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RESEARCH ARTICLE

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Integral Picard group of moduli of polarized K3 surfaces

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Abstract

We compute the integral Picard group of the moduli stack of polarized K3 surfaces of fixed degree whose singularities are at most rational double points, and of its coarse moduli space. We also compute the integral Picard group of the stack of quasi-polarized K3 surfaces, and of the stacky period domain.

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INTRODUCTION

A very interesting invariant of a moduli stack is its Picard group. It was introduced by Mumford in [22], where he also computed the Picard group of the moduli stack of elliptic curves. This calculation prompted a great amount of research in this topic, which eventually leaded to a complete understanding of the Picard group of the moduli stack of curves over fields of almost every characteristic (see [1, 10, 14, 16, 26]). In particular, knowing the Picard group of a Deligne–Mumford stack with finite inertia also gives a description of the rational Picard group of the coarse moduli space. Integral Picard group of other interesting moduli stacks has also been computed in recent years [3, 7] and [11].

Another quite relevant moduli stack is the moduli stack of polarized K3 surfaces. In particular, the rational Picard group of the moduli space M_d of (primitively) polarized K3 surfaces of degree d with at most rational double points has been the subject of much research [6, 21], eventually culminated in the proof of the so-called Noether–Lefschetz conjecture [5], from which one can deduce the rank of $\mathrm{Pic}(M_d) \otimes \mathbb{Q}$. On the other hand, not much is known on the integral Picard group of the associated moduli stack $\mathscr{M}_d$. In this paper, we prove the following (we work over $\mathbb{C}$).

Theorem (Theorem 2.2). *Let $\mathscr{M}_d$ be the moduli stack of primitively polarized K3 surfaces of degree d with at most rational double points. Then, we have*

$$\mathrm{Pic}(\mathscr{M}_d) \simeq \mathbb{Z}^{\rho(d)},$$

where $\rho(d)$ is the rank of $\mathrm{Pic}(M_d) \otimes \mathbb{Q}$ computed in [6].

Furthermore, we prove that the integral Picard group of the moduli space M_d is torsion free.

Theorem (Corollary 2.10). *Let M_d be the moduli space of primitively polarized K3 surfaces of degree d with at most rational double points. Then, we have*

$$\mathrm{Pic}(M_d) \cong \mathbb{Z}^{\rho(d)}.$$

There are other two stacks that are closely related to $\mathscr{M}_d$, namely, the stack $\mathscr{K}_d$ of primitively quasi-polarized K3 surfaces, and the stacky period domain $\mathscr{P}_d$. At the level of schemes, the differences between these stacks do not appear (indeed, $\mathscr{P}_d$ and $\mathscr{M}_d$ have the same coarse moduli space), but as stacks they are all nonisomorphic. Therefore, it makes sense to also ask what their integral Picard groups are. We give an answer in the following.

Theorem (Theorem 3.2, Theorem 3.3, and Theorem 3.4). *The following hold true:*

(1) *As an abstract group,* $\mathrm{Pic}(\mathscr{P}_d) \simeq \mathbb{Z}^{\rho(d)} \oplus \mathbb{Z}/2$.
(2) *The morphism $\mathscr{K}_d \to \mathscr{P}_d$ induces an isomorphism* $\mathrm{Pic}(\mathscr{P}_d) \simeq \mathrm{Pic}(\mathscr{K}_d)$.
(3) *Suppose that* $\frac{d}{2} \not\equiv 1 \pmod 4$*: then we have a split short exact sequence*

$$0 \longrightarrow \mathrm{Pic}(\mathscr{M}_d) \longrightarrow \mathrm{Pic}(\mathscr{P}_d) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

(4) *Suppose* $\frac{d}{2} \equiv 1 \pmod 4$*: then we have a nonsplit short exact sequence*

$$0 \longrightarrow \mathrm{Pic}(\mathscr{M}_d) \times \mathbb{Z}/2 \longrightarrow \mathrm{Pic}(\mathscr{P}_d) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

The generator of the torsion part in the Picard groups above is made explicit in the paper.

Notice that our proof does not give any hint as to what the generators of $\mathrm{Pic}(\mathscr{M}_d)$ are. For $d \leqslant 8$, this is worked out in [9], but for higher values of d, the problem is wide open.

Structure of the paper

The paper is organized as follows. In Section 1, after introducing the moduli stacks we are interested in and after discussing some of their properties, we first show that there exists a morphism $\mathscr{P}_d \to \mathscr{M}_d$ from the stacky period domain (Lemma 1.12) to the stack of polarized K3 surfaces with rational double points, and we show that it induces an injection of Picard groups.

Then, in Section 2, we compute the torsion part of $\mathrm{Pic}(\mathscr{P}_d)$ by looking at the fundamental group of this stack (Proposition 2.4), and then, we prove that the torsion line bundle on $\mathscr{P}_d$ does not come from $\mathscr{M}_d$ (Lemma 2.5).

After proving that the Picard group of $\mathscr{M}_d$ is finitely generated, we obtain the desired conclusion. We then leverage the result just obtained to compute in Section 3 the Picard groups of $\mathscr{K}_d$ and $\mathscr{P}_d$ by means of certain localization exact sequences (Theorem 3.3).

Assumptions

In what follows, we always work over $\mathbb{C}$.

1 | SOME MODULI STACKS OF K3 SURFACES

1.1 | Overview of the section

In this section, we introduce three different stacks, all of which in a sense parameterize polarized K3 surfaces of a fixed degree.

1.2 | The stack of primitively quasi-polarized K3 surfaces

Let $\mathscr{K}_d$ be the stack of primitively quasi-polarized K3 surfaces of degree d. That is, the objects of $\mathscr{K}_d$ over a scheme S are pairs $(X \to S, L)$, where:

- $X \to S$ is a proper, finitely presented and flat morphism whose geometric fibers are smooth K3 surfaces;
- L is a section of $\underline{\mathrm{Pic}}_{X/S} \to S$ that on the geometric fibers is represented by a primitive, numerically effective line bundle of degree d; we also require that if $\langle L_s, C_s \rangle = 0$ for a curve $C_s \subset X_s$, where s is a geometric point of S, then $(C_s^2) = -2$.

