



Distributional Solutions of Burgers' type Equations for Intrinsic Graphs in Carnot Groups of Step 2

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Received: 25 September 2020 / Accepted: 24 February 2022 / Published online: 30 March 2022
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Abstract

We prove that in arbitrary Carnot groups \mathbb{G} of step 2, with a splitting $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$ with \mathbb{L} one-dimensional, the intrinsic graph of a continuous function $\varphi: U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ is $C^1_{\mathbb{H}}$ -regular precisely when φ satisfies, in the distributional sense, a Burgers' type system $D^\varphi \varphi = \omega$, with a continuous ω . We stress that this equivalence does not hold already in the easiest step-3 Carnot group, namely the Engel group. We notice that our results generalize previous works by Ambrosio-Serra Cassano-Vittone and Bigolin-Serra Cassano in the setting of Heisenberg groups. As a tool for the proof we show that a continuous distributional solution φ to a Burgers' type system $D^\varphi \varphi = \omega$, with ω continuous, is actually a broad solution to $D^\varphi \varphi = \omega$. As a by-product of independent interest we obtain that all the continuous distributional solutions to $D^\varphi \varphi = \omega$, with ω continuous, enjoy 1/2-little Hölder regularity along vertical directions.

Keywords Carnot groups · Step-2 Carnot groups · Intrinsically C^1 -surfaces · Broad solutions · Burgers' equation · Distributional solutions to non-linear first order PDEs

Mathematics Subject Classification (2010) 53C17 · 22E25 · 28A75 · 49N60 · 49Q15 · 26A16 · 58J60 · 35F20 · 35F50

D.D.D., S.D. were partially supported by the Academy of Finland (grant 288501 'Geometry of subRiemannian groups' and by grant 322898 'Sub-Riemannian Geometry via Metric-geometry and Lie-group Theory'). G.A., D.D.D., S.D. were partially supported by the European Research Council (ERC Starting Grant 713998 GeoMeG 'Geometry of Metric Groups'). S. Don was also supported by the Swiss National Science Foundation, Grant Nr. 200020 191978. The authors wish to express their gratitude to the anonymous referee for useful feedback on the manuscript.

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1 Introduction

Due to the multitude of applications, sub-Riemannian geometry has attracted a lot of attention in the mathematical community in the recent years. A sub-Riemannian manifold is a generalization of Riemannian manifold for which the metric is induced by a smooth scalar product only defined on a sub-bundle of the tangent bundle. The infinitesimal model of a sub-Riemannian manifold, namely the class of its Gromov-Hausdorff tangents, is represented by the class of (quotients of) Carnot groups [29, 39]. Carnot groups are connected and simply connected Lie groups \mathbb{G} whose Lie algebra \mathfrak{g} admits a stratification, namely a decomposition into nontrivial complementary linear subspaces V_1, \dots, V_s such that

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_s, \quad [V_j, V_1] = V_{j+1}, \quad \text{for } j = 1, \dots, s - 1, \quad [V_s, V_1] = \{0\},$$

where $[V_j, V_1]$ denotes the subspace of \mathfrak{g} generated by the commutators $[X, Y]$ with $X \in V_j$ and $Y \in V_1$. Carnot groups have been studied from very different point of views such as Differential Geometry [10], subelliptic Differential Equations [7, 21, 22, 38], Complex Analysis [40] and Neuroimaging [12].

Concerning Geometric Measure Theory in the setting of Carnot groups, one of the most studied problems in the past twenty years is represented by the rectifiability problem for finite perimeter sets: is it possible to cover the boundary of a finite perimeter set with a countable union of C^1 -regular surfaces? The answer to this question is affirmative in the Euclidean case and it was studied in [17, 18] via a blow-up analysis. The proof of De Giorgi has then been adapted in the framework of step-2 Carnot groups in [24, 25] and then generalized to the so-called Carnot groups of type \star in [35], see also the recent [30]. When dealing with Carnot groups of step 3 or higher, only partial results concerning this question are available in the literature. One of the main difficulty is represented by the fact that it is not known in general if C^1 rectifiability is equivalent to a Lipschitz-type rectifiability. Concerning Heisenberg groups, see [41] for a Rademacher-type theorem for intrinsic Lipschitz graphs of any codimension. Different notions of rectifiability have also been recently investigated, see [4, 19].

The rectifiability problem represents an example that underlines the importance of a fine understanding of intrinsic surfaces inside Carnot groups. The study of different notions of surfaces in Carnot groups has been quite extensive in the recent years and we mention [26] for a definition of C^1 regular submanifold in the Heisenberg groups, [20, 23] for intrinsic Lipschitz graphs and their connection to C^1 -hypersurfaces, [34] for a notion of transversal submanifold and [27, 33] for a notion of C^1 -surface with Carnot group target, but the list is far from being complete.

We focus our attention on codimension-one intrinsic graphs. A codimension-one intrinsic graph Γ inside a Carnot group \mathbb{G} comes with a couple of homogeneous and complementary subgroups \mathbb{W} and \mathbb{L} with \mathbb{L} one-dimensional, see Section 2, and a map $\varphi: U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ such that $\Gamma = \{x \in \mathbb{G} : x = w \cdot \varphi(w), w \in U\}$. It turns out that the regularity of the graph Γ is strictly related to the regularity of φ and its intrinsic gradient $\nabla^\varphi \varphi$, see Section 2. As a geometric pointwise approach, we just say that φ is intrinsically differentiable if its graph has a homogeneous subgroup as blow-up. However, one can define some different notions of regularity that rely on some φ -dependent operators D_W^φ whenever $W \in \text{Lie}(\mathbb{W})$, see Definition 2.6. If an adapted basis of the Lie algebra (X_1, \dots, X_n) is fixed and is such that $\mathbb{L} := \exp(\text{span}\{X_1\})$ and $\mathbb{W} := \exp(\text{span}\{X_2, \dots, X_n\})$, then D^φ is a vector valued operator $(D_{X_2}^\varphi, \dots, D_{X_n}^\varphi) =: (D_2^\varphi, \dots, D_m^\varphi)$. The regularity of Γ is related to the validity of the equation $D^\varphi \varphi = \omega$ in an open subset $U \subseteq \mathbb{W}$, for some $\omega: U \rightarrow \mathbb{R}^{m-1}$, which can be understood in different ways. We briefly present some of them here.

Distributional sense. Since \mathbb{L} is one-dimensional, $D^\varphi\varphi$ is a well-defined distribution, see the last part of Definition 2.13. Thus we could interpret $D^\varphi\varphi = \omega$ in the distributional sense.

Broad sense.* For every $j = 2, \dots, m$ and every point $a \in U$, there exists a C^1 integral curve of $D_{X_j}^\varphi$ starting from a for which the Fundamental Theorem of Calculus with derivative ω holds, see Definition 2.13.

Broad sense. For every $j = 2, \dots, m$ and every point $a \in U$, all the integral curves of $D_{X_j}^\varphi$ starting from a are such that the Fundamental Theorem of Calculus with derivative ω holds, see Definition 2.13.

Approximate sense. For every $a \in U$, there exist $\delta > 0$ and a family $\{\varphi_\varepsilon \in C^1(B(a, \delta)) : \varepsilon \in (0, 1)\}$ such that $\varphi_\varepsilon \rightarrow \varphi$ and $D_j^{\varphi_\varepsilon} \varphi_\varepsilon \rightarrow \omega_j$ uniformly on $B(a, \delta)$ as ε goes to zero.

When \mathbb{G} has step 2 and \mathbb{L} is one-dimensional, the following theorem holds, see [3, Theorem 6.17] for a proof and [3, Theorem 1.7] for an equivalent and coordinate-independent statement. Notice that the statement of the result below needs a choice of coordinates as explained in Section 2.2, see also Eq. 5. We also refer the interested reader to the preliminary section of [3] for the notions used in the statement below that are not treated in the current paper, i.e., the ones in items (a), (b), (c). We stress that we are not going to use these items in the proof of the theorems of this paper.

Theorem 1.1 ([3, Theorem 6.17]) *Let \mathbb{G} be a Carnot group of step 2 and rank m , and let \mathbb{W} and \mathbb{L} be two complementary subgroups of \mathbb{G} , with \mathbb{L} horizontal and one-dimensional. Let $U \subseteq \mathbb{W}$ be an open set, and let $\varphi : U \rightarrow \mathbb{L}$ be a continuous function. Then the following conditions are equivalent*

- (a) *graph(φ) is a $C^1_{\mathbb{H}}$ -hypersurface with tangents complemented by \mathbb{L} ;*
- (b) *φ is uniformly intrinsically differentiable on U ;*
- (c) *φ is intrinsically differentiable on U and its intrinsic gradient is continuous;*
- (d) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that, for every $a \in U$, there exist $\delta > 0$ and a family of functions $\{\varphi_\varepsilon \in C^1(B(a, \delta)) : \varepsilon \in (0, 1)\}$ such that*

$$\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon = \varphi, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} D_j^{\varphi_\varepsilon} \varphi_\varepsilon = \omega_j \quad \text{in } L^\infty(B(a, \delta)),$$

for every $j = 2, \dots, m$;

- (e) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi\varphi = \omega$ in the broad sense on U ;*
- (f) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi\varphi = \omega$ in the broad* sense on U .*

Moreover if any of the previous holds, ω is the intrinsic gradient of φ .

The main result of the current paper is given by the following implication

$$D^\varphi\varphi = \omega \quad \text{in the sense of distributions} \Rightarrow D^\varphi\varphi = \omega \quad \text{in the broad* sense,} \quad (1)$$

in every Carnot group \mathbb{G} of step 2 and for every continuous $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$, with U open, and $\omega \in C(U; \mathbb{R}^{m-1})$ with \mathbb{L} one-dimensional, see Theorem 4.1. This result allows us to improve Theorem 1.1 adding a seventh equivalent condition to the list above¹:

- (g) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi\varphi = \omega$ holds in the distributional sense on U .*

¹To complete the chain of implication one also needs (a) \Rightarrow (g) and this follows from [3, Proposition 4.10].

Item (g) allows us to complete the chain of implications of Theorem 1.1 in the setting of step-2 Carnot groups generalizing the results scattered in [5, 8, 9] where the authors study the same problem in the Heisenberg groups, and [15, 16] where partial generalizations of the results in [5, 8, 9] are obtained in the case of step-2 Carnot groups.

The strategy of the proof of Eq. 1 goes as follows. Given a Carnot group \mathbb{G} of step 2, we consider the free Carnot group \mathbb{F} with step 2 and the same rank of \mathbb{G} , see Section 2.3 for the precise choice of identifications. We show in Proposition 3.3 that if $D^\varphi\varphi = \omega$ in distributional sense inside \mathbb{G} with some continuous $\omega \in C(U; \mathbb{R}^{m-1})$, then also $D^\psi\psi = \omega \circ \pi$ in the distributional sense in \mathbb{F} , where $\psi := (\pi|_{\mathbb{L}_{\mathbb{F}}})^{-1} \circ \varphi \circ \pi$, and $\pi : \pi^{-1}(U) \subseteq \mathbb{F} \rightarrow U \subseteq \mathbb{G}$ is the projection of \mathbb{F} onto \mathbb{G} , where the notation $\mathbb{L}_{\mathbb{F}}$ is explained, e.g., in Proposition 3.3. Then, we prove Proposition 3.2 that tells us that $D^\psi\psi = \tilde{\omega}$ in the distributional sense in \mathbb{F} with $\tilde{\omega} \in C(\pi^{-1}(U); \mathbb{R}^{m-1})$ implies that $D^\psi\psi = \tilde{\omega}$ in the broad* sense, which is exactly implication in Eq. 1 in the setting of free Carnot groups of step 2. Finally, we prove in Proposition 3.4 that $D^\psi\psi = \omega \circ \pi$ in the broad* sense in \mathbb{F} implies $D^\varphi\varphi = \omega$ in the broad* sense in \mathbb{G} . The global strategy of lifting the problem to the free Carnot groups resembles the one used in [3, Section 6] and [31].

The main difficulty arises in the proof of Proposition 3.2 where we have to combine the dimensional reduction given by Lemma 3.1 and the translation invariance of Proposition 2.12 to reduce ourselves to the Burgers' equation of the first Heisenberg group, and then apply the arguments used for this case in [14, Eqq. (3.4) and (3.5)] and [8, Step 1, proof of Theorem 1.2]. We point out that this argument is essentially different by the one used in [8]. One of the reasons for this is that the distributional equation $D^\varphi\varphi = \omega$ in arbitrary Carnot groups of step 2 has a significantly different structure compared to the one in the Heisenberg groups. For example, consider a Carnot group of dimension 5, step 2 and rank 3 with Lie algebra $\mathfrak{g} = \text{span}\{X_1, X_2, X_3, X_4, X_5\}$, horizontal layer $V_1 := \text{span}\{X_1, X_2, X_3\}$ and where the only nonvanishing commutators are given by $[X_1, X_2] = X_4 + X_5$ and $[X_1, X_3] = X_4 - X_5$. Define, in exponential coordinates, $\mathbb{W} := \{x_1 = 0\}$ and $\mathbb{L} := \{x_2 = x_3 = x_4 = x_5 = 0\}$. Then, given a continuous $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ on an open set U , the operators $D_j^\varphi := D_{X_j}^\varphi$ for $j = 2, 3$ have the following form, see Eq. 12,

$$\begin{aligned} D_2^\varphi &= \partial_2 + \varphi\partial_4 + \varphi\partial_5, \\ D_3^\varphi &= \partial_3 + \varphi\partial_4 - \varphi\partial_5, \end{aligned}$$

which show a nonlinearity in two vertical directions, instead of only one as in the Heisenberg groups.²

We remark that Proposition 3.2 and Theorem 4.1 have also an interesting PDE point of view which allows to see the problem independently of the Carnot group structure. Indeed, the theorem can be read to obtain the following regularity result. Assume that the Burgers' type system $D^\varphi\varphi = \omega$ holds in the distributional sense for a continuous map φ and with the continuous datum ω . Then, from each single equation of the system, we infer the following property: for every $j = 2, \dots, m$, φ is (uniformly) Lipschitz continuous on all the integral curves of the operator D_j^φ . In addition, the Fundamental Theorem of Calculus with derivative ω holds on some particular local family of integral curves of D_j^φ , namely the broad* condition holds, and as a consequence the Fundamental Theorem of Calculus with derivative ω holds on all integral curves of D_j^φ , namely also the broad condition holds,

²In this case this double nonlinearity can be removed by considering the Lie algebra automorphism such that $\Psi(X_1) = X_1, \Psi(X_2) = \frac{1}{2}X_2 + \frac{1}{2}X_3, \Psi(X_3) = \frac{1}{2}X_2 - \frac{1}{2}X_3$. This is basically our idea of properly lifting step-2 Carnot groups to free Carnot groups with the same rank.

see (f)⇒(e) of Theorem 4.2. Moreover, when we consider all the equations together, we obtain a remarkable piece of information: φ is 1/2-little Hölder continuous on the vertical coordinates, see Theorem 4.7.

