that $B_{q,q}^{1+\theta}(\mathcal{O}; \mathbb{R}^2) \hookrightarrow H^{\theta+1-\varepsilon,q}(\mathcal{O}; \mathbb{R}^2)$.

(2): Follows from (1) and $L^q(\Gamma_u) \hookrightarrow B_{q,q}^0(\Gamma_u)$ as $q \geq 2$. $\qquad\qquad\square$

2.3 Deterministic Navier–Stokes equations

Let us consider the deterministic Navier–Stokes equations with homogeneous boundary conditions

$$\begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{\pi} = \Delta \overline{u} + \overline{f}, & \text{on } (0, T) \times \mathcal{O}, \\ \operatorname{div} \overline{u} = 0, & \text{on } (0, T) \times \mathcal{O}, \\ \overline{u} = 0, & \text{on } (0, T) \times \Gamma_b, \\ \partial_z \overline{u}_1 = 0, & \text{on } (0, T) \times \Gamma_u, \\ \overline{u}_2 = 0, & \text{on } (0, T) \times \Gamma_u, \\ \overline{u}(0) = \overline{u}_0, & \text{on } \mathcal{O}. \end{cases} \qquad (2.15)$$

Define the trilinear form $b : \mathbb{L}^4 \times V \times \mathbb{L}^4 \to \mathbb{R}$ as

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \partial_i v_j w_j \, \mathrm{d}x\mathrm{d}z = \int_{\mathcal{O}} (u \cdot \nabla v) \cdot w \, \mathrm{d}x\mathrm{d}z \qquad (2.16)$$

which is well-defined and continuous on $\mathbb{L}^4 \times V \times \mathbb{L}^4$ by the Hölder inequality. Since by Sobolev embedding theorem $V \subset \mathbb{L}^4$, b is also defined and continuous on $V \times V \times V$. Moreover, by standard interpolation inequalities,

$$\|f\|_{L^4(\mathcal{O})}^2 \le C \|f\|_{L^2(\mathcal{O})} \|f\|_{H^1(\mathcal{O})} \quad \text{for all } f \in H^1(\mathcal{O}). \tag{2.17}$$

Integrating by parts, the standard oddity relation below holds

$$b(u, v, w) = -b(u, w, v)$$

if $u \in \mathbb{L}^4$, $v, w \in V$.

Lastly we introduce the operator

$$B : \mathbb{L}^4 \times \mathbb{L}^4 \to V^*$$

defined by the identity

$$\langle B(u, v), \phi \rangle = -b(u, \phi, v) = -\int_{\mathcal{O}} (u \cdot \nabla \phi) \cdot v \, dx dz$$

for all $\phi \in V$. When $v \in V$, we may also write

$$\langle B(u, v), \phi \rangle = b(u, v, \phi).$$

Moreover, when $u \cdot \nabla v \in L^2(\mathcal{O}; \mathbb{R}^2)$, it is explicitly given by

$$B(u, v) = \mathbb{P}(u \cdot \nabla v).$$

We have to define our notion of weak solution for problem (2.15).

Definition 2.4 Given $\overline{u}_0 \in H$ and $\overline{f} \in L^2(0, T; V^*)$, we say that

$$\overline{u} \in C([0, T]; H) \cap L^2(0, T; V)$$

is a weak solution of equation (2.15) if for all $\phi \in \mathsf{D}(A)$ and $t \in [0, T]$,

$$\langle \overline{u}(t), \phi \rangle - \int_0^t b(\overline{u}(s), \phi, \overline{u}(s)) \, ds$$

$$= \langle \overline{u}_0, \phi \rangle - \int_0^t \langle \overline{u}(s), A\phi \rangle \, ds + \int_0^t \langle \overline{f}(s), \phi \rangle_{V^*, V} \, ds.$$

The following results are simple adaptations of classical results, see for instance [27, 37, 50, 51].

Lemma 2.5 *If $u, v \in L^4(0, T; \mathbb{L}^4)$ then*

$$B(u, v) \in L^2(0, T; V^*). \tag{2.18}$$

Moreover,

$$|b(u, v, w)| \le \varepsilon \|v\|_V^2 + \varepsilon' \|u\|_V^2 + \frac{C}{\varepsilon^2 \varepsilon'} \|u\|^2 \|w\|_{\mathbb{L}^4}^4 \tag{2.19}$$

$$|b\left(u, v, w\right)| \leq \varepsilon \|v\|_V^2 + \varepsilon' \|w\|_V^2 + \frac{C}{\varepsilon^2 \varepsilon'} \|w\|^2 \|u\|_{\mathbb{L}^4}^4, \qquad (2.20)$$

where C is a constant independent of ε and ε'.

Theorem 2.6 *For every $\overline{u}_0 \in H$ and $\overline{f} \in L^2(0, T; V^*)$ there exists a unique weak solution of Eq. (2.15). It satisfies*

$$\|\overline{u}(t)\|^2 + 2\nu \int_0^t \|\nabla \overline{u}(s)\|_{L^2}^2 \, \mathrm{d}s = \|\overline{u}_0\|^2 + 2 \int_0^t \langle \overline{u}(s), \overline{f}(s) \rangle_{V^*, V} \, \mathrm{d}s.$$

If $(\overline{u}_0^n)_{n \in \mathbb{N}}$ is a sequence in H converging to $\overline{u}_0 \in H$ and $\left(\overline{f}^n\right)_{n \in \mathbb{N}}$ is a sequence in $L^2(0, T; V^*)$ converging to $\overline{f} \in L^2(0, T; V^*)$, then the corresponding unique solutions $(\overline{u}^n)_{n \in \mathbb{N}}$ converge to the corresponding solution $\overline{u}$ in $C([0, T]; H)$ and in $L^2(0, T; V)$.

2.4 Stochastic maximal L^p-regularity

Let $\mathcal{H}$ and $(W_{\mathcal{H}}(t))_{t \geq 0}$ be a Hilbert space and a cylindrical $\mathcal{F}-$Brownian motion on $\mathcal{H}$, respectively. The following result was proven in [53], see also [52, Section 7] and [9, Section 3] for additional references. Below, for a Banach space Y, $H^{s,q}(\mathbb{R}_+; Y)$ denotes the Y-valued Bessel potential space on $\mathbb{R}_+$ with smoothness $s \in \mathbb{R}$ and integrability q; such space can be defined either by complex interpolation (see e.g. [43, Chapter 3, Section4.5]) or by restriction from $\mathbb{R}$ (see e.g. [3, Subsection 3.1]). For the notion of H^∞-calculus and γ-radonifying operators $\gamma(\mathcal{H}, Y)$ we refer to [32, Chapter 9 and10].

Theorem 2.7 *Let X be a Banach space isomorphic to a closed subspace of $L^q(D, \mu)$ where $q \in [2, +\infty)$ and $(D, \mathscr{A}, \mu)$ is a σ-finite measure space. Let $\mathcal{A}$ be an invertible operator and assume that it admits a bounded H^∞ calculus of angle $< \pi/2$ on X and let $(\mathcal{S}(t))_{t \geq 0}$ the bounded analytic semigroup generated by $-\mathcal{A}$. For all $\mathcal{F}-$adapted $G \in L^p(\mathbb{R}_+ \times \Omega; \gamma(\mathcal{H}; X))$ the stochastic convolution process*

$$U(t) = \int_0^t \mathcal{S}(t - s) G(s) \, \mathrm{d}W_{\mathcal{H}}(s) \quad t \geq 0,$$

is well defined in X, takes values in the fractional domain $\mathsf{D}(\mathcal{A}^{1/2})$ almost surely and for all $2 < p < +\infty$ the following space-time regularity estimate holds: $\forall \theta \in [0, \frac{1}{2})$

$$\mathbf{E}\left[\|U(t)\|_{H^{\theta, p}(\mathbb{R}_+; \mathsf{D}(\mathcal{A}^{1/2 - \theta}))}^p\right] \leq C_\theta^p \mathbf{E}\left[\|G\|_{L^p(\mathbb{R}_+; L^q(O; \mathcal{H}))}^p\right] \qquad (2.21)$$

with a constant C_θ independent of G.

For extensions of the above result we refer to [2, 39] for the weighted case, and to [3, Subsection 6.2] for the case of homogeneous spaces. However, the latter situations will not be considered here.

3 Well-posedness

3.1 Stokes equations

As discussed in Sect. 1.1, we start by considering the linear problem (1.3). According to [18, 19], the mild solution w of the former problem is formally given by

$$w(t) = A_q \int_0^t S_q(t - s) \mathcal{N}[h_b(s)] \, dW_{\mathcal{H}}(s). \tag{3.1}$$

Here A_q is (minus) the Stokes operator with homogeneous boundary conditions, and $(S_q(t))_{t \geq 0}$ its corresponding semigroup (cf.Theorem 2.1).

Next step is to prove that $w(t)$ is well defined in some functional spaces and has some regularities useful to treat the nonlinearity of the Navier–Stokes equations.

Proposition 3.1 *Let $\alpha \in [0, \frac{1}{q}]$ and assume that $h_b : (0, T) \times \Omega \to W^{-\alpha, q}(\Gamma_u; \mathcal{H})$ is $\mathcal{F}$-progressive measurable with $\mathbf{P} - a.s.$ paths in $L^p(0, T; W^{-\alpha, q}(\Gamma_u))$. Then the process w defined in (3.1) is a well defined process with $\mathbf{P} - a.s.$ paths in*

$$H^{\theta, p}(0, T; \mathsf{D}(A_q^{\frac{1}{2q} - \frac{\alpha}{2} - \theta - \varepsilon})) \quad \text{for all } \theta \in [0, \tfrac{1}{2}), \ \varepsilon > 0.$$

In particular, if h_b satisfies Hypothesis 1.1, then w has $\mathbf{P} - a.s.$ trajectories in $C([0, T]; H) \cap L^4(0, T; \mathbb{L}^4)$.

Proof By replacing h_b by $\mathbf{1}_{[0,\tau_n] \times \Omega} h_b$, τ_n being the following stopping time

$$\tau_n := \{t \in [0, T] : \|h\|_{L^p(0, t; W^{-\alpha, q}(\Gamma_u; \mathcal{H}))} \geq n\} \quad \text{where} \quad \inf \varnothing := T,$$

for all $n \geq 1$, it is enough to consider the case $h_b \in L^p((0, T) \times \Omega; W^{-\alpha, q}(\Gamma_u; \mathcal{H}))$.