The morphisms in $\mathscr{K}_d$ are given by S-isomorphisms $f : X \xrightarrow{\simeq} X'$ such that $f^*L' = L$. The fibred category $\mathscr{K}_d$ is a smooth Deligne–Mumford stack [24, (1.2.1), (1.2.2)] (note that in *loc. cit.* the stack $\mathscr{K}_d$ is denoted as $\mathbb{M}_d^{\mathrm{sm}}$). From now on, we will refer to the objects of $\mathscr{K}_d$ as quasi-polarized K3 surfaces instead of primitively quasi-polarized K3 surfaces.

1.3 | The stack of primitively polarized K3 surfaces of degree d with at most rational double points

Let $\mathscr{M}_d$ denote the stack of primitively polarized K3 surfaces of degree d with at most rational double points. That is, the objects of $\mathscr{M}_d$ over a scheme S are pairs $(X \to S, L)$ where:

- $X \to S$ is a proper, finitely presented and flat morphism whose geometric fibers are K3 surfaces with at most rational double points;
- L is a section of $\underline{\mathrm{Pic}}_{X/S} \to S$ that on the geometric fibers is represented by an ample, primitive line bundle of degree d.

The morphisms in $\mathscr{M}_d$ are given by S-isomorphisms $f : X \xrightarrow{\simeq} X'$ such that $f^*L' = L$. The fibred category $\mathscr{M}_d$ is a smooth Deligne–Mumford stack with a coarse moduli space, which we denote M_d [18, 84] (note that in *loc. cit.*, the stack $\mathscr{M}_d$ is the one denoted as $\bar{\mathscr{M}}_d$).

1.4 | Lattice theory of K3 surfaces

Let Λ denote the lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. Given a smooth K3 surface X, this lattice arises as $H^2(X, \mathbb{Z})$ together with the cohomological pairing.

Let ℓ be the element in Λ defined as $e + \frac{d}{2} f$, where e and f form a basis for the first copy of U. Then, we denote $\Lambda_d := \ell^{\perp}$ the sublattice of Λ orthogonal to ℓ. Given a smooth K3 surface X with a primitive quasi-polarization L of degree d, then Λ_d arises as the orthogonal of $c_1(L)$ in $H^2(X, \mathbb{Z})$.

1.5 | Period domains

Let Ω_d be the period domain of (primitively) quasi-polarized K3 surfaces of degree d, that is,

$$\Omega_d := \left\{ \omega \in \mathbf{P}(\Lambda_d \otimes \mathbf{C}) \text{ such that} \langle \omega, \omega \rangle > 0, \langle \omega, \overline{\omega} \rangle > 0 \right\}.$$

This is a complex manifold, it has two connected components [19, (1.2)]. Let D_d denote one connected component. Then D_d is a bounded symmetric domain of type IV, hence simply connected [19, (1.2)].

1.6 | Orthogonal transformations

Let $O(\Lambda_d)$ be the group of orthogonal transformations of Λ_d. Set

$$\widetilde{O}(\Lambda_d) = \ker(O(\Lambda_d) \longrightarrow O(\Lambda_d^{\vee}/\Lambda_d)),$$

where $\Lambda_d^{\vee}$ is the lattice formed by $x \in \Lambda_d \otimes \mathbf{Q}$ such that $\langle x, \Lambda \rangle \subset \mathbb{Z}$. Let $O^+(\Lambda_d)$ be the subgroup of orthogonal transformations having positive spinor norm. We set $\Gamma_d := \widetilde{O}(\Lambda_d) \cap O^+(\Lambda_d)$; this arithmetic group can be regarded as the group of orthogonal transformations of Λ that fix ℓ, and it acts on the connected component D_d with a properly discontinuous action [19, (1.2)], [15, p. 16].

1.7 | The period stack

We define the analytic quotient stack

$$\mathscr{P}_d := [\Gamma_d \diagdown D_d],$$

and we refer to this stack as the *period stack*. It is actually a smooth Deligne–Mumford stack [4, Theorem 10.11].

1.8 | The period map

There is a morphism of stacks

$$p_d : \mathscr{K}_d \longrightarrow \mathscr{P}_d,$$

which is the stacky version of the usual period map. Indeed, given a quasi-polarized K3 surface $(\pi : X \to S, L)$, consider the associated analytic morphism $\pi^{an} : X^{an} \to S^{an}$ and the family of lattices $R^2\pi_*^{an}\mathbb{Z}$ together with the cohomological pairing. The quasi-polarization L defines a section of $R^2\pi_*^{an}\mathbb{Z}$.

We can use this object to define a Γ_d-torsor $U_d \to \mathscr{K}_d$: its objects are triples $(\pi : X \to S, L, \alpha)$, where $\alpha : (R^2\pi_*^{an}\mathbb{Z}, L) \simeq (\Lambda, \ell)$ is an isomorphism of lattices that sends L to ℓ (the marking of the K3 surface). We can then construct a Γ_d-equivariant morphism $U_d \to D_d$ by sending a triple $(\pi : X \to S, L, \alpha)$ to the line subbundle $\alpha(\pi_*\Omega) \subset \Lambda_d \otimes \mathscr{O}_S$. The resulting morphism $p_d : \mathscr{K}_d \to [\Gamma_d \diagdown D_d]$ is étale and representable [13, (1.2)].

Étaleness can also be verified directly by proving that for every geometric point $x \in \mathscr{K}_d$ and $y = p_d(x) \in \mathscr{P}_d$, the induced homomorphism of complete rings $\widehat{\mathscr{O}}_{\mathscr{K}_d,x} \to \widehat{\mathscr{O}}_{\mathscr{P}_d,y}$ is an isomorphism.

As explained in [20, Proof of 5.8], this blows down to verify that the induced morphism of tangent spaces is an isomorphism. If the point x corresponds to a quasi-polarized K3 surface (X, L), its tangent space corresponds to the deformation space $\mathrm{Ext}^1(\Omega_X, L^\vee)$, which is isomorphic to the subspace of primitive classes in $H^{1,1}(X)$.