We remark that Theorem 1.1 complemented with (g) is optimal in step-2 Carnot groups for the following reason. Already in the Engel group, which is the easiest step-3 Carnot group, we can find a continuous map φ that solves $D^\varphi\varphi = \omega$ in the sense of distributions for a constant ω whose graph is not uniformly intrinsically differentiable (UID). We however notice that we do not know at present to what extent implication in Eq. 1 holds in Carnot groups of higher step, see Remark 4.5.

We briefly describe the situation in which ω is less regular. In the paper [6], the authors show that, in Heisenberg groups, $D^\varphi\varphi = \omega$ holds in the sense of distributions for some $\omega \in L^\infty(U; \mathbb{R}^{m-1})$ if and only if φ is intrinsically Lipschitz. The validity of Eq. 1 with $\omega \in L^\infty(U; \mathbb{R}^{m-1})$ in the setting of step-2 Carnot groups would open to a slightly modified version of Theorem 1.1 where $\omega \in L^\infty(U; \mathbb{R}^{m-1})$ and (a) is replaced by

(a') graph(φ) is intrinsically Lipschitz for the splitting given by \mathbb{W} and \mathbb{L} .

This topic is out of the aims of this paper and will be target of future investigations.

We notice here that if a generalization of the a priori estimate [37, Lemma 3.1] would hold in any step-2 Carnot group, then, arguing as explained in [3, Remark 4.13], we could improve Theorem 1.1 replacing (d) with

(d') There exists $\omega \in C(U; \mathbb{R}^{m-1})$ and a family of functions $\{\varphi_\varepsilon \in C^1(U) : \varepsilon \in (0, 1)\}$ such that, for every compact set $K \subseteq U$ and every $j = 2, \dots, m$, one has

$$\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon = \varphi \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} D_j^{\varphi_\varepsilon} \varphi_\varepsilon = \omega_j \quad \text{in } L^\infty(K).$$

We refer the reader to [3, Remark 4.13] for a discussion of the literature and of the difference between item (d) and item (d'). We also remark that a smooth approximation that does not involve the intrinsic gradient holds in any Carnot group for co-horizontal intrinsic Lipschitz graphs, see [41, Theorem 1.6].

Intrinsic surfaces of higher codimensions have been studied in the Heisenberg groups in [11, 13]. For what concerns the approach via distributional solutions, finding a meaning of the distributional system $D^\varphi\varphi = \omega$ in higher codimension is still open. The main difficulty comes from the fact that it is not known how to give meaning to mixed terms of the form $\varphi_j \partial_x \varphi_j$. This was already noticed in [28, Remark 4.3.2]. A weak formulation that goes in this direction is collected in [36], where the authors relate zero-level sets of maps in $C_H^{1,\alpha}(\mathbb{H}; \mathbb{R}^2)$ with curves that satisfy certain “Level Set Differential Equations”, see [36, Theorem 5.6].

2 Preliminaries

2.1 Carnot Groups

We give a very brief introduction to Carnot groups. We refer the reader to e.g., [7, 29, 39] for a comprehensive introduction to Carnot groups. A Carnot group \mathbb{G} is a connected and simply connected Lie group, whose Lie algebra \mathfrak{g} is stratified. Namely, there exist subspaces V_1, \dots, V_s of the Lie algebra \mathfrak{g} such that

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_s, \quad [V_j, V_1] = V_{j+1} \quad \forall j = 1, \dots, s - 1, \quad [V_s, V_1] = \{0\}.$$

The integer s is called *step* of the group \mathbb{G} , while $m := \dim(V_1)$ is called *rank* of \mathbb{G} . We set $n := \dim(\mathbb{G})$ to be the topological dimension of \mathbb{G} . We equivalently denote by e or 0 the identity element of the group \mathbb{G} .

Every Carnot group has a one-parameter family of *dilations* that we denote by $\{\delta_\lambda : \lambda > 0\}$ defined as the unique linear maps on \mathfrak{g} such that $\delta_\lambda(X) = \lambda^j X$, for every $X \in V_j$. We denote by δ_λ both the dilations on \mathbb{G} and on \mathfrak{g} , with the usual identification given by the exponential map $\exp: \mathfrak{g} \rightarrow \mathbb{G}$ which is a global diffeomorphism.

We fix a homogeneous norm $\|\cdot\|$ on \mathbb{G} that induces a distance, namely such that $\|\delta_\lambda x\| = \lambda\|x\|$ for every $\lambda > 0$ and $x \in \mathbb{G}$, $\|xy\| \leq \|x\| + \|y\|$ for every $x, y \in \mathbb{G}$, $\|x\| = \|x^{-1}\|$ for every $x \in \mathbb{G}$, and $\|x\| = 0$ if and only if $x = e$. The norm $\|\cdot\|$ induces the left-invariant homogeneous distance according to the equality $d(x, y) := \|y^{-1}x\|$, for every $x, y \in \mathbb{G}$. We denote with $B(a, r)$ the open ball of center a and radius $r > 0$ according to this distance. We stress that on a Carnot group a homogeneous norm that induces a distance always exists, and every two left-invariant homogeneous distances are bi-Lipschitz equivalent.

Definition 2.1 (Complementary subgroups) Given a Carnot group \mathbb{G} , we say that two subgroups \mathbb{W} and \mathbb{L} are *complementary subgroups* in \mathbb{G} if they are *homogeneous*, i.e., closed under the action of δ_λ for every $\lambda > 0$, $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$ and $\mathbb{W} \cap \mathbb{L} = \{e\}$.

We say that the subgroup \mathbb{L} is *horizontal and k -dimensional* if there exist linearly independent $X_1, \dots, X_k \in V_1$ such that $\mathbb{L} = \exp(\text{span}\{X_1, \dots, X_k\})$. Given two complementary subgroups \mathbb{W} and \mathbb{L} , we denote the *projection maps* from \mathbb{G} onto \mathbb{W} and onto \mathbb{L} by $\pi_{\mathbb{W}}$ and $\pi_{\mathbb{L}}$, respectively. Defining $g_{\mathbb{W}} := \pi_{\mathbb{W}}g$ and $g_{\mathbb{L}} := \pi_{\mathbb{L}}g$ for any $g \in \mathbb{G}$, one has

$$g = (\pi_{\mathbb{W}}g) \cdot (\pi_{\mathbb{L}}g) = g_{\mathbb{W}} \cdot g_{\mathbb{L}}. \tag{2}$$

Remark 2.2 If \mathbb{W} and \mathbb{L} are complementary subgroups of \mathbb{G} and \mathbb{L} is one-dimensional, then it is easy to see that \mathbb{L} is horizontal. For the sake of clarity, we will always write \mathbb{L} horizontal and one-dimensional even if one-dimensional is technically sufficient. Notice also that, if \mathbb{W} and \mathbb{L} are complementary subgroups and \mathbb{L} is horizontal, then \mathbb{W} is a normal subgroup of \mathbb{G} .

Definition 2.3 (Adapted basis) Denote by $n_j := \sum_{i=1}^j \dim(V_i)$, for $j = 1, \dots, s$ and $n_0 := 0$. Observe that $n_s = n$. We say that a basis (X_1, \dots, X_n) of \mathfrak{g} is *adapted* if the following facts hold

- For every $j = 1, \dots, s$, the set $\{X_{n_{j-1}+1}, \dots, X_{n_j}\}$ is a basis for V_j .
- For any $j = 1, \dots, s$, the vectors $X_{n_{j-1}+1}, \dots, X_{n_j}$ are chosen among linear combinations of the iterated commutators of length $j - 1$ of the vectors X_1, \dots, X_m .

Definition 2.4 (Exponential coordinates) Let \mathbb{G} be a Carnot group of dimension n and let (X_1, \dots, X_n) be an adapted basis of its Lie algebra. The *exponential coordinates of the first kind* associated with (X_1, \dots, X_n) are given by the one-to-one correspondence

$$\begin{aligned} \mathbb{R}^n &\leftrightarrow \mathbb{G} \\ (x_1, \dots, x_n) &\leftrightarrow \exp(x_1 X_1 + \dots + x_n X_n). \end{aligned}$$

It is well known that this defines a diffeomorphism from \mathbb{R}^n to \mathbb{G} that allows us to identify \mathbb{G} with \mathbb{R}^n .

2.2 Carnot Groups of Step 2 in Exponential Coordinates.

We here introduce Carnot groups of step 2 in exponential coordinates. We adopt as a general reference [7, Chapter 3], but the interested reader could also read the beginning of [3, Subsection 6.2]. In this subsection \mathbb{G} will always be an arbitrary Carnot group of step 2.

We denote with m the rank of \mathbb{G} and we identify \mathbb{G} with $(\mathbb{R}^{m+h}, \cdot)$ by means of exponential coordinates associated with an adapted basis $(X'_1, \dots, X'_m, Y'_1, \dots, Y'_h)$ of the Lie algebra \mathfrak{g} . In this coordinates, we will identify any point $q \in \mathbb{G}$ with $q \equiv (x_1, \dots, x_m, y_1^*, \dots, y_h^*)$. The group operation \cdot between two elements $q = (x, y^*)$ and $q' = (x', (y^*)')$ is given by

$$q \cdot q' = \left(x + x', y^* + (y^*)' - \frac{1}{2} \langle \mathcal{B}x, x' \rangle \right), \tag{3}$$

where $\langle \mathcal{B}x, x' \rangle := (\langle \mathcal{B}^{(1)}x, x' \rangle, \dots, \langle \mathcal{B}^{(h)}x, x' \rangle)$ and $\mathcal{B}^{(i)}$ are linearly independent and skew-symmetric matrices in $\mathbb{R}^{m \times m}$, for $i = 1, \dots, h$. For any $i = 1, \dots, h$ and any $j, \ell = 1, \dots, m$, we set $(\mathcal{B}^{(i)})_{j\ell} := (b_{j\ell}^{(i)})$, and we stress that $\langle \mathcal{B}^{(i)}x, x' \rangle := \sum_{j,\ell=1}^m b_{j\ell}^{(i)}(x')_{jx_\ell}$. It is standard to observe that in these coordinates we can write

$$\begin{aligned} X'_j(p) &= \partial_{x_j} - \frac{1}{2} \sum_{i=1}^h \sum_{\ell=1}^m b_{j\ell}^{(i)} x_\ell \partial_{y_i^*}, & \text{for } j = 1, \dots, m, \\ Y'_i(p) &= \partial_{y_i^*}, & \text{for } i = 1, \dots, h. \end{aligned}$$

We stress that the operation in Eq. 3 is precisely the one obtained by means of the Baker-Campbell-Hausdorff formula in exponential coordinates of the first kind associated with the adapted basis $(X'_1, \dots, X'_m, Y'_1, \dots, Y'_h)$. We also stress that

$$[X'_j, X'_\ell] = \sum_{i=1}^h b_{j\ell}^{(i)} Y'_i, \quad \text{and} \quad [X'_j, Y'_i] = 0, \quad \forall j, \ell = 1, \dots, m, \quad \text{and} \quad \forall i = 1, \dots, h, \tag{4}$$

so that it is clear that $b_{j\ell}^{(i)}$, with $i = 1, \dots, h$, and $1 \leq j, \ell \leq m$, are the so-called *structure coefficients*.

In the sequel we denote by $\mathbb{W}_{\mathbb{G}}$ and $\mathbb{L}_{\mathbb{G}}$ two arbitrary complementary subgroups of \mathbb{G} with $\mathbb{L}_{\mathbb{G}}$ horizontal and one-dimensional. Up to choosing a proper adapted basis of the Lie algebra \mathfrak{g} , we may suppose that $\mathbb{L}_{\mathbb{G}} = \exp(\text{span}\{X'_1\})$ and $\mathbb{W}_{\mathbb{G}} = \exp(\text{span}\{X'_2, \dots, X'_m, Y'_1, \dots, Y'_h\})$. Thus, by means of exponential coordinates we can identify $\mathbb{W}_{\mathbb{G}}$ and $\mathbb{L}_{\mathbb{G}}$ with \mathbb{R}^{m+h-1} and \mathbb{R} , respectively, as follows

$$\begin{aligned} \mathbb{L}_{\mathbb{G}} &\equiv \{(x_1, 0 \dots, 0) : x_1 \in \mathbb{R}\}, \\ \mathbb{W}_{\mathbb{G}} &\equiv \{(0, x_2, \dots, x_m, y_1^*, \dots, y_h^*) : x_i, y_k^* \in \mathbb{R} \text{ for } i = 2, \dots, m; k = 1, \dots, h\}. \end{aligned} \tag{5}$$

2.3 Free Carnot Groups of Step 2 in Exponential Coordinates.

We here introduce free Carnot groups of step 2 in exponential coordinates. We adopt as a general reference [7, Chapter 3], but the interested reader could also read the beginning of [3, Subsection 6.1]. In this subsection \mathbb{F} will always denote a free Carnot group of step 2 and rank m . Recall that the topological dimension of \mathbb{F} is $m + \frac{m(m-1)}{2}$ and denote by $(X_1, \dots, X_m, Y_{21}, \dots, Y_{m(m-1)})$ an adapted basis of the Lie algebra of \mathbb{F} such that $[X_\ell, X_s] = Y_{\ell s}$ for every $1 \leq s < \ell \leq m$.

