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Exponential mixing of all orders on Kähler manifolds: (quasi-)plurisubharmonic observables

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Abstract

Let f be a holomorphic automorphism of a compact Kähler manifold with simple action on cohomology and μ its unique measure of maximal entropy. We prove that μ is exponentially mixing of all orders for all d.s.h. observables, i.e., functions that are locally differences of plurisubharmonic functions. As a consequence, every d.s.h. observable satisfies the central limit theorem with respect to μ.

Keywords Kähler manifolds · Equilibrium measure · Exponential mixing · Central limit theorem · Super-potentials

Mathematics Subject Classification 37F80 · 32H50 · 32U05 · 60F05

1 Introduction

Let (X, ω) be a compact Kähler manifold of dimension k and f a holomorphic automorphism of X. We refer to [8, 10, 16, 19] for the general properties of such maps. In particular, f admits a unique invariant probability measure μ of maximal entropy, called the *equilibrium measure* of f. A natural question is then to study the statistical properties of μ. From [16, 19] we know that μ is mixing. In general, the control of the speed of mixing is a challenging problem. The major difficulties in this setting are the presence of both attractive and repelling directions and the non uniform hyperbolicity of the system. The main goal of this work is to prove that the measure μ, under suitable assumptions on the automorphism f, is exponentially mixing of all orders with respect to a large class of observables which are naturally adapted to holomorphic dynamics. As a consequence, we obtain that the central limit theorem, another largely studied statistical property of dynamical systems, is satisfied for this large class of observables.

The simplest holomorphic dynamical systems displaying both the difficulties above are given by complex Hénon maps, see, e.g., [1, 2, 23]. In this case, the exponential mixing

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for two Hölder-continuous observables was first established by Dinh in [14]. It was recently extended by Bianchi and Dinh in [5] to any number of observables, and by the authors in [26, 30] to all plurisubharmonic (p.s.h.) observables. We also refer to [9, 31] for the case of generic birational maps of $\mathbb{P}^k$ and to [4, 20] for the case of holomorphic endomorphisms of $\mathbb{P}^k$.

On a compact Kähler manifold, p.s.h. functions are constant. So we consider in this paper *quasi-p.s.h.* observables, i.e., functions which are locally differences of a p.s.h. and a smooth function, and *d.s.h.* observables, i.e., functions which are differences of two quasi-p.s.h. functions, see also [17] and Sect. 2.1. In order to prove the mixing, we will need the following quantitative approximation for bounded quasi-p.s.h. functions, which is our first main result. See Definition 2.3 for the definition of the norm $\|\cdot\|_{\mathrm{qpsh}}$.

Theorem 1.1 *Let (X, ω) be a compact Kähler manifold and φ a bounded quasi-p.s.h. function. There exist constants c depending only on (X, ω) and c_m depending on (X, ω) and $m \geq 1$ such that, for every $0 < \varepsilon \leq 1/2$, there exists a smooth function φ_ε with $\varphi_\varepsilon \geq \varphi$ and such that*:

(i) $\|\varphi_\varepsilon\|_{\mathrm{qpsh}} \leq c\|\varphi\|_{\mathrm{qpsh}}$;
(ii) $\|\varphi_\varepsilon - \varphi\|_{L^1(\omega^k)} \leq -c\|\varphi\|_{\mathrm{qpsh}}/\log\varepsilon$;
(iii) $\|\varphi_\varepsilon\|_{\mathscr{C}^m} \leq c_m\|\varphi\|_{\mathrm{qpsh}}\varepsilon^{-m}(-\log\varepsilon)^{m-1}$ *for every integer $m \geq 1$.*

This result is a quantified version of a particular case of the results obtained in [7]. See also [21, Theorem 2.1], of which we will follow the proof, for the case of more regular functions. Observe that in Theorem 1.1 φ may even not be continuous, and so a uniform approximation is not possible. We have instead an estimate in $L^1(\omega^k)$, see also Lemma 2.2, and Proposition 3.3 for the application to the dynamic.

We now move to the dynamical results. We denote by f^n the n-th iterate of f. For $0 \leq q \leq k$, the *dynamical degree of order q* of f is the spectral radius of the pull-back operator f^* acting on the Hodge cohomology group $H^{q,q}(X, \mathbb{R})$. It is denoted by $d_q(f)$, or simply by d_q if there is no confusion. By Poincaré duality, the dynamical degree d_q of f is equal to the dynamical degree $d_{k-q}(f^{-1})$ of f^{-1}. We have $d_0 = d_k = 1$ and $d_q(f^n) = d_q^n$ for all q.

A theorem by Khovanskii [27], Teissier [28], and Gromov [24] implies that the sequence $q \mapsto \log d_q$ is concave. So, there are integers $0 \leq p \leq p' \leq k$ such that

$$1 = d_0 < \cdots < d_p = \cdots = d_{p'} > \cdots > d_k = 1.$$

We assume that f has *simple action on cohomology*, i.e., that we have $p = p'$ and f^*, acting on $H^{p,p}(X, \mathbb{R})$, admits only one eigenvalue of maximal modulus d_p. We call d_p the *main dynamical degree* of f.

The following is our second main result, which settles the problem of mixing for d.s.h. observables on compact Kähler manifolds. See Sect. 3.1 for the precise choice of δ'.

Theorem 1.2 *Let f be a holomorphic automorphism of a compact Kähler manifold (X, ω) of dimension k. Assume that f has simple action on cohomology, let μ be its equilibrium measure and d_p be its main dynamical degree. Then, μ is exponentially mixing of all orders for all observables in* $\mathrm{DSH}(X)$. *More precisely, there exists $0 < \delta' < d_p$ such that for every $\delta' < \delta < d_p$, every integers $\kappa \in \mathbb{N}^*$, $0 = n_0 \leq n_1 \leq \cdots \leq n_\kappa$ and every $\varphi_0, \varphi_1, \ldots, \varphi_\kappa \in$* $\mathrm{DSH}(X)$*, we have*

$$\left|\int \varphi_0(\varphi_1 \circ f^{n_1}) \cdots (\varphi_\kappa \circ f^{n_\kappa})\, \mathrm{d}\mu - \prod_{j=0}^{\kappa} \int \varphi_j\, \mathrm{d}\mu\right| \leq C_{\delta,\kappa} \left(\frac{\delta}{d_p}\right)^{\min_{0\leq j\leq\kappa-1}(n_{j+1}-n_j)/2} \prod_{j=0}^{\kappa} \|\varphi_j\|_{\mathrm{DSH}},$$

where $C_{\delta,\kappa} > 0$ *is a constant independent of* $n_1, \ldots, n_\kappa, \varphi_0, \ldots, \varphi_\kappa$.

We refer to [3, 13] for the more regular case of $\mathscr{C}^2$-continuous observables and to [25] for the case $\kappa = 1$. Observe that all d.s.h. functions are in $L^s(\mu)$ for every $s \geq 1$ [20], hence all the integrals above are well defined.

Our proof in [30] for the case of Hénon maps relies on precise estimates for p.s.h. functions and on the homogeneous structure of $\mathbb{P}^2$. As non-trivial p.s.h. functions do not exist on compact Kähler manifolds, both these ingredients are not available now. Instead, we will make a crucial use of the theory of *super-potentials* by Dinh and Sibony [18, 19], which permits to quantify the regularity of currents of arbitrary bidegree when seen as operators on appropriate spaces of forms.

Moreover, as in [30] we will have to regularize d.s.h. functions. On a Kähler manifold, this yields an additional factor that needs to be carefully estimated, see Proposition 3.6 and Lemma 3.7. This difficulty is new both with respect to [3] and [30].

A consequence of mixing is that all d.s.h. observables satisfy the central limit theorem. More precisely, fix an observable $\varphi \in \mathrm{DSH}(X)$ and set $S_n(\varphi) := \varphi + \varphi \circ f + \cdots + \varphi \circ f^{n-1}$. By Birkhoff's ergodic theorem, we have $n^{-1} S_n(\varphi)(x) \to \langle \mu, \varphi \rangle$ for μ-almost every $x \in X$. As in [30], the following control of the rate of the convergence is a consequence of Theorem 1.2 and [30, Theorem 4.1], which is an adapted version of the criterion in [6]. We let $\mathcal{N}(0, \sigma^2)$ denote the Gaussian distribution with mean 0 and variance σ^2 (when $\sigma = 0$, we mean that $\mathcal{N}(0, \sigma^2)$ is the trivial point distribution at 0).

Corollary 1.3 *Let* X, f *and* μ *be as in Theorem* 1.2. *Then, every* $\varphi \in \mathrm{DSH}(X)$ *satisfies the central limit theorem with respect to* μ. *Namely, we have*

$$\frac{S_n(\varphi) - n\langle \mu, \varphi \rangle}{\sqrt{n}} \longrightarrow \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty \quad \text{in law},$$

where

$$\sigma^2 := \lim_{n \to \infty} \frac{1}{n} \int \big(S_n(\varphi) - \langle \mu, \varphi \rangle\big)^2 \, \mathrm{d}\mu.$$

Notations. The symbols $\lesssim$ and $\gtrsim$ stand for inequalities up to a positive multiplicative constant, and a subscript means that said constant can depend on some variables, e.g., $\lesssim_t$ means that the implicit constant can depend on the variable t. The pairing $\langle \cdot, \cdot \rangle$ is used for the integral of a function with respect to a measure or, more generally, the value of a current at a test form. The mass of a positive closed current S of bidegree (q, q) on a compact Kähler manifold (X, ω) of dimension k is defined as $\|S\| := \langle S, \omega^{k-q} \rangle$. If U is an open set in $\mathbb{C}^k$, we denote by bU the topological boundary of U, i.e., $bU := \overline{U} \backslash U$.

