SCUOLA
NORMALE
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Classe di Lettere e Filosofia
Corso di perfezionamento in Filosofia

Unification and structural completeness in intermediate logics

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Anno Accademico 2021-2022

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Chapter 0

Introduction

Although the existence of (infinitely many) intermediate propositional logics was already discovered by Gödel [66] in 1932, it was only in 1955, with the works of Umezawa [155, 156], that the investigation on intermediate logics was undertaken in a systematic way. Since then, the research on intermediate logics has flourished and followed two distinct, but complementary, approaches: the first deals with single intermediate propositional logics, while the second with the system of intermediate logics as a whole by investigating properties enjoyed by large classes of logics and developing a general theory for them[1].

As it is common in science in general, after having analyzed a number of single particular objects of the same nature, one tries to abstract from them in order to study the nature itself: by making various generalizations and classifications of different kind, a new knowledge of phenomena is acquired. It thus appears evident that this approach is more interesting and that is why, by the middle of the sixties, the second approach became central and led the research on intermediate propositional logics to its acme around the eighties with the introduction, made by Zakharyaschev [171], of the apparatus of canonical formulas. However, the centrality which has been given to this approach must not make us forget the importance of the former, which, not only allowed the development of a more general investigation on the matter, but was also driven by a more philosophical oriented attitude which tends to lack in the second[2].

Over the past three decades, research on intermediate propositional logics has gone through a period of stagnation: most of the questions concerning the main properties of the system of intermediate propositional logics (such as (finite) axiomatizability, finite approximability, tabularity and decidability, only to number a few of them) were answered, reaching, in many cases, a level of

[1]Examples of works following the first approach include those of Kreisel and Putnam [94] on **KP**, Dummett [34] on **LC** and Jankov [87] on **KC**, to name only a few; while Jankov [85] and Maksimova [105], respectively showing that the cardinality of all intermediate logics is $2^{\aleph_0}$ and that there are only 8 intermediate logics with the Craig's interpolation property, are amongst the works following the second approach.

[2]The philosophical attitude was particularly evident when, immediately after Heyting's formalization of intuitionistic propositional logic [72] in 1928, logicians sought a semantic interpretation which, not only was mathematically adequate, but also "intuitionistically" satisfactory: Kleene's realizability logic [90] and Medvedev's logic of finite problems [114] are two different attempts that go in this direction and turn out to be, still nowadays, the most interesting and mysterious known intermediate propositional logics.

technical sophistication so high as to make it difficult for a beginner to even approach the subject, and the few open problems seem to be far from being solved with the current status of knowledge. Still, a few interesting contributions have appeared, in particular in connection with the investigation on the admissibility of inference rules ([78, 79]) and in the field related to unification theory ([62, 64]). With this thesis, we want to give our humble contribution to the study of intermediate propositional logics mostly along these two new lines of research[3].

Main results

The present work follows the tradition of the second approach both by investigating specific issues related to various subclasses of intermediate propositional logics and by using mainly semantical tools such as the existing (dual) equivalence between Heyting algebras and a particular class of ordered-topological spaces now known as Esakia spaces [41, 43].

Esakia duality has become a common theme in the research of the last periods: the possibility of translating algebraic or logical properties in the more intuitive form of topology (and viceversa) and the employment of geometrical techniques in order to understand and to solve both new and old problems has made such a duality an essential tool for the study of intermediate propositional logics. The intimate connection between intermediate propositional logics and modal logic touches also this topic and the proofs of Esakia duality are usually obtained as a byproduct of a duality established in the modal framework. For this reason, we give a full direct proof of the Esakia duality and unify in a coherent exposition all the correspondences of the duality between algebraic and order-topological notions scattered in the literature. Furthermore, we introduce the notion of *partial Esakia equivalence* and show that there exists a one-to-one correspondence between the partial Esakia equivalence on a given Esakia space $\mathcal{X}$ and the partial Esakia morphism from $\mathcal{X}$.

As we mentioned above, one of the most recent line of investigation in intermediate propositional logics deals with unification issues. *Unification theory* ([4, 113]) is the abstract theory of unification, a fundamental process upon which many methods for Automated Deduction are based. Generally speaking, unification can be described as the attempt to identify two given symbolic expressions by replacing certain sub-expressions in them by other expressions and it is thus concerned with equation solving (in the context of free algebras). An important contribution of unification theory to the research on intermediate propositional logics (and, in general, on equational theories) is the classification of logics according to the their *unification type*: every intermediate propositional logic can be characterized as having "bad unification properties" or "nice unification properties".

In order to determine the unification type of a given intermediate propositional logic the *finitely presented Heyting algebras* and the *regular projective Heyting algebras* play an essential rôle. Both notions have been extensively studied: already in 1970 Balbes and Horn [5] gave a complete description of

[3]Some historical notes are scattered through the thesis, but in particular in Chapter 1. The reader interested in the history of both propositional and predicative intermediate logics is referred to [121], in which an historical outline of intermediate logics and a guided bibliography of the results published until 1983 are presented.

the finite projective Heyting algebras, whereas for finitely presented Heyting algebras there exists even a categorical dual representation in terms of sheaves [65] which turned out to be more suitable than the usual order-topological one to address some specific problems on such algebras. However, here we stick to the Esakia duality and, following [21], we give a dual representation of finitely presented Heyting algebras as those Esakia spaces obtained as the categorical limit of some particular finite Esakia spaces. Furthermore, we present a useful characterization of the finitely generated regular projective Heyting algebras for any finitely approximable subvariety of Heyting algebras.

The characterization of the finitely generated regular projective Heyting algebras we obtained allows us to generalize a result of Ghilardi [64] from the locally finite to the finitely approximable context. As a consequence, we show that many interesting intermediate propositional logics do not have finitary unification type. These negative results also mean that we can not expect to acquire any information connected with the problem of the admissibility of inference rules for such logics within the unification framework. Indeed, only if the unification type of a given intermediate propositional logic is at least finitary, we can expect to obtain a decision procedure for the problem of the admissibility of inference rules based on unification (cfr. [62]).

The location in the lattice of intermediate propositional logics of most of the logics having good unification properties is well known. In particular, any intermediate propositional logic having unitary unification type must be an extension of the Jankov's logic **KC**. If, furthermore, an intermediate propositional logic L is an extension of the Gödel-Dummet logic **LC**, then L enjoys a stronger form of unitary unification, called *projective unification*. As well as providing new semantic proofs of the two previous facts, we give an abstract characterization of projective unification: an equational theory E has projective unification if and only if E has *transparent unification* and its corresponding variety $\mathcal{V}_E$ is actively structurally complete.

Structural completeness is not only a remarkable property that a logic can possess but also a significant philosophical concept. Very roughly, we can say that an inference rule ρ is admissible in a given logical calculus L if it can be used in a consistent and conservative way in any derivation in L. Now, if ρ is not derivable in L, then there is no way to provide a justification for its use within the calculus L itself: the phenomenon of admissible but not derivable inference rules in a given logical calculus L thus represents a form of incompleteness of L. That is why we say that a logic is structurally complete whenever every admissible inference rule is also derivable. The importance of structurally complete logical calculi lies in the fact that they are in a certain sense contained within themselves: the system itself makes it possible to derive all the rules of inference that are consistent with it and it is therefore completely transparent.

Investigations on structural completeness in intermediate propositional logics have focused, on the one hand, on proving structural completeness for single intermediate propositional logics ([39, 137]) or for appropriate fragment thereof ([135, 141]) and, on the other hand, on formulating adequate conditions for an intermediate propositional logic to be structurally complete ([139, 27, 30, 28]). In particular, a nice characterization of structurally complete finitely approximable intermediate propositional logics has been obtained by Rybakov [142]. In this work, we study structural completeness within the framework of canonical formulas.

Together with the disjunction property, structural completeness is also the other essential feature involved in the 41st problem posed by Harvey Friedman in [53] and characterizing those intermediate propositional logics that we called *Friedman logics*. Such logics, not only are interesting per se, but are also relevant from a philosophical point of view. Indeed, on the basis of Miglioli and Usberti's account of knowledge [120] and on further considerations on structural completeness, it turns out that *the logic of knowledge* might be a Friedman logic. After locating Friedman logics within the lattice of intermediate propositional logics, we investigate the part at finite depth of the n-canonical frames for such logics and, in particular, we give a complete characterization of the points at depth ≤ 3 of the 2-canonical frame $\mathfrak{F}_L(2)$ for any Friedman logic L. Furthermore, since the only known example of Friedman logic is the Medvedev's logic of finite problem **ML**, we study some of its frames and we prove that the 2-letter fragment of **ML** is decidable.

Contents and new contributions

The present thesis is organized as follows.

Chapter 1 is a chapter of preliminaries: we give the basic definitions and tools of intermediate propositional logics. In particular, we discuss both the Kripke and the algebraic semantics as well as the relationship between the two.

Chapter 2 is entirely devoted to Esakia duality. After introducing the category of Esakia spaces, we shall prove the categorical dual equivalence between such a category and the category of Heyting algebras and list most of the dual correspondences between algebraic and order-topological notions. Furthemore, we introduce the equivalent notion of descriptive frame. Giving a full direct proof of the Esakia duality and unifying in a coherent exposition most of the correspondences of the duality between algebraic and order-topological notions scattered in the literature is surely a novelty aspect of this overviewing chapter.

In Chapter 3 we take into considerations three fundamental classes of Esakia spaces, namely the finitely generated Esakia spaces, the finitely copresented Esakia spaces and the regular injective Esakia spaces. In particular, in Theorem 3.37 we describe the finitely generated regular injective Esakia spaces for any finitely approximable intermediate propositional logic.

In Chapter 4 we study structural completeness and some of its related notions both from a logical and an algebraic point of view. We introduce Zakharyaschev's apparatus of canonical formulas and study structural complete intermediate propositional logics within this framework. More in detail, in Theorem 4.21 we give a new simple proof of Jankov's theorem stating that any extension of **KC** has the same positive fragment of intuitionistic logic using canonical formulas; in §4.3.1 we define the notion of partial Esakia equivalence and show in Proposition 4.25 that there exists a one-to-one correspondence between the partial Esakia equivalence on a given Esakia space $\mathcal{X}$ and the partial Esakia morphism from $\mathcal{X}$.

Chaper 5 deals with unification theory. After giving in Lemma 5.9 an abstract characterization of projective unification, we discuss unification issues in intermediate propositional logics. In particular, in Proposition 5.14 and 5.15 we give new semantic proofs respectively that any intermediate logic with unitary unification must be an extension of Jankov's logic **KC** and that the extensions of the Gödel-Dummet logic **LC** are exactly the intermediate logics with projective unification. Furthermore, in §5.3.2, we shall use the characterization of the finitely generated regular projective Heyting algebras obtained in Chapter 3 in order to to generalize a result of Ghilardi [64] from the locally finite to the finitely approximable context thus proving in Corollary 5.27 that a wide class of intermediate propositional logics does not enjoy finitary unification.

In Chapter 6 we take into considerations Friedman logics. After introducing negatively stable logics, in §6.1.1 we define a strictly increasing sequence of negatively stable logics that are characterized by frames which arise as a variation on the construction of the frames for the logic of rhombuses **RH**, introduced by Maksimova in [106]. In §6.3 we investigate Medvedev's logic of finite problem **ML** by means of its frames: we show in Lemma 6.21 that, for every $n < \omega$, the Medvedev frame $\mathfrak{P}_n$ is $\lceil \log_2 n \rceil$-generated and we discuss some p-morphic images of such frames. In §6.4 we investigate the structure of the upper part of the canonical frames for Friedman logics and we shall give complete characterization of the points at depth ≤ 3 of the 2-canonical frame $\mathfrak{F}_L(2)$ for any Friedman logic L (cfr. Figure 6.10). Finally, in §6.5 we take advantage of the results obtained so far in order to prove in Corollary 6.50 that the 2-letter fragment of Medvedev logic is decidable.

We conclude this thesis with Appendix A in which we introduce Miglioli and Usberti's analysis of knowledge and we argue in §A.3 that the logic of knowledge has to be a Friedman logic.

Chapter 1

Syntax and Semantics

In this chapter we review the basic facts about intermediate propositional logics. In particular, we recall their relational (Kripke) semantics, their algebraic semantics as well as the relationship between the two.

Most of the material of this chapter can be found in any good book in which the author deals with intermediate propositional logics and wants to give the reader a (as much as possible) complete and self-contained overview of the subject. The best references of such a kind are certainly [14], [142] and, especially, [23], from which we have drawn ample inspiration.

1.1 Intuitionistic and classical logic

The *propositional language* $\mathcal{L}$ consists of the following alphabet:

- the *propositional variables* $p_0, p_1, p_2, \ldots$;
- the *propositional constant* $\bot$;
- the *propositional connectives*: $\wedge, \vee, \to$;
- the *punctuation symbols*: (and).

We denote the set of all variables of $\mathcal{L}$ by $\mathbf{Var}\mathcal{L}$ and we assume $\mathbf{Var}\mathcal{L}$ to be countable. We will denote propositional variables by small Roman letters $p, q, r, \ldots$, possibly with subscripts or superscripts.

Definition 1.1. The set of *formulas* of $\mathcal{L}$ (or $\mathcal{L}$-formulas), denoted by $\mathbf{For}\mathcal{L}$, is inductively defined as follows:

- all the variables of $\mathcal{L}$ and the propositional constant are $\mathcal{L}$-formulas, called the *atomic $\mathcal{L}$-formulas*;
- if φ and ψ are $\mathcal{L}$-formulas, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \to \psi)$ are $\mathcal{L}$-formulas;
- nothing else is a $\mathcal{L}$-formula.

We will denote formulas by small Greek letters $\varphi, \psi, \xi, \dots$ and we reserve capital Greek letters $\Gamma, \Delta, \Sigma, \dots$ for denoting sets of $\mathcal{L}$-formulas.
We call all the $\mathcal{L}$-formulas used in the construction of a given $\mathcal{L}$-formula φ, including φ itself, the *subformulas* of φ and we denote the set of the subformulas of φ by $\mathbf{Sub}\,\varphi$; we denote of all the variables in $\mathbf{Sub}\,\varphi$ by $\mathbf{Var}\,\varphi$. Moreover, we will use the notation $\varphi(p_1, \dots, p_n)$ to indicate that $\{p_1, \dots, p_n\} \supseteq \mathbf{Var}\,\varphi$.

We will make use of the following abbreviations: for any $\mathcal{L}$-formula φ,

$$
\begin{aligned}
(\neg\varphi) &= (\varphi \to \bot) \\
(\varphi \leftrightarrow \psi) &= (\varphi \to \psi) \wedge (\psi \to \varphi) \\
\top &= (\bot \to \bot),
\end{aligned}
$$

and for any finite set $\Gamma = \{\psi_1, \dots, \psi_n\}$ of $\mathcal{L}$-formulas,

$$
\begin{aligned}
\bigwedge \Gamma &= \bigwedge_{i=1}^{n} \psi_i = \psi_1 \wedge \dots \wedge \psi_n && (\textstyle\bigwedge \Gamma = \top, \text{ if } n = 0) \\
\bigvee \Gamma &= \bigvee_{i=1}^{n} \psi_i = \psi_1 \vee \dots \vee \psi_n && (\textstyle\bigvee \Gamma = \bot, \text{ if } n = 0).
\end{aligned}
$$

Each ψ_i in a $\mathcal{L}$-formula of the form $\bigwedge_{i=1}^{n} \psi_i$ or $\bigvee_{i=1}^{n} \psi_i$ is called respectively a *conjunct* or a *disjunct* of the $\mathcal{L}$-formula.

Definition 1.2. An (structural) *inference rule* is a pair $\langle \Gamma, \varphi \rangle$ given by a finite set $\Gamma = \{\psi_1, \dots, \psi_n\}$ of $\mathcal{L}$-formulas, called the *premises* of the rule, and a $\mathcal{L}$-formula φ, called the *conclusion* of the rule. We will denote inference rules by Γ/φ and sometimes we will express them in the following form: $\dfrac{\psi_1, \dots, \psi_n}{\varphi}$.

Standard examples of inference rules are the following *congruence rules*:

$$
\frac{p \leftrightarrow q}{p \odot r \leftrightarrow q \odot r} \qquad\qquad \frac{p \leftrightarrow q}{r \odot p \leftrightarrow r \odot q}
$$

where $\odot \in \{\wedge, \vee, \to\}$.

Definition 1.3. A *substitution* σ is a function $\sigma\colon \mathbf{Var}\mathcal{L} \to \mathbf{For}\mathcal{L}$ that can be extended in a unique way to a map $\sigma\colon \mathbf{For}\mathcal{L} \to \mathbf{For}\mathcal{L}$ in the following way:

$$
\begin{aligned}
\sigma(p) &= \sigma(p) && \text{for every } p \in \mathbf{Var}\mathcal{L} \\
\sigma(\psi \odot \xi) &= \sigma(\psi) \odot \sigma(\xi) && \text{for } \odot \in \{\wedge, \vee, \to\} \\
\sigma(\bot) &= \bot.
\end{aligned}
$$

For a substitution σ such that $\sigma(p_i) = \psi_i$ for each $i \in \{1, \dots, n\}$ and $\sigma(r) = r$ for all $r \in \mathbf{Var}\mathcal{L} \setminus \{p_1, \dots, p_n\}$, we will also use the notation $\{\psi_1/p_1, \dots, \psi_n/p_n\}$ and, given a $\mathcal{L}$-formula $\varphi(p_1, \dots, p_n)$, the result of applying the substitution σ to φ is called an *istance* of φ and will also be denoted by $\varphi(\psi_1, \dots, \psi_n)$. Moreover, substitutions compose in the standard way, that is, given two substitution σ and τ, their *composition* is simply the composition $\sigma \circ \tau$.

1.1.1 The intuitionistic and classical propositional calculi

We are now ready to give a representation of intuitionistic logic. For our purposes, we will consider only its Hilbert-style calculus formulation.

Definition 1.4. The *intuitionistic propositional calculus* Int in the language $\mathcal{L}$ consists of the following axioms and inference rules

Axioms:

(A1) $p_0 \to (p_1 \to p_0)$;

(A2) $(p_0 \to (p_1 \to p_2)) \to ((p_0 \to p_1) \to (p_0 \to p_2))$;

(A3) $p_0 \wedge p_1 \to p_0$;

(A4) $p_0 \wedge p_1 \to p_1$;

(A5) $p_0 \to (p_1 \to p_0 \wedge p_1)$;

(A6) $p_0 \to p_0 \vee p_1$;

(A7) $p_1 \to p_0 \vee p_1$;

(A8) $(p_0 \to p_2) \to ((p_1 \to p_2) \to (p_0 \vee p_1 \to p_2))$;

(A9) $\bot \to p_0$;

Inference rules:

Modus Ponens (MP): $\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$;

Substitution (Subst): $\dfrac{\varphi}{\sigma(\varphi)}$ for any substitution σ.

A *derivation* of a $\mathcal{L}$-formula φ in Int is a sequence $\varphi_1, \dots, \varphi_n$ of $\mathcal{L}$-formulas such that $\varphi_n = \varphi$ and, for every $i \in \{1, \dots, n\}$, φ_i is either an axiom or is obtained from some of the preceding $\mathcal{L}$-formulas in the sequence by means of one of the inference rules. The number n is said to be the *length* of the derivation. Furthermore, we say that a $\mathcal{L}$-formula φ is *derivable* in Int if there is a derivation of φ in Int and we write $\vdash_{Int} \varphi$, or simply $\vdash \varphi$ if no confusion arises.

Given a set of $\mathcal{L}$-formulas Γ, a *derivation of φ from the set of assumption* Γ is a sequence $\varphi_1, \dots, \varphi_n$ of $\mathcal{L}$-formulas such that $\varphi_n = \varphi$ and, for every $i \in \{1, \dots, n\}$, φ_i is either an axiom or an assumption in Γ or is obtained from some of the preceding $\mathcal{L}$-formulas in the sequence by means of one of the inference rules, with (Subst) being applied only to axioms. If there is a derivation of φ from Γ, we say that φ is *derivable from* Γ and write $\Gamma \vdash_{Int} \varphi$, or simply $\Gamma \vdash \varphi$ if it is clear from the context. We will abbreviate the expression $\Gamma \cup \{\psi_1, \dots, \psi_n\} \vdash \varphi$ by $\Gamma, \psi_1, \dots, \psi_n \vdash \varphi$ and we will write $\Gamma, \Delta \vdash \varphi$ instead of $\Gamma \cup \Delta \vdash \varphi$.

It follows directly from the previous definition that

1. if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
2. if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$;
3. if $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for all $\psi \in \Gamma$, then $\Delta \vdash \varphi$;
4. if $\Gamma \vdash \varphi$, then $\sigma(\Gamma) \vdash \sigma(\varphi)$ for every substitution σ;

5. $\Gamma \vdash \varphi$ iff $\exists \Delta \subseteq \Gamma$, Δ finite, such that $\Delta \vdash \varphi$.

The properties just stated are not just merely peculiar features of the intuitionistic propositional calculus, but they are indeed characteristic of any propositional calculus based on some set of axioms and inference rules (that includes Subst), that is, of any *axiomatic system* on a given propositional language $\mathcal{L}$. By defining the relation of derivation from assumption as in the case of the calculs Int, we can associate with every axiomatic system AS a relation $\vdash_{AS} \subseteq \mathcal{P}(\mathbf{For}\mathcal{L}) \times \mathbf{For}\mathcal{L}$[1].

Definition 1.5. A binary relation $\vdash \subseteq \mathcal{P}(\mathbf{For}\mathcal{L}) \times \mathbf{For}\mathcal{L}$ is said to be a *consequence relation* on $\mathbf{For}\mathcal{L}$ if $\vdash$ satisfies properties (1)-(4), called respectively *Identity, Monotonicity, Transitivity* and *Structurality.* If, moreover, $\vdash$ satisfies (5), then $\vdash$ is said to be a *finitary* consequence relation[2].

Theorem 1.1 (Łoś-Suzko, [101])**.** *A relation* $\vdash$ *is a finitary consequence relation on* $\mathbf{For}\mathcal{L}$ *iff there exists an axiomatic system* AS *on* $\mathcal{L}$ *such that*

$$\vdash \, = \, \vdash_{AS} .$$

Thus for any axiomatic system AS, $\vdash_{AS}$ is a finitary consequence relation on $\mathbf{For}\mathcal{L}$. We say that a $\mathcal{L}$-formula φ is a *theorem* of AS if $\emptyset \vdash_{AS} \varphi$ (we will write simply $\vdash_{AS} \varphi$) and we call a set of $\mathcal{L}$-formulas $\Gamma \subseteq \mathbf{For}\mathcal{L}$ a *theory* of AS if Γ is closed under $\vdash_{AS}$, that is, if Γ satisfies the following property:

$$\Gamma \vdash_{AS} \varphi \Longrightarrow \varphi \in \Gamma.$$

It is easy to see that the set of theorems is the smallest theory of AS, i.e. the smallest set of formulas containing the axioms of AS and closed under the inference rules of AS. This last property gives us the grounds for the following

Definition 1.6. Let AS be an axiomatic system and let $\vdash_{AS}$ be its finitary consequence relation. The *logic of AS*, denote by **AS**, is the set of theorems of AS, that is

$$\mathbf{AS} = \{\varphi \in \mathbf{For}\mathcal{L} \mid \; \vdash_{AS} \varphi\}.$$

Hence intuitionistic logic **Int** is identified with the set of theorems of the intuitionistic propositional calculus Int.

Definition 1.7. The *classical propositional calculus Cl* in the language $\mathcal{L}$ is obtained from Int by adding the following $\mathcal{L}$-formula to the axioms:

$$\text{(A10)} \quad p_0 \vee (p_0 \to \bot).$$

If we had followed a historical perspective, we should have introduced first the classical propositional calculus Cl and only then the intuitionistic one, by descarding (A10) from the axioms. Indeed, the formula (A10), called the *law of excluded middle*, was strongly criticised by the Dutch mathematician and philosopher L. E. J. Brouwer ([17, 18]), who proposed, at the beginning of the

[1] Cfr. also §4.1 for some other considerations on axiomatic systems.

[2] The origin of this notion traces back to Tarski and its study on logical consequence in [150]. He identified conditions (1)-(3) and (5), while (4) was introduced later by Łoś and Suzko in [101]. Both Tarski and Łoś and Suzko, however, equivalently expressed these conditions by using the formalism of closure operators.

20^{th} century, a new philosophy of mathematics, called *intuitionism*, that moves away from the classical conception of mathematics based on classical logic and from which intuitionistic logic was born[3].

The notions of derivations and derivation from a set of assumptions are defined exactly in the same way as for Int. We denote the fact that a $\mathcal{L}$-formula φ is derivable in Cl by $\vdash_{Cl} \varphi$, while by $\Gamma \vdash_{Cl} \varphi$ we will express the fact that φ is derivable from the set of assumption Γ in Cl.

The law of excluded middle (A10) is equivalent in **Cl** with the following $\mathcal{L}$-formula, commonly called the *law of double negation*

$$\text{(A10}^*\text{)} \quad \neg\neg p_0 \to p_0,$$

and it is well known that intuitionistic logic is properly contained in classical logic, that is $\mathbf{Int} \subsetneq \mathbf{Cl}$, since it is the case that both (A10) and (A10*) belong to **Cl** but not to **Int**. Nevertheless, it is exactly the double negation law that gives a way of connecting intuitionistic to classical logic: it is indeed possible to embed **Cl** in **Int** through what is called the *double-negation translation* which associates to any given $\mathcal{L}$-formula φ its double-negated form $\neg\neg\varphi$.
The possibility of such an embedding stems from the following theorem of the Soviet logician Valery Glivenko:

Theorem 1.2 (Glivenko, 1929)**.** *For every* $\varphi \in \mathbf{For}\mathcal{L}$*,*

$$\varphi \in \mathbf{Cl} \iff \neg\neg\varphi \in \mathbf{Int}.$$

Thus, even though $\mathbf{Int} \subsetneq \mathbf{Cl}$, in view of the previous theorem it is sometimes suggested that **Int** should be thought of as "stronger" than **Cl**: inside **Int** one can see everything that one can see within **Cl** and, furthermore, one can also make many distinctions that classical logic overlooks.

Corollary 1.3. *For every* $\mathcal{L}$*-formula* φ*,*

$$\neg\varphi \in \mathbf{Int} \iff \neg\varphi \in \mathbf{Cl}.$$

Proof. Since $\neg\gamma \leftrightarrow \neg\neg\neg\gamma \in \mathbf{Int}$, for every $\mathcal{L}$-formula γ, the result follows from Glivenko's theorem. □

1.2 Kripke Semantics for Int

In this section we are going to present the well-known Kripke semantics for intuitionistic logic, for which the respective calculus is sound and complete. This kind of semantics takes into account an epistemic feature of the notion of truth that classical reasoning neglects. By accepting that each proposition must be either true or false, classical reasoning adheres to a platonic view of the reality that completely abstracts from the fact that it may be actually *a priori* unknown whether a given proposition is true or false. Given a proposition whose

[3] A good compendium of Brouwer's intuitionism can be found in [73]. For the connection between Brouwer's intuitionism and intuitionistic logic in the broader context of constructivism in 20^{th} century mathematics, cfr. [154], while for the early history of intuitonistic logic, its formalization and the genesis of the so-called Brouwer-Heyting-Kolmogorov interpretation, cfr. [153].

truth-value is not yet known, there is however the possibility that we can settle the question about the truth or falsity of this proposition in the future, by *acquiring new information* on the world around us. It is indeed the process of gaining information that Kripke semantics for intuitionistic logic takes into account.

1.2.1 Kripke frames and models

Definition 1.8. An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where $W \neq \emptyset$ and R is a partial order on W, i.e. $R \subseteq W \times W$ is a relation on W satisfying the following three conditions for all $x, y, z \in W$:

$$\begin{aligned} &xRx && \text{(reflexivity)}\\ xRy \wedge yRz &\Longrightarrow xRz && \text{(transitivity)}\\ xRy \wedge yRx &\Longrightarrow x = y && \text{(anti-symmetry)} \end{aligned}$$

Therefore, $\mathfrak{F}$ is just a partially ordered set. Elements of W are called *points* in $\mathfrak{F}$ and we read xRy as "y is R-accessible from x" or "x sees y".

Definition 1.9. A *valuation* of $\mathcal{L}$ in a Kripke frame $\mathfrak{F} = \langle W, R\rangle$ is a map $\mathfrak{V}$ associating to each $p \in \mathbf{Var}\mathcal{L}$ some (possibly empty) subset $\mathfrak{V}(p) \subseteq W$ satisfying the following condition: for any $u, w \in W$,

$$u \in \mathfrak{V}(p) \wedge uRw \Longrightarrow w \in \mathfrak{V}(p)$$

Subsets of W satisfying the previous condition are called *upward closed subset* of W (or, briefly, *upsets*). Dually, we say that X is a *downward closed subset* of W (or a *downset*) if we have $v \in X$ and wRv imply $w \in X$, for every $v, w \in W$. We denote the set of all upward closed subsets of W by $\mathrm{Up}(W)$. Thus a valuation in $\mathfrak{F}$ is just a function $\mathfrak{V}\colon \mathbf{Var}\mathcal{L} \to \mathrm{Up}(W)$.

Definition 1.10. An *intuitionistic Kripke model* of the language $\mathcal{L}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ where $\mathfrak{F}$ is an intuitionistic Kripke frame and $\mathfrak{V}$ a valuation in $\mathfrak{F}$.

Since we are dealing only with intuitionistic Kripke frames and models, we will often forget the adjective "intuitionistic" in the phrases "intuitionistic Kripke frame" and "intuitionistic Kripke model".

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ be a Kripke model and x a point in the frame $\mathfrak{F} = \langle W, R\rangle$. We define, by induction on the complexity of a formula $\varphi \in \mathbf{For}\mathcal{L}$, the relation $(\mathfrak{M}, x) \models \varphi$ of a formula φ *being true at x in $\mathfrak{M}$*:

$$\begin{aligned} (\mathfrak{M}, x) \models p &\iff x \in \mathfrak{V}(p)\\ (\mathfrak{M}, x) \models \psi \wedge \eta &\iff (\mathfrak{M}, x) \models \psi \text{ and } (\mathfrak{M}, x) \models \eta\\ (\mathfrak{M}, x) \models \psi \vee \eta &\iff (\mathfrak{M}, x) \models \psi \text{ or } (\mathfrak{M}, x) \models \eta\\ (\mathfrak{M}, x) \models \psi \to \eta &\iff \text{for all } y \in W \text{ such that } xRy,\\ &\qquad (\mathfrak{M}, x) \models \psi \text{ implies } (\mathfrak{M}, y) \models \eta\\ (\mathfrak{M}, x) &\not\models \bot \end{aligned}$$

Since $\neg\varphi$ is defined as $\varphi \to \bot$, it follows from the above definition that

$$\begin{aligned} (\mathfrak{M}, x) \models \neg\psi \quad &\iff \quad \text{for all } y \in W \text{ such that } xRy, (\mathfrak{M}, y) \not\models \psi \\ (\mathfrak{M}, x) \models \neg\neg\psi \quad &\iff \quad \text{for all } y \in W \text{ such that } xRy, \\ &\qquad \text{there exists } z \text{ such that } yRz \text{ and } (\mathfrak{M}, z) \models \psi \end{aligned}$$

If the model $\mathfrak{M}$ is clear from the context we will write $x \models \varphi$ instead of $(\mathfrak{M}, x) \models \varphi$ and we will also say that "x satisfies the formula φ in $\mathfrak{M}$". The truth set of a formula φ in $\mathfrak{M}$ is the set $\mathfrak{V}(\varphi) := \{x \in W \mid x \models \varphi\}$. It can be proved, by an easy induction on the complexity of φ, that $\mathfrak{V}(\varphi)$ is upward closed and, dually, that the set of point in which a formula φ is not true, i.e. $\{x \in W \mid x \not\models \varphi\}$, is downward closed.

Definition 1.11. Two Kripke frames $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle U, S\rangle$ are said to be *isomorphic* if there exists a surjective map $f \colon W \to U$ such that, for every $v, w \in W$,

$$vRw \iff f(v)Sf(w).$$

Such a map f is called an *isomorphism* of $\mathfrak{F}$ onto $\mathfrak{G}$. Moreover, two Kripke models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ are said to be *isomorphic* if there exists an isomorphism f of $\mathfrak{F}$ onto $\mathfrak{G}$ such that, for every $p \in \mathbf{Var}\mathcal{L}$,

$$\mathfrak{U}(p) = f(\mathfrak{V}(p)).$$

In this case, we say that f is an isomorphism f of $\mathfrak{M}$ onto $\mathfrak{N}$.

The next proposition allows us not to distinguish between isomorphic models and isomorphic frames.

Proposition 1.4. *Let f be an isomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$. Then, for every point $x \in \mathfrak{M}$ and every $\varphi \in \mathbf{For}\mathcal{L}$,*

$$(\mathfrak{M}, x) \models \varphi \quad \iff \quad (\mathfrak{N}, f(x)) \models \varphi.$$

Definition 1.12. Let $\varphi \in \mathbf{For}\mathcal{L}$, $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame, $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ be a Kripke model on $\mathfrak{F}$ and $\mathcal{K}$ be a class of Kripke frame.

1. We say that φ is *satisfied* in $\mathfrak{M}$ if $x \models \varphi$ for some $x \in W$.
2. We say that φ is *true* in $\mathfrak{M}$ and we write $\mathfrak{M} \models \varphi$ if $x \models \varphi$ for all $x \in W$. If φ is not true in $\mathfrak{M}$, then we say that φ is *refuted* in $\mathfrak{M}$ or $\mathfrak{M}$ is a *countermodel* for φ and we write $\mathfrak{M} \not\models \varphi$.
3. We say that φ is *satisfied* in $\mathfrak{F}$ if φ is satisfied in some model based on $\mathfrak{F}$.
4. We say that φ is *true at a point* x in $\mathfrak{F}$, and we write $(\mathfrak{F}, x) \models \varphi$, if φ is true at x in every model based on $\mathfrak{F}$.
5. We say that φ is *valid* in $\mathfrak{F}$ and we write $\mathfrak{F} \models \varphi$ if φ is true in all models based on $\mathfrak{F}$. If φ is not true in $\mathfrak{F}$, then we say that φ is *refuted* in $\mathfrak{F}$ and we write $\mathfrak{F} \not\models \varphi$.
6. We say that φ is *valid* in $\mathcal{K}$ and we write $\mathcal{K} \models \varphi$ if $\mathfrak{F} \models \varphi$ for all $\mathfrak{F} \in \mathcal{K}$.

The following theorem shows the adequacy of the intuitionistic propositional calculus *Int* to the intuitionistic Kripke semantics. Its proof is quite standard and uses what is commonly called the *canonical model* argument[4].

Theorem 1.5 (Soundness and completeness of *Int*). *For all* $\varphi \in \mathbf{For}\mathcal{L}$,

$$\vdash_{Int} \varphi \quad \Longleftrightarrow \quad \mathfrak{F} \models \varphi \qquad \text{for every frame } \mathfrak{F}.$$

Corollary 1.6. $\mathbf{Int} = \{\varphi \in \mathbf{For}\mathcal{L} \mid \mathfrak{F} \models \varphi$ *for all frames* $\mathfrak{F}\}$.

We also have the the following strengthening of the completeness theorem.

Theorem 1.7 (Strong completeness of *Int*). *For every* $\Gamma \cup \{\varphi\} \subseteq \mathbf{For}\mathcal{L}$,

$$\Gamma \vdash_{Int} \varphi \quad \Longleftrightarrow \quad (\mathfrak{M}, x) \models \Gamma \Longrightarrow (\mathfrak{M}, x) \models \varphi,$$

for every Kripke model $\mathfrak{M}$ *and point* $x \in \mathfrak{M}$.

Remark 1. Note that classical validity is nothing but validity in the single point frame $\imath = \langle\{\circ\}, \langle\circ, \circ\rangle\rangle$ and thus we also obtain the inclusion $\mathbf{Int} \subseteq \mathbf{Cl}$ by semantic means.

1.2.2 Truth-preserving operations

We recall in this section the three main operations on Kripke frames and models which preserve truth and validity. For the proofs of the theorems, cfr. [23].

Generated subframes and generated submodels

A frame $\mathfrak{G} = \langle U, S\rangle$ is called a *subframe* of $\mathfrak{F} = \langle W, R\rangle$ (we write $\mathfrak{G} \subseteq \mathfrak{F}$) if $U \subseteq W$ and S is the restriction of R to U, i.e. $S = R \cap U^2$. Moreover, if U is an upset of W, we call $\mathfrak{G}$ a *generated subframe* of $\mathfrak{F}$ (notation: $\mathfrak{G} \subsetneq \mathfrak{F}$). Finally, if U is the minimal upset containing a subset $V \subseteq W$, we say that U and $\mathfrak{G}$ are *generated by the set* V.

We now introduce a special notation for the operation of downward and upward closure. If $\mathfrak{F} = \langle W, R\rangle$ is a Kripke frame and $X \subseteq W$, then we let

$$X{\uparrow}R := \{x \in W \mid \exists y \in X \ yRx\}$$
$$X{\downarrow}R := \{x \in W \mid \exists y \in X \ xRy\}$$

In the case the frame $\mathfrak{F}$ is understood from the context, we drop R and write simply $X\uparrow$ and $X\downarrow$; if X is a singleton $\{q\}$, we use $q\uparrow$ and $q\downarrow$ instead of $\{q\}\uparrow$ and $\{q\}\downarrow$, respectively[5]. The elements of the set $q\uparrow$ ($q\downarrow$) are called *successors* (*predecessors*) of q; a successor (predecessor) s of q is *proper* if $s \neq q$ and a proper successor (predecessor) s of q is said to be an *immediate successor* (*immediate predecessor*) if $qRzRs$ ($sRzRq$) implies $z = s$ or $z = q$, for every $z \in W$, i.e. there is no element stricly between s and q with respect to the order R. In such a case, we will also write $q \lessdot s$. An element $x \in W$ is *maximal* in $\mathfrak{F}$ if $x\uparrow = \{x\}$; instead, x is said to be the *greatest* point of $\mathfrak{F}$ if $x\downarrow = W$. For every $v \in W$, we denote by $max(v)$ the set of all maximal elements y such that vRy.

[4]Cfr., for instance, [23, Theorem 1.16, 2.43 and 5.12].

[5]In the literature, the set $q\uparrow$ is usually called the *cone over* q.

A model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ is a *submodel* of a $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ (and we write $\mathfrak{N} \subseteq \mathfrak{M}$) if $\mathfrak{G} = \langle U, S \rangle$ is a subframe of $\mathfrak{F} = \langle W, R \rangle$ and, for every $p \in \mathbf{Var}\mathcal{L}$,

$$\mathfrak{U}(p) = \mathfrak{V}(p) \cap U$$

If, moreover, $\mathfrak{G} \subsetneq \mathfrak{F}$, then we say that $\mathfrak{N}$ is a *generated submodel* of $\mathfrak{M}$ and we write $\mathfrak{N} \subsetneq \mathfrak{M}$.

Let $\mathfrak{F} = \langle W, R \rangle$ be a Kripke frame and let $w \in W$. The *subframe of* $\mathfrak{F}$ *generated by* w is the frame $\mathfrak{F}_w := \langle w{\uparrow}, S \rangle$ where S is the restriction of R to $w{\uparrow}$. Kripke frames of this kind, that is, generated by a singleton $\{w\}$, are called *rooted* and w is called the *root* of the frame. Moreover, given a Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, we denote by $\mathfrak{M}_w := \langle \mathfrak{F}_w, \mathfrak{U} \rangle$, where $\mathfrak{U}(p) = \mathfrak{V}(p) \cap w{\uparrow}$ for each $p \in \mathbf{Var}\mathcal{L}$, the *submodel of* $\mathfrak{M}$ *generated by* w.

All the previously defined submodels satisfy the following

Theorem 1.8 (Generation). *Let* $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ *be a generated submodel of a model* $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. *Then, for every* $\varphi \in \mathbf{For}\mathcal{L}$ *and every* $x \in \mathfrak{G}$,

$$(\mathfrak{N}, x) \models \varphi \iff (\mathfrak{M}, x) \models \varphi$$

Corollary 1.9. *If* $\mathfrak{G} \subsetneq \mathfrak{F}$, *then, for every* $\varphi \in \mathbf{For}\mathcal{L}$, *the following hold:*

(i) $(\mathfrak{G}, x) \models \varphi \iff (\mathfrak{F}, x) \models \varphi$, *for all* $x \in \mathfrak{G}$;

(ii) $\mathfrak{F} \models \varphi \Longrightarrow \mathfrak{G} \models \varphi$.

p-morphisms

Let $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle U, S \rangle$ be Kripke frames. A map $f : W \to U$ is said to be *order-preserving*, or *monotone*, if the following condition holds: for every $v, w \in W$

$$vRw \Longrightarrow f(v)Sf(w)$$

Definition 1.13. A map $f : W \to U$ is said to be a *p-morphism between* $\mathfrak{F}$ *and* $\mathfrak{G}$ if it is order-preserving and, moreover, it satisfies the following condition:
for every $w \in W, u \in U$,

$$f(w)Su \Longrightarrow \exists v \in W(wRv \wedge f(v) = u)$$

Furthermore, if f is a surjective p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$, then we say that $\mathfrak{G}$ is a *p-morphic image* of $\mathfrak{F}$[6].

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ be Kripke models. A p-morphism f between $\mathfrak{F}$ and $\mathfrak{G}$ is called a *p-morphism between* $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ *and* $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ if, for every $p \in \mathbf{Var}\mathcal{L}$,

$$\mathfrak{V}(p) = f^{-1}(\mathfrak{U}(p))$$

that is, if for every point $x \in \mathfrak{F}$, $(\mathfrak{M}, x) \models p$ iff $(\mathfrak{N}, f(x)) \models p$. Futhermore, if f is surjective, then $\mathfrak{N}$ is said to be a *p-morphic image* of $\mathfrak{M}$.

Theorem 1.10 (p-morphic image). *Let* $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ *be a p-morphic image of the model* $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ *via the map* f. *Then, for every* $x \in \mathfrak{F}$ *and every* $\varphi \in \mathbf{For}\mathcal{L}$,

$$(\mathfrak{M}, x) \models \varphi \iff (\mathfrak{N}, f(x)) \models \varphi$$

[6]Some authors, cfr. for example [23], call such onto maps *reductions*.

Corollary 1.11. *If $\mathfrak{G}$ is a p-morphic image of $\mathfrak{F}$ then, for every $\varphi \in \mathbf{For}\mathcal{L}$,*

$$\mathfrak{F} \models \varphi \Longrightarrow \mathfrak{G} \models \varphi$$

Now consider a Kripke frame $\mathfrak{F} = \langle W, R\rangle$ and let $q \in W$. We denote by $q^>$ the set of all immediate successors of q. We say that a subset $X \subseteq W$ *covers* $w \in W$ (notation: $w \preceq X$) if $X \subseteq w\uparrow$ and $w^> \subseteq X\uparrow$ and we say that $X \subseteq W$ *totally covers* $w \in W$ (notation: $w \prec X$) if $X = w^>$.

Notice that both $\preceq$ and $\prec$ are relations relating points and sets and that $w \preceq X$ iff $w\uparrow = X\uparrow \cup \{w\}$. As above, we will write $w \preceq v$ and $w \prec v$ as shorthands for, respectively, $w \preceq \{v\}$ and $w \prec \{v\}$. Therefore, $w \preceq v$ is equivalent to $w = v \vee v \in w^>$ and $w \prec v$ means that v is the only immediate successor of w. Notice furthermore the two following special cases with respect to the covering relation: for any $w \in W$, $w \preceq \emptyset \iff w$ is maximal; $\{w\}$ and $w^>\uparrow$ always cover w.

The next result is a slight generalization of a characterization of p-morphisms in terms of the covering relation given by Ghilardi [64] for finite frames. Say that a Kripke frame $\mathfrak{F} = \langle W, R\rangle$ is *Noetherian* if there are no infinite strictly ascending chain $x_0 R x_1 R \ldots$ of elements in W, or, equivalently, if the relation R is *converse well-founded*[7].

Lemma 1.12. *Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ be Kripke frames and $f\colon W \to V$ be an order-preserving map. If f is a p-morphism, then, for all $w \in W$, $X \subseteq W$,*

$$w \preceq X \Longrightarrow f(w) \preceq f(X).$$

Furthermore, if $\mathfrak{F}$ is Noetherian, than the converse also holds.

We conclude this section with the following useful remark on p-morphisms and generated subframes. Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame, $\mathfrak{D} = \langle V, R\restriction_V\rangle \subsetneq \mathfrak{F}$ a generated subframe of $\mathfrak{F}$ and let the function $h\colon \mathfrak{D} \to \mathfrak{G} = \langle Q, S\rangle$ be an onto p-morphism. Denote by $\mathfrak{F}[\mathfrak{D}/\mathfrak{G}] = \langle W[V/Q], R'\rangle$ the Kripke frame defined as follows:

$$W[V/Q] = W \setminus V \uplus Q$$

and the partial ordering R' is defined by letting

$$\begin{aligned} xR'y \quad \iff \quad & x = y, \text{ or} \\ & x, y \in W \setminus V \text{ and } xRy, \text{ or} \\ & x, y \in Q \text{ and } xSy, \text{ or} \\ & x \in W \setminus V,\, y \in Q \text{ and } \exists s \in h^{-1}(y)(xRs). \end{aligned}$$

It is immediately seen that $\mathfrak{F}[\mathfrak{D}/\mathfrak{G}]$ is a well defined Kripke frame. Now, consider the map $f\colon W \to W[V/Q]$ defined by letting

$$f(x) = \begin{cases} x & \text{if } x \in W \setminus V, \\ h(x) & \text{if } x \in V. \end{cases}$$

Lemma 1.13. *The map $f\colon W \to W[V/Q]$ is a p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}[\mathfrak{D}/\mathfrak{G}]$.*

[7] Actually, the two stated conditions are equivalent modulo the axiom of dependent choice.

Proof. Let $x, y \in W$ be such that xRy. We have the following possibilities:

(a) $x \in V$: therefore also $y \in V$ since $V \subsetneq W$ and consequently we have $f(x) = h(x)R'h(y) = f(y)$;

(b) $x, y \in W \setminus V$: thus $f(x) = xR'y = f(y)$;

(c) $x \in W \setminus V$, $y \in V$: hence $f(x) = x$ and $f(y) = h(y) \in Q$. Since $y \in h^{-1}(h(y))$, we have $f(x)R'f(y)$.

Hence f is monotone. So let $w \in W$, $u \in W[V/Q]$ and suppose that $f(w)R'u$. Then, according to the definition of R', we have the following cases:

(a) both $f(w), u \in W \setminus V$ and $f(w)Ru$. Consequently $w \notin V$, so $f(w) = w$ and thus $u \in w{\uparrow}$ is such that $f(u) = u$.

(b) both $f(w), u \in Q$ and $f(w)Su$. Therefore it must be the case that $w \in V$. Hence $f(w) = h(w)$ and since h is a p-morphism there exists $v \in w{\uparrow}$ such that $h(v) = u$. But then $f(v) = h(v) = u$.

(c) $f(w) \in W \setminus V$, $u \in Q$ and there exists $s \in h^{-1}(u)$ such that $f(w)Rs$. Thus $w \notin W$ and $f(w) = w$. Then $s \in V$ is such that $s \in w{\uparrow}$ and $f(s) = h(s) = u$.

We can thus conclude that f is indeed a p-morphism. □

Disjoint unions

Let $\{\mathfrak{F}_i = \langle W_i, R_i\rangle \mid i \in I\}$ be a family of Kripke frames. The *disjoint union of the family* $\{\mathfrak{F}_i\}_{i\in I}$ is the frame $\biguplus_{i\in I} \mathfrak{F}_i := \langle \biguplus_{i\in I} W_i, \bigcup_{i\in I} R_i\rangle$, where $\biguplus_{i\in I} W_i$ is the disjoint union of the W_i's. If the set I is finite, let's say $I = \{1, \ldots, n\}$, then we will also write $\mathfrak{F}_1 + \ldots + \mathfrak{F}_n$.

Let $\{\mathfrak{M}_i = \langle \mathfrak{F}_i, \mathfrak{V}_i\rangle \mid i \in I\}$ be a family of Kripke models. The *disjoint union of the family* $\{\mathfrak{M}_i\}_{i\in I}$ is the model $\biguplus_{i\in I} \mathfrak{M}_i := \langle \biguplus_{i\in I} \mathfrak{F}_i, \bigcup_{i\in I} \mathfrak{V}_i\rangle$, where $(\bigcup_{i\in I} \mathfrak{V}_i)(p) = \bigcup_{i\in I} \mathfrak{V}_i(p)$, for each $p \in \mathbf{Var}\mathcal{L}$.

It is clear that each $\mathfrak{F}_i$ is a generated subframe of $\biguplus_{i\in I} \mathfrak{F}_i$ and each $\mathfrak{M}_i$ is a generated submodel of $\biguplus_{i\in I} \mathfrak{M}_i$. Moreover, we have the following

Theorem 1.14 (Disjoint union)**.** *Let $\biguplus_{i\in I} \mathfrak{M}_i$ be the disjoint union of the family of models $\{\mathfrak{M}_i = \langle \mathfrak{F}_i, \mathfrak{V}_i\rangle \mid i \in I\}$. Then, for each $i \in I$, every $x \in \mathfrak{F}_i$ and every formula $\varphi \in \mathbf{For}\mathcal{L}$,*

$$(\biguplus_{i\in I} \mathfrak{M}_i, x) \models \varphi \iff (\mathfrak{M}_i, x) \models \varphi$$

Corollary 1.15. *Let $\biguplus_{i\in I} \mathfrak{F}_i$ be the disjoint union of a family of frames $\{\mathfrak{F}_i\}_{i\in I}$. Then, for each $i \in I$ and every formula $\varphi \in \mathbf{For}\mathcal{L}$,*

$$\biguplus_{i\in I} \mathfrak{F}_i \models \varphi \iff \mathfrak{F}_i \models \varphi$$

1.3 Superintuitionistic and intermediate logics

Definition 1.14. A *superintuitionistic logic* (*si-logic*, for short) in the language $\mathcal{L}$ is any set L of $\mathcal{L}$-formulas satisfying the following conditions:

- $\mathbf{Int} \subseteq L$;
- L is closed under modus ponens (MP);
- L is closed under uniform substitution (Subst).

We say that a si-logic L is *consistent* if $\bot \notin L$, and *inconsistent* otherwise. Notice that by (A9) and (MP) a si-logic L is inconsistent iff $L = \mathbf{For}\mathcal{L}$. Therefore, according to the previous definition, the set of all $\mathcal{L}$-formulas $\mathbf{For}\mathcal{L}$ is a si-logic and we call it the *incosistent si-logic.*

It is clear that **Int** and $\mathbf{For}\mathcal{L}$ are respectively the smallest and the greatest si-logics with respect to inclusion. For any si-logics L_1 and L_2, we say that L_2 is an *extension* of L_1, or L_1 is a *sublogic* of L_2, if $L_1 \subseteq L_2$ and, moreover, if $L_1 \neq L_2$ then we say that L_2 is a *proper* extension of L_1, or L_1 is a *proper* sublogic of L_2.

Theorem 1.16. *For every consistent si-logic L, $\mathbf{Int} \subseteq L \subseteq \mathbf{Cl}$.*

For this reason consistent si-logics are often call *intermediate logics*, since they are "intermediate" between intuitionistic and classical propositional logics[8].