Given an isomorphism $\alpha : (H^2(X, \mathbb{Z}), L) \simeq (\Lambda, \ell)$, we have an induced identification of $H^{1,1}(X)_{\mathrm{prim}}$ with the subspace orthogonal to the linear span of $\{\omega, \overline{\omega}\}$ in $\Lambda_d \otimes \mathbb{C}$, where ω is any class in $\alpha_{\mathbb{C}}(H^{2,0}(X))$. The latter is exactly the tangent space of D_d at $[\omega]$, which is isomorphic to the tangent space of $\mathscr{P}_d$ at y, as $D_d \to \mathscr{P}_d$ is étale.

Remark 1.9. The period map p_d is not an isomorphism. Indeed, the induced map of automorphism groups is not always surjective: consider a quasi-polarized K3 surface (X, L) whose quasi-polarization *is not* a polarization. Then, in $H^{1,1}(X)$, there is an element δ which is the class of a (-2)-curve. The automorphism of $H^2(X, \mathbb{Z})$ given by the reflection with respect to δ defines then an automorphism of $p_d(X)$ that does not come from an automorphism of (X, L) [13, (1.2)]

1.10 | The contraction map

There is a morphism

$$\varphi_d : \mathscr{K}_d \longrightarrow \mathscr{M}_d,$$

which sends a quasi-polarized K3 surface $(X \to S, L)$ to the image of X via the map associated to the linear system $|L^{\otimes N}|$ for $N \geqslant 3$. The image is a polarized K3 surface with at most rational double points. The rational double points arise because of the (-2)-curves that get contracted by the polarization.

1.11 | Relation between the period map and the contraction map

Call $\mathscr{M}_d^{\mathrm{sm}}$ the open substack of $\mathscr{M}_d$ corresponding to smooth surfaces. The complement $\mathscr{M}_d^{\mathrm{sing}}$ of $\mathscr{M}_d^{\mathrm{sm}}$, with the reduced scheme structure, is a closed substack and a divisor. Set

$$\mathscr{M}_d^{\mathrm{sm}} := \mathscr{M}_d \setminus \mathscr{M}_d^{\mathrm{sing}}, \quad \mathscr{K}_d^{\mathrm{sm}} := \varphi_d^{-1}(\mathscr{M}_d^{\mathrm{sm}}).$$

Let $D_d^{\mathrm{sm}} \subset D_d$ be the open subset of D_d formed by those $[\omega]$ such that the in the sublattice $\omega^\perp \cap \overline{\omega}^\perp \subset \Lambda_d$, there are no elements δ such that $\delta^2 = (-2)$. This open subset is Γ_d-invariant; hence, we can define

$$\mathscr{P}_d^{\mathrm{sm}} := [\Gamma_d \diagdown D_d^{\mathrm{sm}}].$$

Lemma 1.12. *There exists a factorization*

$$\begin{tikzcd} \mathscr{K}_d \arrow[r, "p_d"] \arrow[rr, bend left, "\varphi_d"] & \mathscr{P}_d \arrow[r, "\psi_d"] & \mathscr{M}_d \end{tikzcd}$$

Proof. We want to show that φ_d descends along p_d. The latter is étale, so all we have to do is to check that there is an isomorphism $\mathrm{pr}_1^* \varphi_d \simeq \mathrm{pr}_2^* \varphi_d$ on $\mathscr{K}_d \times_{\mathscr{P}_d} \mathscr{K}_d$, where the pr_i denote the two projections.

As $\mathscr{M}_d$ is separated, if $\mathrm{pr}_1^* \varphi_d$ is isomorphic to $\mathrm{pr}_2^* \varphi_d$ on the generic point, they are isomorphic everywhere. Therefore, in order to conclude is enough to observe that $\mathscr{K}_d^{\mathrm{sm}} \to \mathscr{M}_d^{\mathrm{sm}}$ and $\mathscr{K}_d^{\mathrm{sm}} \to \mathscr{P}_d^{\mathrm{sm}}$ are both isomorphisms, so we have that φ_d descends to $\varphi_d \circ p_d^{-1}$ along $\mathscr{K}_d^{\mathrm{sm}} \to \mathscr{P}_d^{\mathrm{sm}}$.

This implies that there is an isomorphism $\mathrm{pr}_1^* \varphi_d \simeq \mathrm{pr}_2^* \varphi_d$ on the generic point. □

Denote by $\mathscr{M}_d^{\mathrm{ss}}$ the open subset of surfaces with a single A_1-singularity. Since the deformation theory of a K3 surface with rational double points is unobstructed, and the map from the deformation space of the surface to that of the singularities is smooth, we have that $\mathscr{M}_d^{\mathrm{sing}}$ is a reduced divisor, and $\mathscr{M}_d^{\mathrm{ss}}$ is a dense open substack contained in the smooth locus of $\mathscr{M}_d^{\mathrm{sing}}$.

Lemma 1.13. *The divisor $\mathscr{M}_d^{\mathrm{sing}}$ has two irreducible components if $\frac{d}{2} \equiv 1 \pmod 4$, and is irreducible otherwise.*

Proof. First, we prove a similar statement for $\mathscr{P}_d^{\mathrm{sing}}$: indeed, if we look at the action of Γ_d on the set of generic points of the Γ_d-invariant divisor $D_d^{\mathrm{sing}} \subset D_d$, we see that when $\frac{d}{2} \equiv 1 \pmod 4$, this set is made up of two orbits, and is made up of one orbit otherwise [8, Proposition 2.11]. This implies that the substack $\mathscr{P}_d^{\mathrm{sing}}$ has either two or one irreducible components.

To conclude, observe now that $\mathscr{P}_d^{\mathrm{sing}}$ and $\mathscr{M}_d^{\mathrm{sing}}$ share the same coarse space, and hence, they must have the same number of irreducible components. □

2 | COMPUTATION OF THE PICARD GROUP OF $\mathscr{M}_d$

2.1 | Overview of the section

In this section, we compute the Picard group of $\mathscr{M}_d$, the stack of polarized K3 surfaces of degree d with at most rational double points. Let $\rho(d)$ be the rank of the rational Picard group of M_d [5, Corollary 1.3]. Then, the main result of this section is the following.