If we set $n := m + \frac{m(m-1)}{2}$, we can identify \mathbb{F} with \mathbb{R}^n by means of exponential coordinates associated with the adapted basis $(X_1, \dots, X_m, Y_{21}, \dots, Y_{m(m-1)})$. In this coordinates, we will identify any point $q \in \mathbb{F}$ with $q \equiv (x_1, \dots, x_m, y_{21}, \dots, y_{m(m-1)})$. It is readily seen that, in such coordinates, we have

$$\begin{aligned}
 X_j &= \partial_{x_j} + \frac{1}{2} \sum_{j < \ell \leq m} x_\ell \partial_{y_{\ell j}} - \frac{1}{2} \sum_{1 \leq \ell < j} x_\ell \partial_{y_{j\ell}}, & \text{if } 1 \leq j \leq m, \\
 Y_{\ell s} &= \partial_{y_{\ell s}}, & \text{if } 1 \leq s < \ell \leq m.
 \end{aligned}
 \tag{6}$$

Moreover, for any $q = (x, y)$ and $q' = (x', y')$ in \mathbb{F} , the product $q \cdot q'$ is given by the Baker-Campbell-Hausdorff formula, and yields

$$\begin{aligned}
 (q \cdot q')_j &= x_j + x'_j, & \text{if } 1 \leq j \leq m, \\
 (q \cdot q')_{\ell s} &= y_{\ell s} + y'_{\ell s} + \frac{1}{2}(x_\ell x'_s - x'_\ell x_s), & \text{if } 1 \leq s < \ell \leq m.
 \end{aligned}$$

2.4 Projection from \mathbb{F} to \mathbb{G} .

Fix a Carnot group \mathbb{G} of step 2 and rank m as in Section 2.2 and let \mathbb{F} be the free Carnot group of rank m and step 2. By definition of free Carnot groups, there exists a Lie group surjective homomorphism $\pi : \mathbb{F} \rightarrow \mathbb{G}$ such that

$$\pi_*(X_\ell) = X'_\ell, \tag{7}$$

for any $\ell = 1, \dots, m$, see e.g., [31, Section 6]. We identify \mathbb{F} and \mathbb{G} with \mathbb{R}^n and \mathbb{R}^{m+h} , respectively, by means of exponential coordinates as explained above and in Section 2.2 and Section 2.3. From Eq. 7, jointly with the very definition of exponential coordinates, we notice that for any $(x, y) \in \mathbb{R}^n$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{m(m-1)/2}$, there exists $y^* \in \mathbb{R}^h$ such that

$$\pi(x, y) = (x, y^*). \tag{8}$$

Since π is a Lie group homomorphism, its differential is a Lie algebra homomorphism. Hence, for any $1 \leq s < \ell \leq m$, we have that

$$\pi_*(Y_{\ell s}) = \pi_*([X_\ell, X_s]) = [\pi_*(X_\ell), \pi_*(X_s)] = [X'_\ell, X'_s] = \sum_{i=1}^h b_{\ell s}^{(i)} Y'_i,$$

where we used Eqs. 4, and 7, and the fact that for $1 \leq s < \ell \leq m$ one has $[X_\ell, X_s] = Y_{\ell s}$. We can therefore write the following formula

$$\begin{aligned}
 \pi(x_1, \dots, x_m, y_{21}, \dots, y_{m(m-1)}) &= (x_1, \dots, x_m, y_1^*, \dots, y_h^*), & \text{where} \\
 y_i^* &= \sum_{1 \leq s < \ell \leq m} b_{\ell s}^{(i)} y_{\ell s}, & \forall i = 1, \dots, h.
 \end{aligned}
 \tag{9}$$

Remark 2.5 (Main identification) Given a Carnot group \mathbb{G} of step 2 and rank m , and the free Carnot group \mathbb{F} of step 2 and rank m we work in the coordinates of Section 2.2 and Section 2.3. Let $\mathbb{W}_\mathbb{G}$ and $\mathbb{L}_\mathbb{G}$ be two complementary subgroups of \mathbb{G} , with $\mathbb{L}_\mathbb{G}$ one-dimensional. Up to a proper choice of an adapted basis, we can assume we are working in a basis in which Eq. 5 holds. Thus, taking into account Eq. 8, we are in a position to lift $\mathbb{W}_\mathbb{G}$ and $\mathbb{L}_\mathbb{G}$ to two complementary subgroups $\mathbb{W}_\mathbb{F}$ and $\mathbb{L}_\mathbb{F}$ of \mathbb{F} such that $\pi|_{\mathbb{L}_\mathbb{F}} : \mathbb{L}_\mathbb{F} \rightarrow \mathbb{L}_\mathbb{G}$ is an isomorphism and $\pi|_{\mathbb{W}_\mathbb{F}} : \mathbb{W}_\mathbb{F} \rightarrow \mathbb{W}_\mathbb{G}$ is onto. In this way we have the following

identification

$$\begin{aligned} \mathbb{L}_{\mathbb{F}} &\equiv \{(x_1, \dots, x_m, y_{21}, \dots, y_{m(m-1)}) \in \mathbb{R}^n : x_2 = \dots = x_m = y_{21} = \dots = y_{m(m-1)} = 0\}, \\ \mathbb{W}_{\mathbb{F}} &\equiv \{(x_1, \dots, x_m, y_{21}, \dots, y_{m(m-1)}) \in \mathbb{R}^n : x_1 = 0\}. \end{aligned} \tag{10}$$

2.5 Projected Vector Fields in Carnot Groups of Step 2

We recall here the definition of projected vector fields [3, Definition 3.1].

Definition 2.6 (Projected vector fields) Given two complementary subgroups \mathbb{W} and \mathbb{L} in a Carnot group \mathbb{G} , and a continuous function $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ defined on an open set U of \mathbb{W} , we define, for every $W \in \text{Lie}(\mathbb{W})$, the continuous *projected vector field* D_W^φ , by setting

$$(D_W^\varphi)|_w (f) := W|_{w \cdot \varphi(w)} (f \circ \pi_{\mathbb{W}}), \tag{11}$$

for all $w \in U$ and all $f \in C^\infty(\mathbb{W})$. When W is an element X_j of an adapted basis (X_1, \dots, X_n) we also write $D_j^\varphi := D_{X_j}^\varphi$.

Let us fix \mathbb{G} a Carnot group of step 2 and rank m along with two complementary subgroups $\mathbb{W}_{\mathbb{G}}$ and $\mathbb{L}_{\mathbb{G}}$ such that $\mathbb{L}_{\mathbb{G}}$ is horizontal and one-dimensional. Assume we have chosen a basis in such a way that Eq. 5 is satisfied. Take \mathbb{F} the free step-2 Carnot group of rank m and introduce $\mathbb{W}_{\mathbb{F}}$ and $\mathbb{L}_{\mathbb{F}}$ as in Remark 2.5. In this subsection we work in exponential coordinates and we use the identifications and the coordinate representations discussed in Sections 2.2, 2.3, and Remark 2.5. We recall that from the computations in [3, Example 3.6 & Remark 6.9] the projected vector fields relative to a continuous function $\varphi : U \subseteq \mathbb{W}_{\mathbb{G}} \rightarrow \mathbb{L}_{\mathbb{G}}$, with U open, are given by

$$\begin{aligned} D_j^\varphi &= \partial_{x_j} - \sum_{i=1}^h \left(b_{j1}^{(i)} \varphi + \frac{1}{2} \sum_{k=2}^m x_k b_{jk}^{(i)} \right) \partial_{y_i^*} = X'_{j|U} - \sum_{i=1}^h b_{j1}^{(i)} \varphi Y'_{i|U}, \quad \text{for } j = 2, \dots, m, \\ D_i^\varphi &= \partial_{y_i^*} = Y'_{i|U}, \quad \text{for } i = 1, \dots, h. \end{aligned} \tag{12}$$

In the particular case of the free Carnot group \mathbb{F} , given $V \subseteq \mathbb{W}_{\mathbb{F}}$ an open set, and given a continuous map $\psi : V \subseteq \mathbb{W}_{\mathbb{F}} \rightarrow \mathbb{L}_{\mathbb{F}}$, the projected vector fields are given by

$$\begin{aligned} D_j^\psi &= \partial_{x_j} - \psi \partial_{y_{j1}} + \frac{1}{2} \sum_{j < \ell \leq m} x_\ell \partial_{y_{\ell j}} - \frac{1}{2} \sum_{1 < s < j} x_s \partial_{y_{js}} = X_{j|V} - \psi Y_{j1|V}, \quad \text{for } j = 2, \dots, m, \\ D_{\ell s}^\psi &= \partial_{y_{\ell s}} = Y_{\ell s|V}, \quad \text{for } 1 \leq s < \ell \leq m. \end{aligned} \tag{13}$$

Then for each $j = 2, \dots, m$, every integral curve $\gamma_j : I \rightarrow \mathbb{W}_{\mathbb{F}} \equiv \mathbb{R}^{n-1}$ of D_j^ψ defined on an interval $I \subseteq \mathbb{R}$ has vertical components $y := (y_{\ell s})_{1 \leq s < \ell \leq m} : I \rightarrow \mathbb{R}^{\frac{m(m-1)}{2}}$ satisfying the

following equations

$$\begin{aligned}
 \dot{y}_{j1}(t) &= -\psi(x_2, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_m, y(t)), \\
 \dot{y}_{\ell j}(t) &= \frac{1}{2}x_\ell, & \text{if } j < \ell \leq m, \\
 \dot{y}_{js}(t) &= -\frac{1}{2}x_s, & \text{if } 1 < s < j, \\
 \dot{y}_{\ell s}(t) &= 0, & \text{otherwise,}
 \end{aligned}
 \tag{14}$$

where the horizontal components of $\gamma_j(0)$ are $(0, x_2, \dots, x_m)$.

Remark 2.7 (Projection on codimension-one subgroups in Carnot groups of step 2) Notice that, if $\mathbb{W}_\mathbb{G}$ and $\mathbb{L}_\mathbb{G}$ are complementary subgroups defined as in Eq. 5, then $\pi_{\mathbb{W}_\mathbb{G}} : \mathbb{G} \cong \mathbb{R}^{m+h} \rightarrow \mathbb{W}_\mathbb{G} \cong \mathbb{R}^{m+h-1}$ is given by

$$\pi_{\mathbb{W}_\mathbb{G}}(x_1, \dots, x_m, y_1^*, \dots, y_h^*) = \left(0, x_2, \dots, x_m, \dots, y_i^* - \frac{1}{2} \sum_{j=1}^m b_{j1}^{(i)} x_j x_1, \dots \right), \text{ with } i = 1, \dots, h. \tag{15}$$

Indeed, it is enough to observe that, thanks to the explicit expression of the product in Eq. 3, the following equality holds:

$$\pi_{\mathbb{W}_\mathbb{G}}(x_1, \dots, x_m, y_1^*, \dots, y_h^*) \cdot (x_1, 0, \dots, 0) = (x_1, \dots, x_m, y_1^*, \dots, y_h^*).$$

For every $q \in \mathbb{G}$, we define the map

$$\begin{aligned}
 P_q : \mathbb{W}_\mathbb{G} &\rightarrow \mathbb{W}_\mathbb{G} \\
 w &\mapsto \pi_{\mathbb{W}_\mathbb{G}}(q \cdot w).
 \end{aligned}
 \tag{16}$$

Set $q = (q_1, \dots, q_m, q_{m+1}, \dots, q_{m+h}) \in \mathbb{G}$ and $w = (w_1 := 0, w_2, \dots, w_m, w_{m+1}, \dots, w_{m+h}) \in \mathbb{W}_\mathbb{G}$. By using Eqs. 15 and 3, one has, being $i = 1, \dots, h$, that the following chain of equalities holds

$$\begin{aligned}
 P_q(w) &= \pi_{\mathbb{W}_\mathbb{G}}(q \cdot w) \\
 &= \left(0, q_2 + w_2, \dots, q_m + w_m, \dots, q_{m+i} + w_{m+i} + \frac{1}{2} \sum_{j=1}^m \sum_{\ell=2}^m b_{j\ell}^{(i)} q_j w_\ell - \frac{1}{2} \sum_{j=1}^m b_{j1}^{(i)} (q_j + w_j) q_1, \dots \right) \\
 &= \left(0, q_2 + w_2, \dots, q_m + w_m, \dots, \right. \\
 &\quad \left. \dots, q_{m+i} + w_{m+i} + \frac{1}{2} \sum_{j=1}^m \sum_{\ell=2}^m b_{j\ell}^{(i)} q_j w_\ell + \frac{1}{2} \sum_{\ell=2}^m b_{1\ell}^{(i)} w_\ell q_1 - \frac{1}{2} \sum_{\ell=2}^m b_{\ell 1}^{(i)} q_\ell q_1, \dots \right) \\
 &= \left(0, q_2 + w_2, \dots, q_m + w_m, \dots, \right. \\
 &\quad \left. \dots, q_{m+i} + w_{m+i} + \frac{1}{2} \sum_{\ell=2}^m w_\ell \left(\sum_{j=1}^m b_{j\ell}^{(i)} q_j + b_{1\ell}^{(i)} q_1 \right) - \frac{1}{2} \sum_{\ell=2}^m b_{\ell 1}^{(i)} q_\ell q_1, \dots \right),
 \end{aligned}
 \tag{17}$$

where we used the fact that the first component of w is zero and that $\mathcal{B}^{(i)}$ is skew-symmetric and therefore $b_{11}^{(i)} = 0$. If we see P_q as a map from \mathbb{R}^{m+h-1} to \mathbb{R}^{m+h-1} , the differential of P_q at a point $w \in \mathbb{W}$ is identified with a $(m+h-1) \times (m+h-1)$ matrix with the following

components

$$\begin{aligned}
 (dP_q)(w)_{ii} &= 1, & \forall i = 1, \dots, m + h - 1, \\
 (dP_q)(w)_{m+i-1, \ell-1} &= \left(\sum_{j=2}^m \frac{1}{2} b_{j\ell}^{(i)} q_j + b_{1\ell}^{(i)} q_1 \right), & \forall i = 1, \dots, h; \ell = 2, \dots, m, \\
 (dP_q)(w)_{j\ell} &= 0, & \text{otherwise.}
 \end{aligned}
 \tag{18}$$

In particular, $\det(dP_q)(w) = 1$ for any $w \in \mathbb{W}$.

2.6 Invariance Properties of Projected Vector Fields

We collect here some invariance properties that we will use later on. We introduce the operation of q -translation of a function.

Definition 2.8 (Intrinsic graph of a function) Given two complementary subgroups \mathbb{W} and \mathbb{L} of a Carnot group \mathbb{G} , and a function $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$, we define the graph of φ by setting

$$\text{graph}(\varphi) := \{ \Phi(w) := w \cdot \varphi(w) : w \in U \} = \Phi(U).$$

Definition 2.9 (Intrinsic translation of a function) Given two complementary subgroups \mathbb{W} and \mathbb{L} of a Carnot group \mathbb{G} and a map $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$, we define, for every $q \in \mathbb{G}$,

$$U_q := \{ a \in \mathbb{W} : \pi_{\mathbb{W}}(q^{-1} \cdot a) \in U \},$$

and $\varphi_q : U_q \subseteq \mathbb{W} \rightarrow \mathbb{L}$ by setting

$$\varphi_q(a) := \left(\pi_{\mathbb{L}}(q^{-1} \cdot a) \right)^{-1} \cdot \varphi \left(\pi_{\mathbb{W}}(q^{-1} \cdot a) \right).
 \tag{19}$$

Notice that $U_q = P_q(U)$, where P_q is defined as in Eq. 16. This easily comes from the fact that for every $q \in \mathbb{G}$ $P_q \circ P_{q^{-1}} = \text{Id}_{\mathbb{W}}$, see e.g., the proof of Proposition 2.12.