We now see some methods for constructing si-logics.

Theorem 1.17. *Let $\mathcal{C}$ be a class of intuitionistic frames. Then the set of $\mathcal{L}$-formulas valid in all frames $\mathcal{C}$ is a si-logic.*

We will call the si-logic defined in the previous theorem the *logic of the class* $\mathcal{C}$ and denote it by $\operatorname{Log}\mathcal{C}$. Moreover, if $\mathcal{C} = \{\mathfrak{F}\}$, then we will write $\operatorname{Log}\mathfrak{F}$ instead of $\operatorname{Log}\mathcal{C}$ and call it the *logic of* $\mathfrak{F}$. Notice that

$$\operatorname{Log}\mathcal{C} = \bigcap_{\mathfrak{F}\in\mathcal{C}} \operatorname{Log}\mathfrak{F}.$$

Another way of constructing a si-logic is given by the following

Theorem 1.18. *Let $\{L_i \mid i \in I\}$ be a family of si-logics. Then the intersection $\bigcap_{i\in I} L_i$ is also a si-logic.*

Finally, another method of constructing si-logics follows directly from the definition: given any set Γ of $\mathcal{L}$-formulas, it suffices to add this set to **Int** and close the resulting set under MP and Subst. The obtained si-logic L is then denoted by $\mathbf{Int} + \Gamma$ and $\mathcal{L}$-formulas in Γ are called the *additional axiom of L over* **Int**, while L itself is called the *extension of* **Int** *with* Γ. If $\Gamma = \{\varphi_1, \ldots, \varphi_k\}$,

[8]The adjective "intermediate", used to characterize logics between the intuitionistic and classical, was introduced by Umewaza in [155] and [156]. Moreover, it is useful to say that in the context of first-order logic, not every extension of intuitionistic logic is contained in classical logic. Indeed, it is known that classical first-order logic has continuum many proper extensions and intuitionistic first-order logic is a sublogic of each of these extensions. Therefore, the notions of consistent si-logic and intermediate logic do not coincide in the first-order framework, contrary to the propositional case.

then we will also denote $\mathbf{Int} + \Gamma$ by $\mathbf{Int} + \varphi_1 + \ldots + \varphi_k$. So, for instance, we have $\mathbf{Cl} = \mathbf{Int} + p \vee \neg p = \mathbf{Int} + \neg\neg p \to p$.

Given si-logics $L_1 = \mathbf{Int} + \Gamma_1$ and $L_2 = \mathbf{Int} + \Gamma_2$, the logic $L = \mathbf{Int} + \Gamma_1 \cup \Gamma_2$ is called the *sum* of L_1 and L_2 and denoted by $L_1 + L_2$. If $\{L_i \mid i \in I\}$ is a family of si-logics, their sum is the closure under MP and Subst of $\bigcup_{i \in I} L_i$ and is denoted by $\sum_{i \in I} L_i$.

Derivations and *derivation from a set of assumption* in a si-logic $\mathbf{Int} + \Gamma$ are defined as in the case of Int with the only difference that in this case one can also use the extra axiom of L together with the axiom of Int. If a formula φ is derivable in L or derivable from a set of assumption Δ, we write $\vdash_L \varphi$ and $\Delta \vdash_L \varphi$ respectively. The deduction theorem, as well as the replacement theorem, hold for si-logics:

Theorem 1.19. *Let L be a si-logic, Γ a set of $\mathcal{L}$-formulas and $\eta, \psi, \varphi \in \mathbf{For}\mathcal{L}$.*

(i) $\Gamma, \psi \vdash_L \varphi \Longleftrightarrow \Gamma \vdash_L \psi \to \varphi$;

(ii) $\eta \leftrightarrow \psi \in L \Longrightarrow \varphi(\eta) \leftrightarrow \varphi(\psi) \in L$.

The operation of sum between si-logics behaves similarly to the union between two sets. Indeed, we have the following

Theorem 1.20. *The sum of si-logics is idempotent, commutative, associative and distributes over the intersection. Moreover, the intersection of si-logics distributes over the (infinite) sum, that is*

$$L \cap \sum_{i \in I} L_i = \sum_{i \in I} (L \cap L_i).$$

The content of the previous theorem can be restated by saying that the class of si-logics, together with the operations $\cap$ and $+$ of intersection and sum of si-logics, forms a complete bounded lattice with $\mathbf{Int}$ and $\mathbf{For}\mathcal{L}$ as bottom and top elements[9]. This structure is called the *lattice of si-logics* and denoted by Ext**Int**[10].

1.3.1 Basic properties of intermediate logics

Finite Axiomatizability. An intermediate logic L is said to be *finitely axiomatizable* if it can be represented as $L = \mathbf{Int} + \Gamma$ where Γ is a finite set of $\mathcal{L}$-formulas.

Notice that, by (A3)-(A5), we have

$$\mathbf{Int} + \varphi_1 + \ldots + \varphi_k = \mathbf{Int} + \varphi_1 \wedge \ldots \wedge \varphi_k$$

Therefore, an intermediate logic L is finitely axiomatizable iff it is axiomatizable by a single additional formula.

[9] For the definition of complete bounded lattice see Definition 1.18 and 1.19 of §1.4. Moreover, as a consequence of Lemma 1.45, such a class actually carries a Heyting algebra structure.

[10] Notice that, in the language of finitary consequence relations, si-logics are nothing but the theories of Int. Therefore, the lattice of si-logics can also be thought as the lattice of the theories of Int.

(STRONG) KRIPKE COMPLETENESS. We say that an intermediate logic L is *Kripke complete* if there exists a class $\mathcal{K}$ of Kripke frames such that $L = \mathrm{Log}\,\mathcal{K}$. If this is the case, we say that L is *characterized* by $\mathcal{K}$.

We have already seen two examples of Kripke complete logics, namely **Int** and **Cl**, respectively complete with respect to the class of all Kripke frames and the one point frame. Moreover, we have also seen that these two logics satisfy a stronger version of completeness, i.e. completeness with respect to the relation of derivability from assumption. Let us isolate this property.

We say that an intermediate logic is *strongly Kripke complete* if there exists a class $\mathcal{K}$ of Kripke frames such that, for any set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}$-formulas, $\Gamma \vdash_L \varphi$ iff, for every $\mathfrak{F} \in \mathcal{K}$ and point $x \in \mathfrak{F}$, $(\mathfrak{F}, x) \models \Gamma$ implies $(\mathfrak{F}, x) \models \varphi$. In this case, we say that L is *strongly characterized* by $\mathcal{K}$.

It is worth pointing out that not all intermediate logics are strongly Kripke complete and, more importantly, that not all intermediate logics are Kripke complete[11].

As a consequence of Theorem 1.8, we have the following

Corollary 1.21. *Let L be a Kripke complete intermediate logic. Then L is Kripke complete with respect to the class of its rooted frames.*

TABULARITY. An intermediate logic L is called *tabular* if there is a finite frame $\mathfrak{F}$ such that $L = \mathrm{Log}\,\mathfrak{F}$.

Therefore, **Cl** is tabular. However, this is not the case for **Int**, as the following theorem, originally proved by Gödel [66] by algebraic methods, shows.

Theorem 1.22. Int *is not tabular.*

Proof. Suppose, for reduction, that $\mathbf{Int} = \{\varphi \in \mathbf{For}\mathcal{L} \mid \mathfrak{F} \models \varphi\}$ for some finite frame $\mathfrak{F}$, that, say, contains n points. Consider the following inductively defined sequence of formulas $\boldsymbol{bd}_n$:

$$\boldsymbol{bd}_n = p_1 \vee (p_1 \to \bot),$$
$$\boldsymbol{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \boldsymbol{bd}_n).$$

It can be shown that $\mathfrak{F} \not\models \boldsymbol{bd}_n$ only if there is a chain of $n+1$ points in $\mathfrak{F}$. Therefore, $\mathfrak{F} \models \boldsymbol{bd}_n$, contrary to the fact that $\boldsymbol{bd}_n \notin \mathbf{Int}$. ☐

FINITE APPROXIMABILITY. An intermediate logic L is said to be *finitely approximable* (or to have the *finite frame property*) if there exists a class $\mathcal{K}$ of finite frames such that $L = \mathrm{Log}\,\mathcal{K}$[12].

Theorem 1.23. Int *is finitely approximable, in particular* **Int** *is complete with respect to the class of finite rooted frames*[13].

[11] The first example of a Kripke incomplete (finitely axiomatizable) intermediate logic was discovered in 1977 by Shehtman [145].

[12] The finite frame property is equivalent, for intermediate logics, to the much more common *finite model property* of model theory and universal algebra, whose definition is the following: L has the finite model property if there exists a class $\mathcal{M}$ of finite models (structures) such that, for every formula φ, $\varphi \in L$ iff $\mathfrak{N} \models \varphi$ for every $\mathfrak{N} \in \mathcal{M}$. Hence, we will mostly use the phrase "finite model property" for expressing finite approximability. For the equivalence of the two properties in the case of intermediate logics, cfr. [23, Theorem 8.47].

[13] One can strengthen the result by showing, with the use of Theorem 1.10, that **Int** is complete with respect to the class of finite trees, where a frame $\mathfrak{F} = \langle W, R\rangle$ is a *tree* if it is rooted and, for every $x \in W$, $x\!\downarrow$ is finite and linearly ordered by R.

The finite model property is a very important notion that, as we shall see, plays a very important rôle in the study of non-classical logics. The standard technique in order to prove the finite approximability of an intermediate logic is the *filtration* method (cfr. [23, Chapter 5]).

DECIDABILITY. A logic L is called *decidable* if, for every given $\mathcal{L}$-formula φ, there exists an algorithm deciding whether $\varphi \in L$ or not.

It is well known that both **Cl** and **Int** are decidable. For instance, the decidability of **Cl** can be easily acknowledged by noting that the truth-value of a formula $\varphi(p_1, \ldots, p_n)$ depends only on the truth-values of the variables $p_1, \ldots, p_n$. Hence, by writing down all the 2^n possible assignment of the truth-values T and F to the variables, in order to check whether $\varphi \in \mathbf{Cl}$ one has only to verify that the truth-value of φ is always T. Of course, such a verification can always be done. Moreover, the decidability of **Cl** and **Int** follows from the following result, due to Harrop:

Theorem 1.24 (Harrop)**.** *Let L be a finitely axiomatizable and finitely approximable intermediate logic. Then L is decidable.*

It is known that there are continuum many intermediate logics[14] and since there can be only countably many of them that are decidable, there exist also undecidable intermediate logics. A concrete example of a finitely axiomatizable undecidable intermediate logic was first given by Shehtman in [146].

LOCAL TABULARITY. Two $\mathcal{L}$-formulas φ and ψ are said to be *equivalent* in an intermediate logic L (or *L-equivalent*) if $\varphi \leftrightarrow \psi \in L$. An intermediate logic L is called *locally tabular* if, for every $n < \omega$, L contains only a finite number of pairwise non-equivalent $\mathcal{L}$-formulas built from the variables $q_1, \ldots, q_n$.

Every tabular intermediate logic is locally tabular and therefore **Cl** is locally tabular. Moreover, every locally tabular intermediate logic is finitely approximable. However, these three notions are not equivalent since there are locally tabular intermediate logics that are not tabular and finitely approximable intermediate logics that are not locally tabular. A classical example of a logic with the finite model property which is not locally tabular is, as we shall see later, **Int** itself.

STRUCTURAL COMPLETENESS. Consider an inference rule r: $\dfrac{\psi_1, \ldots, \psi_n}{\varphi}$ We say that the rule r is *admissible* in an intermediate logic L if, for every substitution σ, $\sigma(\varphi) \in L$ whenever $\sigma(\psi_1), \ldots, \sigma(\psi_n) \in L$. Moreover, we say that the rule r is *derivable* in L if $\{\psi_1, \ldots, \psi_n\} \vdash_L \varphi$. An intermediate logic L is said to be *structurally complete* if every admissible rule in L is derivable in L.

For any intermediate logic L and for every finite set of $\mathcal{L}$-formuals Γ, it can be proved that $\Gamma \vdash_L \varphi$ iff $\bigwedge\Gamma \to \varphi \in L$, therefore, the rule r is derivable in L iff $\psi_1 \wedge \ldots \wedge \psi_n \to \varphi \in L$. Moreover, it is also clear that if a rule r is derivable in L, then it is also admissible in L.

[14] Already in 1932, Kurt Gödel [66] proved that there exist at least countably many different intermediate logics, but it was only in 1968 that Jankov [85] showed that there are uncountably many of them.

Theorem 1.25. **Cl** *is structurally complete.*

It is clear that, for any intermediate logic L, if L is decidable and structurally complete, then the problem of checking whether a given inference rule is admissible in L, called the *admissibility problem for* L, is decidable as well. Therefore, we have

Corollary 1.26. *The admissibility problem for* **Cl** *is decidable.*

In contrast to **Cl**, **Int** is not structurally complete. For instance, the two following inference rule, called respectively the *Scott rule* and the *Kreisel-Putnam rule*,

$$\frac{(\neg\neg p \to p) \to p \vee \neg p}{\neg p \vee \neg\neg p} \qquad \frac{\neg p \to (q \vee r)}{(\neg p \to q) \vee (\neg p \to r)}$$

are admissible but not derivable in **Int**.

The admissibility problem for **Int** is much more complicated with respect to that of **Cl**. However, it can still be shown that

Theorem 1.27 (Rybakov[15]). *The admissibility problem for* **Int** *is decidable.*

Disjunction Property. An intermediate logic L is said to have the *disjunction property* if, for every $\mathcal{L}$-formulas φ and ψ,

$$\varphi \vee \psi \in L \Longleftrightarrow \varphi \in L \text{ or } \psi \in L.$$

Logics with the disjunction property are also known in the literature as *constructive logics*. It is cleat that **Cl** does not have the disjunction property, since the law of excluded middle $p \vee \neg p$ is a theorem of Cl. A semantic analogue of the disjunction property for Kripke complete intermediate logic is the following

Theorem 1.28. *Let L be an intermediate logic such that $L = Log\mathcal{C}$ for some class $\mathcal{C}$ of (rooted) Kripke frame. Then L has the disjunction property iff, for every $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}$, $\mathfrak{F}_1 + \mathfrak{F}_2$ is a generated subframe of a (rooted) frame $\mathfrak{F} \in \mathcal{C}$.*

From the previous theorem, we immediately get the following

Corollary 1.29. **Int** *has the disjunction property.*

1.3.2 Some examples of intermediate logics

We now introduce some of the most known intermediate logics in the literature and state a few properties about them. In particular, we will say when these logics are *elementary.*

We say that a class of Kripke frames $\mathcal{C}$ is *elementary* if there is a set Φ of first-order sentences in the language $\langle R, = \rangle$ such that, for every Kripke frame $\mathfrak{F}$, $\mathfrak{F} \in \mathcal{C}$ iff $\mathfrak{F}$ is a (classical) model for Φ. A si-logic L is *elementary* if the class of all Kripke frames for L is elementary[16].

[15] For a proof of the theorem, cfr. [142, Chapter 3] or [23, Section 16.7].

[16] An important theorem of Kit Fine (cfr. [48]) says that if an intermediate logic L is characterized by an elementary class of Kripke frames, then it is canonical, i.e. the canonical frame $\mathfrak{F}_L$ is a frame for L; since strong Kripke completeness is in turn implied by canonicity, by showing that a Kripke complete intermediate logic L is not strongly Kripke complete, one would prove that L is not elementary.

The Gödel-Dummett logic

The *Dummett formula* $\boldsymbol{da}$ is the following $\mathcal{L}$-formula:

$$\boldsymbol{da} = (p \to q) \vee (q \to p).$$

The *Gödel-Dummett logic* (or the *logic of chains*) is the following intermediate logic:

$$\mathbf{LC} = \mathbf{Int} + \boldsymbol{da}.$$

This logic has been deeply investigated in the literature and bears its name from the works of Gödel [66] and, especially, of Dummett [34], who first introduced the logic **LC** proving also its decidability.

We say that a Kripke frame $\mathfrak{F} = \langle W, R \rangle$ is *strongly connected* if it satisfies the following condition:

$$\forall x, y, z (xRy \wedge xRz \Rightarrow yRz \vee zRy).$$

It is not hard to prove that the frames for **LC** can be described as follows

Proposition 1.30. *A frame $\mathfrak{F} = \langle W, R \rangle$ validates $\boldsymbol{da}$ iff $\mathfrak{F}$ is strongly connected.*

Notice that every rooted strongly connected Kripke frame $\mathfrak{F}$ is a chain. Moreover, it is possible to show that the logic **LC** is complete with respect to the class of chains and thus is elementary.

The Jankov logic

Consider the $\mathcal{L}$-formula

$$\boldsymbol{wem} = \neg p \vee \neg\neg p,$$

which is known as the *law of the weak excluded middle.* The *Jankov logic*, so called after the Soviet logician V. A. Jankov - who first introduced and studied it extensively in [82] and [87] -, is the following intermediate logic:

$$\mathbf{KC} = \mathbf{Int} + \boldsymbol{wem}.$$

Jankov logic is also known in the literature with the name of *De Morgan logic*, since, over **Int**, the following unprovable instance of the De Morgan law

$$\neg(p \wedge q) \to (\neg p \vee \neg q)$$

is equivalent to $\boldsymbol{wem}$. Therefore, we have

$$\mathbf{KC} = \mathbf{Int} + \neg p \vee \neg\neg p = \mathbf{Int} + \neg(p \wedge q) \to (\neg p \vee \neg q).$$

A Kripke frame $\mathfrak{F} = \langle W, R \rangle$ is called *strongly directed* or *convergent* if satisfies the following condition:

$$\forall x, y, z (xRy \wedge xRz \Rightarrow \exists u (yRu \wedge zRu).$$

Proposition 1.31. *A frame $\mathfrak{F} = \langle W, R \rangle$ validates $\boldsymbol{wem}$ iff $\mathfrak{F}$ is strongly directed.*

Also in the case of Jankov logic, it is possible to show, by a canonical model argument, that the class of strongly directed frames characterizes **KC** and thus that it is elementary.

The Kreisel-Putnam logic

The *Kreisel-Putnam formula* is the following $\mathcal{L}$-formula

$$\boldsymbol{kp} = \neg p \to (q \vee r) \to (\neg p \to q) \vee (\neg p \to r),$$

which is nothing but the Krieisel-Putnam rule, introduced in the previous section, considered as an implicational formula. We define the *Kreisel-Putnam logic* to be the intermediate logic obtained from intuitionistic logic by adding the formula $\boldsymbol{kp}$, i.e.

$$\mathbf{KP} = \mathbf{Int} + \boldsymbol{kp}.$$

This logic has been introduced by Kreisel and Putnam in [94] as a counterexample to a conjecture of the Polish logician Jan Łukasiewicz, who, in 1952 ([102]), asserted that **Int** was the greatest propositional system closed under Subst and MP and enjoying the disjunction property[17].

With respect to the Kreisel-Putnam rule, Prucnal in [138] proved the following

Theorem 1.32 (Prucnal)**.** *The Kreisel-Putnam rule is admissible in every intermediate logic.*

As an immediate corollary, we obtain

Corollary 1.33. *Let L be a structurally complete intermediate logic. Then*

$$\mathbf{KP} \subseteq L.$$

It can be shown, by a filtration argument, that **KP** is finitely approximable and thus, by Theorem 1.24, decidable. Moreover, the Kreisel-Putnam logic is elementary and we conclude this brief description of this logic with the following soundness result:

Proposition 1.34. *A frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{kp}$ iff $\mathfrak{F}$ satisfies the following condition:*

$$\begin{aligned}\forall x, y, z(xRy \wedge xRz \wedge \neg yRz \wedge \neg zRy \to \exists u(xRu \wedge uRy \wedge uRz \wedge \\ \forall v(uRv \to \exists w(vRw \wedge (yRw \vee zRw))))).\end{aligned}$$

The Scott logic

In analogy with the Kreisel-Putnam logic, we define the *Scott logic* **SL** to be the following intermediate logic

$$\mathbf{SL} = \mathbf{Int} + \boldsymbol{sa},$$

where $\boldsymbol{sa}$ is the $\mathcal{L}$-formula, obtained from the Scott rule and known as the *Scott formula*,

$$\boldsymbol{sa} = ((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p.$$

The Scott logic was mentioned in the paper [94] as another counterexample to Łukasiewicz's conjecture. In contrast to **KP**, **SL** is not elementary, since it can

[17] To be precise, Łukasiewicz stated a conjecture concerning the set of the unprovable formulas of **Int** that is equivalent to the above stated assertion. Cfr. [102, pp. 208-209].

be shown that $\boldsymbol{sa}$ is not first-order definable over partial orders[18]. However, if we restrict ourselves to the class of Kripke frames of finite depth, then the following characterization of the **SL**-frames is possible[19].

Let $\mathfrak{F} = \langle W, R \rangle$ be a Kripke frame and let $x, y, z \in W$ be such that y and z are both maximal in $\mathfrak{F}$ while x is not. We say that x is *prefinal* if, for every proper successor w of x, w is maximal in $\mathfrak{F}$ and we say that y and z are *prefinally connected* in $\mathfrak{F}$ if $y = z$ or there is a sequence $v_1, \ldots, v_n$ $(n > 1)$ of maximal elements in $\mathfrak{F}$ such that

(i) $y = v_1$ and $z = v_n$;

(ii) for every $1 \leq i < n$, there exists $k \in W$ such that k is prefinal and $\{v_i, v_{i+1}\} \subseteq max(k)$.

Proposition 1.35. *Let $\mathfrak{F} = \langle W, R \rangle$ be a Kripke frame of finite depth. Then $\mathfrak{F}$ validates $\boldsymbol{sa}$ iff, for every $w \in W$ and $x, y \in max(w)$, x and y are prefinally connected in $\mathfrak{F}_w$.*

Another way of charactering **SL** is as follows:

Proposition 1.36. *A Kripke frame $\mathfrak{F} = \langle W, R \rangle$ validates $\boldsymbol{sa}$ iff the Kripke frame $\mathfrak{S}$ is not a p-morphic image of every generated subframe of $\mathfrak{F}$, where $\mathfrak{S}$ is the following frame*[20]

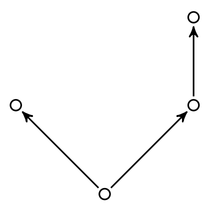


Figure 1.1: *The Kripke frame $\mathfrak{S}$.*

Medvedev's logic

The logics considered so far were introduced axiomatically by providing a (finite) axiomatization for them. We will instead present the following intermediate logic through its semantical characterization. The *Medvedev's logic* **ML**,

[18]This fact was first proved by Van Benthem in [159] by the use of the downward Löwenheim-Skolem Theorem of first-order logic, cfr. [159, Theorem 86]. For a sketch of a proof which use the compactness theorem of fisrt-order logic, cfr. [23, p. 187].

[19]For the following characterization of **SL**-frames of finite depth, cfr. [49]. Cfr. also [46], where the same characterization is used to prove the Kripke completeness and the finite approximability of the Scott logic with respect to the so characterized class of finite frames, thus disproving a conjecture of Minari [123]. Finally, over trees, it can be proved that $\boldsymbol{sa}$ defines the following first-order condition:

$$\forall x \neg \exists yzv(xRy \wedge xRz \wedge zRv \wedge z \neq v \wedge \neg \exists w(yRw \wedge zRw)).$$

[20]The class of frames $\mathcal{S} = \{\mathfrak{F} \mid \forall \mathfrak{G} \subsetneq \mathfrak{F}(\mathfrak{S} \text{ is not a p-morphic image of } \mathfrak{G})\}$ characterizes **SL** too. Moreover, it can be shown that if we restrict $\mathcal{S}$ to those frames $\mathfrak{F}$ such that, for each $x \in \mathfrak{F}$, there exists y maximal in $\mathfrak{F}$ such that xRy, then $\mathcal{S}$ contains exactly those frames satisfying the sufficient condition validating $\boldsymbol{sa}$ of Proposition 1.35.

also known in the literature as the *logic of finite problems*, was first introduced in 1962 by the Soviet logician Y. Medvedev ([114, 116]) as an attempt to properly formalize the informal interpretation of intuitionistic logic as a calculus of problem proposed by Kolmogorov in 1932 ([91]).

For every $n > 0$ considered as a set $n = \{0, 1, \ldots, n-1\}$, let the *Medvedev frame* based on n be the following Kripke frame

$$\mathfrak{P}_n = \langle \mathcal{P}(n) \setminus \{\emptyset\}, \supseteq \rangle$$

For instance, the Medvedev frames $\mathfrak{P}_2$ and $\mathfrak{P}_3$ are respectively as follows:

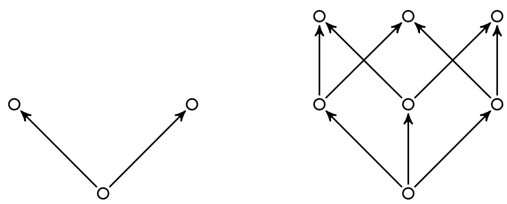


Figure 1.2: *The Medvedev frames* $\mathfrak{P}_2$ *and* $\mathfrak{P}_3$.

Therefore, these frames are nothing but n-ary Boolean cubes without the top element. *Medvedev's logic* **ML** is then defined as

$$\mathbf{ML} = \bigcap_{0<n} \mathrm{Log}\,\mathfrak{P}_n$$

From this semantic characterization, **ML** is finitely approximable and one can easily show that both **SL** and **KP** are included in **ML**. Furthermore, by Theorem 1.28, it follows that **ML** has the disjunction property. But more than this is known: actually, Medvedev's logic is a *maximal* intermediate logic with the disjunction property[21].

Thus Medvedev's logic is a very interesting and rich intermediate logic. Indeed, in [137], it is also proved that

Theorem 1.37 (Prucnal)**.** *Medvedev's logic* **ML** *is structurally complete.*

Finally, it is no coincidence that this logic has been introduced by providing only a Kripke semantics for it, since no axiomatization is known. As to the topic of **ML**'s axiomatizability, it is only known that it is not finitely axiomatizable ([110]) and therefore the problem of the decidability of **ML** is still an open issue.

The logics of bounded depth

Consider the sequence of formulas $\boldsymbol{bd}_n$ defined in the previous section in the proof of Theorem 1.22. For each $n > 0$, the *logic of bounded depth* $\mathbf{BD}_n$ (also known as *logic of the n^{th} slice*, [76]) is the following intermediate logic

$$\mathbf{BD}_n = \mathbf{Int} + \boldsymbol{bd}_n.$$

[21]The interest on maximal constructive intermediate logic was aroused by [89], where it is shown that there is no greatest intermediate logic with the disjunction property. The existence of maximal constructive intermediate logics follows easily from Zorn's lemma, but **ML** was the first concrete example of an intermediate logic of that kind and this result was first proved by Levin [98] (cfr. also [106]). In [45], it is shown that the set of all maximal intermediate logics with the disjunction property has the power of continuum.

Notice that $\mathbf{BD}_1 = \mathbf{Int} + p_1 \vee \neg p_1 = \mathbf{Cl}$.

Definition 1.15. Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame.

1. The frame $\mathfrak{F}$ is of *depth* $n < \omega$, and we write $d(\mathfrak{F}) = n$, if there exists a chain of n points in $\mathfrak{F}$ and every other chain in $\mathfrak{F}$ contains at most n points; moreover, we say that the frame $\mathfrak{F}$ is of *finite depth* if $d(\mathfrak{F}) < \omega$.

2. The frame $\mathfrak{F}$ is of *infinite depth*, and we write $d(\mathfrak{F}) = \omega$, if, for every $n < \omega$, $\mathfrak{F}$ contains a chain of n points.

3. For $w \in W$, the *depth of* w is the depth of $\mathfrak{F}_w$ and we denote it by $d(w)$.

Proposition 1.38. *A frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{bd}_n$ iff $d(\mathfrak{F}) \leq n$, that is iff $\mathfrak{F}$ satisfies the following condition:*

$$\forall x_0, \ldots, x_n (\bigwedge_{i=0}^{n-1} x_i R x_{i+1} \Rightarrow \bigvee_{i \neq j} x_i = x_j).$$

Therefore, for each $n > 0$, the logic $\mathbf{BD}_n$ is elementary.

The logics of bounded width

Let us consider the following sequence of $\mathcal{L}$-formulas $\boldsymbol{bw}_n$:

$$\boldsymbol{bw}_n = \bigvee_{i=0}^{n} (p_i \rightarrow \bigvee_{j \neq i} p_j), \qquad n \geq 1.$$

For each $n > 0$, the *logic of bounded width* $\mathbf{BW}_n$ is the following intermediate logic

$$\mathbf{BW}_n = \mathbf{Int} + \boldsymbol{bw}_n.$$

Given a Kripke frame $\mathfrak{F} = \langle W, R\rangle$, a subset $X \subseteq W$ is said to be an *antichain* in $\mathfrak{F}$ if, for every $x, y \in X$, $xRy \Longrightarrow x = y$, that is, distinct point in X do not see each other.

Definition 1.16. Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame.

1. The frame $\mathfrak{F}$ is of *width* $n < \omega$, and we write $w(\mathfrak{F}) = n$, if there exists an antichain of n points in $\mathfrak{F}$ and every other antichain in $\mathfrak{F}$ contains at most n points; moreover, we say that the frame $\mathfrak{F}$ is of *finite width* if $w(\mathfrak{F}) < \omega$.

2. The frame $\mathfrak{F}$ is of *infinite width*, and we write $w(\mathfrak{F}) = \omega$, if, for every $n < \omega$, $\mathfrak{F}$ contains an antichain of n points.

Proposition 1.39. *A frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{bw}_n$ iff, for every rooted subframe $\mathfrak{G} \subsetneq \mathfrak{F}$, $w(\mathfrak{G}) \leq n$, that is iff $\mathfrak{G}$ satisfies the following condition:*

$$\forall x, x_0, \ldots, x_n (\bigwedge_{i=0}^{n-1} x R x_i \Rightarrow \bigvee_{i \neq j} x_i R x_j).$$

Thus the $\mathcal{L}$-formulas $\boldsymbol{bw}_n$ bound the width of *rooted frames*. However, we can not find an intuitionistic formula which bounds the width of an arbitrary frame, since, for instance, the frame $\langle\{0,1,2,\dots\},=\rangle$, being the disjoint union of ω single-point frames, validates all formulas in **Cl**. In particular, the logics $\mathbf{BW}_n$ are not elementary. However, a classical theorem of Fine shows that these logics are Kripke complete.

We say that an intermediate logic L is *of width* n if $\boldsymbol{bw}_n \in L$ but $\boldsymbol{bw}_{n+1} \notin L$.

Theorem 1.40 (Fine[22])**.** *Any intermediate logic of width n is characterized by a class of Noetherian Kripke frames of width $\leq n$. In particular, for each $n \geq 1$, $\mathbf{BW}_n$ is Kripke complete.*

The logics of bounded branching

Consider the following family of $\mathcal{L}$-formulas:

$$\boldsymbol{bb}_n = \bigwedge_{i=0}^{n}((p_i \to \bigvee_{j\neq i} p_j) \to \bigvee_{j\neq i} p_j) \to \bigvee_{i=0}^{n} p_i, \qquad n \geq 1.$$

The *logics of bounded branching* $\mathbf{T}_n$ (also known as the *Gabbay - De Jongh logics*, cfr. [56]), for $n \geq 1$, are defined as follows:

$$\mathbf{T}_n = \mathbf{Int} + \boldsymbol{bb}_n.$$

A finite Kripke frame $\mathfrak{F} = \langle W, R\rangle$ is said to be of *branching* $\leq n$ if every point in $\mathfrak{F}$ has at most n distinct immediate successors.

Proposition 1.41. *A finite Kripke frame $\mathfrak{F} = \langle W, R\rangle$ validates $\boldsymbol{bb}_n$ iff $\mathfrak{F}$ is of branching $\leq n$.*

The restriction to finite frames is actually necessary, since, analogously to the Scott formula, the arbitrary validating frames for the formulas $\boldsymbol{bb}_n$ can not be characterized by any first-order condition on the accessibility relation R and thus the logics $\mathbf{T}_n$ ($n \geq 2$) are not elementary.

In [56], Gabbay and de Jongh proved that the logics $\mathbf{T}_n$ are decidable and have the disjunction property. Moreover, they also showed that these logics are complete with respect to the class of finite frames of branching $\leq n$. As a consequence, one can show, for every $n > 1$, that the class $\mathcal{T}_n$ of n-ary trees, that is, trees whose non maximal points have exactly n distinct immediate successors, also characterizes $\mathbf{T}_n$.

1.4 Lattices and Heyting Algebras

Definition 1.17. Let $\mathfrak{P} = \langle P, \leq\rangle$ be a partially ordered set and let $A \subseteq P$. An element $p \in P$ is an *upper bound* for A if $a \leq p$ for every $a \in A$, while $p \in P$ is said to be the *least upper bound* of A (or *supremum* of A), and we write $supA$, if p is an upper bound for A and if $a \leq b$ for every $a \in A$, then $p \leq b$, that is, p is the smallest among the upper bounds of A. Dually, an element $p \in P$ is a *lower bound* for A if p is an upper bound in the order dual poset $\mathfrak{P}^{\mathrm{op}} = \langle P, \geq\rangle$ and p is the *greatest lower bound* of A (or *infimum* of A), and we write $infA$, if p is the supremum of A in $\mathfrak{P}^{\mathrm{op}}$.

[22]For a proof of this theorem, cfr. [23, Section 10.4] or [142, Section 2.10].

Definition 1.18. A partially ordered set $\langle L, \leq \rangle$ is called a *lattice* if, for every $a, b \in L$, both $sup\{a, b\}$ and $inf\{a, b\}$ exist (in L). Moreover, L is said to be *bounded* if L has a least and a greatest element denoted by 0 and 1 respectively.

Now, given a lattice $\langle L, \leq \rangle$, one can define two binary operations $\wedge$ and $\vee$ on L by setting, for every $a, b \in L$,

$$a \wedge b = inf\{a, b\}$$
$$a \vee b = sup\{a, b\}$$

and show that lattices can also be defined axiomatically as algebraic structures. Indeed, the following holds (cfr. [20, p. 8])

Proposition 1.42. *An algebraic structure $\langle L, \wedge, \vee, 0, 1 \rangle$, with the following similarity type $\langle 2, 2, 0, 0 \rangle$, is a bounded lattice iff L satisfies the following identities:*

$$\begin{aligned} x \wedge y &\approx y \wedge x & x \vee y &\approx y \vee x \\ x \wedge (y \wedge z) &\approx (x \wedge y) \wedge z & x \vee (y \vee z) &\approx (x \vee y) \vee z \\ x \wedge x &\approx x & x \vee x &\approx x \\ x \wedge 1 &\approx x & x \vee 0 &\approx x \\ x \wedge (y \vee x) &\approx x & x \vee (y \wedge x) &\approx x \end{aligned}$$

Therefore a bounded lattice can be viewed as a structure $\langle L, \wedge, \vee, 0, 1 \rangle$ where $\langle L, \wedge, 1 \rangle$ and $\langle L, \vee, 0 \rangle$ are both commutative monoids and $\wedge$ and $\vee$ are connected by the *absorption law* : $x \wedge (y \vee x) \approx x \approx x \vee (y \wedge x)$.

Definition 1.19. A lattice $\langle L, \wedge, \vee \rangle$ is said to be *complete* if, for every subset $A \subseteq L$, there exist $\bigvee A = supA$ and $\bigwedge A = infA$.

Obviously, every finite lattice is a bounded complete lattice.

Definition 1.20. A *distributive lattice* is a lattice $\langle L, \wedge, \vee \rangle$ which satisfies the following *distributive laws*[23],

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$$

We are now finally ready to define the main notion of this section.

Definition 1.21. A bounded distributive lattice $\mathfrak{H} = \langle H, \wedge, \vee, 0, 1 \rangle$ is a *Heyting algebra* or a *pseudo-Boolean algebra* if, for every $a, b \in H$, there exists an element, denoted by $a \to b$, such that, for every $c \in H$,

$$c \leq a \to b \Leftrightarrow a \wedge c \leq b$$

We call the binary operation $\to$ the operation of *implication* and for every $a \in H$ we define $\neg a := a \to 0$ and call it the *pseudo-complement* of a.

[23]More precisely, for a lattice to be distributive it is only requested that it satisfies at least one of the two distributive laws, since it can be shown that each of these two identities implies the other.

Given a lattice $\mathfrak{L} = \langle L, \wedge, \vee \rangle$ and $a, b, c \in L$, the element c is said to be the *relative pseudo-complement* of a with respect to b if c is the greatest element among elements $x \in L$ such that $a \wedge x \leq b$, that is,

$$c = \bigvee \{x \in L \mid a \wedge x \leq b\}.$$

Moreover, it is not hard to show that, for any Heyting algebra $\mathfrak{H} = \langle H, \wedge, \vee, 0, 1 \rangle$ and elements $a, b \in H$, the element $a \to b$ is the relative pseudo-complement of a with respect to b. The relative pseudo-complement of a to a, that is, $a \to a$ is the greatest element of $\mathfrak{H}$, hence we can present any Heyting algebra with the following similarity type: $\langle \wedge, \vee, \to, 0 \rangle$. Thus from now on, we will denote a Heyting algebra as $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0 \rangle$.

In complete analogy with the case of lattices, Heyting algebras too can be given an algebraic axiomatic characterization.

Proposition 1.43. *A bounded lattice $\mathfrak{L} = \langle L, \wedge, \vee, 0, 1 \rangle$ is a Heyting algebra iff there is a binary operation $\to$ on L that satisfies the following identities:*

$$\begin{aligned} x \to x &\approx 1 \\ x \wedge (x \to y) &\approx x \wedge y \\ y \wedge (x \to y) &\approx y \\ x \to (y \wedge z) &\approx (x \to y) \wedge (x \to z) \end{aligned}$$

Let $\mathcal{HA}$ denote the class of all Heyting algebras. Then, by Birkohoff theorem, we have the following

Corollary 1.44. *$\mathcal{HA}$ is a variety.*

We now give an alternative characterization of *complete* Heyting algebras, i.e. Heyting Algebras which are complete as a lattice.

Lemma 1.45. *A complete bounded lattice $\mathfrak{L} = \langle L, \wedge, \vee, 0, 1 \rangle$ is a Heyting algebra iff it satisfies the following* infinite distributive law

$$x \wedge \bigvee_{i \in I} y_i \approx \bigvee_{i \in I} (x \wedge y_i)$$

for every index set I.

We conclude this section with some examples of Heyting algebras.

Example 1. Let $\mathfrak{L} = \langle L, \wedge, \vee \rangle$ be a finite distributive lattice. Since L is finite, $\mathfrak{L}$ is bounded and complete. Moreover, since $\mathfrak{L}$ is distributive, it satisfies the infinite distributive law. Therefore, by Lemma 1.45, every finite distributive lattice is a Heyting algebra.

Example 2. A *chain* is a poset $\langle P, \leq \rangle$ where the relation $\leq$ is a *total order*, that is, for every $a, b \in P$, either $a \leq b$ or $b \leq a$. Let $\mathfrak{C} = \langle C, \leq \rangle$ be a chain with least and greatest element. Define, for $a, b \in C$,

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b. \end{cases}$$

Then it is easy to show that $\mathfrak{C}$ with the implication $\to$ is a Heyting algebra. Hence, every chain with a least and greatest element is a Heyting algebra.

Example 3. By Theorem 1.20 and 1.45, it follows that the lattice of si-logics Ext**Int** is a Heyting algebra.

Example 4. A bounded lattice $\mathfrak{L} = \langle L, \wedge, \vee, 0, 1\rangle$ is called *complemented* if every element $a \in L$ has a *complement*, that is, an element $b \in L$ such that

$$a \vee b = 1 \qquad \text{and} \qquad a \wedge b = 0.$$

A *Boolean algebra* or *Boolean lattice* is a bounded complemented distributive lattice. It can be shown that in distributive lattices complements are unique and in a Boolean algebra $\mathfrak{B}$ the complement of an element a is usually denoted by $\neg a$.
Now, let a Boolean algebra $\mathfrak{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ be given. By setting

$$a \to b := \neg a \vee b$$

we turn $\mathfrak{B}$ into a Heyting algebra $\langle B, \wedge, \vee, \to, 0\rangle$. Therefore, every Boolean algebra is a Heyting algebra.

Example 5. Let $\langle X, \tau\rangle$ be a topological space. Then the structure $\langle \tau, \cap, \cup, \sqsupset, \emptyset\rangle$ provides an example of a complete Heyting algebra, where the implication is defined as follows:

$$A \sqsupset B := (X \setminus A \cup B)^{\circ}$$

Indeed, $(X \setminus A \cup B)^{\circ} = \bigcup\{Z \in \tau \mid Z \subseteq X \setminus A \cup B\} = \bigcup\{Z \in \tau \mid A \cap Z \subseteq B\}$.

Example 6. Consider the propositional intuitionistic logic **Int** with his language $\mathcal{L}$. For every $\varphi, \psi \in \mathbf{For}\mathcal{L}$, define the relation $\sim_{\mathbf{Int}} \subseteq \mathbf{For}\mathcal{L} \times \mathbf{For}\mathcal{L}$ as follows

$$\varphi \sim_{\mathbf{Int}} \psi \iff \varphi \leftrightarrow \psi \in \mathbf{Int}.$$

It is easily seen that $\sim_{\mathbf{Int}}$ is an equivalence relation on the set of $\mathcal{L}$-formulas. Denote by $[\varphi]$ the equivalence class $\{\psi \mid \psi \sim_{\mathbf{Int}} \varphi\}$ of φ by the equivalence relation $\sim_{\mathbf{Int}}$ and let $\mathbf{For}\mathcal{L}/_{\sim_{\mathbf{Int}}} = \{[\varphi] \mid \varphi \in \mathcal{L}\}$ be the quotient set of $\mathcal{L}$-formulas by $\sim_{\mathbf{Int}}$. Now, the logical connectives in $\mathcal{L}$ naturally induce operations on $\mathbf{For}\mathcal{L}/_{\sim_{\mathbf{Int}}}$ as follows[24]:

$$[\varphi] \circ [\psi] := [\varphi \circ \psi] \qquad \text{for } \circ \in \{\wedge, \vee, \to\}$$

The structure $\mathfrak{L}_{\mathbf{Int}} = \langle \mathbf{For}\mathcal{L}/_{\sim_{\mathbf{Int}}}, \wedge, \vee, \to, [\bot]\rangle$ is called the *Lindenbaum-Tarksi algebra* of **Int** and this algebra constitutes an example of Heyting algebra. In particular, the Lindenbaum algebra of **Int** is the $\mathcal{HA}$-free algebra over ω generators.

1.4.1 Filters and congruences in Heyting algebras

We now consider an algebraic analog of a set of formulas closed under *modus ponens*, that is, the notion of *filter*.

Definition 1.22. Let $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ be a Heyting algebra. A subset $\nabla \subseteq H$ is called a *filter* in $\mathfrak{H}$ if, for every $a, b \in H$,

- $1 \in \nabla$;

[24] The operations are indeed well defined since the Equivalent Replacement Theorem 1.19 holds for **Int**.

- $a \in \nabla$ and $a \to b \in \nabla \Longrightarrow b \in \nabla$.

Thus we trivially have that $\{1\}$ and H are filters. A filter different from H is called *proper*. The previous definition is clearly equivalent to the more common formulation of a filter in a lattice.

Lemma 1.46. *Let $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ be a Heyting algebra. A subset $\nabla \subseteq H$ is filter in $\mathfrak{H}$ iff, for every $a, b \in H$*

1. $\nabla \neq \emptyset$;
2. $a, b \in \nabla \Longrightarrow a \wedge b \in \nabla$;
3. $a \in \nabla$ and $a \leq b \Longrightarrow b \in \nabla$.

Consider a lattice $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ and let $X \subseteq L$. Since the intersection of an arbitrary family of filters containing X is again a filter containing X and, in particular, is the smallest filter containing X, the notion of the *filter generated by X* makes sense and we denote such a filter by $[X)$. The next lemma gives a constructive characterization of $[X)$.

Lemma 1.47. *Let $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ be a lattice and let X be a non-empty subset of L. Then the filter generated by X is the following set:*

$$[X) = \{b \in L \mid a_1 \wedge \ldots \wedge a_n \leq b, \text{ for some } a_1, \ldots, a_n \in X\}.$$

If the set X is a singleton, i.e. $X = \{a\}$ for some $a \in L$, then we denote the filter generated by X with $[a)$. Filters of this form are called *principal filters*. Note that in a finite lattice, every filter is a principal filter, because it is generated by the (finite) conjunction of its elements.

The dual notion of filter is the notion of *ideal*. Indeed, an ideal ∇ in a lattice $\mathfrak{L}$ is a filter in $\mathfrak{L}^{\mathrm{op}}$.

Proposition 1.48. *Let $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ be a Heyting algebra and let $\nabla \subseteq H$ be a filter in $\mathfrak{H}$. The set of filters in $\mathfrak{H}$ containing ∇ forms a complete distributive lattice with the infimum and supremum defined respectively by*

$$\bigwedge\{\nabla_i \mid i \in I\} = \bigcap_{i\in I} \nabla_i \qquad \text{and} \qquad \bigvee\{\nabla_i \mid i \in I\} = [\bigcup_{i\in I} \nabla_i).$$

The previous proposition allows us to give the following

Definition 1.23. The lattice of filters in $\mathfrak{H}$ containing $\nabla = \{1\}$ is called the *lattice of filters* in $\mathfrak{H}$ and we denote it by $\mathbf{Fi}(\mathfrak{H})$.

We now show a close correspondence between the lattice of congruence relation $\mathbf{Con}(\mathfrak{H})$ and the lattice of filters $\mathbf{Fi}(\mathfrak{H})$ of a given Heyting algebra $\mathfrak{H}$. For a full proof of the following lemmas, the reader is referred to [142, pp. 117-119].

Lemma 1.49. *Let $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ be a Heyting algebra and ∇ be a filter on $\mathfrak{H}$. Then the relation $\sim_\nabla \subseteq H \times H$ defined by*

$$a \sim_\nabla b \Leftrightarrow (a \to b) \wedge (b \to a) \in \nabla$$

is a congruence relation on $\mathfrak{H}$. Moreover, for every $a \in H$, $a \in \nabla$ iff $a \sim_\nabla 1$.

Lemma 1.50. *Let $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ be a Heyting algebra and $\sim$ be a congruence on $\mathfrak{H}$. Then $\nabla_{\sim} := \{a \in H \mid a \sim 1\}$ is a filter on $\mathfrak{H}$. Moreover, $\sim = \sim_{\nabla_{\sim}}$.*

From the two above lemmas, we thus obtain the following

Lemma 1.51. *Let $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ be a Heyting algebra, ∇ be a filter on $\mathfrak{H}$ and $\sim$ be a congruence on $\mathfrak{H}$. Then $\sim_{\nabla_{\sim}} = \sim$ and $\nabla_{\sim_{\nabla}} = \nabla$.*

We are now ready for the main theorem of this section.

Theorem 1.52. *Let $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ be a Heyting algebra. The function $\nabla\colon \mathbf{Con}(\mathfrak{H}) \to \mathbf{Fi}(\mathfrak{H})$ given by $\sim \mapsto \nabla_{\sim}$ is a lattice isomorphism. Hence, for every Heyting algebra $\mathfrak{H}$, $\mathbf{Con}(\mathfrak{H}) \cong \mathbf{Fi}(\mathfrak{H})$.*

Due to this correspondence between congruences and filters, given any filter ∇ of a Heyting algebra $\mathfrak{H}$, we will also write $\mathfrak{H}/\nabla$ to denote the quotient algebra of $\mathfrak{A}$ by the congruence $\sim_{\nabla}$ associated to ∇. Moreover, as a direct consequence of the previous theorem, we have

Corollary 1.53. *The variety $\mathcal{HA}$ of Heyting algebras is congruence-distributive and has the Congruence Extension Property (CEP).*

The existing isomorphism between the lattice of congruence relation and the lattice of filters of a given Heyting algebra allows us also to give a simple proof of the following useful characterization of subdirectly irriducibles Heyting algebras, first established by Jankov in [83].

Theorem 1.54. *A Heyting algebra $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ is subdirectly irriducible iff $\mathfrak{H}$ has a second greatest element, i.e. an element ω which is the greatest element in the set $\{a \in H \mid a < 1\}$.*

Proof. ($\Longrightarrow$) Let $\mathfrak{H}$ be subdirectly irriducible. Therefore there is a minimum congruence in $\mathbf{Con}(\mathfrak{H}) \setminus \{\Delta\}$, where Δ is the diagonal relation on H. By Theorem 1.52, $\mathfrak{H}$ has a minimum filter in $\mathbf{Fi}(\mathfrak{H}) \setminus \{\{1\}\}$, denote it by ∇. Then there is an element $\omega \in \nabla$ such that $\omega \neq 1$. We claim that $\nabla = [\omega)$. Suppose not, then since $[\omega) \subseteq \nabla$, there is $b \in \nabla$ such that $\omega \nleq b$. But then $[b) \subset \nabla$, which is not possible. Moreover, if there exists $b \in [\omega)$ such that $b \neq \omega$ and $b \neq 1$, then $[b)$ again would be a proper subset of ∇. Hence $\nabla = [\omega) = \{\omega, 1\}$. Now, for every $1 \neq b \in H$, $[\omega) \subseteq [b)$, that is $b \leq \omega$. Hence ω is the second greatest element of $\mathfrak{H}$.
($\Leftarrow$) If $\mathfrak{H}$ has a second greatest element ω, then $[\omega)$ would be the minimum filter in $\mathbf{Fi}(\mathfrak{H}) \setminus \{\{1\}\}$. By Theorem 1.52, there would be a minimum congruence in $\mathbf{Con}(\mathfrak{H}) \setminus \{\Delta\}$, i.e. $\mathfrak{H}$ would be subdirectly irriducible. □

1.5 Algebraic Completeness of Int and its extensions

First notice that there is no real difference between the propositional language $\mathcal{L}$ and the similarity type $\Omega = \langle \wedge, \vee, \to, 0\rangle$ of Heyting algebras with a countable set of variables X. Therefore, we can identify the set of formulas $\mathbf{For}\mathcal{L}$ with the set of terms $Tm(X)$ of type Ω over X and a valuation $h\colon \mathbf{Tm}(X) \to \mathfrak{A}$ into a given Heyting algebra $\mathfrak{A}$ as a function $h\colon \mathbf{For}\mathcal{L} \to \mathfrak{A}$.

Definition 1.24. Let $\mathfrak{A}$ be a Heyting algebra. We say that a $\mathcal{L}$-formula φ is *valid* in $\mathfrak{A}$ if $\mathfrak{A}$ satisfies the identity $\varphi \approx \top$, that is, if $h(\varphi) = 1$ for every valuation $h\colon \mathbf{For}\mathcal{L} \to \mathfrak{A}$.
Moreover, we say that an inference rule $r : \{\psi_1, \dots, \psi_n\}/\varphi$ is *valid* in $\mathfrak{A}$ if $\mathfrak{A}$ satisfies the quasi-identity $q(r) : \psi_1 \approx 1 \,\&\dots\&\, \psi_n \approx 1 \Rightarrow \varphi \approx 1$.