Theorem 2.2. *We have*

$$\mathrm{Pic}(\mathscr{M}_d) \simeq \mathbb{Z}^{\rho(d)}.$$

2.3 | Proof of the first main theorem

By definition, $D_d \to \mathscr{P}_d$ is a Γ_d-torsor. We can use it to define an analytic line bundle on $\mathscr{P}_d$ as follows: take $D_d \times \mathbf{A}^1$ and let Γ_d acts diagonally, where the action on $\mathbf{A}^1$ is given by $A \cdot \lambda := \det(A)\lambda$. The resulting quotient $\mathscr{L}_d := [\Gamma_d \backslash D_d \times \mathbf{A}^1]$ is a line bundle over $\mathscr{P}_d$, which is not trivial because the determinant of an element in Γ_d is not trivial in general. Observe also that $\mathscr{L}_d^{\otimes 2} \simeq \mathscr{O}_{\mathscr{P}_d}$, because the determinant of an element in Γ_d is a square root of the unity.

Proposition 2.4. *The analytic line bundle $\mathscr{L}_d$ is algebraic, and we have*

$$\mathrm{Pic}(\mathscr{P}_d)[n] \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \cdot [\mathscr{L}_d] & \text{ifn is even,} \\ 0 & \text{ifn is odd,} \end{cases}$$

where $\mathrm{Pic}(\mathscr{P}_d)[n]$ *denotes the* $\mathbb{Z}$*-submodule of elements annihilated by* n*.*

Proof. Let μ_n denote the group of n-roots of unity. Then in the étale topology, we have a short exact sequence of sheaves

$$0 \longrightarrow \mu_n \longrightarrow \mathscr{O}^* \xrightarrow{(-)^n} \mathscr{O}^* \longrightarrow 0,$$

where the morphism $\mathscr{O}^* \to \mathscr{O}^*$ sends x to x^n. By looking at the induced long exact sequence in étale cohomology, we have

$$H^0(\mathscr{P}_d, \mathscr{O}^*) \to H^0(\mathscr{P}_d, \mathscr{O}^*) \to H^1(\mathscr{P}_d, \mu_n) \to H^1(\mathscr{P}_d, \mathscr{O}^*) \to H^1(\mathscr{P}_d, \mathscr{O}^*). \tag{2.1}$$

Observe that $H^0(\mathscr{P}_d, \mathscr{O}^*) = \mathbb{C}^*$. Indeed, consider the coarse moduli space $\pi : \mathscr{P}_d \to \mathscr{M}_d \to M_d$, and its Baily–Borel compactification $\overline{M}_d$: the latter is a normal projective variety [4, Theorem 10.11], and $\overline{M}_d \setminus M_d$ has codimension > 2. This implies that $\mathscr{O}(M_d) \simeq \mathscr{O}(\overline{M}_d) = \mathbb{C}$; as M_d is a coarse space for $\mathscr{P}_d$, we have $\pi_* \mathscr{O}_{\mathscr{P}_d} \simeq \mathscr{O}_{M_d}$, from which our claim follows.

In particular, the first arrow in (2.1) is surjective because $\mathbb{C}$ is algebraically closed. As $H^1(\mathscr{P}_d, \mathscr{O}^*) \simeq \mathrm{Pic}(\mathscr{P}_d)$, we deduce that $\mathrm{Pic}(\mathscr{P}_d)[n] \simeq H^1(\mathscr{P}_d, \mu_n)$. The latter group classifies cyclic covers of $\mathscr{P}_d$, which are also classified by surjective homomorphisms $\pi_1(\mathscr{P}_d) \to \mathbb{Z}/n\mathbb{Z}$.

As D_d is simply connected, we deduce that $\pi_1(\mathscr{P}_d) \simeq \Gamma_d$. Any morphism $\Gamma_d \to \mathbb{Z}/n\mathbb{Z}$ factors through the abelianization of Γ_d, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [15, Theorem 1.7]. From this, we deduce that $\mathrm{Pic}(\mathscr{P}_d)[n]$ is trivial if n is odd, and isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if n is even.

Let $\mathscr{P}_d(\mathbb{C})$ denote the analytic stack associated to $\mathscr{P}_d$, and let $\mathscr{O}_{an}^*$ denote the sheaf of invertible holomorphic functions on $\mathscr{P}_d(\mathbb{C})$. Then, we have a commutative diagram

$$\begin{tikzcd} H^1(\mathscr{P}_d, \mu_2) \arrow[r] \arrow[d] & H^1(\mathscr{P}_d, \mathscr{O}^*) \arrow[d] \\ H^1_{cl}(\mathscr{P}_d(\mathbb{C}), \mu_2) \arrow[r] & H^1_{cl}(\mathscr{P}_d(\mathbb{C}), \mathscr{O}^*_{an}) \end{tikzcd}$$

where $H_{cl}(-,-)$ on the bottom row are the sheaf cohomology groups with respect to the analytic topology. By Artin's comparison theorem [2, Exposé XI, Theorem 4.4.(iii)], the first vertical arrow is an isomorphism: this implies that $\mathscr{L}_d$ is in the image of $H^1(\mathscr{P}_d, \mu_2) \to H^1_{cl}(\mathscr{P}_d(\mathbb{C}), \mathscr{O}^*_{an})$, from which we deduce that it is in the image of the right vertical arrow, that is, $\mathscr{L}_d$ comes from an algebraic line bundle, which has to be unique because of our previous computation. □

Lemma 2.5. *The line bundle $\mathscr{L}_d$ on $\mathscr{P}_d$ does not descend to a line bundle on $\mathscr{M}_d$.*

Proof. The stabilizer of a generic point of $\mathscr{P}_d^{\mathrm{sing}}$ is μ_2, generated by the automorphism given by the reflection σ with respect to the unique (up to scalar) element $\delta \in \Lambda_d$ with $\delta^2 = (-2)$. This automorphism does not come from an automorphism of the associated singular K3 surface [13, Remark 1.3]. Therefore, if we show that σ acts nontrivially on a generic fiber of $\mathscr{L}_d|_{\mathscr{P}_d^{\mathrm{sing}}}$, we can conclude that $\mathscr{L}_d$ does not come from $\mathscr{M}_d$.