2 Preliminaries

2.1 Quasi-plurisubharmonic and d.s.h. functions

We fix in this section a compact Kähler manifold (X, ω). A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is called *quasi-plurisubharmonic* (*quasi-p.s.h.* for short) if, locally, it is the difference of a p.s.h. function and a smooth one. A function $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ is *d.s.h.* [12, 17] if it is the difference of two quasi-p.s.h. functions outside of a pluripolar set. Denote by $\mathrm{DSH}(X)$

the space of d.s.h. functions on X. If φ is d.s.h., there are two positive closed $(1,1)$-currents $R^{\pm}$ on X such that $\mathrm{dd^c}\varphi = R^+ - R^-$. As these two currents are cohomologous, they have the same mass. We define a norm on $\mathrm{DSH}(X)$ by

$$\|\varphi\|_{\mathrm{DSH}} := \left|\int \varphi\, \omega^k\right| + \inf \|R^{\pm}\|,$$

where the infimum is taken over all $R^{\pm}$ as above. We obtain an equivalent norm if, instead of ω^k, we take any measure ν that is *PB*, i.e., such that all d.s.h. functions are integrable with respect to ν. We will need the following decomposition result for d.s.h. functions, see for instance [17] and [30, Lemma 2.1].

Lemma 2.1 *Let φ be a d.s.h. function on X with $\|\varphi\|_{\mathrm{DSH}} \le 1$. There exist two functions φ_+ and φ_- which are quasi-p.s.h. and such that*

$$\mathrm{dd^c}\varphi_\pm \ge -C\omega, \qquad \|\varphi_\pm\|_{\mathrm{DSH}} \le C, \qquad \varphi_\pm \le 0, \qquad \textit{and} \qquad \varphi = \varphi_+ - \varphi_-,$$

where C is a positive constant that depends on (X, ω) but is independent of φ.

Let $\rho(z) := \widetilde{\rho}(|z|)$ be a radial function on $\mathbb{C}^k$ such that

$$\widetilde{\rho} \ge 0, \qquad \widetilde{\rho}(t) = 0 \ \text{for } t \ge 1, \qquad \text{and} \qquad \int_{\mathbb{C}^k} \rho \,\mathrm{dLeb} = 1.$$

For $\varepsilon > 0$, we set $\rho_\varepsilon(z) := \varepsilon^{-2k}\rho(z/\varepsilon)$. For every function u on an open set $U \subset \mathbb{C}^k$ and every subset $U' \Subset U$, define the convolution

$$u_\varepsilon(z) := (u * \rho_\varepsilon)(z) = \int_{|w|\le 1} u(z - \varepsilon w)\rho(w)\,\mathrm{dLeb}(w) \quad \text{for} \quad z \in U' \tag{2.1}$$

provided that $0 < \varepsilon < \mathrm{dist}(U', bU)$. If u is bounded, it follows from the definition that

$$\|u_\varepsilon\|_{\mathscr{C}^m(U')} \lesssim \|u\|_{L^\infty(U)}\varepsilon^{-m} \tag{2.2}$$

for every positive integer m. If u is p.s.h., then u_ε is also p.s.h. and u_ε is decreasing to u as $\varepsilon \searrow 0$. We also have the following estimate.

Lemma 2.2 *Let $U' \Subset U$ be open subsets of $\mathbb{C}^k$ and u a bounded p.s.h. function on U. For every $0 < \varepsilon <$ dist(U', bU), we have*

$$\|u_\varepsilon - u\|_{L^1(U',\mathrm{Leb})} \lesssim_{U,U'} \|u\|_{L^\infty(U)}\varepsilon.$$

Proof The proof uses standard arguments, but we give it for the reader's convenience. We will proceed in three steps.

Step 1. For every compact set $K \subseteq \mathbb{C}$ and every finite positive measure ν on $\mathbb{C}$ whose support is compactly contained in a ball B of radius R containing K, we have that

$$\int_K \left|u_\nu(z - w) - u_\nu(z)\right| \mathrm{dLeb}(z) \lesssim_{K,R} \nu(B)\varepsilon \quad \text{for every} \quad |w| \le \varepsilon, \tag{2.3}$$

where $u_\nu(z) := \int_{\mathbb{C}} \log|z - \zeta|\,\mathrm{d}\nu(\zeta)$.

Proof of Step 1. We have the following estimate:

$$\int_K \big|\log|z - w| - \log|z|\big|\,\mathrm{dLeb}(z) \lesssim_K \varepsilon \quad \text{for every} \quad |w| \le \varepsilon.$$

Combining it with the definition of u_ν, we get (2.3).

Step 2. For every open set $V \subseteq \mathbb{C}$, every compact set $K \Subset V$, and every function u which is subharmonic and bounded in V, we have that

$$\int_K \big|u(z-w)-u(z)\big|\,\mathrm{dLeb}(z) \lesssim_{K,V} \|u\|_{L^\infty(V)}\varepsilon. \tag{2.4}$$

Proof of Step 2. Assume without loss of generality that $\|u\|_{L^\infty(V)} = 1$. Let K_η be the η-neighborhood of K. Choose η sufficiently small to have $K_{3\eta} \Subset V$, and take χ_η a positive smooth cut-off function with $\chi_\eta|_{K_{2\eta}} \equiv 1$ and $\operatorname{supp}\chi_\eta \Subset K_{3\eta}$. Define ν to be equal to $\chi_\eta \cdot \mathrm{dd^c}u$ on V and to 0 outside of V. We have $\nu(B) \lesssim_{K,\eta,V} \|u\|_{L^\infty(V)}$, where B is a large ball containing $K_{3\eta}$. Consider u_ν defined as in Step 1. Since ν satisfies the hypothesis of Step 1, u_ν satisfies inequality (2.3). An integration by parts gives

$$\begin{aligned} u_\nu(z) = \int_{\mathbb{C}} \log|z-\zeta|\chi_\eta(\zeta)\,\mathrm{dd^c}u(\zeta) &= \int_{\mathbb{C}} \delta_z\chi_\eta(\zeta)u(\zeta) \\ &+ \int_{\mathbb{C}} \Big(\log|z-\zeta|\mathrm{dd^c}\chi_\eta(\zeta) + \mathrm{d}\log|z-\zeta| \wedge \mathrm{d^c}\chi_\eta(\zeta) + \mathrm{d}\,\chi_\eta(\zeta) \wedge \mathrm{d^c}\log|z-\zeta|\Big)u(\zeta), \end{aligned}$$

from which it follows that, for every $z \in K_{2\eta}$, $u_\nu(z) - u(z)$ is equal to

$$\int_{\zeta \in K_{2\eta}^c \cap K_{3\eta}} \Big(\log|z-\zeta|\mathrm{dd^c}\chi_\eta(\zeta) + \mathrm{d}\log|z-\zeta| \wedge \mathrm{d^c}\chi_\eta(\zeta) + \mathrm{d}\,\chi_\eta(\zeta) \wedge \mathrm{d^c}\log|z-\zeta|\Big)u(\zeta). \tag{2.5}$$

Differentiating (2.5) under the integral sign, we get $\|u-u_\nu\|_{\mathscr{C}^1(K_\eta)} \lesssim_{K,\eta} 1$. It follows that

$$\int_K \big|(u-u_\nu)(z-w) - (u-u_\nu)(z)\big|\,\mathrm{dLeb}(z) \lesssim_{K,\eta} \varepsilon \quad \text{for every} \quad |w| \le \varepsilon.$$

Writing $u = u_\nu + (u_\nu - u)$, we then obtain (2.4).

Conclusion. Let u be as in the statement. Assume without loss of generality that $\|u\|_{L^\infty(U)} = 1$. Denote by $\pi : \mathbb{C}^k \to \mathbb{C}^{k-1}$ the projection on the first $k-1$ coordinates and by $\pi_k : \mathbb{C}^k \to \mathbb{C}$ the projection on the last one. Take $w \in \mathbb{C}^k$ with $|w| \le \varepsilon$. Setting $z = (\hat{z}, z_k)$ with $\hat{z} \in \mathbb{C}^{k-1}$ and $z_k \in \mathbb{C}$, taking R sufficiently large (depending on U and U'), and assuming without loss of generality that w has the form $w = (0, w_k)$, we have

$$\begin{aligned} &\int_{U'} \big|u(z-w)-u(z)\big|\,\mathrm{dLeb}(z) \\ &\le \int_{\mathbb{D}_R^{k-1}\cap\pi(U')} \left(\int_{\pi_k\big((\{\hat{z}\}\times\mathbb{C})\cap U'\big)} \big|u(\hat{z}, z_k - w_k) - u(\hat{z}, z_k)\big|\,\mathrm{dLeb}(z_k) \right) \mathrm{dLeb}(\hat{z}) \\ &\lesssim_{U,U'} \int_{\mathbb{D}_R^{k-1}\cap\pi(U')} \varepsilon\,\mathrm{dLeb}(\hat{z}) \le \int_{\mathbb{D}_R^{k-1}} \varepsilon\,\mathrm{dLeb}(\hat{z}) \lesssim_{U,U'} \varepsilon, \end{aligned} \tag{2.6}$$

where in the second inequality we used (2.4). The assertion follows from (2.6) and the definition of u_ε. □

We will use the following norm for bounded quasi-p.s.h. functions.

Definition 2.3 For every bounded quasi-p.s.h. functions φ, define the norm

$$\|\varphi\|_{\mathrm{qpsh}} := \|\varphi\|_\infty + \min\big\{c \ge 0 \mid \mathrm{dd^c}\varphi \ge -c\omega\big\}.$$

We are now ready to prove Theorem 1.1. The second item corrects an inaccuracy in [25, first inequality in (3.1)], which affects the estimate in [25, Lemma 3.2]. Those estimates should be $\|\phi_\varepsilon - \phi\|_{L^1(\omega^k)} \lesssim -1/\log\varepsilon$ and $|g(\varepsilon) - g(0)| \lesssim (-1/\log\varepsilon)^\alpha$ respectively.