Theorem 1.55 (Algebraic completeness of **Int**)**.** *Let* $\varphi \in \mathbf{For}\mathcal{L}$*. Then*

$$\varphi \in \mathbf{Int} \iff \varphi \textit{ is valid in } \mathfrak{A}, \textit{ for every Heyting algebra } \mathfrak{A}.$$

Proof. ($\Longrightarrow$) Let $\mathfrak{A}$ be a Heyting algebra. We have to show that all the axiom (A1)-(A9) and all inference rule of Int are valid in $\mathfrak{A}$. As an example, let us show that (A1) is valid in $\mathfrak{A}$. Consider a valuation $h\colon \mathbf{For}\mathcal{L} \to \mathfrak{A}$, then

$$\begin{aligned} h(p_0 \to (p_1 \to p_0)) &= h(p_0) \to (h(p_1) \to h(p_0)) \\ &= \bigvee\{z \in A \mid z \wedge h(p_0) \leq h(p_1) \to h(p_0)\} \\ &= \bigvee\{z \in A \mid z \wedge h(p_0) \wedge h(p_1) \leq h(p_0)\} = 1. \end{aligned}$$

Now consider the rule MP. Suppose that φ and $\varphi \to \psi$ are valid in $\mathfrak{A}$ and let $h\colon \mathbf{For}\mathcal{L} \to \mathfrak{A}$ be a valuation. Then, from our hypothesis, we have

$$\begin{aligned} 1 = h(\varphi) \to h(\psi) &\iff 1 \wedge h(\varphi) = h(\psi) \\ &\iff 1 \wedge 1 = h(\psi) \\ &\iff 1 = h(\psi). \end{aligned}$$

($\Longleftarrow$) Let $\varphi \notin \mathbf{Int}$ and consider the Lindenbaum-Tarski algebra of **Int** $\mathfrak{L}_{\mathbf{Int}}$ (cfr. Example 6). Suppose, towards a contradiction, that φ is valid in $\mathfrak{L}_{\mathbf{Int}}$. Then, consider the natural projection $\pi\colon \mathbf{For}\mathcal{L} \to \mathfrak{L}_{\mathbf{Int}}$ associating to each $\mathcal{L}$-formula ψ its equivalence class $[\psi]$. Since π is a valuation, $\pi(\varphi) = 1$, that is, $[\varphi] = [\top]$, and thus $\varphi \sim_{\mathbf{Int}} \top$. But by definition of $\sim_{\mathbf{Int}}$, we then have $\varphi \leftrightarrow \top \in \mathbf{Int}$, whence $\varphi \in \mathbf{Int}$, contrary to our hypothesis. □

Now consider a si-logic L. We can associate with L the class of Heyting algebra $\mathcal{V}_L$ validating all the formulas from L, that is, such that $\mathcal{V}_L \models \varphi \approx 1$, for all $\varphi \in L$. Clearly $\mathcal{V}_L$ forms a variety. Moreover, since the Lindenbaum-Tarski construction goes through for every si-logic L (just substitute every occurrence of **Int** with L in Example 6) and thus delivers a Heyting algebra $\mathfrak{L}_L \in \mathcal{V}_L$, we can easily generalize Theorem 1.55 to every si-logic L and thus get the following

Theorem 1.56 (Algebraic completeness of Ext**Int**)**.** *For every si-logic L and every* $\varphi \in \mathbf{For}\mathcal{L}$*,*

$$\varphi \in L \iff \varphi \textit{ is valid in } \mathfrak{A}, \textit{ for every Heyting algebra } \mathfrak{A} \in \mathcal{V}_L.$$

Conversely, with every non-empty class of Heyting algebra $\mathcal{C}$, let $\mathrm{Log}\,\mathcal{C}$ be the set of $\mathcal{L}$-formulas validated by every algebra in $\mathcal{C}$, that is,

$$\mathrm{Log}\,\mathcal{C} = \{\varphi \in \mathbf{For}\mathcal{L} \mid \mathfrak{A} \models \varphi \approx 1, \text{ for every } \mathfrak{A} \in \mathcal{C}\}.$$

Theorem 1.57. *Let $\mathcal{C}$ be a non-empty class of Heyting algebra. Then Log$\mathcal{C}$ is a si-logic.*

As a consequence of the algebraic completeness theorem for si-logic, we get

Theorem 1.58. *Let $\mathcal{V}$ be a variety of Heyting algebras and L be a si-logic. Then*

$$\mathcal{V}_{Log\,\mathcal{V}} = \mathcal{V} \qquad Log\,\mathcal{V}_L = L$$

The variety of Heyting algebras $\mathcal{V}_L$ is called the *characteristic variety* for the si-logic L, while $\mathrm{Log}\,\mathcal{V}$ is called the *logic* of the variety $\mathcal{V}$. Then the previous theorem tells us that there is a one-to-one correspondence between the class of all si-logics and the class of subvarieties of the variety $\mathcal{HA}$ of Heyting algebras. But there is something more.

First, let us define on the class of of subvarieties of Heyting algebras the two following lattice operations: for any subvariety $\mathcal{V}_1$ and $\mathcal{V}_2$,

$$\mathcal{V}_1 \wedge \mathcal{V}_2 := \mathcal{V}_1 \cap \mathcal{V}_2 \qquad \mathcal{V}_1 \vee \mathcal{V}_2 := \mathbb{HSP}(\mathcal{V}_1 \cup \mathcal{V}_2).$$

Then the class of all subvarieties of $\mathcal{HA}$ forms a complete bounded lattice with respect to the operations $\wedge$ and $\vee$ with greatest element $\mathcal{HA}$ and smallest element the variety containing the trivial algebra. We call this structure the *lattice of subvarieties of Heyting algebras* and denote it by $\mathrm{Sub}\mathcal{HA}$.

Definition 1.25. Two lattice $\mathfrak{A} = \langle A, \wedge, \vee\rangle$ and $\mathfrak{B} = \langle B, \wedge, \vee\rangle$ are said to be *dually isomorphic* if there exists a isomorphism $f\colon \mathfrak{A} \to \mathfrak{B}$ that dually preserves the operations, that is, for every $x, y \in A$,

$$f(x \wedge y) = f(x) \vee f(y) \qquad f(x \vee y) = f(x) \wedge f(y).$$

Such an f is then called a *dual isomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$.

Theorem 1.59. *The lattice of si-logic* $\mathrm{Ext}\mathbf{Int}$ *and the lattice of subvarieties of Heyting algebras* $\mathrm{Sub}\mathcal{HA}$ *are dually isomorphic, with the map $L \mapsto \mathcal{V}_L$ being a dual isomorphism.*

1.6 Heyting algebras and Kripke frames

Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame and consider the set of upsets $\mathrm{Up}(W)$ of W. If we endow $\mathrm{Up}(W)$ with the operation of set-theoretic intersection and union, then $\langle \mathrm{Up}(W), \cap, \cup\rangle$ forms a bounded lattice with $\emptyset$ and W as bottom and top element respectively. Can we define an operation of implication on $\mathrm{Up}(W)$ in order to turn $\langle \mathrm{Up}(W), \cap, \cup\rangle$ into a Heyting algebra? The answer is positive. Indeed, define $\supset\colon \mathrm{Up}(W)^2 \to \mathrm{Up}(W)$ as follows: for every $X, Y \in \mathrm{Up}(W)$,

$$X \supset Y := \{x \in W \mid \forall y(xRy \wedge y \in X \to y \in Y)\}.$$

It is easily seen that $\supset$ is a Heyting implication, therefore the algebra

$$\mathfrak{F}^+ = \langle \mathrm{Up}(W), \cap, \cup, \supset, \emptyset\rangle$$

is a Heyting algebra and we call it the *dual* of $\mathfrak{F}$. Moreover, notice that every valuation $\mathfrak{V}$ in $\mathfrak{F}$ is at the same time a valuation in $\mathfrak{F}^+$ and, in particular, for every formula φ, we have $\mathfrak{F} \models \varphi$ iff $\mathfrak{F}^+ \models \varphi$.

The following Figure 1.3 shows the intuitionistic frames $\mathfrak{S}$ (on the left) and its dual $\mathfrak{S}^+$ (on the right):

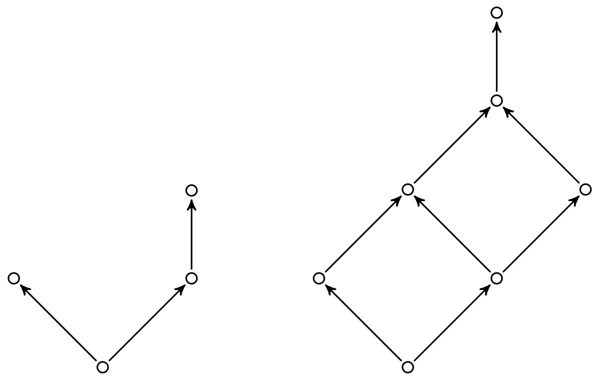


Figure 1.3: *The frame $\mathfrak{G}$ and its dual $\mathfrak{G}^+$.*

Thus, for every Kripke frame $\mathfrak{F}$, we get a Heyting algebra $\mathfrak{F}^+$. We now show that the converse is also true, that is, we show how to associate with every Heyting algebra a Kripke frame. But before doing that, we need some preliminary definitions.

Definition 1.26. Let $\mathfrak{H} = \langle H, \wedge, \vee, \rightarrow, 0\rangle$ be a Heyting algebra. A proper filter $\nabla \subsetneq H$ of $\mathfrak{H}$ is said to be

- *prime* if $x \vee y \in \nabla$ implies $x \in \nabla$ or $y \in \nabla$;
- *maximal* if ∇ is not contained in a proper filter in $\mathfrak{H}$ different from ∇;
- an *ultrafilter* if, for every element $a \in H$, either $a \in \nabla$ or $\neg a \in \nabla$.

Definition 1.27. Let $\mathfrak{H} = \langle H, \wedge, \vee, \rightarrow, 0\rangle$ be a Heyting algebra. An element $a \in H$ is said to be

- *prime* if $a \neq 0$ and, for every $b, c \in H$, $a = b \vee c$ implies $a = b$ or $a = c$[25];
- an *atom* if $a \neq 0$ and, for every $b \in H$, $b \leq a$ implies $b = 0$ or $b = a$.

For Heyting algebras, ultrafilters and maximal filters coincide, while in Boolean algebras all the three previous notions of filters extensionally determine the same class.

Proposition 1.60. *Let $\mathfrak{H} = \langle H, \wedge, \vee, \rightarrow, 0\rangle$ be a Heyting algebra Then*

(i) *a principal filter in $\mathfrak{H}$ is prime iff it is generated by a prime element;*

(ii) *a principal filter $\mathfrak{H}$ is an ultrafilter iff it is generated by an atom.*

We are going now to prove a theorem that will be useful in what follows.

Theorem 1.61. *Every filter ∇ in an Heyting algebra $\mathfrak{H}$ such that $a \notin \nabla$, for some $a \in \mathfrak{H}$, can be extended to a prime filter ∇' such that $a \notin \nabla'$.*

[25]More precisely, in any lattice $\mathfrak{L} = \langle L, \wedge, \vee\rangle$, an element $a \in L$ satisfying the previous condition would be called *join-irreducible*, while if we replace the equality relation with the lattice partial ordering $\leq$, we then get the notion of a *join-prime* element. When the lattice $\mathfrak{L}$ is distributive, as in the case of the lattice reduct of an Heyting algebra, the two notion collapse.

Proof. Consider the set $\mathcal{F}$ of filters Δ in $\mathfrak{H}$ such that $\nabla \subseteq \Delta$ and $a \notin \Delta$ ordered by $\subseteq$ and let $\mathcal{Q} \subseteq \mathcal{F}$ be a chain. Then $\bigcup \mathcal{Q} \in \mathcal{F}$. Therefore, by Zorn's lemma[26], there exists a maximal filter ∇' in $\mathfrak{H}$ such that $\nabla \subseteq \nabla'$ and $a \notin \nabla'$. We now show that ∇' is prime.

Suppose otherwise. Then there are $b, c \in \mathfrak{H}$ such that $b \vee c \in \nabla'$ but $b \notin \nabla'$ and $c \notin \nabla'$. Let $\nabla_b = [\nabla' \cup \{b\})$ and $\nabla_c = [\nabla' \cup \{c\})$. By maximality of ∇, we have $a \in \nabla_b \cap \nabla_c$ and so there are elements $b_1, c_1 \in \nabla'$ such that $b_1 \wedge b \leq a$ and $c_1 \wedge c \leq a$. Therefore, $b_1 \wedge c_1 \wedge b \leq a$ and $b_1 \wedge c_1 \wedge c \leq a$ and thus

$$(b_1 \wedge c_1 \wedge b) \vee (b_1 \wedge c_1 \wedge c) = (b_1 \wedge c_1) \wedge (b \vee c) \leq a.$$

But then, since both $b_1 \wedge c_1$ and $b \vee c$ belong to ∇', by definition of filter it follows that $a \in \nabla'$, which is a contradiction. □

Corollary 1.62. *Let $\mathfrak{H}$ be an Heyting algebra and let $a, b \in \mathfrak{A}$ such that $a \nleq b$. Then there exist a prime filter ∇ such that $a \in \nabla$ and $b \notin \nabla$.*

Proof. Just consider the principal filter $\Delta = [b)$ and apply Theorem 1.61. □

Notice that the proof of Theorem 1.61 can be easily modified in order to show the following

Lemma 1.63 (Stone)**.** *Let ∇ be a filter and Δ an ideal in an Heyting algebra $\mathfrak{H}$ such that $\nabla \cap \Delta = \emptyset$. Then ∇ can be extended to a prime filter ∇' such that $\nabla' \cap \Delta = \emptyset$.*

Now consider a Heyting algebra $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$. We define the *dual* frame of $\mathfrak{H}$ to be the Kripke frame $\mathfrak{H}_+ = \langle W_{\mathfrak{H}}, R_{\mathfrak{H}}\rangle$, where

$$W_{\mathfrak{H}} = \{\nabla \in 2^H \mid \nabla \text{ prime filter of } \mathfrak{H}\},$$
$$\nabla R_{\mathfrak{H}} \nabla' \iff \nabla \subseteq \nabla'.$$

Notice that if $\mathfrak{H}$ is finite, then $\mathfrak{H}_+$ is finite as well; moreover, since in a finite Heyting algebra every filter is principal, by Proposition 1.60 (i), every element $\nabla \in W_{\mathfrak{H}}$ is of the form $[a)$ for some prime element $a \in H$ and $[a) \subseteq [b)$ iff $b \leq a$. Therefore, the frame $\mathfrak{H}_+$ is isomorphic to the frame $\langle W, \geq\rangle$ where W is the set of all prime elements of H and $\leq$ is the lattice order of $\mathfrak{H}$.

The previous comment gives us the right insights in order to prove the following

Theorem 1.64. *Every finite Heyting algebras is isomorphic to the dual of some finite Kripke frame. In particular, $\mathfrak{A} \cong \mathfrak{F}^+ \cong (\mathfrak{A}_+)^+$, where $\mathfrak{F} = \langle W, \geq\rangle$ is the frame of the prime elements of $\mathfrak{A}$ ordered by the opposite lattice order of $\mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be a finite Heyting algebra and $\mathfrak{F} = \langle W, \geq\rangle$ be the frame of the prime elements of $\mathfrak{A}$. Since every $a \in A$ can be represented as $\bigvee\{b \in W \mid b \leq a\}$, the map $f \colon A \to \mathrm{Up}(W)$ defined by $f(a) = \{b \in W \mid b \leq a\}$ turns out to be an isomorphism of $\mathfrak{A}$ onto $\mathfrak{F}^+$[27]. □

[26] Zorn's lemma states that, given any partially ordered set $\langle P, \leq\rangle$, if every chain in P has an upper bound, then there exist a $\leq$-maximal element in P. It is a well known fact that Zorn's lemma is equivalent to the Axiom of Choice.

[27] For a full proof of the theorem, cfr. [23, Theorem 7.30] or [128, §1].

Actually, something even stronger is true. The previous theorem can indeed be strengthened to the following duality.

Theorem 1.65. *The category $\mathcal{HA}_{<\omega}$ of finite Heyting algebras and related morphism is dually equivalent to the category $\mathbf{KF}_{<\omega}$ having finite Kripke frames as objects and p-morphisms as arrows.*

Proof. Here we will only give a sketch of the proof. For more details, cfr. the proof of Theorem 2.14 and, more generally, §2.
We have to define two contravariant functors which are pseudo-inverses. So let $\Psi\colon \mathcal{HA}_{<\omega} \to \mathbf{KF}_{<\omega}$ be defined as follows:

$$\Psi(\mathfrak{A}) = \mathfrak{A}_+,$$
$$\Psi(f)\colon \Psi(\mathfrak{B}) \to \Psi(\mathfrak{A}) \text{ given by } \Psi(f) = f^{-1},$$

for every finite Heyting algebra $\mathfrak{A}$ and for every morphism $f\colon \mathfrak{A} \to \mathfrak{B}$ of $\mathcal{HA}_{<\omega}$. Then $\Psi(f)$ is shown to be a p-morphism and thus Ψ is a well defined contravariant functor.
Now let $\Phi\colon \mathbf{KF}_{<\omega} \to \mathcal{HA}_{<\omega}$ be the functor defined as follows:

$$\Phi(\mathfrak{F}) = \mathfrak{A}^+,$$
$$\Psi(h)\colon \Psi(\mathfrak{G}) \to \Psi(\mathfrak{F}) \text{ given by } \Psi(h) = h^{-1},$$

for every finite Kripke frame $\mathfrak{F}$ and for every p-morphism $h\colon \mathfrak{F} \to \mathfrak{G}$. It is possible to show that $\Phi(h)$ is indeed a Heyting morphism and therefore Φ is well defined.
In order to establish the dual equivalence of $\mathcal{HA}_{<\omega}$ and $\mathbf{KF}_{<\omega}$, the natural transformations we need are the following: for every finite Heyting algebra $\mathfrak{A}$,

$$\alpha\colon 1_{\mathcal{HA}_{<\omega}} \xrightarrow{\sim} \Phi \circ \Psi \quad \text{defined by} \quad \begin{array}{l} \alpha_{\mathfrak{A}}\colon \mathfrak{A} \to \Phi \circ \Psi(\mathfrak{A}) \\ \alpha_{\mathfrak{A}}(a) = \{\nabla \in W_{\mathfrak{A}} \mid a \in \nabla\} \end{array}$$

and, for every finite Kripke frame $\mathfrak{F} = \langle W, R\rangle$,

$$\beta\colon 1_{\mathbf{KF}_{<\omega}} \xrightarrow{\sim} \Psi \circ \Phi \quad \text{defined by} \quad \begin{array}{l} \beta_{\mathfrak{F}}\colon \mathfrak{F} \to \Psi \circ \Phi(\mathfrak{F}) \\ \beta_{\mathfrak{F}}(w) = \{U \in \mathrm{Up}(W) \mid w \in U\}. \end{array}$$

□

One could wonder whether the previous duality can be extended to the categories of Heyting algebras $\mathcal{HA}$ and Kripke frames $\mathbf{KF}$. The answer is negative: in the previous section we have seen that every intermediate logic L has a sound and complete algebraic semantics, given by the variety of Heyting algebras $\mathcal{V}_L$; now if there was such a duality, every intermediate logic L would be characterized by the class of frames $\{\mathfrak{A}_+ \mid \mathfrak{A} \in \mathcal{V}_L\}$, and thus every logic would be Kripke complete. However, as we mentioned in §1.3.1 (cfr. footnote n.11), there are intermediate logics that are not Kripke complete.

Furthermore, as dealing with the infinite is much more complex that dealing with the finite, the methodology and the proof techniques often have to be changed. Indeed, a simple example shows that the category $\mathbf{KF}$ is not suitable anymore for establishing a duality: for every Kripke frame $\mathfrak{F} = \langle W, R\rangle$, $\mathfrak{F}^+$ is a complete Heyting algebra, since $\langle \mathrm{Up}(W), \cap, \cup\rangle$ is complete as a lattice. Hence, every Heyting algebra that is not complete can not be obtained from any Kripke frame.

Chapter 2

Esakia duality

Esakia duality is the dual equivalence between the category of Heyting algebras and the category of Esakia spaces, which was first established in 1974 by the Georgian logician Leo Esakia in the greatly influential paper [43]. A monograph on the subject was then published in 1985 by the Georgian publishing house *Metsniereba* and became very popular among Soviet logicians. Due to the fact that it was written in Russian, the book was not easily available to the Western logicians, who had to wait the recent publication of [41] to see Esakia's original presentation finally translated in English.

As we had already mentioned, proofs of Esakia duality are usually obtained as a byproduct of a duality established in a more encompassing modal framework and the proof contained in [41] makes no difference. So, even if most of the material of this chapter can be found in [41], we will give here a full direct proof of the Esakia duality. Furthermore, we will unify in a coherent exposition most of the correspondences of the duality between algebraic and order-topological notions which are scattered in the literature and collect them in Table 2.1 in a way which is similar to the duality dictionary for Heyting algebras of [10].

2.1 Priestley and Esakia spaces

Definition 2.1. A topological space $\mathcal{X} = \langle X, \tau \rangle$ is said to be a *Stone space* if $\mathcal{X}$ is compact, Hausdorff and zero-dimensional.

A triple $\mathcal{X} = \langle X, \tau, R \rangle$, where $\langle X, \tau \rangle$ is a topological space and $\langle X, R \rangle$ is a poset, is called an *ordered topological space*. Given two ordered topological spaces $\mathcal{X} = \langle X, \tau, R \rangle$ and $\mathcal{Y} = \langle Y, \tau', S \rangle$, we say that $\mathcal{X}$ is *order-homeomorphic* to $\mathcal{Y}$ if there exists a map $f \colon X \to Y$ such that f is a homeomorphism and f is an order-isomorphism.
Furthermore, we say that the partial order R on X satisfies the *Priestley separation axiom* if the following condition holds: for every $x, y \in X$,

$$\neg(xRy) \Longrightarrow \exists\, U \subseteq X \text{ such that } U \text{ is a clopen upset, } x \in U \text{ and } y \notin U, \qquad \text{(P)}$$

and we say that R is *point-closed* if $x{\uparrow}$ is closed for every $x \in X$.

Definition 2.2. Let $\mathcal{X} = \langle X, \tau, R \rangle$ be an ordered topological space. Then $\mathcal{X}$ is called a *Priestley space* if $\langle X, \tau \rangle$ is compact and R satisfies the Priestley separation axiom.

It is not difficult to show that, for any Priestley space $\mathcal{X} = \langle X, \tau, R\rangle$, the topological space $\langle X, \tau\rangle$ is a Stone space. Furthermore, we have the following

Lemma 2.1. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be a Priestley space. Then, for each closed subset $F \subseteq X$, both $F\!\uparrow$ and $F\!\downarrow$ are closed. In particular, R is point-closed.*

Now, let $\mathcal{X} = \langle X, \tau, R\rangle$ be an ordered topological space. We say that R is *clopen* if, for every clopen set U, $U\!\downarrow$ is also clopen. Now we will introduce the central notion of this chapter.

Definition 2.3. An *Esakia space* is a a Priestley space $\mathcal{X} = \langle X, \tau, R\rangle$ such that R is clopen relation.

It is worth mentioning that Esakia spaces can be characterized without any reference to Priestley spaces. Indeed, in [43] Esakia spaces have been defined as particular ordered topological spaces $\mathcal{X} = \langle X, \tau, R\rangle$ such that $\langle X, \tau\rangle$ is a Stone space, R is point-closed and the map $\rho\colon X \to \mathcal{C}(X)$ determined by R and given by $x \mapsto x\!\uparrow$ is a continuous map between $\langle X, \tau\rangle$ and the topological space $\langle \mathcal{C}(X), \tau_V\rangle$ given by the non-empty closed subset of X with the Vietoris topology. For more information on the original definition of Esakia spaces, cfr. [43, 41].

2.2 Towards a complete duality for Heyting algebras

We are now ready to lay the foundations for a full duality for Heyting algebras. Recall that, given a Heyting algebra $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$, we denote by $\langle W_{\mathfrak{H}}, R_{\mathfrak{H}}\rangle$ the poset given by the set of prime filters of $\mathfrak{H}$ ordered by inclusion, that is

$$W_{\mathfrak{H}} = \{\nabla \in 2^H \mid \nabla \text{ prime filter of } \mathfrak{H}\},$$
$$\nabla R_{\mathfrak{H}} \nabla' \iff \nabla \subseteq \nabla'.$$

Furthermore, we denote by $\widehat{\cdot}\colon H \to 2^{W_{\mathfrak{H}}}$ the map given by

$$\widehat{a} = \{\nabla \in W_{\mathfrak{H}} \mid a \in \nabla\}$$

and by $\mathcal{P}_{\mathfrak{H}}$ the range of $\widehat{\cdot}$, that is,

$$\mathcal{P}_{\mathfrak{H}} = \{\widehat{a} \mid a \in H\}.$$

Now, for $a \in H$, let $-\widehat{a}$ denote the complement of $\widehat{a}$ in $W_{\mathfrak{H}}$, i.e. $-\widehat{a} = W_{\mathfrak{H}} \setminus \widehat{a}$ and let $-\mathcal{P}_{\mathfrak{H}}$ be the set $\{-\widehat{a} \mid a \in H\}$. We can now define on $W_{\mathfrak{H}}$ a topology $\tau_{\mathfrak{H}}$ by letting $\mathcal{S} = \mathcal{P}_{\mathfrak{H}} \cup -\mathcal{P}_{\mathfrak{H}}$ be a subbasis and consider the ordered topological space $\mathcal{W}_{\mathfrak{H}} = \langle W_{\mathfrak{H}}, \tau_{\mathfrak{H}}, R_{\mathfrak{H}}\rangle$. We will prove that $\mathcal{W}_{\mathfrak{H}}$ is an Esakia space, but first, let us analyze the topology $\tau_{\mathfrak{H}}$.

Lemma 2.2. *For every $a, b \in H$,*

(i) $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$;

(ii) $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$.

Proof. Let us prove (ii). Let $\nabla \in \widehat{a \vee b}$. Then $a \vee b \in \nabla$. Since ∇ is a prime filter, either $a \in \nabla$ or $b \in \nabla$. Therefore, either $\nabla \in \widehat{a}$ or $\nabla \in \widehat{b}$, that is, $\nabla \in \widehat{a} \cup \widehat{b}$. Conversely, if $\nabla \in \widehat{a} \cup \widehat{b}$, then either $a \in \nabla$ or $b \in \nabla$ and since $a, b \leq a \vee b$, we have $a \vee b \in \nabla$ by definition of filter. Thus $\nabla \in \widehat{a \vee b}$. □

Notice, moreover that $\widehat{0} = \emptyset$ and $\widehat{1} = W_{\mathfrak{H}}$. Since, by the previous lemma, any open set $O \in \tau_{\mathfrak{H}}$, given by a finite intersection of member of $\mathcal{S}$, is of the form $\widehat{a} \cap -\widehat{b}$, we have that $\mathcal{B} = \{\widehat{a} \cap -\widehat{b} \,|\, a, b \in H\}$ is a basis for the topology $\tau_{\mathfrak{H}}$. Furthermore, notice that, by construction, $\mathcal{B}$ is a basis consisting of clopen set of $W_{\mathfrak{H}}$, that is, $\langle W_{\mathfrak{H}}, \tau_{\mathfrak{H}} \rangle$ is a zero-dimensional topological space, and every element of the form $\widehat{a}$ is clopen, since $\widehat{a} = \widehat{a} \cap -\widehat{0} \in \mathcal{B}$.

Lemma 2.3. *$\langle W_{\mathfrak{H}}, \tau_{\mathfrak{H}} \rangle$ is a compact topological space.*

Proof. By Alexander's Subbase Theorem we need to show that every cover of $W_{\mathfrak{H}}$ by elements from $\mathcal{S}$ has a finite subcover. So let $\{\widehat{a_i} \,|\, i \in I\} \cup \{-\widehat{b_j} \,|\, j \in J\}$ be a cover of $W_{\mathfrak{H}}$, that is,

$$\bigcup_{i \in I} \widehat{a_i} \cup \bigcup_{j \in J} -\widehat{b_j} = W_{\mathfrak{H}}.$$

Now consider the smallest filter ∇ containing $\{b_i \,|\, j \in J\}$ and the smallest ideal Δ containing $\{a_i \,|\, i \in I\}$. If $\nabla \cap \Delta = \emptyset$, then by Lemma 1.63, there exist $\nabla' \in W_{\mathfrak{H}}$ such that $\nabla \subseteq \nabla'$ and $\nabla' \cap \Delta = \emptyset$. Therefore, $\nabla' \in \widehat{b_j}$ for each $j \in J$ and $\nabla' \notin \widehat{a_j}$ for each $i \in I$ and, consequently, $\nabla' \notin \bigcup_{i \in I} \widehat{a_i} \cup \bigcup_{j \in J} -\widehat{b_j}$, contrary to the fact that $\bigcup_{i \in I} \widehat{a_i} \cup \bigcup_{j \in J} -\widehat{b_j}$ is a cover of $W_{\mathfrak{H}}$. Thus, there exists some element $c \in \nabla \cap \Delta$ and, by definition, we have

$$\begin{aligned} &b_{j_1} \wedge \ldots \wedge b_{j_n} \leq c && \text{for some finite } \{j_1, \ldots, j_n\} \subseteq J, \\ &c \leq a_{i_1} \vee \ldots \vee a_{i_k} && \text{for some finite } \{i_1, \ldots, i_k\} \subseteq I. \end{aligned}$$

But then, by Lemma 2.2, we have

$$\widehat{b_{j_1}} \cap \ldots \cap \widehat{b_{j_n}} \subseteq \widehat{c} \qquad \text{and} \qquad \widehat{c} \subseteq \widehat{a_{i_1}} \cup \ldots \cup \widehat{a_{i_k}}$$

and thus $\widehat{b_{j_1}} \cap \ldots \cap \widehat{b_{j_n}} \subseteq \widehat{a_{i_1}} \cup \ldots \cup \widehat{a_{i_k}}$. Therefore, it follows that

$$\bigcup_{s=1}^{k} \widehat{a_{i_s}} \cup \bigcup_{d=1}^{n} -\widehat{b_{j_d}} = W_{\mathfrak{H}},$$

which is a finite subcover of $\{\widehat{a_i} \,|\, i \in I\} \cup \{-\widehat{b_j} \,|\, j \in J\}$ for $W_{\mathfrak{H}}$. □

Proposition 2.4. *The triple $\mathcal{W}_{\mathfrak{H}} = \langle W_{\mathfrak{H}}, \tau_{\mathfrak{H}}, R_{\mathfrak{H}} \rangle$ is a Priestley space.*

Proof. Thanks to the previous lemma, we only need to prove that $R_{\mathfrak{H}}$ satisfies the Priestley separation axiom. So, let $\nabla, \nabla' \in W_{\mathfrak{H}}$ be such that $\neg(\nabla R_{\mathfrak{H}} \nabla')$, that is $\nabla \nsubseteq \nabla'$. Then there exists an element $a \in \nabla$ such that $a \notin \nabla'$. Thus $\nabla \in \widehat{a}$ and $\nabla' \notin \widehat{a}$. Since $\widehat{a}$ is clopen, we only need to show that it is an upset. So, let $\Sigma \in \widehat{a}$ and $\Sigma \subseteq \Sigma'$. Since we have $a \in \Sigma$, $a \in \Sigma'$ and therefore $\Sigma' \in \widehat{a}$. □

Thus, in order to show that $\mathcal{W}_{\mathfrak{H}} = \langle W_{\mathfrak{H}}, \tau_{\mathfrak{H}}, R_{\mathfrak{H}} \rangle$ is an Esakia space, we only need to show that $R_{\mathfrak{H}}$ is a clopen relation. Before to do so, we need another preliminary lemma.

Lemma 2.5. *For every $a, b \in H$,*

$$\widehat{a \to b} = W_{\mathfrak{H}} \setminus (\widehat{a} \cap -\widehat{b})\downarrow.$$

Proof. Let $\nabla \in \widehat{a \to b}$. Then $a \to b \in \nabla$. Now if $\nabla \in (\widehat{a} \cap -\widehat{b})\downarrow$, there exists $\nabla' \in \widehat{a} \cap -\widehat{b}$ such that $\nabla \subseteq \nabla'$. Thus, $a, a \to b \in \nabla'$ and, since ∇' is a filter, $b \in \nabla'$, contrary to the fact that $\nabla' \in -\widehat{b}$. So, $\widehat{a \to b} \subseteq W_{\mathfrak{H}} \setminus (\widehat{a} \cap -\widehat{b})\downarrow$. For the converse inclusion, let $\nabla \notin \widehat{a \to b}$. So, $a \to b \notin \nabla$. We have to prove that $\nabla \notin W_{\mathfrak{H}} \setminus (\widehat{a} \cap -\widehat{b})\downarrow$, that is, we have to find a prime filter Δ such that $a \in \Delta$, $b \notin \Delta$ and $\nabla \subseteq \Delta$. Consider the filter Σ generated by $\nabla \cup \{a\}$. If $a \to b \in \Sigma$, then there are $c_1, \ldots, c_n \in \nabla$ such that $c_1 \wedge \ldots \wedge c_n \wedge a \leq a \to b$. By letting $c = \bigwedge_{i=1}^{n} c_i$, by definition of $\to$, we then have $c \wedge a \leq b$ and so $c \leq a \to b$. But then we have $a \to b \in \nabla$, contrary to our assumption. Therefore, $a \to b \notin \Sigma$ and, by Theorem 1.61, there exists a prime filter Σ' such that $\Sigma \subseteq \Sigma'$ and $a \to b \notin \Sigma'$. Hence $b \notin \Sigma'$ and thus $W_{\mathfrak{H}} \setminus (\widehat{a} \cap -\widehat{b})\downarrow \subseteq \widehat{a \to b}$. □

Proposition 2.6. *The triple $\mathcal{W}_{\mathfrak{H}} = \langle W_{\mathfrak{H}}, \tau_{\mathfrak{H}}, R_{\mathfrak{H}} \rangle$ is an Esakia space.*

Proof. Let $U \subseteq W_{\mathfrak{H}}$ be a clopen set. Thus $U = \bigcup_{i=1}^{n} \widehat{a_i} \cap -\widehat{b_i}$ for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in H$. Then

$$U\downarrow = (\bigcup_{i=1}^{n} \widehat{a_i} \cap -\widehat{b_i})\downarrow = \bigcup_{i=1}^{n} (\widehat{a_i} \cap -\widehat{b_i})\downarrow.$$

But, by the previous lemma, $(\widehat{a_i} \cap -\widehat{b_i})\downarrow = -\widehat{a_i \to b_i}$ for all $i \in \{1, \ldots, n\}$ and thus

$$U\downarrow = \bigcup_{i=1}^{n} -\widehat{a_i \to b_i},$$

which, as a finite union of clopen, is clopen. □

Thus we have just shown that, for every Heyting algebra $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0 \rangle$, the ordered topological space $\mathcal{W}_{\mathfrak{H}} = \langle W_{\mathfrak{H}}, \tau_{\mathfrak{H}}, R_{\mathfrak{H}} \rangle$ is an Esakia space. We call the space $\mathcal{W}_{\mathfrak{H}}$ the *dual topological space* of $\mathfrak{H}$ and we denote it by $\mathfrak{H}_+$.

We are now going to show that the converse is also true: given an Esakia space $\mathcal{X} = \langle X, \tau, R \rangle$, we can find an Heyting algebra related to the space $\mathcal{X}$.

Proposition 2.7. *Let $\mathcal{X} = \langle X, \tau, R \rangle$ be an Esakia space and denote by X^{CU} the set of clopen upsets of X. Then $\langle X^{CU}, \cap, \cup, \supset, \emptyset \rangle$, where the operation $\supset$ is defined as*

$$U \supset V := X \setminus (U \setminus V)\downarrow,$$

is an Heyting algebra.

Proof. It is clear that the intersection and the union of two clopen upsets is again a clopen upset. Furthermore, since the operations of union and intersections distributes over each other, we have that the reduct $\langle X^{CU}, \cap, \cup, \emptyset \rangle$ forms a distributive lattice. Moreover, since the relation R of $\mathcal{X}$ is clopen, for every $U, V \in X^{CU}$, the set $U \supset V$ is again clopen and, being the complement of a downset, also an upset. So, in order to prove that $\langle X^{CU}, \cap, \cup, \supset, \emptyset \rangle$ is an

Heyting algebra, we only have to show that, for any $W \in X^{CU}$, the following equivalence holds:

$$W \cap U \subseteq V \iff W \subseteq U \supset V.$$

($\Longleftarrow$): suppose $W \subseteq U \supset V$ and let $x \in W \cap U$. Thus $x \in U \supset V$ which implies $x \notin U \setminus V$. But since $x \in U$, then $x \in V$.
($\Longrightarrow$): let $W \cap U \subseteq V$ and $x \in W$. If $x \notin U \supset V$, then $x \in (U \setminus V)\downarrow$ and so there exists $y \in U \setminus V$ such that xRy. Since W is an upset, $y \in U$ and thus $y \in V$ contrary to the fact that $y \notin V$. □

We call the Heyting algebra $\mathcal{X}^+ = \langle X^{CU}, \cap, \cup, \supset, \emptyset\rangle$ the *dual* of the Esakia space $\mathcal{X}$. We now show that the relation between an Heyting algebra $\mathfrak{H}$ and its double dual $(\mathfrak{H}_+)^+$ is a very close one. Indeed we have the following:

Proposition 2.8. *Let $\mathfrak{H} = \langle H, \wedge, \vee, \to, 0\rangle$ be an Heyting algebra. Then $\mathfrak{H}$ is isomorphic to $(\mathfrak{H}_+)^+ = \langle W_{\mathfrak{H}}^{CU}, \cup, \cap, \supset, \emptyset\rangle$ and the map $\alpha_{\mathfrak{H}}\colon H \to W_{\mathfrak{H}}^{CU}$ given by $\alpha_{\mathfrak{H}}(a) = \widehat{a}$ is the witnessing isomorphism.*

Proof. By Lemma 2.2 and Lemma 2.5, it is clear that $\alpha_{\mathfrak{H}}$ is a homomorphism. Furthermore, if $a, b \in H$ are such that $a \neq b$, then either $a \nleq b$ or $b \nleq a$. Suppose for the sake of definiteness that $a \nleq b$ and consider the filter $[a)$. Since $b \notin [a)$, by Theorem 1.61, we can extend $[a)$ to a prime filter ∇ such that $b \notin \nabla$. Then $\nabla \in \widehat{a}$ and $\nabla \notin \widehat{b}$ and thus $\alpha_{\mathfrak{H}}(a) \neq \alpha_{\mathfrak{H}}(b)$. Therefore, the map $\alpha_{\mathfrak{H}}$ is an embedding. Let us show that it is also surjective.
Consider an arbitrary clopen upset $U \in W_{\mathfrak{H}}^{CU}$ and let $\nabla \in U$. For every $\Delta \notin U$, we have $\nabla \nsubseteq \Delta$, since $U = U\uparrow$. So, there exists $a_\Delta \in \nabla$ such that $a_\Delta \notin \Delta$. Thus $\nabla \in \widehat{a_\Delta}$ and $\Delta \notin \widehat{a_\Delta}$ or, equivalently, $\nabla \notin -\widehat{a_\Delta}$ and $\Delta \in -\widehat{a_\Delta}$. This means that the set $\{-\widehat{a_\Delta} \mid \Delta \notin U\}$ is a cover of $W_{\mathfrak{H}} \setminus U$. Since U is clopen by assumption, $W_{\mathfrak{H}} \setminus U$ is also clopen and therefore compact. Hence there exists a finite subcover $\{-\widehat{a_{\Delta_i}} \mid i \in \{1, \ldots, n\}\}$, where $\Delta_i \in W_{\mathfrak{H}} \setminus U$ for each i, and therefore

$$W_{\mathfrak{H}} \setminus U \subseteq \bigcup_{i=1}^{n} -\widehat{a_{\Delta_i}}.$$

This implies

$$\nabla \in \bigcap_{i=1}^{n} \widehat{a_{\Delta_i}} \subseteq U$$

and thus, by letting $a_\nabla = \bigwedge_{i=1}^{n} a_{\Delta_i}$, we have $\nabla \in \widehat{a_\nabla} \subseteq U$. Therefore, we have that $\{\widehat{a_\nabla} \mid \widehat{a_\nabla} \subseteq U\}$ is an open cover of U. But since U is clopen and thus compact, there exists a finite subcover $\{\widehat{a_{\nabla_j}} \mid j \in J\}$, with J finite and $\widehat{a_{\nabla_j}} \subseteq U$ for each j, that is

$$U = \bigcup_{j \in J} \widehat{a_{\nabla_j}}.$$

Finally, by letting $a_U = \bigvee_{j \in J} a_{\nabla_j}$, we have $U = \widehat{a_U}$ and $\alpha_{\mathfrak{H}}$ is onto. □

One could wonder whether an analogous result holds for an Esakia space $\mathcal{X}$ and its double dual $(\mathcal{X}^+)_+$. The answer is positive, as shown in the next

Proposition 2.9. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space. Then $\mathcal{X}$ is order-homeomorphic to $(\mathcal{X}^+)_+ = \langle W_{\mathcal{X}^+}, \tau_{\mathcal{X}^+}, R_{\mathcal{X}^+}\rangle$ and the order-homeomorphism is given by the map $\beta_{\mathcal{X}}\colon X \to W_{\mathcal{X}^+}$ defined as*

$$\beta_{\mathcal{X}}(x) = \{U \in X^{CU} \mid x \in U\}.$$

Proof. First let us show that the map $\beta_{\mathcal{X}}$ is well defined, that is, $\beta_{\mathcal{X}}(x)$ is a prime filter of $\mathcal{X}^+$, for every $x \in X$. So, consider an arbitrary $x \in X$. Clearly, $\beta_{\mathcal{X}}(x) \neq \emptyset$, since $x \in X \in X^{CU}$. Now, let $U, V \in \beta_{\mathcal{X}}(x)$. Then $x \in U$ and $x \in V$. Hence $x \in U \cap V$ and $U \cap V \in \beta_{\mathcal{X}}(x)$. Moreover, if $U \in \beta_{\mathcal{X}}(x)$ and $U \subseteq V$, then $x \in U \subseteq V$ and thus $V \in \beta_{\mathcal{X}}(x)$. So $\beta_{\mathcal{X}}(x)$ is a filter. Now, suppose that $U \cup V \in \beta_{\mathcal{X}}(x)$. Then $x \in U \cup V$ and so either $x \in U$ or $x \in V$. This means that either $U \in \beta_{\mathcal{X}}(x)$ or $V \in \beta_{\mathcal{X}}(x)$, i.e. $\beta_{\mathcal{X}}(x)$ is a prime filter.
Now suppose xRy. If $U \in \beta_{\mathcal{X}}(x)$, then $x \in U$ and, since U is an upset, $y \in U$. Hence $U \in \beta_{\mathcal{X}}(y)$. Therefore, $\beta_{\mathcal{X}}(x) \subseteq \beta_{\mathcal{X}}(y)$ and the map $\beta_{\mathcal{X}}$ is order-preserving. Conversely, if $\neg(xRy)$, then by (P), there exists a clopen upset U such that $x \in U$ and $y \notin U$. This means that $U \in \beta_{\mathcal{X}}(x)$ and $U \notin \beta_{\mathcal{X}}(y)$, that is, $\beta_{\mathcal{X}}(x) \nsubseteq \beta_{\mathcal{X}}(y)$ and the map $\beta_{\mathcal{X}}$ is order-reversing.
Let us show that $\beta_{\mathcal{X}}$ is continuous. Since the topology $\tau_{W_{\mathcal{X}^+}}$ is given by the basis $\mathcal{B} = \{\widehat{U} \cap -\widehat{V} \mid U, V \in X^{CU}\}$, we just have to show that $\beta_{\mathcal{X}}^{-1}(\widehat{U} \cap -\widehat{V}) \in \tau$. Thus, if we show that, for every clopen upset $U \in X^{CU}$, $\beta_{\mathcal{X}}^{-1}(\widehat{U})$ is clopen, then we are done. So, we have

$$\begin{aligned}
\beta_{\mathcal{X}}^{-1}(\widehat{U}) &= \{x \in X \mid \beta_{\mathcal{X}}(x) \in \widehat{U}\} \\
&= \{x \in X \mid U \in \beta_{\mathcal{X}}(x)\} \\
&= \{x \in X \mid x \in U\} \\
&= U
\end{aligned}$$

and so $\beta_{\mathcal{X}}^{-1}(\widehat{U})$ is clopen.

Finally, let us show that $\beta_{\mathcal{X}}$ is a bijection. By the above reasoning on (P), one can show that $\beta_{\mathcal{X}}$ is an injection. Now, suppose that $\beta_{\mathcal{X}}$ is not surjective. Then there exists a prime filter $\nabla \in \mathcal{X}^+$ such that, for all $x \in X$, $\beta_{\mathcal{X}}(x) \neq \nabla$. This means that $\nabla \notin \beta_{\mathcal{X}}(X)$. Notice that, since $\beta_{\mathcal{X}}$ is a continuous map between Hausdorff spaces, $\beta_{\mathcal{X}}$ is closed. Moreover, since X is a closed set, $\beta_{\mathcal{X}}(X)$ is closed too. Hence, ∇ belongs to the open set $W_{\mathcal{X}^+} \setminus \beta_{\mathcal{X}}(X)$, that is, $W_{\mathcal{X}^+} \setminus \beta_{\mathcal{X}}(X)$ is a neighbourhood of ∇. Hence, there exists a clopen V of $W_{\mathcal{X}^+}$ such that $\nabla \in V \subseteq W_{\mathcal{X}^+} \setminus \beta_{\mathcal{X}}(X)$. Since V is clopen, V is a finite union of element of $\mathcal{B}$ and thus we may assume that it is of the form $\widehat{U_1} \cap -\widehat{U_2}$ for some $U_1, U_2 \in X^{CU}$. So, $\emptyset = \beta_{\mathcal{X}}^{-1}(V) = \beta_{\mathcal{X}}^{-1}(\widehat{U_1}) \cap \beta_{\mathcal{X}}^{-1}(-\widehat{U_2})$. Since $\beta_{\mathcal{X}}^{-1}(\widehat{U}) = U$ for every $U \in X^{CU}$, we have $U_1 \cap X \setminus U_2 = \emptyset$ and therefore $U_1 \subseteq U_2$. But then we have $V = \widehat{U_1} \cap -\widehat{U_2} = \emptyset$, contrary to the fact that $\nabla \in V$.
Therefore, $\beta_{\mathcal{X}}$ is a bijection and, in particular, it is an order-isomorphism between $\langle X, R\rangle$ and $\langle W_{\mathcal{X}^+}, R_{\mathcal{X}^+}\rangle$ and a homeomorphism between $\langle X, \tau\rangle$ and $\langle W_{\mathcal{X}^+}, \tau_{\mathcal{X}^+}\rangle$. □

2.3 The categories $\mathcal{HA}$ and $\mathcal{ES}$

We have just shown that given an Heyting algebra $\mathfrak{H} = \langle H, \wedge, \vee, \rightarrow, 0\rangle$ and an Esakia space $\mathcal{X} = \langle X, \tau, R\rangle$,

$$\mathfrak{H} \cong (\mathfrak{H}_+)^+ \qquad \text{and} \qquad \mathcal{X} = (\mathcal{X}^+)_+.$$

We now show that we can extend the previous duality to a full categorical duality. Let $\mathcal{HA}$ be the algebraic category of Heyting algebras and related homomorphisms and denote by $\boldsymbol{\mathcal{ES}}$ the category having Esakia spaces as objects and continuous p-morphism as arrows.

In order to establish a duality between these two categories, we have to define two contravariant functors which are pseudo-inverses. So let $\Psi\colon \boldsymbol{\mathcal{HA}} \to \boldsymbol{\mathcal{ES}}$ be defined as follows:

$$\Psi(\mathfrak{A}) = \mathfrak{A}_+,$$
$$\Psi(f)\colon \Psi(\mathfrak{B}) \to \Psi(\mathfrak{A}) \text{ given by } \Psi(f) = f^{-1},$$

for every Heyting algebra $\mathfrak{A}$ and for every homomorphism $f\colon \mathfrak{A} \to \mathfrak{B}$ of $\boldsymbol{\mathcal{HA}}$. Moreover, let $\Phi\colon \boldsymbol{\mathcal{ES}} \to \boldsymbol{\mathcal{HA}}$ be the functor defined as follows:

$$\Phi(\mathcal{X}) = \mathcal{X}^+,$$
$$\Phi(h)\colon \Phi(\mathcal{Y}) \to \Phi(\mathcal{X}) \text{ given by } \Phi(h) = h^{-1},$$

for every Esakia space $\mathcal{X}$ and for every continuous p-morphism $h\colon \mathcal{X} \to \mathcal{Y}$.

We first need to show that the maps Ψ and Φ are indeed functors. Notice that by Propositions 2.6 and 2.7 we have that Ψ and Φ are indeed well defined on objects. So we only need to show that the maps Ψ and Φ are also well defined on morphisms. To achieve this, we need some preliminary lemmas.

Lemma 2.10. *Let $f\colon W \to V$ be a monotone map between the two posets $\langle W, R\rangle$ and $\langle V, S\rangle$. The following are equivalent:*

(i) *f is a p-morphism;*

(ii) *$f^{-1}(A{\downarrow}) = f^{-1}(A){\downarrow}$, for every $A \subseteq V$;*

(iii) *$f^{-1}(v{\downarrow}) = f^{-1}(v){\downarrow}$, for every $v \in V$.*

Now consider an Heyting algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, 0\rangle$ and its dual Esakia space $\mathfrak{A}_+$. Let us fix a prime filter $\nabla \in W_{\mathfrak{A}}$ and consider the family $\mathcal{C}$ of all clopen set U_i in $W_{\mathfrak{A}}$ such that $\nabla \in U_i$. Notice that if $U_i, U_j \in \mathcal{C}$, then $U_i \cap U_j$ is again a clopen containing ∇ and thus $U_i \cap U_j \in \mathcal{C}$.

More generally, we say that a family $\mathcal{C}$ of non-empty subsets of an Esakia space $\mathcal{X}$ is *downward directed* if, for every $U, V \in \mathcal{C}$, there exists $Z \in \mathcal{C}$ such that $Z \subseteq U \cap V$. Notice moreover that every downward directed family has the finite intersection property.