The following invariance properties with respect to intrinsic translations are matter of simple computations. The complete proofs can be found in [3, Proposition 2.10] and [3, Lemma 3.13], respectively.

Proposition 2.10 ([3, Proposition 2.10]) *Let \mathbb{W} and \mathbb{L} be two complementary subgroups of a Carnot group \mathbb{G} and let $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ be a function. Then, for every $q \in \mathbb{G}$, the following facts hold.*

- (a) $\text{graph}(\varphi_q) = q \cdot \text{graph}(\varphi)$;
- (b) $(\varphi_q)_{q^{-1}} = \varphi$;
- (c) *If \mathbb{W} is normal, then $U_q = q_{\mathbb{W}} \cdot (q_{\mathbb{L}} \cdot U \cdot (q_{\mathbb{L}})^{-1})$ and*

$$\varphi_q(a) = q_{\mathbb{L}} \cdot \varphi((q_{\mathbb{L}})^{-1} \cdot q_{\mathbb{W}}^{-1} \cdot a \cdot q_{\mathbb{L}}),$$

for any $a \in U_q$;

- (d) *If $q = \varphi(a)^{-1} \cdot a^{-1}$ for some $a \in U$, then*

$$\varphi_q(e) = e.$$

Lemma 2.11 ([3, Lemma 3.13]) *Let \mathbb{W} and \mathbb{L} be two complementary subgroups of a Carnot group \mathbb{G} , with \mathbb{L} k -dimensional and horizontal and let $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ be a continuous*

function defined on U open. Take $W \in \text{Lie}(\mathbb{W})$, and let us denote $D^\varphi := D_W^\varphi$. Let $T > 0$, $w \in \mathbb{W}$, and let $\gamma : [0, T] \rightarrow U$ be a C^1 -regular solution of the Cauchy problem

$$\begin{cases} \gamma'(t) = D^\varphi \circ \gamma(t), \\ \gamma(0) = w. \end{cases} \tag{20}$$

Then for every $q \in \mathbb{G}$ there exists a unique C^1 map $\gamma_q : [0, T] \rightarrow U_q$ such that

$$\pi_{\mathbb{W}}(q^{-1} \cdot \gamma_q(t)) = \gamma(t), \quad \forall t \in [0, T]. \tag{21}$$

In addition, γ_q is a solution of the Cauchy problem

$$\begin{cases} \gamma'_q(t) = D^{\varphi_q} \circ \gamma_q(t), \\ \gamma_q(0) = q_{\mathbb{W}} \cdot q_{\mathbb{L}} \cdot w \cdot (q_{\mathbb{L}})^{-1}. \end{cases} \tag{22}$$

Moreover the following equality holds

$$\varphi_q(\gamma_q(0))^{-1} \cdot \varphi_q(\gamma_q(t)) = \varphi(\gamma(0))^{-1} \cdot \varphi(\gamma(t)), \quad \forall t \in [0, T]. \tag{23}$$

In the following proposition we prove the invariance of being a distributional solution with respect to q -translation.

Proposition 2.12 *Let \mathbb{W} and \mathbb{L} be two complementary subgroups of a step-2 Carnot group \mathbb{G} with \mathbb{L} one-dimensional. Let Ω be an open set in \mathbb{W} and let $\omega \in L^1_{\text{loc}}(\Omega)$. Let us choose coordinates on \mathbb{G} as explained in Section 2.2, see also Eq. 5. Assume that for some $\ell = 2, \dots, m$ the map $\varphi : U \rightarrow \mathbb{L}$ is a distributional solution of the equation $D^\varphi_\ell \varphi = \omega$ on U . Then, for every $q \in \mathbb{G}$, the map φ_q defined in Definition 2.9 is a distributional solution of*

$$D^{\varphi_q}_\ell \varphi_q = \omega \circ P_{q^{-1}},$$

on the open set $P_q(\Omega)$.

Proof By item (c) of Proposition 2.10, we know that in exponential coordinates $\varphi_q(w) = q_1 + \varphi(P_{q^{-1}}(w))$, where $P_{q^{-1}}$ is defined in Eq. 16. Indeed, since \mathbb{W} is normal, the following equality holds

$$P_{q^{-1}}(w) = \pi_{\mathbb{W}}(q^{-1} \cdot w) = \pi_{\mathbb{W}}((q_{\mathbb{L}})^{-1} \cdot (q_{\mathbb{W}})^{-1} \cdot w \cdot q_{\mathbb{L}} \cdot (q_{\mathbb{L}})^{-1}) = q^{-1} \cdot w \cdot q_{\mathbb{L}}. \tag{24}$$

Moreover we claim $P_{q^{-1}} = P_q^{-1}$, for all $q \in \mathbb{G}$. Indeed, since \mathbb{W} is normal, the following equality holds

$$P_q(w) = \pi_{\mathbb{W}}(q \cdot w) = \pi_{\mathbb{W}}(q \cdot w \cdot q^{-1} \cdot q_{\mathbb{W}} \cdot q_{\mathbb{L}}) = q \cdot w \cdot q^{-1} \cdot q_{\mathbb{W}} = q \cdot w \cdot (q_{\mathbb{L}})^{-1}, \tag{25}$$

and hence it is clear by comparing Eqs. 24 and 25 that $P_q \circ P_{q^{-1}} = P_{q^{-1}} \circ P_q = \text{Id}_{\mathbb{W}}$, for all $q \in \mathbb{G}$. Moreover notice that from item (c) of Proposition 2.10 and Eq. 25 we get that

$$P_q(\Omega) = \Omega_q, \quad \text{for all } q \in \mathbb{G}. \tag{26}$$

For every $\xi \in C^\infty_c(P_q(\Omega))$, using Eq. 12 we can write the action of the distribution $D^{\varphi_q}_\ell \varphi_q$ on ξ , where we mean that the coordinates are $w = (x_2, \dots, x_m, y_1^*, \dots, y_h^*) \in \mathbb{W}$, as

follows

$$\begin{aligned}
 \langle D_\ell^{\varphi_q} \varphi_q, \xi \rangle &= \int_{P_q(\Omega)} \left(-\varphi_q \frac{\partial \xi}{\partial x_\ell} + \frac{1}{2} \sum_{i=1}^h b_{\ell 1}^{(i)} \varphi_q^2 \frac{\partial \xi}{\partial y_i^*} + \frac{1}{2} \sum_{i=1}^h \sum_{j=2}^m x_j b_{\ell j}^{(i)} \varphi_q \frac{\partial \xi}{\partial y_i^*} \right) dw \\
 &= \int_{P_q(\Omega)} \left(-(q_1 + \varphi \circ P_{q-1}) \frac{\partial \xi}{\partial x_\ell} + \frac{1}{2} \sum_{i=1}^h b_{\ell 1}^{(i)} (q_1 + \varphi \circ P_{q-1})^2 \frac{\partial \xi}{\partial y_i^*} \right) dw \\
 &\quad + \int_{P_q(\Omega)} \left(\frac{1}{2} \sum_{i=1}^h \sum_{j=2}^m x_j b_{\ell j}^{(i)} (q_1 + \varphi \circ P_{q-1}) \frac{\partial \xi}{\partial y_i^*} \right) dw \\
 &= \int_{P_q(\Omega)} \left(-\varphi \circ P_{q-1} \frac{\partial \xi}{\partial x_\ell} + \sum_{i=1}^h b_{\ell 1}^{(i)} \left[\frac{1}{2} (\varphi \circ P_{q-1})^2 + q_1 \varphi \circ P_{q-1} \right] \frac{\partial \xi}{\partial y_i^*} \right) dw \\
 &\quad + \int_{P_q(\Omega)} \left(\frac{1}{2} \sum_{i=1}^h \sum_{j=2}^m x_j b_{\ell j}^{(i)} (\varphi \circ P_{q-1}) \frac{\partial \xi}{\partial y_i^*} \right) dw,
 \end{aligned} \tag{27}$$

where in the third equality we used the fact that ξ has compact support in $P_q(\Omega)$ and q_1 does not depend on w . Taking into account that, by Remark 2.7, $\det(dP_q) = 1$ everywhere on \mathbb{W} , we perform in Eq. 27 the change of variable $w' = P_{q-1}(w)$. Thus, recalling that $P_q \circ P_{q-1} = \text{Id}_{\mathbb{W}}$, and by exploiting Eq. 17, we obtain the following equality

$$\begin{aligned}
 \langle D_\ell^{\varphi_q} \varphi_q, \xi \rangle &= \int_{\Omega} \left(-\varphi \frac{\partial \xi}{\partial x_\ell} \circ P_q + \sum_{i=1}^h b_{\ell 1}^{(i)} \left[\frac{1}{2} \varphi^2 + q_1 \varphi \right] \frac{\partial \xi}{\partial y_i^*} \circ P_q \right) dw \\
 &\quad + \int_{\Omega} \left(\frac{1}{2} \sum_{i=1}^h \sum_{j=2}^m (x_j + q_j) b_{\ell j}^{(i)} \varphi \frac{\partial \xi}{\partial y_i^*} \circ P_q \right) dw.
 \end{aligned} \tag{28}$$

We can now use Eq. 18 to compute the derivatives of $\xi \circ P_q$ as follows

$$\begin{aligned}
 \frac{\partial}{\partial x_\ell} (\xi \circ P_q) &= \frac{\partial \xi}{\partial x_\ell} \circ P_q + \sum_{i=1}^h \left(\sum_{j=2}^m \frac{1}{2} b_{j\ell}^{(i)} q_j + b_{1\ell}^{(i)} q_1 \right) \frac{\partial \xi}{\partial y_i^*} \circ P_q, \quad \forall \ell = 2, \dots, m, \\
 \frac{\partial}{\partial y_i^*} (\xi \circ P_q) &= \frac{\partial \xi}{\partial y_i^*} \circ P_q, \quad \forall i = 1, \dots, h.
 \end{aligned} \tag{29}$$

By using Eqs. 29 into 28 we get

$$\begin{aligned}
 \langle D_\ell^{\varphi_q} \varphi_q, \xi \rangle &= \int_{\Omega} -\varphi \left(\frac{\partial (\xi \circ P_q)}{\partial x_\ell} - \sum_{i=1}^h \left(\sum_{j=2}^m \frac{1}{2} b_{j\ell}^{(i)} q_j + b_{1\ell}^{(i)} q_1 \right) \frac{\partial (\xi \circ P_q)}{\partial y_i^*} \right) dw \\
 &\quad + \int_{\Omega} \sum_{i=1}^h \left(b_{\ell 1}^{(i)} \left[\frac{1}{2} \varphi^2 + q_1 \varphi \right] + \frac{1}{2} \sum_{j=2}^m (x_j + q_j) b_{\ell j}^{(i)} \varphi \right) \frac{\partial (\xi \circ P_q)}{\partial y_i^*} dw \\
 &= \int_{\Omega} \left(-\varphi \frac{\partial (\xi \circ P_q)}{\partial x_\ell} + \frac{1}{2} \sum_{i=1}^h b_{\ell 1}^{(i)} \varphi^2 \frac{\partial (\xi \circ P_q)}{\partial y_i^*} + \frac{1}{2} \sum_{i=1}^h \sum_{j=2}^m x_j b_{\ell j}^{(i)} \varphi \frac{\partial (\xi \circ P_q)}{\partial y_i^*} \right) dw \\
 &= \langle D_\ell^\varphi \varphi, \xi \circ P_q \rangle,
 \end{aligned}$$

where, in order to write the second equality, we used that the matrices \mathcal{B} are skew-symmetric. Now, exploiting the last identity, the assumption and the fact that for every $\xi \in C_c^\infty(P_q(\Omega))$ we have $\xi \circ P_q \in C_c^\infty(\Omega)$, we conclude

$$\langle D_\ell^{\varphi_q} \varphi_q, \xi \rangle = \langle D_\ell^\varphi \varphi, \xi \circ P_q \rangle = \int_\Omega \omega(\xi \circ P_q) \, d\omega = \int_{P_q(\Omega)} (\omega \circ P_{q^{-1}}) \xi \, d\omega,$$

where in the last equality we changed the variables and exploited the fact that $\det(dP(q)) = 1$. By the arbitrariness of $\xi \in C_c^\infty(P_q(\Omega))$, the proof is complete. \square

2.7 Broad, Broad* and Distributional Solutions of $D^\varphi \varphi = \omega$

We recall the following definition as a particular case of [3, Section 3.4, Definition 3.24]. For discussions about the dependence of the definition of broad* regularity on the chosen adapted basis, that is not important for this paper, we refer the interested reader to [3, Remark 3.26 & Remark 4.4].

Definition 2.13 (Broad*, broad and distributional solutions) Let \mathbb{W} and \mathbb{L} be complementary subgroups of a Carnot group \mathbb{G} , with \mathbb{L} one-dimensional. Let $U \subseteq \mathbb{W}$ be open and let $\varphi : U \rightarrow \mathbb{L}$ be a continuous function. Consider an adapted basis (X_1, \dots, X_n) of the Lie algebra of \mathbb{G} such that $\mathbb{L} = \exp(\text{span}\{X_1\})$ and $\mathbb{W} = \exp(\text{span}\{X_2, \dots, X_n\})$. Let $\omega := (\omega_j)_{j=2, \dots, m} : U \rightarrow \mathbb{R}^{m-1}$ be a continuous function. Up to identifying \mathbb{L} with \mathbb{R} by means of exponential coordinates, we say that $\varphi \in C(U)$ is a *broad* solution* of $D^\varphi \varphi = \omega$ in U if for every $a_0 \in U$ there exist $0 < \delta_2 < \delta_1$ such that $\overline{B(a_0, \delta_1)} \cap \mathbb{W} \subseteq U$ and there exist $m - 1$ maps $E_j^\varphi : (\overline{B(a_0, \delta_2)} \cap \mathbb{W}) \times [-\delta_2, \delta_2] \rightarrow \overline{B(a_0, \delta_1)} \cap \mathbb{W}$ for $j = 2, \dots, m$, satisfying the following two properties.

- (a) For every $a \in \overline{B(a_0, \delta_2)} \cap \mathbb{W}$ and every $j = 2, \dots, m$, the map $E_j^\varphi(a) := E_j^\varphi(a, \cdot)$ is C^1 -regular and it is a solution of the Cauchy problem

$$\begin{cases} \dot{\gamma} = D_j^\varphi \circ \gamma, \\ \gamma(0) = a, \end{cases}$$

in the interval $[-\delta_2, \delta_2]$, where the vector field $D_j^\varphi := D_{X_j}^\varphi$ is defined in Eq. 11.