Proof of Theorem 1.1. We follow the proof of [21, Theorem 2.1]. Fix a function $\vartheta \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ with support in $[-1, 1]$ such that $\int_\mathbb{R} \vartheta(x)\mathrm{d}x = 1$ and $\int_\mathbb{R} x\vartheta(x)\mathrm{d}x = 0$. For each $0 < \varepsilon \le 1$ and each integer $l \ge 1$, consider the function $\max_\varepsilon : \mathbb{R}^l \to \mathbb{R}$ defined by

$$\max_\varepsilon(t_1, \dots, t_l) := \int_{\mathbb{R}^l} \max(t_1 + x_1, \dots, t_l + x_l)\varepsilon^{-l} \prod_{i=1}^{l} \vartheta(x_i/\varepsilon)\mathrm{d}x_1 \dots \mathrm{d}x_l.$$

We will need the following properties of $\max_\varepsilon$, see [11, Lemma I.5.18] and [21, Lemma 2.2]. The notation $(t_1, \dots, \widehat{t_i}, \dots, t_l)$ below means that the component t_i is omitted in the expression.

Lemma 2.4 *We have the following properties*:

(a) $\max_\varepsilon(t_1, \dots, t_l)$ *is non-decreasing in all variables, smooth and convex on* $\mathbb{R}^l$;
(b) $\max(t_1, \dots, t_l) \le \max_\varepsilon(t_1, \dots, t_l) \le \varepsilon + \max(t_1, \dots, t_l)$;
(c) $\max_\varepsilon(t_1, \dots, t_l) = \max_\varepsilon(t_1, \dots, \widehat{t_i}, \dots, t_l)$ *if* $t_i + 2\varepsilon \le \max(t_1, \dots, \widehat{t_i}, \dots, t_l)$;
(d) $\max_\varepsilon(t_1 + a, \dots, t_l + a) = \max_\varepsilon(t_1, \dots, t_l) + a$ *for every* $a \in \mathbb{R}$;
(e) *if* $u_1, \dots, u_l$ *are p.s.h. functions defined on some domain* D *in* $\mathbb{C}^n$, *then so is* $\max_\varepsilon(u_1, \dots, u_l)$;
(f) *if* $u_1, \dots, u_l$ *are real-valued functions in* $\mathscr{C}^m(D)$, *where* $m \ge 1$ *is an integer and* D *is a domain in* $\mathbb{C}^n$, *then there is a constant* $c_{l,m} > 0$ *depending only on* l, m *and* ϑ *such that*

$$\|\max_\varepsilon(u_1, \dots, u_l)\|_{\mathscr{C}^m} \le \varepsilon + \sup_{1\le i\le l} \|u_i\|_\infty + c_{l,m} \sum_{r_{ij}} \varepsilon^{1-\sum r_{ij}} \prod_{i,j} \|u_i\|_{\mathscr{C}^j}^{r_{ij}},$$

the sum being taken over all $r_{ij} > 0$ *with* $1 \le i \le l$ *and* $j \ge 1$ *such that* $\sum jr_{ij} \le m$.

We will also need the following lemma. Recall that u_δ is the regularization given by convolution, see (2.1).

Lemma 2.5 *Let* $F : W \to W'$ *be a biholomorphic map between two open subsets* W *and* W' *of* $\mathbb{C}^n$. *Let* u *be a bounded p.s.h. function on* W. *Then, for every open set* $U \Subset W$ *we can find a constant* $\delta_U > 0$ *such that for every* $0 < \delta < \delta_U$, *the function* $u_\delta^F := (u \circ F^{-1})_\delta \circ F$ *is well-defined on a neighborhood of* $\overline{U}$. *Moreover, there are constants* $c_{W,W',U} > 0$ *and* $c_{W,W',U,m} > 0$ *(for integers* $m \ge 1$*) such that, when* $0 < \delta < \delta_U$, *we have*

$$\|u_\delta^F - u_\delta\|_{L^\infty(U)} \le c_{W,W',U} \|u\|_{L^\infty(W)} \frac{1}{|\log\delta|} \quad \text{and} \quad \|u_\delta^F\|_{\mathscr{C}^m(U)} \le c_{W,W',U,m} \|u\|_{L^\infty(W)} \delta^{-m}.$$

Proof Since u is bounded, the first inequality follows from explicit estimates of the expressions in the proof of [7, Lemma 4]. The second one is a direct consequence of the estimate on the $\mathscr{C}^m$-norm of the convolution and the fact that F is a diffeomorphism. □

By linearity, up to rescaling we can assume $\|\varphi\|_{\mathrm{qpsh}} = 1$. This implies $\|\varphi\|_\infty \le 1$ and $\mathrm{dd^c}\varphi \ge -\omega$. Observe that it suffices to construct a smooth function φ_ε with $\mathrm{dd^c}\varphi_\varepsilon \ge -(1 - c'/\log\varepsilon)\omega$ and such that

$$\varphi_\varepsilon \ge \varphi + \frac{C}{\log\varepsilon}, \quad \|\varphi_\varepsilon\|_\infty \le \tilde{c}$$

and

$$\|\varphi_\varepsilon-\varphi\|_{L^1(\omega^k)}\le -c/\log\varepsilon,\quad \|\varphi_\varepsilon\|_{\mathscr{C}^m}\le c_m\varepsilon^{-m}(-\log\varepsilon)^{m-1}, \tag{2.7}$$

where $C, c, \tilde{c}$ and c' are constants that depend only on X and ω, while c_m depends on X, ω and m. Indeed, the function $\varphi_\varepsilon - C/\log\varepsilon$ will satisfy all the conditions in the statement of Theorem 1.1.

Fix finite covers $(V_j)_{j\in J}$, $(U_j)_{j\in J}$ and $(W_j)_{j\in J}$ of X by local charts such that $V_j \Subset U_j \Subset W_j$ for every $j\in J$, for some finite index set $J=\{1,\dots,l\}$.

We can choose the W_j's small enough, so that for each $j\in J$ we can fix a smooth function g_j, defined on a small neighbourhood of W_j, such that

$$\mathrm{dd^c} g_j=\omega \quad\text{on}\quad W_j. \tag{2.8}$$

Since the W_j's depend only on X, we have $\|g_j\|_{\mathscr{C}^2(W_j)}\lesssim 1$, where the implicit constant depends only on X and ω. Then the function

$$u_j:=\varphi+g_j \tag{2.9}$$

satisfies $\|u_j\|_{L^\infty(W_j)}\lesssim 1$ and $\mathrm{dd^c}u_j=\mathrm{dd^c}\varphi+\mathrm{dd^c}g_j=\mathrm{dd^c}\varphi+\omega\ge 0$. So, u_j is p.s.h. on W_j.

Let $j,k\in J$ be such that $U_j\cap U_k\neq\varnothing$. One can regularize the restriction $u_j|_{U_j\cap U_k}$ by convolution in two ways.

The first one is to use the local chart of W_j, i.e., W_j will play the role of U in Lemma 2.2, and we get a function $u_{j,\varepsilon}$. The second way is to use the local chart of W_k.

Let F be the change of coordinates on $W_j\cap W_k$ from the local chart of W_j to the local chart of W_k. Denote by $u^F_{j,\delta}$ the function given by Lemma 2.5, which corresponds to the regularization of u_j using the local chart of W_k. We have

$$u_{j,\varepsilon}-u_{k,\varepsilon}=u_{j,\varepsilon}-u^F_{j,\varepsilon}+(u_j-u_k)_\varepsilon \quad\text{on}\quad U_j\cap U_k,$$

where the term $(u_j-u_k)_\varepsilon$ is the regularization of u_j-u_k by convolution using the local chart of W_k. Recall from (2.9) that $u_j-u_k=g_j-g_k$. This, together with the previous equality and Lemma 2.5, implies

$$\|(u_{j,\varepsilon}-u_{k,\varepsilon})-(g_j-g_k)\|_{L^\infty(U_j\cap U_k)}\lesssim -1/\log\varepsilon. \tag{2.10}$$

Fix a constant $c>0$ large enough. For each $j\in J$, let η_j be a smooth function defined in U_j such that $\eta_j=0$ on V_j and that $\eta_j=-c$ away from a compact subset of U_j. We have that $\mathrm{dd^c}\eta_j\ge -c'\omega$ for some constant $c'>0$. For each $\varepsilon>0$ and $j\in J$, consider the function

$$v_{j,\varepsilon}:=u_{j,\varepsilon}-g_j-\eta_j/\log\varepsilon \quad\text{on}\quad U_j. \tag{2.11}$$

We set

$$\varphi_\varepsilon:=-\frac{1}{\varepsilon\log\varepsilon}\max\nolimits_\varepsilon\big(-(\varepsilon\log\varepsilon)v_{1,\varepsilon},\dots,-(\varepsilon\log\varepsilon)v_{l,\varepsilon}\big). \tag{2.12}$$

Note that to define $\varphi_\varepsilon(x)$, $x\in X$, we remove $-v_{j,\varepsilon}\varepsilon\log\varepsilon$ from the last formula if $x\notin U_j$. From the definition of φ_ε and Lemma 2.4 (b), we have $\|\varphi_\varepsilon\|_\infty\lesssim 1$, where the implicit constant depends only on X and ω. Observe also that, since $u_{j,\varepsilon}\ge u_j$ on U_j, we have

$$v_{j,\varepsilon}\ge u_j-g_j-\eta_j/\log\varepsilon\ge\varphi+C/\log\varepsilon \quad\text{on}\quad U_j$$

for some constant $C>0$. By Lemma 2.4 (b), we get $\varphi_\varepsilon\ge\varphi+C/\log\varepsilon$.

We will show that the function φ_ε is smooth on X. For this purpose, we only need to prove the property in a neighborhood of an arbitrary fixed point of X. Since each $v_{j,\varepsilon}$ is well-defined and smooth on U_j, using (2.12) and Lemma 2.4 (a), it is enough to prove the following claim.