Lemma 2.11 (Esakia's lemma)**.** *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space. Then, for every downward directed family $\mathcal{C}$ of non-empty closed sets of $\mathcal{X}$,*

$$(\bigcap_{U\in\mathcal{C}} U){\downarrow} = \bigcap_{U\in\mathcal{C}} (U{\downarrow}).$$

Proof. The left-to-right inclusion $\subseteq$ is immediate. For the converse inclusion, let $x \in \bigcap_{U\in\mathcal{C}}(U{\downarrow})$. Then $x \in U{\downarrow}$ and so $x{\uparrow} \cap U \neq \emptyset$ for every $U \in \mathcal{C}$. Since $\mathcal{C}$ is downward directed and R is point-closed, it follows that $\{x{\uparrow} \cap U \mid U \in \mathcal{C}\}$ is a family of non-empty closed sets with the finite intersection property. Since $\mathcal{X}$ is compact, it follows that $\bigcap_{U\in\mathcal{C}}(x{\uparrow} \cap U) \neq \emptyset$. Hence $x{\uparrow} \cap \bigcap_{U\in\mathcal{C}} U \neq \emptyset$, which implies $x \in (\bigcap_{U\in\mathcal{C}} U){\downarrow}$. □

We are now ready to prove the following

Lemma 2.12. *Let $f\colon \mathfrak{A} \to \mathfrak{B}$ be a morphism between Heyting algebras. Then $\Psi(f)$ is a morphism between Esakia spaces.*

Proof. For notational issues, let us denote $\Psi(f)$ by $g\colon \mathfrak{B}_+ \to \mathfrak{A}_+$. First notice that g is indeed a well defined function, since, for every prime filter ∇ of $\mathfrak{B}$, $g(\nabla) = f^{-1}(\nabla)$ is a prime filter of $\mathfrak{A}$. Furthermore, g is monotone: indeed suppose $\nabla \subseteq \nabla'$, then if $a \in g(\nabla)$, $f(a) \in \nabla$ and so $f(a) \in \nabla'$ by our hypothesis. Hence $a \in f^{-1}(\nabla') = g(\nabla')$.
In order to show that g is continuous, we just have to show that $g^{-1}(\widehat{a})$ is clopen, for every clopen set in $W_{\mathfrak{A}}$ of the form $\widehat{a}$, for $a \in A$. But

$$\begin{aligned} g^{-1}(\widehat{a}) &= \{\nabla \in W_{\mathfrak{B}} \mid g(\nabla) \in \widehat{a}\} \\ &= \{\nabla \in W_{\mathfrak{B}} \mid a \in g(\nabla)\} \\ &= \{\nabla \in W_{\mathfrak{B}} \mid a \in f^{-1}(\nabla)\} \\ &= \{\nabla \in W_{\mathfrak{B}} \mid f(a) \in \nabla\} \\ &= \widehat{f(a)} \end{aligned}$$

and thus $g^{-1}(\widehat{a})$ is clopen.
Finally, let us show that g is a p-morphism. By Lemma 2.10, it is enough to show that $g^{-1}(\nabla\downarrow) = g^{-1}(\nabla)\downarrow$ for every $\nabla \in W_{\mathfrak{A}}$. First notice that, for every $a, b \in A$, we have

$$\begin{aligned} g^{-1}((\widehat{a} \cap -\widehat{b})\downarrow) &= g^{-1}(-(\widehat{a \to b})) && \text{by Lemma 2.5} \\ &= W_{\mathfrak{B}} \setminus g^{-1}(\widehat{a \to b}) \\ &= W_{\mathfrak{B}} \setminus f(\widehat{a \to b}) \\ &= -(\widehat{f(a) \to f(b)}) && \text{since } f \text{ is a homomorphism} \\ &= (\widehat{f(a)} \cap -\widehat{f(b)})\downarrow && \text{by Lemma 2.5} \end{aligned}$$

which is clopen since $R_{\mathfrak{B}}$ is a clopen relation. Furthermore, we have

$$\begin{aligned} g^{-1}(\widehat{a} \cap -\widehat{b}) &= \{\nabla \in W_{\mathfrak{B}} \mid g(\nabla) \in \widehat{a} \cap -\widehat{b}\} \\ &= \{\nabla \in W_{\mathfrak{B}} \mid g(\nabla) \in \widehat{a}\} \cap \{\nabla \in W_{\mathfrak{B}} \mid g(\nabla) \notin \widehat{b}\} \\ &= \{\nabla \in W_{\mathfrak{B}} \mid a \in g(\nabla)\} \cap \{\nabla \in W_{\mathfrak{B}} \mid b \notin g(\nabla)\} \\ &= \{\nabla \in W_{\mathfrak{B}} \mid a \in f^{-1}(\nabla)\} \cap \{\nabla \in W_{\mathfrak{B}} \mid b \notin f^{-1}(\nabla)\} \\ &= \{\nabla \in W_{\mathfrak{B}} \mid f(a) \in \nabla\} \cap \{\nabla \in W_{\mathfrak{B}} \mid f(b) \notin \nabla\} \\ &= \widehat{f(a)} \cap -\widehat{f(b)} \end{aligned}$$

and therefore, for all $a, b \in A$,

$$g^{-1}((\widehat{a} \cap -\widehat{b})\downarrow) = g^{-1}(\widehat{a} \cap -\widehat{b})\downarrow. \qquad (\star)$$

Now consider an arbitrary prime filter ∇ of $\mathfrak{A}$. Then the singleton $\{\nabla\}$ is a closed set in $W_{\mathfrak{A}}$ and it is the intersection of all the basis elements $\widehat{a} \cap -\widehat{b}$ in $\mathcal{B}$ such that $\nabla \in \widehat{a} \cap -\widehat{b}$. Let us denote by $\mathcal{C}$ such a family and by $\mathcal{D}$ the family $\{g^{-1}(\widehat{a} \cap -\widehat{b}) \mid \nabla \in \widehat{a} \cap -\widehat{b}\}$. It can be easily shown that both $\mathcal{C}$ and

$\mathcal{D}$ are downward directed families of non-empty closed sets and we can thus apply Esakia's Lemma 2.11 to them. Then, since preimages commute with the set-theoretic operations, we have

$$\begin{aligned}
g^{-1}(\nabla{\downarrow}) &= g^{-1}(\{\nabla\}{\downarrow}) \\
&= g^{-1}((\bigcap_{\nabla\in\widehat{a}\cap-\widehat{b}} \widehat{a}\cap-\widehat{b}){\downarrow}) \\
&= g^{-1}(\bigcap_{\nabla\in\widehat{a}\cap-\widehat{b}} (\widehat{a}\cap-\widehat{b}){\downarrow}) && \text{by Esakia's lemma} \\
&= \bigcap_{\nabla\in\widehat{a}\cap-\widehat{b}} g^{-1}((\widehat{a}\cap-\widehat{b}){\downarrow}) \\
&= \bigcap_{\nabla\in\widehat{a}\cap-\widehat{b}} g^{-1}(\widehat{a}\cap-\widehat{b}){\downarrow} && \text{by } (\star) \\
&= (\bigcap_{\nabla\in\widehat{a}\cap-\widehat{b}} g^{-1}(\widehat{a}\cap-\widehat{b})){\downarrow} && \text{by Esakia's lemma} \\
&= g^{-1}(\bigcap_{\nabla\in\widehat{a}\cap-\widehat{b}} (\widehat{a}\cap-\widehat{b})){\downarrow} \\
&= g^{-1}(\{\nabla\}){\downarrow} \\
&= g^{-1}(\nabla){\downarrow}
\end{aligned}$$

which finally shows that g is indeed a p-morphism. □

Lemma 2.13. *Let $h\colon \mathcal{X} \to \mathcal{Y}$ be a morphism between Esakia spaces. Then $\Phi(h)$ is a morphism between Heyting algebras.*

Proof. For notational issues, let us denote $\Phi(h)$ by $g\colon \mathcal{Y}^+ \to \mathcal{X}^+$. Notice that g is indeed a well defined function. Indeed, since h is continuous, its inverse image $g = h^{-1}$ maps clopen sets to clopen sets and, since h is also monotone, the inverse image of an upset is again an upset. Furthermore, g is a lattice homomorphism, since inverse images commutes with the set-theoretic operations. So, in order to show that g is a homomorphism between the Heyting algebras $\mathcal{Y}^+$ and $\mathcal{X}^+$, we only have to check that g preserves the operation $\supset$ of implication. Let U, V be clopen upset of Y and suppose $x \in g(U \supset V)$. Thus $h(x) \in U \supset V = Y \setminus (U \setminus V){\downarrow}$. If $x \in (g(U) \setminus g(V)){\downarrow}$, there exists $z \in g(U) \setminus g(V)$ and $x \in z{\downarrow}$. Thus $h(z) \in U$ and $h(z) \notin V$ which implies$h(z) \in (U \setminus V){\downarrow}$. But since h is monotone, $h(z) \in U \supset V$, contradiction. Thus $x \notin (g(U) \setminus g(V)){\downarrow}$, that is, $x \in g(U) \supset g(V)$. Conversely, if $x \notin g(U \supset V)$, then $h(x) \in (U \setminus V){\downarrow}$ and so there exist $z \in U \setminus V$ such that $h(x) \in z{\downarrow}$. Since h is a p-morphism, there is $x' \in x{\uparrow}$ such that $h(x') = z$. Therefore, $x' \in g(U \setminus V) = g(U) \setminus g(V)$ and $x \in (g(U) \setminus g(V)){\downarrow}$. Hence, $x \notin g(U) \supset g(V)$. □

We have just proved that $\Psi\colon \mathcal{HA} \to \mathcal{ES}$ and $\Phi\colon \mathcal{ES} \to \mathcal{HA}$ are well defined contravariant functors. We are now ready to show that they establish a duality between the above categories.

Theorem 2.14 (Esakia's Duality Theorem)**.** *The categories $\mathcal{HA}$ of Heyting algebras and $\mathcal{ES}$ of Esakia spaces are dually equivalent.*

Proof. Consider the contravariant functors $\Psi\colon \boldsymbol{\mathcal{HA}} \to \boldsymbol{\mathcal{ES}}$ and $\Phi\colon \boldsymbol{\mathcal{ES}} \to \boldsymbol{\mathcal{HA}}$ defined as above. In order to establish the dual equivalence of $\boldsymbol{\mathcal{HA}}$ and $\boldsymbol{\mathcal{ES}}$, the natural transformations we need are the following:
for every Heyting algebra $\mathfrak{A}$, let

$$\alpha\colon 1_{\boldsymbol{\mathcal{HA}}} \xrightarrow{\sim} \Phi \circ \Psi \quad \text{be defined by} \quad \begin{array}{l} \alpha_{\mathfrak{A}}\colon \mathfrak{A} \to \Phi \circ \Psi(\mathfrak{A}), \\ \alpha_{\mathfrak{A}}(a) = \{\nabla \in W_{\mathfrak{A}} \mid a \in \nabla\}, \end{array}$$

and, for every Esakia space $\mathcal{X}$, let

$$\beta\colon 1_{\boldsymbol{\mathcal{ES}}} \xrightarrow{\sim} \Psi \circ \Phi \quad \text{be defined by} \quad \begin{array}{l} \beta_{\mathcal{X}}\colon \mathcal{X} \to \Psi \circ \Phi(\mathcal{X}), \\ \beta_{\mathcal{X}}(x) = \{U \in X^{CU} \mid x \in U\}. \end{array}$$

Notice that, for every Heyting algebra $\mathfrak{A}$ and every Esakia space $\mathcal{X}$, we defined the components $\alpha_{\mathfrak{A}}$ and $\beta_{\mathcal{X}}$ of α and β as in Proposition 2.8 and Proposition 2.9 respectively and therefore $\alpha_{\mathfrak{A}}$ and $\beta_{\mathcal{X}}$ are isomorphisms. Thus, if we can show that α and β are indeed natural transformations, then what we have actually defined are two natural isomorphisms and the required duality will thus be established.
Let $f\colon \mathfrak{A} \to \mathfrak{B}$ be a morphism in $\boldsymbol{\mathcal{HA}}$. In order to see that the following diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\alpha_{\mathfrak{A}}} & \Phi \circ \Psi(\mathfrak{A}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Phi\circ\Psi(f)} \\ \mathfrak{B} & \xrightarrow[\alpha_{\mathfrak{B}}]{} & \Phi \circ \Psi(\mathfrak{B}) \end{array}$$

commutes, consider $a \in \mathfrak{A}$. Then,

$$\begin{aligned} ((\Phi \circ \Psi(f)) \circ \alpha_{\mathfrak{A}})(a) &= \Phi \circ \Psi(f)(\alpha_{\mathfrak{A}}(a)) \\ &= (\Psi(f))^{-1}(\widehat{a}) \\ &= \widehat{f(a)} && \text{by Lemma 2.12} \\ &= \alpha_{\mathfrak{B}}(f(a)) \\ &= (\alpha_{\mathfrak{B}} \circ f)(a). \end{aligned}$$

Now, let $h\colon \mathcal{X} \to \mathcal{Y}$ be a morphism in $\boldsymbol{\mathcal{ES}}$. In order to see that the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\beta_{\mathcal{X}}} & \Psi \circ \Phi(\mathcal{X}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Psi\circ\Phi(h)} \\ \mathcal{Y} & \xrightarrow[\beta_{\mathcal{Y}}]{} & \Psi \circ \Phi(\mathcal{Y}) \end{array}$$

commutes, consider $x \in \mathcal{X}$. Then,

$$\begin{aligned}((\Psi \circ \Phi(h)) \circ \beta_{\mathcal{X}})(x) &= \Psi \circ \Phi(h)(\beta_{\mathcal{X}}(x)) \\ &= (\Phi(h))^{-1}(\beta_{\mathcal{X}}(x)) \\ &= \{V \in Y^{CU} \mid h^{-1}(V) \in \beta_{\mathcal{X}}(x)) \\ &= \{V \in Y^{CU} \mid x \in h^{-1}(V)\} \\ &= \{V \in Y^{CU} \mid h(x) \in V\} \\ &= \beta_{\mathcal{Y}}(h(x)) \\ &= (\beta_{\mathcal{Y}} \circ h)(x)\end{aligned}$$

Therefore, α and β are natural isomorphisms, which, together with the functors Ψ and Φ, yield a co-equivalence between $\mathcal{HA}$ and $\mathcal{ES}$. □

Consider the full subcategory $\mathcal{BA}$ of $\mathcal{HA}$ given by Boolean algebras and related morphisms. It is well known that, in any Boolean algebra $\mathfrak{B}$, prime filters and ultrafilters coincide and therefore the poset of prime filters $\langle W_{\mathfrak{B}}, R_{\mathfrak{B}}\rangle$ of $\mathfrak{B}$ turns out to be discrete, that is, $R_{\mathfrak{B}}$ is the identity relation $=$. Therefore the Esakia space $\mathfrak{B}_+ = \langle W_{\mathfrak{B}}, \tau_{\mathfrak{B}}, R_{\mathfrak{B}}\rangle$ reduces to the Stone space $\langle W_{\mathfrak{B}}, \tau_{\mathfrak{B}}\rangle$. Conversely, every Stone space $\langle X, \tau\rangle$ can be considered as the Esakia space $\langle X, \tau, R\rangle$ with R the discrete order and thus the Heyting algebra $\mathcal{X}^+$ is simply the lattice of clopen sets of X, which is a Boolean algebra, since it is clearly closed under set-theoretic complementation.

These considerations allows us to recover from Esakia's duality as a particular case the celebrated *Stone's representation theorem for Boolean algebras* and the following *Stone's duality* for Boolean algebra:

Corollary 2.15. *Let* $\mathcal{S}\boldsymbol{tone}$ *be the category of Stone spaces and continuous maps. Then the categories* $\mathcal{BA}$ *and* $\mathcal{S}\boldsymbol{tone}$ *are dually equivalent.*

Remark 2. In giving a proof of Esakia's duality in the previous paragraphs, we actually proved something more general. In fact, we proved a categorical duality between the category $\mathcal{DL}$ of bounded distributive lattices with related homomorphisms and the category $\mathcal{PS}$ of Priestley spaces with continuous order-preserving maps as arrows:

Theorem 2.16 (Priestley's Duality)**.** *The categories* $\mathcal{DL}$ *of bounded distributive lattices and* $\mathcal{PS}$ *of Priestley spaces are dually equivalent.*

2.4 Some basic properties of Esakia spaces

In this section, following §3.2 of [41], we will state, mostly without proof, a few important properties of Esakia spaces which show the usefullness of the topological approach to Heyting algebras.

Theorem 2.17. *Let* $\mathcal{X} = \langle X, \tau, R\rangle$ *be an Esakia space and* $F \subseteq X$ *a closed subset of* X*. The following hold:*

(1) *for every* $x \in F$*, there exists a maximal point* $y \in F$ *such that* xRy*;*

(2) *for every* $x \in F$*, there exists a minimal point* $y \in F$ *such that* yRx*.*

Proof. (1) Let $x \in F$ and consider the poset $x{\uparrow} \cap F$. Now, let C be a chain contained in $x{\uparrow} \cap F$ and consider the family $\mathcal{A} = \{y{\uparrow} \mid y \in C\}$. Since C is a chain and R is point-closed, we have that $\mathcal{A}$ is a family of closed sets with the finite intersection property. Therefore, by compactness, $\bigcap \mathcal{A} \neq \emptyset$ and so there exists a point a which is greater that every element of C. Hence, every chain in $x{\uparrow} \cap F$ has an upper-bound and by Zorn's lemma, the set $x{\uparrow} \cap F$ has a maximal element y.
(2) The proof is similar to (1) and uses Lemma 2.1. □

As immediate consequences of the previous theorem, we then get

Corollary 2.18. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and $F \subseteq X$ a non-empty closed subset of X. Then $max(F) \neq \emptyset$ and $min(F) \neq \emptyset$.*

Corollary 2.19. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and $F \subseteq X$ a non-empty closed upset of X. Then $F = min(F){\uparrow}$.*

Notice moreover that by choosing X for F in Theorem 2.17, the set of maximal elements $max(X)$ of every Esakia space $\mathcal{X} = \langle X, \tau, R\rangle$ is non-empty. Furthermore, the following holds

Theorem 2.20. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space. Then the set $max(X)$ is closed in X.*

Now, for every Esakia space $\mathcal{X} = \langle X, \tau, R\rangle$ and $Y \subseteq X$, we consider the ordered topological space $\mathcal{Y} = \langle Y, \tau_Y, R_Y\rangle$ where $\langle Y, \tau_Y\rangle$ is the subspace of the Stone space $\langle X, \tau\rangle$ given by the subspace topology τ_Y and $\langle Y, R_Y\rangle$ is the poset obtained by restricting the ordering relation R to Y, that is, $y{\uparrow}_Y = y{\uparrow}_X \cap Y$ for all $y \in Y$.

Theorem 2.21. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and $U \subseteq X$ be a closed upset of X. Then $\mathcal{U} = \langle U, \tau_U, R_U\rangle$ is an Esakia space.*

Proof. First, let us show that $\mathcal{U}$ is a Priestley space. Since $\langle U, \tau_U\rangle$ is a closed subspace of the compact space $\langle X, \tau\rangle$, $\langle U, \tau_U\rangle$ is compact as well. Then suppose that $\neg(u_1 R_U u_2)$. Then $\neg(u_1 R u_2)$ and by the Priestley separation axiom, there exists a clopen set Q of X such that $u_1 \in Q$ and $u_2 \notin Q$. But then $Q \cap U$ is a clopen upset in U separating u_1 and u_2. Hence $\mathcal{U}$ is a Priestley space.
Now, let S be a clopen of U. Then $S = Q \cap U$ for some clopen Q in X. Then

$$\begin{aligned} S{\downarrow}_U &= (Q \cap U){\downarrow}_U \\ &= \bigcup\{u{\downarrow}_U \mid u \in Q \cap U\} \\ &= \bigcup\{u{\downarrow}_X \cap U \mid u \in Q \cap U\} \\ &= U \cap \bigcup\{u{\downarrow}_X \mid u \in Q \cap U\} \\ &= U \cap (Q \cap U){\downarrow}_X. \end{aligned}$$

Clearly $U \cap (Q \cap U){\downarrow}_X \subseteq U \cap Q{\downarrow}_X$. Now, let $v \in U \cap Q{\downarrow}_X$. Then $v \in U$ and there exists $q \in Q$ such that vRq. But since U is an upset, $q \in U$ and therefore $q \in Q \cap U$ and $v \in (Q \cap U){\downarrow}_X$. Hence $U \cap (Q \cap U){\downarrow}_X = U \cap Q{\downarrow}_X$ and

$$S{\downarrow}_U = U \cap Q{\downarrow}_X.$$

But since Q is clopen and R is a clopen relation, $Q{\downarrow_X}$ is clopen in X and therefore $S{\downarrow_U}$ is clopen in U. Therefore R_U is a clopen relation and $\mathcal{U}$ is an Esakia space. □

The previous theorem gives us the justification for the following

Definition 2.4. Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and $U \subseteq X$ a closed upset of X. We call the Esakia subspace $\mathcal{U} = \langle U, \tau_U, R_U\rangle$ the *generated subspace* of $\mathcal{X}$ induced by U.

Theorem 2.22. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and $U \subseteq X$ be a clopen set of X. Then $\mathcal{U} = \langle U, \tau_U, R_U\rangle$ is an Esakia space.*

Proof. As above, one can show that $\mathcal{U}$ is a Priestley space. Now, let S be a clopen of U. Then $S = Q \cap U$ for some clopen Q in X and thus S is clopen in X as well. Moreover, since R is a clopen relation, $S{\downarrow_X}$ is also clopen in X. But

$$\begin{aligned} S{\downarrow_U} &= (Q \cap U){\downarrow_U} \\ &= \bigcup\{u{\downarrow_U} \mid u \in Q \cap U\} \\ &= \bigcup\{u{\downarrow_X} \cap U \mid u \in Q \cap U\} \\ &= U \cap \bigcup\{u{\downarrow_X} \mid u \in Q \cap U\} \\ &= U \cap S{\downarrow_X}, \end{aligned}$$

which implies that $S{\downarrow_U}$ is clopen in U. Therefore R_U is a clopen relation and $\mathcal{U}$ is an Esakia space. □

2.5 Duality's correspondences

In this section we will use Esakia duality to translate the most basic algebraic concepts about Heyting algebras in the language of Esakia's spaces. All the results of this section are scattered through the literature, however the main references are certainly [70, 8, 41] and [13].

2.5.1 Filters, Ideals and Congruences

Let be $\mathfrak{A} = \langle A, \wedge, \vee, \to, 0\rangle$ a Heyting algebra and let $\mathfrak{A}_+ = \langle X, \tau, R\rangle$ be its dual Esakia space. Consider the lattice of filters $\boldsymbol{\mathcal{F}i}(\mathfrak{A})$ and the lattice of ideals $\boldsymbol{\mathcal{I}d}(\mathfrak{A})$ of $\mathfrak{A}$. For $\nabla \in \boldsymbol{\mathcal{F}i}(\mathfrak{A})$ and $\Delta \in \boldsymbol{\mathcal{I}d}(\mathfrak{A})$, define the maps

$$\begin{aligned} \chi(\nabla) &= \bigcap\{\widehat{a} \mid a \in \nabla\} \\ \vartheta(\Delta) &= \bigcup\{\widehat{a} \mid a \in \Delta\}. \end{aligned}$$

It is clear that the range of χ is included in the set of closed sets of $\mathfrak{A}_+$. Now let us denote the lattice of closed upsets of $\mathfrak{A}_+$, ordered by inclusion, by $\boldsymbol{\mathcal{FU}}(\mathfrak{A}_+)$. Dually, it is immediately seen that the range of ϑ is included in the set of opens of $\mathfrak{A}_+$. The set of the open upsets of $\mathfrak{A}_+$ ordered by the inclusion relation forms again a lattice that we denote by $\boldsymbol{\mathcal{OU}}(\mathfrak{A}_+)$.

Lemma 2.23. *The map* $\chi\colon \boldsymbol{\mathcal{F}i}(\mathfrak{A}) \to \boldsymbol{\mathcal{FU}}(\mathfrak{A}_+)$ *is a dual isomorphism between the (complete) lattice of filters of* $\mathfrak{A}$ *and the (complete) lattice of closed upsets of* $\mathfrak{A}_+$ *ordered by inclusion, while the map* $\vartheta\colon \boldsymbol{\mathcal{I}d}(\mathfrak{A}) \to \boldsymbol{\mathcal{OU}}(\mathfrak{A}_+)$ *is a dual isomorphism between the (complete) lattice of ideals of* $\mathfrak{A}$ *and the (complete) lattice of open upsets of* $\mathfrak{A}_+$ *ordered by inclusion.*

Proof. We prove the lemma only for the part regarding χ and we leave the rest to the reader. Given a filter $\nabla \subseteq A$, we have

$$\begin{aligned}\chi(\nabla) &= \bigcap\{\widehat{a} \,|\, a \in \nabla\}\\ &= \{\nabla' \in W_{\mathfrak{A}} \,|\, \forall a \in \nabla(\nabla' \in \widehat{a})\}\\ &= \{\nabla' \in W_{\mathfrak{A}} \,|\, \forall a \in \nabla(a \in \nabla')\}\\ &= \{\nabla' \in W_{\mathfrak{A}} \,|\, \nabla \subseteq \nabla'\},\end{aligned}$$

which shows that $\chi(\nabla)$ is indeed a closed upset. Conversely, given a closed upset U, then $U = \bigcap\{\widehat{a} \,|\, U \subseteq \widehat{a}\}$. Indeed, if $\nabla \notin U$, we have $\Delta \nsubseteq \nabla$, for every $\Delta \in U$. Thus, let a_Δ be such that $a_\Delta \in \Delta$ and $a_\Delta \notin \nabla$, for each $\Delta \in U$. Then, $\Delta \in \widehat{a_\Delta}$ and $\nabla \notin \widehat{a_\Delta}$ for every $\Delta \in U$. Thus the family $\mathcal{C} = \{\widehat{a_\Delta} \,|\, \Delta \in U\}$ is an open cover of U and since U is closed and the space is Hausdorff, by compactness, there is a finite subcover $\{\widehat{a_{\Delta_i}} \,|\, i = 1, \dots n\}$ of U. Let $a_U = \bigvee_{i=1}^n a_{\Delta_i}$, then $U \subseteq \widehat{a_U}$ and $\nabla \notin \widehat{a_U}$. Moreover, it can be easily shown that the set $\{a \in A \,|\, U \subseteq \widehat{a}\}$ is a filter and the image of such a set under χ is U. Finally, it is evident that $\nabla \subseteq \nabla' \implies \chi(\nabla') \subseteq \chi(\nabla)$. Now, if $\nabla \nsubseteq \nabla'$, let $a \in \nabla \setminus \nabla'$ and consider a prime filter Σ that extend ∇' and such that $a \notin \Sigma$, which exists by Theorem 1.61. Then $\Sigma \in \chi(\nabla')$ but $\Sigma \notin \chi(\nabla)$. Therefore $\nabla \subseteq \nabla' \impliedby \chi(\nabla') \subseteq \chi(\nabla)$. ☐

Notice that if ∇ is a prime filter, then $\chi(\nabla) = \nabla\!\uparrow$, that is, $\chi(\nabla)$ is a principal upset in X. Every prime ideal Δ is mapped to the open upset $X \setminus \nabla\!\downarrow$, where ∇ is the prime filter $A \setminus \Delta$. Moreover, if $\nabla = [a)$ is a principal filter, or $\Delta = (a]$ is a principal ideal, then $\chi(\nabla) = \widehat{a} = \vartheta(\Delta)$, which is a clopen upset. Furthermore, if ∇ is an ultrafilter, then, since it is a prime filter, $\chi(\nabla)$ is the principal upset $\nabla\!\uparrow$; however, since ∇ is not contained in any other proper filter different from ∇, it follows that $\chi(\nabla) = \nabla\!\uparrow = \{\nabla\}$ and ∇ is a R-maximal element in X. By an analogous reasoning, it follows that, for every maximal ideal Δ, $\vartheta(\Delta) = W_{\mathfrak{A}} \setminus \nabla\!\downarrow$, where ∇ is the maximal element in X corresponding to the maximal filter $A \setminus \Delta$.

Finally, since by Theorem 1.52 we have that the lattice $\boldsymbol{\mathcal{F}i}(\mathfrak{A})$ of filters of $\mathfrak{A}$ is isomorphic to the lattice $\boldsymbol{\mathcal{C}on}(\mathfrak{A})$ of congruences of $\mathfrak{A}$, we also have the following immediate

Corollary 2.24. *The (complete) lattice* $\boldsymbol{\mathcal{C}on}(\mathfrak{A})$ *of congruences of* $\mathfrak{A}$ *and the (complete) lattice* $\boldsymbol{\mathcal{FU}}(\mathfrak{A}_+)$ *of closed upsets of* $\mathfrak{A}_+$ *are dually isomorphic.*

2.5.2 Infima and suprema

Let $\mathcal{X} = \langle X, \tau, R\rangle$ be a Priestley space and let $S \subseteq X$. We define the operators $\mathbf{J}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ and $\mathbf{D}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ as follows:

- $\mathbf{J}(S)$ is the largest open upset contained in S;
- $\mathbf{D}(S)$ is the smallest closed upset containing S.

For a proof of the following lemma, cfr. [70, Lemma 3.1]:

Lemma 2.25. *Let $\mathfrak{L}$ be a bounded distributive lattice and let $\mathcal{X} = \langle X, \tau, R\rangle$ be its Priestley space. Then, for any $S \subseteq X$, the following hold:*

(i) $\mathbf{J}(S) = \bigcup\{\widehat{a} \,|\, \widehat{a} \subseteq S\} = X \setminus (X \setminus S^\circ){\downarrow}$;

(ii) $\mathbf{D}(S) = \bigcap\{\widehat{a} \,|\, S \subseteq \widehat{a}\} = \overline{S}{\uparrow}$.

Corollary 2.26. *Let $\mathfrak{A}$ be a Heyting algebra, $\mathcal{X} = \langle X, \tau, R\rangle$ its dual Esakia space and let $S \subseteq X$ be an upset. Then $\mathbf{D}(S) = \overline{S}$.*

Proof. By definition of closure, we have $\overline{S} \subseteq \mathbf{D}(S)$. For the converse inclusion, if $x \notin \overline{S}$, then $x \in X \setminus \overline{S}$ and since X is zero-dimensional, there exists a clopen set U such that $x \in U \subseteq X \setminus \overline{S}$. Hence, $U \cap S = \emptyset$. Moreover, since S is an upset, we have $U{\downarrow} \cap S = \emptyset$ and, since R is a clopen relation, $U{\downarrow}$ is clopen. Therefore, $X \setminus U{\downarrow}$ is a clopen upset and thus it is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$. Consequently, we have $S \subseteq \widehat{a}$ and $x \notin \widehat{a}$. Therefore, by (ii) of the previous Lemma, $x \notin \mathbf{D}(S)$. □

Theorem 2.27. *Let $\mathfrak{A}$ be a Heyting algebra, $\mathcal{X} = \langle X, \tau, R\rangle$ its dual Esakia space and $S \subseteq \mathfrak{A}$. Then the following hold:*

(i) $\bigvee S$ *exists in* $\mathfrak{A} \iff \overline{\bigcup_{s\in S}\widehat{s}}$ *is clopen in* X*;*

(ii) $\bigwedge S$ *exists in* $\mathfrak{A} \iff \mathbf{J}(\bigcap_{s\in S}\widehat{s})$ *is clopen in* X*.*

Proof. Let us prove (i). Then (ii) can be proved by a dual argument.
($\Longrightarrow$) Suppose $q = \bigvee S$ exists in $\mathfrak{A}$. Since, for all $s \in S$, $s \leq q$, we have $\widehat{s} \subseteq \widehat{q}$ and thus $\bigcup_{s\in S}\widehat{s} \subseteq \widehat{q}$. Since $\bigcup_{s\in S}\widehat{s}$ is an upset and $\widehat{q}$ is a closed upset, we have $\overline{\bigcup_{s\in S}\widehat{s}} = \mathbf{D}(\bigcup_{s\in S}\widehat{s}) \subseteq \widehat{q}$ by Corollary 2.26. Now, if $x \notin \overline{\bigcup_{s\in S}\widehat{s}}$, then by Lemma 2.25 (ii) there exists $c \in \mathfrak{A}$ such that $x \notin \widehat{c} \supseteq \bigcup_{s\in S}\widehat{s}$. Therefore, by Corollary 1.62, $s \leq c$ for all $s \in S$ and so $q \leq c$. Hence $\widehat{q} \subseteq \widehat{c}$ and thus $x \notin \widehat{q}$. This means that $\overline{\bigcup_{s\in S}\widehat{s}} = \widehat{q}$ and $\overline{\bigcup_{s\in S}\widehat{s}}$ is clopen.
($\Longleftarrow$) Assume that $\overline{\bigcup_{s\in S}\widehat{s}}$ is clopen in X. Since $\bigcup_{s\in S}\widehat{s}$ is an upset, we have, by Corollary 2.26, $\overline{\bigcup_{s\in S}\widehat{s}} = \mathbf{D}(\bigcup_{s\in S}\widehat{s})$, which is again an upset. Therefore, $\overline{\bigcup_{s\in S}\widehat{s}} = \widehat{q}$ for some $q \in \mathfrak{A}$ and, by Lemma 2.25 (ii),

$$\bigcap\{\widehat{a} \,|\, \bigcup_{s\in S}\widehat{s} \subseteq \widehat{a}\} = \widehat{q}.$$

Therefore q is the least upper bound of S, that is, $\bigvee S = q \in \mathfrak{A}$. □

2.5.3 Completely join-prime elements

An element a in a bounded lattice $\mathfrak{L}$ is said to be *completely join-prime* if $a \neq 0$ and, for every non-empty subset B of $\mathfrak{L}$ such that $\bigvee B$ exists in $\mathfrak{L}$, $a \leq \bigvee B$ implies that there exists $b \in B$ such that $a \leq b$. Furthermore, a is said to be *completely join-irreducible* if $a \neq 0$ and, for every non-empty subset B of $\mathfrak{L}$ such that $\bigvee B$ exists in $\mathfrak{L}$, $a = \bigvee B$ implies that there exists $b \in B$ such that $a = b$[1]. For a bounded lattice $\mathfrak{L}$, we will use the following notation:

- $J(\mathfrak{L})$ is the set of join-prime elements of $\mathfrak{L}$;

[1] Cfr. also the footnote 25 of §1.6.

- $J^\infty(\mathfrak{L})$ is the set of completely join-prime elements of $\mathfrak{L}$;
- $JI(\mathfrak{L})$ is the set of join-irreducible elements of $\mathfrak{L}$;
- $JI^\infty(\mathfrak{L})$ is the set of completely join-irreducible elements of $\mathfrak{L}$.

We know that $J(\mathfrak{L}) \subseteq JI(\mathfrak{L})$ and that $J(\mathfrak{L}) = JI(\mathfrak{L})$, if the lattice $\mathfrak{L}$ is distributive. Furthermore, it is clear that $J^\infty(\mathfrak{L}) \subseteq JI^\infty(\mathfrak{L})$. The converse holds if $\mathfrak{L}$ satisfies the infinite distributive law

$$x \wedge \bigvee_{i\in I} y_i \approx \bigvee_{i\in I}(x \wedge y_i)$$

for every index set I. In particular, we have that $J^\infty(\mathfrak{L}) = JI^\infty(\mathfrak{L})$ if $\mathfrak{L}$ is a complete Heyting algebra by Lemma 1.45. Notice, moreover, that we have $J^\infty(\mathfrak{L}) \subseteq J(\mathfrak{L})$ and if $\mathfrak{L}$ is finite, then the converse inclusion $J(\mathfrak{L}) \subseteq J^\infty(\mathfrak{L})$ also holds. Thus, for a finite distributive lattice (finite Heyting algebra), all the four sets above coincide.

Now, recall that a point x in a topological space $\langle X, \tau\rangle$ is isolated if $\{x\} \in \tau$. We denote the set of isoleted point of X by X_{iso}.

Theorem 2.28. *Let $\mathfrak{A}$ be a Heyting algebra and let $\mathcal{X} = \langle X, \tau, R\rangle$ be its dual Esakia space. Then*

(i) $a \in J(\mathfrak{A}) \iff \exists x \in X$ *such that* $x\!\uparrow = \widehat{a}$*;*

(ii) $a \in J^\infty(\mathfrak{A}) \iff \exists x \in X_{iso}$ *such that* $x\!\uparrow = \widehat{a}$*.*

Proof. (i) Suppose $a \in J(\mathfrak{A})$. Then, since $J(\mathfrak{A}) = JI(\mathfrak{A})$, the principal filter $[a)$ is a prime filter of $\mathfrak{A}$. Hence $[a) = x \in X$ for some $x \in X$ and $\widehat{a} = x\!\uparrow$. Conversely, suppose $\widehat{a} = x\!\uparrow$ for some $x \in X$. Then, if $a \leq b \vee c$ we have $x\!\uparrow \subseteq \widehat{b} \cup \widehat{c}$ and thus $x \in \widehat{b}$ or $x \in \widehat{c}$. Therefore $x\!\uparrow \subseteq \widehat{b}$ or $x\!\uparrow \subseteq \widehat{c}$ and, consequently, $a \leq b$ or $a \leq c$, i.e. $a \in J(\mathfrak{A})$.
(ii) ($\Longrightarrow$) Suppose $a \in J^\infty(\mathfrak{A})$. Since $J^\infty(\mathfrak{A}) \subseteq J(\mathfrak{A})$, by (i) there exists $x \in X$ such that $\widehat{a} = x\!\uparrow$. In order to show that $x \in X_{\text{iso}}$, suppose for reductio that it is not. Let $Q = \widehat{a} \setminus \{x\}$. Then, since $\widehat{a} \setminus \{x\} \subseteq \widehat{a}$ and $\widehat{a}$ is closed, $\overline{Q} \subseteq \widehat{a}$. Moreover notice that, since $\widehat{a}$ is open, x cannot be an isolated point of $\widehat{a}$ and thus it must be a limit point of $\widehat{a}$. Then we have $x \in \overline{Q}$ and $\overline{Q} = \widehat{a}$. Now, since X is Hausdorff, $\{x\}$ is closed and thus $Q = \widehat{a} \setminus \{x\}$ is an open upset. Then by Lemma 2.25 (i), $Q = \mathbf{J}(Q) = \bigcup\{\widehat{b} \mid \widehat{b} \subseteq Q\}$ and thus

$$\widehat{a} = \overline{\bigcup_{\widehat{b}\subseteq Q} \widehat{b}}.$$

By Theorem 2.27, we then have $a = \bigvee B$ where $B = \{b \in \mathfrak{A} \mid \widehat{b} \subseteq Q\}$. But the fact that $x \notin Q$ implies that $x \notin \widehat{b}$ for all $b \in B$. Hence $a \nleq b$ for all $b \in B$ and thus $a \notin J^\infty(\mathfrak{A})$.
($\Longleftarrow$) Let $x\!\uparrow = \widehat{a}$ for some $x \in X_{\text{iso}}$ and suppose $a \leq \bigvee B$ for some non-empty set $B \subseteq \mathfrak{A}$. Then, by Theorem 2.27, $\widehat{a} \subseteq \overline{\bigcup_{b\in B} \widehat{b}}$ and $\overline{\bigcup_{b\in B} \widehat{b}}$ is clopen. Therefore $x \in \overline{\bigcup_{b\in B} \widehat{b}}$ and since x is also an isolated point of $\overline{\bigcup_{b\in B} \widehat{b}}$ we have $x \in \bigcup_{b\in B} \widehat{b}$. Thus $x \in \widehat{b}$ for some $b \in B$, $\widehat{a} = x\!\uparrow \subseteq \widehat{b}$ and, consequently, $a \leq b$. Hence $a \in J^\infty(\mathfrak{A})$. ∎

2.5.4 Homomorphic images, subalgebras and direct products

Lemma 2.29. *Let $h\colon \mathcal{X} \to \mathcal{Y}$ be a morphism between the Esakia spaces $\mathcal{X} = \langle X, \tau, R\rangle$ and $\mathcal{Y} = \langle Y, \tau', S\rangle$ and let $g\colon \mathcal{Y}^+ \to \mathcal{X}^+$ be the morphism dual to h.*

(i) *h is injective $\iff$ g is surjective;*

(ii) *h is surjective $\iff$ g is injective.*

Proof. (i) Let h be injective. Since h is a continuous function from a compact space to a Hausdorff space, h is closed. Now let $U \in X^{CU}$ be a clopen upset of X. Then $X \setminus U$ is a clopen downset and $h(X \setminus U)$ is a closed downset. Hence $h(X \setminus U)$ is the intersection of the family $\mathcal{C}$ of all the clopens of Y containing it, that is,

$$h(X \setminus U) = \bigcap\{Q \in Y^C \mid h(X \setminus U) \subseteq Q\}.$$

Since $\mathcal{C}$ is a downward directed family of non-empty closed sets, by Esakia's Lemma we get

$$h(X \setminus U) = (\bigcap_{Q\in\mathcal{C}} Q){\downarrow} = \bigcap_{Q\in\mathcal{C}} Q{\downarrow},$$

which is again an intersection of clopens since S is a clopen relation. Consequently, by injectivity of h,

$$X \setminus U = h^{-1}(h(X \setminus U)) = \bigcap_{Q\in\mathcal{C}} h^{-1}(Q{\downarrow}).$$

Hence,

$$U = \bigcup_{Q\in\mathcal{C}} (X \setminus h^{-1}(Q{\downarrow})) = \bigcup_{Q\in\mathcal{C}} h^{-1}(Y \setminus Q{\downarrow}).$$

Since h is continuous, all the $h^{-1}(Y \setminus Q{\downarrow})$'s are clopen upsets in X and, since U is closed, we can apply compactness and find a finite subcover $\mathcal{D} \subseteq \mathcal{C}$ that covers U. Therefore,

$$U = \bigcup_{Q\in\mathcal{D}} h^{-1}(Y \setminus Q{\downarrow}) = h^{-1}(Z) = g(Z)$$

for some clopen upset Z in Y. So g is surjective.
Conversely, if g is surjective, let x, y be two different point of X. Then either $\neg(xRy)$ or $\neg(yRx)$. Suppose for the sake of definiteness that the former holds. Then, by Priestley separation axiom, there exists a clopen upset $U \subseteq X$ such that $x \in U$ and $y \notin U$. Since g is surjective, $U = g(V)$ for some clopen upset $V \subseteq Y$. Since $g(V) = h^{-1}(V)$, it follows that $h(x) \in V$ and $h(y) \notin V$ which implies that they are distinct.
(ii) Assume h is surjective and let $U, V \in Y^{CU}$ be two clopen upset of Y such that $U \neq V$. So, there exists $y \in U \setminus V$ and by surjectivity of h, $y = h(x)$ for some $x \in X$. But then $x \in h^{-1}(U) = g(U)$ and $x \notin h^{-1}(V) = g(V)$, that is, g is injective.
Conversely, suppose that g is injective and let $y \in Y$. Suppose that $h^{-1}(y{\downarrow}) = \emptyset$. Then, for all $x \in X$, $\neg(h(x)Sy)$ and by Priestley separation axiom there exists a clopen upset Q_x such that $h(x) \in Q_x$ and $y \notin Q_x$. Hence $h(X) \subseteq \bigcup_{x\in X} Q_x$

and, since h is closed, by compactness we can find a finite subset $Z \subseteq X$ such that $h(X) \subseteq \bigcup_{x\in Z} Q_x$. By letting $\bigcup_{x\in Z} Q_x = Q$, we have that Q is a clopen upset of Y such that $h(X) \subseteq Q$ and $y \notin Q$. Then $h^{-1}(Q) = X = h^{-1}(Y)$ and since $g = h^{-1}$ is injective, we conclude $Q = Y$, contradiction. Therefore $h^{-1}(y{\downarrow}) \neq \emptyset$ and there is $z \in X$ such that $h(z)Sy$. But since h is a p-morphism, there exists $q \in z{\uparrow}$ such that $h(q) = y$. Thus h is surjective. □

The previous lemma, together with the Esakia's Duality Theorem 2.14, imply the following

Lemma 2.30. *Let $f\colon \mathfrak{A} \to \mathfrak{B}$ be a morphism between Heyting algebras and let $g\colon \mathfrak{B}_+ \to \mathfrak{A}_+$ be the morphism dual to f.*

(i) *f is injective $\Longleftrightarrow$ g is surjective;*

(ii) *f is surjective $\Longleftrightarrow$ g is injective.*

Let us now turn our focus to direct products of Heyting algebras. We will show that an appropriate topological sum for Esakia spaces is the dual notion of direct product of Heyting algebras. We start with the following

Lemma 2.31. *Let $\{\mathcal{X}_i = \langle X_i, \tau_i, R_i\rangle \mid i \in I\}$, $|I| < \aleph_0$, be a finite family of disjoint Esakia spaces. Then the topological sum $\sum_{i\in I} \mathcal{X}_i = \langle X, \tau, R\rangle$ of the $\mathcal{X}_i$'s, where $R = \bigcup_{i\in I} R_i$, is an Esakia space.*

Proof. It is well known that the topological sum $\sum_{i\in I} \mathcal{X}_i$ is compact. Then let $\neg(xRy)$. Then $x \in X_i$ and $y \in X_j$ for some $i, j \in I$. If $i \neq j$, then clearly X_i is a clopen upset of X separating x and y. If $i = j$, then since $\mathcal{X}_j$ is an Esakia space, there exist a clopen upset $U \subseteq X_j$ separating x and y. But, for all $i \in I$, $U \cap X_i$ is either U or $\emptyset$ and therefore U is also a clopen upset of X. Finally, if U is a clopen of X, then $U \cap X_i$ is a clopen subset of X_i for all $i \in I$. Hence, since the R_i's are clopen relations, $(U \cap X_i){\downarrow_i} = U{\downarrow_i}$ are also clopen in each $i \in I$. But then $U{\downarrow} = \bigcup_{i\in I} U{\downarrow_i}$ is clopen in X and R is a clopen relation. Therefore $\sum_{i\in I} \mathcal{X}_i$ is an Esakia space. □

Lemma 2.32. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Heyting algebras and let $\mathcal{X}_1 = \langle X_1, \tau_1, R_1\rangle$ and $\mathcal{X}_2 = \langle X_2, \tau_2, R_2\rangle$ be two disjoint Esakia spaces.*

1. *The Esakia space $\mathfrak{A}_{1+} + \mathfrak{A}_{2+}$, which is the topological sum of the Esakia spaces $\mathfrak{A}_{1+}$ and $\mathfrak{A}_{2+}$, is order-homeomorphic to $(\mathfrak{A}_1 \times \mathfrak{A}_2)_+$, the Esakia space dual to the direct product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.*

2. *The Heyting algebra $(\mathcal{X}_1 + \mathcal{X}_2)^+$, dual to the topological sum of $\mathcal{X}_1$ and $\mathcal{X}_2$, is isomorphic to the direct product $\mathcal{X}_1^+ \times \mathcal{X}_2^+$ of the Heyting algebras $\mathcal{X}_1^+$ and $\mathcal{X}_2^+$.*

Proof. (1) Define a map $f\colon W_{\mathfrak{A}_1} \cup W_{\mathfrak{A}_2} \to W_{\mathfrak{A}_1\times\mathfrak{A}_2}$ by letting, for every $\nabla_1 \in W_{\mathfrak{A}_1}$ and $\nabla_2 \in W_{\mathfrak{A}_2}$,

$$f(\nabla_1) = \{\langle a_1, a_2\rangle \in A_1 \times A_2 \mid a_1 \in \nabla_1, a_2 \in A_2\}$$

and

$$f(\nabla_1) = \{\langle a_1, a_2\rangle \in A_1 \times A_2 \mid a_1 \in A_1, a_2 \in \nabla_2\}.$$

Then it can be shown that f is a both a homeomorphism between the topological spaces $\langle W_{\mathfrak{A}_1} \cup W_{\mathfrak{A}_2}, \tau\rangle$ and $\langle W_{\mathfrak{A}_1\times\mathfrak{A}_2}, \tau_{\mathfrak{A}_1\times\mathfrak{A}_2}\rangle$ and a order-isomorphism between the posets $\langle W_{\mathfrak{A}_1} \cup W_{\mathfrak{A}_2}, R_{\mathfrak{A}_1} \cup R_{\mathfrak{A}_2}\rangle$ and $\langle W_{\mathfrak{A}_1\times\mathfrak{A}_2}, R_{\mathfrak{A}_1\times\mathfrak{A}_2}\rangle$. For a proof of the Lemma, cfr. [23, Theorem 8.76].
(2) Let $f\colon (\mathcal{X}_1 + \mathcal{X}_2)^+ \to \mathcal{X}_1^+ \times \mathcal{X}_2^+$ be defined as follows: for every clopen upset $U \subseteq X$,

$$f(U) = \langle U \cap X_1, U \cap X_2\rangle.$$

Notice that, since both $U \cap X_1$ and $U \cap X_2$ are clopen upset of X_1 and X_2 respectively, the function f is well defined. Clearly f is a bijection and since the operations in $\mathcal{X}_1^+ \times \mathcal{X}_2^+$ are defined componentwise, it can be easily shown that f is indeed a homomorphism. As an example, let us show that f preserves intersections. Let U, V be clopen upset of X, then

$$\begin{aligned} f(U \cap V) &= \langle U \cap V \cap X_1, U \cap V \cap X_2\rangle \\ &= \langle U \cap X_1, U \cap X_2\rangle \cap \langle V \cap X_1, V \cap X_2\rangle \\ &= f(U) \cap f(V). \end{aligned}$$

□

Notice that Lemma 2.32 (1) can not be extend to arbitrary infinite families of Heyting algebras. The reason why such an extension is not possible is the fact that the topological sum of an infinite family of compact spaces is not necessarily compact and therefore can not yield an Esakia space. However, we still get the following

Lemma 2.33. *Let $\{\mathcal{X}_i = \langle X_i, \tau_i, R_i\rangle \mid i \in I\}$ be a family of disjoint Esakia spaces and let $\sum_{i\in I} \mathcal{X}_i = \langle X, \tau, R\rangle$ be the topological sum of the $\mathcal{X}_i$'s. Then*

$$(\sum_{i\in I} \mathcal{X}_i)^+ \cong \prod_{i\in I} \mathcal{X}_i^+.$$

Proof. Define $f\colon (\sum_{i\in I} \mathcal{X}_i)^+ \to \prod_{i\in I} \mathcal{X}_i^+$ by letting, for every clopen upset U of X and each $i \in I$,

$$f(U)(i) = U \cap X_i.$$

Notice that f is well defined and bijective. Moreover it can be shown that f is a Heyting morphism by using the fact that the operations in $(\sum_{i\in I} \mathcal{X}_i)^+$ are computed componentwise. □

Lemma 2.34. *Let $\{\mathfrak{A}_i \mid i \in I\}$ be a family of finite Heyting algebras. Then*

$$\prod_{i\in I} \mathfrak{A}_i \cong (\sum_{i\in I} (\mathfrak{A}_i)_+)^+.$$

Proof. Since, for each $i \in I$, $\mathfrak{A}_i \cong ((\mathfrak{A}_i)_+)^+$, we have $\prod_{i\in I} \mathfrak{A}_i \cong \prod_{i\in I}((\mathfrak{A}_i)_+)^+$ and, by the previous lemma, $\prod_{i\in I}((\mathfrak{A}_i)_+)^+ \cong (\sum_{i\in I}(\mathfrak{A}_i)_+)^+$. □

Now we can recollect the results obtained in Lemmas 2.29, 2.30 and 2.32 in the following

Theorem 2.35. *Let I be a finite index set and let $\{\mathfrak{A}_i \mid i \in I\}$ be a finite family of Heyting algebras. Moreover let $\{\mathcal{X}_i \mid i \in I\}$ be a finite family of Esakia space.*

1. (i) *$\mathfrak{A}_i$ is a homomorphic image of $\mathfrak{A}_j$ iff $\mathfrak{A}_{i+}$ is isomorphic to a generated subspace of $\mathfrak{A}_{j+}$;*

(ii) $\mathfrak{A}_i$ *is a subalgebra of* $\mathfrak{A}_j$ *iff* $\mathfrak{A}_{i+}$ *is order-homeomorphic to a continuous p-morphic image of* $\mathfrak{A}_{j+}$*;*

(iii) $(\prod_{i\in I}\mathfrak{A}_i)_+$ *is order-homeomorphic to the topological sum* $\sum_{i\in I}\mathfrak{A}_{i+}$.