Let $\varepsilon > 0$ be fixed later. From Corollary 2.3 and Theorem 2.7 we have that $\mathbf{P} - a.s.$ and for each $\theta \in [0, \frac{1}{2})$

$$\widetilde{w} = \int_0^\cdot S_q(\cdot - s) A_q^{\frac{1-\alpha}{2} + \frac{1}{2q} - \varepsilon} \mathcal{N}[h_b(s)] \, dW_{\mathcal{H}}(s) \in H^{\theta, p}(0, T; \mathsf{D}(A_q^{1/2 - \theta}))$$

Therefore, $\mathbf{P} - a.s.$,

$$w = A_q^{\frac{1+\alpha}{2} - \frac{1}{2q} + \varepsilon} \widetilde{w} \in H^{\theta, p}(0, T; \mathsf{D}(A_q^{\frac{1}{2q} - \frac{\alpha}{2} - \theta - \varepsilon})).$$

Finally, note that, by Hypothesis 1.1, Theorem 2.1 and the Sobolev embeddings (see e.g. [9, Proposition 2.7]) we can find $\theta_1, \theta_2 \in [0, \frac{1}{2})$ and $\varepsilon > 0$ such that

$$H^{\theta_1, p}(0, T; \mathsf{D}(A_q^{\frac{1}{2q} - \frac{\alpha}{2} - \theta_1 - \varepsilon})) \hookrightarrow C([0, T]; H),$$

$$H^{\theta_2, p}(0, T; \mathsf{D}(A_q^{\frac{1}{2q} - \frac{\alpha}{2} - \theta_2 - \varepsilon})) \hookrightarrow L^4(0, T; \mathbb{L}^4).$$

where the first embedding follows from $\alpha < \frac{1}{q} - \frac{2}{p}$ and the second one from the remaining conditions in Hypothesis 1.1. Hence the proof is complete. $\qquad\square$

3.2 Auxiliary Navier–Stokes type equations

Having solved the Stokes problem we introduce the auxiliary variable

$$v(t) = u(t) - w(t),$$

which satisfies (1.4), i.e.

$$\begin{cases} \partial_t v + (v + w) \cdot \nabla (v + w) + \nabla (P - \rho) = \nu \Delta v, & \text{on } (0, T) \times \mathcal{O}, \\ \operatorname{div} v = 0, & \text{on } (0, T) \times \mathcal{O}, \\ v = 0, & \text{on } (0, T) \times \Gamma_b, \\ \partial_z v_1 = 0, & \text{on } (0, T) \times \Gamma_u, \\ v_2 = 0, & \text{on } (0, T) \times \Gamma_u, \\ v(0) = u_0, & \text{on } \mathcal{O}. \end{cases}$$

This first equation in the above system has the form

$$\partial_t v + v \cdot \nabla v + \nabla \pi = \nu \Delta v - L(v, w)$$

with the affine function

$$L(v, w) = v \cdot \nabla w + w \cdot \nabla v + w \cdot \nabla w.$$

For each $\omega \in \Omega$ fixed, the Navier–Stokes structure is preserved and the auxiliary equation for v with homogeneous boundary conditions is solvable similarly to the classical Navier–Stokes equations. Therefore, let us introduce the notion of weak solution of the deterministic problem (1.4) with random coefficients. Recall that A and b are (minus) the Stokes operator on $\mathbb{L}^2$ and defined in (2.16), respectively.

Definition 3.2 Given $u_0 \in H$ and $w \in L^4(0, T; \mathbb{L}^4)$, we say that

$$v \in C([0, T]; H) \cap L^2(0, T; V)$$

is a weak solution of Eq. (1.4) if

$$\langle v(t), \phi \rangle - \int_0^t b(v(s) + w(s), \phi, v(s) + w(s)) \, \mathrm{d}s$$
$$= \langle u_0, \phi \rangle - \int_0^t \langle v(s), A\phi \rangle \, \mathrm{d}s$$

for every $\phi \in D(A)$ and $t \in [0, T]$.

Theorem 3.3 *For every $u_0 \in H$ and $w \in L^4\left(0, T; \mathbb{L}^4\right)$, there exists a unique weak solution v of Eq. (1.4). Moreover, v satisfies for all $t \in [0, T]$*

$$\|v(t)\|^2 + 2 \int_0^t \|\nabla v(s)\|_{L^2}^2 \, ds = \|u_0\|^2 + 2 \int_0^t (b(v, v, w) + b(w, v, w))(s) \, ds. \tag{3.2}$$

If $\left(u_0^n\right)_{n \in \mathbb{N}}$ is a sequence in H converging to $u_0 \in H$ and $(w^n)_{n \in \mathbb{N}}$ is a sequence in $L^4\left(0, T; \mathbb{L}^4\right)$ converging to $w \in L^4\left(0, T; \mathbb{L}^4\right)$, then the corresponding unique solutions $(v^n)_{n \in \mathbb{N}}$ converge to the corresponding solution v in $C\left([0, T]; H\right)$ and in $L^2(0, T; V)$.

Proof We split the proof into several steps.

Step 1: Uniqueness. Let $v^{(i)}$ be two solutions. The function $z = v^{(1)} - v^{(2)}$ satisfies

$$\langle z(t), \phi \rangle - \int_0^t \left(b\left(v^{(1)} + w, \phi, v^{(1)} + w\right) - b\left(v^{(2)} + w, \phi, v^{(2)} + w\right) \right) \, ds$$
$$= - \int_0^t \langle z(s), A\phi \rangle \, ds$$

for every $\phi \in D(A)$. A simple manipulation gives us

$$b\left(v^{(1)} + w, \phi, v^{(1)} + w\right) - b\left(v^{(2)} + w, \phi, v^{(2)} + w\right) - b(z, \phi, z)$$
$$= b\left(v^{(2)} + w, \phi, z\right) + b\left(z, \phi, v^{(2)} + w\right)$$

hence

$$\langle z(t), \phi \rangle - \int_0^t b(z(s), \phi, z(s)) \, ds = - \int_0^t \langle z(s), A\phi \rangle \, ds + \int_0^t \langle \widetilde{f}(s), \phi \rangle \, ds$$

where

$$\widetilde{f} = -B\left(v^{(2)} + w, z\right) - B\left(z, v^{(2)} + w\right).$$

By Lemma 2.5, $\widetilde{f} \in L^2(0, T; V^*)$. Then, by Theorem 2.6,

$$\|z(t)\|^2 + 2 \int_0^t \|\nabla z(s)\|_{L^2}^2 \, ds = 2 \int_0^t b\left(z, z, v^{(2)} + w\right)(s) \, ds.$$

Again by Lemma 2.5, we have

$$\left| b\left(z, z, v^{(2)} + w\right) \right| \leq \left| b\left(z, z, v^{(2)}\right) \right| + |b(z, z, w)|$$
$$\leq \varepsilon \|z\|_V^2 + \varepsilon \|z\|_V^2 + \frac{C}{\varepsilon^3} \|z\|^2 \|v^{(2)}\|_{\mathbb{L}^4}^4$$

$$+ \varepsilon \|z\|_V^2 + \varepsilon \|z\|_V^2 + \frac{C}{\varepsilon^3} \|z\|^2 \|w\|_{\mathbb{L}^4}^4$$

$$= 4\varepsilon \|z\|_V^2 + \frac{C}{\varepsilon^3} \|z\|^2 \left(\|v^{(2)}\|_{\mathbb{L}^4}^4 + \|w\|_{\mathbb{L}^4}^4 \right).$$

Summarizing, with $4\varepsilon = 1$, using the fact that $\|z\|_V^2 = \|\nabla z\|_{L^2}^2$, renaming the constant C,

$$\|z(t)\|^2 + \int_0^t \|\nabla z(s)\|_{L^2}^2 \, ds \le C \int_0^t \|z(s)\|^2 \left(1 + \|v^{(2)}(s)\|_{\mathbb{L}^4}^4 + \|w(s)\|_{\mathbb{L}^4}^4 \right) \, ds.$$

We conclude $z = 0$ by the Gronwall lemma, using the assumption on w and the integrability properties of $v^{(2)}$.

Step 2: Existence. Define the sequence (v^n) by setting $v^0 = 0$ and for every $n \ge 0$, given $v^n \in C([0, T]; H) \cap L^2(0, T; V)$, let v^{n+1} be the solution of equation (2.15) with initial condition u_0 and with

$$-B(v^n, w) - B(w, v^n) - B(w, w)$$

in place of f. In particular

$$\left\langle v^{n+1}(t), \phi \right\rangle - \int_0^t b\left(v^{n+1}(s), \phi, v^{n+1}(s)\right) \, ds$$

$$= \langle u_0, \phi \rangle - \int_0^t \left\langle v^{n+1}(s), A\phi \right\rangle \, ds$$

$$- \int_0^t \left\langle \left(B(v^n, w) + B(w, v^n) + B(w, w)\right)(s), \phi \right\rangle \, ds$$

for every $\phi \in D(A)$. In order to claim that this definition is well done, we notice that

$$B(v^n, w), B(w, v^n), B(w, w) \in L^2(0, T; V^*)$$

by Lemma 2.5.

Then let us investigate the convergence of (v^n). First, let us prove a bound. From the previous identity and Theorem 2.6 we get

$$\|v^{n+1}(t)\|^2 + 2\int_0^t \|\nabla v^{n+1}(s)\|_{L^2}^2 \, ds$$

$$= \|u_0\|^2 + 2\int_0^t \left(b\left(v^n, v^{n+1}, w\right) + b\left(w, v^{n+1}, v^n\right) + b\left(w, v^{n+1}, w\right) \right)(s) \, ds.$$

It gives us (using Lemma 2.5)

$$\|v^{n+1}(t)\|^2 + \int_0^t \|\nabla v^{n+1}(s)\|_{L^2}^2 \, ds$$

$$= \|u_0\|^2 + \varepsilon \int_0^t \|v^n(s)\|_V^2 \, ds$$

$$+ C_\varepsilon \int_0^t \|v^n(s)\|^2 \left(1 + \|w(s)\|_{\mathbb{L}^4}^4\right) ds + C_\varepsilon \int_0^t \|w(s)\|_{\mathbb{L}^4}^4 \, ds.$$

Choosing a small constant ε, one can find $R > \|u_0\|^2$ and $\overline{T}$ small enough, depending only from $\|u_0\|$ and $\|w\|_{L^4(0,T;\mathbb{L}^4)}$, such that if

$$\sup_{t\in[0,\overline{T}]} \|v^n(t)\|^2 \le R, \qquad \int_0^{\overline{T}} \|v^n(s)\|_V^2 \, ds \le R \tag{3.3}$$

then the same inequalities hold for v^{n+1}.