Claim 1. For all $x \in U_j$ close enough to bU_j, we have

$$\begin{aligned}&\max\nolimits_\varepsilon\big(-(\varepsilon\log\varepsilon)v_{1,\varepsilon},\ldots,-(\varepsilon\log\varepsilon)v_{l,\varepsilon}\big)(x)\\&\quad=\max\nolimits_\varepsilon\big(-(\varepsilon\log\varepsilon)v_{1,\varepsilon},\ldots,\widehat{-(\varepsilon\log\varepsilon)v_{j,\varepsilon}},\ldots,-(\varepsilon\log\varepsilon)v_{l,\varepsilon}\big)(x).\end{aligned}$$

Proof of Claim 1 Let $k\in J$ be such that $x\in V_k$. We deduce from (2.11) and the equality $\eta_k(x)=0$ that

$$v_{k,\varepsilon}(x)=u_{k,\varepsilon}(x)-g_k(x).$$

The same argument using the equality $\eta_j(x)=-c$ gives

$$v_{j,\varepsilon}(x)=u_{j,\varepsilon}(x)-g_j(x)+c/\log\varepsilon.$$

Putting the two last equalities together with (2.10), and using that $c>0$ is large enough, we get

$$v_{k,\varepsilon}(x)\geq v_{j,\varepsilon}(x)-2/\log\varepsilon.$$

This, combined with Lemma 2.4 (c), implies Claim 1. □

The next claim gives the estimate of $dd^c\varphi_\varepsilon$ from below.

Claim 2. We have $dd^c\varphi_\varepsilon\geq-(1-c'/\log\varepsilon)\omega$.

Proof of Claim 2 It is enough to work in a small open set W in X. By Claim 1, we can remove from the definition (2.12) of φ_ε all functions $-(\varepsilon\log\varepsilon)v_{j,\varepsilon}$ such that $W\not\subset U_j$. So we have $W\subset U_j$ for the indexes j considered below. Since u_j is p.s.h., so is $u_{j,\varepsilon}$. Therefore, we deduce from (2.8) and (2.11) that

$$dd^c v_{j,\varepsilon}=dd^c u_{j,\varepsilon}-\omega-dd^c\eta_j/\log\varepsilon\geq-(1-c'/\log\varepsilon)\omega.$$

Choose a function g on W such that $dd^c g=-(\varepsilon\log\varepsilon)(1-c'/\log\varepsilon)\omega$. We deduce from (2.12) and Lemma 2.4 (d) that

$$\varphi_\varepsilon=-\frac{1}{\varepsilon\log\varepsilon}\max\nolimits_\varepsilon\big(-(\varepsilon\log\varepsilon)v_{1,\varepsilon}+g,\ldots,-(\varepsilon\log\varepsilon)v_{l,\varepsilon}+g\big)+\frac{1}{\varepsilon\log\varepsilon}g.$$

Since $-(\varepsilon\log\varepsilon)v_{j,\varepsilon}+g$ is p.s.h. on W, applying Lemma 2.4 (e) we obtain that $dd^c\varphi_\varepsilon\geq-(1-c'/\log\varepsilon)\omega$ thus proving Claim 2. □

We now conclude the proof of Theorem 1.1. By (2.9), (2.11) and Lemma 2.2, we get

$$\begin{aligned}\|\varphi-v_{j,\varepsilon}\|_{L^1(U_j)}&=\|(u_j-g_j)-(u_{j,\varepsilon}-g_j-\eta_j/\log\varepsilon)\|_{L^1(U_j)}\\&\leq\|u_j-u_{j,\varepsilon}\|_{L^1(U_j)}-\|\eta_j\|_\infty/\log\varepsilon\lesssim-1/\log\varepsilon.\end{aligned}\tag{2.13}$$

Consider the partition of X given by the sets $S_{J'} = \big(\bigcap_{j\in J'} U_j\big) \cap \big(\bigcap_{j\notin J'} U_j^c\big)$, where J' varies among all subsets of J. We have

$$\begin{aligned}\|\varphi - \max\{v_{1,\varepsilon},\dots,v_{l,\varepsilon}\}\|_{L^1(\omega^k)} &= \int_X |\varphi - \max\{v_{1,\varepsilon},\dots,v_{l,\varepsilon}\}| \mathrm{d}\omega^k = \sum_{J'\subseteq J}\int_{S_{J'}} |\varphi - \max_{j\in J'}\{v_{j,\varepsilon}\}|\mathrm{d}\omega^k \\ &\le \sum_{J'\subseteq J}\int_{S_{J'}}\Bigg(\sum_{j\in J'}|\varphi - v_{j,\varepsilon}|\Bigg)\mathrm{d}\omega^k = \sum_{J'\subseteq J}\sum_{j\in J'}\|\varphi - v_{j,\varepsilon}\|_{L^1(S_{J'})} \\ &\le \sum_{J'\subseteq J}\sum_{j\in J'}\|\varphi - v_{j,\varepsilon}\|_{L^1(U_j)} \lesssim -1/\log\varepsilon,\end{aligned}$$

where at every point $x \in X$ we take the maximum only over the functions $v_{j,\varepsilon}$ such that $x \in U_j$, and we used inequality (2.13) in the last step. This and Lemma 2.4 (b) prove the first estimate in (2.7). For the second estimate, we deduce from Lemma 2.4 (f) that

$$\begin{aligned}\|\varphi_\varepsilon\|_{\mathscr{C}^m} &= -\frac{1}{\varepsilon\log\varepsilon}\left\|\max\nolimits_\varepsilon\big(-(\varepsilon\log\varepsilon)v_{1,\varepsilon},\dots,-(\varepsilon\log\varepsilon)v_{l,\varepsilon}\big)\right\|_{\mathscr{C}^m} \\ &\lesssim -1/\log\varepsilon + \sup_{1\le i\le l}\|v_i\|_\infty - \frac{1}{\varepsilon\log\varepsilon}\sum_{r_{ij}}\varepsilon^{1-\sum r_{ij}}\prod_{i,j}\big(-(\varepsilon\log\varepsilon)\|v_i\|_{\mathscr{C}^j}\big)^{r_{ij}},\end{aligned} \tag{2.14}$$

the sum being taken over all $r_{ij} > 0$ with $1 \le i \le l$ and $j \ge 1$ such that $\sum jr_{ij} \le m$. On the other hand, by (2.2) and (2.11), we have

$$\|v_i\|_{\mathscr{C}^j} = \|u_{i,\varepsilon} - g_i - \eta_i/\log\varepsilon\|_{\mathscr{C}^j} \lesssim \varepsilon^{-j}.$$

Inserting these estimates into (2.14), we obtain that φ_ε satisfies the second inequality in (2.7). The theorem follows. □

A positive measure ν on X is said to be *moderate* if, for every bounded family $\mathscr{F}$ of d.s.h. functions on X, there exist constants $\alpha > 0$ and $c > 0$ such that

$$\nu\big\{z \in X : |\psi(z)| > M\big\} \le ce^{-\alpha M} \qquad \text{for every } M \ge 0 \text{ and } \psi \in \mathscr{F},$$

see [12, 20]. Moderate measures are PB. We have the following result, which is proven in the case of $\mathbb{P}^k$ in [30, Lemma 2.3]. The same proof applies in the general case of compact Kähler manifolds.

Lemma 2.6 *Let ν be a moderate measure on X. There exists a constant $\alpha > 0$ such that for every $q \ge 1$ there exists a constant $C_q > 0$ such that for every non-positive quasi-p.s.h. function φ on X satisfying $\|\varphi\|_{\mathrm{DSH}} \le 1$ and $\mathrm{dd^c}\varphi \ge -\omega$, and for every $N \ge 0$, we can write $\varphi = \varphi_1^{(N)} + \varphi_2^{(N)}$, where $\varphi_1^{(N)}$ is quasi-p.s.h., with:*

$$\mathrm{dd^c}\varphi_1^{(N)} \ge -\omega, \quad \|\varphi_1^{(N)}\|_\infty \le N, \quad \textit{and} \quad \|\varphi_2^{(N)}\|_{L^q(\nu)} \le C_q e^{-\alpha N/q}.$$

2.2 Super-potentials of currents on compact Kähler manifolds

Denote by $\mathscr{D}_q$ the real space generated by all positive closed (q,q)-currents on X. Define a norm $\|\cdot\|_*$ on $\mathscr{D}_q$ by

$$\|\Omega\|_* := \min\big\{\|\Omega^+\| + \|\Omega^-\|\big\},$$

where the minimum is taken over all positive closed currents $\Omega^\pm$ such that $\Omega = \Omega^+ - \Omega^-$. Observe that $\|\Omega^\pm\|$ only depend on the cohomology classes of $\Omega^\pm$ in $H^{q,q}(X,\mathbb{R})$. In particular, if φ is such that $\mathrm{dd^c}\varphi \geq -c\omega$, by writing $\mathrm{dd^c}\varphi = (\mathrm{dd^c}\varphi + c\omega) - c\omega$ we find $\|\mathrm{dd^c}\varphi\|_* \leq 2c$. It follows that for every bounded quasi-p.s.h. function φ we have

$$\|\mathrm{dd^c}\varphi\|_* \leq 2\|\varphi\|_{\mathrm{qpsh}}.$$

We will consider the following topology on $\mathscr{D}_q$: given a sequence of currents $(S_n)_{n\geq 0}$ and a current S, we say that the S_n's converge to S if they converge in the sense of currents and $\|S_n\|_*$ is uniformly bounded. We call this topology the $*$-*topology*. By [15], smooth forms are dense in $\mathscr{D}_q$ with respect to the $*$-topology. They are also dense in the space $\mathscr{D}_q^0$ given by those currents $S \in \mathscr{D}_q$ which are exact, i.e., whose cohomology class $\{S\}$ in $H^{q,q}(X, \mathbb{R})$ is 0.