2. (i) $\mathcal{X}_i$ *is isomorphic to a generated subspace of* $\mathcal{X}_j$ *iff* $\mathcal{X}_i^+$ *is a homomorphic image of* $\mathcal{X}_j^+$*;*

(ii) $\mathcal{X}_i$ *is a continuous p-morphic image of* $\mathcal{X}_j$ *iff* $\mathcal{X}_i^+$ *is isomorphic to a subalgebra of* $\mathcal{X}_j^+$*;*

(iii) $(\sum_{i\in I}\mathcal{X}_i)^+$ *is isomorphic to* $\prod_{i\in I}\mathcal{X}_i^+$.

Subalgebras and Esakia equivalences

The previous theorem tells us that the subalgebras of a Heyting algebra $\mathfrak{A}$ correspond to the Esakia spaces which are continuous p-morphic images of the dual space $X = \langle X, \tau, R\rangle$ of $\mathfrak{A}$. However, there is another way to characterize the subalgebras of a given Heyting algebras by means of their dual Esakia spaces. Indeed, notice that given any onto Esakia morphism $h\colon \mathcal{X} \to \mathcal{Q} = \langle Q, \tau', S\rangle$, since h is a continuous map between compact Hausdorff spaces, it is closed and thus it is a *quotient map*. We can then recover an equivalence relation $\sim$ on X by letting, for all $x, y \in X$,

$$x \sim_h y \Longleftrightarrow h(x) = h(y).$$

Now, for any $x \in X$ and any $U \subseteq X$, denote by $[x]$ the $\sim_h$-equivalence class of x and by $[U]$ the set of the equivalence classes $[u]$ for $u \in U$, that is,

$$[x] = \{y \in X \mid x \sim_h y\} \qquad\qquad [U] = \{[u] \mid u \in U\}.$$

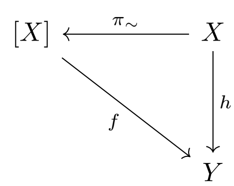
We can then consider the *quotient space* $[\mathcal{X}] = \langle [X], [\tau]\rangle$ where $[\tau]$ is the quotient topology defined as

$$[\tau] = \{[U] \subseteq [X] \mid \bigcup [U] \in \tau\}.$$

Notice that $[\tau]$ is the finest topology with respect to which the projection map $\pi_{\sim_h} : X \to [X]$, associating to each $x \in X$ its equivalence class $[x]$, is continuous. Now, the topological analogue of the first isomorphism theorems tells us that the topological space $\langle Q, \tau'\rangle$ is homeomorphic to the quotient space $\langle [X], [\tau]\rangle$. Indeed, the following universal property for quotients holds:
for any topological spaces $\langle X, \tau_X\rangle$, $\langle Y, \tau_Y\rangle$, and any equivalence relation $\sim$ on X, if $h\colon X \to Y$ is a continuous map such that, for all $a, b \in X$,

$$a \sim b \Longrightarrow h(a) = h(b),$$

then there exists a unique continuous map $f\colon [X] \to Y$ such that $h = f \circ \pi_\sim$, that is, the following diagram commutes:



Now, even if the quotient space of a compact space is always compact, the quotient of a Hausdorff space (and thus of a Stone space) need not to be Haudorff (or Stone). Therefore, in order to establish the duality between the subalgebras of a Heyting algebra and the quotients of its dual space, we need to characterize and further specify the equivalence relations on the dual space that actually give an Esakia space.

Reconsider the equivalence relation $\sim_h$ obtained from the Esakia morphism $h\colon \mathcal{X} \to \mathcal{Q}$ of the paragraphs above and call a subset $Y \subseteq X$ *$\sim_h$-saturated* if $Y = \bigcup[Y]$. Now, let $u, w \in X$ be such that uRw. Then, if $v \in [u]$, we have $h(v) = h(u)$. Since h is monotone, we then have $h(v)Sh(w)$ and, since h is a p-morphism, there exists $z \in X$ such that vRz and $h(z) = h(w)$. Hence $z \in [w]$ and $v \in [w]{\downarrow}$. Furthermore, if $w, v \in X$ are such that $\neg(w \sim_h v)$, then $h(w) \neq h(v)$. Therefore, either $\neg(h(w)Sh(v))$ or $\neg(h(v)Sh(w))$. Since $\mathcal{Q}$ is an Esakia space, by the Priestley separation axiom there exists a clopen upset V in Q such that either $h(w) \in V$ and $h(v) \notin V$ or $h(v) \in V$ and $h(w) \notin V$. Therefore, either $w \in h^{-1}(V)$ and $v \notin h^{-1}(V)$ or $v \in h^{-1}(V)$ and $w \notin h^{-1}(V)$ for $h^{-1}(V)$ a clopen upset of X. Moreover, notice that, given any $u \in h^{-1}(V)$, if $x \in [u]$, then $h(x) = h(u) \in V$ so that $x \in h^{-1}(V)$. Thus $\bigcup[h^{-1}(V)] = h^{-1}(V)$, that is, $h^{-1}(V)$ is $\sim_h$-saturated.

Therefore the equivalence relation $\sim_h$ satisfies the two following properties:

1. For every $w, v \in X$, wRv implies $[w] \subseteq [v]{\downarrow}$;
2. For every $w, v \in X$, if $\neg(w \sim_h v)$, then w and v are separated by an $\sim_h$-saturated clopen upset of X, i.e. there is a clopen upset $U \subseteq X$ such that $\bigcup[U] = U$ and either $w \in U$ and $v \notin U$ or $w \notin U$ and $v \in U$.

We will show that an equivalence relation E on X that satisfies the two above properties is exactly the equivalence relation we are looking for in order to establish our duality.

Definition 2.5. Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space. An equivalence relation E on X is called an *Esakia equivalence* or a *correct partition* on $\mathcal{X}$ if the following conditions hold:

1. For every $w, v \in X$, wRv implies $[w] \subseteq [v]{\downarrow}$;
2. For every $w, v \in X$, if $\neg(wEv)$, then w and v are separated by an *E-saturated* clopen upset of X[2].

Now, let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and let E be an Esakia equivalence on X. Consider the quotient space $\langle [X], [\tau]\rangle$ induced by E and define on $[X]$ a relation $[R]$ as follows:

$$[R] = \{\langle [x], [y]\rangle \mid \exists u, v \in X\, ([x] \ni uRv \in [y])\}.$$

Clearly $[R]$ is reflexive. Moreover, with the help of (2), it can be shown that $[R]$ is antisymmetric and finally one can show that $[R]$ is transitive by using property (1). Therefore $[R]$ is a partial ordering on $[X]$. We can thus consider the ordered topological space $[\mathcal{X}] = \langle [X], [\tau], [R]\rangle$ and we call such a space $[\mathcal{X}]$ the *Esakia quotient space* of $\mathcal{X}$ given by the Esakia equivalence E.

[2]If E satisfies only this second condition, then we say that E is a *Priestley equivalence.*

Lemma 2.36. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and E an equivalence relation on X. Then the Esakia quotient space $[\mathcal{X}] = \langle [X], [\tau], [R]\rangle$ is an Esakia space iff the relation E is an Esakia equivalence on $\mathcal{X}$.*

Proof. ($\Longrightarrow$) Suppose that the Esakia quotient $[\mathcal{X}]$ is an Esakia space. We show that E satisfies the properties of Definition 2.5.
(1) Let $w, v \in X$ be such that wRv. We have to show that $[w] \subseteq [v]{\downarrow}$, that is, if $u \in [w]$, then uRq for some $q \in [v]$. By contraposition, this is equivalent to:

$$\forall q \in X(q \in [v] \to \neg(uRq)) \Longrightarrow u \notin [w].$$

So let assume the antecedent holds and let $q \in [v]$. Then $\neg(uRq)$ and there exists a clopen upset U_q in X such that $u \in U_q$ and $q \in X \setminus U_q$. Therefore we have $[v] \subseteq \bigcup_{q\in[v]} X \setminus U_q$. Since $[X]$ is an Esakia space, $\{[v]\}$ is closed in $[X]$ and this implies that $[v]$ is a closed subset of X. Since X is compact, there exists a finite subcover of $\{X \setminus U_q \mid q \in [v]\}$, say $\{X \setminus U_{q_i} \mid i = 1, \dots, k\}$, that covers $[v]$. Hence $[v] \subseteq X \setminus U$ and $u \in U$ for $U = \bigcap_{i=1}^{k} U_{q_i}$, a clopen upset of X. Now, since by our hypothesis $w \in [v]{\downarrow}$ and $X \setminus U$ is a downset, $w \in X \setminus U$. But then, since $[U] \cap [X \setminus U] = \emptyset$, $[u] \in [U]$ and $[w] \in [X \setminus U]$, necessarily $[u] \neq [w]$, that is, $u \notin [w]$.
(2) Suppose that $\neg(wEv)$, that is, $[w] \neq [v]$. Hence either $\neg([w][R][v])$ or $\neg([v][R][w])$. Assume that the former holds. Then, by Priestley separation axiom, there is a clopen upset $[U] \subseteq [X]$ such that $[w] \in [U]$ and $[v] \notin [U]$. By definition of $[\tau]$, $\bigcup[U]$ and $\bigcup([X]\setminus[U]) = X\setminus\bigcup[U]$ are open in X, that is, $\bigcup[U]$ is a E-saturated clopen subset of X such that $w \in \bigcup[U]$ and $v \notin \bigcup[U]$. Moreover, if $x \in \bigcup[U]$ and xRy, then $x \in [u]$ for some $u \in U$. Therefore, $[x][R][y]$ and $[x] \in [U]$ and, since $[U]$ is an upset, we have $[y] \in [U]$. Consequently $y \in \bigcup[U]$, that is, $\bigcup[U]$ is a E-saturated clopen upset of X separating w and v.

($\Longleftarrow$) Suppose that E is an Esakia equivalence on $\mathcal{X}$. In order to see that $[\mathcal{X}]$ is indeed an Esakia space, first notice that $[X]$ is compact, being the image of a compact space under the continuous function π_E. Moreover, if $\neg([x][R][y])$, then $\neg(xEy)$. Since E is a correct partition, there exists a E-saturated clopen upset $U \subseteq X$ such that $x \in U$ and $y \notin U$. Therefore $[U]$ is a clopen set in $[X]$ such that $[x] \in [U]$ and $[y] \notin [U]$. Moreover, if $[u] \in [U]$ and $[u][R][v]$, then xRv for some $x \in [u], y \in [v]$. Since U is saturated, $x \in [u] \subseteq U$ and therefore $y \in U$, since U is an upset. Consequently, $[v] = [y] \in [U]$, that is, $[U]$ is a clopen upset of $[X]$ separating $[x]$ and $[y]$. Hence we have just shown that $[\mathcal{X}]$ is a Priestley space.
Finally, if $[U]$ is a clopen set in $[X]$, then $\bigcup[U]$ is a clopen in X. Then $(\bigcup[U]){\downarrow}$ is also clopen in X, since R is clopen. But it can be shown that the following equation holds:

$$(\bigcup[U]){\downarrow} = \bigcup([U][{\downarrow}]).$$

Indeed, if $x \in (\bigcup[U]){\downarrow}$, then xRs for some $q \in U$ such that $s \in [q]$. But then $[x][R][s]$ and $[s] \in [U]$, hence $[x] \in [U][{\downarrow}]$ and $x \in \bigcup([U][{\downarrow}])$. Conversely, if $x \in \bigcup([U][{\downarrow}])$, then there exist $[p] \in [X]$ and $[q] \in [V]$ such that $[p][R][q]$ and $x \in [p]$. By property (1) of E, it follows that $[p] \subseteq [q]{\downarrow}$. Therefore xRz for some $z \in [q]$ and so $x \in (\bigcup[U]){\downarrow}$. We then conclude that $[U][{\downarrow}]$ is clopen in $[X]$ by definition of the quotient topology $[\tau]$. This means that $[R]$ is a clopen relation and that $[\mathcal{X}]$ is an Esakia space. $\square$

We can now prove the following

Proposition 2.37. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space. There exists a one-to-one correspondence between the Esakia equivalences on $\mathcal{X}$ and the onto Esakia morphisms with $\mathcal{X}$ as domain*[3].

Proof. From the reasoning at the beginning of this section we have that if $h\colon \mathcal{X} \to \mathcal{Q}$ is a surjective Esakia morphism, then the relation E_h defined by

$$xE_h y \Longleftrightarrow h(x) = h(y),$$

for all $x, y \in X$, is an Esakia equivalence on $\mathcal{X}$.
Conversely, let E be an Esakia equivalence on $\mathcal{X}$ and consider the Esakia quotient space $[\mathcal{X}] = \langle [X], [\tau], [R]\rangle$. We show that the canonical projection $\pi_E\colon X \to [X]$ is a surjective Esakia morphism. By the previous lemma, π_E is a continuous map between Esakia spaces which is surjective. Thus we only need to show that π_E is a p-morphism. Clearly π_E is monotone by definition of $[R]$. So, suppose that $\pi_E(x)[R][y]$. Then $[x][R][y]$, which implies, by (1), $[x] \subseteq [y]\!\downarrow$. Therefore, since $x \in [x]$, we have that xRz for some $z \in [y]$, that is, there exists $z \in X$ such that xRz and $\pi_E(z) = [y]$. Thus $\pi_E\colon X \to [X]$ is a p-morphism and π_E is a well defined onto Esakia morphism.
Finally, by considering the universal property for quotients, it is easy to show that the mappings $h \mapsto E_h$ and $E \mapsto \pi_E$ are inverse to each other. □

Corollary 2.38. *Let $\mathfrak{A}$ be an Heyting algebra. Then there exists a one-to-one correspondence between the subalgebras of $\mathfrak{A}$ and the Esakia equivalence on $\mathfrak{A}_+$*[4].

2.5.5 Further examples of duality's correspondence

We will now use the duality's correspondences established in the previous section to further extend our list of dual notions.

Definition 2.6. Let $\mathcal{X} = \langle X, \tau, R\rangle$ be a Priestley space. We say that $\mathcal{X}$ is *extremally order-disconnected* if, for every open upset $U \subseteq X$, $\mathbf{D}(U)$ is clopen.

Notice that if $\mathcal{X}$ is an Esakia space, Corollary 2.26 tells us that $\mathcal{X}$ is extremally order-disconnected iff, for every open upset $U \subseteq X$, $\overline{U}$ is clopen.

Proposition 2.39. *Let $\mathfrak{A}$ be a Heyting algebra and $\mathcal{X} = \langle X, \tau, R\rangle$ its dual Esakia space. Then $\mathfrak{A}$ is complete iff $\mathcal{X}$ is extremally order-disconnected.*

Proof. ($\Longrightarrow$) Assume that $\mathfrak{A}$ is complete and let $U \subseteq X$ be an open upset. Then $\mathbf{J}(U) = U$ and thus $U = \bigcup\{\widehat{a} \mid \widehat{a} \subseteq S\}$, by Lemma 2.25. Let $B = \{a \in A \mid \widehat{a} \subseteq S\}$ and, since $\mathfrak{A}$ is complete, $\bigvee B$ exists in $\mathfrak{A}$. But then, by Theorem 2.27, $\overline{\bigcup_{b\in B} \widehat{b}} = \overline{U}$ is clopen in X. Hence $\mathcal{X}$ is extremally order-disconnected.
($\Longleftarrow$) Suppose that $\mathcal{X}$ is extremally order-disconnected and let $B \subseteq A$. Since $\bigcup_{b\in B} \widehat{b}$ is an open upset of X, it follows that $\overline{\bigcup_{b\in B} \widehat{b}}$ is clopen and thus, by Theorem 2.27, $\bigvee B$ exists in $\mathfrak{A}$. Since B was arbitrary, $\mathfrak{A}$ is complete. □

[3] The correspondence is modulo order-homeomorphic spaces in the codomain of h.
[4] Actually, such a one-to-one correspondence is a dual-isomorphism between the lattice of subalgebras of A and the lattice of correct partitions on $\mathfrak{A}_+$.

Now, recall that a Heyting algebra $\mathfrak{A}$ is called *completely join-prime generated* if $J^\infty(\mathfrak{A})$ is *join-dense* in $\mathfrak{A}$, that is, if every element $a \in A$ is the least upper bound of a (possibly empty) set of completely join-prime elements. Equivalently, $\mathfrak{A}$ is completely join-prime generated iff the following property holds:
for each $a, b \in A$,

$$a \nleq b \Longrightarrow \exists q \in J^\infty(\mathfrak{A})(q \leq a \;\&\; q \nleq b). \tag{*}$$

Definition 2.7. Let $\mathcal{X} = \langle X, \tau, R\rangle$ be a Priestley space. We say that an element $x \in X$ is an *order-isolated point* if both $x{\uparrow}$ and $x{\downarrow}$ are clopen. We denote by $X_{\mathrm{iso}\uparrow}$ the set of order-isolated point of X.

Clearly if $x \in X_{\mathrm{iso}\uparrow}$, then $\{x\} = x{\uparrow} \cap x{\downarrow}$ is clopen and thus $x \in X_{\mathrm{iso}}$. The converse, however, does not hold. Moreover, since for all $x \in X$, both $x{\uparrow}$ and $x{\downarrow}$ are closed in a Priestley space, x is order-isolated if both $x{\uparrow}$ and $x{\downarrow}$ are open. If, instead, $\mathcal{X} = \langle X, \tau, R\rangle$ is also an Esakia space and $x \in X_{\mathrm{iso}}$, then x is order-isolated iff $x{\uparrow}$ is (cl)open, formally,

$$X_{\mathrm{iso}\uparrow} = \{x \in X_{\mathrm{iso}} \mid x{\uparrow} \in \tau\}.$$

Proposition 2.40. *Let $\mathfrak{A}$ be a Heyting algebra and $\mathcal{X} = \langle X, \tau, R\rangle$ its dual Esakia space. Then $\mathfrak{A}$ is completely join-prime generated iff $X_{iso\uparrow}$ is dense in X.*

Proof. ($\Longrightarrow$) If $\mathfrak{A}$ is completely join-prime generated then condition (*) holds. We know that $\mathcal{B} = \{\widehat{a} \cap -\widehat{b} \mid a, b \in A\}$ is a basis for X. So let $Q \in \mathcal{B}$ be a non-empty basic element. Then $Q = \widehat{a} \cap -\widehat{b} \neq \emptyset$ for some $a, b \in A$. Thus $\widehat{a} \nsubseteq \widehat{b}$, that is, $a \nleq b$. By (*), there exists $q \in J^\infty(\mathfrak{A})$ such that $q \leq a$ and $q \nleq b$. Then by Theorem 2.28 (ii), there exists $x \in X_{\mathrm{iso}\uparrow}$ such that $x{\uparrow} = \widehat{q}$ and consequently $x \in \widehat{a}$ and $x \notin \widehat{b}$. Thus $X_{\mathrm{iso}\uparrow} \cap Q \neq \emptyset$ which implies that $X_{\mathrm{iso}\uparrow}$ is dense in X.
($\Longleftarrow$) Suppose that $X_{\mathrm{iso}\uparrow}$ is dense in X and let $a, b \in A$ be such that $a \nleq b$. Hence $\widehat{a} \nsubseteq \widehat{b}$ and consequently $\widehat{a} \cap -\widehat{b} \neq \emptyset$. Since $X_{\mathrm{iso}\uparrow}$ is dense, $X_{\mathrm{iso}\uparrow} \cap (\widehat{a} \cap -\widehat{b}) \neq \emptyset$. Let $x \in X_{\mathrm{iso}\uparrow} \cap (\widehat{a} \cap -\widehat{b})$. Since $x{\uparrow}$ is a clopen upset, it is of the form $\widehat{q}$ for some $q \in A$ and thus, by Theorem 2.28 (ii), $q \in J^\infty(\mathfrak{A})$. Furthermore, since $x \in \widehat{a}$ and $x \notin \widehat{b}$, $\widehat{q} \subseteq \widehat{a}$ and $\widehat{q} \nsubseteq \widehat{b}$, that is, $q \leq a$ and $q \nleq b$. Therefore (*) holds and we conclude that $\mathfrak{A}$ is completely join-prime generated. ☐

Given a Heyting algebra $\mathfrak{A}$, we say that $\mathfrak{A}$ is *well-connected* if, for every $a, b \in A$, $a \vee b = 1$ implies $a = 1$ or $b = 1$. Moreover, recall that by Theorem 1.54, $\mathfrak{A}$ is subdirectly irreducible iff it has a second greatest element. Clearly $\mathfrak{A}$ is well-connected iff 1 is join-irreducible, which, in the context of distributive lattice is equivalent to join-prime. Furthermore, we have the following

Lemma 2.41. *Let $\mathfrak{A}$ be a Heyting algebra. Then $\mathfrak{A}$ is subdirectly irreducible iff 1 is completely join-prime.*

Proof. If $\mathfrak{A}$ is subdirectly irreducible, then there exists a second greatest element $\omega \in A$. Now, let $B \subseteq A$ a non-empty subset such that $\bigvee B$ exists in $\mathfrak{A}$ and $1 \leq \bigvee B$. If $1 \notin B$, then $\bigvee B \leq \omega < 1$, contradiction. Hence $1 \in B$ and $1 \leq 1$. Thus 1 is completely join-prime. Conversely, if $\mathfrak{A}$ has not a second greatest element, then $\bigvee A \setminus \{1\} = 1$ but $1 \nleq a$ for all $a \in A \setminus \{1\}$, that is, 1 is not completely join-prime. ☐

Therefore, from Theorem 2.28, we immediately have

Proposition 2.42. *Let $\mathfrak{A}$ be a Heyting algebra and $\mathcal{X} = \langle X, \tau, R\rangle$ its dual Esakia space.*

(i) *$\mathfrak{A}$ is well-connected iff $X = x{\uparrow}$ for some $x \in X$;*

(ii) *$\mathfrak{A}$ is subdirectly irreducible iff $X = x{\uparrow}$ for some $x \in X_{iso}$.*

Remark 3. The previous proposition implies that the dual space $\mathfrak{A}_+$ of a given subdirectly irreducible Heyting algebra $\mathfrak{A}$ is rooted and, moreover, if we remove the root from the space, we still get a clopen upset.

The following example of duality correspondence is very instructive since it employs many of the previous established dualities. Let an Esakia space $\mathcal{X} = \langle X, \tau, R\rangle$ be given. Define the subset X_{fin} of X as follows:

$$X_{\text{fin}} = \{x \in X \mid x{\uparrow} \text{ is finite}\},$$

that is, X_{fin} is the union of all the finite upsets of X. The following proposition, as well as the previous one, have been first discovered by Esakia (cfr. [41, Appendix A]).

Proposition 2.43. *Let $\mathfrak{A}$ be a Heyting algebra and $\mathcal{X} = \langle X, \tau, R\rangle$ its dual Esakia space. Then $\mathfrak{A}$ is finitely approximable iff X_{fin} is dense in X.*

Proof. ($\Longrightarrow$) If $\mathfrak{A}$ is finitely approximable, then, by a classical result of universal algebra, we can assume that $\mathfrak{A}$ is a subdirect product of its finite homomorphic images. So, let $\{\mathfrak{A}_i\}_{i\in I}$ be the family of the finite homomorphic images of $\mathfrak{A}$ and let $\iota\colon \mathfrak{A} \to \prod_{i\in I} \mathfrak{A}_i$ be the inclusion embedding such that $\pi_j \circ \iota\colon \mathfrak{A} \to \mathfrak{A}_j$ is onto. Now consider the family $\{\mathcal{X}_i\}_{i\in I}$ of the dual Esakia spaces of the $\mathfrak{A}_i$'s. Then, for each $i \in I$, by Theorem 2.35 1(i) we can assume, wlog, that $\mathcal{X}_i$ is a finite generated subspace of $\mathcal{X}$, that is, each X_i is a finite upset of X. Hence $\bigcup_{i\in I} X_i \subseteq X_{\text{fin}}$. Now, consider a non-empty basic element $\widehat{a} \setminus \widehat{b}$ of X, where $a, b \in A$. Then $\widehat{a} \nsubseteq \widehat{b}$ and thus $a \nleq b$. Since ι is the inclusion map, $a \nleq b$ in $\prod_{i\in I} A_i$ and therefore there exists $j \in I$ such that $\pi_j(a) \nleq \pi_j(b)$. This means that $\widehat{a} \cap X_j \nsubseteq \widehat{b} \cap X_j$ and thus there exists $x \in (\widehat{a} \setminus \widehat{b}) \cap X_j \subseteq (\widehat{a} \setminus \widehat{b}) \cap \bigcup_{i\in I} X_i$. Therefore the set $\bigcup_{i\in I} X_i$ is dense in X, since it intersects every basic element, and consequently X_{fin} is also dense in X.
($\Longleftarrow$) Assume that X_{fin} is dense in X and let $\{X_i\}_{i\in I}$ be the family of the finite upsets of X. Clearly $X_{\text{fin}} = \bigcup_{i\in I} X_i$. Notice moreover that, for each $i \in I$, $X_i = \bigcup_{x\in X_i} \{x\}$ is closed, being a finite union of closed sets, and thus we can consider the generated subspaces $\mathcal{X}_i$ of $\mathcal{X}$. Now, consider, for each $i \in I$, the dual Heyting algebra $\mathcal{X}_i^+$ of $\mathcal{X}_i$ and denote it by $\mathfrak{A}_i$. Then, by Theorem 2.35, it follows that each $\mathcal{X}_i^+$ is a finite homomorphic image of $\mathcal{X}^+ \cong \mathfrak{A}$. Now define a function $h\colon X^{CU} \to \prod_{i\in I} X_i^{CU}$ by letting, for all clopen upset $U \in X^{CU}$,

$$h(U) = f_U \iff f_U(i) = U \cap X_i.$$

Now, it can be easily seen that h is indeed a homomorphism. For instance, $h(\emptyset) = f_\emptyset$ is the least element of $\prod_{i\in I} X_i^{CU}$ and, given two clopen upset U, V of

X and $i \in I$, we have

$$\begin{aligned} h(U \cap V)(i) &= f_{U\cap V}(i) \\ &= (U \cap V) \cap X_i \\ &= (U \cap X_i) \cap (V \cap X_i) \\ &= f_U(i) \cap f_V(i) \\ &= h(U)(i) \cap h(V)(i) \\ &= (h(U) \cap h(V))(i). \end{aligned}$$

Furthermore, for any $j \in I$, we have

$$\begin{aligned} (\pi_j \circ h)(X^{CU}) &= \{\pi_j(f_U) \,|\, U \in X^{CU}\} \\ &= \{f_U(j) \,|\, U \in X^{CU}\} \\ &= \{U \cap X_j \,|\, U \in X^{CU}\} \\ &= X_j^{CU} \end{aligned}$$

that is, every induced projection $\pi_j \circ h\colon X^{CU} \to X_j^C$ is onto. We now show that h is injective. So, let U, V be two distinct clopen upsets of X. Then $U \cap (X \setminus V)$ is a non-empty clopen of X and since $\bigcup_{i\in I} X_i$ is dense in X, $(U \cap (X \setminus V)) \cap \bigcup_{i\in I} X_i \neq \emptyset$. Therefore $(U \cap (X \setminus V)) \cap X_j \neq \emptyset$ for some $j \in I$ and consequently, $h(U) = U \cap X_j \neq V \cap X_j = h(V)$.
Therefore $\mathcal{X}^+ \cong \mathfrak{A}$ is a subdirect product of the family $\{\mathfrak{A}_i\}_{i\in I}$ of finite homomorphic images of $\mathcal{X}^+$, that is, $\mathfrak{A}$ is finitely approximable. □

Let us conclude this section with another easy correspondence.

Definition 2.8. Let $\mathfrak{A}$ be a Heyting algebra. A filter ∇ of $\mathfrak{A}$ is said to be of *finite index* if the quotient algebra $\mathfrak{A}/\nabla$ is finite.

Thus, for a filter ∇ of finite index, it follows that the corresponding closed upset $\chi(\nabla)$ in $\mathfrak{A}_+$ is finite and, conversely, if U is a closed upset of $\mathfrak{A}_+$, then $\chi^{-1}(U)$ is a filter of finite index.

Lemma 2.44. *Let $\mathfrak{A}$ be a Heyting algebra and $\mathcal{X} = \langle X, \tau, R\rangle$ its dual Esakia space. Then the following are equivalent:*

(i) $X_{fin} \subseteq X_{iso\uparrow}$;

(ii) *every filter ∇ of $\mathfrak{A}$ of finite index is principal.*

Proof. (i)$\Longrightarrow$(ii). Let ∇ be a filter of $\mathfrak{A}$ of finite index. Then $\chi(\nabla)$ is a finite closed upset in X. By Corollary 2.19, $\chi(\nabla) = min(\chi(\nabla))\!\uparrow = \bigcup_{x\in min(\chi(\nabla))} x\!\uparrow$ and since $\chi(\nabla)$ is finite, $min(\chi(\nabla))$ is finite as well. Moreover, $min(\chi(\nabla)) \subseteq X_{\text{fin}}$. By (i), it then follows that each $x \in min(\chi(\nabla))$ is an isolated point such that $x\!\uparrow$ is clopen. Hence $\chi(\nabla)$ is a clopen upset of X and, consequently, ∇ is a principal filter.
(ii)$\Longrightarrow$(i). Let $x \in X_{\text{fin}}$. Then $x\!\uparrow$ is a finite closed upset of X. Then, the corresponding filter $\chi^{-1}(x\!\uparrow)$ is of finite index and therefore principal by (ii). Hence $x\!\uparrow$ is a finite clopen upset of X. By finiteness of $x\!\uparrow$, it follows that x is an isolated point of X and thus $x \in X_{\text{iso}\uparrow}$. □

Here is the final list of the dualities correspondences established so far.

$\mathcal{HA}$	$\mathcal{ES}$
Heyting algebra	Esakia space
homomorphism	continuous p-morphism
filter	closed upset
ideal	open upset
principal filter / ideal	clopen upset
prime filter ∇	principal upset $\nabla\uparrow$
ultrafilter ∇	$\{\nabla\}$, ∇ maximal element
finite index filter	finite closed upset
congruence	closed upset
homomorphic image	generated subspace
subalgebra	Esakia quotient
direct product	topological sum
complete algebra	e.o.d. Esakia space
completely join-prime generated algebra	$X_{\mathrm{iso}\uparrow}$ dense
well-connected algebra	rooted space
subdirectly irreducible algebra	rooted space with isolated root
finitely approximable algebra	X_{fin} dense

Table 2.1: *Esakia Duality's Correspondences*

2.6 Descriptive frames and models

In this section we introduce the notion of descriptive frame, which is nothing but a generalization of the notion of Kripke frame, and show the tight connection existing between descriptive frame and Esakia space.

Definition 2.9. An *intuitionistic general frame* is a triple $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$, where $\langle W, R\rangle$ is an intuitionistic Kripke frame and $\mathcal{P}$, the set of *admissible sets* of $\mathfrak{F}$, is a subset of $\mathrm{Up}(W)$ such that

- $\emptyset \in \mathcal{P}$;
- $W \in \mathcal{P}$;
- $\mathcal{P}$ is closed under $\cap, \cup$ and the following operation $\supset$:

$$\begin{aligned} U \supset V :&= \{x \in W \mid \forall y \in W(xRy \wedge y \in U \Rightarrow y \in V\} \\ &= W \setminus (U \setminus V)\downarrow, \end{aligned}$$

  for every $U, V \subseteq W$.

Therefore, every Kripke frame $\mathfrak{F} = \langle W, R\rangle$ can be seen as a general frame where all the upsets are admissible, that is, with $\mathcal{P}$ equal to $\mathrm{Up}(W)$.

Notice that, given a general frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$, the set $\mathcal{P}$ can be seen as a particular Alexandrov topology; furthermore, it should be noted that considering a Kripke frame $\mathfrak{F} = \langle W, R\rangle$ as a general frame is nothing more than endowing W with the Alexandrov 's topology. Hence, one could wonder whether there exists a connection between general frames and Esakia spaces. The answer is clearly positive but in order to spell out such a connection we need the following

Definition 2.10. Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a general frame.

- $\mathfrak{F}$ is said to be *refined* if, for every $u, v \in W$, $\neg(uRv)$ implies that there exists a $U \in \mathcal{P}$ such that $u \in U$ and $v \notin U$.
- $\mathfrak{F}$ is said to be *compact* if, for every $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq -\mathcal{P} = \{W\backslash U \mid U \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property, then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.
- $\mathfrak{F}$ is said to be *descriptive* if $\mathfrak{F}$ is both refined and compact.

It should be evident that the refinedness condition on a given general frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ is a sort of Priestley separation axiom with respect to the class $\mathcal{P}$ of the admissible sets, while the compactness condition on $\mathfrak{F}$ is the requirement that the class $\mathcal{P} \cup -\mathcal{P}$ is compact in the topological sense. With this insights, the reader should be able to follow easily the following correspondence between descriptive general frame and Esakia spaces.

Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space. Then let $\mathcal{P}_{\mathcal{X}}$ be the set of clopen upsets of X. It is clear that $\mathfrak{F}_{\mathcal{X}} = \langle X, R, \mathcal{P}_{\mathcal{X}}\rangle$ is a descriptive frame. Conversely, given any descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$, consider the set $\mathcal{P} \cup -\mathcal{P}$. Then defined a topology $\tau_{\mathcal{P}}$ on W by declaring $\mathcal{P} \cup -\mathcal{P}$ as a subbasis. Then one can show that $\mathcal{X}_{\mathfrak{F}} = \langle W, \tau_{\mathcal{P}}, R\rangle$ is an Esakia space and that every clopen of W is a finite union of finite intersections of elements of $\mathcal{P} \cup -\mathcal{P}$. This means that the clopen upsets of W are exactly the elements of $\mathcal{P}$[5].

Clearly, this correspondence also holds for the Esakia spaces which are dual to Heyting algebras. In particular, given a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$, the dual $\mathfrak{F}^+$ is the Heyting algebra $\langle \mathcal{P}, \cup, \cap, \supset, \emptyset\rangle$, which we know is the Heyting algebra of the clopen upset of the Esakia space corresponding to $\mathfrak{F}$. Conversely, given a Heyting algebra $\mathfrak{A}$, its dual descriptive frame is the frame $\mathfrak{A}_+ = \langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, \mathcal{P}_{\mathfrak{A}}\rangle$, where $\mathcal{P}_{\mathfrak{A}} = \{\widehat{a} \mid a \in A\}$, which, as we have seen, coincides with the set of clopen upset of the Esakia space $\mathfrak{A}_+$.

Therefore, the following duality holds:

Theorem 2.45. *Let $\mathfrak{A}$ be a Heyting algebra and let $\mathfrak{F}$ be a descriptive frame. Then*

$$(\mathfrak{A}_+)^+ \cong \mathfrak{A} \qquad \textit{and} \qquad (\mathfrak{F}^+)_+ \cong \mathfrak{F}.$$

It should be remarked that, for a general frame $\mathfrak{F}$, the equivalence $(\mathfrak{F}^+)_+ \cong \mathfrak{F}$ does not hold unless the frame is descriptive. Therefore, since our main interest lies in Heyting algebras and we want to take advantages of the established dualities, we will mostly be concerned with descriptive frames. Hence in what follows it may happen that we use the word "frame" as a synonym of "descriptive frame".

Definition 2.11. Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a descriptive frame. We call a map $\mathfrak{V}\colon \mathbf{Var}\mathcal{L} \to \mathcal{P}$ a *descriptive valuation* and the pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ a *descriptive model.*

[5]Thus $\tau_{\mathcal{P}}$ is the *patch topology* of the topology given by $\mathcal{P}$.

The notions of a $\mathcal{L}$-formula φ "being true at a point x in a descriptive model $\mathfrak{M}$" and all the related definitions concerning truth for a descriptive model (and frame) (cfr. Definition 1.12) are defined exactly in the same way as for Kripke models (and frame): just replace everywhere the word "Kripke model" or "Kripke frame" with "descriptive model" or "descriptive frame", respectively.

2.6.1 Truth-preserving operations on general frames

In the next subsections we generalize the truth preserving operations defined on Kripke frames in order to be apt for descriptive frames. For a (descriptive) frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$, we denote by $\boldsymbol{\kappa}\mathfrak{F}$ the Kripke frame $\langle W, R\rangle$.

Generated subframes, p-morphism and disjoint unions

Definition 2.12. Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ be (descriptive) frames. Let also $\{\mathfrak{F}_i = \langle W_i, R_i, \mathcal{P}_i\rangle\}_{i\in I}$, where $I = \{1, \dots, k\}$, be a finite family of (descriptive) frames.

(i) $\mathfrak{G}$ is said to be a *generated subframe* of $\mathfrak{F}$, and we write $\mathfrak{G} \subsetneq \mathfrak{F}$, if $\boldsymbol{\kappa}\mathfrak{G} \subsetneq \boldsymbol{\kappa}\mathfrak{F}$ and $\mathcal{Q} = \{U \cap V \mid U \in \mathcal{P}\}$.

(ii) A function $f\colon W \to V$ is said to be a *p-morphism* between $\mathfrak{F}$ and $\mathfrak{G}$ if f is a p-morphism between $\boldsymbol{\kappa}\mathfrak{F}$ and $\boldsymbol{\kappa}\mathfrak{F}$ and, moreover, for every $U \in \mathcal{Q}$, $f^{-1}(U) \in \mathcal{P}$.

(iii) The *disjoint union of the family* $\{\mathfrak{F}_i\}_{i\in I}$ is the (descriptive) frame defined as follows: $\biguplus_{i\in I} \mathfrak{F}_i := \langle W', R', \mathcal{P}'\rangle$, where $W' = \biguplus_{i\in I} W_i$, $R' = \bigcup_{i\in I} R_i$ and $\mathcal{P}' = \{\bigcup_{i\in I} U_i \mid U_i \in \mathcal{P}_i \text{ for all } i \in I\}$.

The notions of generated submodels, p-morphisms between descriptive models as well as the notion of a finite disjoint union of (descriptive) models are defined in the same way as in the case of Kripke models. Therefore the analogues of Theorems 1.8, 1.10 and 1.14 hold for (descriptive) frame as well, that is, we have the following

Theorem 2.46. *Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ be (descriptive) models. Let also $\{\mathfrak{M}_i = \langle \mathfrak{F}_i, \mathfrak{V}_i\rangle\}_{i\in I}$, where $I = \{1, \dots, k\}$, be a finite family of (descriptive) models.*

1. *If $\mathfrak{N} \subsetneq \mathfrak{M}$, then, for every $\varphi \in \mathbf{For}\mathcal{L}$ and every $x \in \mathfrak{G}$,*

$$(\mathfrak{N}, x) \models \varphi \iff (\mathfrak{M}, x) \models \varphi.$$

2. *If $\mathfrak{N}$ is a p-morphic image of the model $\mathfrak{M}$ via the map f, then, for every $x \in \mathfrak{F}$ and every $\varphi \in \mathbf{For}\mathcal{L}$,*

$$(\mathfrak{M}, x) \models \varphi \iff (\mathfrak{N}, f(x)) \models \varphi.$$

3. *If $\biguplus_{i\in I} \mathfrak{M}_i$ is the disjoint union of the family of models $\{\mathfrak{M}_i \mid i \in I\}$, then, for each $i \in I$, every $x \in \mathfrak{F}_i$ and every formula $\varphi \in \mathbf{For}\mathcal{L}$,*

$$(\biguplus_{i\in I} \mathfrak{M}_i, x) \models \varphi \iff (\mathfrak{M}_i, x) \models \varphi.$$

Finally, taking into account the correspondence established between descriptive frames and Esakia spaces, we have the following analogous of Theorem 2.35:

Theorem 2.47. *Let I be a finite index set and let $\{\mathfrak{A}_i \,|\, i \in I\}$ be a finite family of Heyting algebras. Moreover let $\{\mathfrak{F}_i \,|\, i \in I\}$ be a finite family of descriptive frames.*

1. (i) *$\mathfrak{A}_i$ is a homomorphic image of $\mathfrak{A}_j$ iff $\mathfrak{A}_{i+}$ is isomorphic to a generated subframe of $\mathfrak{A}_{j+}$;*
   (ii) *$\mathfrak{A}_i$ is a subalgebra of $\mathfrak{A}_j$ iff $\mathfrak{A}_{i+}$ is isomorphic to a p-morphic image of $\mathfrak{A}_{j+}$;*
   (iii) *$(\prod_{i\in I} \mathfrak{A}_i)_+$ is isomorphic to the disjoint union $\sum_{i\in I} \mathfrak{A}_{i+}$.*
2. (i) *$\mathfrak{F}_i$ is isomorphic to a generated subframe of $\mathfrak{F}_j$ iff $\mathfrak{F}_i^+$ is a homomorphic image of $\mathfrak{F}_j^+$;*
   (ii) *$\mathfrak{F}_i$ is a p-morphic image of $\mathfrak{F}_j$ iff $\mathfrak{F}_i^+$ is isomorphic to a subalgebra of $\mathfrak{F}_j^+$;*
   (iii) *$(\sum_{i\in I} \mathfrak{F}_i)^+$ is isomorphic to $\prod_{i\in I} \mathfrak{F}_i^+$.*

Descriptive congruences

Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a descriptive frame and $\sim$ an equivalence relation on W. Denote by $[x]$ the equivalence class generated by x under $\sim$ and let $[X]$ be the set $\{[x] \mid x \in X\}$.

Definition 2.13. Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a descriptive frame. An equivalence relation $\sim$ on W is said to be a *descriptive congruence* on $\mathfrak{F}$ if the following hold:

1. $xRy \Longrightarrow [x] \subseteq [y]{\downarrow}$, for every $x, y \in W$;
2. for every $w, v \in W$, if $\neg(w \sim v)$, then w and v are separated by an *E-saturated* admissible upset, i.e. there is $U \in \mathcal{P}$ such that $\bigcup[U] = U$ and either $w \in U$ and $v \notin U$ or $w \notin U$ and $v \in U$.

Given a congruence relation $\sim$ on a descriptive frame $\mathfrak{F}$, we define the frame $[\mathfrak{F}] = \langle [W], [R], [\mathcal{P}]\rangle$, where

$$[R] = \{\langle [x], [y]\rangle \mid \exists u, v \in W\,([x] \ni uRv \in [y])\}$$

and

$$[\mathcal{P}] = \{[U] \,|\, \bigcup[U] = U \in \mathcal{P}\}.$$

The frame $[\mathfrak{F}]$ is called the *descriptive quotient frame* of $\mathfrak{F}$ under $\sim$[6]. Furthermore, if $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is a descriptive model, then $[\mathfrak{M}] = \langle [\mathfrak{F}], [\mathfrak{V}]\rangle$, where $[\mathfrak{V}](p) = [\mathfrak{V}(p)]$, for every variable p, is called the *descriptive quotient model* of $\mathfrak{M}$ under $\sim$.

Theorem 2.48. *Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ be descriptive frames.*

[6]The fact that $[\mathfrak{F}]$ is indeed a descriptive frame follows from the definition of $\sim$ and the fact that $[P]$ is closed under the intuitionistic operation. In fact, one can easily show that the following equalities hold: $[U \odot V] = [U] \odot [V]$ for $\odot \in \{\cap, \cup, \supset\}$.

(i) *If $\sim$ is a descriptive congruence on the frame $\mathfrak{F}$, then the canonical map $f_\sim : W \to [W]$ defined by*

$$f_\sim(w) = [w]$$

is a p-morphism from $\mathfrak{F}$ onto $[\mathfrak{F}]$ and of $\mathfrak{M}$ onto $[\mathfrak{M}]$.

(ii) *If f is a surjective p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$, then the relation $\sim_f$ on W defined by*

$$w \sim_f v \Leftrightarrow f(w) = f(v)$$

is a descriptive congruence on $\mathfrak{F}$ and $[\mathfrak{F}]$ is isomorphic to $\mathfrak{G}$ via the isomorphism $[x] \mapsto f(x)$.

Theorem 2.48 tells us that there exists a one-to-one correspondence between descriptive congruences on a descriptive frame $\mathfrak{F}$ and its descriptive p-morphic images. This is clearly not a surprise, since it is evident that a descriptive congruence on a descriptive frame $\mathfrak{F}$ corresponds to an Esakia equivalence on the corresponding Esakia space. Therefore, we have the following analogous of Corollary 2.38

Corollary 2.49. *Let $\mathfrak{A}$ be an Heyting algebra. There exists a one-to-one correspondence between the subalgebras of $\mathfrak{A}$ and the descriptive congruences on $\mathfrak{A}_+$.*

The notion of descriptive congruence is particularly fruitful because it enables us to define the limit of an infinite chain of p-morphisms. For every $i \in \omega$, let f_i be a p-morphism from $\mathfrak{F}_i = \langle W_i, R_i, \mathcal{P}_i \rangle$ onto $\mathfrak{F}_{i+1} = \langle W_{i+1}, R_{i+1}, \mathcal{P}_{i+1} \rangle$. Now, the composition $g_i = f_{i-1} \circ f_{i-2} \circ \ldots \circ f_0$ is a p-morphism from $\mathfrak{F}_0$ onto $\mathfrak{F}_i$ and we can consider the descriptive congruence $\sim_i$ on $\mathfrak{F}_0$ corresponding to g_i. Let $\mathcal{Q}_i = \{g_i^{-1}(U) \mid U \in \mathcal{P}_i\}$. Then we know that $[\mathcal{P}_0]_{\sim_i}$ is equal to the set

$$[\mathcal{Q}_i]_{\sim_i} = \{[g_i^{-1}(U)]_{\sim_i} \mid U \in \mathcal{P}_i\} = \{[U]_{\sim_i} \mid \bigcup [U]_{\sim_i} = U \in \mathcal{P}_0\}.$$

Moreover, for every $i \in \omega$, we have $\sim_i \subseteq \sim_{i+1}$ and one can easily verify that $\sim := \bigcup_{i \in \omega} \sim_i$ is again a descriptive congruence on $\mathfrak{F}_0$. Then we define the *limit* of the chain of p-morphism $(f_i)_{i \in \omega}$ as the p-morphism $f_\sim$ from the frame $\mathfrak{F}_0$ onto the quotient frame $[\mathfrak{F}_0]$ under $\sim$ defined by $f_\sim(x) = [x]$. Furthermore, if we let $\mathcal{Q} = \bigcap_{i \in \omega} \mathcal{Q}_i$, then it is possible to show that

$$[\mathcal{P}_0]_\sim = [\mathcal{Q}]_\sim = \{[U] \mid U \in \mathcal{Q}\}.$$

Clearly, the definition extends naturally to chains of p-morphism between descriptive models: in this case, $f_\sim$ is a p-morphism between $\mathfrak{M}_0 = \langle \mathfrak{F}_0, \mathfrak{V}_0 \rangle$ onto the quotient model $[\mathfrak{M}] = \langle [\mathfrak{F}_0], [\mathfrak{V}_0] \rangle$.

2.6.2 Sums of Heyting algebras and descriptive frames

Let us briefly recall a useful operation on Heyting algebras as well as the corresponding dual operation on descriptive frames, that is, the *vertical sum* of Heyting algebras and the *linear sum* of descriptive frames.

Definition 2.14. Let $\mathfrak{F}_1 = \langle W_1, R_1, \mathcal{P}_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, \mathcal{P}_2\rangle$ be descriptive frames. The *linear sum* of $\mathfrak{F}_1$ and $\mathfrak{F}_1$ is the descriptive frame $\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \langle W, R, \mathcal{P}\rangle$ defined as follows:

$$\begin{aligned} W =& W_1 \uplus W_2, \\ R =& R_1 \cup R_2 \cup (W_2 \times W_1), \\ \mathcal{P} =& \{U \mid U \in \mathcal{P}_1 \text{ or } U = W_1 \cup Q, \text{ where } Q \in \mathcal{P}_2.\} \end{aligned}$$

The visual representation of the operation $\oplus$ is very simple: indeed $\oplus$ just puts $\kappa\mathfrak{F}_1$ on top of $\kappa\mathfrak{F}_2$. Furthermore, by considering the corresponding Esakia spaces, we have that the topology $\tau_{\mathcal{P}}$ of $\mathcal{X}_{\mathfrak{F}}$ is given by the topological sums of the topologies $\tau_{\mathcal{P}_1}$ and $\tau_{\mathcal{P}_2}$ of $\mathcal{X}_{\mathfrak{F}_1}$ and $\mathcal{X}_{\mathfrak{F}_2}$ respectively.

Now let us introduce the dual operation on Heyting algebras.

Definition 2.15. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Heyting algebras. The *vertical sum* of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is the Heyting algebra $\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2$ given by the linear sum of $\mathfrak{A}_2$ and $\mathfrak{A}_1$, considered as partially ordered sets, by identifying $1_{\mathfrak{A}_1}$ with $0_{\mathfrak{A}_2}$.

The fact that $\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2$ is indeed a Heyting algebras was notice by Troelstra in [152], where the operation $\overline{\oplus}$ was first introduced[7]. Figuratively speaking, notice that the operation $\overline{\oplus}$ puts the Heyting algebra $\mathfrak{A}_2$ on top of $\mathfrak{A}_1$ identifying the greatest element of $\mathfrak{A}_1$ with the least element of $\mathfrak{A}_2$.

The following theorem shows that the operations of vertical and linear sum are dual to each other:

Theorem 2.50. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Heyting algebras and let $\mathfrak{F}_1 = \langle W_1, R_1, \mathcal{P}_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, \mathcal{P}_2\rangle$ be descriptive frames. Then the following hold:*

(i) $(\mathfrak{F}_1 \oplus \mathfrak{F}_2)^+ \cong \mathfrak{F}_1^+ \overline{\oplus} \mathfrak{F}_2^+$;

(ii) $(\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2)_+ \cong (\mathfrak{A}_1)_+ \oplus (\mathfrak{A}_2)_+$.

Proof. We limit ourselves to defining the corresponding isomorphisms, leaving it to the reader to prove that the defined functions are really such.
(i) Define $h\colon (\mathfrak{F}_1 \oplus \mathfrak{F}_2)^+ \to \mathfrak{F}_1^+ \overline{\oplus} \mathfrak{F}_2^+$ by letting

$$h(u) = \begin{cases} U & \text{if } U \subseteq W_1 \\ U \cap W_2 & \text{otherwise} \end{cases}$$

for every admissible set U of $\mathfrak{F}_1 \oplus \mathfrak{F}_2$.
(ii) Define $f\colon (\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2)_+ \to (\mathfrak{A}_1)_+ \oplus (\mathfrak{A}_2)_+$ by letting

$$f(\nabla) = \begin{cases} \nabla & \text{if } \nabla \subsetneq A_2 \\ \nabla \cap A_1 & \text{otherwise} \end{cases}$$

for every prime filter ∇ of $(\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2)_+$. □

[7] The operation $\overline{\oplus}$ just introduced is a special case of another operation $\boxplus$, defined on the class of lattices, introduced by Wroński in [166] and defined as follows: for lattices $\mathfrak{A} = \langle A, \leq_A\rangle$ and $\mathfrak{B} = \langle B, \leq_B\rangle$ such that $A \cap B$ is a filter in $\mathfrak{A}$ and an ideal in $\mathfrak{B}$ and such that the orderings $\leq_A$ and $\leq_B$ coincide on $A \cap B$, let $\mathfrak{A} \boxplus \mathfrak{B} = \langle A \cup B, \leq_A \cup \leq_B, \leq_A \circ \leq_B\rangle$. An interesting results of Kotas and Wojtylak [92] states that, for every finite distributive lattice $\mathfrak{D}$, there exists a finite family $\{\mathfrak{B}_i\}_{i\in I}$ of Boolean algebras such that $\mathfrak{D} = \boxplus_{i\in I}\mathfrak{B}_i$.