Set $w_n = v^n - v^{n-1}$, for $n \ge 1$. From the identity above,

$$\langle w_{n+1}(t), \phi \rangle - \int_0^t \left(b\left(v^{n+1}, \phi, v^{n+1}\right) - b\left(v^n, \phi, v^n\right) \right)(s) \, ds$$

$$= -\int_0^t \langle w_{n+1}(s), A\phi \rangle \, ds - \int_0^t \left\langle \left(B\left(v^n, w\right) - B\left(v^{n-1}, w\right)\right)(s), \phi \right\rangle ds$$

$$- \int_0^t \left\langle \left(B\left(w, v^n\right) - B\left(w, v^{n-1}\right)\right)(s), \phi \right\rangle ds.$$

Again as above, since

$$b\left(v^{n+1}, \phi, v^{n+1}\right) - b\left(v^n, \phi, v^n\right) - b\left(w_{n+1}, \phi, w_{n+1}\right)$$

$$= b\left(v^n, \phi, w_{n+1}\right) + b\left(w_{n+1}, \phi, v^n\right)$$

we may rewrite it as

$$\langle w_{n+1}(t), \phi \rangle - \int_0^t b\left(w_{n+1}(s), \phi, w_{n+1}(s)\right) ds$$

$$= -\int_0^t \langle w_{n+1}(s), A\phi \rangle \, ds - \int_0^t \langle (B(w_n, w) + B(w, w_n))(s), \phi \rangle \, ds$$

$$+ \int_0^t \left(b \left(v^n, \phi, w_{n+1} \right) + b \left(w_{n+1}, \phi, v^n \right) \right) (s) \, ds.$$

One can check as above the applicability of Theorem 2.6 and get

$$\| w_{n+1} (t) \|^2 + 2 \int_0^t \| \nabla w_{n+1} (s) \|_{L^2}^2 \, ds$$

$$= 2 \int_0^t \left(b \left(w_n, w_{n+1}, w \right) + b \left(w, w_{n+1}, w_n \right) \right) (s) \, ds$$

$$+ 2 \int_0^t b \left(w_{n+1}, w_{n+1}, v^n \right) (s) \, ds.$$

As above we deduce

$$\left| b \left(w_{n+1}, w_{n+1}, v^n \right) \right| \leq \frac{1}{2} \| w_{n+1} \|_V^2 + C \| w_{n+1} \|^2 \| v^n \|_{\mathbb{L}^4}^4.$$

But

$$\left| b \left(w_n, w_{n+1}, w \right) + b \left(w, w_{n+1}, w_n \right) \right| \leq \frac{1}{2} \| w_{n+1} \|_V^2 + \frac{1}{4} \| w_n \|_V^2 + C \| w_n \|^2 \| w \|_{\mathbb{L}^4}^4.$$

Hence

$$\| w_{n+1} (t) \|^2 + \int_0^t \| \nabla w_{n+1} (s) \|_{L^2}^2 \, ds$$

$$\leq C \int_0^t \| w_{n+1} (s) \|^2 \left(1 + \| v^n (s) \|_{\mathbb{L}^4}^4 \right) \, ds$$

$$+ \frac{1}{4} \int_0^t \| w_n (s) \|_V^2 \, ds + C \int_0^t \| w_n (s) \|^2 \| w (s) \|_{\mathbb{L}^4}^4 \, ds.$$

Now we work under the bounds (3.3) and deduce, using the Gronwall lemma, for $\overline{T}$, depending only from $\| u_0 \|$ and $\| w \|_{L^4(0,T;\mathbb{L}^4)}$, possibly smaller than the previous one,

$$\sup_{t \in [0, \overline{T}]} \| w_{n+1} (t) \|^2 + \int_0^{\overline{T}} \| w_{n+1} (s) \|_V^2 \, ds$$

$$\leq \frac{1}{2} \left(\sup_{t \in [0, \overline{T}]} \| w_n (t) \|^2 + \int_0^{\overline{T}} \| w_n (s) \|_V^2 \, ds \right).$$

It implies that the sequence (v^n) is Cauchy in $C \left([0, \overline{T}] ; H \right) \cap L^2 \left(0, \overline{T}; V \right)$. The limit v has the right regularity to be a weak solution and satisfies the weak formulation; in the identity above for v^{n+1} and v^n we may prove that

$$\int_0^t b \left(v^{n+1} (s), \phi, v^{n+1} (s) \right) \, ds \to \int_0^t b \left(v (s), \phi, v (s) \right) \, ds$$

$$\int_0^t b\left(v^n\left(s\right), \phi, w\left(s\right)\right) \mathrm{d}s \to \int_0^t b\left(v\left(s\right), \phi, w\left(s\right)\right) \mathrm{d}s$$

$$\int_0^t b\left(w\left(s\right), \phi, v^n\left(s\right)\right) \mathrm{d}s \to \int_0^t b\left(w\left(s\right), \phi, v\left(s\right)\right) \mathrm{d}s.$$

All these convergences can be proved easily by recalling the definition of b. Similarly, we can pass to the limit in the energy identity. After proving existence and uniqueness in $[0, \overline{T}]$ we can reiterate the existence procedure and in a finite number of steps cover the interval $[0, T]$.

Step 3: Continuous dependence on the data. Let v^n (resp. v) the unique solution of (1.4) with data u_0^n, w^n (resp. u_0, w). Since $u_0^n \to u_0$ in H (resp. $w^n \to w$ in $L^4(0, T; \mathbb{L}^4)$) the family $(u_0^n)_{n \in \mathbb{N}}$ is bounded in H (resp. the family $(w^n)_{n \in \mathbb{N}}$ is bounded in $L^4(0, T; \mathbb{L}^4)$), by (3.2) one can show easily that the family $(v^n)_{n \in \mathbb{N}}$ is bounded in $C([0, T]; H) \cap L^2(0, T; V) \hookrightarrow L^4(0, T; \mathbb{L}^4)$. Moreover for each $t \in [0, T]$, $z^n := v^n - v$ satisfies the energy relation

$$
\begin{aligned}
\frac{1}{2}\|z^n(t)\|^2 + \int_0^t \|\nabla z^n(s)\|_{L^2}^2 \, \mathrm{d}s = {} & \frac{1}{2}\|u_0^n - u_0\|^2 \\
& + \int_0^t b(v^n(s) + w^n(s), z^n(s), w^n(s) - w(s)) \, \mathrm{d}s \\
& + \int_0^t b(z^n(s), z^n(s), v(s) + w(s)) \, \mathrm{d}s \\
& + \int_0^t b(w^n(s) - w(s), z^n(s), v(s) + w(s)) \, \mathrm{d}s.
\end{aligned}
\tag{3.4}
$$

We can easily bound the right hand side of relation (3.4) by Young's inequality and Hölder's inequality obtaining

$$
\begin{aligned}
\frac{1}{2}\|z^n(t)\|^2 + \frac{1}{2}\int_0^t \|\nabla z^n(s)\|_{L^2}^2 \, \mathrm{d}s \leq {} & \frac{1}{2}\|u_0^n - u_0\|^2 \\
& + C\int_0^t \|z^n(s)\|^2 \left(\|v(s)\|_{\mathbb{L}^4}^4 + \|w(s)\|_{\mathbb{L}^4}^4\right) \, \mathrm{d}s \\
& + C\|w^n - w\|_{L^4(0,T;\mathbb{L}^4)}^2 \left(\|w^n\|_{L^4(0,T;\mathbb{L}^4)}^2 + \|w\|_{L^4(0,T;\mathbb{L}^4)}^2\right) \\
& + C\|w^n - w\|_{L^4(0,T;\mathbb{L}^4)}^2 \left(\|v^n\|_{L^4(0,T;\mathbb{L}^4)}^2 + \|v\|_{L^4(0,T;\mathbb{L}^4)}^2\right). \tag{3.5}
\end{aligned}
$$

Applying Gronwall's inequality to relation (3.5) the claim follows immediately. □

Remark 3.4 Freezing the variable $\omega \in \Omega$ and solving (1.4) for each ω does not allow to obtain information about the measurability properties of v. However, measurability of v with respect of the progressive σ-algebra follows from the continuity of the solution map with respect to u_0 and w. Therefore we have the required measurability

properties for v with w being the mild solution of (1.3). In particular v has **P**-a.s. paths in $C(0, T; H) \cap L^2(0, T; V)$, it is progressively measurable with respect to these topologies and

$$\langle v(t), \phi \rangle - \int_0^t b(v(s) + w(s), \phi, v(s) + w(s)) \, \mathrm{d}s$$

$$= \langle u_0, \phi \rangle - \int_0^t \langle v(s), A\phi \rangle \, \mathrm{d}s \quad \mathbf{P} - a.s. \tag{3.6}$$

for every $\phi \in D(A)$ and $t \in [0, T]$.

Proof of Theorem 1.4 It follows immediately combining Proposition 3.1, Theorem 3.3 and Remark 3.4.

4 Interior regularity

4.1 Stokes equations

As in the proof of Theorem 1.4, by a stopping time argument we may assume that h_b is also $L^p(\Omega)$-integrable, cf. the beginning of the proof of Proposition 3.1. This fact will be used below without further mentioning it. We start showing a lemma, concerning the relation between the mild and the weak formulation of (1.3) defined below.

Definition 4.1 Let Hypothesis 1.1 be satisfied. A stochastic process w is a weak solution of (1.3) if it is $\mathcal{F}$-progressively measurable with $\mathbf{P} - a.s.$ paths in

$$w \in C([0, T]; H) \cap L^4(0, T; \mathbb{L}^4)$$

and for each $\phi \in D(A)$

$$\langle w(t), \phi \rangle = - \int_0^t \langle w(s), A\phi \rangle \, \mathrm{d}s + \int_0^t \langle h_b(s), \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, \mathrm{d}W_{\mathcal{H}}(s) \tag{4.1}$$

for each $t \in [0, T]$, $\mathbf{P} - a.s.$

Note that the last term in (4.1) is well-defined as $\alpha < 1/2$ and $q' < 2$.