- (b) For every $a \in \overline{B(a_0, \delta_2)} \cap \mathbb{W}$, for every $t \in [-\delta_2, \delta_2]$, and every $j = 2, \dots, m$ one has

$$\varphi(E_j^\varphi(a, t)) - \varphi(a) = \int_0^t \omega_j(E_j^\varphi(a, s)) \, ds.$$

Moreover, we say that $D^\varphi \varphi = \omega$ in the *broad sense* on U if for every $W \in \text{Lie}(\mathbb{W}) \cap V_1$ and every $\gamma : I \rightarrow U$ integral curve of D_W^φ , it holds that

$$\frac{d}{ds} \Big|_{s=t} (\varphi \circ \gamma)(s) = \langle \omega(\gamma(t)), W \rangle, \quad \forall t \in I,$$

where $\langle \omega, W \rangle$ is the standard scalar product on \mathbb{R}^{m-1} in exponential coordinates.

Finally, let us notice that, for every $j = 2, \dots, m$, D_j^φ is a continuous vector field with coefficients that might depend polinomially on φ and on some of the coordinates, see Section 2.5 for the case in which \mathbb{G} is of step 2, that is the only one treated in this paper, and [3, Proposition 3.9] for the general case. We say that $D^\varphi \varphi = \omega$ holds in the *distributional sense* on U if for every $j = 2, \dots, m$ one has $D_j^\varphi \varphi = \omega_j$ in the distributional sense. Notice

that the distribution $(D_j^\varphi)\varphi$ is well defined since the coefficients of D_j^φ just contain polynomial terms in φ and terms depending on the coordinates. The interested reader could read [3, Item (a) of Proposition 4.10]. For the aims of this paper, we stress that when we write $\varphi\partial_x\varphi$ for a continuous φ we mean the distribution $1/2\partial_x(\varphi^2)$.

3 Reduction of the Main Theorems to Free Carnot Groups of Step 2

In this section we analyze the link between distributional and broad* solutions to $D^\varphi\varphi = \omega$ with a continuous datum ω . We first show that a distributional solution of $D^\varphi\varphi = \omega$ with a continuous datum ω is a broad* solution inside free Carnot groups \mathbb{F} of step 2, see Proposition 3.2 (b). In this proof, a crucial role is played by the particular structure of the projected vector fields inside free Carnot groups of step 2, which produces Burgers' type operators in higher dimensions, see Eq. 13. Indeed, combining the invariance result in Proposition 2.12 and the dimensional reduction of Lemma 3.1, we can reduce ourselves to deal with the Burgers' distributional equation with continuous datum on the first Heisenberg group \mathbb{H}^1 , and then exploit the arguments used by Dafermos in [14] and by Bigolin and Serra Cassano in [8].

Secondly, by the fact that we can write the explicit expression of the projection π from \mathbb{F} to a Carnot group \mathbb{G} of step 2 of the same rank, we prove that being a distributional solution to $D^\varphi\varphi = \omega$ on \mathbb{G} lifts to \mathbb{F} , see Proposition 3.3. Finally, Proposition 3.4 states that the notion of broad* solution is preserved by π , i.e., a broad* solution on \mathbb{F} becomes a broad* solution on \mathbb{G} . The resulting strategy resembles the one used in [31]. The forthcoming lemma is a variation of the Fundamental Theorem of Calculus of Variations, i.e., du Bois-Reymond's lemma. We provide the complete proof of the forthcoming lemma for the reader's convenience.

Lemma 3.1 *Let $n_1, n_2, k \in \mathbb{N}$ and let Ω be an open set in $\mathbb{R}^{n_1+n_2+k}$. Let $f_0, f_1, \dots, f_{n_1} \in C(\Omega)$ and assume that, for every $\varphi \in C_c^\infty(\Omega)$, one has*

$$\iiint \left(f_0(x, y, z)\varphi(x, y, z) + \sum_{i=1}^{n_1} f_i(x, y, z) \frac{\partial\varphi}{\partial x_i}(x, y, z) \right) dx dy dz = 0,$$

where $(x, y, z) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^k$. Then, for every $z_0 \in \mathbb{R}^k$ such that $\Omega_0 := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (x, y, z_0) \in \Omega\}$ is nonempty, and any $\widehat{\varphi} \in C_c^\infty(\Omega_0)$, one has

$$\iint \left(f_0(x, y, z_0)\widehat{\varphi}(x, y) + \sum_{i=1}^{n_1} f_i(x, y, z_0) \frac{\partial\widehat{\varphi}}{\partial x_i}(x, y) \right) dx dy = 0.$$

Proof By translation invariance, we can assume without loss of generality that $z_0 = 0$. Up to iterate the argument k times, we can also assume without loss of generality that $k = 1$. Let $\widehat{\varphi} := \widehat{\varphi}(x, y)$ be such that $\text{supt}(\widehat{\varphi}) \Subset \Omega_0$. Choose $\varepsilon_0 > 0$ small enough and consider, for any $\varepsilon \in (0, \varepsilon_0]$, the map $\varphi^\varepsilon(x, y, z) := \frac{1}{2\varepsilon}\varphi_0^\varepsilon(z)\widehat{\varphi}(x, y)$ with $\text{supt}(\varphi_0^\varepsilon) \subseteq [-\varepsilon - \varepsilon^2, \varepsilon + \varepsilon^2]$, $\varphi_0^\varepsilon \geq 0$ and $\varphi_0^\varepsilon = 1$ on $[-\varepsilon, \varepsilon]$, and such that $\text{supt}(\widehat{\varphi}) \times \text{supt}(\varphi_0^{\varepsilon_0}) \Subset \Omega$. Then, by the hypothesis and Fubini's Theorem we may write

$$\frac{1}{2\varepsilon} \int_{-\varepsilon-\varepsilon^2}^{\varepsilon+\varepsilon^2} \varphi_0^\varepsilon(z) \left(\iint \left(f_0(x, y, z)\widehat{\varphi}(x, y) + \sum_{i=1}^{n_1} f_i(x, y, z) \frac{\partial\widehat{\varphi}}{\partial x_i}(x, y) \right) dx dy \right) dz = 0. \tag{30}$$

Notice that the function

$$F(z) := \iint \left(f_0(x, y, z)\widehat{\varphi}(x, y) + \sum_{i=1}^{n_1} f_i(x, y, z) \frac{\partial \widehat{\varphi}}{\partial x_i}(x, y) \right) dx dy,$$

is continuous on $[-\varepsilon_0 - \varepsilon_0^2, \varepsilon_0 + \varepsilon_0^2]$. We can then decompose the left-hand side of Eq. 30 in the following way

$$\frac{1}{2\varepsilon} \int_{-\varepsilon-\varepsilon^2}^{\varepsilon+\varepsilon^2} \varphi_0^\varepsilon(z)F(z) dz = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} F(z) dz + \frac{1}{2\varepsilon} \int_{[-\varepsilon-\varepsilon^2, \varepsilon+\varepsilon^2] \setminus [-\varepsilon, \varepsilon]} \varphi_0^\varepsilon(z)F(z) dz. \tag{31}$$

Since $\varphi_0^\varepsilon \leq 1$, we have

$$\left| \frac{1}{2\varepsilon} \int_{[-\varepsilon-\varepsilon^2, \varepsilon+\varepsilon^2] \setminus [-\varepsilon, \varepsilon]} \varphi_0^\varepsilon(z)F(z) dz \right| \leq M \frac{\varepsilon^2}{\varepsilon} = M\varepsilon,$$

where M is the maximum of $|F|$ in $[-\varepsilon_0 - \varepsilon_0^2, \varepsilon_0 + \varepsilon_0^2]$. Letting $\varepsilon \rightarrow 0$ in Eq. 31, the thesis follows by means of the continuity of F at 0. □

Proposition 3.2 *Let \mathbb{F} be the free Carnot group of step 2, rank m and topological dimension n , and let \mathbb{W} and \mathbb{L} be two complementary subgroups of \mathbb{F} such that \mathbb{L} is one-dimensional. Let Ω be an open subset of \mathbb{W} and $\psi : \Omega \rightarrow \mathbb{L}$ be a continuous function. Choose an adapted basis and exponential coordinates on \mathbb{F} as in Section 2.3, see also Eq. 10. Assume there exists $\omega \in C(\Omega; \mathbb{R}^{m-1})$ such that $D^\psi \psi = \omega$ holds in the distributional sense on Ω . Then, the following facts hold.*

- (a) *For every $j = 2, \dots, m$ and for every integral curve $\gamma : [0, T] \rightarrow \Omega$ of D_j^ψ , the map $\psi \circ \gamma : [0, T] \rightarrow \mathbb{L}$ is Lipschitz and the Lipschitz constant only depends on j and ω .*
- (b) *$D^\psi \psi = \omega$ holds in the broad* sense on Ω .*

Proof Preliminary dimensional reduction. Fix $j = 2, \dots, m$. Assume $0 \in \Omega$, $\psi(0) = 0$ and $D_j^\psi \psi = \omega_j$ in the sense of distributions on Ω . Taking Eq. 13 into account, this amounts to saying that

$$\int \left(-\psi \partial_{x_j} \varphi + \frac{\psi^2}{2} \partial_{y_{j1}} \varphi - \frac{1}{2} \sum_{j < \ell \leq m} x_\ell \psi \partial_{y_{\ell j}} \varphi + \frac{1}{2} \sum_{1 < s < j} x_s \psi \partial_{y_{js}} \varphi \right) d\mathcal{L}^{n-1} = \int \varphi \omega_j d\mathcal{L}^{n-1}, \tag{32}$$

for every $\varphi \in C_c^\infty(\Omega)$. Since ψ and ω are continuous, we are in a position to apply Lemma 3.1 to the variables $z = (x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_m)$ at $x_2^0 = \dots = x_{j-1}^0 = x_{j+1}^0 = \dots = x_m^0 = 0$. More precisely n_1 is the number of variables that are differentiated, namely $\{x_j, y_{j1}, y_{\ell j}, y_{js} : j < \ell \leq m, 1 < s < j\}$, $k = m - 2$ and n_2 is the number of remaining variables. Equation 32 then becomes

$$\begin{aligned} & \int \left(-\psi(0, \dots, 0, x_j, 0, \dots, 0, y) \partial_{x_j} \varphi(x_j, y) + \frac{\psi^2}{2}(0, \dots, 0, x_j, 0, \dots, 0, y) \partial_{y_{j1}} \varphi(x_j, y) \right) dx_j dy \\ & = \int \omega_j(0, \dots, 0, x_j, 0, \dots, 0, y) \varphi(x_j, y) dx_j dy. \end{aligned} \tag{33}$$

for any $\varphi \in C_c^\infty(\widetilde{\Omega})$, where $\widetilde{\Omega} := \{(x_j, y) \in \mathbb{R} \times \mathbb{R}^{m(m-1)/2} : (0, \dots, 0, x_j, 0, \dots, 0, y) \in \Omega\}$. Let us apply again Lemma 3.1 with $n_1 = 2$ being the number of variables along which the test φ is differentiated in Eq. 33, namely x_j and y_{j1} , $n_2 = 0$, and k being the number of

the remaining variables $y_{s\ell}$ with $\ell < s \leq m$ and $(s, \ell) \neq (j, 1)$. We apply the Lemma 3.1 at values $y_{s\ell}^0 = 0$ and we get

$$\int \left(-\widehat{\psi}(x_j, y_{j1}) \partial_{x_j} \varphi(x_j, y_{j1}) + \frac{\widehat{\psi}^2}{2}(x_j, y_{j1}) \partial_{y_{j1}} \varphi(x_j, y_{j1}) \right) dx_j dy_{j1} = \int \widehat{\omega}_j(x_j, y_{j1}) \varphi(x_j, y_{j1}) dx_j dy_{j1}, \tag{34}$$

for every $\varphi \in C_c^\infty(\widehat{\Omega})$ with $\widehat{\Omega} := \{(x_j, y_{j1}) \in \mathbb{R}^2 : (0, \dots, 0, x_j, 0, \dots, 0, y_{j1}, 0, \dots, 0) \in \Omega\}$, where

$$\begin{aligned} \widehat{\psi}(x_j, y_{j1}) &:= \psi(0, \dots, 0, x_j, 0, \dots, 0, y_{j1}, 0, \dots, 0), \\ \widehat{\omega}_j(x_j, y_{j1}) &:= \omega_j(0, \dots, 0, x_j, 0, \dots, 0, y_{j1}, 0, \dots, 0). \end{aligned} \tag{35}$$

Observe that the dimensional reduction shown above can be performed in the same way to the distributional equation $D^{\psi_q} \psi_q = \omega \circ P_{q-1}$ in the set $P_q(\Omega)$, where for any $w \in \Omega$ we have set $q = (w \cdot \psi(w))^{-1}$ and the map ψ_q is defined in Definition 2.9. Indeed in this case, if $w \in \Omega$, then by item (c) of Proposition 2.10 we have $0 \in P_q(\Omega)$ and, thanks to item (d) of Proposition 2.10, $\psi_q(0) = 0$.

(a) It is sufficient to show that, for any $2 \leq j \leq m$, there exists a constant $C > 0$ such that

$$|\psi(\gamma(T)) - \psi(\gamma(0))| \leq CT,$$

whenever $\gamma : [0, T] \rightarrow \Omega$ is an integral curve of D_j^ψ with $\gamma(0) = w$, where w is any point in Ω .

Fix $2 \leq j \leq m$. We first assume $0 \in \Omega$ with $\psi(0) = 0$ and we consider an integral curve $\gamma : [0, T] \rightarrow \Omega$ of D_j^ψ such that $\gamma(0) = 0$.