Chapter 3

Fundamental Esakia spaces

In this chapter, we are going to take into consideration three fundamental classes of Esakia spaces which are of the outmost importance for the present work. It is worth mentioning that Section 3.1 is essentially based on Nick Bezhanishvili's Ph.D. thesis [14], in particular the 3$^{\text{rd}}$ chapter, Section 3.2 is based on the technical report [21] of Carsten Butz and, finally, the main sources for Section 3.3 are certainly [62], [64] and [69].

3.1 Finitely generated Heyting algebras

We start this section recalling from [31] two lemmas which will prove to be very useful in order to check whether there exists a p-morphism between two finite rooted Kripke frames.

Lemma 3.1. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a descriptive frame and $w, v \in W$*

1. *Let $R(w) \setminus \{w\} = R(v)$, i.e. v is the only immediate successor of w, and let E be the smallest equivalence relation on $\mathfrak{F}$ that identifies w and v:*

$$E = \{(u, u) \mid u \in W\} \cup \{(wEv), (vEw)\}.$$

   Then E is a descriptive congruence and we call the corresponding map $f_E \colon W \to [W]$ an α-reduction.

2. *Let $R(w) \setminus \{w\} = R(v) \setminus \{v\}$, i.e. the set of immediate successors of w and v coincide, and let E be the smallest equivalence relation that identifies w and v. The E is a descriptive congruence and we call the corresponding map $f_E \colon W \to [W]$ a β-reduction.*

Now, recall that a map $f \colon W \to V$ is said to be *proper* if there exist distinct $u, v \in W$ such that $f(u) = f(v)$. We have the following

Lemma 3.2. *Let $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$ be finite Kripke frames. Suppose $f \colon W \to V$ is a proper p-morphism. Then there exists a sequence $f_1, \ldots, f_n$ of α- or β-reductions such that $f = f_1 \circ \cdots \circ f_n$.*

Proof. Let f be a proper p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$. Since f is proper, the set $Q = \{v \in V \mid \exists w, u \in W (w \neq u \wedge f(w) = v = f(u))\}$ is non-empty. Let v be a maximal point in Q and let $w, u \in max(f^{-1}(v))$. Since f is a p-morphism, $R(w) \setminus \{w, u\} = R(u) \setminus \{w, u\}$. Then we have the two following possibilities:

(α) w is the unique successor of u or u is the unique successor of w, i.e. $R(u) \setminus \{u\} = R(w)$ or $R(w) \setminus \{w\} = R(u)$. Let E be the smallest equivalence relation identifying w and u. By Lemma 3.1, $f_E \colon W \to [W]$ is an α-reduction from $\mathfrak{F}$ onto $[\mathfrak{F}] = \langle [W], [R] \rangle$. Now let $h \colon [W] \to V$ be defined as follows:

$$h([x]) = f(x)$$

for every $[x] \in [W]$. Then it can be easily shown that h is a p-morphism from $[\mathfrak{F}]$ onto $\mathfrak{G}$ and $h \circ f_E = f$.

(β) w and u are incomparable. Therefore $R(w) \setminus \{w\} = R(u) \setminus \{u\}$ and, by letting E be the smallest equivalence relation identifying w and u, Lemma 3.1 tells us that $f_E \colon W \to [W]$ is a β-reduction from $\mathfrak{F}$ onto $[\mathfrak{F}]$. As above, define $h \colon [W] \to V$ by

$$h([x]) = f(x)$$

for every $[x] \in [W]$. Then h is a p-morphism from $[\mathfrak{F}]$ onto $\mathfrak{G}$ and moreover $h \circ f_E = f$.

Now, if h is not proper, h is injective and thus the frames $[\mathfrak{F}]$ and $\mathfrak{G}$ are isomorphic and f itself is an α- or β-reduction. If h is proper, then repeat the above proof for h. Since $\mathfrak{F}$ and $\mathfrak{G}$ are finite frames, the process will eventually end and we get a sequence $f_1, \ldots, f_n$ of α- or β-reductions such that $f = f_1 \circ \cdots \circ f_n$. $\square$

Definition 3.1. Let $\mathfrak{F}$ be a descriptive frame. $\mathfrak{F}$ is said to be *κ-generated* if $\mathfrak{F}^+$ is κ-generated. $\mathfrak{F}$ is said to be *finitely generated* if $\mathfrak{F}^+$ is finitely generated. Moreover, the generators of $\mathfrak{F}^+$ will be regarded as the *generators* of $\mathfrak{F}$ as well.

Let $\mathfrak{A}$ be a Heyting algebra and $\mathfrak{F}$ its corresponding dual descriptive frame. From the previous definition, it follows that $\mathfrak{F}$ is κ-generated iff $\mathfrak{A}$ is κ-generated. Now, for each $n \in \omega$, let $\mathbf{Var}\mathcal{L}_n$ be the set $\{p_1, \ldots, p_n\}$ of propositional variables. Fix elements $g_1, \ldots, g_n$ of $\mathfrak{A}$ and consider a valuation $v \colon \mathbf{Var}\mathcal{L}_n \to \mathfrak{A}$ such that $v(p_i) = g_i$ for every $i \in \{1, \ldots, n\}$. From now on we will not distinguish between a Heyting algebra $\mathfrak{A}$ with some fixed elements $g_1, \ldots, g_n$ and the pair $\langle \mathfrak{A}, v \rangle$ given by $\mathfrak{A}$ with a valuation v defined as above. Since $\mathfrak{A} \cong \mathfrak{F}^+$, let $G_1, \ldots, G_n$ be the elements of $\mathfrak{F}^+$ corresponding to the g_i's and let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be the descriptive model where $\mathfrak{V}(p_i) = G_i$ for all $i \in \{1, \ldots, n\}$. We call $\mathfrak{M}$ the dual descriptive model corresponding to $\langle \mathfrak{A}, v \rangle$. Furthermore, if the set $\{g_1, \ldots, g_n\} \subseteq \mathfrak{A}$ corresponds with the set of generators of $\mathfrak{A}$, then we call v the standard valuation and the corresponding dual descriptive model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ of $\langle \mathfrak{A}, v \rangle$ the *standard model* on $\mathfrak{F}$.

Definition 3.2. Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be the dual descriptive model of $\langle \mathfrak{A}, v \rangle$. We associate, to each $w \in \mathfrak{F}$, a sequence $\langle c_1^w, \ldots, c_n^w \rangle$ such that, for $j \in \{1, \ldots, n\}$,

$$c_j^w = \begin{cases} 0 & \text{if } w \notin \mathfrak{V}(p_j) \\ 1 & \text{if } w \in \mathfrak{V}(p_j) \end{cases}$$

The sequence $\langle c_1^w, \ldots, c_n^w \rangle$ associated to w is called the *colour* of w and we denote it with $col(w)$.

Theorem 3.3 (Colouring Theorem, [42])**.** *Let $\mathfrak{A}$ be a Heyting algebra and let $g_1, \ldots, g_n$ be fixed elements of $\mathfrak{A}$. Moreover, let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be its dual descriptive model. Then the following are equivalent:*

1. *$\mathfrak{A}$ is generated by $g_1, \dots, g_n$;*

2. *for every proper p-morphism f from $\mathfrak{F}$ onto $\mathfrak{F}'$, there exist points u and v in $\mathfrak{F}$ such that $f(u) = f(v)$ and $col(u) \neq col(v)$;*

3. *for every proper descriptive congruence $\sim$ on $\mathfrak{F}$, there exists a $\sim$-equivalence class containing points of different colours.*

Proof. The equivalence between (2) and (3) follows from Theorem 2.48, therefore it suffices to show the equivalence between (1) and (3).
(1)⇒(3) Let $\mathfrak{A}$ be generated by $g_1, \dots, g_n$ and let $\sim$ be a proper descriptive congruence of $\mathfrak{F}$. Consider the Heyting algebra $\mathfrak{A}_\sim$ corresponding to $\sim$, that is, $\mathfrak{A}_\sim$ is the Heyting algebra of all the $\sim$-saturated admissible subset of $\mathfrak{F}$. Since $\sim$ is proper, $\mathfrak{A}_\sim$ is a proper subalgebra of $\mathfrak{A}$ and therefore there is $i \leq n$ such that $g_i \notin \mathfrak{A}_\sim$. Hence, $\mathfrak{V}(p_i)$ is not $\sim$-saturated, that is $\bigcup[\mathfrak{V}(p_i)] \nsubseteq \mathfrak{V}(p_i)$. Hence there is $u \in \bigcup[\mathfrak{V}(p_i)]$ such that $u \notin \mathfrak{V}(p_i)$, i.e. there is $v \in \mathfrak{V}(p_i)$ such that $u \sim v$ and $u \notin \mathfrak{V}(p_i)$. Hence $col(u) \neq col(v)$ and $[u]$ is a $\sim$-equivalence class containing points of different colours.
(3)⇒(1) Suppose $\mathfrak{A}$ is not generated by $g_1, \dots, g_n$ and let $\mathfrak{A}'$ be the subalgebra generated by these elements. Clearly, $\mathfrak{A}'$ is a proper subalgebra of $\mathfrak{A}$. Let $\sim_{\mathfrak{A}'}$ be the descriptive congruence corresponding to $\mathfrak{A}'$. Since, for all $i \leq n$, $g_i \in \mathfrak{A}'$, every $\mathfrak{V}(p_i)$ is $\sim_{\mathfrak{A}'}$-saturated. Suppose $[u]$ is a $\sim_{\mathfrak{A}'}$-equivalence class containing points of different colours, say u and v. Then, for some $j \in \{1, \dots, n\}$, $c_j^u \neq c_j^v$ and we can assume $u \in \mathfrak{V}(p_j)$ and $v \notin \mathfrak{V}(p_j)$. It follows that $\mathfrak{V}(p_j)$ is not $\sim_{\mathfrak{A}'}$-saturated and therefore $g_j \notin \mathfrak{A}'$, contradicting our assumption. □

For every descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, denote by $W^{=d}$, $W^{\leq d}$ and $W^{<d}$ the set of all point in $\mathfrak{F}$ of depth d, $\leq d$ and $< d$, respectively. Define the sets $W^{\geq d}$ and $W^{>d}$ in an analogous way. Furthermore let $\mathfrak{F}^{\leq d}$ be the descriptive subframe of $\mathfrak{F}$ generated by the set $W^{\leq d}$. It is evident that $W^{<\omega} = \bigcup_{k \in \omega} W^{=k}$, $W = W^{<\omega} \cup W^{=\omega}$ and $W^{<\omega} \cap W^{=\omega} = \emptyset$. Notice moreover that $W^{>d} = W \setminus W^{\leq d}$ and that $W^{\leq 1} = max(\mathfrak{F})$.

Definition 3.3. Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a descriptive frame and let $x \in W$. Then x is said to be an *atom* of $\mathfrak{F}$ if both $W \setminus x{\downarrow}$ and $\{x\} \cup W \setminus x{\downarrow}$ belongs to $\mathcal{P}$.

Remark 4. Notice that a point x in a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ is an atom iff x is an isolated point in the corresponding Esakia space $\langle W, \tau_{\mathcal{P}}, R \rangle$. Indeed, if x is an atom, then $W \setminus x{\downarrow}$ and $\{x\} \cup W \setminus x{\downarrow}$ are clopen upsets of W. Hence $x{\downarrow}$ is clopen and consequently $x{\downarrow} \cap (\{x\} \cup W \setminus x{\downarrow}) = \{x\}$ is clopen. Hence x is an isolated point. Conversely, if $x \in W_{\text{iso}}$, then, by definition, $\{x\} \in \tau_{\mathcal{P}}$. Since the space is Hausdorff, $\{x\}$ is closed and thus $\{x\}$ is clopen. Since R is a clopen relation, $x{\downarrow}$ is clopen and therefore both $W \setminus x{\downarrow}$ and $\{x\} \cup W \setminus x{\downarrow}$ are clopen upsets, i.e. they belongs to $\mathcal{P}$. Thus x is an atom.

We will now prove that the structure of a finitely generated descriptive frame has the following form: for each n such that $0 < n \leq d(\mathfrak{F})$, the *n-layer* $W^{=n}$ is non-empty and finite; every point in $W^{=n}$ is an atom; and $W^{=n}{\downarrow} = W^{\geq n}$. Frames having the previously described structure are also called *top heavy.*

Theorem 3.4. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a finitely generated descriptive frame. Then the set $max(\mathfrak{F})$ of the maximal points of $\mathfrak{F}$ is a finite admissible subset of $\mathfrak{F}$ of cardinality at most 2^n, where n is the number of the generators of $\mathfrak{F}$.*

Proof. Assume that $\mathfrak{F}$ is n-generated. Consider its dual Heyting algebra $\mathfrak{A}$ and let $g_1, \ldots, g_n$ be the generators of $\mathfrak{A}$. Let $v\colon \mathbf{Var}\mathcal{L}_n \to \mathfrak{A}$ be the standard valuation such that $v(p_i) = g_i$ for every $i \in \{1, \ldots, n\}$ and consider the corresponding standard dual coloured model $\langle \mathfrak{F}, \mathfrak{V} \rangle$. Now, suppose that there exist $u, v \in max(\mathfrak{F})$ such $col(u) = col(v)$. Then the assumption of Lemma 3.1 (ii) apply and we can consider the descriptive congruence E that identifies u and v. But then E is a proper descriptive congruence the equivalence classes of which contain points of the same colours and this implies that $\mathfrak{A}$ is not generated by $g_1, \ldots, g_n$ by Theorem 3.3, contrary to our assumption. Thus different points of $max(\mathfrak{F})$ are of different colours. Since there are 2^n different colours, it follows that $|max(\mathfrak{F})| \leq 2^n$.
We now show that $max(\mathfrak{F})$ is an admissible subset of $\mathfrak{F}$ or, equivalently, that $max(\mathfrak{F})$ is clopen in the corresponding Esakia space. By Theorem 2.20, $max(\mathfrak{F})$ is closed. Now, let $x \in W \setminus max(\mathfrak{F})$. Then clearly $\neg(vRx)$ for all $v \in max(\mathfrak{F})$ and, by Priestley's axiom, there exist clopen upsets $U_v \subseteq W$ such that $v \in U_v$ and $x \notin U_v$. Since $max(\mathfrak{F})$ is finite, $U_x = \bigcup_{v \in max(\mathfrak{F})} U_v$ is a clopen upset of W such that $x \notin U_x$. Therefore, we have

$$W \setminus max(\mathfrak{F}) \subseteq \bigcup_{x \in W \setminus max(\mathfrak{F})} W \setminus U_x.$$

If we show that the converse inclusion holds, then we are done. So, consider $x \in W \setminus max(\mathfrak{F})$ such that $z \in W \setminus U_x$. Then, by definition of U_x, $z \notin U_v$ for all $v \in max(\mathfrak{F})$ and therefore z can not be a maximal point of $\mathfrak{F}$. We conclude that $max(\mathfrak{F})$ is a finite clopen upset of W, that is, $max(\mathfrak{F})$ is a finite admissible subset of $\mathfrak{F}$ of cardinality at most 2^n. □

Given a descriptive model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on the frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, call a subset $U \subseteq W$ *definable in* $\mathfrak{M}$ if there exists a $\mathcal{L}$-formula φ such that $U = \mathfrak{V}(\varphi)$ and denote by $\mathcal{D}_{\mathfrak{M}}$ the set of all definable subset of $\mathfrak{M}$. It is clear that $\mathcal{D}_{\mathfrak{M}} \subseteq \mathcal{P}$. Furthermore, if $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is the dual model of the n-generated Heyting algebra $\mathfrak{A}$ with the valuation v of Theorem 3.4, then clearly

$$\mathcal{D}_{\mathfrak{M}} = \mathcal{P} = \{\mathfrak{V}(\varphi) \mid \varphi \in \mathbf{For}\mathcal{L}_n\}.$$

Now, for every $d \in \omega$, let $\mathfrak{F}^{>d}$ be the general frame defined as follows:

$$\mathfrak{F}^{>d} = \langle W^{>d}, R^{>d}, \mathcal{P}^{>d} \rangle,$$

where $R^{>d} = R \cap W^{>d}$ and $\mathcal{P}^{>d} = \{U \cap W^{>d} \mid U \in \mathcal{P}\}$.

Lemma 3.5. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a finitely generated descriptive frame. Then the general frame $\mathfrak{F}^{>1}$ is a descriptive clopen subframe of $\mathfrak{F}$.*

Proof. Let $\mathfrak{A}$ be the dual Heyting algebra of $\mathfrak{F}$, $g_1, \ldots, g_n$ the generators of $\mathfrak{A}$ and $v\colon \mathbf{Var}\mathcal{L}_n \to \mathfrak{A}$ the valuation associating to each p_i the generator g_i for every $i \in \{1, \ldots, n\}$. Now consider the dual coloured model $\langle \mathfrak{F}, \mathfrak{V} \rangle$. By Theorem 3.4, $max(\mathfrak{F})$ is a finite admissible subset of $\mathfrak{F}$. This means that $max(\mathfrak{F}) = W^{\leq 1}$ is a clopen upset of the corresponding Esakia space of $\mathfrak{F}$ and, since $W^{>1} = W \setminus W^{\leq 1}$, we have that $W^{>1}$ is clopen in W and therefore the subspace induced by it is a clopen subspace of the Esaka space corresponding to $\mathfrak{F}$ that is also an Esakia space by Theorem 2.22. □

The following theorem has first been proved by Kuznetsov by algebraic means; cfr. also [12, Lemma 2.2(3)] for an algebraic proof and [14, Claim 3.1.11] for a semantic proof.

Theorem 3.6. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a finitely generated descriptive frame. Then the subframe $\mathfrak{F}^{>1}$ is also finitely generated and the cardinality of the set of generators is $\leq 2^{2^n} + n$, where n is the number of the generators of $\mathfrak{F}$.*

Since the relation of being a subframe is transitive, by an easy induction we immediately get from Lemma 3.5 and Theorem 3.6 the following

Corollary 3.7. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a finitely generated infinite descriptive frame. Then, for every $k \in \omega$, $\mathfrak{F}^{>k}$ is a finitely generated clopen subframe of $\mathfrak{F}$.*

We are now ready to give the characterization of the structure of a finitely generated descriptive frame mentioned above.

Theorem 3.8. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a finitely generated infinite descriptive frame. Then the following hold:*

(i) *for every $k \in \omega$, $W^{=k}$ is a finite set;*

(ii) *for every $k \in \omega$, $W^{\leq k}$ is an admissible set of $\mathfrak{F}$;*

(iii) *every point in $W^{<\omega}$ is an atom;*

(iv) *for every $u \in W^{=\omega}$ and $k \in \omega$, there exists $v \in W^{=k}$ such that uRv.*

Proof. We prove (i) and (ii) simultaneously. The base case $k = 1$ follows from Theorem 3.4. So assume by induction hypothesis that (i) and (ii) holds for $k = n$. Consider the subframe $\mathfrak{F}^{>n}$. By Corollary 3.7, $\mathfrak{F}^{>n}$ is finitely generated and $max(\mathfrak{F}^{>n}) = W^{=n+1}$ is a finite admissible set of $\mathfrak{F}^{>n}$ by Theorem 3.4. This means that $W^{=n+1}$ is a clopen upset of the Esakia space corresponding to $\mathfrak{F}^{>n}$ and thus also a clopen of $\mathfrak{F}$ by Corollary 3.7. Since by induction hypothesis $W^{\leq n}$ is a finite clopen upset of $\mathfrak{F}$, we have that $W^{\leq n} \cup W^{=n+1} = W^{\leq n+1}$ is again a clopen upset of $\mathfrak{F}$, that is, a finite admissible set in $\mathfrak{F}$.
(iii) We show that each point in $W^{<\omega}$ is an isolated point in the Esakia space $\mathcal{X}_{\mathfrak{F}}$ corresponding to $\mathfrak{F}$. Let $u \in W^{<\omega}$. Then we can assume that $u \in W^{=k}$ for some $1 \neq k \in \omega$, since if u is a maximal point, then Theorem 3.4 implies that u is isolated. Since by (ii) $W^{\leq k-1}$ and $W^{\leq k}$ are both finite clopen upsets of W, each upset contained in them is also clopen and thus in particular $u{\uparrow}$ is clopen in W. Furthermore, since $W \setminus W^{\leq k-1} = W^{>k-1}$ is also clopen, $W^{>k-1} \cap u{\uparrow} = \{u\}$ is clopen as well and thus u is an isolated point of W.
(iv) Let $u \in W^{=\omega}$ and $k \in \omega$. Since by (ii) $W^{\leq k-1}$ is admissible in $\mathfrak{F}$, $W^{\leq k-1}$ is clopen in the Esakia space $\mathcal{X}_{\mathfrak{F}}$. Thus $W^{>k-1}$ is a closed subset of W containing u. But then, by Theorem 2.17 (i), there exists $v \in max(W^{>k-1}) = W^{=k}$ such that uRv. ☐

Remark 5. Notice that we actually proved something more of what is merely stated in (iii) of the previous theorem. Indeed, we showed that, for any given finitely generated descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, every point of finite depth is order-isolated in the corresponding Esakia space, that is, $W_{\text{fin}} \subseteq W_{\text{iso}\uparrow}$. Therefore, by Lemma 2.44, it follows that each filter of $\mathfrak{F}^+$ of finite index is principal.

Remark 6. For any given finitely generated descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and $n \in \omega$, the generated subframe $\mathfrak{F}^{\leq n}$ is actually a finite Kripke frame since all the upsets of $W^{\leq n}$ are admissible in $\mathfrak{F}$. Notice moreover that $\mathfrak{F}^{<\omega}$ is a Noetherian frame and a Kripke frame as well: indeed, since all its points are isolated in the corresponding Esakia space $\langle W, \tau_{\mathcal{P}}, R \rangle$, the subspace topology of the ordered-topological space corresponding to $\mathfrak{F}^{<\omega}$ is the discrete one and thus all the upsets are indeed clopen in $W^{<\omega}$.

Remark 7. Notice that if ∇ is a non-principal prime filter of a finitely generated Heyting algebra $\mathfrak{A}$, then ∇ is a point at infinite depth in the dual descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ of $\mathfrak{A}$. Indeed, if it were the case that $\nabla \in W^{=d}$ for some $d \in \omega$, then $\nabla{\uparrow}$ would be a finite upset. Since all point of $W^{<\omega}$ are isolated, $\nabla{\uparrow}$ would be in particular a clopen upset of W and thus of the form $\widehat{q}$ for some $q \in \mathfrak{A}$. Therefore, $q \in \nabla$ and if $c \in \nabla$, then $\nabla \in \widehat{c}$, consequently $\widehat{q} = \nabla{\uparrow} \subseteq \widehat{c}$ which in turn implies $q \leq c$. Therefore $\nabla = [q)$ would be principal, contrary to our assumption. Thus $\nabla \in W^{=\omega}$.

3.1.1 Finitely-generated free Heyting algebras

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a (Kripke) model and $\mathbf{Var}\mathcal{L}_n$ a finite set of propositional variables. Consider the colouring of $\mathfrak{M}$ induced by $\mathfrak{V}$: each point $w \in \mathfrak{M}$ is associated with its colour, $col(w) = \langle c_1^w, \dots, c_n^w \rangle$, a finite binary sequence of length n which gives us all the relevant informations on whether $p \in \mathbf{Var}\mathcal{L}_n$ is true at w or not.

Now, consider the set $\mathcal{C}ol_n$ of colours of lenght n, that is, the set of all binary sequence of lenght n. We can then tranform $\mathcal{C}ol_n$ into a Boolean algebra by defining on it a partial ordering $\trianglelefteq$ by letting, for any two given elements $\langle c_1, \dots, c_n \rangle$ and $\langle d_1, \dots, d_n \rangle$ of $\mathcal{C}ol_n$,

$$\langle c_1, \dots, c_n \rangle \trianglelefteq \langle d_1, \dots, d_n \rangle \iff c_i \leq d_i \text{ for all } i \in \{1, \dots, n\}.$$

Indeed it is easy to see that $\langle \mathcal{C}ol_n, \trianglelefteq \rangle$ forms a 2^n-element Boolean algebra. Moreover, we will write $\langle c_1, \dots, c_n \rangle \triangleleft \langle d_1, \dots, d_n \rangle$ iff $\langle c_1, \dots, c_n \rangle \trianglelefteq \langle d_1, \dots, d_n \rangle$ and $\langle c_1, \dots, c_n \rangle \neq \langle d_1, \dots, d_n \rangle$.

Recall that given a Kripke frame $\mathfrak{F} = \langle W, R \rangle$, $w \in \mathfrak{F}$ and $X \subseteq W$, we say that X totally covers w and we write $w \prec X$ if $X = w^{>}$, that is, X is the set of all immediate successor of w. Now, if every point in W has only finitely many successor, then R is the reflexive and transitive closure of the immediate successor relation and thus is completely determined by such a relation. Therefore, since for any finitely generated descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, for each $k \in \omega$, $W^{\leq k}$ is finite, every point in $W^{<\omega}$ has only finitely many successor and thus the relation $R \cap W^{<\omega}$ on $\mathfrak{F}^{<\omega}$ is completely determined by the relation $\prec$ relating point and subsets of $W^{<\omega}$.

Definition 3.4. Let $\mathbf{F}_{\mathcal{HA}}(n)$ be the free Heyting algebra over n generators and let $g_1, \dots, g_n$ be the generators of $\mathbf{F}_{\mathcal{HA}}(n)$. We denote by $\mathfrak{F}_{\mathbf{Int}}(n)$ the dual descriptive frame $\langle W_{\mathbf{Int}}(n), R_{\mathbf{Int}}(n), \mathcal{P}_{\mathbf{Int}}(n) \rangle$ of $\mathbf{F}_{\mathcal{HA}}(n)$ and call it the *n-canonical frame for* **Int**. Moreover, we denote by $\mathfrak{M}_{\mathbf{Int}}(n) = \langle \mathfrak{F}_{\mathbf{Int}}(n), \mathfrak{V}_{\mathbf{Int}}(n) \rangle$ the dual descriptive coloured model of $\langle \mathfrak{F}_{\mathbf{Int}}(n), v \rangle$, where v is the standard valuation, and call it the *n-canonical model for* **Int**.

Before moving on, let us remark the following well known properties of the κ-canonical frame and model for **Int**.

Lemma 3.9. *Let $\mathfrak{F}$ be a λ-generated descriptive frame. Then $\mathfrak{F}$ is (isomorphic to) a generated subframe of $\mathfrak{F}_{\mathbf{Int}}(\kappa)$ and the standard model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ on $\mathfrak{F}$ is a generated submodel of $\mathfrak{M}_{\mathbf{Int}}(\kappa)$, for any $\kappa \geq \lambda$.*

We are now going to give a complete description of the upper part of $\mathfrak{F}_{\mathbf{Int}}(n)$, namely of the points at finite depth of the n-canonical frame for **Int**, for every $n \in \omega$. In particular, we are going to define a model $\mathfrak{N}(n) = \langle \mathfrak{U}(n), \mathfrak{V} \rangle$ based on the frame $\mathfrak{U}(n) = \langle U(n), R, \mathcal{Q} \rangle$ and show that $\mathfrak{U}(n) \cong \mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$.

Let $n \in \omega$ be fixed and consider the Boolean algebra $\langle \mathcal{C}ol_n, \trianglelefteq \rangle$. We construct $\mathfrak{N}(n)$ by induction on layers $U(n)^{=k}$, where $1 \leq k < \omega$.

($k = 1$) Let $U(n)^{=1}$ consists of 2^n points $u_1, \dots, u_{2^n}$ and let $h \colon U(n)^{=1} \to \mathcal{C}ol_n$ be a bijection. Moreover, let $R(u_i) = \{u_i\}$ and $col(u_i) = h(u_i)$ for all $i \in \{1, \dots, 2^n\}$;

($k = m + 1$) Assume that the model $\mathfrak{N}^{=d}$ has been defined for all $d \leq m$.

(a) For every point $w \in U(n)^{=m}$ and each colour $\langle c_1, \dots, c_n \rangle \lhd col(w)$, add to the model a unique point v such that $R(v) = R(w) \cup \{v\}$ and let $col(v) = \langle c_1, \dots, c_n \rangle$.

(b) For every antichain $X \subseteq U(n)^{\leq m}$ with ≥ 2 points at least one of which in $U(n)^{=m}$ and each colour $\langle c_1, \dots, c_n \rangle \trianglelefteq col(u)$ for every $u \in X$, add to $\mathfrak{N}^{=m+1}$ a unique point v and let $R(v) = R(X) \cup \{v\}$ and let $col(v) = \langle c_1, \dots, c_n \rangle$.

Then let $U(n) = \bigcup_{i=1}^{\omega} U(n)^{=i}$, R be the relation determined by the previous construction and finally let $\mathfrak{V}(p_j) = \{w \in U(n) \mid c^w_j = 1\}$ for all $p_j \in \mathbf{Var}\mathcal{L}_n$.

From the previous construction, it is clear that the model $\mathfrak{N}(n) = \langle \mathfrak{U}(n), \mathfrak{V} \rangle$ based on the general frame $\mathfrak{U}(n) = \langle U(n), R, \mathcal{Q} \rangle$, where $\mathcal{Q} = \{\mathfrak{V}(\varphi) \mid \varphi \in \mathbf{For}\mathcal{L}_n\}$ is a well defined model. Furthermore, $\mathfrak{N}$ is the minimal model such that $max(\mathfrak{U}(n))$ has 2^n points of different colours and that satisfies the items (a) and (b) of the construction above, namely, there are no proper submodels $\mathfrak{N}'$ of $\mathfrak{N}$ with the same features and thus it is unique up to isomorphisms.

Before showing that $\mathfrak{N}$ is isomorphic to the upper part of the n-canonical model $\mathfrak{M}_{\mathbf{Int}}(n)$, notice the following

Lemma 3.10. *For every $k, n \in \omega$, $\mathfrak{U}(n)^{\leq k}$ is a n-generated descriptive frame.*

Proof. Since, for every $k \in \omega$, $U(n)^{\leq k}$ is finite, $\mathfrak{U}(n)^{\leq k}$ is clearly descriptive. Now, consider the generated submodel $\mathfrak{N}^{\leq k} = \langle \mathfrak{U}(n)^{\leq k}, \mathfrak{V}' \rangle$ of $\mathfrak{N}$, that is,

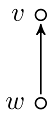
$$\mathfrak{V}'(p_j) = \mathfrak{V}(p_j) \cap U(n)^{\leq k}, \text{ for all } j \in \{1, \dots, n\},$$

and let f be a proper p-morphism from from $\mathfrak{U}(n)^{\leq k}$ onto $\mathfrak{F}$. Since $U(n)^{\leq k}$ is finite, $\mathfrak{F}$ is also finite and thus, by Lemma 3.2, f is a finite composition of α- or β-reductions. But then, by the construction of $\mathfrak{U}(n)$, it follows that any α- or β-reduction identifies points of different colours and therefore we have that $U(n)^{\leq k}$ is n-generated by Theorem 3.3. □

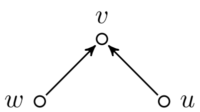
We are now ready to prove the following

Theorem 3.11. *For every $n \in \omega$, $\mathfrak{U}(n) \cong \mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$.*

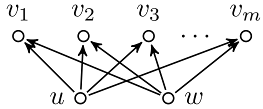
Proof. Let $n \in \omega$ be fixed. We are going to show, by induction on $k \in \omega$, that $\mathfrak{U}(n)^{\leq k} \cong \mathfrak{F}^{\leq k}_{\mathbf{Int}}(n)$. By construction $\mathfrak{U}(n)^{\leq 1}$ has 2^n distinct point. Moreover, by Theorem 3.4, $\mathfrak{F}^{\leq 1}_{\mathbf{Int}}(n)$ has at most 2^n points. Since, by Lemma 3.10, $\mathfrak{U}(n)^{\leq 1}$ is a n-generated descriptive frame, it follows from Lemma 3.9 that $\mathfrak{F}_{\mathbf{Int}}(n)$ must contain a isomorphic copy of $\mathfrak{U}(n)^{\leq 1}$ as a generated subframe. Therefore, we can conclude that $\mathfrak{U}(n)^{\leq 1} \cong \mathfrak{F}^{\leq 1}_{\mathbf{Int}}(n)$.
Now let $k > 1$ and suppose, for induction hypothesis, that $\mathfrak{U}(n)^{\leq m} \cong \mathfrak{F}^{\leq m}_{\mathbf{Int}}(n)$ for all $m \leq k$. Again by Lemma 3.10 and Lemma 3.9, we can assume wlog that $\mathfrak{F}_{\mathbf{Int}}(n)$ contains $\mathfrak{U}(n)^{\leq k+1}$ as a generated subframe. Now, suppose towards a contradiction that $\mathfrak{U}(n)^{\leq k+1} \ncong \mathfrak{F}^{\leq k+1}_{\mathbf{Int}}(n)$. Since $\mathfrak{U}(n)^{\leq k} \cong \mathfrak{F}^{\leq k}_{\mathbf{Int}}(n)$, there exists a point $w \in \mathfrak{F}^{=k+1}_{\mathbf{Int}}(n)$ such that $w \notin \mathfrak{U}(n)^{=k+1}$. Consider the set $w^{>}$ of the immediate successor of w. Since, w is not a maximal element of $\mathfrak{F}_{\mathbf{Int}}(n)$, $w^{>} \neq \emptyset$ and, moreover, by Theorem 3.8 (i), $w^{>}$ is finite. Notice furthermore that $w^{>} \subseteq \mathfrak{F}^{=k}_{\mathbf{Int}}(n) = \mathfrak{U}(n)^{=k}$. Now, first suppose that $|w^{>}| = 1$. Then $w^{>} = \{v\}$, for some $v \in \mathfrak{U}(n)^{=k}$. If $col(w) = col(v)$ then we are in the following situation:



and we can consider the α-reduction on $\mathfrak{F}_{\mathbf{Int}}(n)$ that identifies w and v. Therefore, by the Colouring Theorem, it follows that $\mathfrak{F}_{\mathbf{Int}}(n)$ is not n-generated, which is a contradiction.
If, otherwise, $col(w) \lhd col(v)$, then, by the point (a) of the construction of $\mathfrak{U}(n)$, it follows that there exists $u \in \mathfrak{U}(n)^{=k+1}$, and thus in $\mathfrak{F}^{\leq k+1}_{\mathbf{Int}}(n)$, such that $u \prec v$ and $col(u) = col(w)$. Then we are in the following situation:



and we can consider the β-reduction on $\mathfrak{F}_{\mathbf{Int}}(n)$ that identifies w and u. Therefore, by the Colouring Theorem, it follows that $\mathfrak{F}_{\mathbf{Int}}(n)$ is not n-generated, which is again a contradiction.
Finally, suppose that $|w^{>}| = m$ and let $w^{>} = \{v_1, \ldots, v_m\}$, for some points $v_j \in \mathfrak{U}(n)^{=k}$ for all $j \leq m$. Then $col(w) \unlhd col(v_j)$ for all $j \leq m$ and thus, by the point (b) of the construction of $\mathfrak{U}(n)$, it follows that there exists $u \in \mathfrak{U}(n)^{=k+1}$, and thus in $\mathfrak{F}^{\leq k+1}_{\mathbf{Int}}(n)$, such that $u \prec w^{>}$ and $col(u) = col(w)$. Then, we are in the following situation:



Then again consider the β-reduction on $\mathfrak{F}_{\mathbf{Int}}(n)$ that identifies w and u. By the Colouring Theorem, we still get a contradiction.

Therefore $\mathfrak{U}(n)^{\leq k+1} \cong \mathfrak{F}_{\mathbf{Int}}^{\leq k+1}(n)$. We can thus conclude that

$$\mathfrak{U}(n) = \bigcup_{i\in\omega} \mathfrak{U}(n)^{\leq i} \cong \bigcup_{i\in\omega} \mathfrak{F}_{\mathbf{Int}}^{\leq i}(n) = \mathfrak{F}_{\mathbf{Int}}^{<\omega}(n). \qquad \square$$

The next picture shows the generated submodel $\mathfrak{M}_{\mathbf{Int}}^{\leq 2}(2)$ of $\mathfrak{M}_{\mathbf{Int}}(2)$ with its colouring, where we let $\langle 1,1\rangle = a$, $\langle 1,0\rangle = b$, $\langle 0,1\rangle = c$ and $\langle 0,0\rangle = d$[1].

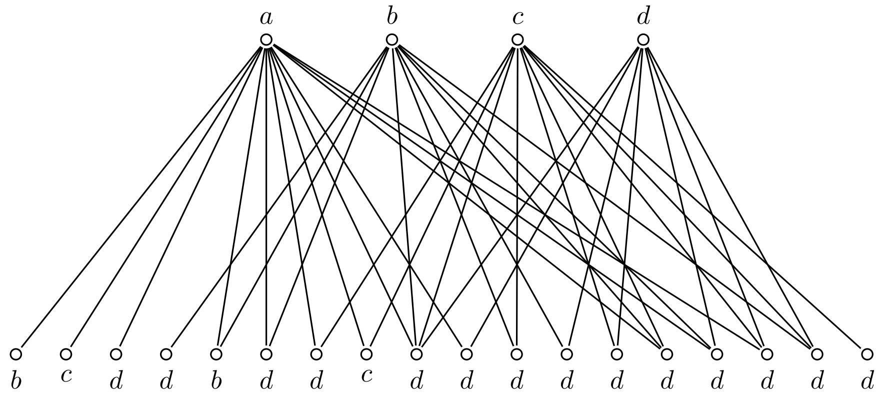


Figure 3.1: *The generated submodel* $\mathfrak{M}_{\mathbf{Int}}^{\leq 2}(2)$.

Let us now introduce an important tool due to de Jongh which allows us to identify and characterize the principal cones of finite depth of the n-canonical model.

Definition 3.5. Let us consider the n-canonical model $\mathfrak{M}_{\mathbf{Int}}(n) = \langle \mathfrak{F}_{\mathbf{Int}}(n), \mathfrak{V}\rangle$ and let w be a point of $\mathfrak{F}_{\mathbf{Int}}^{<\omega}(n)$. We define, by induction on $d(w)$ the $\mathcal{L}_n$-formulas ϕ_w and ψ_w as follows:

$(d(w) = 1)$

$$\phi_w = \bigwedge_{w\in\mathfrak{V}(p_j)} p_j \wedge \bigwedge_{w\notin\mathfrak{V}(p_k)} \neg p_k$$
$$\psi_w = \neg\phi_w.$$

$(d(w) > 1)$ Let $S(w) \subseteq \mathbf{Var}\mathcal{L}_n$ be the following set of variables

$$S(w) = \{p_j \mid w \notin \mathfrak{V}(p_j), v \in \mathfrak{V}(p_j) \text{ for all } v \in w^{>}\}$$

and then let

$$\phi_w = \bigwedge_{w\in\mathfrak{V}(p_j)} p_j \wedge \left((\bigvee S(w) \vee \bigvee_{v\in w^{>}} \psi_v) \to \bigvee_{v\in w^{>}} \phi_v \right)$$
$$\psi_w = \phi_w \to \bigvee_{v\in w^{>}} \phi_v.$$

[1] Please notice that both the drawings [23, Fig. 8.12] and [69, Fig. 2] are misleading. Indeed, they both miss a point in $\mathfrak{F}_{\mathbf{Int}}^{=2}(2)$ and there are points with the same colours and successors.

We call the $\mathcal{L}_n$-formulas ϕ_w and ψ_w the *de Jongh formulas.*

Theorem 3.12. *For every $w \in \mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$, the following hold:*

(i) $w{\uparrow} = \mathfrak{V}(\phi_w)$;

(ii) $W_{\mathbf{Int}}(n) \setminus w{\downarrow} = \mathfrak{V}(\psi_w)$.

Proof. We show (i) and (ii) simultaneously by induction on the depth of w.

$d(w) = 1$ Then $w \in max(\mathfrak{F}_{\mathbf{Int}}(n))$ and thus $w{\uparrow} = \{w\} \subseteq \mathfrak{V}(\phi_w)$. Conversely, let v be a point in $W_{\mathbf{Int}}(n)$ such that $v \neq w$. If $v \in max(\mathfrak{F}_{\mathbf{Int}}(n))$, then $col(w) \neq col(v)$ and thus $v \notin \mathfrak{V}(\phi_w)$. Hence, if $v \in u{\downarrow}$, for some $u \in max(\mathfrak{F}_{\mathbf{Int}}(n))$ different from w, then also $v \notin \mathfrak{V}(\phi_w)$. Finally, if $v \in \mathfrak{F}^{>1}_{\mathbf{Int}}(n)$ and $max(v) = \{w\}$, then it must be the case that $col(v) \lhd col(w)$ and, consequently $v \notin \mathfrak{V}(\phi_w)$. Therefore, $w{\uparrow} = \mathfrak{V}(\phi_w)$ and, since by the definition of intuitionistic negation we have

$$\mathfrak{V}(\psi_w) = \mathfrak{V}(\neg\phi_w) = W_{\mathbf{Int}}(n) \setminus \mathfrak{V}(\phi_w){\downarrow},$$

it follows that $\mathfrak{V}(\psi_w) = W_{\mathbf{Int}}(n) \setminus w{\downarrow}$.

$d(w) > 1$ Assume that the theorem holds for all points of depth $< d(w)$. In particular, the theorem holds for the set $w^{>}$ of all the immediate successors of w. Since, for all $v \in w^{>}$, $w \in v{\downarrow}$, it follows that $w \notin W_{\mathbf{Int}}(n) \setminus v{\downarrow}$ and thus $w \not\models \bigvee_{v\in W^{>}} \psi_v$. Moreover, by definition of $S(w)$, it follows that $w \not\models \bigvee S(w) \vee \bigvee_{v\in W^{>}} \psi_v$ and consequently $w \in \mathfrak{V}(\phi_w)$. Since $\mathfrak{V}(\phi_w)$ is an upset, we thus have $w{\uparrow} \subseteq \mathfrak{V}(\phi_w)$. Conversely, let $v \in \mathfrak{V}(\phi_w)$. Then, since $v \models \bigwedge_{w\in\mathfrak{V}(p_j)} p_j$, it follows that $col(w) \unlhd col(v)$. Now, if $v \not\models \bigvee S(w) \vee \bigvee_{v\in W^{>}} \psi_v$, then $col(w) = col(v)$ and $v \in u{\downarrow}$ for all $u \in w^{>}$. Consequently, by the structure of $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$, it follows that $v = w \in w{\uparrow}$. If, instead, $v \models \bigvee S(w) \vee \bigvee_{v\in W^{>}} \psi_v$, then also $v \models \bigvee_{v\in w^{>}} \phi_v$. Hence $v \in \mathfrak{V}(\phi_u)$ for some $u \in w^{>}$ and thus $v \in u{\uparrow} \subseteq w{\uparrow}$, by the induction hypothesis. Therefore, $\mathfrak{V}(\phi_w) = w{\uparrow}$. Moreover, notice that $q \notin \mathfrak{V}(\psi_w)$ iff $q \not\models \phi_w \to \bigvee_{v\in w^{>}} \phi_v$. The previous relation holds exactly when there exists $u \in q{\uparrow}$ such that $u \models \phi_w$ and $u \not\models \bigvee_{v\in w^{>}} \phi_v$, that is, iff $u \in w{\uparrow}$ and $u \notin v{\uparrow}$ for all $v \in w^{>}$. But this is equivalent to u being equal to w and thus we have $q \notin \mathfrak{V}(\psi_w)$ iff $q \in w{\downarrow}$. We can thus conclude that $W_{\mathbf{Int}}(n) \setminus w{\downarrow} = \mathfrak{V}(\psi_w)$. □

The previous theorem tells us that the principal cones as well as the complements of the principal downsets of $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$ are definable in $\mathfrak{M}_{\mathbf{Int}}(n)$. One could wonder whether it is the case that all the upsets of $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$ are definable. The answer to this question is negative. Indeed, it is possible to show that, for every $n \geq 1$, the cardinality of the set $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$ is $2^{\aleph_0}$ (cfr. [14, Theorem 3.2.19]); but since there are only countably many distinct $\mathcal{L}_n$-formulas, it follows that not all the upsets of $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$ are definable.

Now, if we replace $\mathcal{HA}$ with any subvariety $\mathcal{V}_L$, which is the characteristic variety of a si-logic L, in Definition 3.4, then we get the *n-canonical frame* $\mathfrak{F}_L(n)$ and the *n-canonical model* $\mathfrak{M}_L(n)$ for L. By Lemma 3.9, $\mathfrak{F}_L(n)$ is a generated subframe of $\mathfrak{F}_{\mathbf{Int}}(n)$ and its upper part $\mathfrak{F}^{<\omega}_L(n)$ can be obtained by removing from $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$ all the points in which some formulas in L are refuted (under $\mathfrak{V}_{\mathbf{Int}}(n)$).

For instance, the (upper part) of the n-canonical frame for **LC**, $\mathfrak{F}^{<\omega}_{\mathbf{LC}}(n)$, can be obtained from $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$ by removing all the points which have at least two immediate successors and their predecessors. The following pictures shows the 2-canonical model for **LC**. Notice that $\mathfrak{F}_{\mathbf{LC}}(2)$ is finite, confirming the well known fact that **LC** is locally tabular[2].

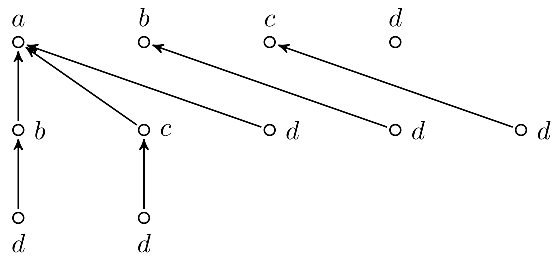


Figure 3.2: *The 2-canonical model $\mathfrak{M}_{\mathbf{LC}}(2)$.*

A straightforward generalization of Lemma 3.9 yields the following

Lemma 3.13. *Let L be an intermediate logic and let $\mathfrak{F}$ be a λ-generated descriptive frame for L. Then $\mathfrak{F}$ is (isomorphic to) a generated subframe of $\mathfrak{F}_L(\kappa)$ and the standard model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ on $\mathfrak{F}$ is a generated submodel of $\mathfrak{M}_L(\kappa)$, for any $\kappa \geq \lambda$.*

It is well know that each intermediate logic L is characterized by its n-canonical models, that is, for every $\mathcal{L}_n$-formula φ, the following holds:

$$\varphi \in L \iff \mathfrak{M}_L(n) \models \varphi.$$

If, furthermore, L is finitely approximable, then we have the following

Theorem 3.14. *Let L be a finitely approximable intermediate logic. Then, for every $\mathcal{L}_n$-formula φ,*

$$\varphi \in L \iff \mathfrak{M}^{<\omega}_L(n) \models \varphi.$$

Proof. The direction ($\Longrightarrow$) clearly holds. For ($\Longleftarrow$), suppose that $\varphi \notin L$. Since L is finitely approximable, it is characterized by its finite Kripke models and therefore there exists a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{V} \rangle$ for L based on the finite frame $\mathfrak{G}$ such that $\mathfrak{N} \not\models \varphi$. Therefore, $\mathfrak{V}(\varphi) \neq 1_{\mathfrak{G}^+}$ in the dual finite Heyting algebra $\mathfrak{G}^+$. Now let $\mathfrak{A}$ be the subalgebra of $\mathfrak{G}^+$ generated by $\mathfrak{V}(p_1), \ldots, \mathfrak{V}(p_n)$. Then $\mathfrak{A}$ is a finite, n-generated Heyting algebra belonging to the characteristic variety $\mathcal{V}_L$ of L and such that $\mathfrak{V}(\varphi) \neq 1_{\mathfrak{A}}$. Then, by Lemma 3.13, it follows that its dual standard model $\langle \mathfrak{A}_+, \mathfrak{V} \rangle$ is a generated submodel of $\mathfrak{M}_L(n)$. Since $\mathfrak{A}_+$ is finite, $\langle \mathfrak{A}_+, \mathfrak{V} \rangle$ is a generated submodel of $\mathfrak{M}^{<\omega}_L(n)$ and thus $\mathfrak{M}^{<\omega}_L(n) \not\models \varphi$. $\square$

Therefore, if L is an intermediate logic that is finitely approximable, then L (in the language of κ variables $\mathcal{L}_\kappa$) is characterized by the model $\mathfrak{M}^{<\omega}_L(\kappa)$. This means that the κ-canonical frame $\mathfrak{F}_L(\kappa)$ is completely determined by the upper part $\mathfrak{F}^{<\omega}_L(\kappa)$ and that $(\mathfrak{F}^{<\omega}_L(\kappa)^+)_+ \cong \mathfrak{F}_L(\kappa)$. This latter fact is also a consequence of the following more general

[2] Cfr. also [75].

Lemma 3.15. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and let $U \subseteq X$ be an upset. Then the double dual $(\mathcal{U}^+)_+$ of the ordered-topological subspace $\mathcal{U} = \langle U, \tau_U, R_U\rangle$ is an Esakia space order-homeomorphic to the generated subspace of $\mathcal{X}$ induced by $\overline{U}$. In particular, if moreover U is dense in X, then $(\mathcal{U}^+)_+ \cong \mathcal{X}$.*

Proof. The ordered-topological subspace $\mathcal{U} = \langle U, \tau_U, R_U\rangle$ may not be an Esakia space, since it may not be compact. However, the relation R_U is still a clopen relation (cfr. the proof of Theorem 2.21) and thus the algebra of the clopen upsets of $\mathcal{U}$, namely $\mathcal{U}^+$, is a Heyting algebra. Therefore, $(\mathcal{U}^+)_+$ is an Esakia space. Furthermore, by Corollary 2.26, $\overline{U}$ is a closed upset and thus the generated subspace induced by $\overline{U}$, $\overline{\mathcal{U}}$, is an Esakia space. Now, notice that, for each upset $V \subseteq X$ and for every subsets $P, Q \subseteq X$, we have

$$\begin{aligned}
[X \setminus (P \setminus Q){\downarrow}_X] \cap V &= (X \cap V) \setminus [(P \setminus Q){\downarrow}_X \cap V] \\
&= V \setminus [(P \setminus Q) \cap V]{\downarrow}_V \qquad \text{by Thorem 2.21} \\
&= V \setminus [(P \cap V) \setminus (Q \cap V)]{\downarrow}_V.
\end{aligned}$$

and thus

$$(P \supset_X Q) \cap U = (P \cap U) \supset_U (Q \cap U).$$

Therefore, since every element of $\mathcal{U}^+$ is of the form $V \cap U$ for some clopen upset V of X, the map $V \cap U \mapsto V \cap \overline{U}$ is an isomorphism between $\mathcal{U}^+$ and $\overline{\mathcal{U}}^+$. Consequently $(\mathcal{U}^+)_+ \cong (\overline{\mathcal{U}}^+)_+ \cong \overline{\mathcal{U}}$. □

It is also worth to point out that if we consider intermediate logics L with the disjunction property, we can get more information on the n-canonical models for L. In particular, we have the following

Theorem 3.16. *Let L be an intermediate logic. Then L has the disjunction property iff, for every $n \in \omega$, the n-canonical frame for L, $\mathfrak{F}_L(n)$, is rooted.*

Proof. Consider, for every $n \in \omega$, the Lindenbaum-Tarski algebra for L over the language $\mathcal{L}_n$, $\mathfrak{L}_L(n)$. Now, suppose that L has the disjunction property and let $[\varphi] \vee [\psi] = [1]$ for some arbitrary $[\varphi], [\psi] \in \mathfrak{L}_L(n)$. Then $\varphi \vee \psi \in L$ and thus either $\varphi \in L$ or $\psi \in L$. Consequently, either $[\varphi] = [1]$ or $[\psi] = [1]$, that is, $\mathfrak{L}_L(n)$ is well connected. Conversely, suppose that $\mathfrak{L}_L(n)$ is well connected for every $n \in \omega$ and let $\varphi \vee \psi \in L$. Then $\mathbf{Var}\varphi \cup \mathbf{Var}\psi \subseteq \mathbf{Var}\mathcal{L}_k$ for some $k \in \omega$. Thus $[\varphi] \vee [\psi] = [1]$ holds in $\mathfrak{L}_L(k)$ and since $\mathfrak{L}_L(k)$ is well connected, we have either $[\varphi] = [1]$ or $[\psi] = [1]$. Hence either $\varphi \in L$ or $\psi \in L$, i.e. L has the disjunction property. Therefore, L has the disjunction property iff, for every $n \in \omega$, $\mathfrak{L}_L(n)$ is well connected. But now notice that $\mathfrak{F}_L(n)$ is rooted iff the Esakia space $\mathcal{X}_{\mathfrak{F}_L(n)}$ corresponding to it is rooted. By Proposition 2.42, $\mathcal{X}_{\mathfrak{F}_L(n)}$ is rooted iff the $\mathcal{V}_L$-free algebra $\mathbf{F}_{\mathcal{V}_L}(n)$ over n-generators is well connected. Since $\mathbf{F}_{\mathcal{V}_L}(n) \cong \mathfrak{L}_L(n)$, the theorem follows. □

It is well know that **Int** has the disjunction property and, consequently, the n-canonical frames for **Int** are rooted. Moreover, it can be shown (cfr. [14, Theorem 3.2.13]) that, for every $n \in \omega$, $\mathfrak{F}_{\mathbf{Int}}(n) \setminus \mathfrak{F}^{<\omega}_{\mathbf{Int}}(n) \neq \emptyset$ and, in particular, that $\mathfrak{F}_{\mathbf{Int}}(1) \setminus \mathfrak{F}^{<\omega}_{\mathbf{Int}}(1) = \{\mathfrak{r}\}$ is a singleton set consisting of the root of $\mathfrak{F}_{\mathbf{Int}}(1)$. The next picture shows the 1-canonical model $\mathfrak{M}_{\mathbf{Int}}(1)$. In the literature, the upper part $\mathfrak{M}^{<\omega}_{\mathbf{Int}}(1)$ is also called the *Rieger-Nishimura ladder*.