Remark 4.2 In the definition above, the term $\langle h_b(s), \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)}$ is given by the following linear operator on $\mathcal{H}$:

$$\mathcal{H} \ni h' \mapsto \langle h_b(s)h', \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} = L_\phi(h_b(s)h')$$

where $L_\phi := \langle \cdot, \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)}$. By the ideal property of γ-spaces and $\gamma(\mathcal{H}, W^{-\alpha,q}(\Gamma_u)) = W^{-\alpha,q}(\Gamma_u; \mathcal{H})$ we have

$$\|\langle h_b(s), \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)}\|_{\mathcal{H}^*} \lesssim \|h_b(s)\|_{W^{-\alpha,q}(\Gamma_u; \mathcal{H})} \|\phi\|_{W^{\alpha,q'}(\Gamma_u)},$$

a.e. on $(0, T) \times \Omega$. Whence, the stochastic integral in (4.1) is well-defined with scalar value as

$$
\mathbf{E}\left[\left|\int_0^T \langle h_b(s), \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{H}}(s)\right|^2\right]
$$

$$
= \mathbf{E}\left[\int_0^T \|\langle h_b(s), \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)}\|^2_{\mathcal{H}^*} \, ds\right]
$$

$$
\lesssim \mathbf{E}\left[\int_0^T \|h_b(s)\|^2_{W^{-\alpha,q}(\Gamma_u;\mathcal{H})} \|\phi\|^2_{W^{-\alpha,q'}(\Gamma_u)} \, ds\right] < \infty,
$$

where the last estimate follows from Hypothesis 1.1.

Lemma 4.3 *Let Hypothesis 1.1 be satisfied. There exists a unique weak solution of* (1.3) *in the sense of Definition* 4.1 *and it is given by the formula* (3.1).

Proof We split the proof into two steps.

Step 1: There exists a unique weak solution of (1.3) *and it is necessarily given by the mild formula* (3.1). Let $\psi \in C^1([0, T]; D(A))$. Arguing as in the first step of the proof of [27, Theorem 1.7], see also [26, Proposition 17], one can readily check that w satisfies

$$
\langle w(t), \psi(t) \rangle = \int_0^t \langle w(s), \partial_s \psi(s) \rangle \, ds - \int_0^t \langle w(s), A\psi(s) \rangle \, ds
$$

$$
+ \int_0^t \langle h_b(s), \psi(s) \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{H}}(s) \tag{4.2}
$$

for each $t \in [0, T]$, $\mathbf{P}-a.s.$ The stochastic integral in the relation above is well-defined arguing as in Remark 4.2. Now consider $\phi \in D(A^2)$ and use $\psi_t(s) = S_{q'}(t-s)\phi$, $s \in [0, t]$ as test function in (4.2) obtaining, since $S_{q'}(t)|_H = S(t)$,

$$
\langle w(t), \phi \rangle = \int_0^t \langle h_b(s), S_{q'}(t-s)\phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{H}}(s). \tag{4.3}
$$

Recalling the definition of the Neumann map $\mathcal{N}$, (4.3) can be rewritten as

$$
\langle w(t), \phi \rangle = \int_0^t \langle \mathcal{N}[h_b(s)], A_{q'} S_{q'}(t-s)\phi \rangle \, dW_{\mathcal{H}}(s). \tag{4.4}
$$

Then, exploiting the self-adjointness property of S_q and A_q we have that weak solutions of (1.3) satisfy the mild formulation. Therefore they are unique.

Step 2: The mild formula (3.1) *is a weak solution of* (1.3) *in the sense of* Definition 4.1. We begin by noticing that w has the required regularity due to Proposition 3.1. Let us test our mild formulation (3.1) against functions $\phi \in D(A^2) \subseteq D(A_{q'}^2)$. It holds, since $S_{q'}(t)|_H = S(t)$, $A_{q'}|_{D(A)} = A$ and exploiting self-adjointness property of S_q and A_q

$$\langle w(t), \phi \rangle = \int_0^t \langle \mathcal{N}[h_b(s)], AS(t-s)\phi \rangle \, dW_{\mathcal{H}}(s)$$

$$= \int_0^t \langle h_b(s), S(t-s)\phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{H}}(s) \quad \mathbf{P} - a.s.,$$

where in the last step we used the definition of Neumann map. In order to complete the proof of this step it is enough to show that

$$\int_0^t \langle h_b(s), S(t-s)\phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{H}}(s) = - \int_0^t \langle w(s), A\phi \rangle \, ds$$

$$+ \int_0^t \langle h_b(s), \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{H}}(s) \quad \mathbf{P} - a.s. \quad (4.5)$$

Relation (4.5) is true. Indeed,

$$\int_0^t \langle w(s), A\phi \rangle \, ds = \int_0^t ds \int_0^s \langle \mathcal{N}[h_b(r)], S(s-r)A^2\phi \rangle \, dW_{\mathcal{H}}(r) \quad \mathbf{P} - a.s. \quad (4.6)$$

The double integrals in (4.6) can be exchanged via stochastic Fubini's Theorem, see [20], since

$$\int_0^t ds \left(\mathbf{E} \left[\int_0^s dr \| \langle \mathcal{N}[h_b(r)], S(s-r)A^2\phi \rangle \|_{\mathcal{H}}^2 \right] \right)^{1/2}$$

$$\leq C(T, q) \|A^2\phi\|_{\mathbb{L}^2} \mathbf{E} \left[\|h_b\|_{L^2(0,T; W^{-\alpha,q}(\Gamma_u; \mathcal{H}))}^2 \right]^{1/2} < +\infty$$

Therefore the double integral in the right hand side of (4.6) can be rewritten as

$$\int_0^t ds \int_0^s \langle \mathcal{N}[h_b(r)], S(s-r)A^2\phi \rangle \, dW_{\mathcal{H}}(r)$$

$$= \int_0^t dW_{\mathcal{H}}(r) \int_r^t \langle \mathcal{N}[h_b(r)], S(s-r)A^2\phi \rangle \, ds$$

$$= \int_0^t \langle \mathcal{N}[h_b(r)], A\phi \rangle \, dW_{\mathcal{H}}(r)$$

$$- \int_0^t \langle \mathcal{N}[h_b(r)], AS(t-r)\phi \rangle \, dW_{\mathcal{H}}(r)$$

$$= \int_0^t \langle h_b(r), \phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{H}}(r)$$

$$- \int_0^t \langle h_b(r), S(t-r)\phi \rangle_{W^{-\alpha,q}(\Gamma_u), W^{\alpha,q'}(\Gamma_u)} \, dW_{\mathcal{H}}(r) \quad \mathbf{P} - a.s.$$

Inserting this expression in (4.6), (4.5) holds and the proof is complete. □

Thanks to the weak formulation guaranteed by Lemma 4.3 we can easily obtain the interior regularity of the linear stochastic problem (1.3). Let N_0 be the $\mathbf{P}$ null measure

set where at least one between $w \notin C([0, T]; H) \cap L^4(0, T; \mathbb{L}^4)$, $v \notin C([0, T]; H) \cap L^2(0, T; V)$, (4.1) and (3.6) is not satisfied. In the following we will work pathwise in $\Omega \setminus N_0$ even if not specified.

Corollary 4.4 *Let Hypothesis* 1.1 *be satisfied. Let w be the unique weak solution of* (1.3) *in the sense of Definition* 4.1. *Then, for all $0 < t_1 \le t_2 < T$, $x_0 \in \mathcal{O}$, $r > 0$ such that* $\mathrm{dist}(B(x_0, r), \partial\mathcal{O}) > 0$,

$$w \in C([t_1, t_2], C^\infty(B(x_0, r); \mathbb{R}^2)) \quad \mathbf{P} - a.s.$$

Proof Denote $\omega_w = \mathrm{curl}\, w \in C([0, T]; H^{-1}(\mathcal{O})) \, \mathbf{P}-a.s.$ Since $\mathrm{dist}(B(x_0, r), \partial\mathcal{O}) > 0$, $0 < t_1 \le t_2 < T$ we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_2 + 2\varepsilon < T$, $\mathrm{dist}(B(x_0, r + 2\varepsilon), \partial\mathcal{O}) > 0$. Let us consider $\psi \in C_c^\infty(\mathcal{O})$ and use $\nabla^\perp\psi$ as test function in (4.1). This implies that ω_w is a distributional solution of the heat equation

$$\partial_t \omega_w = \Delta\omega_w.$$

Since ω_w solves the heat equation in distributions, a standard localization argument and regularity results for the heat equation (see e.g. [49, Chapter 6, Section 1]) imply that

$$\omega_w \in C([t_1 - \varepsilon, t_2 + \varepsilon], C^\infty(B(x_0, r + \varepsilon))) \quad \mathbf{P} - a.s.$$

Let us now consider a test function $\phi \in C_c^\infty(B(x_0, r + \varepsilon))$ identically equal to one on $B(x_0, r + \varepsilon/2)$. Since $\mathrm{div}\, w = 0$, we have that $\hat{w} = \phi w$ solves the elliptic problem

$$\Delta\hat{w} = \nabla^\perp\omega_w\phi + \Delta\phi w + 2\nabla\phi \cdot \nabla w, \quad \hat{w}|_{\partial B(x_0, r+\varepsilon)} = 0.$$

Since $w \in C([t_1 - \varepsilon, t_2 + \varepsilon]; L^2(B(x_0, r + \varepsilon))) \, \mathbf{P}-a.s.$ by Proposition 3.1, it follows that

$$\nabla^\perp\omega_w\phi + \Delta\phi w + 2\nabla\phi \cdot \nabla w \in C([t_0 - \varepsilon, T]; H^{-1}(B(x_0, r + \varepsilon))) \quad \mathbf{P} - a.s.$$

Therefore, by standard elliptic regularity theory

$$\hat{w} \in C([t_1 - \varepsilon, t_2 + \varepsilon]; H^1(B(x_0, r + \varepsilon))) \quad \mathbf{P} - a.s.$$

From the fact that $\phi \equiv 1$ on $B(x_0, r + \varepsilon/2)$ it follows that

$$w \in C([t_1 - \varepsilon/4, t_2 + \varepsilon/4]; H^1(B(x_0, r + \varepsilon/4))) \quad \mathbf{P} - a.s.$$

Therefore, the required regularity of w is established by inductively reiterating this argument and by considering test functions $\phi \in C_c^\infty(B(x_0, r + \frac{\varepsilon}{2^{2n}}))$ identically equal to one on $B(x_0, r + \frac{\varepsilon}{2^{2n+1}})$. $\qquad\square$

4.2 Auxiliary Navier–Stokes type equations

In order to deal with the interior regularity of (1.4) we perform a Serrin type argument, see [36, 47]. The regularity of w guaranteed by Corollary 4.4 will play a crucial role to treat the linear terms appearing in (1.4). We start with the following lemma.