Taking Eqs. 13 and 14 into account, we can explicitly write all the components of $\gamma(t)$ as follows:

$$\begin{cases} \gamma_j(t) = t, \\ \gamma_{j1}(t) = -\int_0^t \widehat{\psi}(\tau, \gamma_{j1}(\tau)) d\tau, \\ \gamma_i(t) = 0, & \forall i = 1, \dots, m, i \neq j, \\ \gamma_{\ell s}(t) = 0, & \forall (\ell, s) \text{ with } 1 \leq s < \ell \leq m \text{ and } (\ell, s) \neq (j, 1), \end{cases}$$

where $\widehat{\psi}$ is defined in Eq. 35. We can thus define $\widehat{\gamma} : [0, T] \rightarrow \mathbb{R}^2$ by letting $\widehat{\gamma}(t) := (t, \gamma_{j1}(t))$. By the same choice of test functions given in [14, Equations (2.5) and (2.6)], from Eq. 34 we can derive

$$\begin{aligned} \int_{\gamma_{j1}(T)-\varepsilon}^{\gamma_{j1}(T)} \widehat{\psi}(T, x) dx - \int_{-\varepsilon}^0 \widehat{\psi}(0, x) dx - \int_0^T \int_{\gamma_{j1}(t)-\varepsilon}^{\gamma_{j1}(t)} \widehat{\omega}_j(t, x) dx dt \\ = -\frac{1}{2} \int_0^T (\widehat{\psi}(t, \gamma_{j1}(t) - \varepsilon) - \widehat{\psi}(t, \gamma_{j1}(t)))^2 dt, \end{aligned} \tag{36}$$

for every sufficiently small $\varepsilon > 0$, see [14, Equation (3.4)] with the choice $g = \widehat{\omega}_j$, $u = \widehat{\psi}$, $\sigma = 0$, $\tau = T$, $\xi = \gamma_{j1}$ and the change of sign of the right hand side with respect to the reference comes by the fact that in our case $f(u) = -\frac{1}{2}u^2$ instead of $f(u) = \frac{1}{2}u^2$. Since the right hand side of Eq. 36 is negative, we can write

$$\int_{\gamma_{j1}(T)-\varepsilon}^{\gamma_{j1}(T)} \widehat{\psi}(T, x) dx - \int_{-\varepsilon}^0 \widehat{\psi}(0, x) dx \leq \int_0^T \int_{\gamma_{j1}(t)-\varepsilon}^{\gamma_{j1}(t)} \widehat{\omega}_j(t, x) dx dt \leq \varepsilon \|\omega_j\|_{L^\infty(\Omega)} T.$$

Dividing both sides by ε and letting $\varepsilon \rightarrow 0$, by the continuity of ψ we get

$$\widehat{\psi}(T, \gamma_{j1}(T)) - \widehat{\psi}(0, 0) \leq \|\omega_j\|_{L^\infty(\Omega)} T. \tag{37}$$

Similarly, by mimicking [14, Equation (3.5)] we can write for every sufficiently small $\varepsilon > 0$ the equation

$$\begin{aligned} \int_{\gamma_{j1}(T)}^{\gamma_{j1}(T)+\varepsilon} \widehat{\psi}(T, x) \, dx - \int_0^\varepsilon \widehat{\psi}(0, x) \, dx - \int_0^T \int_{\gamma_{j1}(t)}^{\gamma_{j1}(t)+\varepsilon} \widehat{\omega}_j(t, x) \, dx \, dt \\ = \frac{1}{2} \int_0^T (\widehat{\psi}(t, \gamma_{j1}(t) + \varepsilon) - \widehat{\psi}(t, \gamma_{j1}(t)))^2 \, dt. \end{aligned}$$

Noticing that the right hand side is positive one gets

$$\int_{\gamma_{j1}(T)}^{\gamma_{j1}(T)+\varepsilon} \widehat{\psi}(T, x) \, dx - \int_0^\varepsilon \widehat{\psi}(0, x) \, dx \geq \int_0^T \int_{\gamma_{j1}(t)}^{\gamma_{j1}(t)+\varepsilon} \widehat{\omega}_j(t, x) \, dx \, dt \geq -\varepsilon \|\omega_j\|_{L^\infty(\Omega)} T.$$

Dividing both sides by ε and letting $\varepsilon \rightarrow 0$ we get

$$\widehat{\psi}(T, \gamma_{j1}(T)) - \widehat{\psi}(0, 0) \geq -\|\omega_j\|_{L^\infty(\Omega)} T. \tag{38}$$

Combining Eqs. 37 and 38 we finally obtain

$$|\widehat{\psi}(T, \gamma_{j1}(T)) - \widehat{\psi}(0, 0)| = |\widehat{\psi}(\widehat{\gamma}(T)) - \widehat{\psi}(\widehat{\gamma}(0))| = |\psi \circ \gamma(T) - \psi \circ \gamma(0)| \leq \|\omega_j\|_{L^\infty(\Omega)} T,$$

for any integral curve $\gamma : [0, T] \rightarrow \Omega$ of D_j^ψ with $\gamma(0) = 0$.

For the general case, assume $w \in \Omega$ and let $\gamma : [0, T] \rightarrow \Omega$ be an integral curve of D_j^ψ with $\gamma(0) = w$. Setting $q := (w \cdot \psi(w))^{-1}$, by Lemma 2.11 and in particular Eq. 23, there exists an integral curve $\gamma_q : [0, T] \rightarrow P_q(\Omega)$ of $D_j^{\psi_q}$ such that $\gamma_q(0) = 0$ and

$$\psi_q(\gamma_q(t)) - \psi_q(\gamma_q(0)) = \psi(\gamma(t)) - \psi(\gamma(0)), \quad \forall t \in [0, T]. \tag{39}$$

We also know by Proposition 2.1 that $D_j^{\psi_q} \psi_q = \omega_j \circ P_{q-1}$ in the distributional sense in $P_q(\Omega)$. Since $w \in \Omega$, then $0 \in P_q(\Omega)$ and $\psi_q(0) = 0$, see items (c) and (d) of Proposition 2.10. We can therefore run the same argument used in the preliminary dimensional reduction and the first part of (a) to $\psi_q, P_q(\Omega), \gamma_q$ and $\omega_j \circ P_{q-1}$, to get that

$$|\psi_q \circ \gamma_q(T) - \psi_q \circ \gamma_q(0)| \leq \|\omega_j \circ P_{q-1}\|_{L^\infty(P_q(\Omega))} T.$$

The proof of (a) is complete if we use Eq. 39 and we observe that the Lipschitz constant is uniform by the fact that

$$\|\omega_j\|_{L^\infty(\Omega)} = \|\omega_j \circ P_{q-1}\|_{L^\infty(P_q(\Omega))}.$$

(b) Fix $a_0 \in \Omega$, $2 \leq j \leq m$ and let $\delta_1 > 0$ be such that $B(a_0, 2\delta_1) \cap \mathbb{W} \subseteq \Omega$. Up to reducing δ_1 , recalling the explicit expression of P_q in Eq. 16, we can assume that for every $w \in B(a_0, \delta_1) \cap \mathbb{W}$ one has $B(0, 2\delta_1) \cap \mathbb{W} \subseteq P_q(\Omega)$ where, as before, $q := (w \cdot \psi(w))^{-1}$.

Let $w \in B(a_0, \delta_1) \cap \mathbb{W}$. From the fact that $D_j^\psi \psi = \omega_j$ in the distributional sense on Ω , we conclude that $D_j^{\psi_q} \psi_q = \omega_j \circ P_{q-1}$ in the distributional sense on $P_q(\Omega)$, where $q := (w \cdot \psi(w))^{-1}$, see Proposition 2.12. Moreover $0 \in P_q(\Omega)$ and $\psi_q(0) = 0$, see items (c) and (d) of Proposition 2.10. Thus from the preliminary result on the reduction of dimension, see Eq. 34 and Eq. 35, we conclude that $D_j^{\widehat{\psi}_q} \widehat{\psi}_q = \widehat{\omega}_j \circ \widehat{P}_{q-1}$ holds in the distributional sense on $\widehat{P}_q(\Omega)$. Here we recall that by $D_j^{\widehat{\psi}_q}$ we mean the classical Burgers' operator $\partial_j - \widehat{\psi}_q \partial_{j1}$ on the open subset $\widehat{P}_q(\Omega)$ of $\mathbb{R}^2 := \{(x_j, y_{j1}) : x_j, y_{j1} \in \mathbb{R}\}$.

Then we exploit this information and the argument in [8, Step 1 of proof of Theorem 1.2] to find $0 < \delta_2 < \delta_1$ and a C^1 -smooth integral curve $\widehat{\gamma}: [-\delta_2, \delta_2] \rightarrow B(0, \delta_1) \cap \widehat{P}_q(\Omega)$ of $D_j^{\widehat{\psi}_q}$ such that $\widehat{\gamma}(0) = 0$ and

$$\widehat{\psi}_q(\widehat{\gamma}(t)) - \widehat{\psi}_q(\widehat{\gamma}(0)) = \int_0^t (\omega_j \circ \widehat{P}_{q^{-1}})(\widehat{\gamma}(s)) \, ds, \quad \forall t \in [-\delta_2, \delta_2]. \tag{40}$$

Moreover, by the same argument used in [8, Step 1 of proof of Theorem 1.2], we can choose $\delta_2 := \min\{\delta_1/4, \delta_1/(2M_q)\}$, where $M_q := \sup_{B(0, 2\delta_1) \cap \widehat{P}_q(\Omega)} |\widehat{\psi}_q|$. In particular, if $w \in B(a_0, \delta_1) \cap \mathbb{W}$ and δ_1 is small enough, M_q has a uniform bound depending on the supremum of ψ in some a priori fixed neighborhood of a_0 , since ψ_q is explicit in terms of q , see item (c) of Proposition 2.10. As a consequence, up to eventually reducing and fixing δ_1, δ_2 has a positive lower bound independent of $q = (w \cdot \psi(w))^{-1}$, when we allow w to run in $B(a_0, \delta_1) \cap \mathbb{W}$. We still denote this lower bound with δ_2 .

Recalling Eq. 14 and the first part of the proof of item (a), we can write $\widehat{\gamma}(t) = (t, \gamma_{j1}(t))$ for some $\gamma_{j1}: [-\delta_2, \delta_2] \rightarrow \mathbb{R}$. For any $w \in B(a_0, \delta_1) \cap \mathbb{W}$ we can hence define a w -dependent $\gamma: [-\delta_2, \delta_2] \rightarrow B(0, \delta_1) \cap \mathbb{W} \subseteq P_q(\Omega)$ by letting

$$\gamma(t) := (0, \dots, 0, t, 0, \dots, 0, \gamma_{j1}(t), 0, \dots, 0), \quad \forall t \in [-\delta_2, \delta_2]. \tag{41}$$

Then, since $\gamma(0) = 0$, from the particular expression of $D_j^{\psi_q}$, see Eqs. 13 and 14, and by the fact that $\widehat{\gamma}$ is an integral curve of $D_j^{\widehat{\psi}_q}$, we get that γ is an integral curve of $D_j^{\psi_q}$, and from Eqs. 41, 40, and Eq. 35 the following equality holds

$$\psi_q(\gamma(t)) - \psi_q(\gamma(0)) = \int_0^t (\omega_j \circ P_{q^{-1}})(\gamma(s)) \, ds, \quad \forall t \in [-\delta_2, \delta_2].$$

Thanks to Lemma 2.11 and to item (b) of Proposition 2.10, we can translate the integral curve γ to an integral curve $\gamma_{q^{-1}}: [-\delta_2, \delta_2] \rightarrow \Omega$ of D_j^{ψ} with $\gamma_{q^{-1}}(0) = w$, such that, exploiting Eq. 23, the following equality holds

$$\psi_q(\gamma(t)) - \psi_q(\gamma(0)) = \psi(\gamma_{q^{-1}}(t)) - \psi(\gamma_{q^{-1}}(0)) = \int_0^t \omega_j(\gamma_{q^{-1}}(s)) \, ds, \quad \forall t \in [-\delta_2, \delta_2],$$

where the last equality is true since $P_{q^{-1}} \circ \gamma = \gamma_{q^{-1}}$, see Eq. 21 and Eq. 16. Thus we have shown that if we fix $a_0 \in \Omega, j = 2, \dots, m$, and δ_1 sufficiently small, we can find δ_2

such that for every $w \in B(a_0, \delta_1)$ there exists an integral curve $\gamma: [-\delta_2, \delta_2] \rightarrow \Omega$ of D_j^{ψ} such that $\gamma(0) = w$ and

$$\psi(\gamma(t)) - \psi(\gamma(0)) = \int_0^t \omega_j(\gamma(s)) \, ds, \quad \forall t \in [-\delta_2, \delta_2].$$

By the continuity of ψ this suffices to conclude that $D^{\psi} \psi = \omega$ in the broad* sense on Ω , see Definition 2.13. □

Proposition 3.3 *Let \mathbb{G} be a Carnot group of step 2, rank m and topological dimension $m + h$, and let $\mathbb{W}_{\mathbb{G}}$ and $\mathbb{L}_{\mathbb{G}}$ be two complementary subgroups of \mathbb{G} , with $\mathbb{L}_{\mathbb{G}}$ one-dimensional. Let \mathbb{F} be the free Carnot group of step 2, and rank m and topological dimension n , and choose coordinates on \mathbb{G} and \mathbb{F} as explained in Remark 2.5. Denote with $\mathbb{W}_{\mathbb{F}}$ and $\mathbb{L}_{\mathbb{F}}$ the complementary subgroups of \mathbb{F} with $\mathbb{L}_{\mathbb{F}}$ one-dimensional such that $\pi(\mathbb{W}_{\mathbb{F}}) = \mathbb{W}_{\mathbb{G}}$ and $\pi_{|\mathbb{L}_{\mathbb{F}}}: \mathbb{L}_{\mathbb{F}} \rightarrow \mathbb{L}_{\mathbb{G}}$, see Eq. 10. Let U be an open set in $\mathbb{W}_{\mathbb{G}}$ and denote by $V \subseteq \mathbb{W}_{\mathbb{F}}$ the open*

set defined by $V := \pi^{-1}(U)$. Let $\varphi : U \rightarrow \mathbb{L}_{\mathbb{G}}$ be a continuous map and let $\psi : V \rightarrow \mathbb{L}_{\mathbb{F}}$ be the map defined as

$$\psi := (\pi|_{\mathbb{L}_{\mathbb{F}}})^{-1} \circ \varphi \circ \pi|_V.$$

Assume there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi \varphi = \omega$ holds in the distributional sense on U .

Then, ψ is a distributional solution to $D^\psi \psi = \omega \circ \pi$ on V .