Recall that $\mathscr{L}_d$ is constructed using the determinant representation of Γ_d: then it follows that σ_d acts via the determinant on a generic fiber of $\mathscr{L}_d|_{\mathscr{P}_d^{\mathrm{sing}}}$, and $\det(\sigma) = -1$; we deduce that the action of σ_d is not trivial, and thus, the lemma is proved. □

Lemma 2.6. *We have*

$$\varphi_{d*}\mathscr{O}_{\mathscr{K}_d} = \mathscr{O}_{\mathscr{M}_d}, \quad p_{d*}\mathscr{O}_{\mathscr{K}_d} = \mathscr{O}_{\mathscr{P}_d}.$$

Proof. The first statement follows from the fact that ψ_d is proper and birational, and $\mathscr{M}_d$ is smooth.

For the second, the point is that p_d is representable, étale, surjective and birational. Let $U \to \mathscr{P}_d$ be an étale map, where U is a scheme. Set $V \overset{\mathrm{def}}{=} U \times_{\mathscr{P}_d} \mathscr{K}_d$; then V is an algebraic space, the map $V \to U$ is étale and a homeomorphism; we need to show that the induced homomorphism $\mathscr{O}(U) \to \mathscr{O}(V)$ is an isomorphism.

If U^{sing} and V^{sing} are the inverse images of $\mathscr{P}^{\mathrm{sing}}$ in U and V, respectively, and set $U' \overset{\mathrm{def}}{=} U \setminus U^{\mathrm{sing}}$ and $V' \overset{\mathrm{def}}{=} V \setminus V^{\mathrm{sing}}$. Then, the restriction $V' \to U'$ of the projection $V \to U$ is an isomorphism; hence, $\mathscr{O}(U') \to \mathscr{O}(V')$ is an isomorphism. But $\mathscr{O}(U)$ is the subring of $\mathscr{O}(U')$ of function without poles on U^{sing}, and analogously, $\mathscr{O}(V$ is the subring of $\mathscr{O}(V')$ of function without poles on V^{sing}. Since $V \to U$ is étale and surjective, a function in $\mathscr{O}(U')$ has poles on U^{sing} if and only if its pullback to V' has poles along V^{sing}; this completes the proof. □

Proposition 2.7. *The pullback homomorphism $\psi_d^* : \mathrm{Pic}(\mathscr{M}_d) \to \mathrm{Pic}(\mathscr{P}_d)$ is injective.*

Proof. Lemma 2.6 implies that $\mathscr{O}_{\mathscr{M}_d} \to \psi_{d*}\mathscr{O}_{\mathscr{P}_d}$ is an isomorphism. Therefore, given a line bundle $\mathscr{L}$ such that $\psi_d^*\mathscr{L} \simeq \mathscr{O}_{\mathscr{P}_d}$, applying the projection formula, we have

$$\mathscr{L} \simeq \mathscr{L} \otimes \mathscr{O}_{\mathscr{M}_d} \simeq \mathscr{L} \otimes \psi_{d*}\mathscr{O}_{\mathscr{P}_d} \simeq \psi_{d*}(\psi_d^*\mathscr{L} \otimes \mathscr{O}_{\mathscr{P}_d}) \simeq \psi_{d*}\mathscr{O}_{\mathscr{P}_d} \simeq \mathscr{O}_{\mathscr{M}_d}.$$ □

Corollary 2.8. *The Picard group of $\mathscr{M}_d$ is torsion free.*

Proof. By Proposition 2.7, the pullback of any nontrivial torsion line bundle on $\mathscr{M}_d$ is a nontrivial torsion line bundle on $\mathscr{P}_d$. The only nontrivial torsion line bundle on $\mathscr{P}_d$ is $\mathscr{L}_d$, which by Lemma 2.5 does not come from $\mathscr{M}_d$; thus, there are no nontrivial torsion line bundles on $\mathscr{M}_d$. □

Lemma 2.9. *The Picard group of $\mathscr{M}_d$ is finitely generated.*

Proof. Applying excision to the pair $\mathscr{M}_d^{\mathrm{sing}} \subset \mathscr{M}_d$, we see that there is an exact sequence

$$\mathbb{Z}^{\oplus e} \longrightarrow \operatorname{Pic}(\mathscr{M}_d) \longrightarrow \operatorname{Pic}(\mathscr{M}_d^{\mathrm{sm}}) \longrightarrow 0,$$

where e is the number of irreducible components of $\mathscr{M}_d^{\mathrm{sing}}$, which is either 1 or 2 by Lemma 1.13. Therefore, is enough to prove that $\operatorname{Pic}(\mathscr{M}_d^{\mathrm{sm}})$ is finitely generated.

We have an isomorphism $\mathscr{M}_d^{\mathrm{sm}} \simeq [H_d/\mathrm{PGL}_n]$, where H_d is a smooth quasi-projective variety [18, Example 4.5]. It always exists a PGL_n-representation V such that PGL_n acts freely on an open subset $U \subset V$ whose complement has codimension $\geqslant 2$ [12, Lemma 9]. Consider then $X_d := [H_d \times U/\mathrm{PGL}_d]$: we claim that (1) X_d is a scheme and (2) $\operatorname{Pic}(X_d) \simeq \operatorname{Pic}(\mathscr{M}_d^{\mathrm{sm}})$.

Claim (1) can be proved as follows: the quotient stack $[U/\mathrm{PGL}_n]$ is actually a scheme [12, Lemma 9], the group PGL_n is connected and H_d is smooth, hence normal; from this, it follows that $[H_d \times U/\mathrm{PGL}_n]$ is a (smooth) scheme [12, Proposition 23.(2)].

Claim (2) follows from the fact that $\mathscr{M}_d^{\mathrm{sm}} \simeq [H_d/\mathrm{PGL}_n]$ is a smooth quotient stack; hence, we can identify its Picard group with its equivariant Picard group $\operatorname{Pic}^{\mathrm{PGL}_n}(H_d)$ [12, Proposition 18]. By homotopy invariance of equivariant Picard groups, we have $\operatorname{Pic}^{\mathrm{PGL}_n}(H_d) \simeq \operatorname{Pic}^{\mathrm{PGL}_n}(H_d \times V)$ [12, Lemma 2.(b)]. As the complement of $H_d \times U$ in $H_d \times V$ has codimension $\geqslant 2$, by excision, we deduce $\operatorname{Pic}^{\mathrm{PGL}_n}(H_d \times V) \simeq \operatorname{Pic}^{\mathrm{PGL}_n}(H_d \times U)$, and the latter is isomorphic to $\operatorname{Pic}([H_d \times U/\mathrm{PGL}_n]) \simeq \operatorname{Pic}(X_d)$.