For every $0 < l < +\infty$, denote by $\|\cdot\|_{\mathscr{C}^l}$ the standard $\mathscr{C}^l$ norm on the space of differential forms. We consider a norm $\|\cdot\|_{\mathscr{C}^{-l}}$ defined by

$$\|S\|_{\mathscr{C}^{-l}} := \sup_{\|\Phi\|_{\mathscr{C}^l}\leq 1} |\langle S, \Phi\rangle|,$$

where the supremum is on smooth $(k-q, k-q)$-forms Φ on X. Observe that, by interpolation [29], for every $0 < l < l' < +\infty$ and $m > 0$ there exists a positive constant $c_{l,l',m}$ such that

$$\|S\|_{\mathscr{C}^{-l'}} \leq \|S\|_{\mathscr{C}^{-l}} \leq c_{l,l',m}\|S\|_{\mathscr{C}^{-l'}}^{l/l'} \quad \text{for all } S \text{ such that } \|S\|_* \leq m. \tag{2.15}$$

Remark 2.7 Observe that point (ii) in Theorem 1.1 implies

$$\|\varphi_\varepsilon - \varphi\|_{\mathscr{C}^{-2}} \lesssim -\|\varphi\|_{\mathrm{qpsh}}/\log\varepsilon \quad \text{and} \quad \|\mathrm{dd^c}\varphi_\varepsilon - \mathrm{dd^c}\varphi\|_{\mathscr{C}^{-2}} \lesssim -\|\varphi\|_{\mathrm{qpsh}}/\log\varepsilon.$$

Following [18, 19], we now recall the definition of the *super-potential* of a current $S \in \mathscr{D}_q$. Fix a basis $\{\alpha\} := \big\{\{\alpha_1\}, \ldots, \{\alpha_t\}\big\}$ of $H^{q,q}(X, \mathbb{R})$. We can take all the α_j's to be smooth. For any $R \in \mathscr{D}_{k-q+1}^0$, there exists a real $(k-q, k-q)$-current U_R such that $\mathrm{dd^c}U_R = R$. We call U_R a *potential* of R. After adding some smooth real closed form to U_R, we can assume that U_R is α-*normalized*, i.e., that $\langle U_R, \alpha_j\rangle = 0$ of all $1 \leq j \leq t$. We can choose U_R smooth if R is smooth. The α-*normalized super-potential* $\mathscr{U}_S$ of S is the linear functional on the smooth forms in $\mathscr{D}_{k-q+1}^0$ which is defined by

$$\mathscr{U}_S(R) := \langle S, U_R\rangle.$$

Note that $\mathscr{U}_S(R)$ does not depend on the choice of U_R.

We say that S has a *continuous super-potential* if $\mathscr{U}_S$ can be extended continuously to a linear functional on all of $\mathscr{D}_{k-q+1}^0$ with respect to the $*$-topology. If $S \in \mathscr{D}_q^0$, then $\mathscr{U}_S$ does not depend on the choice of α. If S is smooth, then it has a continuous super-potential and for every $R \in \mathscr{D}_{k-q+1}^0$ we have $\mathscr{U}_S(R) = \mathscr{U}_R(S)$, where $\mathscr{U}_R$ is the super-potential of R. The equality still holds if we only assume that S has a continuous super-potential, see [19].

Definition 2.8 Take $S \in \mathscr{D}_q$. For $l > 0$, $0 < \lambda \leq 1$, and $M > 0$, we say that a super-potential $\mathscr{U}_S$ of S is (l, λ, M)-*Hölder-continuous* if it is continuous and we have

$$|\mathscr{U}_S(R)| \leq M\|R\|_{\mathscr{C}^{-l}}^{\lambda} \quad \text{for every } R \in \mathscr{D}_{k-q+1}^0 \text{ with } \|R\|_* \leq 1.$$

If S is such that $\mathscr{U}_S$ is (l, λ, M)-Hölder-continuous, (2.15) implies that $\mathscr{U}_S$ is also (l', λ', M')-Hölder-continuous for every $l' > 0$ and some constants λ' and M' which depend on λ, M, l', l, but are independent of S. Definition 2.8 does not depend on the normalization of the super-potential.

3 Mixing for d.s.h. functions

In this section, we are going to prove Theorem 1.2. We follow the general strategy of [30], but we cannot use results about p.s.h. functions in the Kähler setting. We use instead the techniques from Sect. 2.

3.1 Mixing for bounded quasi-p.s.h. functions

Recall that f is a holomorphic automorphism of X with simple action on cohomology and d_p is its main dynamical degree. We fix a constant $\max\{d_{p-1}, d_{p+1}\} < \delta_0 < d_p$ such that all the eigenvalues of f^* acting on $H^{p,p}(X,\mathbb{R})$, except for d_p, have modulus smaller than δ_0. We call δ_0 the *auxiliary dynamical degree* of f. We denote by μ the equilibrium measure of f. From [16, 19] we know that μ is the intersection of a positive closed (p,p)-current T_+ and a positive closed $(k-p,k-p)$-current T_- (the *Green currents* of f and f^{-1}, respectively). We have $f^*(T_+) = d_p T_+$ and $f_*(T_-) = d_p T_-$. Moreover, for every positive closed (p,p)-current (respectively, $(k-p,k-p)$-current) S of mass 1, we have that $d_p^{-n}(f^n)^*(S)$ converges to T_+ (respectively, $d_{k-p}^{-n} f^n_*(S)$ converges to T_-). We also have that T_+ (respectively, T_-) is the unique positive closed current in the class $\{T_+\}$ (respectively, $\{T_-\}$).

By [19, Theorem 4.4.2], we have that μ has a $(2,M,r)$-Hölder-continuous super-potential for some $M > 0$ and $0 < r \leq 1$. We fix

$$\delta' := d_p^{\frac{1}{r+1}} \delta_0^{\frac{r}{r+1}}.$$

Observe that $\delta_0 < \delta' < d_p$. For $\delta' < \tilde{\delta} < d_p$ we have

$$\tilde{\delta}/d_p > (\delta_0/d_p)^{\frac{r}{r+1}}.$$

We will prove the mixing with rate $(\delta/d_p)^{\frac{r}{2(r+1)}}$ for every $\delta_0 < \delta < d_p$. From the above inequality, this implies Theorem 1.2 with our choice of δ'.

Remark 3.1 One can actually prove that $r \geq \frac{1}{8}\left(\frac{\log(d_p/\delta'')}{\log(d_p/\delta'') + \log A}\right)^2$, where $A = \|f^*\|_{\mathscr{C}^1}$ and δ'' is any real number between δ_0 and d_p, see for instance [19, Proposition 3.4.2 and Lemma 4.2.5]. Hence, r depends only on the dynamical degrees and the Lipschitz constant of f. In particular, it can be taken to depend continuously on f.

We start with establishing a weaker version of Theorem 1.2 for bounded quasi-p.s.h. functions.

Proposition 3.2 *For every $\delta_0 < \delta < d_p$ and every $\kappa \in \mathbb{N}^*$ there exists a constant $C_{\kappa,\delta} > 0$ such that, for every $\kappa+1$ bounded quasi-p.s.h. functions $g_0, g_1, \dots, g_\kappa$, and every $0 = n_0 \leq n_1 \leq \cdots \leq n_\kappa$, we have*

$$\left| \int g_0(g_1 \circ f^{n_1}) \cdots (g_\kappa \circ f^{n_\kappa})\, \mathrm{d}\mu - \prod_{j=0}^{\kappa} \int g_j \,\mathrm{d}\mu \right| \leq C_{\kappa,\delta} \left(\frac{\delta}{d_p}\right)^{\frac{r}{2(r+1)} \min_{0\leq j\leq \kappa-1}(n_{j+1}-n_j)} \prod_{j=0}^{\kappa} \|g_j\|_{\mathrm{qpsh}}.$$

Observe that, by linearity, Proposition 3.2 also holds if we assume that either g_j or $-g_j$ is quasi-p.s.h. for every j. Observe also that, since for every $\mathscr{C}^2$-continuous function g we have $\|g\|_{\mathrm{qpsh}} \lesssim \|g\|_{\mathscr{C}^2}$, Proposition 3.2 is already stronger than [3, Theorem 1.2].

We will need the following version of Theorem 1.1 (ii), where ω^k is replaced with μ.

Proposition 3.3 *Let φ a bounded quasi-p.s.h. function on X and φ_ε the functions given by Theorem 1.1 for $0 < \varepsilon \leq 1/2$. We have*

$$\|\varphi_\varepsilon - \varphi\|_{L^1(\mu)} \lesssim \|\varphi\|_{\mathrm{qpsh}}(-1/\log\varepsilon)^r,$$

where the implicit constant depends only on (X, ω).

Proof By linearity, up to rescaling we can assume $\|\varphi\|_{\mathrm{qpsh}} = 1$ without loss of generality. We have

$$\|\mathrm{dd^c}\varphi\|_* \lesssim 1 \quad \text{and} \quad \|\mathrm{dd^c}\varphi_\varepsilon\|_* \leq 2\|\varphi_\varepsilon\|_{\mathrm{qpsh}} \lesssim 1.$$

Since $\varphi_\varepsilon \geq \varphi$, we have

$$\|\varphi_\varepsilon - \varphi\|_{L^1(\mu)} = \langle \mu, \varphi_\varepsilon - \varphi\rangle = \langle S, \varphi_\varepsilon - \varphi\rangle + \langle \omega^k, \varphi_\varepsilon - \varphi\rangle, \tag{3.1}$$

where $S := \mu - \omega^k \in \mathscr{D}_k^0$. We just have to bound the two terms of the sum in the right hand side of (3.1). Theorem 1.1 gives

$$|\langle \omega^k, \varphi_\varepsilon - \varphi\rangle| = \|\varphi_\varepsilon - \varphi\|_{L^1(\omega^k)} \lesssim -1/\log\varepsilon \lesssim (-1/\log\varepsilon)^r,$$

where the last inequality is true because $\varepsilon \leq 1/2$ implies $-1/\log\varepsilon \lesssim 1$.