Proof Fix $j = 2, \dots, m$. Let us identify any element in $\mathbb{W}_{\mathbb{G}}$ with (x^*, y^*) where $x^* \in \mathbb{R}^{m-1}$ and $y^* \in \mathbb{R}^h$, and let us identify any element in $\mathbb{W}_{\mathbb{F}}$ with (x, y) where $x \in \mathbb{R}^{m-1}$ and $y \in \mathbb{R}^{m(m-1)/2}$. Then, taking Eq. 12 into account, we have that

$$\int_U \left(-\varphi \frac{\partial \xi}{\partial x_j^*} + \frac{1}{2} \sum_{i=1}^h b_{j1}^{(i)} \varphi^2 \frac{\partial \xi}{\partial y_i^*} + \frac{1}{2} \sum_{i=1}^h \sum_{\ell=2}^m x_\ell^* b_{j\ell}^{(i)} \varphi \frac{\partial \xi}{\partial y_i^*} \right) dx^* dy^* = \int_U \omega_j \xi dx^* dy^*, \tag{42}$$

for every $\xi \in C_c^\infty(U)$. Hence, by exploiting Eq. 13, we would like to show that

$$\begin{aligned} \langle D_j^\psi \psi, \tilde{\xi} \rangle &:= \int_V \left(-\psi \frac{\partial \tilde{\xi}}{\partial x_j} + \frac{1}{2} \psi^2 \frac{\partial \tilde{\xi}}{\partial y_{j1}} - \frac{1}{2} \sum_{j < \ell \leq m} x_\ell \psi \frac{\partial \tilde{\xi}}{\partial y_{\ell j}} + \frac{1}{2} \sum_{1 < s < j} x_s \psi \frac{\partial \tilde{\xi}}{\partial y_{js}} \right) dx dy \\ &= \int_V (\omega_j \circ \pi) \tilde{\xi} dx dy, \end{aligned} \tag{43}$$

for every $\tilde{\xi} \in C_c^\infty(V)$. We consider the change of variables in $\mathbb{W}_{\mathbb{F}} \cong \mathbb{R}^{n-1}$ given by

$$\left(x_2^*, \dots, x_m^*, y_1^*, \dots, y_h^*, \hat{y}_{h+1}^*, \dots, \hat{y}_{\frac{m(m-1)}{2}}^* \right)^\top := M (x_2, \dots, x_m, y_{21}, \dots, y_{m(m-1)})^\top,$$

M being a matrix of order $n - 1$ defined as

$$M := \begin{pmatrix} I_{m-1} & 0 & \dots & 0 \\ 0 & b_{21}^{(1)} & \dots & b_{m(m-1)}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & b_{21}^{(h)} & \dots & b_{m(m-1)}^{(h)} \\ 0 & & & \tilde{M} \end{pmatrix}, \tag{44}$$

where I_{m-1} is the identity matrix of order $m - 1$ and \tilde{M} is a $(\frac{m(m-1)}{2} - h) \times \frac{m(m-1)}{2}$ matrix such that M is invertible. We denote the elements of \tilde{M} by $b_{\ell s}^{(i)}$ with $1 \leq s < \ell \leq m$ and $i = h + 1, \dots, \frac{m(m-1)}{2}$. Such a matrix \tilde{M} exists thanks to the fact that the matrices $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(h)}$ as in Eq. 3 are linearly independent and then the matrix

$$\begin{pmatrix} b_{21}^{(1)} & \dots & b_{m(m-1)}^{(1)} \\ \vdots & \ddots & \vdots \\ b_{21}^{(h)} & \dots & b_{m(m-1)}^{(h)} \end{pmatrix},$$

has maximum rank equal to h . Denote for shortness $x^* := (x_2^*, \dots, x_m^*)$, $y^* := (y_1^*, \dots, y_h^*)$ and $\hat{y}^* := (\hat{y}_{h+1}^*, \dots, \hat{y}_{\frac{m(m-1)}{2}}^*)$. By Eq. 9 we get that, for every $(0, x_2, \dots, x_m, y_{21}, \dots, y_{m(m-1)}) \in V$, the following equality holds

$$\begin{aligned} \psi(x_2, \dots, x_m, y_{21}, \dots, y_{m(m-1)}) &= \varphi\left(x_2, \dots, x_m, \sum_{1 \leq s < \ell \leq m} b_{\ell s}^{(1)} y_{\ell s}, \dots, \sum_{1 \leq s < \ell \leq m} b_{\ell s}^{(h)} y_{\ell s}\right) \\ &= \varphi(x^*, y^*). \end{aligned}$$

Given $\tilde{\xi} \in C_c^\infty(V)$, we define $\xi(x^*, y^*, \hat{y}^*) := \tilde{\xi} \circ M^{-1}(x^*, y^*, \hat{y}^*)^\top \in C_c^\infty(M(V))$, and then, by using the chain rule in order to write the partial derivatives of ξ with respect to ξ , we can write

$$\begin{aligned} \langle D_j^\psi \psi, \tilde{\xi} \rangle &= \int d\hat{y}^* \int \left(-\varphi \frac{\partial \xi}{\partial x_j^*} + \frac{1}{2} \varphi^2 \sum_{i=1}^h b_{j1}^{(i)} \frac{\partial \xi}{\partial y_i^*} - \frac{1}{2} \sum_{j < \ell \leq m} \sum_{i=1}^h b_{\ell j}^{(i)} x_\ell^* \varphi \frac{\partial \xi}{\partial y_i^*} \right. \\ &\quad \left. + \frac{1}{2} \sum_{1 < s < j} \sum_{i=1}^h b_{js}^{(i)} x_s^* \varphi \frac{\partial \xi}{\partial y_i^*} \right) \frac{1}{|\det(M)|} dx^* dy^* \\ &\quad + \int \sum_{i=h+1}^{\frac{m(m-1)}{2}} \left(\frac{1}{2} \varphi^2 b_{j1}^{(i)} - \frac{1}{2} \sum_{j < \ell \leq m} b_{\ell j}^{(i)} x_\ell^* \varphi + \frac{1}{2} \sum_{1 < s < j} b_{js}^{(i)} x_s^* \varphi \right) \frac{\partial \xi}{\partial \hat{y}_i^*} \frac{1}{|\det(M)|} dx^* dy^* d\hat{y}^* \\ &= \int d\hat{y}^* \int \left(-\varphi \frac{\partial \xi}{\partial x_j^*} + \frac{1}{2} \varphi^2 \sum_{i=1}^h b_{j1}^{(i)} \frac{\partial \xi}{\partial y_i^*} + \frac{1}{2} \sum_{k=2}^m \sum_{i=1}^h b_{jk}^{(i)} x_k^* \varphi \frac{\partial \xi}{\partial y_i^*} \right) \frac{1}{|\det(M)|} dx^* dy^*, \end{aligned} \tag{45}$$

where $1/|\det(M)|$ is the determinant of the change of variables; we stress that in the last equality we used the fact that $\mathcal{B}^{(i)}$ is a skew-symmetric matrix for every $i = 1, \dots, h$, and

$$\begin{aligned} &\int \sum_{i=h+1}^{\frac{m(m-1)}{2}} \left(\frac{1}{2} \varphi^2 b_{j1}^{(i)} - \frac{1}{2} \sum_{j < \ell \leq m} b_{\ell j}^{(i)} x_\ell^* \varphi + \frac{1}{2} \sum_{1 < s < j} b_{js}^{(i)} x_s^* \varphi \right) \frac{\partial \xi}{\partial \hat{y}_i^*} \frac{1}{|\det(M)|} dx^* dy^* d\hat{y}^* \\ &=: \int \sum_{i=h+1}^{\frac{m(m-1)}{2}} A_i(x^*, y^*) \frac{\partial \xi}{\partial \hat{y}_i^*} \frac{1}{|\det(M)|} dx^* dy^* d\hat{y}^* = 0, \end{aligned}$$

because $\xi \in C_c^\infty(M(V))$, together with the use of Fubini's Theorem and the fact that the terms of $A_i(x^*, y^*)$ only depend on the variables x^* and y^* .

Since, by construction, the projection of $M(V)$ onto the variables (x^*, y^*) is precisely $\pi(V) = U$, we get that $\xi(\cdot, \hat{y}_0^*) \in C_c^\infty(U)$ for every $\hat{y}_0^* \in \mathbb{R}^{m(m-1)/2-h}$, and thus by using Eq. 42 and Fubini's Theorem, we get

$$\langle D_j^\psi \psi, \tilde{\xi} \rangle = \int d\hat{y}^* \int \omega_j \xi \frac{1}{|\det(M)|} dx^* dy^* = \int \omega_j \xi \frac{1}{|\det(M)|} dx^* dy^* d\hat{y}^*.$$

Finally, if we consider the reversed change of variables $(x_2, \dots, x_m, y_{21}, \dots, y_{m(m-1)})^\top = M^{-1}(x^*, y^*, \hat{y}^*)^\top$ where M^{-1} is the inverse matrix of M , see Eq. 44, and recalling Eq. 9,

it follows

$$\begin{aligned}
 \langle D_j^\psi \psi, \tilde{\xi} \rangle &= \int \omega_j \left(x, \sum_{1 \leq s < \ell \leq m} b_{\ell s}^{(1)} y_{\ell s}, \dots, \sum_{1 \leq s < \ell \leq m} b_{\ell s}^{(h)} y_{\ell s} \right) \tilde{\xi}(x, y) \frac{|\det(M)|}{|\det(M)|} dx dy. \\
 &= \int (\omega_j \circ \pi)(x, y) \tilde{\xi}(x, y) dx dy,
 \end{aligned}
 \tag{46}$$

for every $\tilde{\xi} \in C_c^\infty(V)$, where $|\det(M)|$ is the determinant of the change of variables. Hence Eq. 43 holds and the proof is complete. \square

Proposition 3.4 *Let \mathbb{G} be a Carnot group of step 2, rank m and topological dimension $m+h$ and let $\mathbb{W}_{\mathbb{G}}$ and $\mathbb{L}_{\mathbb{G}}$ be two complementary subgroups of \mathbb{G} , with $\mathbb{L}_{\mathbb{G}}$ one-dimensional. Let \mathbb{F} be the free Carnot group of step 2, rank m and topological dimension n , and choose coordinates on \mathbb{G} and \mathbb{F} as explained in Remark 2.5. Denote by $\mathbb{W}_{\mathbb{F}}$ and $\mathbb{L}_{\mathbb{F}}$ the complementary subgroups of \mathbb{F} with $\mathbb{L}_{\mathbb{F}}$ one-dimensional such that $\pi(\mathbb{W}_{\mathbb{F}}) = \mathbb{W}_{\mathbb{G}}$ and $\pi|_{\mathbb{L}_{\mathbb{F}}} := \mathbb{L}_{\mathbb{F}} \rightarrow \mathbb{L}_{\mathbb{G}}$, see Eq. 10. Let U be an open set in $\mathbb{W}_{\mathbb{G}}$ and denote with $V \subseteq \mathbb{W}_{\mathbb{F}}$ the open set defined by $V := \pi^{-1}(U)$. Let $\varphi: U \rightarrow \mathbb{L}_{\mathbb{G}}$ be a continuous map and let $\psi: V \rightarrow \mathbb{L}_{\mathbb{F}}$ be the map defined as*

$$\psi := (\pi|_{\mathbb{L}_{\mathbb{F}}})^{-1} \circ \varphi \circ \pi|_V.$$

Assume there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\psi \psi = \omega \circ \pi$ holds in the broad* sense on V . Then, φ is a broad* solution to $D^\varphi \varphi = \omega$ on U .

Proof In order to give the proof of the statement we first show the following intermediate result: for every $j = 2, \dots, m$, every point $q := (0, x_2, \dots, x_m, y_{21}, \dots, y_{m(m-1)}) \in V$, and every integral curve $\gamma: [0, T] \rightarrow V$ of D_j^ψ starting from q we have that $\pi \circ \gamma: [0, T] \rightarrow U$ is an integral curve of D_j^φ starting from $\pi(q) = (0, x_2, \dots, x_m, y_1^*, \dots, y_h^*)$, see Eq. 8. Moreover we stress that from Eq. 9 we have $y_i^* = \sum_{1 \leq s < \ell \leq m} b_{\ell s}^{(i)} y_{\ell s}$, for all $i = 1, \dots, h$.

Take an integral curve $\gamma: [0, T] \rightarrow U$ of D_j^ψ starting from q . Then, the vertical components of γ satisfy the system of ODEs in Eq. 14, while the horizontal components stay the same apart from the j -th component for which $\gamma_j(t) = x_j + t$ for every $t \in [0, T]$. From the explicit expression of the projection in Eqs. 8 and 9, we can write the vertical components of $\pi \circ \gamma$ as a linear combination of the vertical components of γ , while the horizontal components stay the same. Then, exploiting the ODEs in Eq. 14, taking the derivatives of those linear expressions, and by using the definition of ψ in terms of φ in the statement, one simply obtains that $\pi \circ \gamma: [0, T] \rightarrow U$ is an integral curve of D_j^φ starting from $\pi(q)$. We omit the simple computations since they are similar to the ones already done before.

In order to conclude, notice that, from the relation between ψ and φ in the statement, we obtain the following equivalence

$$\varphi(\pi \circ \gamma(t)) - \varphi(\pi \circ \gamma(0)) = \int_0^t \omega_j(\pi \circ \gamma(s)) ds \Leftrightarrow \psi(\gamma(t)) - \psi(\gamma(0)) = \int_0^t (\omega_j \circ \pi)(\gamma(s)) ds, \tag{47}$$

for every integral curve $\gamma: [0, T] \rightarrow V$ of D_j^ψ , with $j = 2, \dots, m$, and every $t \in [0, T]$. Thus, from the previous observation on the projection of the integral curves and the equivalence in Eq. 47, we get the thesis by taking into account that the definition of broad* solution in Definition 2.13 is local. \square

4 Main Theorems

We are ready to prove the main theorem of this paper, by making use of the invariance results proved in Section 3. We stress that the following theorem is the converse of [3, Corollary 6.15].

Theorem 4.1 *Let \mathbb{G} be a Carnot group of step 2 and rank m , and let \mathbb{W} and \mathbb{L} be two complementary subgroups of \mathbb{G} , with \mathbb{L} horizontal and one-dimensional. Let $U \subseteq \mathbb{W}$ be an open set, and let $\varphi: U \rightarrow \mathbb{L}$ be a continuous function. Choose coordinates on \mathbb{G} as explained in Section 2.2, see also Eq. 5. Assume there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi \varphi = \omega$ holds in the distributional sense on U . Then $D^\varphi \varphi = \omega$ holds in the broad* sense on U .*

Proof It directly follows by joining together Proposition 3.3, Proposition 3.2, and Proposition 3.4. □

By making use of the previous theorem and [3, Theorem 6.17] we obtain the following characterization of $C^1_{\mathbb{H}}$ -hypersurfaces in Carnot groups of step 2. For the notion of intrinsic differentiability we refer the interested reader to [3, Definition 2.17], while for the notion of intrinsic gradient we refer the interested reader to [3, Definition 2.20 & Remark 2.21]. For the definition of $C^1_{\mathbb{H}}$ -hypersurface we refer the interested reader to [25, Definition 1.6]. For a detailed account on the latter notion we refer the interested reader to the introduction of [3] and in particular to [3, Definition 2.27] for the definition of co-horizontal $C^1_{\mathbb{H}}$ -regular surfaces with complemented tangents.