Lemma 4.5 *Let Hypothesis 1.1 be satisfied. Let v be the unique solution of (1.4) in the sense of Definition 3.2, where w is as in Corollary 4.4. Then, for all $0 < t_1 \leq t_2 < T$, $x_0 \in \mathcal{O}$, $r > 0$ such that $\mathrm{dist}(B(x_0, r), \partial\mathcal{O}) > 0$,*

$$v \in C([t_1, t_2], H^{3/2}(B(x_0, r); \mathbb{R}^2)) \quad \mathbf{P} - a.s.$$

Proof As described in Lemma 4.3, arguing as in the proof of [27, Theorem 1.7], we can extend the weak formulation satisfied by v to time dependent test functions $\phi \in C^1([0, T]; H) \cap C([0, T]; D(A))$ obtaining that for each $t \in [0, T]$

$$\langle v(t), \phi(t) \rangle - \langle u_0, \phi(0) \rangle = \int_0^t \langle v(s), \partial_s \phi(s) \rangle \, ds - \int_0^t \langle v(s), A\phi(s) \rangle \, ds$$
$$+ \int_0^t b(v(s) + w(s), \phi(s), v(s) + w(s)) \, ds \quad \mathbf{P} - a.s.$$

Choosing $\phi = -\nabla^\perp \chi$, $\chi \in C_c^\infty((0, T) \times \mathcal{O})$ in the weak formulation above and denoting by

$$\omega = \mathrm{curl}\, v \in C([0, T]; H^{-1}) \cap L^2((0, T) \times \mathcal{O}) \quad \mathbf{P} - a.s.,$$
$$\omega_w = \mathrm{curl}\, w \in C([t_1, t_2], C^\infty(B(x_0, r))) \quad \mathbf{P} - a.s.$$

it follows that

$$-\int_0^T \langle \omega(s), \partial_s \chi(s) \rangle + \langle \omega(s), \Delta\chi(s) \rangle \, ds = \int_0^T \langle \mathrm{curl}(w(s) \otimes w(s)), \nabla\chi(s) \rangle \, ds$$
$$+ \int_0^T \langle \mathrm{curl}(w(s) \otimes v(s)), \nabla\chi(s) \rangle \, ds$$
$$+ \int_0^T \langle \mathrm{curl}(v(s) \otimes w(s)), \nabla\chi(s) \rangle \, ds$$
$$+ \int_0^T \langle \omega(s), v(s) \cdot \nabla\chi(s) \rangle \, ds.$$

This means that ω is a distributional solution in $(0, T) \times \mathcal{O}$ of the partial differential equation

$$\partial_t \omega + v \cdot \nabla\omega = \Delta\omega - \mathrm{div}\bigg(\mathrm{curl}(w(s) \otimes w(s))$$
$$+ \mathrm{curl}(w(s) \otimes v(s)) + \mathrm{curl}(v(s) \otimes w(s)) \bigg).$$

Since dist$(B(x_0, r), \partial\mathcal{O}) > 0$, $0 < t_1 \leq t_2 < T$ we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_1 \leq t_2 < t_2 + 2\varepsilon < T$, dist$(B(x_0, r + 2\varepsilon), \partial\mathcal{O}) > 0$. Let us consider $\psi \in C_c^\infty((0, T) \times \mathcal{O})$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, r + \varepsilon)$ such that it is equal to one in $[t_1 - \varepsilon/2, t_2 + \varepsilon/2] \times B(x_0, r + \varepsilon/2)$. Let us denote by $\tilde{\omega} = \omega\psi \in L^2((0, T) \times \mathbb{R}^2)$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, r + \varepsilon)$, then $\tilde{\omega}$ is a distributional solution in $(0, T) \times \mathbb{R}^2$ of

$$\partial_t \tilde{\omega} = \Delta\tilde{\omega} - v \cdot \nabla\tilde{\omega} - w \cdot \nabla\tilde{\omega} + g \tag{4.7}$$

with

$$g = \partial_t\psi\omega - 2\nabla\psi \cdot \nabla\omega - \Delta\psi\omega + v \cdot \nabla\psi\omega$$
$$- \psi w \cdot \nabla\omega_w - \psi v \cdot \nabla\omega_w + w \cdot \nabla\psi\omega.$$

Due to Corollary 4.4 the terms

$$-\psi w \cdot \nabla\omega_w - \psi v \cdot \nabla\omega_w + w \cdot \nabla\psi\omega \in L^2((0, T) \times \mathbb{R}^2) \quad \mathbf{P} - a.s.$$

Therefore $g \in L^2(0, T; H^{-1}(\mathbb{R}^2)) + L^1(0, T; L^2(\mathbb{R}^2))$ $\mathbf{P} - a.s.$ Then, arguing as in the first step of the proof of [36, Theorem 13.2], the fact that $\tilde{\omega}$ is a distributional solution of (4.7) implies that $\tilde{\omega} \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2))$. Therefore

$$\omega \in C([t_1 - \varepsilon/4, t_2 + \varepsilon/4]; L^2(B(x_0, r + \varepsilon/4)))$$
$$\cap L^2(t_1 - \varepsilon/4, t_2 + \varepsilon/4; H^1(B(x_0, r + \varepsilon/4))) \quad \mathbf{P} - a.s.$$

Introducing $\phi \in C_c^\infty(B(x_0, r + \varepsilon/4))$ equal to one in $B(x_0, r + \varepsilon/8)$, since $\omega = \text{curl } v$, then ϕv satisfies

$$\Delta(\phi v) = \nabla^\perp\omega\phi + \Delta\phi v + 2\nabla\phi \cdot \nabla v, \quad (\phi v)|_{\partial B(x_0, r + \varepsilon/4)} = 0. \tag{4.8}$$

From the regularity of ω, by standard elliptic regularity theory (see for example [6]), it follows that $\phi v \in C([t_1 - \varepsilon/4, t_2 + \varepsilon/4]; H^1(B(x_0, r + \varepsilon/4); \mathbb{R}^2)) \cap L^2(t_1 - \varepsilon/4, t_2 + \varepsilon/4; H^2(B(x_0, r + \varepsilon/4); \mathbb{R}^2))$ $\mathbf{P} - a.s.$ Therefore, since $\phi \equiv 1$ on $B(x_0, r + \varepsilon/8)$

$$v \in C([t_1 - \frac{\varepsilon}{16}, t_2 + \frac{\varepsilon}{16}]; H^1(B(x_0, r + \frac{\varepsilon}{16}); \mathbb{R}^2))$$
$$\cap L^2(t_1 - \frac{\varepsilon}{16}, t_2 + \frac{\varepsilon}{16}; H^2(B(x_0, r + \frac{\varepsilon}{16}); \mathbb{R}^2)) \quad \mathbf{P} - a.s. \tag{4.9}$$

Let us now consider $\hat{\psi} \in C_c^\infty((t_1 - \frac{\varepsilon}{16}, t_2 + \frac{\varepsilon}{16}) \times B(x_0, r + \frac{\varepsilon}{16}))$ such that it is equal to one in $[t_1 - \frac{\varepsilon}{32}, t_2 + \frac{\varepsilon}{32}] \times B(x_0, r + \frac{\varepsilon}{32})$. Let us denote by $\hat{\omega} = \omega\hat{\psi} \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2))$ supported in $(t_1 - \frac{\varepsilon}{16}, t_2 + \frac{\varepsilon}{16}) \times B(x_0, r + \frac{\varepsilon}{16})$, then $\hat{\omega}$ is a distributional solution in $(0, T) \times \mathbb{R}^2$ of

$$\partial_t\hat{\omega} = \Delta\hat{\omega} + \hat{g} \tag{4.10}$$

with

$$\hat{g} = -v \cdot \nabla\hat{\omega} - w \cdot \nabla\hat{\omega} + \partial_t\hat{\psi}\omega - 2\nabla\hat{\psi} \cdot \nabla\omega - \Delta\hat{\psi}\omega + v \cdot \nabla\hat{\psi}\omega$$
$$- \hat{\psi}w \cdot \nabla\omega_w - \hat{\psi}v \cdot \nabla\omega_w + w \cdot \nabla\hat{\psi}\omega.$$

By Corollary 4.4 and relation (4.9) it follows that $\hat{g} \in L^2(0, T; H^{-1/2}(\mathbb{R}^2))$ $\mathbf{P} - a.s.$ Therefore $\hat{\omega} \in C([0, T]; H^{1/2}(\mathbb{R}^2)) \cap L^2(0, T; H^{3/2}(\mathbb{R}^2))$ $\mathbf{P} - a.s.$ and arguing as above

$$v \in C([t_1 - \frac{\varepsilon}{64}, t_2 + \frac{\varepsilon}{64}], H^{3/2}(B(x_0, r + \frac{\varepsilon}{64}); \mathbb{R}^2))$$
$$\cap L^2(t_1 - \frac{\varepsilon}{64}, t_2 + \frac{\varepsilon}{64}, H^{5/2}(B(x_0, r + \frac{\varepsilon}{64}); \mathbb{R}^2)) \quad \mathbf{P} - a.s.$$

This concludes the proof of Lemma 4.5. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 4.6 *Let Hypothesis* 1.1 *be satisfied. Let v be the unique weak solution of* (1.4) *in the sense of Definition* 3.2, *where w is as in* 4.4. *Then, for all* $0 < t_1 \leq t_2 < T$, $x_0 \in \mathcal{O}, r > 0$ *such that* $\mathrm{dist}(B(x_0, r), \partial\mathcal{O}) > 0$,

$$v \in C([t_1, t_2]; C^\infty(B(x_0, r); \mathbb{R}^2)) \quad \mathbf{P} - a.s.$$