For the other term, since μ has a $(2, M, r)$-Hölder-continuous super-potential, then S has a $(2, M', r)$-Hölder-continuous super-potential for some $M' > 0$. Using $\{S\} = 0$ and Remark 2.7, we deduce that

$$|\langle S, \varphi_\varepsilon - \varphi\rangle| = |\mathscr{U}_S(\mathrm{dd^c}\varphi_\varepsilon - \mathrm{dd^c}\varphi)| \lesssim \|\mathrm{dd^c}\varphi_\varepsilon - \mathrm{dd^c}\varphi\|^r_{\mathscr{C}^{-2}} \lesssim (-1/\log\varepsilon)^r.$$

This concludes the proof. □

Consider now the Kähler manifold $X \times X$ equipped with the Kähler form $\tilde{\omega} = \pi_1^*\omega + \pi_2^*\omega$, where π_1, π_2 are the canonical projections of $X \times X$ onto its factors. Define a new automorphism of $X \times X$ by

$$F(z, w) := \big(f(z), f^{-1}(w)\big).$$

Using Künneth formula, one can show that the dynamical degree of order k of F is equal to d_p^2 (see also [13, Section 4]), which is an eigenvalue of multiplicity 1 of F^*, and that all the others dynamical degrees and the eigenvalues of F^* on $H^{k,k}(X \times X, \mathbb{R})$, except for d_p^2, are strictly smaller than $d_p\delta_0$. Hence F and $d_p\delta_0$ satisfy the same conditions as f and δ_0.

One can see that the Green (k, k)-currents of F and F^{-1} are $\mathbb{T}_+ := T_+ \otimes T_-$ and $\mathbb{T}_- := T_- \otimes T_+$ respectively (see [22, Section 4.1.8] for the tensor product of currents) and that they satisfy

$$F^*(\mathbb{T}_+) = d_p^2\mathbb{T}_+ \quad \text{and} \quad F_*(\mathbb{T}_-) = d_p^2\mathbb{T}_-.$$

In particular, they have $(1, \lambda, M)$-Hölder-continuous super-potentials for some $M > 0$ and $0 < \lambda \leq 1$, see [19, Lemma 4.2.5]. Let Δ denote the diagonal of $X \times X$. Then $[\Delta]$ is a positive closed (k, k)-current on $X \times X$.

When proving Proposition 3.2, we can assume without loss of generality that $\|g_j\|_{\mathrm{qpsh}} \leq 1$ for every j, which implies $\|g_j\|_\infty \leq 1$ and $\mathrm{dd^c}g_j \geq -\omega$. Applying Theorem 1.1 and Proposition 3.3 with ε sufficiently small, we can assume that the g_j's are smooth with

$$\|g_j\|_{\mathscr{C}^2} \leq -\varepsilon^{-2}\log\varepsilon \lesssim \varepsilon^{-3}. \tag{3.2}$$

One can check that ε can be chosen to satisfy

$$-\frac{1}{\log\varepsilon}=\min\left\{\left(\frac{\delta_0}{d_p}\right)^{\frac{1}{2(r+1)}\max_{0\le j\le\kappa-1}(n_{j+1}-n_j)},\ \frac{1}{\log 2}\right\}. \tag{3.3}$$

Let j_0 be such that $n_{j_0+1}-n_{j_0}=\max_{0\le j\le\kappa-1}\{n_{j+1}-n_j\}$. On $X\times X$, we define

$$G_j(z,w):=g_j(w)\ \text{ for }\ j\le j_0\quad\text{and}\quad G_j(z,w)=g_j(z)\ \text{ for }\ j\ge j_0+1.$$

Notice that the G_j's are smooth for every j, they satisfy $\|G_j\|_\infty=\|g_j\|_\infty\le 1$ and $\|G_j\|_{\mathscr{C}^2}\lesssim\|g_j\|_{\mathscr{C}^2}\lesssim\varepsilon^{-3}$. Since $dd^c g_j\ge-\omega$ for every j, we also have that $dd^c G_j\ge-\tilde{\omega}$.

Set $l_j:=n_{j_0}-n_j$ for $j\le j_0$ and $l_j:=n_j-n_{j_0+1}$ for $j\ge j_0+1$, and set $\widetilde{G}_j:=G_j\circ F^{l_j}$ for every j. Define the auxiliary quasi-p.s.h. function Φ on $X\times X$ by

$$\Phi:=\Phi_{n_0,\dots,n_\kappa}=\sum_{j=0}^{\kappa}\left((\kappa+1)\widetilde{G}_j+\frac{\kappa}{2}\widetilde{G}_j^2\right)+\prod_{j=0}^{\kappa}\widetilde{G}_j. \tag{3.4}$$

As in [30], this function will play a very important role in the proof of Proposition 3.2. We have the following estimate for $dd^c\Phi$.

Lemma 3.4 *We have*

$$dd^c\Phi\gtrsim_\kappa-\sum_{j=0}^{\kappa}(F^{l_j})^*\tilde{\omega},$$

where the implicit constant is independent of $n_1,\dots,n_\kappa,g_0,\dots,g_\kappa$.

Proof Remember that the inequality $i\partial(g\pm h)\wedge\overline{\partial}(g\pm h)\ge 0$, which is valid for all smooth functions g and h, implies

$$\pm\,(i\partial g\wedge\overline{\partial}h+i\partial h\wedge\overline{\partial}g)\ge-(i\partial g\wedge\overline{\partial}g+i\partial h\wedge\overline{\partial}h). \tag{3.5}$$

From (3.5), it follows that we have

$$\begin{aligned}i\partial\overline{\partial}\Phi&=\sum_{j=0}^{\kappa}i\partial\overline{\partial}\widetilde{G}_j\left(\kappa+1+\kappa\widetilde{G}_j+\prod_{s\ne j}\widetilde{G}_s\right)+\kappa\sum_{j=0}^{\kappa}i\partial\widetilde{G}_j\wedge\overline{\partial}\widetilde{G}_j+\sum_{j\ne s}\left(i\partial\widetilde{G}_j\wedge\overline{\partial}\widetilde{G}_s\prod_{t\ne j,s}\widetilde{G}_t\right)\\&\ge\sum_{j=0}^{\kappa}i\partial\overline{\partial}\widetilde{G}_j\left(\kappa+1+\kappa\widetilde{G}_j+\prod_{s\ne j}\widetilde{G}_s\right)+\sum_{j=0}^{\kappa}i\partial\widetilde{G}_j\wedge\overline{\partial}\widetilde{G}_j\left(\kappa-\sum_{s\ne j}\left(\prod_{t\ne j,s}|\widetilde{G}_t|\right)\right).\end{aligned} \tag{3.6}$$

Using the fact that $\|G_j\|_\infty=\|g_j\|_\infty\le 1$ and $dd^c G_j\ge-\tilde{\omega}$ for every j, we get

$$\begin{aligned}&\sum_{j=0}^{\kappa}i\partial\overline{\partial}\widetilde{G}_j\left(\kappa+1+\kappa\widetilde{G}_j+\prod_{s\ne j}\widetilde{G}_s\right)+\sum_{j=0}^{\kappa}i\partial\widetilde{G}_j\wedge\overline{\partial}\widetilde{G}_j\left(\kappa-\sum_{s\ne j}\left(\prod_{t\ne j,s}|\widetilde{G}_t|\right)\right)\\&\quad\ge\sum_{j=0}^{\kappa}i\partial\overline{\partial}\widetilde{G}_j\left(\kappa+1+\kappa\widetilde{G}_j+\prod_{s\ne j}\widetilde{G}_s\right)\gtrsim_\kappa-\sum_{j=0}^{\kappa}(F^{l_j})^*\tilde{\omega}.\end{aligned} \tag{3.7}$$

The assertion follows from (3.6) and (3.7). □

We have the following estimate for the function Φ.

Lemma 3.5 *We have* $\|dd^c\Phi\wedge\mathbb{T}_+\|_*\le c_\kappa$ *for some constant* $c_\kappa>0$ *which is independent of* $n_1,\dots,n_\kappa,g_0,\dots,g_\kappa$.

As in [30, Lemma 3.4], a delicate point in Lemma 3.5 is the independence of c_κ from the n_j's. Here, we will see explicitly the crucial role of the assumption of simple action on cohomology, which was implicitly used in [30] as every Hénon-Sibony map satisfies this condition.

Proof of Lemma 3.5. We deduce from Lemma 3.4 and the positivity of $\mathbb{T}_+$ that we have

$$\mathrm{dd^c}\Phi \wedge \mathbb{T}_+ \gtrsim_\kappa -\sum_{j=0}^{\kappa} (F^{l_j})^*\tilde{\omega} \wedge \mathbb{T}_+ =: -\Omega. \tag{3.8}$$

We will show that, for every j, the mass of $(F^{l_j})^*\tilde{\omega} \wedge \mathbb{T}_+$ is bounded independently of $n_1, \dots, n_\kappa$. Using that $F^*(\mathbb{T}_+) = d_p^2 \mathbb{T}_+$, we have

$$(F^{l_j})^*\tilde{\omega} \wedge \mathbb{T}_+ = d_p^{-2l_j} (F^{l_j})^*(\tilde{\omega} \wedge \mathbb{T}_+). \tag{3.9}$$

If $m \geq 0$ is the multiplicity of $d_{k+1}(F)$ as an eigenvalue of F^* acting on $H^{k+1,k+1}(X \times X)$, for every current R in $\mathscr{D}_{k+1}(X \times X)$ we have $\|(F^n)^*(R)\|_* \lesssim n^m d_{k+1}(F)^n \|R\|_*$. Since $d_{k+1}(F) \leq d_p \max\{d_{p+1}(f), d_{p-1}(f)\} < d_p \delta_0$, we get $\|(F^n)^*(R)\|_* \lesssim (d_p\delta_0)^n \|R\|_*$. For every j, it follows that we have

$$\left\|(F^{l_j})^*(\tilde{\omega} \wedge \mathbb{T}_+)\right\| \lesssim (d_p\delta_0)^{l_j} \|\tilde{\omega} \wedge \mathbb{T}_+\|_* \lesssim (d_p\delta_0)^{l_j}. \tag{3.10}$$

We can write $\mathrm{dd^c}\Phi \wedge \mathbb{T}_+$ as $(\mathrm{dd^c}\Phi \wedge \mathbb{T}_+ + \tilde{c}_\kappa \Omega) - \tilde{c}_\kappa \Omega$, which is the difference of two positive currents, where $\tilde{c}_\kappa$ is the implicit constant in (3.8). Since $\mathrm{dd^c}\Phi \wedge \mathbb{T}_+$ is exact, the mass of $\mathrm{dd^c}\Phi \wedge \mathbb{T}_+ + \tilde{c}_\kappa\Omega$ is equal to $\|\tilde{c}_\kappa \Omega\|$. Hence, combining (3.9) and (3.10) and using the definitions of Ω and $\|\cdot\|_*$ gives the statement. □

Let now S be a fixed positive closed (k, k)-current of mass 1 on $X \times X$. We will need the following estimate, see also [25, Proposition 3.3].