Theorem 4.2 *Let \mathbb{G} be a Carnot group of step 2 and rank m , and let \mathbb{W} and \mathbb{L} be two complementary subgroups of \mathbb{G} , with \mathbb{L} horizontal and one-dimensional. Let $U \subseteq \mathbb{W}$ be an open set and let $\varphi: U \rightarrow \mathbb{L}$ be a continuous function. Choose coordinates on \mathbb{G} as explained in Section 2.2, see also Eq. 5. Then the following conditions are equivalent:*

- (a) *graph(φ) is a $C^1_{\mathbb{H}}$ -hypersurface with tangents complemented by \mathbb{L} ;*
- (b) *φ is uniformly intrinsically differentiable on U ;*
- (c) *φ is intrinsically differentiable on U and its intrinsic gradient is continuous;*
- (d) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that, for every $a \in U$, there exist $\delta > 0$ and a family of functions $\{\varphi_\varepsilon \in C^1(B(a, \delta)) : \varepsilon \in (0, 1)\}$ such that*

$$\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon = \varphi \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} D_j^{\varphi_\varepsilon} \varphi_\varepsilon = \omega_j \quad \text{in } L^\infty(B(a, \delta)),$$

for every $j = 2, \dots, m$;

- (e) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi \varphi = \omega$ holds in the broad sense on U ;*
- (f) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi \varphi = \omega$ holds in the broad* sense on U ;*
- (g) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi \varphi = \omega$ holds in the distributional sense on U .*

Moreover, if any of the previous holds, ω is the intrinsic gradient of φ .

Proof The equivalence between (a),(b),(c),(d),(e), and (f) follows from [3, Theorem 6.17]. The implication (g) \Rightarrow (f) follows from Theorem 4.1, where Theorem 4.1 is the one in the paper. The implication (b) \Rightarrow (g) follows from [3, Item (c) of Proposition 4.10]. □

Remark 4.3 (Intrinsic normal and area formula) We stress that if any of the hypotheses in Theorem 4.2 is satisfied, we can write the intrinsic normal to $\text{graph}(\varphi)$ and an area formula for $\text{graph}(\varphi)$ explicitly in terms of the intrinsic gradient ω , see [3, Item (d) of Proposition 4.10 and Remark 4.11].

Remark 4.4 (Approximation of a distributional solution to $D^\varphi\varphi = \omega$) The approximating sequence in item (d) of Theorem 4.2 is a priori dependent on the point $a \in U$ we choose. This happens because in order to obtain [3, Theorem 6.17], from which Theorem 4.2 follows, we use [3, Item (b) of Proposition 4.10], in which the approximating sequence is constructed in a way that is a priori dependent on the point $a \in U$. Nevertheless the upgrade of such approximation from a local one on balls to an approximation on arbitrary compact sets, with a sequence of functions that is not dependent on the compact set itself, is very likely to be true in the setting of Carnot groups of step 2 by exploiting the same technique explained in [3, Remark 4.13] and based on [37]. Since this topic does not fit in this paper we will not treat it here, and it will subject of further investigations.

Remark 4.5 (Counterexample to Theorem 4.2 on the Engel group) Consider the Engel group \mathbb{E} , i.e., the Carnot group whose Lie algebra \mathfrak{e} admits an adapted basis (X_1, X_2, X_3, X_4) such that

$$\mathfrak{e} := \text{span}\{X_1, X_2\} \oplus \text{span}\{X_3\} \oplus \text{span}\{X_4\},$$

where $[X_1, X_2] = X_3$, and $[X_1, X_3] = X_4$. We identify \mathbb{E} with \mathbb{R}^4 by means of exponential coordinates, and we define the couple of homogeneous complementary subgroups $\mathbb{W} := \{x_1 = 0\}$, and $\mathbb{L} := \{x_2 = x_3 = x_4 = 0\}$ in such coordinates. Then, by explicit computations that can be found in [28, Section 4.4.1], we get that, for a continuous function $\varphi: U \subseteq \mathbb{W} \rightarrow \mathbb{L}$, with U open, the projected vector fields on \mathbb{W} are

$$D_{X_2}^\varphi = \partial_{x_2} + \varphi\partial_{x_3} + \frac{\varphi^2}{2}\partial_{x_4}, \quad D_{X_3}^\varphi = \partial_{x_3} + \varphi\partial_{x_4}, \quad D_{X_4}^\varphi = \partial_{x_4}. \tag{48}$$

Thus, if we consider the function $\varphi(0, x_2, x_3, x_4) := x_4^{1/3}$ on \mathbb{W} , we get that $D_{X_2}^\varphi\varphi = \frac{1}{6}\partial_{x_4}(\varphi^3) = \frac{1}{6}$ in the distributional sense on \mathbb{W} . On the other hand $\varphi: \mathbb{W} \rightarrow \mathbb{L}$ is not uniformly intrinsically differentiable, since it is not 1/3-little Hölder continuous along the coordinate x_4 , see Definition 4.6, while for a function to be uniformly intrinsically differentiable this is a necessary condition, see [3, Example 5.3] and [3, (a) \Rightarrow (c) of Theorem 4.17]. Then we conclude that the chain of equivalences of Theorem 4.2 cannot be extended already in the easiest step-3 Carnot group.

Nevertheless we do not know whether Theorem 4.1 holds in some cases beyond the setting of step-2 Carnot groups. In particular we do not know whether Theorem 4.1 holds in the Engel group with the splitting previously discussed. Interesting developments in the direction of studying whether distributional solutions to Burgers' type equations with non-convex fluxes are also broad solutions are given in [2] and [1].

We conclude with the following Hölder property that happens to be a consequence of φ being a distributional solution to $D^\varphi\varphi = \omega$ with a continuous datum ω . For the purpose, we here recall the definition of little Hölder continuity.

Definition 4.6 (little Hölder functions, [32]) Let $U \subseteq \mathbb{R}^n$ be an open set. We denote by $h^\alpha(U; \mathbb{R}^k)$ the set of all α -little Hölder continuous functions of order $0 < \alpha < 1$, i.e., the

set of maps $\varphi \in C(U; \mathbb{R}^k)$ satisfying

$$\lim_{r \rightarrow 0} \left(\sup \left\{ \frac{|\varphi(b') - \varphi(b)|}{|b' - b|^\alpha} : b, b' \in U, 0 < |b' - b| < r \right\} \right) = 0. \quad (49)$$

Theorem 4.7 *Let \mathbb{G} be a Carnot group of step 2 and rank m , and let \mathbb{W} and \mathbb{L} be two complementary subgroups of \mathbb{G} , with \mathbb{L} horizontal and one-dimensional. Let $U \subseteq \mathbb{W}$ be an open set and let $\varphi: U \rightarrow \mathbb{L}$ be a continuous function. Choose coordinates on \mathbb{G} as explained in Section 2.2, see also Eq. 5. If one of the items of Theorem 4.2 holds, then φ is 1/2-little Hölder continuous along the vertical coordinates on every open subset $U' \Subset U$.*

Proof It is a consequence of Theorem 4.2 and [3, Remark 3.23 & Theorem 6.12]. \square

Funding Open access funding provided by Scuola Normale Superiore within the CRUI-CARE Agreement.

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References

1. Alberti, G., Bianchini, S., Caravenna, L.: Eulerian, lagrangian and broad continuous solutions to a balance law with non-convex flux II. arXiv:[SISSA 32/2016/MATE](https://arxiv.org/abs/1603.03437)
2. Alberti, G., Bianchini, S., Caravenna, L.: Eulerian, Lagrangian and broad continuous solutions to a balance law with non-convex flux I. *J. Differential Equations* **261**(8), 4298–4337 (2016)
3. Antonelli, G., Di Donato, D., Don, S., Le Donne, E.: Characterizations of uniformly differentiable co-horizontal intrinsic graphs in Carnot groups. Accepted in *Annales de l'Institut Fourier*. Preprint, available at arXiv:[2005.11390](https://arxiv.org/abs/2005.11390) (2020)
4. Antonelli, G., Le Donne, E.: Pauls rectifiable and purely Pauls unrectifiable smooth hypersurfaces. *Nonlinear Anal.* **200**(30), 111983 (2020)
5. Ambrosio, L., Cassano, F.S., Vittone, D.: Intrinsic regular hypersurfaces in Heisenberg groups. *J. Geom. Anal.* **16**(2), 187–232 (2006)
6. Bigolin, F., Caravenna, L., Cassano, F.S.: Intrinsic Lipschitz graphs in Heisenberg groups and continuous solutions of a balance equation. *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **32**(5), 925–963 (2015)
7. Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin (2007)
8. Bigolin, F., Cassano, F.S.: Distributional solutions of Burgers' equation and intrinsic regular graphs in Heisenberg groups. *J. Math. Anal. Appl.* **366**(2), 561–568 (2010)
9. Bigolin, F., Cassano, F.S.: Intrinsic regular graphs in Heisenberg groups vs. weak solutions of non-linear first-order PDEs. *Adv. Calc. Var.* **3**(1), 69–97 (2010)
10. Capogna, L., Danielli, D., Pauls, S.D., Tyson, J.T.: *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Volume 259 of *Progress in Mathematics*. Basel, Birkhäuser Verlag (2007)
11. Corni, F., Magnani, V.: Area formula for regular submanifolds of low codimension in Heisenberg groups. Preprint, available at arXiv:[2002.01433](https://arxiv.org/abs/2002.01433) (2020)
12. Citti, G., Manfredini, M., Sarti, A.: Neuronal oscillations in the visual cortex: Γ -convergence to the Riemannian Mumford-Shah functional. *SIAM J. Math. Anal.* **35**(6), 1394–1419 (2004)
13. Corni, F.: Intrinsic regular surfaces of low codimension in Heisenberg groups. Accepted paper on *Ann. Acad. Sci. Fenn. Math.* (2020)
14. Dafermos, C.M.: Continuous solutions for balance laws. *Ric. Mat.* **55**(1), 79–91 (2006)

15. Di Donato, D.: Intrinsic differentiability and intrinsic regular surfaces in carnot groups Accepted paper in Potential Anal. (2020)
16. Di Donato, D.: Intrinsic lipschitz graphs in carnot groups of step 2. Ann. Acad. Sci. Fenn. Math, pp. 1–51 (2020)
17. De Giorgi, E.: Su una teoria generale della misura $(r - 1)$ -dimensionale in uno spazio ad r dimensioni. Ann. Mat Pura Appl. **36**(4), 191–213 (1954)
18. De Giorgi, E.: Nuovi teoremi relativi alle misure $(r - 1)$ -dimensionali in uno spazio ad r dimensioni. Ricerche Mat. **4**, 95–113 (1955)
19. Don, S., Le Donne, E., Moisala, T., Vittone, D.: A rectifiability result for finite-perimeter sets in Carnot groups. Accepted in Indiana Univ. Math. J. Preprint, available at arXiv:[1912.00493](https://arxiv.org/abs/1912.00493) (2019)
20. Franchi, B., Marchi, M., Serapioni, R.: Differentiability and approximate differentiability for intrinsic lipschitz functions in carnot groups and a rademacher theorem. Anal. Geom. Metr. Spaces **2**(3), 258–281 (2014)
21. Folland, G.B.: A fundamental solution for a subelliptic operator. Bull. Amer. Math. Soc. **79**, 373–376 (1973)
22. Folland, G.B.: Subelliptic estimates and function spaces on nilpotent Lie groups. Ark Mat. **13**(2), 161–207 (1975)
23. Franchi, B., Serapioni, R.P.: Intrinsic Lipschitz graphs within Carnot groups. J. Geom. Anal. **26**(3), 1946–1994 (2016)
24. Franchi, B., Serapioni, R., Serapioni, R.P., Cassano, F.S.: Rectifiability and perimeter in the Heisenberg group. Math. Ann. **321**(3), 479–531 (2001)
25. Franchi, B., Serapioni, R., Cassano, F.S.: On the structure of finite perimeter sets in step 2 Carnot groups. The Journal of Geometric Analysis **13**(3), 421–466 (2003)
26. Franchi, B., Serapioni, R., Cassano, F.S.: Regular submanifolds, graphs and area formula in Heisenberg groups. Adv. Math. **211**(1), 152–203 (2007)
27. Julia, A., Golo, S.N., Vittone, D.: Area of intrinsic graphs and coarea formula in Carnot groups. Accepted in Mathematische Zeitschrift. Preprint, available at arXiv:[1811.05457](https://arxiv.org/abs/1811.05457) (2020)
28. Kozhevnikov, A.: Propriétés Métriques Des Ensembles De Niveau Des Applications Différenciables Sur Les Groupes De Carnot. PhD Thesis Université Paris Sud - Paris XI (2015)
29. Le Donne, E.: A primer on Carnot groups: homogenous groups, carnot-Carathéodory spaces, and regularity of their isometries. Anal. Geom. Metr. Spaces **5**, 116–137 (2017)
30. Le Donne, E., Moisala, T.: Semigenerated Carnot algebras and applications to sub-riemannian perimeter. Accepted in Mathematische Zeitschrift. Preprint, available at arXiv:[2004.08619](https://arxiv.org/abs/2004.08619) (2020)
31. Le Donne, E., Pinamonti, A., Speight, G.: Universal differentiability sets and maximal directional derivatives in Carnot groups. J. Math. Pures Appl. **121**(9), 83–112 (2019)
32. Lunardi, A.: Analytic semigroups and optimal regularity in parabolic problems. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel. [2013 reprint of the 1995 original] (1995)
33. Magnani, V.: Towards differential calculus in stratified groups. J. Aust. Math. Soc. **95**(1), 76–128 (2013)
34. Magnani, V.: Towards a theory of area in homogeneous groups. Calc. Var. Partial Differential Equations, 58(3):Paper No. 91, 39 (2019)
35. Marchi, M.: Regularity of sets with constant intrinsic normal in a class of Carnot groups. Ann. Inst. Fourier (Grenoble) **64**(2), 429–455 (2014)
36. Magnani, V., Stepanov, E., Trevisan, D.: A rough calculus approach to level sets in the Heisenberg group. J. Lond. Math Soc. (2) **97**(3), 495–522 (2018)
37. Monti, R., Vittone, D.: Sets with finite \mathbb{H} -perimeter and controlled normal. Math. Z. **270**(1–2), 351–367 (2012)
38. Sánchez-Calle, A.: Fundamental solutions and geometry of the sum of squares of vector fields. Invent. Math. **78**(1), 143–160 (1984)
39. Cassano, F.S.: Some topics of geometric measure theory in Carnot groups. In: Geometry, analysis and dynamics on sub-Riemannian manifolds. vol. 1, EMS Ser. Lect. Math., pp. 1–121. Eur. Math. Soc., Zürich (2016)
40. Stein, E.M., Shakarchi, R.: Complex analysis, volume 2 of Princeton Lectures in Analysis. Princeton University Press, Princeton (2003)
41. Vittone, D.: Lipschitz graphs and currents in Heisenberg groups. Accepted in Forum of Mathematics, Sigma. Preprint, available at arXiv:[2007.14286](https://arxiv.org/abs/2007.14286) (2020)