This implies that $\operatorname{Pic}(\mathscr{M}_d) \simeq \operatorname{Pic}(X_d)$, so we reduce to proving the lemma in the case of a smooth quasi-projective variety.

There exists a smooth compactification $Y_d \supset X_d$ ([23] and [17]), so if we prove that $\operatorname{Pic}(Y_d)$ is finitely generated, we are done.

For this, observe that $\underline{\operatorname{Pic}}_{Y_d}$ is an abelian group scheme over $\mathbb{C}$, and we claim that $\underline{\operatorname{Pic}}^0_{Y_d}$ is an abelian variety with finitely many torsion points: if this is the case, then we can conclude that it is trivial.

To see that $\underline{\operatorname{Pic}}^0_{Y_d}$ has finitely many torsion points, consider the open embedding $X_d \hookrightarrow Y_d$ and the induced pullback homomorphism of groups $\operatorname{Pic}^0(Y_d) \to \operatorname{Pic}^0(X_d)$: if we prove that the latter has finitely many torsion points, we are done, because the complement of X_d in Y_d is made of finitely many divisors. But we just proved that $\operatorname{Pic}(X_d) \simeq \operatorname{Pic}(\mathscr{M}_d^{\mathrm{sm}})$ that is torsion free; hence, our claim holds true.

The claim implies that $\operatorname{Pic}(Y_d)$ injects into the Neron–Severi group, which is finitely generated. This concludes the proof. □

Proof of Theorem 2.2. By Lemma 2.9 and Corollary 2.8, we know that $\operatorname{Pic}(\mathscr{M}_d)$ is a finitely generated, torsion-free abelian group. Its rank is equal to the rank of $\operatorname{Pic}(\mathscr{M}_d) \otimes \mathbf{Q}$, and the latter group is isomorphic to $\operatorname{Pic}(M_d) \otimes \mathbf{Q}$, whose rank is known [5, Corollary 1.3]. □

We can also easily compute the Picard group of the moduli space M_d of (primitively) polarized K3 surfaces of degree d with at most rational double points, leveraging Corollary 2.8.

Corollary 2.10. *We have*

$$\operatorname{Pic}(M_d) \cong \mathbb{Z}^{\rho(d)}.$$

Proof. Since $\rho(d)$ is the rank of $\mathrm{Pic}(M_d) \otimes \mathbb{Q}$, it is enough to show that the Picard group of M_d is torsion free. By [25, Proposition 6.1], the coarse moduli space $\pi : \mathcal{M}_d \to M_d$ induces an injection of Picard groups $\pi^* : \mathrm{Pic}(M_d) \hookrightarrow \mathrm{Pic}(\mathcal{M}_d)$. By Corollary 2.8, the latter group is torsion-free, and so, $\mathrm{Pic}(M_d)$ is also torsion-free. $\square$

3 | THE PICARD GROUPS OF $\mathcal{P}_d$ and $\mathcal{K}_d$

3.1 | Overview of the section

In this last section, we leverage our knowledge of the Picard group of $\mathcal{M}_d$ to compute the Picard groups of $\mathcal{K}_d$ and $\mathcal{P}_d$.

From the fact that $\mathrm{Pic}(\mathcal{P}_d)$ is finitely generated, which is proved exactly like in the case of $\mathcal{M}_d$ (see Lemma 2.9), and from Proposition 2.4, we immediately obtain the following.

Theorem 3.2. *As an abstract group,* $\mathrm{Pic}(\mathcal{P}_d) \simeq \mathbb{Z}^{\rho(d)} \oplus \mathbb{Z}/2$.

The exact relation between $\mathrm{Pic}(\mathcal{M}_d)$, $\mathrm{Pic}(\mathcal{P}_d)$, and $\mathrm{Pic}(\mathcal{K}_d)$ depends on whether $\frac{d}{2} \equiv 1 \pmod 4$ or not.

Theorem 3.3. *Suppose that* $\frac{d}{2} \not\equiv 1 \pmod 4$. *Then, the pullback* $p_d^* : \mathrm{Pic}(\mathcal{P}_d) \to \mathrm{Pic}(\mathcal{K}_d)$ *is an isomorphism and we have a split exact sequence*

$$0 \longrightarrow \mathrm{Pic}\,\mathcal{M}_d \xrightarrow{\varphi_d^*} \mathrm{Pic}\,\mathcal{P}_d \longrightarrow \mathbb{Z}/2 \longrightarrow 0\,.$$

where the splitting is given by $1 \mapsto [\mathcal{L}_d]$.

Theorem 3.4. *Suppose* $\frac{d}{2} \equiv 1 \pmod 4$. *Then* $p_d^* : \mathrm{Pic}(\mathcal{P}_d) \to \mathrm{Pic}(\mathcal{K}_d)$ *is an isomorphism; furthermore, we have a nonsplit short exact sequence*

$$0 \longrightarrow \mathrm{Pic}\,\mathcal{M}_d \times \mathbb{Z}/2 \xrightarrow{\varphi_d^* \times \mathrm{id}} \mathrm{Pic}\,\mathcal{P}_d \longrightarrow \mathbb{Z}/2 \longrightarrow 0\,.$$

and neither class $[\mathcal{P}_{d,1}^{\mathrm{sing}}]$ *or* $[\mathcal{P}_{d,2}^{\mathrm{sing}}]$ *sent to zero by the last map.*

3.5 | Proof of the remaining main theorems

Call $\mathcal{P}_d^{\mathrm{sing}}$ and $\mathcal{K}_d^{\mathrm{sing}}$ the inverse images of $\mathcal{M}_d^{\mathrm{sing}}$ in $\mathcal{P}_d$ and $\mathcal{K}_d$, respectively, with their reduced scheme structure. Since $\mathcal{K}_d \to \mathcal{P}_d$ is étale, $\mathcal{K}_d^{\mathrm{sing}}$ is the scheme-theoretic inverse image of $\mathcal{P}_d^{\mathrm{sing}}$. Moreover, when $\frac{d}{2} \equiv 1 \pmod 4$, let $\mathcal{M}_{d,i}^{\mathrm{sing}}$ (resp., $\mathcal{P}_{d,i}^{\mathrm{sing}}$, $\mathcal{K}_{d,i}^{\mathrm{sing}}$) for $= 1, 2$ be the two irreducible components of $\mathcal{M}_d^{\mathrm{sing}}$ (resp., $\mathcal{P}_d^{\mathrm{sing}}$, $\mathcal{K}_d^{\mathrm{sing}}$) given by Lemma 1.13.