Proof Since $\mathrm{dist}(B(x_0, r), \partial\mathcal{O}) > 0$, $0 < t_1 \leq t_2 < T$ we can find ε small enough such that $0 < t_1 - 2\varepsilon < t_1 \leq t_2 < t_2 + 2\varepsilon < T$, $\mathrm{dist}(B(x_0, r + 2\varepsilon), \partial\mathcal{O}) > 0$ and $\psi \in C_c^\infty((0, T) \times \mathcal{O})$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, r + \varepsilon)$ such that it is equal to one in $[t_1 + \varepsilon/2, t_2 + \varepsilon/2] \times B(x_0, r + \varepsilon/2)$. From Lemma 4.5 and Sobolev embedding theorem we know that $v \in C([t_1 - \varepsilon, t_2 + \varepsilon]; L^\infty(B(x_0, r + \varepsilon); \mathbb{R}^2))$ $\mathbf{P} - a.s.$ Denoting by

$$\omega = \mathrm{curl}\, v \in C([0, T]; H^{-1}) \cap L^2((0, T) \times \mathcal{O}),$$
$$\omega_w = \mathrm{curl}\, w \in C([t_1 - 2\varepsilon, t_2 + \varepsilon], C^\infty(B(x_0, r + 2\varepsilon))) \quad \mathbf{P} - a.s.$$

and $\tilde{\omega} = \omega\psi \in L^2((0, T) \times \mathbb{R}^2)$ supported in $[t_1 - \varepsilon, t_2 + \varepsilon] \times B(x_0, r + \varepsilon)$, then, arguing as in the proof of Lemma 4.5, it follows that $\tilde{\omega}$ is a distributional solution in $(0, T) \times B(x_0, r + \varepsilon)$ of

$$\partial_t\tilde{\omega} = \Delta\tilde{\omega} + \tilde{g} \tag{4.11}$$

with

$$\tilde{g} = -v \cdot \nabla\tilde{\omega} - w \cdot \nabla\tilde{\omega} + \partial_t\psi\omega - 2\nabla\psi \cdot \nabla\omega - \Delta\psi\omega + v \cdot \nabla\psi\omega$$
$$- \psi w \cdot \nabla\omega_w - \psi v \cdot \nabla\omega_w + w \cdot \nabla\psi\omega.$$

From the regularity of ω, v, $\tilde{\omega}$, ω_w, w, then $\tilde{g} \in L^2(t_1 - \varepsilon, t_2 + \varepsilon; H^{-1}(B(x_0, r + \varepsilon)))$ $\mathbf{P} - a.s.$ By standard regularity theory for the heat equation, see for example Step 2 in [36, Theorem 13.1], a solution of (4.11) with $\tilde{g} \in L^2(t_1 - \varepsilon, t_2 + \varepsilon; H^{k-1}(B(x_0, r +$

$\varepsilon)))$, $k \in \mathbb{N}$, belongs to $C([t_1 - \varepsilon/2, t_2 + \varepsilon/2]; H^k(B(x_0, r + \varepsilon/2))) \cap L^2(t_1 - \varepsilon/2, t_2 + \varepsilon/2; H^{k+1}(B(x_0, r + \varepsilon/2)))$. Therefore

$$\tilde{\omega} \in C([t_1 - \varepsilon/2, t_2 + \varepsilon/2]; L^2(B(x_0, r + \varepsilon/2)))$$
$$\cap L^2(t_1 - \varepsilon/2, t_2 + \varepsilon/2; H^1(B(x_0, r + \varepsilon/2))) \quad \mathbf{P} - a.s.$$

which implies

$$\omega \in C([t_1 - \varepsilon/4, t_2 + \varepsilon/4]; L^2(B(x_0, r + \varepsilon/4)))$$
$$\cap L^2(t_1 - \varepsilon/4, t_2 + \varepsilon/4; H^1(B(x_0, r + \varepsilon/4))) \quad \mathbf{P} - a.s.$$

since $\psi \equiv 1$ on $(t_1 - \varepsilon/2, t_2 + \varepsilon/2) \times B(x_0, r + \varepsilon/2)$. Considering now $\phi \in C_c^\infty(\mathcal{O})$ supported on $B(x_0, r + \varepsilon/4)$ such that $\phi \equiv 1$ on $B(x_0, r + \varepsilon/8)$, since $\text{curl } v = \omega$ then ϕv satisfies

$$\Delta(\phi v) = \nabla^\perp \omega \phi + \Delta \phi v + 2 \nabla \phi \cdot \nabla v, \quad (\phi v)|_{\partial B(x_0, r + \varepsilon/4)} = 0. \tag{4.12}$$

Since

$$\nabla^\perp \omega \phi + \Delta \phi v + 2 \nabla \phi \cdot \nabla v \in C([t_1 - \varepsilon/4, t_2 + \varepsilon/4]; H^{-1}(B(x_0, r + \varepsilon/4)))$$
$$\cap L^2(t_1 - \varepsilon/4, t_2 + \varepsilon/4; L^2(B(x_0, r + \varepsilon/4))) \quad \mathbf{P} - a.s.,$$

by standard elliptic regularity theory (see for example [6]),

$$\phi v \in C([t_1 - \varepsilon/4, t_2 + \varepsilon/4]; H^1(B(x_0, r + \varepsilon/4)))$$
$$\cap L^2(t_1 - \varepsilon/4, t_2 + \varepsilon/4; H^2(B(x_0, r + \varepsilon/4))) \quad \mathbf{P} - a.s.$$

Since $\phi \equiv 1$ on $B(x_0, r + \varepsilon/8)$ then

$$v \in C([t_1 - \frac{\varepsilon}{16}, t_2 + \frac{\varepsilon}{16}]; H^1(B(x_0, r + \frac{\varepsilon}{16})))$$
$$\cap L^2(t_1 - \frac{\varepsilon}{16}, t_2 + \frac{\varepsilon}{16}; H^2(B(x_0, r + \frac{\varepsilon}{16}))) \quad \mathbf{P} - a.s.$$

Reiterating the argument as in Step 3 in [36, Theorem 13.1] the thesis follows. $\qquad\square$

Proof of Theorem 1.7 The claim follows by Corollaries 4.4 and 4.6 and a localization argument. Moreover, to obtain the claimed smoothness up to time $t = T$, let us consider the extension by 0 of h_b on $[0, T + 1]$, i.e.

$$\tilde{h}_b(t) = \begin{cases} h_b(t), & \text{if } t \in (0, T), \\ 0, & \text{if } t \in (T, T + 1). \end{cases}$$

Let $\tilde{u}$ be the unique weak solution (1.2) provided by Theorem 1.4 with T replaced by $T + 1$. Then, by Corollary 4.4, Corollary 4.6 and a standard covering argument, for

all $t_0 \in (0, T)$, $\mathcal{O}_0 \subset \mathcal{O}$ such that $\mathrm{dist}(\mathcal{O}_0, \partial\mathcal{O}) > 0$,

$$\widetilde{u} \in C([t_0, T]; C^\infty(\mathcal{O}_0; \mathbb{R}^2)) \quad \mathbf{P} - a.s. \tag{4.13}$$

Now, let u be the unique weak solution of (1.2) provided by Theorem 1.4. By uniqueness, we have $u = \widetilde{u}|_{[0,T]}$ and the conclusion follows from (4.13).

Data availability Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Appendix A: H^∞-calculus for the Stokes operator

In this appendix we prove Theorem 2.1. Here we use the transference result proven in [42]. In this section we also need the concept of $\mathcal{R}$-sectoriality, which can be again found in [43, Chapters 3 and 4] and [32, Chapter 10].

To discuss the main idea in the proof of Theorem 2.1, let us define the operator $B_q v := -\Delta v$ on $L^q(\mathcal{O}; \mathbb{R}^2)$ with domain

$$\mathsf{D}(B_q) := \big\{ f = (f_1, f_2) \in W^{2,q}(\mathcal{O}; \mathbb{R}^2) : f|_{\Gamma_b} = 0,$$
$$f_2|_{\Gamma_u} = \partial_z f_1|_{\Gamma_u} = 0 \big\}.$$

Note that B_q has the same boundary conditions of A_q. However, B_q is considerably more simple than A_q since, to study its spectral properties, it is possible to use reflection arguments which are not available for the Stokes operator, cf. Appendix 1.

We prove Theorem 2.1 by using the transference techniques developed in [42]. By [42, Theorem 9], we divide the proof into the following steps:

(1) Boundedness of the H^∞-calculus for A_2 and B_2 (i.e. in the Hilbertian case).
(2) Boundedness of the H^∞-calculus for B_q for all $q \in (1, \infty)$.
(3) $\mathcal{R}$-sectoriality of A_q.
(4) Conclusion via transference and interpolation [42, Theorems 5 and 9].

A.1 The Hilbertian case

Here we analyse the L^2-case of Theorem 2.1, i.e. the operators $A := A_2$ and $B := B_2$ acting on $\mathbb{L}^2$ and $l^2(\mathcal{O}; \mathbb{R}^2)$, respectively.

Proposition A.1 *A and B are positive self-adjoint operators, and*

$$\mathsf{D}(A^\gamma) = \mathsf{D}((A^*)^\gamma) = \mathbb{H}^{2\gamma}(\mathcal{O}), \quad \mathsf{D}(B^\gamma) = \mathsf{D}((B^*)^\gamma) = H^{2\gamma}(\mathcal{O}; \mathbb{R}^2) \quad \text{(A.1)}$$

for all $\gamma \in (0, \frac{1}{4})$.

The above result and [32, Proposition 10.2.23] imply that A and B have bounded H^∞-calculus of angle 0. Below we mainly focus on the operator A. The argument to treat B is analogous and simpler.

Consider the elliptic problem associated to A, i.e.

$$\begin{cases} -\Delta u + \nabla \pi = f, & \text{on } \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } \mathcal{O}, \\ u = 0, & \text{on } \Gamma_b, \\ \partial_z u_1 = 0, & \text{on } \Gamma_u, \\ u_2 = 0, & \text{on } \Gamma_u. \end{cases} \quad \text{(A.2)}$$

If $f \in V^*$, the definition of weak solutions $u \in V$ is standard and similar to the one of (2.1). The well-posedness of (A.2) is analysed below.