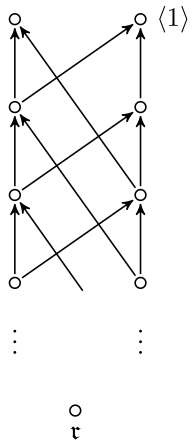


Figure 3.3: *The* 1*-canonical model* $\mathfrak{M}_{\mathbf{Int}}(1)$.

We now point up some features of the Esakia space $\mathcal{X}_{\mathfrak{F}_{\mathbf{Int}}(n)} = \langle X, \tau, R\rangle$ corresponding to $\mathfrak{F}_{\mathbf{Int}}(n)$. First notice that, since $\mathcal{HA}$ is finitely approximable, for every $n \in \omega$, the n-generated $\mathcal{HA}$-free algebra $\mathbf{F}_{\mathcal{HA}}(n)$ is finitely approximable. Hence by Proposition 2.43, we have that X_{fin} is a dense subset of the space $\mathcal{X}_{\mathfrak{F}_{\mathbf{Int}}(n)}$. By Remark 5, we also have $X_{\text{fin}} \subseteq X_{\text{iso}\uparrow} \subseteq X_{\text{iso}}$. Therefore, we have

Lemma 3.17. *The isolated points of the Esakia space* $\mathcal{X}_{\mathfrak{F}_{\mathbf{Int}}(n)} = \langle X, \tau, R\rangle$ *are exactly the points at finite depth. Equivalently,* $X^{<\omega}$ *coincide with the sets of atoms of* $\mathfrak{F}_{\mathbf{Int}}(n)$.

Proof. It suffice to show that $X_{\text{iso}} \subseteq X_{\text{fin}}$. If $x \notin X_{\text{fin}}$, then $X_{\text{fin}} \setminus \{x\} = X_{\text{fin}}$ and thus $x \in \overline{X_{\text{fin}} \setminus \{x\}} = X$, since X_{fin} is dense in X and, consequently, x is a limit point of X_{fin}. Now, if x is an isolated point of X, then x is also an isolated point of X_{fin} and thus x is not a limit point of X_{fin}. This contradiction shows that $x \notin X_{\text{iso}}$. □

Therefore, for $\mathcal{X}_{\mathfrak{F}_{\mathbf{Int}}(n)}$, we have $X_{\text{fin}} = X_{\text{iso}\uparrow} = X_{\text{iso}}$, consequently $X_{\text{iso}\uparrow}$ is dense in X and thus, by Proposition 2.40, we can conclude that

Lemma 3.18. *For every* $n \in \omega$*, the* n*-generated* $\mathcal{HA}$*-free algebra* $\mathbf{F}_{\mathcal{HA}}(n)$ *is completely join-prime generated.*

Furthermore, since all the principal prime filters of $\mathbf{F}_{\mathcal{HA}}(n)$ are generated by a prime element and all the prime elements are also completely join-prime, it follows, by Theorem 2.28 (ii), that the set of all principal prime filters of $\mathbf{F}_{\mathcal{HA}}(n)$ is included in X_{fin}. But since all the non-principal prime filters of $\mathbf{F}_{\mathcal{HA}}(n)$ are elements of $X^{=\omega} = X \setminus X_{\text{fin}}$ by Remark 7, we have that X_{fin} is exactly the set of all principal prime filters of $\mathbf{F}_{\mathcal{HA}}(n)$.

3.2 Finitely presented Heyting algebras

Recall that an Heyting algebra $\mathfrak{A}$ is said to be *finitely presented* if $\mathfrak{A}$ is isomorphic to the quotient of a free Heyting algebra $\mathbf{F}_{\mathcal{HA}}(n)$, for some $n < \omega$, under a compact congruence, which can be always generated by a pair $\langle \varphi, \top\rangle$.

3.2.1 Implicational complexity

Definition 3.6. Let φ be a $\mathcal{L}$-formula. The *implicational complexity* $c(\varphi)$ of φ is defined inductively as follows:

- $c(\varphi) = 0$ if φ is atomic;
- $c(\eta \circ \xi) = max(c(\eta), c(\xi))$, for $\circ \in \{\wedge, \vee\}$;
- $c(\eta \to \xi) = max(c(\eta), c(\xi)) + 1$;

For every $n, k < \omega$, let us denote by Ξ_n^k the set of $\mathcal{L}_n$-formulas, constructed from the variables $p_1, \ldots, p_n$, of implicational complexity at most k, that is,

$$\Xi_n^k := \{\varphi \in \mathbf{For}\mathcal{L}_n \mid c(\varphi) \leq k\}.$$

Lemma 3.19. *For any $n, k < \omega$, the set Ξ_n^k contains only finitely-many non-equivalent in* **Int** *formulas.*

Proof. Fix an arbitrary $n < \omega$ and proceed by induction on k.

($k = 0$) Notice that the elements of Ξ_n^0 can be brought into disjunctive normal form, that is, any $\varphi \in \Xi_n^0$ is equivalent to $\bigvee_i \bigwedge_j \Gamma_{ij}$, where $\Gamma_{ij} \subseteq \mathbf{Var}\mathcal{L}_n$. Therefore the number of non-equivalent formulas in Ξ_n^0 is $\leq 2^{2^n}$.

($k = m + 1$) By induction hypothesis, the number of non-equivalent formulas in Ξ_n^m is finite, say $\kappa < \aleph_0$. Since any $\varphi \in \Xi_n^k$ is equivalent in **Int** to $\psi_1 \to \psi_2$ for some $\psi_1, \psi_2 \in \Xi_n^m$, it follows that the cardinality of the set of non-equivalent formulas in Ξ_n^k is bounded by $2^\kappa < \aleph_0$. ☐

Consider the n-canonical frame $\mathfrak{F}_{\mathbf{Int}}(n) = \langle W_{\mathbf{Int}}, R_{\mathbf{Int}}, \mathcal{P}_{\mathbf{Int}}\rangle$ for some fixed $n < \omega$. Recall that the admissible sets of $\mathfrak{F}_{\mathbf{Int}}(n)$ are exactly the upsets $\mathfrak{V}_{\mathbf{Int}}(\varphi)$ for some $\mathcal{L}_n$-formula φ, where $\mathfrak{V}_{\mathbf{Int}}$ is the standard valuation on $\mathfrak{F}_{\mathbf{Int}}(n)$. Now let $w \in W_{\mathbf{Int}}$. For any $k < \omega$, we define the *k-bounded theory of w* to be the set $\mathbb{T}_k(w)$ defined as follows

$$\mathbb{T}_k(w) := \{\varphi \in \Xi_n^k \mid w \in \mathfrak{V}_{\mathbf{Int}}(\varphi)\}.$$

Define a preorder $\precsim_k$ on $W_{\mathbf{Int}}$ by letting

$$w \precsim_k u \iff \mathbb{T}_k(w) \subseteq \mathbb{T}_k(u),$$

and then let $\sim_k \subseteq W_{\mathbf{Int}} \times W_{\mathbf{Int}}$ be the equivalence relation defined by

$$w \sim_k u \iff w \precsim_k u \;\;\&\;\; u \precsim_k w.$$

Therefore, by definition, we have $\precsim_{k+1} \subseteq \precsim_k$, for every $k < \omega$. Furthermore, notice that, since $\mathfrak{F}_{\mathbf{Int}}(n)$ is refined, it follows that

$$\bigcap_k \precsim_k = R_{\mathbf{Int}}.$$

Finally, remark that, by the previous lemma, for each $k < \omega$, there are only finitely many $\sim_k$-equivalence classes, that is, $\sim_k$ has finite index.

For every $w \in W_{\mathbf{Int}}$ and each $k < \omega$, let φ_w^k and ψ_w^k be the $\mathcal{L}_n$-formulas defined as follows:

$$\varphi_w^k := \bigwedge \mathbb{T}_k(w)$$
$$\psi_w^k := \bigvee (\Xi_n^k \setminus \mathbb{T}_k(w)).$$

It is understood that in the above formulas, the sets $\mathbb{T}_k(w)$ and $\Xi_n^k \setminus \mathbb{T}_k(w)$ are considered up to **Int**-equivalence and thus, by Lemma 3.19, φ_w^k and ψ_w^k are well defined and belong to Ξ_n^k.

Remark 8. For every $w, v \in W_{\mathbf{Int}}$ and each $k < \omega$,

$$v \models \varphi_w^k \quad \Longleftrightarrow \quad w \preceq_k v$$
$$v \not\models \psi_w^k \quad \Longleftrightarrow \quad v \preceq_k w.$$

Therefore,

$$v \not\models \varphi_w^k \to \psi_w^k \quad \Longleftrightarrow \quad \exists z \in v\!\uparrow (z \sim_k w).$$

The following proposition shows the connection between the natural order $R_{\mathbf{Int}}$ on $W_{\mathbf{Int}}$ and the preorders $\precsim_k$.

Proposition 3.20. *Let $w, v \in W_{\mathbf{Int}}$ and let $k < \omega$. The following equivalences hold:*

(i) $w \preceq_0 v \Longleftrightarrow w \in \mathfrak{V}_{\mathbf{Int}}(p)$ *implies* $v \in \mathfrak{V}_{\mathbf{Int}}(p)$*, for every* $p \in \mathbf{Var}\mathcal{L}_n$*;*

(ii) $w \preceq_{k+1} v \Longleftrightarrow$ *for every* $v' \in v\!\uparrow$*, there is* $w' \in w\!\uparrow$ *such that* $w' \sim_k v'$*.*

Proof. (i) The direction ($\Longrightarrow$) follows trivially by the definition of $\precsim_0$. Conversely, since each $\varphi \in \Xi_n^0$ is equivalent to a $\mathcal{L}_n$-formula in disjunctive normal form, the direction ($\Longleftarrow$) follows easily by the definition of $\models$.
(ii) For ($\Longleftarrow$), suppose that $\varphi \in \mathbb{T}_{k+1}(w)$. By assumption, there is $z \in w\!\uparrow$ such that $z \sim_k v$. So, $w \precsim_k z \precsim_k v$ and thus it is clear that if $\varphi \in \Xi_n^k$, then $\varphi \in \mathbb{T}_{k+1}(v)$. So assume that φ is of the form $\psi_1 \to \psi_2$, where $c(\psi_i) = k$, for $i \in \{1, 2\}$ and let $v' \in v\!\uparrow$ be such that $v' \models \psi_1$. Then, by assumption, there is $z \in w\!\uparrow$ such that $z \sim_k v'$. Hence, since $\psi_1 \in \mathbb{T}_k(v')$, $\psi_1 \in \mathbb{T}_k(z)$ as well. But $w \precsim_{k+1} z$ and thus $z \models \varphi$. Hence $z \models \psi_2$, that is, $\psi_2 \in \mathbb{T}_k(z)$. Consequently, $\psi_2 \in \mathbb{T}_k(v')$ and, since v' was arbitrary, we can conclude that $v \models \varphi$. Hence $\varphi \in \mathbb{T}_{k+1}(v)$ and therefore $w \preceq_{k+1} v$.
For ($\Longrightarrow$), assume $w \preceq_{k+1} v$ and consider the following set $\{[u]_k \mid u \in v\!\uparrow\}$. Since there are only finitely many $\sim_k$-equivalence classes, such set is finite. By choosing exactly one representative u_i for each equivalence class, we get a finite set $\{u_1, \ldots, u_j\} \subseteq v\!\uparrow$ such that each element $v' \in v\!\uparrow$ is $\sim_k$-equivalent to one of the u_i's. Now, let ξ_v be the following $\mathcal{L}_n$-formula:

$$\xi_v := \bigvee_{i=1}^{j} (\varphi_{u_i}^k \to \psi_{u_i}^k).$$

Notice that $\xi_v \in \Xi_n^{k+1}$ and, for each $z \in W_{\mathbf{Int}}$,

$$\begin{aligned} z \not\models \xi_v &\Longleftrightarrow \text{ for all } i \in \{1, \ldots, j\},\ z \not\models \varphi_{u_i}^k \to \psi_{u_i}^k \\ &\Longleftrightarrow \text{ for all } i \in \{1, \ldots, j\}, \text{ exists } x \in z\!\uparrow \text{ such that } x \sim_k u_i. \end{aligned}$$

In particular, we have $v \not\models \xi_v$. So, $\xi_v \notin \mathbb{T}_{k+1}(v)$ and, since $w \preceq_{k+1} v$, we also get $\xi_v \notin \mathbb{T}_{k+1}(w)$. Therefore, for all $i \in \{1, \ldots, j\}$, there is $w' \in w{\uparrow}$ such that $w' \sim_k u_i$. Hence, by the choice of the u_i's, for every $v' \in v{\uparrow}$, there is $w' \in w{\uparrow}$ such that $w' \sim_k v'$. □

For any $w \in W_{\mathbf{Int}}$, $X \subseteq W_{\mathbf{Int}}$ and $k < \omega$, we let

$$w{\uparrow}^k := \{v \in W_{\mathbf{Int}} \mid w \precsim_k v\},$$
$$X{\uparrow}^k := \bigcup_{w \in X} w{\uparrow}^k.$$

So, by definition, we have that $X{\uparrow}^k$ is the smallest upward ${\uparrow}^k$-closed subset of $W_{\mathbf{Int}}$ containing X. Furthermore, we have the following

Lemma 3.21. *Let $X \subseteq W_{\mathbf{Int}}$. Then $X = \mathfrak{V}_{\mathbf{Int}}(\varphi)$, for some $\varphi \in \Xi^k_n$, if and only if, $X{\uparrow}^k = X$. Therefore, $X \subseteq W_{\mathbf{Int}}$ is admissible in $\mathfrak{F}_{\mathbf{Int}}(n)$ iff $X = X{\uparrow}^k$ for some $k < \omega$.*

Proof. Suppose $X = \mathfrak{V}_{\mathbf{Int}}(\varphi)$, for some $\varphi \in \Xi^k_n$ and let $w \in X{\uparrow}^k$. Then there is $v \in X$ such that $v \precsim_k w$. Therefore $\varphi \in \mathbb{T}_k(w)$ and so $w \in \mathfrak{V}_{\mathbf{Int}}(\varphi) = X$. Conversely, suppose that $X{\uparrow}^k = X$. Let $\delta := \bigvee_{w \in X} \varphi^k_w \in \Xi^k_n$, which is a finite disjunction, provided we consider the index set modulo $\sim_k$-equivalence. Then it is easily seen that $v \in \mathfrak{V}_{\mathbf{Int}}(\delta)$ iff $v \in X{\uparrow}^k$, that is, $X = \mathfrak{V}_{\mathbf{Int}}(\delta)$. Finally, we can conclude that $X \subseteq W_{\mathbf{Int}}$ is admissible in $\mathfrak{F}_{\mathbf{Int}}(n)$ iff it is of the form $\mathfrak{V}_{\mathbf{Int}}(\varphi)$ for some $\varphi \in \mathbf{For}\mathcal{L}_n$ iff $X = X{\uparrow}^k$ for $k = c(\varphi)$. □

Notice that by the previous lemma, for any $w \in W_{\mathbf{Int}}$,

$$w{\uparrow}^k = \mathfrak{V}_{\mathbf{Int}}(\varphi^k_w) \in \mathcal{P}_{\mathbf{Int}}.$$

Furthermore, it is easily seen that

$$w{\downarrow}_k = W_{\mathbf{Int}} \setminus \mathfrak{V}_{\mathbf{Int}}(\psi^k_w) \in -\mathcal{P}_{\mathbf{Int}}.$$

Consequently it follows that the $\sim_k$-equivalence class of w, $[w]_k = w{\uparrow}^k \cap w{\downarrow}_k$, is a finite intersection of elements of $\mathcal{P}_{\mathbf{Int}} \cup -\mathcal{P}_{\mathbf{Int}}$ and thus a clopen set of the Esakia space corresponding to $\mathfrak{F}_{\mathbf{Int}}(n)$. Let us underline this fact with the following

Remark 9. Let $w \in W_{\mathbf{Int}}$ and $k < \omega$. The $\sim_k$-equivalence class of w,

$$[w]_k = \{v \in W_{\mathbf{Int}} \mid v \sim_k w\},$$

is a clopen set in the Esakia space $\mathcal{X}_{\mathfrak{F}_{\mathbf{Int}}(n)}$ corresponding to $\mathfrak{F}_{\mathbf{Int}}(n)$.

Now, for any $k < \omega$, let $[\mathfrak{F}_{\mathbf{Int}}(n)]_k = \langle [W_{\mathbf{Int}}]_k, [\precsim_k] \rangle$ be the canonical quotient of $\langle W_{\mathbf{Int}}, \precsim_k \rangle$, that is,

$$[W_{\mathbf{Int}}]_k = \{[w]_k \mid w \in W_{\mathbf{Int}}\}$$
$$[w]_k [\precsim_k] [v]_k \iff w \precsim_k v.$$

Recall that, for any subset $A \subseteq W_{\mathbf{Int}}$, $[A]$ denote the set of the $\sim_k$-equivalence classes of the elements of A, namely $[A] := \{[a] \mid a \in A\}$. Therefore, by definition of the partial ordering, for every $A \subseteq W_{\mathbf{Int}}$, we have

$$[A{\uparrow}^k] = [A][{\uparrow}^k],$$

that is, $\uparrow^k$-upsets of $W_{\mathbf{Int}}$ corresponds to upsets in $[W_{\mathbf{Int}}]$. Moreover, notice that, for any $\varphi \in \Xi_n^k$, the admissible set $\mathfrak{V}_{\mathbf{Int}}(\varphi)$ of $\mathfrak{F}_{\mathbf{Int}}(n)$ is $\sim_k$-saturated, that is

$$\bigcup[\mathfrak{V}_{\mathbf{Int}}(\varphi)] \subseteq \mathfrak{V}_{\mathbf{Int}}(\varphi).$$

Conversely, given any upset $[A] \subseteq [W_{\mathbf{Int}}]$, we have $[A] = [A][\uparrow^k] = [A\uparrow^k]$ and consequently $[A] = [\mathfrak{V}_{\mathbf{Int}}(\psi)]$, for some $\psi \in \Xi_n^k$, by Lemma 3.21 (and so $[A]$ is also $\sim_k$-saturated). We can thus conclude that the upsets of $[W_{\mathbf{Int}}]$ correspond exactly to the $\sim_k$-equivalence classes $[\mathfrak{V}_{\mathbf{Int}}(\varphi)]$ for the $\mathcal{L}_n$-formulas φ of implicational complexity $\leq k$. Finally, observe that, for $w, v \in W_{\mathbf{Int}}$, if $w \nsim_k v$, then either $w \not\precsim_k v$ or $v \not\precsim_k w$ and thus, by Remark 8, we have either $w \in \mathfrak{V}_{\mathbf{Int}}(\psi_v^k)$ and $v \notin \mathfrak{V}_{\mathbf{Int}}(\psi_v^k)$ or $v \in \mathfrak{V}_{\mathbf{Int}}(\psi_w^k)$ and $w \notin \mathfrak{V}_{\mathbf{Int}}(\psi_w^k)$, that is, w and v are separated by $\sim_k$-saturated admissible sets. Hence $\sim_k$ is a Priestley equivalence relation on $\mathfrak{F}_{\mathbf{Int}}(n)$. Therefore, we have the following

Proposition 3.22. *For any $k < \omega$, the canonical projection*

$$\begin{aligned}\pi_k^\infty \colon \mathfrak{F}_{\mathbf{Int}}(n) &\to [\mathfrak{F}_{\mathbf{Int}}(n)]_k\\ w &\mapsto [w]_k\end{aligned}$$

is an onto Priestley morphism. Therefore, the Priestley dual of the finite frame $[\mathfrak{F}_{\mathbf{Int}}(n)]_k$ is isomorphic to the finite bounded distributive sublattice $\mathbf{F}^k_{\mathcal{HA}}(n)$ of the elements of implicational complexity $\leq k$ of the free Heyting algebra $\mathbf{F}_{\mathcal{HA}}(n)$.

Since for any $j < k$ and $w, v \in W_{\mathbf{Int}}$ we have $w \precsim_k v \Longrightarrow w \precsim_j v$, it follows that the maps

$$\begin{aligned}\pi_j^k \colon [\mathfrak{F}_{\mathbf{Int}}(n)]_k &\to [\mathfrak{F}_{\mathbf{Int}}(n)]_j\\ [w]_k &\mapsto [w]_j\end{aligned}$$

are monotone surjections. Therefore we have the following diagram

$$\mathfrak{F}_{\mathbf{Int}}(n) \ldots \twoheadrightarrow [\mathfrak{F}_{\mathbf{Int}}(n)]_{k+1} \overset{\pi_k^{k+1}}{\twoheadrightarrow} [\mathfrak{F}_{\mathbf{Int}}(n)]_k \twoheadrightarrow \ldots \twoheadrightarrow [\mathfrak{F}_{\mathbf{Int}}(n)]_1 \overset{\pi_0^1}{\twoheadrightarrow} [\mathfrak{F}_{\mathbf{Int}}(n)]_0$$

which we call the *standard approximation of* $\mathfrak{F}_{\mathbf{Int}}(n)$.

3.2.2 Finitely copresented frames

Definition 3.7. Let $\mathfrak{F}$ be a descriptive frame. Then $\mathfrak{F}$ is said to be *finitely copresented* if its dual Heyting algebra $\mathfrak{F}^+$ is finitely presented.

By the previous definition, it immediately follows that a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ is finitely copresented iff it is isomorphic to an admissible set of the n-canonical frame $\mathfrak{F}_{\mathbf{Int}}(n)$ for some $n < \omega$, that is, if $\mathfrak{F} \cong \mathfrak{F}_{\mathbf{Int}}(n)/\varphi$, where $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi$ is the subframe of $\mathfrak{F}_{\mathbf{Int}}(n)$ generated by $\mathfrak{V}_{\mathbf{Int}}(\varphi)$ for some $\mathcal{L}_n$-formula φ[3]. Furthermore, the universal property of the finitely presented Heyting algebra $\mathfrak{F}^+$ translates into the following dual statement for $\mathfrak{F}$:

[3]In terms of Esakia spaces, we have that an Esakia space $\mathcal{X} = \langle X, \tau, R\rangle$ is finitely copresented if it is order-homeomorphic to a clopen upset of the Esakia space $\mathcal{E}_L(n)$ dual to the free Heyting algebra $\mathbf{F}_{\mathcal{V}_L}(n)$ for some finite set of generators n. In particular, if $\mathcal{X}^+ \cong \mathbf{F}_{\mathcal{V}_L}(n)/\varphi$, then $\mathcal{X}$ is order-homeomorphic to the subspace generated by the clopen upset $\widehat{\varphi}$ of $\mathcal{E}_L(n)$.

(#) for every descriptive frame $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ and every family $\{Q_i\}_{1\leq i\leq n}$ of admissible sets of $\mathfrak{G}$ such that $\varphi(Q_1, \dots, Q_n) = V$, there exists a unique p-morphism $h\colon \mathfrak{G} \to \mathfrak{F}$ such that $h^{-1}(G_i) = Q_i$, where the family $\{G_i\}_{1\leq i\leq n}$ is the set of generators of $\mathfrak{F}$. Furthermore, h is injective if $\{Q_i\}_{1\leq i\leq n}$ generates $\mathfrak{G}$.

The following lemma shows the relationship existing between $\mathcal{L}_n$-formulas and the corresponding finitely copresented frames.

Lemma 3.23. *Let φ and ψ be two $\mathcal{L}_n$-formulas. Then*

$$\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \subsetneq \mathfrak{F}_{\mathbf{Int}}(n)/\psi \iff \varphi \to \psi \in \mathbf{Int}.$$

Proof. Since the universes of $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi$ and $\mathfrak{F}_{\mathbf{Int}}(n)/\psi$ are $\mathfrak{V}_{\mathbf{Int}}(\varphi)$ and $\mathfrak{V}_{\mathbf{Int}}(\psi)$ respectively, we have

$$\begin{aligned}
\mathfrak{V}_{\mathbf{Int}}(\varphi) \subseteq \mathfrak{V}_{\mathbf{Int}}(\psi) &\iff \mathfrak{V}_{\mathbf{Int}}(\varphi) \setminus \mathfrak{V}_{\mathbf{Int}}(\psi) = \emptyset \\
&\iff (\mathfrak{V}_{\mathbf{Int}}(\varphi) \setminus \mathfrak{V}_{\mathbf{Int}}(\psi)){\downarrow} = \emptyset \\
&\iff W_{\mathbf{Int}} \setminus (\mathfrak{V}_{\mathbf{Int}}(\varphi) \setminus \mathfrak{V}_{\mathbf{Int}}(\psi)){\downarrow} = W_{\mathbf{Int}} \\
&\iff \mathfrak{V}_{\mathbf{Int}}(\varphi) \supset \mathfrak{V}_{\mathbf{Int}}(\psi) = W_{\mathbf{Int}} \\
&\iff \mathfrak{V}_{\mathbf{Int}}(\varphi \to \psi) = W_{\mathbf{Int}} \\
&\iff \varphi \to \psi \in \mathbf{Int}.
\end{aligned}$$
□

Recall that every finitely generated frame $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ is isomorphic to a generated subframe of the n-canonical frame $\mathfrak{F}_{\mathbf{Int}}(n)$, where n is the cardinality of the set of generators of $\mathfrak{F}$, that is, V is a closed upset of $W_{\mathbf{Int}}$. Since every closed upset in a Esakia space is an intersection of clopen upsets, it follows that

$$V = \bigcap_{j\in J} Q_j,$$

where $Q_j \in \mathcal{P}_{\mathbf{Int}}$ for each $j \in J$.
We call the family $\mathcal{P}_{\mathfrak{G}} = \{Q_j\}_{j\in J}$ a *presentation* of $\mathfrak{G}$. Furthermore, since each $Q_j = \mathfrak{V}_{\mathbf{Int}}(\varphi_j)$ for some $\mathcal{L}_n$-formula φ_j, we define the *implicational degree of* $\mathcal{P}_{\mathfrak{G}}$ as the maximum between the implicational complexity of the $\mathcal{L}_n$-formulas that appear in $\mathcal{P}_{\mathfrak{G}}$, that is, $max(\{c(\varphi_j) \mid \mathfrak{V}_{\mathbf{Int}}(\varphi_j) \in \mathcal{P}_{\mathfrak{G}}\}$. Relying on these facts, we can use the following

Definition 3.8. Let $\mathfrak{G}$ be a generated subframe of the n-canonical frame $\mathfrak{F}_{\mathbf{Int}}(n)$ for some $n < \omega$. The *rank of* $\mathfrak{G}$, $\rho(\mathfrak{G})$, is the minimum implicational degree of a presentation for $\mathfrak{G}$.

Lemma 3.24. *Let $\mathfrak{G} \subsetneq \mathfrak{F}_{\mathbf{Int}}(n)$, for some $n < \omega$. Then $\mathfrak{G}$ is finitely copresented if and only if the rank of $\mathfrak{G}$ is finite.*

Proof. Let $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ and suppose that $\rho(\mathfrak{G}) = k < \omega$. Let $\{\mathfrak{V}_{\mathbf{Int}}(\varphi_j)\}_{j\in J}$ be a presentation for $\mathfrak{G}$ such that, for each $j \in J$, $c(\varphi_j) \leq k$. Then let $v \in V$ and suppose that $v \precsim_k w$ for some $w \in W_{\mathbf{Int}}$. Since $V = \bigcap_{j\in J} \mathfrak{V}_{\mathbf{Int}}(\varphi_j)$, then $v \in \mathfrak{V}_{\mathbf{Int}}(\varphi_j)$ for all $j \in J$. But $c(\varphi_j) \leq k$. Consequently, $w \in \mathfrak{V}_{\mathbf{Int}}(\varphi_j)$ for all $j \in J$, that is, $w \in V$. Hence $V = V{\uparrow}^k$ and so $V \in \mathcal{P}_{\mathbf{Int}}$ by Lemma 3.21. Thus $\mathfrak{G}$ is finitely copresented. □

Example 7. Let $k < \omega$ and consider the frame $[\mathfrak{F}_{\mathbf{Int}}(n)]_k$. Its dual is a finite bounded distributive lattice and thus a finitely presented Heyting algebra. Therefore $[\mathfrak{F}_{\mathbf{Int}}(n)]_k$ is (isomorphic to) an admissible set of $\mathfrak{F}_{\mathbf{Int}}(n)$. In particular,

$$[W_{\mathbf{Int}}] = \bigcup_{[A]\subseteq[W_{\mathbf{Int}}]} [A][\uparrow^k],$$

that is, $[W_{\mathbf{Int}}]$ can be though as the (finite) union of all its (clopen) upsets. Since, for $[A],[B] \subseteq [W_{\mathbf{Int}}]$, we have that also $[W_{\mathbf{Int}}] \setminus ([A] \setminus [B])[\downarrow_k]$ is an upset of $[W_{\mathbf{Int}}]$ and such upset corresponds to an admissible set of implicational complexity $k+1$, it follows that $\rho([\mathfrak{F}_{\mathbf{Int}}(n)]_k) = k + 1$[4].

Given any finitely presented frame $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$, we have the following diagram, where $\iota\colon \mathfrak{G} \hookrightarrow \mathfrak{F}_{\mathbf{Int}}(n)$ is the inclusion $V \subseteq W_{\mathbf{Int}}$ and the ι_k's are the obvious inclusions between the quotients frames.

$$\begin{array}{ccccccccc}
\mathfrak{F}_{\mathbf{Int}}(n)\ldots \twoheadrightarrow & [\mathfrak{F}_{\mathbf{Int}}(n)]_{k+1} & \overset{\pi_k^{k+1}}{\twoheadrightarrow} & [\mathfrak{F}_{\mathbf{Int}}(n)]_k & \twoheadrightarrow \ldots \twoheadrightarrow & [\mathfrak{F}_{\mathbf{Int}}(n)]_1 & \overset{\pi_0^1}{\twoheadrightarrow} & [\mathfrak{F}_{\mathbf{Int}}(n)]_0 \\
\uparrow \iota & \uparrow \iota_{k+1} & & \uparrow \iota_k & & \uparrow \iota_1 & & \uparrow \iota_0 \\
\mathfrak{G}\ldots \longrightarrow\!\!\!\!\!\twoheadrightarrow & [\mathfrak{G}]_{k+1} & \overset{\chi_k^{k+1}}{\longrightarrow\!\!\!\!\!\twoheadrightarrow} & [\mathfrak{G}]_k & \longrightarrow\!\!\!\!\!\twoheadrightarrow \ldots \longrightarrow\!\!\!\!\!\twoheadrightarrow & [\mathfrak{G}]_1 & \overset{\chi_0^1}{\longrightarrow\!\!\!\!\!\twoheadrightarrow} & [\mathfrak{G}]_0
\end{array}$$

We call the family of finite posets $\{[\mathfrak{G}]_i\}_{i<\omega}$ the *standard approximation of* $\mathfrak{G}$.

Lemma 3.25. *For any $j > \rho(\mathfrak{G})$, $[\mathfrak{G}]_j$ is an $[\uparrow^j]$-upset of $[\mathfrak{F}_{\mathbf{Int}}]_j$. Consequently, the inclusion $\iota_j\colon [\mathfrak{G}]_j \hookrightarrow [\mathfrak{F}_{\mathbf{Int}}]_j$ is a p-morphism.*

Proof. Let $[u]$ be a point of $[\mathfrak{G}]_j$, $[v]$ a point in $[\mathfrak{F}_{\mathbf{Int}}(n)]_j$ and suppose that $[u][\preceq_j][v]$. We then have $u \in V = \mathfrak{V}_{\mathbf{Int}}(\varphi)$, for some $\mathcal{L}_n$-formula φ such that $c(\varphi) = \rho(\mathfrak{G})$. Hence, since $u \precsim_j v$, it follows that $v \in V$. Thus $[v] \in [\mathfrak{G}]_j$. □

Observe that, except for the frames $\mathfrak{G}$ and $[\mathfrak{G}]_j$ for $j > \rho(\mathfrak{G})$, none of the $[\mathfrak{G}]_k$ are $[\uparrow^k]$-upsets of $[\mathfrak{F}_{\mathbf{Int}}(n)]_k$. However they come very close to be so, in the following sense: if $[u]_k \in [\mathfrak{G}]_k$, $[v]_k \in [\mathfrak{F}_{\mathbf{Int}}(n)]_k$, then, for all $i < k$,

$$[u]_k[\preceq_k][v]_k \Longrightarrow [v]_i \in [\mathfrak{G}]_i.$$

Indeed, if $[u]_k[\preceq_k][v]_k$, then $u \precsim_k v$ and, by Proposition 3.20, for every $i < k$, we can find $u' \in u\uparrow$ such that $u' \sim_i v$. Therefore, $[v]_i = [u']_i \in [\mathfrak{G}]_i$.

This situation is described in [60] as the property of the inclusion mapping $\iota_k\colon [\mathfrak{G}]_k \to [\mathfrak{F}_{\mathbf{Int}}(n)]_k$ of being π_i^k-*open* or a π_i^k-*p-morphism.* More generally, let $\mathfrak{F} = \langle W, R\rangle, \mathfrak{G} = \langle V, S\rangle$ and $\mathfrak{D}$ be finite Kripke frames and let $f\colon \mathfrak{F} \to \mathfrak{G}$ and $g\colon \mathfrak{G} \to \mathfrak{D}$ be monotone maps. We say that f is *open relatively to* g (briefly g-*open*) if the following condition holds: for every $w \in W, v \in V$,

$$f(w)Sv \Longrightarrow \exists x \in W\,(wRx \;\;\&\;\; (g \circ f)(x) = g(v)).$$

Equivalently, we have that f is g-open if, for every $A \subseteq \mathfrak{D}$,

$$f^{-1}(g^{-1}(A)\downarrow) = f^{-1}(g^{-1}(A))\downarrow.$$

[4]If $\rho([\mathfrak{F}_{\mathbf{Int}}(n)]_k) < k+1$, then every $U \in \mathcal{P}_{\mathbf{Int}}$ would be an upset of $[W_{\mathbf{Int}}]$, but this is impossible because of the infiniteness of $\mathfrak{F}_{\mathbf{Int}}(n)$.

Whenever f is the inclusion map, we say that $\mathfrak{F} \subseteq \mathfrak{G}$ is itself *g-open*[5]. Furthermore, notice the following fact:

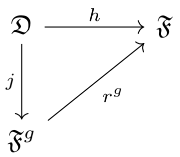
Remark 10. Let $\mathfrak{F} \subseteq \mathfrak{G}$ be g-open and let $w \in \mathfrak{F}$. Then $w{\uparrow}_{\mathfrak{G}} \cap \mathfrak{F}$ is also g-open.

Now, let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G}$ be finite Kripke frames and let a monotone map $g\colon \mathfrak{F} \to \mathfrak{G}$ be given. In this framework, a subset $X \subseteq W$ is said to be *rooted* if X has a least element $x \in X$, which we call the root. We define $\mathfrak{F}^g = \langle W^d, \supseteq\rangle$ to be the Kripke frame whose underlying set is

$$W^d := \{S \subseteq W \mid S \text{ rooted } \ \& \ \ g\text{-open}\}.$$

Furthermore, we let $r^g\colon \mathfrak{F}^g \to \mathfrak{F}$ be the function associating to each element $S \in W^d$ its root. Notice that r^g is order-preserving and it is easily seen that it is also a g-open map by Remark 10. The importance of such a construction is explained in the next proposition, due to Ghilardi [60]:

Proposition 3.26. *Let $g\colon \mathfrak{F} \to \mathfrak{G}$ be a monotone map between the finite Kripke frames $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$. Given a finite Kripke frame $\mathfrak{D} = \langle B, T\rangle$ and a g-open monotone map $h\colon \mathfrak{D} \to \mathfrak{F}$, there exists a unique r^g-open monotone map $j\colon \mathfrak{D} \to \mathfrak{F}^g$ such that the following triangle commutes*



Proof. The map $j\colon \mathfrak{D} \to \mathfrak{F}^g$ is defined as follows: for every $b \in B$,

$$j(b) = \{h(c) \mid c \in b{\uparrow}\}.$$

□

As an immediate consequence of the previous proposition, notice that by choosing the identity on $\mathfrak{F}$ as h, we get a map $r_g\colon \mathfrak{F} \to \mathfrak{F}^g$ such that $r^g \circ r_g = 1_{\mathfrak{F}}$. By definition, we have that $r_g(w) = w{\uparrow}$ and thus we have, for every point $w \in W$ and $Q \in W^g$ with root q,

$$r^g(Q)\, R\, w \Longleftrightarrow Q \supseteq r_g(w),$$

that is, r_g is a right adjoint of r^g which is also a section. Consequently, the map r^g is also co-open[6]. Let us summarize the previous considerations in the following

[5]By considering the dual point of view, the g-openness of $f\colon \mathfrak{F} \to \mathfrak{G}$ says that the dual map $f^+\colon \mathfrak{G}^+ \to \mathfrak{F}^+$ preserves the operation of implication between elements in the image of $g^+\colon \mathfrak{D}^+ \to \mathfrak{G}^+$, that is, for all $D_1, D_2 \in \mathfrak{D}^+$,

$$f^+(g^+(D_1) \to_{\mathfrak{G}^+} g^+(D_2)) = f^+(g^+(D_1)) \to_{\mathfrak{F}^+} f^+(g^+(D_2)).$$

[6]A map $g\colon \mathfrak{F} \to \mathfrak{G}$ between Kripke frames $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ is said to be *co-open* if it is satisfies the following dual condition of p-morphisms: for every $w \in W$ and $q \in V$,

$$f(w)Sq \Longrightarrow \exists v \in W\, (vRw \ \ \& \ \ f(v) = q).$$

Then notice that if $Q \in W^g$ and $w \in W$ are such that $r^g(Q)Rw$, then $r_g(Q) \in W^g$ satisfies such condition.

Corollary 3.27. *Let $g\colon \mathfrak{F} \to \mathfrak{G}$ be an order-preserving map between the finite Kripke frames. Then $r^g\colon \mathfrak{F}^g \to \mathfrak{F}$ is a co-open map that admits a right adjoint $r_g\colon \mathfrak{F} \to \mathfrak{F}^g$ which is a section of it.*

Let a Priestley morphism $g\colon \mathfrak{F} \to \mathfrak{G}$ between finite frames $\mathfrak{F}$ and $\mathfrak{G}$ be given. We iterate the construction above by defining, for all $j \geq 0$, the finite frames $\mathfrak{F}_{j+1}$ and the maps $r_{j+1}\colon \mathfrak{F}_{j+1} \to \mathfrak{F}_j$ inductively as follows:

$$\begin{aligned} \mathfrak{F}_0 &= \mathfrak{F} & \qquad r_0 &= g \\ \mathfrak{F}_{j+1} &= \mathfrak{F}_j^{r_j} & \qquad r_{j+1} &= r^{r_j}. \end{aligned}$$

Notice that, for all $j \geq 1$, each r_j is surjective. Thus we obtain the following diagram:

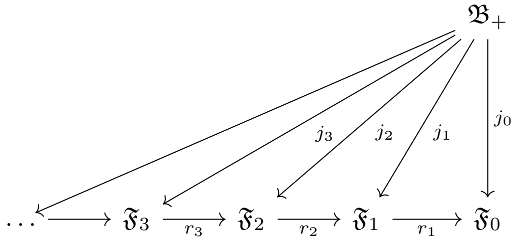
$$\cdots \twoheadrightarrow \mathfrak{F}_{j+1} \xrightarrow{r_{j+1}} \mathfrak{F}_j \xrightarrow{r_j} \cdots \twoheadrightarrow \mathfrak{F}_1 \xrightarrow{r_1} \mathfrak{F}_0 \xrightarrow{r_0} \mathfrak{G} \qquad (\Delta)$$

where, for all $j \geq 0$, r_{j+1} is a r_j-open map. We denote by $\varprojlim_g \mathfrak{F}$ the limit of the above sequence in the category $\boldsymbol{\mathcal{PS}}$ of Priestley spaces and by $r_j^\infty\colon \varprojlim_g \mathfrak{F} \to \mathfrak{F}_j$ the canonical maps. Since $\varprojlim_g \mathfrak{F}$ is a cone on Δ, we have, for all $j \geq 0$, $r_{j+1} \circ r_{j+1}^\infty = r_j^\infty$.

Theorem 3.28. *Let $(\varprojlim_g \mathfrak{F})^+ = \varinjlim_{g^+} \mathfrak{F}^+$ be the dual object of $\varprojlim_g \mathfrak{F}$ in the category $\boldsymbol{\mathcal{DL}}$ of bounded distributive lattices, that is, the colimit of the sequence ∇ dual to Δ. Then $\varinjlim_{g^+} \mathfrak{F}^+$ is a bi-Heyting algebra satisfying the following universal property:*

($\star$) *for every Heyting algebra $\mathfrak{B}$ and any g^+-open $\boldsymbol{\mathcal{DL}}$-morphism $h\colon \mathfrak{F}_0^+ \to \mathfrak{B}$, there exists a unique Heyting morphism $u\colon \varinjlim_{g^+} \mathfrak{F}^+ \to \mathfrak{B}$ such that $u \circ (r_0^\infty)^+ = h$.*

Proof. Notice that, by the very definition of the sequence Δ, the implication operation is well-defined in the colimit $\varinjlim_{g^+} \mathfrak{F}^+$ and thus it is a Heyting algebra. Furthermore, since the r_j's are also co-open for all $j \geq 1$, by Corollary 3.27 it follows that the operation of co-implication is also well-defined in it and thus $\varinjlim_{g^+} \mathfrak{F}^+$ is a bi-Heyting algebra. Now let first consider a finite Heyting algebra $\mathfrak{B}$ and let $h\colon \mathfrak{F}_0^+ \to \mathfrak{B}$ be a g^+-open morphism in $\boldsymbol{\mathcal{DL}}$. Then the dual $j_0 := h_+\colon \mathfrak{B}_+ \to \mathfrak{F}_0$ of h is a g-open Priestley morphism between finite posets. Therefore, by Proposition 3.26, there exists a unique r_1-open monotone map $j_1\colon \mathfrak{B}_+ \to \mathfrak{F}_1$ such that $r_1 \circ j_1 = j_0$. But then we can repeatedly apply Proposition 3.26 to the j_i's and get the following commutative diagram



Therefore $\mathfrak{B}_+$ is a cone on the sequence Δ and thus, by the universal property of limits, there exists a unique Priestley morphism $u\colon \mathfrak{B}_+ \to \varprojlim_g \mathfrak{F}$ such that $r_i^\infty \circ u = j_i$. Let us show that u is in fact an Esakia morphism. So let $c \in \mathfrak{B}_+$, $d \in \varprojlim_g \mathfrak{F}$ and suppose $u(c) \leq d$. By the monotonicity of the r_j^∞'s, we have

$$j_{i+1}(c) \leq_{i+1} r_{i+1}^\infty(d),$$

for all $i \geq 0$, and since j_{i+1} is r_{i+1}-open, it follows that there exists $s_i \in c{\uparrow}$ such that $r_{i+1} \circ j_{i+1}(s_i) = r_{i+1} \circ r_{i+1}^\infty(d)$, that is,

$$r_i^\infty(u(s_i)) = r_i^\infty(d).$$

Now consider the set $S = \{s \in c{\uparrow} \mid r_i^\infty(u(s)) = r_i^\infty(d), \text{ for some } i < \omega\}$. Clearly S is non-empty and finite and thus we can consider a maximal element s^* in S. Then s^* must be a successor of all the s_i and therefore, for all $i \geq 0$, $r_i^\infty(u(s^*)) = r_i^\infty(d)$. Hence $s^* \in c{\uparrow}$ and $u(s^*) = d$; consequently, u is an Esakia morphism such that $r_0^\infty \circ u = j_0$. Dualizing, we have $h = u^+ \circ (r_0^\infty)^+$, so it follows that the Heyting morphism $u^+\colon \varinjlim_{g^+} \mathfrak{F}^+ \to \mathfrak{B}$ is the unique morphism such that $u^+ \circ (r_0^\infty)^+ = h$.

Thus we have proved ($\star$) for every finite Heyting algebra. As a consequence of the fact that the variety $\mathcal{HA}$ of Heyting algebras is finitely approximable, every Heyting algebra can be represented as a subdirect product of finite Heyting algebras and thus, using the universal property of limits, by a standard argument one can finally prove ($\star$) □

Remark 11. By virtue of the universal property ($\star$), $\varinjlim_{g^+} \mathfrak{F}^+$ must be a finitely presented Heyting algebra and it is not difficult to show that the finite set $(r_0^\infty)^+(\mathfrak{F}_0^+)$ is a set of generators for it.

Now we are going to prove that every finitely presented Heyting algebras can be represented as the colimit of a dual sequence of the form Δ.

Theorem 3.29 (Butz [21])**.** *Let $\mathfrak{G}$ be a finitely copresented descriptive frame and let $\rho(\mathfrak{G}) = k$. Then $\mathfrak{G} \cong \varprojlim_{\chi_k^{k+1}} [\mathfrak{G}]_{k+1}$, where $\chi_k^{k+1}\colon [\mathfrak{G}]_{k+1} \to [\mathfrak{G}]_k$.*

Proof. We can assume that $\mathfrak{G}$ is the frame $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi$, where $\varphi \in \Xi_n^k$. Furthermore, we let $G_1, \ldots, G_n$ be the generators of $\mathfrak{G}^+$, so that $\mathfrak{G} = \varphi(G_1, \ldots, G_n)$. Since each G_i is represented by a formula in Ξ_n^0, it follows that each G_i is a $\sim_j$-saturated upset for each $j \geq 0$. Therefore the $[G_i]_{k+1}$'s are upset of $[\mathfrak{G}]_{k+1}$ that generate $([\mathfrak{G}]_{k+1})^+ = \mathfrak{F}_0^+$ and, consequently, the clopen upsets

$$Q_i = (r_0^\infty)^{-1}([G_i]_{k+1}), \qquad i \in \{1, \ldots, n\},$$

are generators of $(\varprojlim_{\chi_k^{k+1}} [\mathfrak{G}]_{k+1})^+$ by the previous Remark. Furthermore, since the canonical projection r_0^∞ is χ_k^{k+1}-open and $[G_i]_{k+1} = (\chi_k^{k+1})^{-1}([G_i]_k)$ for every $i \in \{1, \ldots, n\}$, it follows that,

$$\psi(Q_1, \ldots, Q_n) = (r_0^\infty)^{-1}([\psi(G_1, \ldots, G_n)]_{k+1}),$$

for every $\psi \in \Xi_n^k$. In particular, by the surjectivity of r_0^∞, it follows that $\varphi(Q_1, \ldots, Q_n) = \varprojlim_{\chi_k^{k+1}} [\mathfrak{G}]_{k+1}$. So, by the universal property (#) of $\mathfrak{G}$, there exists an injective Esakia morphism $h\colon \varprojlim_{\chi_k^{k+1}} [\mathfrak{G}]_{k+1} \to \mathfrak{G}$ such that

$$h^{-1}(G_i) = Q_i = (r_0^\infty)^{-1}([G_i]_{k+1}), \qquad \forall i \in \{1, \ldots, n\}.$$

On the other hand, the Priestley morphism $\chi^{\infty}_{k+1}\colon \mathfrak{G} \to [\mathfrak{G}]_{k+1}$ is χ^{k+1}_{k}-open and thus, by the dual of ($\star$), there is a Esakia morphism $u\colon \mathfrak{G} \to \varprojlim_{\chi^{k+1}_k}[\mathfrak{G}]_{k+1}$ such that $r^{\infty}_0 \circ u = \chi^{\infty}_{k+1}$. Hence, for each $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} u^{-1} \circ h^{-1}(G_i) &= (u^{-1} \circ (r^{\infty}_0)^{-1})([G_i]_{k+1}) \\ &= (\chi^{\infty}_{k+1})^{-1}([G_i]_{k+1}) \\ &= G_i, \end{aligned}$$

where the last equation holds since G_i is $\sim_{k+1}$-saturated. But the G_i's are the generators $\mathfrak{G}^+$ and this implies that the composition of the dual maps is the identity on $\mathfrak{G}^+$, that is, $u_+ \circ h_+ = 1_{\mathfrak{G}^+}$. Therefore $h_+\colon \mathfrak{G}^+ \to (\varprojlim_{\chi^{k+1}_k}[\mathfrak{G}]_{k+1})^+$ is injective and, dually, h is onto. Hence $h\colon \varprojlim_{\chi^{k+1}_k}[\mathfrak{G}]_{k+1} \to \mathfrak{G}$ is an order homeomorphisms and so $\mathfrak{G} \cong \varprojlim_{\chi^{k+1}_k}[\mathfrak{G}]_{k+1}$. □

Corollary 3.30. *Every finitely presented Heyting algebra is a bi-Heyting algebra.*

Remark 12. Consider the diagram (Δ). If we let g be the unique map to the one point frame $\mathbb{1}$, then any Priestley morphism from a frame $\mathfrak{G}$ to $\mathfrak{F}_0$ will be g-open. Consequently, by Theorem 3.28, the dual of $\varprojlim_g \mathfrak{F}$ will be the Heyting algebra freely generated by the distributive lattice $\mathfrak{F}_0$. In particular, for $\pi^0_{-1}\colon [\mathfrak{F}_{\mathbf{Int}}(n)]_0 \to \mathbb{1}$ we have $\mathfrak{F}_{\mathbf{Int}}(n) \cong \varprojlim_{\pi^0_{-1}} \mathfrak{F}$.