Lemma 3.6. *We have*

$$\psi_d^*[\mathcal{M}_d^{\mathrm{sing}}] = 2[\mathcal{P}_d^{\mathrm{sing}}], \quad p_d^*[\mathcal{P}_d^{\mathrm{sing}}] = [\mathcal{K}_d^{\mathrm{sing}}].$$

If $\frac{d}{2} \equiv 1 \pmod 4$*, we have*

$$\psi_d^*[\mathscr{M}_{d,i}^{\mathrm{sing}}] = 2[\mathscr{P}_{d,i}^{\mathrm{sing}}], \quad p_d^*[\mathscr{P}_{d,i}^{\mathrm{sing}}] = [\mathscr{K}_{d,i}^{\mathrm{sing}}].$$

Proof. The second equations of both statements follow from the fact that p_d is étale.

For the first, notice that a generic closed point $\xi :\ \operatorname{Spec}\mathbb{C} \to \mathscr{M}_d^{\mathrm{sing}}$ corresponds to a polarized K3 surface (X, L) with only one singular point of type A_1. If $\widetilde{X} \to X$ is the crepant resolution of X, we can fix an isomorphism $\alpha : H^2(\widetilde{X}, \mathbb{Z}) \simeq \Lambda$ sending $\mathrm{c}_1(L)$ in ℓ; by taking $\alpha \otimes \mathrm{id}_{\mathbb{C}}(H^{2,0}(X))$, we obtain a point $\gamma \in D_d$ mapping to ξ, hence a lifting $\eta :\ \operatorname{Spec}\mathbb{C} \to \mathscr{P}_d$ of ξ.

The automorphism group of η is the stabilizer of γ in Λ_d, which contains the reflexion along the element $\delta \in \Lambda$ corresponding to the class in $H^2(X, \mathbb{Z})$ of the (-2)-curve contracted by $\widetilde{X} \to X$. Then the group homomorphism $\operatorname{Aut}\eta \to \operatorname{Aut}\xi$ is surjective, and its kernel is cyclic of order 2, generated by the reflexion along δ.

It follows that ψ_d is ramified of order 2 along the irreducible components $\mathscr{P}_d^{\mathrm{sing}}$, which implies the first equations of both statements. □

Lemma 3.7. *For* $= 1, 2$*, the divisor* $[\mathscr{P}_{d,i}^{\mathrm{sing}}]$ *does not belong to the image of* $\psi_d^* : \operatorname{Pic}(\mathscr{M}_d) \to \operatorname{Pic}(\mathscr{P}_d)$.

Proof. Let us assume that $i = 1$, the other case can be proved in the same way. We argue by contradiction, thus suppose that $[\mathscr{P}_{d,1}^{\mathrm{sing}}] = \psi_d^*[\mathscr{L}']$. This implies that the ideal sheaf $\mathscr{O}(-\mathscr{P}_{d,1}^{\mathrm{sing}})$ comes from $\mathscr{M}_d$.

Let $x \in \mathscr{P}_{d,1}^{\mathrm{sing}}$ be a generic point, and let $\sigma_d \in \operatorname{Aut}(x)$ be the involution that does not come from $\operatorname{Aut}(\psi_d(x))$. Then, $\mathscr{O}(-\mathscr{P}_{d,1}^{\mathrm{sing}})(x)$ is generated by a local equation of $\mathscr{P}_{d,1}^{\mathrm{sing}}$ on which σ_d should act trivially. We now show that this is not the case.

Let $y \in D_d$ be a point mapping to x in $\mathscr{P}_d$: then there exists a (-2)-class $\delta \in \Lambda_d$ such that

$$\ell(z) = \langle \delta, z \rangle = 0$$

is a local equation for the preimage of $\mathscr{P}_{d,1}^{\mathrm{sing}}$ around y. By construction, the involution σ_d corresponds to the reflection with respect to the hyperplane $\delta^{\perp}$; hence, it maps $\delta \mapsto -\delta$. This implies that $\sigma \cdot \ell(z) = -\ell(z)$, and hence, the action on the generator of $\mathscr{O}(-\mathscr{P}_{d,1}^{\mathrm{sing}})$ is not trivial. We have reached a contradiction. □

Proof of Theorem 3.3. The fact that $\psi_d^* :\ \operatorname{Pic}(\mathscr{M}_d) \to \operatorname{Pic}(\mathscr{P}_d)$ and $p_d^* :\ \operatorname{Pic}(\mathscr{P}_d) \to \operatorname{Pic}(\mathscr{K}_d)$ are injective follows from Lemma 2.6 and the projection formula.

By Lemma 1.13, we have that $\mathscr{M}_d^{\mathrm{sing}}$, $\mathscr{P}_d^{\mathrm{sing}}$, and $\mathscr{K}_d^{\mathrm{sing}}$ are irreducible. We have three homomorphisms $\mathbb{Z} \to \operatorname{Pic}(\mathscr{M}_d)$, $\mathbb{Z} \to \operatorname{Pic}(\mathscr{P}_d)$, and $\mathbb{Z} \to \operatorname{Pic}(\mathscr{K}_d)$ sending $1 \in \mathbb{Z}$ in $\mathscr{M}_d^{\mathrm{sing}}$, $\mathscr{P}_d^{\mathrm{sing}}$, and $\mathscr{K}_d^{\mathrm{sing}}$, respectively. From Lemma 3.6, we get two commutative diagrams with exact rows

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & \operatorname{Pic}(\mathscr{M}_d) \arrow[r] \arrow[d, "\psi_d^*"] & \operatorname{Pic}(\mathscr{M}_d^{\mathrm{sm}}) \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z} \arrow[r] & \operatorname{Pic}(\mathscr{P}_d) \arrow[r] & \operatorname{Pic}(\mathscr{M}_d^{\mathrm{sm}}) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, equal] & \mathrm{Pic}(\mathscr{P}_d) \arrow[r] \arrow[d, "p_d^*"] & \mathrm{Pic}(\mathscr{P}_d^{\mathrm{sm}}) \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z} \arrow[r] & \mathrm{Pic}(\mathscr{K}_d) \arrow[r] & \mathrm{Pic}(\mathscr{K}_d^{\mathrm{sm}}) \arrow[r] & 0
\end{tikzcd}$$

From the second, we get that $\psi_d^* : \ \mathrm{Pic}(\mathscr{P}_d) \to \mathrm{Pic}(\mathscr{K}_d)$ is surjective, and hence, that is an isomorphism, as claimed.