Proposition A.2 *For each $f \in V^*$ there exists a unique solution of problem (A.2). Moreover if $f \in H$ then $u \in \mathsf{D}(A)$ and*

$$\|u\|_{\mathsf{D}(A)} + \|\pi\|_{H^1/\mathbb{R}} \leq C\|f\|.$$

Proof Existence of weak solutions follows immediately by Lax–Milgram Lemma, since Poincaré inequality holds in V. Therefore we can endow V with the norm $\|\nabla u\|_V := \|\nabla u\|_{L^2}$ equivalent to the standard H^1 norm. Let now $f \in H$, therefore

the weak formulation satisfied by u reduces to

$$\langle \nabla u, \nabla \phi \rangle = \langle f, \phi \rangle \quad \forall \phi \in V. \tag{A.3}$$

Considering $\phi \in \mathcal{D}(\mathcal{O})$ it follows that

$$\langle \Delta u + f, \phi \rangle_{\mathcal{D}(\mathcal{O})', \mathcal{D}(\mathcal{O})} = 0,$$

therefore by [51, Proposition 1.1, Proposition 1.2] it exists $\pi \in L^2(D)$ such that

$$-\Delta u + \nabla \pi = f \tag{A.4}$$

in the sense of distributions and

$$\|\pi\|_{L^2/\mathbb{R}} \leq C \|\Delta u\|_{H^{-1}} + \|f\| \lesssim \|f\|.$$

The higher regularity follows by the standard Niremberg's method of finite difference quotients. Therefore, fix $h > 0$, extending periodically either u and f in the x direction and consider $\phi = \tau_h \tau_{-h} u$ as a test function in (A.3), where $\tau_h v = \frac{v(x+h, z) - v(x, z)}{h}$. Then by change of variables it follows that

$$\|\tau_{-h} \nabla u\| \leq C \|f\|.$$

The relation above implies that

$$\|\partial_x \nabla u\| \leq C \|f\|.$$

Let us now consider $\phi = \partial_x \psi$, $\psi \in \mathcal{D}(\mathcal{O})$ as test function in (A.3). Therefore arguing as above it follows that $\partial_x \pi \in L^2(\mathcal{O})$ and $\|\partial_x \pi\|_{L^2} \lesssim \|f\|$. Since Eq. (A.4) is satisfied in the sense of distribution and u is divergence free it follows that

$$\partial_z \pi = f_2 + \partial_{xx} u_2 - \partial_{xz} u_1 \in L^2(\mathcal{O}).$$

This implies that $\|\nabla \pi\|_{L^2} \lesssim \|f\|$. Lastly u_1 satisfies

$$-\partial_{zz} u_1 = -\partial_x \pi + \partial_{xx} u_1 + f_1 \in L^2(\mathcal{O}),$$

which completes the proof. $\qquad \square$

We are ready to prove Proposition A.1.

Proof of Proposition A.1 Step 1: A and B are a positive self-adjoint operators. As above we only discuss the operator A. The positivity of A is clear. Next, note that, integrating by parts

$$\langle Au, v \rangle = \langle u, Av \rangle \quad \forall u, v \in D(A).$$

This means that A is symmetric. It remains to show that $\mathsf{D}(A^*) = \mathsf{D}(A)$ and $\forall u \in \mathsf{D}(A)$, $A^*u = Au$. By definition

$$\mathsf{D}(A^*) = \{u \in H : \quad F : \mathsf{D}(A) \subseteq H \to \mathbb{R}, \quad F(v) = \langle u, Av \rangle$$
$$\text{has a linear bounded extension on } H\}.$$

For each $u \in \mathsf{D}(A^*)$, $F(v) = \langle u, Av \rangle = \langle f_u, v \rangle$ therefore $A^*u = f_u$. In particular, $\forall u \in \mathsf{D}(A^*)$ $\langle u, Av \rangle = \langle A^*u, v \rangle$. Thanks to the fact that A is symmetric we have $\mathsf{D}(A) \subseteq \mathsf{D}(A^*)$. Given now $v \in \mathsf{D}(A^*)$, $f_v = A^*v \in H$, let us consider the boundary value problem (A.2) with forcing term equal to f_v. By Proposition A.2 it has a unique solution $(w, \pi) \in \mathsf{D}(A) \times H^1(\mathcal{O})$, this implies that $Aw = f_v = A^*v$. For each $z \in H$, let us consider the boundary value problem (A.2) with forcing term equal to z. By Proposition A.2 it exists a unique $S_z \in \mathsf{D}(A)$ such that $As_z = z$. Therefore $\langle z, w - v \rangle = \langle As_z, w - v \rangle = \langle s_z, Aw - A^*v \rangle = 0$ thanks to the fact that A is symmetric. Since z is arbitrary, then $v = w$ and the claim follows.

Step 2: Proof of (A.1). We begin by proving the first identity in (A.1). Note that $\mathsf{D}(B^\gamma) = \mathsf{D}((B^*)^\gamma)$ for $\gamma < 1/2$ follows from [34, Theorem 1.1] and Step 1. By Step 1 and [32, Proposition 10.2.23], B has bounded H^∞-calculus and in particular B has the bounded imaginary powers property, [43, Subsection 3.4]. By [43, Theorem 3.3.7], $\mathsf{D}(B^\gamma) = [L^2(\mathcal{O}; \mathbb{R}^2), \mathsf{D}(B)]_\gamma$ for all $\gamma < 1$. The latter gives $\mathsf{D}(B^\gamma) = H^{2\gamma}(\mathcal{O}; \mathbb{R}^2)$ in case $\gamma < 1/4$ by [44]. The second identity in (A.1) follows analogously, where one uses the argument in [28] (see also [48, Proposition 5.5, Chapter 17]) to deduce $\mathsf{D}(A^\gamma) = \mathbb{H}^{2\gamma}(\mathcal{O})$ from the first identity in (A.1).

A.2 Bounded H^∞-calculus for Laplace operators

In this subsection we prove the boundedness of the H^∞-calculus for B_q. The basic idea is to use the product structure of the domain $\mathcal{O}$ and to write $B_q u = (L_{q,\mathrm{R}} u_1, L_{q,\mathrm{D}} u_2)$ where

$$\mathsf{D}(L_{q,\mathrm{D}}) := \{f \in W^{2,q}(\mathcal{O}) : f|_{\Gamma_b} = f|_{\Gamma_u} = 0\}, \qquad L_{q,\mathrm{D}} f := \Delta f,$$
$$\mathsf{D}(L_{q,\mathrm{R}}) := \{f \in W^{2,q}(\mathcal{O}) : \partial_z f|_{\Gamma_b} = 0, \ f|_{\Gamma_u} = 0\}, \qquad L_{q,\mathrm{R}} f := \Delta f.$$

Proposition A.3 (Bounded H^∞-calculus for Laplace operators) *Let $q \in (1, \infty)$ and let $\mathcal{O}$ be as above. Then $-L_{q,\mathrm{D}}$ and $-L_{q,\mathrm{R}}$ have a bounded H^∞-calculus of angle 0. In particular B_q has a bounded H^∞-calculus of angle 0.*

The above statement also holds for the Neumann Laplacian, but it will not be needed below.

Proof We divide the proof into three steps. In the first step, we exploit the product structure of our domain to reduce the problem to a one dimensional situation.

Step 1: Reduction to the 1d case. Then the Dirichlet and the Robin Laplacian in 1d are given by

$$\mathsf{D}(\ell_{q,\mathrm{D}}) := \{f \in W^{2,q}(0, a) : f(0) = f(a) = 0\}, \qquad \ell_{q,\mathrm{R}} f := \partial_x^2 f,$$

$$\mathsf{D}(\ell_{q,\mathrm{R}}) := \left\{ f \in W^{2,q}(0,a) \; : \; \partial_x f(0) = f(a) = 0 \right\}, \qquad \ell_{q,\mathrm{R}} f := \partial_x^2 f.$$

Let us consider $\ell_{q,\mathrm{D}}$, the other case is analogue. In this step we assume that $-\ell_{q,\mathrm{D}}$ has a bounded H^∞-calculus of angle 0. Let $\ell_{q,\mathrm{P}}$ be the Laplacian on the periodic torus $\mathbb{T}$ with domain $W^{2,q}(\mathbb{T})$. The boundedness of the H^∞-calculus for $-\ell_{q,\mathrm{P}}$ such operator follows from the periodic version of [32, Theorem 10.2.25] and $\omega_{H^\infty}(\ell_{q,\mathrm{P}}) = 0$.

On $L^2(\mathcal{O})$ considers the operator

$$(\ell_{q,\mathrm{D}}^{(x)} f)(x, z) = (\ell_{q,\mathrm{D}} f(\cdot, z))(x), \qquad (\ell_{q,\mathrm{P}}^{(y)} f)(x, z) = (\ell_{q,\mathrm{D}} f(x, \cdot))(z),$$

with the corresponding natural domains. It is clear that both $-\ell_{q,\mathrm{D}}^{(x)}$ and $-\ell_{q,\mathrm{P}}^{(z)}$ have bounded H^∞-calculus of angle 0. Now by sum of commuting operators [43, Corollary 4.5.8], the sum operator $-A_q := -\ell_{q,\mathrm{D}}^{(x)} - \ell_{q,\mathrm{P}}^{(z)}$ has a bounded H^∞-calculus of angle 0 with domain

$$\mathsf{D}(A_q) = \mathsf{D}(\ell_{q,\mathrm{D}}^{(x)}) \cap \mathsf{D}(\ell_{q,\mathrm{P}}^{(z)}) = \mathsf{D}(L_{q,\mathrm{D}})$$

where the last equality follows from elliptic regularity.

Step 2: $-L_{q,\mathrm{D}}$ has a bounded H^∞-calculus of angle 0. By rescaling and translation we may replace $(0, a)$ by $(-\pi, \pi)$. Let $\ell_{q,\mathrm{P}}$ be the Laplacian on the periodic torus $\mathbb{T} = (-\pi, \pi)$ (as measure space) with domain $W^{2,q}(\mathbb{T})$. Let

$$Y := \Big\{ f \in L^2(\mathbb{T}) : \; f = \sum_{n \geq 0} f_n \sin(nx) \text{ where } (f_n)_{n \geq 1} \in \ell^2 \Big\}.$$

It is clear that $Y \subseteq L^2(\mathbb{T})$ is closed, and

$$(\lambda - \ell_{q,\mathrm{P}})^{-1} : Y \to Y \quad \text{for all} \quad \lambda \in \rho(L_{q,\mathrm{D}}).$$

Now note that $L_{q,\mathrm{D}}$ is the part of $\ell_{q,\mathrm{P}}$ on Y, i.e.

$$\mathsf{D}(L_{q,\mathrm{D}}) = \{ f \in \mathsf{D}(\ell_{q,\mathrm{P}}) \cap Y : \ell_{q,\mathrm{P}} f \in Y \},$$
$$L_{q,\mathrm{D}} f = \ell_{q,\mathrm{P}} f \quad \text{for all } f \in \mathsf{D}(L_{q,\mathrm{D}}).$$

Now the claim of Step 1 follows from [32, Proposition 10.2.18] and the periodic version of [32, Theorem 10.2.25].