Proposition 3.6 *Let S be a positive closed (k, k)-current such that $S_n := d_p^{-2n} F_*^n(S)$ converges to $\mathbb{T}_-$. There exists a constant $c_\kappa > 0$, independent of the g_j's and the n_j's, such that for all n we have*

$$|\langle S_n \wedge \mathbb{T}_+, \Phi\rangle - \langle \mathbb{T}_- \wedge \mathbb{T}_+, \Phi\rangle| \leq -c_\kappa n_\kappa (\delta_0/d_p)^n \log \varepsilon,$$

where ε is as in (3.3).

Proof By definition of super-potentials, we have

$$\begin{aligned} &\langle S_n \wedge \mathbb{T}_+, \Phi\rangle - \langle \mathbb{T}_- \wedge \mathbb{T}_+, \Phi\rangle \\ &\quad = \mathscr{U}_{S_n}(\mathrm{dd^c}\Phi \wedge \mathbb{T}_+) - \mathscr{U}_{\mathbb{T}_-}(\mathrm{dd^c}\Phi \wedge \mathbb{T}_+) + \langle S_n, K\rangle - \langle \mathbb{T}_-, K\rangle \end{aligned} \tag{3.11}$$

where K is a smooth closed (k, k)-form such that $\Phi\mathbb{T}_+ - K$ is a normalized potential of $\mathrm{dd^c}\Phi \wedge \mathbb{T}_+$.

By Lemma 3.5, we may assume $\|\mathrm{dd^c}\Phi \wedge \mathbb{T}_+\|_* \leq 1$. Since $\|G_j\|_{\mathscr{C}^2} \lesssim \varepsilon^{-3}$ by (3.2), we have

$$\|\Phi\|_{\mathscr{C}^2} \lesssim_\kappa \varepsilon^{-3(\kappa+1)} \prod_{j=0}^{k} C^{l_j} \lesssim \varepsilon^{-3(\kappa+1)} C^{(\kappa+1)n_\kappa},$$

where $C = \max\{\|F\|_{\mathscr{C}^2}, 1\}$. Then we deduce from [19, Proposition 3.4.2] that $\mathrm{dd^c}\Phi \wedge \mathbb{T}_+$ has a $\left(2, \lambda, M_\kappa C^{(\kappa+1)n_\kappa} \varepsilon^{-3(\kappa+1)}\right)$-Hölder-continuous super-potential for some $M_\kappa > 0$ and

$0 < \lambda \leq 1$. Thus, applying [25, Proposition 2.4] with $M_\kappa C^{(\kappa+1)n_\kappa} \varepsilon^{-3(\kappa+1)}$ instead of ε^{-2}, yields

$$\left| \mathscr{U}_{S_n}(\mathrm{dd^c}\Phi \wedge \mathbb{T}_+) - \mathscr{U}_{\mathbb{T}_-}(\mathrm{dd^c}\Phi \wedge \mathbb{T}_+) \right| \lesssim_\kappa -n_\kappa (\delta_0/d_p)^n \log \varepsilon. \tag{3.12}$$

Note that $\|\Phi\|_\infty \lesssim_\kappa 1$. From the same proof of [25, Lemma 3.1], we get

$$\left| \langle S_n, K \rangle - \langle \mathbb{T}_-, K \rangle \right| \lesssim (\delta_0/d_p)^n. \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), we conclude

$$|\langle S_n \wedge \mathbb{T}_+, \Phi \rangle - \langle \mathbb{T}_- \wedge \mathbb{T}_+, \Phi \rangle| \lesssim_\kappa -n_\kappa (\delta_0/d_p)^n \log \varepsilon + (\delta_0/d_p)^n \lesssim -n_\kappa (\delta_0/d_p)^n \log \varepsilon.$$

The proof is complete. □

Using the invariance of μ, the inequality of Proposition 3.2 does not change if we replace n_j by $n_j - 1$ for $j \neq j_0$ and g_{j_0} by $g_{j_0} \circ f^{-1}$. Therefore, it is enough to assume that $n_{j_0+1} - n_{j_0}$ is even. We have the following lemma, which we will use to prove Proposition 3.2 by induction. To deal with the additional factor n_κ in Proposition 3.6, we need an inductive step more sophisticated than [30, Lemma 3.5]. Here we make use of our choice of j_0.

Lemma 3.7 *There is a constant $c_\kappa > 0$, independent of $n_1, \ldots, n_\kappa$ and $g_0, \ldots, g_\kappa$, such that*

$$\left| \int \prod_{j=0}^{\kappa} (g_j \circ f^{n_j})\, \mathrm{d}\mu - \int \prod_{j \leq j_0} (g_j \circ f^{n_j - n_{j_0}}) \mathrm{d}\mu \int \prod_{j \geq j_0+1} (g_j \circ f^{n_j - n_{j_0+1}})\, \mathrm{d}\mu \right| \leq c_\kappa \left(\frac{\delta}{d_p} \right)^{\frac{r(n_{j_0+1} - n_{j_0})}{2(r+1)}}.$$

Proof Set $\Psi_L := \prod_{j \leq j_0} (g_j \circ f^{n_j - n_{j_0}})$ and $\Psi_R := \prod_{j \geq j_0+1} (g_j \circ f^{n_j - n_{j_0+1}})$. By the invariance of μ, the statement can be rewritten as

$$\left| \int \Psi_L (\Psi_R \circ f^{n_{j_0+1} - n_{j_0}})\, \mathrm{d}\mu - \int \Psi_L\, \mathrm{d}\mu \int \Psi_R\, \mathrm{d}\mu \right| \leq c_\kappa \left(\frac{\delta}{d_p} \right)^{\frac{r(n_{j_0+1} - n_{j_0})}{2(r+1)}} \tag{3.14}$$

for some $c_\kappa > 0$ independent of $n_1, \ldots, n_\kappa$ and $g_0, \ldots, g_\kappa$. To prove inequality (3.14), we will make use of the function Φ defined in (3.4).

Recall that $\mu = T_+ \wedge T_-$, and that the currents $T_\pm$ and $\mathbb{T}_\pm$ have Hölder-continuous superpotentials. It follows that the intersections of $T_\pm$ and $\mathbb{T}_\pm$ with positive closed currents are well defined. Set $\widetilde{g}_j := g_j \circ f^{n_j - n_{j_0}}$ for $j \leq j_0$ and $\widetilde{g}_j := g_j \circ f^{n_j - n_{j_0+1}}$ for $j \geq j_0 + 1$. Using the invariance of μ, we have

$$\begin{aligned} A :&= \int \Psi_L (\Psi_R \circ f^{n_{j_0+1} - n_{j_0}})\, \mathrm{d}\mu + \int \left((\kappa + 1) \sum_{j=0}^{\kappa} g_j + \frac{\kappa}{2} \sum_{j=0}^{\kappa} g_j^2 \right) \mathrm{d}\mu \\ &= \Bigg\langle T_+ \wedge T_-\,, \sum_{j=0}^{\kappa} \left((\kappa + 1) \widetilde{g}_j \circ f^{(n_{j_0+1} - n_{j_0})/2} + \frac{\kappa}{2} \widetilde{g}_j^2 \circ f^{(n_{j_0+1} - n_{j_0})/2} \right) \\ &\qquad + (\Psi_L \circ f^{-(n_{j_0+1} - n_{j_0})/2})(\Psi_R \circ f^{(n_{j_0+1} - n_{j_0})/2}) \Bigg\rangle \end{aligned}$$

$$= \Big\langle (T_+ \otimes T_-) \wedge [\Delta]\,,\ \sum_{j\le j_0} \Big((\kappa+1)\widetilde{g}_j\big(f^{(n_{j_0+1}-n_{j_0})/2}(w)\big) + \frac{\kappa}{2}\widetilde{g}_j^2\big(f^{(n_{j_0+1}-n_{j_0})/2}(w)\big)\Big)$$

$$+ \sum_{j\ge j_0+1} \Big((\kappa+1)\widetilde{g}_j\big(f^{(n_{j_0+1}-n_{j_0})/2}(z)\big) + \frac{\kappa}{2}\widetilde{g}_j^2\big(f^{(n_{j_0+1}-n_{j_0})/2}(z)\big)\Big)$$

$$+ \Psi_L\big(f^{-(n_{j_0+1}-n_{j_0})/2}(w)\big)\Psi_R\big(f^{(n_{j_0+1}-n_{j_0})/2}(z)\big)\Big\rangle.$$

From the definitions of Ψ_L, Ψ_R and Φ and the fact that $F_*(\mathbb{T}_+) = d_p^{-2}\mathbb{T}_+$, it follows that

$$A = \big\langle \mathbb{T}_+ \wedge [\Delta]\,,\ \Phi \circ F^{(n_{j_0+1}-n_{j_0})/2}\big\rangle = \big\langle \mathbb{T}_+ \wedge [\Delta]\,,\ (F^{(n_{j_0+1}-n_{j_0})/2})^*\Phi\big\rangle$$
$$= \big\langle d_p^{-(n_{j_0+1}-n_{j_0})}\mathbb{T}_+ \wedge (F^{(n_{j_0+1}-n_{j_0})/2})_*[\Delta]\,,\ \Phi\big\rangle.$$