From the first, we obtain that the cokernel of the injective map $\psi_d^* \, \mathrm{Pic}(\mathscr{M}_d) \to \mathrm{Pic}(\mathscr{P}_d)$ is contained in the cokernel of $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$, which is $\mathbb{Z}/2\mathbb{Z}$. Since we have a 2-torsion element of $\mathrm{Pic}(\mathscr{P}_d)$, the class of $\mathscr{L}_d$, which does not come from $\mathrm{Pic}(\mathscr{M}_d)$ (Lemma 2.5), this proves the result. □

Proof of Theorem 3.4. The fact that $\psi_d^* : \ \mathrm{Pic}(\mathscr{M}_d) \to \mathrm{Pic}(\mathscr{P}_d)$ and $p_d^* : \ \mathrm{Pic}(\mathscr{P}_d) \to \mathrm{Pic}(\mathscr{K}_d)$ are injective follows from Lemma 2.6 and the projection formula, as in the previous proof.

We have $\mathscr{M}_d^{\mathrm{sing}} = \mathscr{M}_{d,1}^{\mathrm{sing}} \cup \mathscr{M}_{d,2}^{\mathrm{sing}}$, where the $\mathscr{M}_{d,i}^{\mathrm{sing}}$ are integral divisors. Similarly, we have $\mathscr{P}_d^{\mathrm{sing}} = \mathscr{P}_{d,1}^{\mathrm{sing}} \cup \mathscr{P}_{d,2}^{\mathrm{sing}}$ and $\mathscr{K}_d^{\mathrm{sing}} = \mathscr{K}_{d,1}^{\mathrm{sing}} \cup \mathscr{K}_{d,2}^{\mathrm{sing}}$.

Define a homomorphism $\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2 \to \mathrm{Pic}(\mathscr{M}_d)$ given by $e_i \mapsto [\mathscr{M}_{d,i}^{\mathrm{sing}}]$. We also have homomorphisms $\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2 \to \mathrm{Pic}(\mathscr{P}_d)$ and $\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2 \to \mathrm{Pic}(\mathscr{K}_d)$ defined in a similar way. From the second part of Lemma 3.6, we have commutative diagrams

$$\begin{tikzcd}
\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2 \arrow[r] \arrow[d, "\cdot 2"] & \mathrm{Pic}(\mathscr{M}_d) \arrow[r] \arrow[d, "\psi_d^*"] & \mathrm{Pic}(\mathscr{M}_d^{\mathrm{sm}}) \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2 \arrow[r] & \mathrm{Pic}(\mathscr{P}_d) \arrow[r] & \mathrm{Pic}(\mathscr{M}_d^{\mathrm{sm}}) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2 \arrow[r] \arrow[d, equal] & \mathrm{Pic}(\mathscr{P}_d) \arrow[r] \arrow[d, "p_d^*"] & \mathrm{Pic}(\mathscr{P}_d^{\mathrm{sm}}) \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2 \arrow[r] & \mathrm{Pic}(\mathscr{K}_d) \arrow[r] & \mathrm{Pic}(\mathscr{K}_d^{\mathrm{sm}}) \arrow[r] & 0.
\end{tikzcd}$$

From the second, we get that $\psi_d^* : \ \mathrm{Pic}(\mathscr{P}_d) \to \mathrm{Pic}(\mathscr{K}_d)$ is surjective, and hence, that is is an isomorphism, as claimed.

From the first, we get that there is an exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathscr{M}_d) \longrightarrow \mathrm{Pic}(\mathscr{P}_d) \xrightarrow{f} \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Observe that $f(\mathscr{L}_d) \neq 0$ because $\mathscr{L}_d$ does not come from $\mathrm{Pic}(\mathscr{M}_d)$. From this, we deduce that we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathscr{M}_d) \times \mathbb{Z}/2 \cdot [\mathscr{L}_d] \longrightarrow \mathrm{Pic}(\mathscr{P}_d) \xrightarrow{g} \mathbb{Z}/2.$$

To prove that the last arrow is surjective, we show that $g([\mathscr{P}_{d,1}^{\mathrm{sing}}]) \neq 0$ (the same argument applies also to $[\mathscr{P}_{d,2}^{\mathrm{sing}}]$).

We argue by contradiction: if $g([\mathscr{P}_{d,1}^{\mathrm{sing}}]) = 0$, then

$$[\mathscr{P}_{d,1}^{\mathrm{sing}}] = \psi_d^*[\mathscr{L}'] + a[\mathscr{L}_d], \quad a \in \{0, 1\}.$$

Suppose first $a = 1$. If we restrict everything to the open substack $\mathscr{P}_d \setminus \mathscr{P}_{d,1}^{\mathrm{sing}}$, we deduce that $\mathscr{L}_d|_{\mathscr{P}_d \setminus \mathscr{P}_{d,1}^{\mathrm{sing}}}$ comes from $\mathscr{M}_d \setminus \mathscr{M}_{d,1}^{\mathrm{sing}}$; this is a contradiction, because by construction, the automorphism group of a point $x \in \mathscr{P}_{d,2}^{\mathrm{sing}}$ acts nontrivially on the fiber $\mathscr{L}_d(x)$.

This shows that we must have $a = 0$, but also this cannot be the case because of Lemma 3.7. Therefore, $g([\mathscr{P}_{d,1}^{\mathrm{sing}}]) \neq 0$ as claimed. The fact that g is not split follows from the fact that $\mathrm{Pic}(\mathscr{P}_d)[2] \simeq \mathbb{Z}/2$ by Proposition 2.4. □

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