Step 2: $-L_{q,\mathrm{R}}$ has a bounded H^∞-calculus of angle 0. As in the above step, by rescaling we replace $(0, a)$ by $(0, \pi)$. Consider the reflection map

$$Rf(z) := \begin{cases} f(z) & z \in (0, \pi), \\ f(-z) & z \in (-\pi, 0). \end{cases}$$

Let $L_{q,\mathrm{D}}$ be the Dirichlet Laplacian on $(-\pi, \pi)$. Then one can readily check that $\rho(L_{q,\mathrm{D}}) \subseteq \rho(L_{q,\mathrm{R}})$ and for all $\lambda \in \rho(L_{q,\mathrm{D}})$

$$(\lambda - L_{q,\mathrm{R}})^{-1} f = \left[(\lambda - L_{q,\mathrm{D}})^{-1} R f\right]\Big|_{(0,\pi)}.$$

Now the claim follows from Step 1 and the definition of H^∞-calculus. □

A.3 $\mathcal{R}$-sectoriality for the Stokes operator

For the notion of $\mathcal{R}$-boundedness of a family of linear operators we refer to [32, Chapter 8]. For a family of linear operators $\mathscr{J}$, the $\mathcal{R}$-bound is denoted by $\mathcal{R}(\mathscr{J})$. As in [32, Chapter 10], we said that operator T on a Banach space X is said to be $\mathcal{R}$-sectorial if there exists $\phi \in (0, \pi)$ such that $\rho(A) \subseteq \{\lambda \in \mathbb{C} \,|\, |\arg \lambda| \geq \pi - \phi\}$ and

$$\mathcal{R}(\lambda(\lambda - T)^{-1} \,|\, |\arg \lambda| > \pi - \phi) < \infty.$$

The $\mathcal{R}$-sectoriality angle is the infimum over all $\phi \in (0, \pi)$ for which the above holds. The main result of this subsection reads as follows.

Proposition A.4 *For all $q \in (1, \infty)$, the operator A_q is $\mathcal{R}$-sectorial with $\mathcal{R}$-sectoriality angle $< \pi/2$.*

Proof Fix $q \in (1, \infty)$. For simplicity we first prove the statement for a shifted Stokes operator and in a second step we conclude by a simple translation argument.

Step 1: There exists λ_q such that $\lambda_q + A_q$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-sectoriality angle $< \pi/2$. Due to the the well-known equivalence of maximal L^q-regularity and $\mathcal{R}$-sectoriality proven by L. Weis [54] (see also [43, Subsection 4.2, Chapter 3]), it is enough to show that, for all $f \in L^q(0, 1; \mathbb{L}^q)$, the Stokes problem on $\mathcal{O}$,

$$\begin{cases} \partial_t u = \Delta u - \nabla P + f, & \text{on } (0, 1) \times \mathcal{O}, \\ \operatorname{div} u = 0, & \text{on } (0, 1) \times \mathcal{O}, \\ \quad u = 0, & \text{on } (0, 1) \times \Gamma_b, \\ u_2 = \partial_z u_1 = 0, & \text{on } (0, 1) \times \Gamma_u, \\ u(0) = 0, & \text{on } \mathcal{O}, \end{cases} \tag{A.5}$$

admits a unique solution in the class

$$u \in W^{1,q}(0, 1; \mathbb{L}^q) \cap L^q(0, 1; \mathbb{W}^{2,q}(\mathcal{O})), \quad P \in L^q(0, 1; W^{1,q}(\mathcal{O})). \tag{A.6}$$

The proof follows a standard localization argument. Let $(\phi_j)_{j=1}^N$ be a smooth partition of the unity such that, for all $j \in \{1, \ldots, N\}$, $\operatorname{diam}(\operatorname{supp} \phi_k) < \frac{1}{2}$

$$\text{either} \quad \operatorname{supp} \phi_j \cap (\mathbb{T} \times \{0\}) = \varnothing \quad \text{or} \quad \operatorname{supp} \phi_j \cap (\mathbb{T} \times \{a\}) = \varnothing.$$

Fix $k \in \{1, \ldots, N\}$. Multiplying (A.5) by ϕ_k, we obtain for $u_k := \phi_k u$ and $P_k = \phi_k P$ either

$$
\begin{cases}
\partial_t u_k = \Delta u_k - \nabla P_k + \phi_k f + \mathcal{L}_k(u, P), & \text{on } (0, 1) \times \mathbb{R} \times (0, \infty), \\
\operatorname{div} u_k = \nabla \phi_k \cdot u, & \text{on } (0, 1) \times \mathbb{R} \times (0, \infty), \\
\quad u_k = 0, & \text{on } (0, 1) \times \mathbb{R} \times \{0\}, \\
u_k(0) = 0, & \text{on } \mathbb{R} \times (0, \infty),
\end{cases}
\tag{A.7}
$$

or

$$
\begin{cases}
\quad \partial_t u_k = \Delta u_k - \nabla P_k + \phi_k f + \mathcal{L}_k(u, P), & \text{on } (0, 1) \times \mathbb{R} \times (-\infty, a), \\
\quad \operatorname{div} u_k = \nabla \phi_k \cdot u, & \text{on } (0, 1) \times \mathbb{R} \times (-\infty, a), \\
\quad \phi_k u_2 = 0, & \text{on } (0, 1) \times \mathbb{R} \times \{a\}, \\
\partial_z(\phi_k u_1) = u_1 \partial_z \phi_k, & \text{on } (0, 1) \times \mathbb{R} \times \{a\}, \\
\quad u_k(0) = 0, & \text{on } \mathbb{R} \times (-\infty, a).
\end{cases}
\tag{A.8}
$$

Here $\mathcal{L}_k$ denotes a lower order operator w.r.t. to the maximal regularity space for (u, P) in (A.6).

Maximal $L^p(L^q)$-regularity estimates for (A.7) and (A.8) are proven in [43, Theorem 7.2.1] in the case of no-slip or pure-slip, respectively (see conditions (7.16) and (7.17) on [43, p. 323]). Now a-priori estimates for solutions as in (A.6) in the maximal $L^p(L^q)$-regularity class $W^{1,q}(0, 1; \mathbb{L}^q) \cap L^q(0, 1; \mathbb{W}^{2,q}(\mathcal{O}))$ follows by repeating the localization argument of [43, Subsection 3.4 in Chapter 7] to adsorb the lower order terms.

It remains to discuss the existence of solutions as in (A.6). Arguing as in step 3 of Theorem 2.2 and using L^2-theory, one can prove existence of smooth solutions to equation (A.5) in case of smooth data f. Hence the existence follows from a standard density argument and the a-priori estimates obtained above for solutions in the class $W^{1,q}(0, 1; \mathbb{L}^q) \cap L^q(0, 1; \mathbb{W}^{2,q}(\mathcal{O}))$.

Step 2: Conclusion. By Step 1, it remains to remove the shift λ_q. Arguing as in [1, Proposition 2.2], by holomorphicity of the resolvent and [43, Proposition 4.1.12], it is enough to show that

$$
\rho(A_q) \subseteq \{\lambda \in \mathbb{C} \mid |\arg z| > \psi\}, \quad \text{for some } \psi < \pi/2.
$$

In the case $q > 2$, noticing that $(\lambda - A_q) = (\lambda - \lambda_q - A_q) + \lambda_q$ and that $\rho(\lambda_q + A_q) \subseteq \{\lambda \in \mathbb{C} \mid |\arg z| > \phi\}$ for some $\phi < \pi/2$ by Step 1, the conclusion can be obtained by using a standard bootstrap method via Sobolev embeddings.

In the case $q < 2$ one uses $(A_q)^* = A_{q'}$. $\qquad\square$

A.4 Proof of Theorem 2.1

As the proof of Theorem 2.1 follows the one of [42, Theorem 16], we only provide a sketch.

Proof of Theorem 2.1—Sketch *Step 1: There exists* $\beta > 0$ *for which the following estimates hold*:

$$
\begin{aligned}
\sup_{1 \le t, s \le 2} \mathcal{R}(\varphi(2^{j+\ell} s A_2) \psi(t 2^j B_2)) &\lesssim 2^{-\beta \ell}, \\
\sup_{1 \le t, s \le 2} \mathcal{R}(\varphi(2^{j+\ell} s A_2)^* \mathbb{P}_p^* \psi(t 2^j B_2)^*) &\lesssim 2^{-\beta \ell}.
\end{aligned}
\tag{A.9}
$$

Recall that $\mathcal{R}(\mathscr{J})$ stand for the $\mathcal{R}$-bound of the family of operators $\mathscr{J}$, see [32, Chapter 9] for details on $\mathcal{R}$-boundedness.

By elliptic regularity we have $\mathbb{P} : H^1(\mathcal{O}; \mathbb{R}^2) \to \mathbb{H}^1(\mathcal{O})$. Interpolating we obtain $\mathbb{P} : H^s(\mathcal{O}; \mathbb{R}^2) \to \mathbb{H}^s(\mathcal{O}; \mathbb{R}^2)$ for all $s \in (0, 1)$. Hence $\mathbb{P} : \mathsf{D}(B^\gamma) \to \mathsf{D}(A^\gamma)$ for all $\gamma \in (0, 1/4)$. The estimate (A.9) now follows from [42, Proposition 10] and (A.1).

Step 2: Boundedness of the H^∞-calculus. Next we argue as in the proof of [42, Theorem 5]. Let $q \in (1, \infty)$ be as in the statement of Theorem 2.1 and fix $p \in (q, \infty)$. By $\mathcal{R}$-sectoriality of A_p and B_p (i.e. Proposition A.4) and [32, Proposition 10.3.2],

$$
\mathcal{R}(\varphi(t A_p) : t > 0) \le c_0 \quad \text{and} \quad \mathcal{R}(\psi(s B_p) : s > 0) \le c_0.
\tag{A.10}
$$

Note that $(A_r)_{r \in (1, \infty)}, (B_r)_{r \in (1, \infty)}$ are consistent family of operators. Hence, by complex interpolation and [32, Proposition 8.4.4], we have that (A.9) holds for some $\beta = \beta(r, p) > 0$ and with (A_2, B_2) replaced by (A_q, B_q). Now the boundedness of the H^∞-calculus follows from [42, Theorem 9].

Step 3: Description of the fractional powers. To obtain the description of the fractional powers of A_q and B_q one can argue as in the proof of (A.1) by using the bounded imaginary power property and [28, 44]. $\square$

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