Therefore, we have

$$\int \Psi_L(\Psi_R \circ f^{n_{j_0+1}-n_{j_0}})\,\mathrm{d}\mu + \int \left((\kappa+1)\sum_{j=0}^{\kappa} g_j + \frac{\kappa}{2}\sum_{j=0}^{\kappa} g_j^2\right)\mathrm{d}\mu \tag{3.15}$$
$$= \big\langle d_p^{-(n_{j_0+1}-n_{j_0})}(F^{(n_{j_0+1}-n_{j_0})/2})_*[\Delta] \wedge \mathbb{T}_+\,,\ \Phi\big\rangle.$$

Since $\mu \otimes \mu = \mathbb{T}_+ \wedge \mathbb{T}_- = \mathbb{T}_- \wedge \mathbb{T}_+$, and using also the invariance of μ, we get

$$\int \left((\kappa+1)\sum_{j=0}^{\kappa} g_j + \frac{\kappa}{2}\sum_{j=0}^{\kappa} g_j^2\right)\mathrm{d}\mu + \langle \mu, \Psi_L\rangle\langle \mu, \Psi_R\rangle = \langle \mu \otimes \mu, \Phi\rangle = \big\langle \mathbb{T}_- \wedge \mathbb{T}_+, \Phi\big\rangle. \tag{3.16}$$

Subtracting (3.16) from (3.15) and applying Proposition 3.6 with $S = [\Delta]$, we get

$$\left|\int \Psi_L(\Psi_R \circ f^{n_{j_0+1}-n_{j_0}})\,\mathrm{d}\mu - \int \Psi_L\,\mathrm{d}\mu \int \Psi_R\,\mathrm{d}\mu\right| \lesssim_\kappa -n_\kappa(\delta_0/d_p)^{(n_{j_0+1}-n_{j_0})/2}\log\varepsilon.$$

By our choice of j_0 we have $n_\kappa \le \kappa(n_{j_0+1} - n_{j_0})$. Thus, substituting (3.3), we get

$$-n_\kappa(\delta_0/d_p)^{(n_{j_0+1}-n_{j_0})/2}\log\varepsilon \lesssim_\kappa (n_{j_0+1}-n_{j_0})(\delta_0/d_p)^{\frac{r(n_{j_0+1}-n_{j_0})}{2(r+1)}} \lesssim_\delta (\delta/d_p)^{\frac{r(n_{j_0+1}-n_{j_0})}{2(r+1)}},$$

where the last inequality is valid for every $\delta_0 < \delta < d_p$. This concludes the proof. □

End of the proof of Proposition 3.2 We proceed by induction. The base case $\kappa = 1$ is given by Lemma 3.7. Suppose that the statement holds for l observables for every $1 \le l \le \kappa - 1$. We need to prove that it holds for κ, i.e., that we have

$$\left|\int \prod_{j=0}^{\kappa}(g_j \circ f^{n_j})\,\mathrm{d}\mu - \prod_{j=0}^{\kappa}\int g_j\,\mathrm{d}\mu\right| \lesssim \left(\frac{\delta}{d_p}\right)^{\frac{r}{2(r+1)}\min_{0\le j\le\kappa-1}(n_{j+1}-n_j)}.$$

Recall that we assume that $\|g_j\|_{\mathrm{qpsh}} \le 1$ for every $j \ge 1$. Again by Lemma 3.7, it is enough to show that we have

$$\left|\int \prod_{j\le j_0}(g_j \circ f^{n_j-n_{j_0}})\,\mathrm{d}\mu \int \prod_{j\ge j_0+1}(g_j \circ f^{n_j-n_{j_0+1}})\,\mathrm{d}\mu - \int \prod_{j\le j_0}(g_j \circ f^{n_j-n_{j_0}})\mathrm{d}\mu \prod_{j\ge j_0+1}\int g_j\,\mathrm{d}\mu\right|$$
$$\lesssim \left(\frac{\delta}{d_p}\right)^{\frac{r}{2(r+1)}\min_{j_0+1\le j\le\kappa-1}(n_{j+1}-n_j)}$$

and

$$\left|\int \prod_{j\le j_0} (g_j \circ f^{n_j - n_{j_0}}) \mathrm{d}\mu \prod_{j\ge j_0+1} \int g_j \mathrm{d}\mu - \prod_{j=0}^{\kappa} \int g_j \, \mathrm{d}\mu \right| \lesssim \left(\frac{\delta}{d_p}\right)^{\frac{r}{2(r+1)} \min_{1\le j\le j_0}(n_{j+1}-n_j)}.$$

These inequalities follow from the inductive assumption. The proof is complete. □

3.2 Mixing for all d.s.h. functions

We can now deduce Theorem 1.2 from Proposition 3.1. We use the same arguments as in the proof [30, Theorem 1.2].

Proof of Theorem 1.2 Up to rescaling, we can assume without loss of generality that $\|\varphi_j\|_{\mathrm{DSH}} \le 1$ for every j. Applying Lemma 2.1, and by linearity, we may also assume that we have

$$\varphi_j \le 0, \quad \|\varphi_j\|_{\mathrm{DSH}} \le 1, \quad \text{and} \quad \mathrm{dd^c}\varphi_j \ge -\omega \quad \text{for every } j.$$

Using Lemma 2.6, we can write $\varphi_j = \varphi_{j,1}^{(N)} + \varphi_{j,2}^{(N)}$, where we choose N as

$$N := \frac{r}{2\alpha(r+1)} \min_{0\le j\le \kappa-1} (n_{j+1} - n_j) \log \frac{d_p}{\delta}. \tag{3.17}$$

Observe that if $N = 0$ there is nothing to prove, so we can assume without loss of generality that $N > 0$, i.e., that $\min_{0\le j\le\kappa-1}(n_{j+1} - n_j) \ge 1$. Since N is fixed, we will omit its dependence and write $\varphi_{j,1}^{(N)} = \varphi_{j,1}$ and $\varphi_{j,2}^{(N)} = \varphi_{j,2}$.

Indexing all the possible choices of the v_j's indexes in the φ_{j,v_j}'s with $\mathbf{v} := (v_0, v_1, \dots, v_\kappa) \in \{1,2\}^{\kappa+1}$, as in [30, Section 3.2] we have

$$\begin{aligned}\left|\int \left(\prod_{j=0}^{\kappa} \varphi_j \circ f^{n_j}\right) \mathrm{d}\mu - \prod_{j=0}^{\kappa} \int \varphi_j \, \mathrm{d}\mu \right| &\le \left|\int \left(\prod_{j=0}^{\kappa} \varphi_{j,1} \circ f^{n_j}\right) \mathrm{d}\mu - \prod_{j=0}^{\kappa} \int \varphi_{j,1} \, \mathrm{d}\mu \right| \\ &+ \sum_{\mathbf{v}\ne(1,\dots,1)} \left(\left| \int \left(\prod_{j=0}^{\kappa} \varphi_{j,v_j} \circ f^{n_j}\right) \mathrm{d}\mu \right| + \left| \prod_{j=0}^{\kappa} \int \varphi_{j,v_j} \, \mathrm{d}\mu \right| \right).\end{aligned}$$

To estimate the right hand side of the last expression, we treat two terms separately.

Case $\mathbf{v} = (1,\dots,1)$. Since all the $\varphi_{j,1}$'s are quasi-p.s.h. and from Lemma 2.6 we have $\|\varphi_{j,1}\|_{\mathrm{qpsh}} \le N + 1 \lesssim_{\alpha,d_p,\delta} N$ for every j, we can apply Proposition 3.2 to get

$$\begin{aligned}&\left|\int \varphi_{0,1}(\varphi_{1,1} \circ f^{n_1}) \cdots (\varphi_{\kappa,1} \circ f^{n_\kappa}) \, \mathrm{d}\mu - \prod_{j=0}^{\kappa} \int \varphi_{j,1} \, \mathrm{d}\mu \right| \\ &\le C_\kappa \left(\frac{\delta}{d_p}\right)^{\frac{r}{2(r+1)} \min_{0\le j\le\kappa-1}(n_{j+1}-n_j)} \prod_{j=0}^{\kappa} \|\varphi_{j,1}\|_{\mathrm{qpsh}} \\ &\lesssim_{\alpha,d_p,\delta,\kappa} C_\kappa \left(\frac{\delta}{d_p}\right)^{\frac{r}{2(r+1)} \min_{0\le j\le\kappa-1}(n_{j+1}-n_j)} N^{\kappa+1}.\end{aligned}$$

Case $\mathbf{v} \ne (1,\dots,1)$. Using the estimates on the $\varphi_{j,1}$'s and $\varphi_{j,2}$'s given by Lemma 2.6, and by Hölder's inequality, each of these terms is bounded by $N^\kappa e^{-\alpha N}$, up to a multiplicative constant depending only on κ.

Combining the estimates above, and inserting the value of N as in (3.17), we obtain a constant $\tilde{C}_{\delta,\kappa} > 0$ independent of $n_1, \dots, n_\kappa$ such that

$$\left| \int \varphi_0(\varphi_1 \circ f^{n_1}) \cdots (\varphi_\kappa \circ f^{n_\kappa})\, \mathrm{d}\mu - \prod_{j=0}^{\kappa} \int \varphi_j \, \mathrm{d}\mu \right| \\ \leq \tilde{C}_{\delta,\kappa} \min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)^{\kappa+1} \left(\frac{\delta}{d_p} \right)^{\frac{r}{2(r+1)} \min_{0 \leq j \leq \kappa-1}(n_{j+1}-n_j)}.$$

Thus, the estimate in Theorem 1.2 holds up to choosing a slightly worse δ and corresponding constant $C_{\delta,\kappa} \geq \tilde{C}_{\delta,\kappa}$. This concludes the proof. □

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