The following interesting result gives us a criterion to settle when a finitely presented Heyting algebra is well-connected.

Proposition 3.31. *Let $\mathfrak{G}$ be a finitely copresented frame. Then $\mathfrak{G}$ is rooted iff $[\mathfrak{G}]_{\rho(\mathfrak{G})+1}$ is rooted.*

Proof. Let $\rho(\mathfrak{G}) = k$. Clearly, if $\mathfrak{G}$ is rooted, then so is $[\mathfrak{G}]_{\rho(\mathfrak{G})+1}$. Conversely, suppose that $[\mathfrak{G}]_{\rho(\mathfrak{G})+1}$ is rooted. By Theorem 3.29, we have $\mathfrak{G} \cong \varprojlim_{\chi^{k+1}_k} \mathfrak{F}$, so that if we show that $\varprojlim_{\chi^{k+1}_k} \mathfrak{F}$ is rooted we are done. Now, a root for $\varprojlim_{\chi^{k+1}_k} \mathfrak{F}$ would be a point $w = (w_i)_{i<\omega} \in \varprojlim_{\chi^{k+1}_k} \mathfrak{F}$ such that, for each $i < \omega$,

- $r^{\infty}_i(w) \leq_{\mathfrak{F}_i} v$, for all $v \in \mathfrak{F}_i$;
- $r_{i+1} \circ r^{\infty}_{i+1}(w) = r^{\infty}_i(w)$.

Let w_0 be the root of $[\mathfrak{G}]_{\rho(\mathfrak{G})+1} = \mathfrak{F}_0$ and define $w_{i+1} := w_i{\uparrow}$. Clearly, for each $i < \omega$, $w_i \in \mathfrak{F}_i$ and $w_{i+1} \leq_{\mathfrak{F}_{i+1}} v$ for any $v \in \mathfrak{F}_{i+1}$, since points in $\mathfrak{F}_{i+1}$ are subsets of $\mathfrak{F}_i$, $w_{i+1} = \mathfrak{F}_i$ and the ordering is reverse inclusion. Moreover, we have $r_{i+1}(w_{i+1}) = w_i$, since by definition r_{i+1} maps r_i-open rooted subsets of $\mathfrak{F}_i$ onto their roots. Therefore, the point $w = (w_i)_{i<\omega}$ so identified satisfies the two above conditions and thus it is the root of $\varprojlim_{\chi^{k+1}_k} \mathfrak{F}$. □

We conclude this section by reminding the reader a few more results concerning finitely copresented frames. The following theorem has first been proved by Ghilardi and Zawadowski [65] by employing a sheaf representation of Heyting algebras. Cfr. also van Gool, Reggio [161] for a more topological approach to the issue.

Theorem 3.32. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be finitely copresented descriptive frames and let $h\colon \mathfrak{F} \to \mathfrak{G}$ be a p-morphism. Then h is open in the topological sense.*

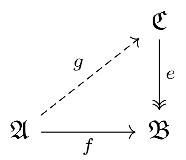
As a consequence of the previous theorem, we get the following remarkable property of Heyting algebras.

Corollary 3.33. *Let $\mathfrak{A}$ be a finitely generated subalgebra of a finitely presented Heyting algebra $\mathfrak{B}$. Then $\mathfrak{A}$ is also finitely presented.*

Proof. Dualizing, we have that $\mathfrak{A}_+$ is a generated subframe of $\mathfrak{F}_{\mathbf{Int}}(n)$, for some $n < \omega$, and a p-morphic image of $\mathfrak{B}_+$. So, let $g\colon \mathfrak{B}_+ \to \mathfrak{A}_+$ be a onto p-morphism. Then $i \circ g\colon \mathfrak{B}_+ \to \mathfrak{F}_{\mathbf{Int}}(n)$, where $i\colon \mathfrak{A}_+ \to \mathfrak{F}_{\mathbf{Int}}(n)$ is the inclusion map, is also a p-morphism. In particular, $i \circ g$ is a p-morphism between finitely copresented descriptive frames and thus it is open by Theorem 3.32. Therefore $i \circ g(\mathfrak{B}_+) = \mathfrak{A}$ is open in $\mathfrak{F}_{\mathbf{Int}}(n)$ and, consequently, $\mathfrak{A}$ is finitely copresented, being a clopen upset of $\mathfrak{F}_{\mathbf{Int}}(n)$. $\square$

3.3 Finitely generated regular projective Heyting algebras

Let L be an intermediate logic and consider its corresponding variety $\mathcal{V}_L$ and the associated equational category $\boldsymbol{\mathcal{V}_L}$. We remind the reader that a Heyting algebra $\mathfrak{A} \in \boldsymbol{\mathcal{V}_L}$ is said to be *regular projective* in $\boldsymbol{\mathcal{V}_L}$ if, for any regular epi $e\colon \mathfrak{C} \to \mathfrak{B}$ and any morphism $f\colon \mathfrak{A} \to \mathfrak{B}$, there exists a morphism $g\colon \mathfrak{A} \to \mathfrak{C}$ such that the following diagram commutes



We now provide a characterization of regular projective finitely generated Heyting algebras for any finitely approximable intermediate logic L. For that purpose, the following result will be useful.

Theorem 3.34. *Let $\boldsymbol{\mathcal{K}}$ be an equational category.*

1. *A retract of a regular projective object is regular projective.*
2. *If $f\colon \mathfrak{A} \to \mathfrak{B}$ is a surjective morphism and $\mathfrak{B}$ is regular projective, then $\mathfrak{B}$ is a retract of $\mathfrak{A}$.*
3. *An algebra $\mathfrak{A}$ in $\boldsymbol{\mathcal{K}}$ is regular projective iff it is a retract of a $\mathcal{K}$-free algebra.*

3.3.1 Finitely generated regular injective frames

Let L be a finitely approximable intermediate logic and consider its corresponding variety $\mathcal{V}_L$ and the associated equational category $\boldsymbol{\mathcal{V}_L}$. Denote by $\boldsymbol{\mathcal{ES}_L}$ and by $\boldsymbol{\mathcal{DF}_L}$ the subcategories of $\boldsymbol{\mathcal{ES}}$ and $\boldsymbol{\mathcal{DF}}$ respectively which are dual to $\boldsymbol{\mathcal{V}_L}$.

Now, by duality, it follows that a Heyting algebra $\mathfrak{A}$ is regular projective in $\boldsymbol{\mathcal{V}_L}$ iff the corresponding dual Esakia space and descriptive frame $\mathfrak{A}_+$ are regular injective in $\mathcal{ES}_L$ and $\mathcal{DF}_L$ respectively. Since we know that in equational categories the regular epimorphism are just the surjective morphisms, we thus have that a descriptive frame $\mathfrak{F}$ for L is regular injective in $\mathcal{DF}_L$ iff, whenever $\mathfrak{G}$ and $\mathfrak{D}$ are descriptive frames for L such that $\mathfrak{G}$ is a generated subframe of $\mathfrak{D}$ and $f\colon \mathfrak{G} \to \mathfrak{F}$ is a p-morphism, then f can be extended to a p-morphism $h\colon \mathfrak{D} \to \mathfrak{F}$. An alternative equivalent formulation of a regular injective frame comes from the dual statement of Theorem 3.34 which tells us that a descriptive frame $\mathfrak{F}$ for L is regular injective iff whenever $\mathfrak{F}$ is a generated subframe of some canonical frame $\mathfrak{F}_L(\kappa)$ for L, then there exists a p-morphism $g\colon \mathfrak{F}_L(\kappa) \to \mathfrak{F}$ which is the identity on $\mathfrak{F}$.

Say that a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ is *finitely approximable* if W_{fin} is dense in the corresponding Esakia space $\mathcal{X}_{\mathfrak{F}} = \langle W, \tau_{\mathcal{P}}, R\rangle$. Furthermore, we denote by $\mathfrak{F}^{\triangledown} = \langle W^{\triangledown}, R^{\triangledown}, \mathcal{P}^{\triangledown}\rangle$ the descriptive frame obtained from $\mathfrak{F}$ by adding a new point below all elements of W, that is,

$$
\begin{aligned}
W^{\triangledown} &= W \cup \{a\};\\
R^{\triangledown} &= R \cup \{\langle a, w\rangle \mid w \in W\};\\
\mathcal{P}^{\triangledown} &= \mathcal{P} \cup \{W^{\triangledown}\};
\end{aligned}
$$

where a is such that $\{a\} \cap W = \emptyset$. Notice that $\mathcal{P}^{\triangledown}$ is the Heyting algebra obtained from $\mathcal{P}$ by adding a new top element or, equivalently, as the vertical sum $\mathcal{P}\overline{\oplus}\,\mathbf{2}$ of $\mathcal{P}$ with the two elements Boolean algebra and it is thus subdirectly irreducible. Moreover, with respect to the corresponding Esakia spaces, we have that $\mathcal{X}_{\mathfrak{F}^{\triangledown}} = \langle W^{\triangledown}, \tau_{\mathcal{P}^{\triangledown}}, R^{\triangledown}\rangle$ is the Esakia space obtained as the *extension topology* of W plus $\{a\}$, which corresponds to the *Alexandroff extension* of W, since $\langle W, \tau_{\mathcal{P}}\rangle$ is a Stone space[7].

Lemma 3.35. *Let L be a finitely approximable intermediate logic and $\mathfrak{F}$ be a regular injective frame in $\mathcal{DF}_L$. Then $\mathfrak{F}$ is finitely approximable.*

Proof. Since $\mathfrak{F}$ is regular injective, then $\mathfrak{F}$ is a p-morphic image of a κ-canonical frame $\mathfrak{F}_L(\kappa)$. Now consider the corresponding Esakia spaces $\mathcal{X}_{\mathfrak{F}} = \langle Y, \tau', S\rangle$ and $\mathcal{X}_{\mathfrak{F}_L(\kappa)} = \langle X, \tau, R\rangle$ of $\mathfrak{F}$ and $\mathfrak{F}_L(\kappa)$ respectively and let $f\colon X \to Y$ be a onto Esakia morphism. From the fact that L is finitely approximable, we deduce that X_{fin} is dense in X and thus $f(X_{\mathrm{fin}})$ is dense in Y, since f is a continuous surjection. Furthermore, since f is a p-morphism, we have that $f(X_{\mathrm{fin}}) \subseteq Y_{\mathrm{fin}}$ which implies that Y_{fin} is a dense subset of Y. Therefore $\mathfrak{F}$ is finitely approximable. □

Lemma 3.36. *Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ be two finitely approximable descriptive frames and let $f\colon W^{<\omega} \to V^{<\omega}$ be a p-morphism between $\mathfrak{F}^{<\omega}$ and $\mathfrak{G}^{<\omega}$. Then there exists a unique p-morphism $h\colon W \to V$ between $\mathfrak{F}$ and $\mathfrak{G}$ that extends f. Furthermore, if f is onto, then also is h.*

Proof. If $f\colon W^{<\omega} \to V^{<\omega}$ is a p-morphism, then it can be shown that the dual map $f^{+}\colon \mathcal{Q}^{<\omega} \to \mathcal{P}^{<\omega}$ of f, defined by $f^{+} = f^{-1}$, is a Heyting algebra

[7] As we mentioned, the dual operation of $^{\triangledown}$ coincide with the operation $\overline{\oplus}\,\mathbf{2}$ and is nothing but the operation Γ introduced by Jaśkowski in [88] where it is also shown that

$$\mathbf{Int} = \{\varphi \in \mathbf{For}\mathcal{L} \mid \mathfrak{J}_n \models \varphi,\ \text{for all } n \geq 1\},$$

where the Kripke frame $\mathfrak{J}_1$ is the one-point frame $\mathbf{1}$ and $\mathfrak{J}_{n+1}$ is the result of the operation $^{\triangledown}$ on the disjoint union of n copies of $\mathfrak{J}_n$.

homomorphism between $(\mathfrak{G}^{<\omega})^+$ and $(\mathfrak{F}^{<\omega})^+$. Then, by duality, the dual map $(f^+)_+$ of f^+, defined by $(f^+)_+ = (f^+)^{-1}$ is a p-morphism from $((\mathfrak{F}^{<\omega})^+)_+$ to $((\mathfrak{G}^{<\omega})^+)_+$. We have thus the following commutative diagram

$$\begin{array}{ccccc}
\mathfrak{F}^{<\omega} & \longrightarrow & \mathcal{Q}^{<\omega} & \longrightarrow & ((\mathfrak{F}^{<\omega})^+)_+ \\
{\scriptstyle f}\downarrow & & {\scriptstyle f^+}\downarrow & & \downarrow{\scriptstyle (f^+)_+} \\
\mathfrak{G}^{<\omega} & \longrightarrow & \mathcal{P}^{<\omega} & \longrightarrow & ((\mathfrak{G}^{<\omega})^+)_+
\end{array}$$

Now, by Lemma 3.15, we have that $((\mathfrak{F}^{<\omega})^+)_+ \cong \mathfrak{F}$ and that $((\mathfrak{G}^{<\omega})^+)_+ \cong \mathfrak{G}$, therefore, by identifying the two frames, we may assume that $(f^+)_+ := h$ is a p-morphism between $\mathfrak{F}$ and $\mathfrak{G}$. By the duality construction, we thus have $h \restriction_{W^{<\omega}} = f$, that is, h extends f. Now, notice that f is a continuous function defined on a dense subset of W and h is a continuous extension of f over W. Suppose that $g \colon W \to V$ is another continuous extension of f over W and consider the set

$$B = \{w \in W \mid g(w) = h(w)\}.$$

Since V is Hausdorff, it not hard to show that B is closed in W. But then, since $W^{<\omega} \subseteq B$, we have $W = \overline{W^{<\omega}} \subseteq \overline{B} = B$. Therefore, $g = h$ and we conclude that h is unique. □

Recall that a subset $X \subseteq W$ of a frame $\mathfrak{F} = \langle W, R\rangle$ covers a point $w \in W$ ($w \preceq X$) if $w{\uparrow} = X{\uparrow} \cup \{w\}$. Moreover, let us state the following

Definition 3.9. Let L be an intermediate logic and let $\mathfrak{F}$ be a descriptive frame for L. Then $\mathfrak{F}$ is said to have the L-*extension property* if, for every descriptive frame $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ such that $\mathfrak{G} \subsetneq \mathfrak{F}$, $d(\mathfrak{G}) < \omega$ and $\mathfrak{G}^{\triangledown} \in \boldsymbol{\mathcal{DF}_L}$, there exists $w \in \mathfrak{F}$ such that $w \preceq V$[8].

We are now ready to provide the (dual) characterization of finitely generated regular projective Heyting algebras mentioned at the beginning of this section.

Theorem 3.37. *Let L be a finitely approximable intermediate logic. A finitely generated descriptive frame $\mathfrak{F}$ for L is regular injective in $\boldsymbol{\mathcal{DF}_L}$ iff $\mathfrak{F}$ is finitely approximable and $\mathfrak{F}$ has the L-extension property.*

Proof. ($\Longrightarrow$) Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a finitely generated regular injective frame in $\boldsymbol{\mathcal{DF}_L}$ and suppose that $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ is a descriptive generated subframe of $\mathfrak{F}$ such that $d(\mathfrak{G}) < \omega$ and $\mathfrak{G}^{\triangledown} \in \boldsymbol{\mathcal{DF}_L}$. By Lemma 3.35, we have that $\mathfrak{F}$ is finitely approximable. Assume that n is the cardinality of the set of generators of $\mathfrak{F}$. Since $\mathfrak{F}$ is regular injective and $\mathfrak{F} \subsetneq \mathfrak{F}_L(n)$, there exists a p-morphism $g \colon \mathfrak{F}_L(n) \to \mathfrak{F}$ which is the identity on $\mathfrak{F}$. Now, since $\mathfrak{G}^{\triangledown} \in \boldsymbol{\mathcal{DF}_L}$ and $\mathfrak{G}^{\triangledown}$ is n-generated as well, $\mathfrak{G}^{\triangledown} \subsetneq \mathfrak{F}_L(n)$ and we let v be the root of $\mathfrak{G}^{\triangledown}$ in $\mathfrak{F}_L(n)$. Then $v \preceq V$ in $\mathfrak{F}_L(n)$ and therefore $g(v) \preceq g(V) = V$, by Lemma 1.12 and the fact that $V \subsetneq W$.
($\Longleftarrow$) Assume that $\mathfrak{F}$ is a finitely approximable, n-generated descriptive frame with the L-extension property. By Lemma 3.13, $\mathfrak{F}$ is a generated subframe of the canonical frame $\mathfrak{F}_L(n) = \langle X, S, \mathcal{Q}\rangle$. In order to prove that $\mathfrak{F}$ is regular

[8]Notice that if $\mathfrak{G}$ is a rooted generated subframe of $\mathfrak{F}$, then the condition is trivial, since w can be taken to be the root of $\mathfrak{G}$.

injective, we need to find a surjective p-morphism h from $\mathfrak{F}_L(n)$ to $\mathfrak{F}$ which is the retract of the inclusion map from $\mathfrak{F}$ to $\mathfrak{F}_L(n)$. So consider the points at finite depth of $\mathfrak{F}_L(n)$, that is, the frame $\mathfrak{F}_L^{<\omega}(n)$. Clearly $\mathfrak{F}^{<\omega} \subsetneq \mathfrak{F}_L^{<\omega}(n)$. We are now going to define the value $f(v)$ of the retract $f\colon X^{<\omega} \to W^{<\omega}$ by induction on the depth of $v \in X^{<\omega}$. First notice that since both the points in $X^{<\omega}$ and $W^{<\omega}$ are atoms of the respective descriptive frames $\mathfrak{F}_L(n)$ and $\mathfrak{F}$, it suffice to define a p-morphism between $\boldsymbol{\kappa}\mathfrak{F}_L^{<\omega}(n)$ and $\boldsymbol{\kappa}\mathfrak{F}^{<\omega}$. So, suppose that f is already defined for all points in $X^{\leq k}$ and let $v \in X^{=k+1}$. Now, if $v \in W^{=k+1}$, then put $f(v) = v$; otherwise, since f is already defined on the set $v^{>}$ of the immediate successors of v, consider the Kripke frame $\mathfrak{G} = \langle f(v^{>}\uparrow), R\restriction_{f(v^{>}\uparrow)}\rangle$. Clearly $\mathfrak{G} \subsetneq \mathfrak{F}^{<\omega} \subsetneq \mathfrak{F}$ and $d(\mathfrak{G}) < \omega$. Furthermore, notice that $\mathfrak{G}^{\triangledown}$ is a p-morphic image of $\mathfrak{F}_L(n)_v = \langle v\uparrow, S\restriction_{v\uparrow}\rangle \in \boldsymbol{\mathcal{DF}_L}$ and thus $\mathfrak{G}^{\triangledown}$ belongs to $\boldsymbol{\mathcal{DF}_L}$ as well. Hence, by hypothesis, there exists a point w in $\mathfrak{F}^{<\omega}$ such that $w \preceq f(v^{>}\uparrow)$ and we let $f(v) = w$. The verification that f is indeed a p-morphism is left to the reader.

Now, since L has the finite model property, the canonical frame $\mathfrak{F}_L(n)$ is finitely approximable and thus the conditions of Lemma 3.36 apply and we get an onto p-morphism $h\colon \mathfrak{F}_L(n) \to \mathfrak{F}$ that extends f. In order to show that h is the morphism we sought, we only need to prove that h is the identity on $\mathfrak{F}$. Take into consideration the corresponding Esakia spaces $\mathcal{X}_{\mathfrak{F}_L(n)}$ and $\mathcal{X}_{\mathfrak{F}}$ of $\mathfrak{F}_L(n)$ and $\mathfrak{F}$ respectively and let $B = \{y \in X \mid h(y) = y\}$. It is easy to prove that B is a closed subset of X and since $\mathcal{X}_{\mathfrak{F}}$ is a generated subspace of $\mathcal{X}_{\mathfrak{F}_L(n)}$, B is also closed in W. Thus, since $W^{<\omega} \subseteq B \subseteq W$ and $W^{<\omega}$ is dense in W, it follows that $B = W$ and so $h\restriction_{\mathfrak{F}} = 1_{\mathfrak{F}}$. □

Remark 13. It absolutely can not be said that Theorem 3.37 above is something really new in the literature on intermediate propositional logics. Something analogous has already been proved by Ghilardi in [62] (cfr. also [65]): a straightforward generalization to finitely approximable intermediate logics of Theorem 5 is basically our result restricted to finitely copresented descriptive frames. In fact, when dealing with finitely copresented frames in any finitely approximable intermediate logic, the requirement of finite approximability can be dropped (cfr. Lemma 5.23) and thus our theorem boils down to the equivalence of (ii) and (iii) of Ghilardi's result.

Of course, our working setting is different from Ghilardi's. In particular, Ghilardi's result relies on a background duality that contains some combinatorial ingredients (some of which are presented in §3.2) which replace the topological ingredients of Esakia duality and force one to work with finitely copresented frames only. Furthermore, Ghilardi's characterization of regular injective finitely copresented frames is mediated by the use of a particular substitution θ_A (the Löwenheim substitution) that makes the proof of such a result more involved and does not allow an immediate translation of the used notions to our setting. Still, the positive side of this approach is its costructiveness and it is thus more appealing when one deals with algorithmic issues.

Theorem 3.37 is more close to Grigolia's Theorem 3.1 of [69] (cfr. Proposition 3.39 below). Indeed, our theorem is a nothing but a generalization to finitely approximable varieties of Heyting algebras of Grigolia's result. Even though Grigolia's proof also makes use of Esakia duality, we still think our proof is more linear and simpler.

To summarize, our Theorem 3.37 can be seen as a slight generalization of some already known results about regular injective frames in the literature on intermediate propositional logics. Its relevance is mainly due to the fact that it relies only on Esakia duality and thus it can be used, alongside with all the topological machinery of Esakia duality, to prove (or provide different proofs of) facts related with regular projective Heyting algebras.

From the previous theorem, by considering only finite frames, we get the following result obtained by Ghilardi for locally tabular intermediate logics [64, Proposition 4].

Proposition 3.38 (Ghilardi)**.** *Let L be a finitely approximable intermediate logic and $\mathfrak{F} = \langle W, R\rangle$ a finite frame for L. Then $\mathfrak{F}$ is regular injective in $\boldsymbol{\mathcal{DF}_L}$ iff, for every generated subframe $\mathfrak{G} = \langle V, S\rangle \subsetneq \mathfrak{F}$ such that $\mathfrak{G}^{\triangledown} \in \boldsymbol{\mathcal{DF}_L}$, there exists $w \in \mathfrak{F}$ such that $w \preceq V$.*

Furthermore, if we consider the variety $\mathcal{HA}$ of Heyting algebras, we get the following results obtained by Grigolia [69, Theorem 3.1].

Proposition 3.39 (Grigolia)**.** *A finitely generated descriptive frame $\mathfrak{F}$ is injective in $\boldsymbol{\mathcal{DF}}$ iff $\mathfrak{F}$ is finitely approximable and has the extension property.*

Notice that in the previous proposition we used the adjective "injective" instead of the phrase "regular injective". For the reason why this change is possible, cfr. the following

Remark 14. Recall that in an equational category $\boldsymbol{\mathcal{K}}$, since all the surjective morphism are epic and the regular epimorphism are exactly the surjective morphisms, the notions of regular epimorphism and epimorphism coincide iff $\boldsymbol{\mathcal{K}}$ has the ES-property, that is, iff $\boldsymbol{\mathcal{K}}$ is a balanced category. For equational categories $\boldsymbol{\mathcal{V}_L}$ corresponding to varieties that algebraize some intermediate logics L, the logical counterpart of the ES-property is the following infinitary version of the so-called *Beth definability property*. Let $X, Y \subseteq \mathbf{Var}\mathcal{L}$ be such that $X \cap Y = \emptyset$ and let $\Gamma \subseteq \mathbf{For}\mathcal{L}_{X\cup Y}$; then Y is said to be *implicitly defined* in terms of X by Γ (in L) if, for all $y \in Y$,

$$\Gamma, \sigma(\Gamma) \vdash_L y \leftrightarrow \sigma(y) \tag{ID}$$

for every substitution σ such that $\sigma(x) = x$ for all $x \in X$. Moreover, we say that Y is *explicitly defined* in terms of X by Γ (in L) if for all $y \in Y$ there exists a $\mathcal{L}_X$-formula φ_y such that

$$\Gamma \vdash_L y \leftrightarrow \varphi_y. \tag{ED}$$

Finally, L is said to have the (infinitary) *Beth property* if (ID) implies (ED)[9].

It is well known that the Beth property is closely related to the *Craig's interpolation property*, namely the former is implied by the latter, and that, for

[9]We say that L has the *finite Beth property* if (ID) implies (ED) when Y is a finite set of variables. Such a property corresponds, algebraically, to the *weak ES-property* that states that all almost onto epis are surjective, where a homomorphism $f\colon \mathfrak{A} \to \mathfrak{B}$ is said to be *almost onto* when $\mathfrak{B}$ is generated by $f(A) \cup \{b\}$ for some $b \in B$. A classical result of Kreisel [93] states that all varieties of Heyting algebras have the weak ES-property. In contrast, Maksimova [108, 109] has shown that a stronger version of the Beth property, called *projective Beth property* and that corresponds algebraically to the request that all strong epis are surjective, holds only for finitely many varieties of Heyting algebras (the exact number of them is 16).

instance, **Int**, **KC**, **LC** and **Cl** do have interpolation[10]. Therefore, for these logics, the corresponding equational categories $\boldsymbol{\mathcal{V}_L}$ are balanced and all the epimorphisms are regular. Hence, by duality it follows that all the monomorphism are regular in $\boldsymbol{\mathcal{ES}_L}$. Now, someone could wonder whether such an equivalence holds for all the subcategories of Esakia spaces. This is not the case: there is a continuum of subcategories $\boldsymbol{\mathcal{ES}_L}$ of $\boldsymbol{\mathcal{ES}}$ for which the equivalence between monomorphism and regular monomorphisms does not hold. Indeed, it has recently been shown in [126] that all the varieties of Heyting algebras corresponding to the logics of bounded width $\mathbf{BW}_n$, for all $n > 2$, as well as a continuum of (locally finite) subvarieties of the variety $\mathbb{V}(\mathbf{RN})$ generated by the Rieger-Nishimura lattice lack the ES-property[11].

The following proposition shows that the variety of Heyting algebras behaves very much like to the variety of lattices when projective algebras are concerned (cfr. Corollary 5.10 of [52]).

Proposition 3.40. *A finitely generated Heyting algebra $\mathfrak{A} \in \mathcal{HA}$ is projective iff $\mathfrak{A}$ is (isomorphic to) a subalgebra of a finitely generated $\mathcal{HA}$-free algebra.*

Proof. Let n be the cardinality of the set of generators of $\mathfrak{A}$. Clearly if $\mathfrak{A}$ is projective, then $\mathfrak{A}$ is the retract of the $\mathcal{HA}$-free algebra $\mathbf{F}_{\mathcal{HA}}(n)$ and thus a subalgebra of it. Conversely, suppose that $\mathfrak{A}$ is a subalgebra of $\mathbf{F}_{\mathcal{HA}}(m)$ for some $m < \omega$ (wlog, we can assume that $n \leq m$). Let $\mathfrak{F} = \langle X, S, \mathcal{Q}\rangle$ be the dual descriptive frame of $\mathfrak{A}$ and consider the m-canonical frame $\mathfrak{F}_{\mathbf{Int}}(m) = \langle W_{\mathbf{Int}}, R_{\mathbf{Int}}, \mathcal{P}_{\mathbf{Int}}\rangle$ for **Int**. By duality, there exists a onto p-morphism $h\colon \mathfrak{F}_{\mathbf{Int}}(m) \to \mathfrak{F}$. By the same reasoning of Lemma 3.35, we have that $\mathfrak{F}$ is finitely approximable. Now let $\mathfrak{G} = \langle V, T, \mathcal{R}\rangle$ be a descriptive frame such that $\mathfrak{G} \subsetneq \mathfrak{F}$ and $d(\mathfrak{G}) < \omega$. Since $\mathfrak{F}$ is finitely generated and the depth of $\mathfrak{G}$ is finite, by letting $min(V) = \{v_1, \ldots, v_k\}$, we have that $\{v_i\}$ is clopen in the Esakia space corresponding to $\mathfrak{F}$, for each $i \in \{1, \ldots, k\}$, and thus $h^{-1}(v_i)$ is clopen in $W_{\mathbf{Int}}$. Since $\mathfrak{F}_{\mathbf{Int}}(m)$ is finitely approximable, the intersection $V_i = h^{-1}(v_i) \cap W^{<\omega}_{\mathbf{Int}}$ is non-empty and so we can consider a point $x_i \in V_i$ for each $i \in \{1, \ldots, k\}$. Now, let $D := \biguplus_{i\in\{1,\ldots,k\}} x_i{\uparrow}$ and notice that $\mathfrak{D} = \langle D, R_{\mathbf{Int}} \restriction_D, \mathcal{P}_{\mathbf{Int}} \cap D\rangle$ is a generated subframe of $\mathfrak{F}_{\mathbf{Int}}(m)$ of finite depth. Hence, by Proposition 3.39, there exists a point $w \in W_{\mathbf{Int}}$ such that $w \preceq D$ and, since p-morphism preserves the covering relation, we have $h(w) \preceq h(D)$. But $h(D) = V$: indeed, if $x \in h(D)$, then $x = h(y)$ for some $y \in D$; so, $y \in x_i{\uparrow}$ for some $i \in \{1, \ldots, k\}$ and, consequently, $v_i = h(x_i)Th(y) = x$, that is, $x \in V$. Conversely, if $x \in V$, then v_jTx for some j. Thus $h(x_j)Tx$ and, since h is a p-morphism, there exists $y \in x_j{\uparrow}$ such that $h(y) = x$; hence $y \in D$ and $x \in h(D)$. Consequently $h(w)$ is a point in X covered by V and therefore $\mathfrak{F}$ is regular injective again by Proposition 3.39. Dualizing, $\mathfrak{A}$ is a projective algebra in $\mathcal{HA}$. □

[10]The algebraic counterpart of Craig's interpolation property for an intermediate logic L is the *amalgamation property* of the corresponding variety $\mathcal{V}_L$: for any $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ in $\mathcal{V}_L$, if $f\colon \mathfrak{A} \to \mathfrak{B}$ and $g\colon \mathfrak{A} \to \mathfrak{C}$ are monomorphisms, then there exists an algebra $\mathfrak{D} \in \mathcal{V}_L$ and monomorphisms $h\colon \mathfrak{B} \to \mathfrak{D}$ and $j\colon \mathfrak{C} \to \mathfrak{D}$ such that $h \circ f = j \circ g$. A classical result of Maksimova [107, 105] states that there are exactly 8 si-logics with the amalgamation property. Cfr. also [41, §A.7] for a proof of the fact that $\mathcal{HA}$ has the amalgamation property based on the duality with Esakia spaces.

[11]The reader interested on the topic of interpolation and definability in modal and intuitionistic logic is referred to the monograph [57].

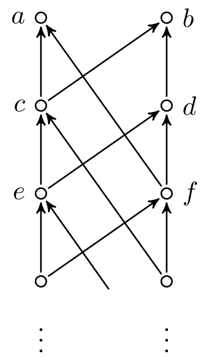
By taking into consideration only the finite frames in Proposition 3.39, we get a different proof of the characterization of finite projective Heyting algebras given by Balbes and Horn [5] in terms of vertical sums of the two- and four-elements Boolean algebras **2** and **4**. Cfr. also [69, Corollary 3.2].

Corollary 3.41. *Let $\mathfrak{A}$ be a finite Heyting algebra. Then $\mathfrak{A}$ is projective in $\boldsymbol{\mathcal{HA}}$ iff $\mathfrak{A}$ is isomorphic to the Heyting algebra*

$$\overline{\bigoplus}_{i=1}^{n} \mathfrak{B}_i,$$

where, for all $i < n$, $\mathfrak{B}_i$ is either isomorphic to the Boolean algebras **2** *or* **4** *and $\mathfrak{B}_n$ is isomorphic to the Boolean algebra* **2***.*

Proof. By duality, it suffice to show that the frame $\mathfrak{A}_+$ is injective in $\boldsymbol{\mathcal{DF}}$ iff it is isomorphic to the frame $\bigoplus_{i=1}^{n} \mathfrak{B}_{i+}$, where, for each $i < n$, $\mathfrak{B}_{i+}$ is either the one point frame $\mathbf{2}_+$: $\circ$ or the two point frame $\mathbf{4}_+ \cong \mathbf{2}_+ \uplus \mathbf{2}_+$: $\circ \quad \circ$ and $\mathfrak{B}_{n+}$ is isomorphic to $\mathbf{2}_+$.
The direction ($\Longleftarrow$) is clear. For ($\Longrightarrow$), suppose that $\mathfrak{A}_+$ is injective and let $\{1, \ldots, n\}$ be the layers of $\mathfrak{A}_+$. Consider the last layer $\mathfrak{A}_+^{=n}$: if $\mathfrak{A}_+^{=n}$ contains more than two points, say v and u, then by Proposition 3.38, there exists w in $\mathfrak{A}_+$ such that $w \preceq \{v, u\}$, thus $w \in \mathfrak{A}_+^{=n+1}$, contrary to the fact that n was the last layer of $\mathfrak{A}_+$. So $\mathfrak{A}_+$ is rooted (and by duality $\mathfrak{A}$ is subdirectly irreducible). Now, for each $k \leq n-1$, $\mathfrak{A}_+^{=k}$ must contain at most 2 points. Otherwise, if for some k, $\mathfrak{A}_+^{=k}$ has $m \geq 3$ elements, then $\mathfrak{A}_+^{=k+1}$ must have at least $2^m - m - 1$ elements (the cardinality of the set of antichains of two or more elements), that is, $|\mathfrak{A}_+^{=k}| < |\mathfrak{A}_+^{=k+1}|$, contrary to the finiteness of $\mathfrak{A}_+$. Therefore $max(\mathfrak{A}_+)$ has at most 2 points. Suppose that $a \neq b$ are the maximal elements of $\mathfrak{A}_+$ and $\mathfrak{A}_+^{=2} = \{c, d\}$. If $c \preceq \{a, b\}$ and $d \preceq \{b\}$, then there must be $e, f \in \mathfrak{A}_+^{=3}$ such that $f \preceq \{d, a\}$ and $e \preceq \{c, d\}$. In the same way $\{f, c\}$ and $\{e, f\}$ must cover the elements of $\mathfrak{A}_+^{=4}$ and thus we eventually get the Rieger-Nishimura ladder



which we know is infinite, getting a contradiction with the finiteness of $\mathfrak{A}_+$. Therefore, if a layer contains two distinct elements a, b, then if the next layer contains two elements as well, these elements are both covered by $\{a, b\}$. If the next layer contains only one element, then this element is covered by $\{a, b\}$. Finally, if $a = b$, then $\{a\}$ covers all the elements of the next layer. We can conclude that $\mathfrak{A}_+ \cong \bigoplus_{i=1}^{n} \mathfrak{B}_{i+}$. □

Corollary 3.42. *Let $A \subsetneq max(\mathfrak{F}_{\mathbf{Int}}(n))$. Then $\mathfrak{F}_{\mathbf{Int}}(n) \setminus A\downarrow$ is injective in $\boldsymbol{\mathcal{DF}}$ for every $n < \omega$.*

Proof. Let $\mathfrak{F}_{\mathbf{Int}}(n) = \langle W_{\mathbf{Int}}, R_{\mathbf{Int}}, \mathcal{P}_{\mathbf{Int}} \rangle$ and denote by $\mathfrak{G} = \langle W_{\mathbf{Int}} \setminus A\downarrow, S, \mathcal{Q} \rangle$ the descriptive generated subframe $\mathfrak{F}_{\mathbf{Int}}(n) \setminus A\downarrow$ of $\mathfrak{F}_{\mathbf{Int}}(n)$. Clearly $\mathfrak{G}$ is also n-generated. Furthermore, since A is a finite union of atoms, $A \in \mathcal{P}_{\mathbf{Int}}$, that is, A is clopen in the corresponding Esakia space $\mathcal{X}_{\mathfrak{F}_{\mathbf{Int}}(n)}$ of $\mathfrak{F}_{\mathbf{Int}}(n)$. Hence since $R_{\mathbf{Int}}$ is a clopen relation, $A\downarrow$ is also clopen and thus $W_{\mathbf{Int}} \setminus A\downarrow$ is a clopen upset of $W_{\mathbf{Int}}$. It follows that $(W_{\mathbf{Int}} \setminus A\downarrow)^{<\omega} = W_{\mathbf{Int}}^{<\omega} \cap (W_{\mathbf{Int}} \setminus A\downarrow)$ is the intersection of a dense subset with an open subset of $W_{\mathbf{Int}}$ and therefore $(W_{\mathbf{Int}} \setminus A\downarrow)^{<\omega}$ is dense in $W_{\mathbf{Int}} \setminus A\downarrow$, that is, $\mathfrak{G}$ is finitely approximable.
Now let $\mathfrak{D} = \langle V, S', \mathcal{Q}' \rangle$ be such that $\mathfrak{D} \subsetneq \mathfrak{G}$ and $d(\mathfrak{D}) < \omega$. Since $\mathfrak{D} \subsetneq \mathfrak{F}_{\mathbf{Int}}(n)$ and $\mathfrak{F}_{\mathbf{Int}}(n)$ is injective in $\boldsymbol{\mathcal{DF}}$, there exists $w \in \mathfrak{F}_{\mathbf{Int}}(n)$ such that $w \preceq V$. If $w \notin \mathfrak{G}$, then $w \in A\downarrow$ and thus $wR_{\mathbf{Int}}m$ for some maximal element $m \in A$. Since $w\uparrow = V \cup \{w\}$, it follows that $m \in V$, contrary to the fact that $V \cap A\downarrow = \emptyset$. Thus w is a point in $\mathfrak{G}$ covered by V and $\mathfrak{G}$ is injective in $\boldsymbol{\mathcal{DF}}$. □

Let L be an intermediate logic and $\mathfrak{F}$ be a descriptive frame. A descriptive generated subframe $\mathfrak{G} \in \boldsymbol{\mathcal{DF}_L}$ of $\mathfrak{F}$ is said to be *L-universal* (*for* $\mathfrak{F}$) if every generated subframe $\mathfrak{D}'$ of $\mathfrak{F}$ which is a p-morphic image of a descriptive frame $\mathfrak{D}$ in $\boldsymbol{\mathcal{DF}_L}$ is also a generated subframe of $\mathfrak{G}$. The following Lemma provides plenty of examples of L-universal generated subframes.

Lemma 3.43. *Let $\mathfrak{F} = \langle V, S, \mathcal{Q} \rangle$ be a generated subframe of the n-canonical frame $\mathfrak{F}_{\mathbf{Int}}(n)$ for* **Int**. *Then, for every intermediate logic L, the generated subframe $\mathfrak{F} \cap \mathfrak{F}_L(n) = \langle V \cap W_L, R, \mathcal{P} \rangle$ of $\mathfrak{F}$, given by the intersection of $\mathfrak{F}$ with the n-canonical frame $\mathfrak{F}_L(n) = \langle W_L, R_L, \mathcal{P}_L \rangle$ for L, is a L-universal subframe for $\mathfrak{F}$.*

Proof. Notice that, since $\mathfrak{F} \cap \mathfrak{F}_L(n) \subsetneq \mathfrak{F}_L(n)$, $\mathfrak{F} \cap \mathfrak{F}_L(n)$ is indeed a generated subframe of $\mathfrak{F}$ in $\boldsymbol{\mathcal{DF}_L}$. So let $\mathfrak{D}'$ be a generated subframe of $\mathfrak{F}$ such that $\mathfrak{D}'$ is a p-morphic image of a frame $\mathfrak{D} \in \boldsymbol{\mathcal{DF}_L}$. Therefore, since $\mathfrak{D}' \subsetneq \mathfrak{F} \subsetneq \mathfrak{F}_{\mathbf{Int}}(n)$, $\mathfrak{D}'$ is also n-generated descriptive frame that belongs to $\boldsymbol{\mathcal{DF}_L}$. But then, by Lemma 3.13, $\mathfrak{D}' \subsetneq \mathfrak{F}_L(n)$. Hence $\mathfrak{D}' \subsetneq \mathfrak{F} \cap \mathfrak{F}_L(n)$. □

The most interesting property of L-universal subframe is stated in the following

Lemma 3.44. *Let L_1, L_2 be two intermediate logics such that $L_1 \subseteq L_2$. Then a L_2-universal generated subframe $\mathfrak{G} \in \boldsymbol{\mathcal{DF}_{L_2}}$ of a regular injective frame $\mathfrak{F}$ in $\boldsymbol{\mathcal{DF}_{L_1}}$ is regular injective in $\boldsymbol{\mathcal{DF}_{L_2}}$.*

Proof. Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be descriptive frames for L_2 such that $\mathfrak{D}_1 \subsetneq \mathfrak{D}_2$ and let $f\colon \mathfrak{D}_1 \to \mathfrak{G}$ be a p-morphism. Since $\mathfrak{G} \subsetneq \mathfrak{F}$, we can consider f as a p-morphism between $\mathfrak{D}_1$ and $\mathfrak{F}$ (more precisely, consider the p-morphism obtained by extending the codomain of f to $\mathfrak{F}$). Since $\mathrm{Ob}(\boldsymbol{\mathcal{DF}_{L_2}}) \subseteq \mathrm{Ob}(\boldsymbol{\mathcal{DF}_{L_1}})$ and $\mathfrak{F}$ is injective, f can be extended to a p-morphism $g\colon \mathfrak{D}_2 \to \mathfrak{F}$. But then, since $\mathfrak{G}$ is L_2-universal for $\mathfrak{F}$ and $g(\mathfrak{D}_2)$ is a generated subframe of $\mathfrak{F}$ which is a p-morphic image of $\mathfrak{D}_2 \in \boldsymbol{\mathcal{DF}_{L_2}}$, $g(\mathfrak{D}_2)$ is also a generated subframe of $\mathfrak{G}$, that is, $g\colon \mathfrak{D}_2 \to \mathfrak{G}$ is a p-morphism and $g\restriction_{\mathfrak{D}_1} = f$. Thus we conclude that $\mathfrak{G}$ is regular injective in $\boldsymbol{\mathcal{DF}_{L_2}}$. □

Now we shall make use of the previous results in order to prove that every descriptive frame for L which is isomorphic to a finitely copresented frame $\mathfrak{F}_L(n)/\varphi$, where $\varphi \in \mathbf{For}\mathcal{L}_n$ is a consistent Harrop formula, is regular injective. Recall that the class $\mathcal{H}$ of *Harrop formulas* is defined inductively as follows:

(i) $\mathbf{Var}\mathcal{L} \cup \{\bot\} \subseteq \mathcal{H}$;

(ii) if $\varphi, \psi \in \mathcal{H}$, then $\varphi \wedge \psi \in \mathcal{H}$;

(iii) if $\varphi \in \mathbf{For}\mathcal{L}$ and $\psi \in \mathcal{H}$, then $\varphi \to \psi \in \mathcal{H}$.

We then have the following

Proposition 3.45. *Let $\varphi \in \mathcal{H}$ be a consistent $\mathcal{L}_n$-formula. Then, for every intermediate logic L, $\mathfrak{F}_L(n)/\varphi$ is regular injective in $\boldsymbol{\mathcal{DF}_L}$.*

Proof. Since, for every formula $\varphi \in \mathbf{For}\mathcal{L}_n$,

$$\mathfrak{F}_L(n)/\varphi = \mathfrak{V}_L(n)(\varphi) = \mathfrak{V}_{\mathbf{Int}}(n)(\varphi) \cap W_L = \mathfrak{F}_{\mathbf{Int}}(n)/\varphi \cap \mathfrak{F}_L(n),$$

by Lemmas 3.43 and 3.44 it suffices to show that $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi$ is injective in $\boldsymbol{\mathcal{DF}}$. Thus we proceed following the inductive definition of $\mathcal{H}$.

(i) Notice that since $\bot$ is excluded by hypothesis we shall deal only with propositional variables. So let $p_i \in \mathbf{Var}\mathcal{L}_n$ and consider the descriptive frame $\mathfrak{F}_{\mathbf{Int}}(n)/p_i = \langle V, R_{\mathbf{Int}} \restriction_V, \mathcal{P}_{\mathbf{Int}} \cap V\rangle$. Since $V = \mathfrak{V}_{\mathbf{Int}}(n)(p_i) \in \mathcal{P}_{\mathbf{Int}}$, V is a clopen upset of $\mathfrak{F}_{\mathbf{Int}}(n)$ and thus $V^{<\omega} = W_{\mathbf{Int}}^{<\omega} \cap V$ is also dense in V, being $W_{\mathbf{Int}}^{<\omega}$ dense in $W_{\mathbf{Int}}$. Therefore $\mathfrak{F}_{\mathbf{Int}}(n)/p_i$ is a finitely approximable frame. Now consider any generated subframe $\mathfrak{G} = \langle V_1, R_1, \mathcal{P}_1\rangle$ of $\mathfrak{F}_{\mathbf{Int}}(n)/p_i$ such that $d(\mathfrak{G}) < \omega$. If V_1 is rooted, that is, if $V_1 = v\uparrow$ for some $v \in V_1$, then $v \preceq V_1$. If V_1 is not rooted, then $min(V_1)$ is an antichain of points at finite depth. Let $c = \langle c_1, \dots, c_n\rangle$ be the minimal colour of the points in $min(V_1)$. By the construction of $\mathfrak{F}_{\mathbf{Int}}^{<\omega}(n)$, there exists a point $w \in W_{\mathbf{Int}}^{=d(\mathfrak{G})+1}$ such that $col(w) = c$ and $w \preceq V_1$. Since $\mathfrak{V}_{\mathbf{Int}}(n)(p_i) = \{x \in W_{\mathbf{Int}} \mid c_i^x = 1\}$, it follows that $c_i^w = c_i = 1$ and thus $w \in V$. Hence, by Proposition 3.39, $\mathfrak{F}_{\mathbf{Int}}(n)/p_i$ is injective in $\boldsymbol{\mathcal{DF}}$[12].

(ii) Let $\varphi, \psi \in \mathbf{For}\mathcal{L}_n$ be two Harrop formulas such that their conjuction is consistent and consider the corresponding finitely copresented frames

$$\mathfrak{F}_{\mathbf{Int}}(n)/\varphi = \langle V_1, R_{\mathbf{Int}} \restriction_{V_1}, \mathcal{P}_{\mathbf{Int}} \cap V_1\rangle$$

and

$$\mathfrak{F}_{\mathbf{Int}}(n)/\psi = \langle V_2, R_{\mathbf{Int}} \restriction_{V_2}, \mathcal{P}_{\mathbf{Int}} \cap V_2\rangle.$$

By duality, we have that

$$\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \wedge \psi = \mathfrak{F}_{\mathbf{Int}}(n)/\varphi \cap \mathfrak{F}_{\mathbf{Int}}(n)/\psi$$

and thus $V_1 \cap V_2 = \mathfrak{V}_{\mathbf{Int}}(n)(\varphi) \cap \mathfrak{V}_{\mathbf{Int}}(n)(\psi)$ is a non-empty clopen upset of $W_{\mathbf{Int}}$. Therefore, it follows that $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \wedge \psi$ is a finitely approximable generated subframe of $\mathfrak{F}_{\mathbf{Int}}(n)$. Now consider any generated subframe

[12] Alternatively, notice that $\mathfrak{F}_{\mathbf{Int}}(n)/p_i$ is isomorphic to the n-canonical frame $\mathfrak{F}_{\mathbf{Int}}(n-1)$ and thus injective by Proposition 3.40.

$\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ of $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \wedge \psi$ such that $d(\mathfrak{G}) < \omega$. As before, if V is rooted, that is, if $V = v{\uparrow}$ for some $v \in V$, then $v \preceq V$. If V is not rooted, then let $c = \langle c_1, \dots, c_n \rangle$ be the minimal colour of the points in $min(V)$. By the construction of $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$, there exists a point $w \in W^{=d(\mathfrak{G})+1}_{\mathbf{Int}}$ such that $col(w) = c$ and $w \preceq V$. Since $col(w) = col(v)$ for some point $v \in min(V) \subseteq V_1 \cap V_2$, it follows that also $w \in V_1 \cap V_2$. Hence, by Proposition 3.39, $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \wedge \psi$ is injective in $\boldsymbol{\mathcal{DF}}$.

(iii) Let $\varphi \in \mathbf{For}\mathcal{L}$ and $\psi \in \mathcal{H}$ and consider the corresponding finitely copresented frames $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi$ and $\mathfrak{F}_{\mathbf{Int}}(n)/\psi$ defined as in (ii) above. Then, by letting $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \to \psi = \langle V, R_{\mathbf{Int}} \restriction_V, \mathcal{P}_{\mathbf{Int}} \cap V \rangle$, we have that $V = W_{\mathbf{Int}} \setminus (V_1 \setminus V_2){\downarrow}$. First notice that if $\psi \leftrightarrow \bot \in \mathbf{Int}$, that is, if $V_2 = \emptyset$, then $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \to \psi \cong \mathfrak{F}_{\mathbf{Int}}(n) \setminus V_1{\downarrow}$. But, for any closed subset U of $W_{\mathbf{Int}}$, we have $U{\downarrow} = max(U){\downarrow}$. Indeed, since $max(U) \subseteq U$, $max(U){\downarrow} \subseteq U{\downarrow}$; conversely, if $x \in U{\downarrow}$, then $xR_{\mathbf{Int}}y$ for some $y \in U$. Since U is closed, by Theorem 2.17 (i), there exists $z \in max(U)$ such that $yR_{\mathbf{Int}}z$ and thus $x \in max(U){\downarrow}$. Therefore, since V_1 is a clopen upset, we have $max(V_1) \subsetneq max(\mathfrak{F}_{\mathbf{Int}}(n))$ and thus $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \to \psi$ is injective in $\boldsymbol{\mathcal{DF}}$ by Corollary 3.42. So, let us assume that ψ is also consistent. Notice that, by the same reasoning as above, $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \to \psi$ is finitely approximable. Now consider any generated subframe $\mathfrak{G} = \langle X, S, \mathcal{Q} \rangle$ of $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \to \psi$ such that $d(\mathfrak{G}) < \omega$ and let $min(X) = \{x_1, \dots, x_m\}$. Suppose that there are no points $k \in V$ such that $k \preceq X$. Since, by the construction of $\mathfrak{F}^{<\omega}_{\mathbf{Int}}(n)$, the set $\{y \in W_{\mathbf{Int}} \,|\, y \preceq X\} \neq \emptyset$, let z be a such a point. Then $z \notin V$, that is, $z \in (V_1 \setminus V_2){\downarrow}$. So, there exists a point $u \in V_1 \setminus V_2$ such that $zR_{\mathbf{Int}}u$. Now, if u is a proper successor of z, then $x_j R_{\mathbf{Int}} u$ for some $x_j \in min(X)$. Therefore $x_j \in (V_1 \setminus V_2){\downarrow}$, but this is impossible because $x_j \in X \subsetneq V$. Consequently, it must be the case that $z = u \in V_1 \setminus V_2$. Since V_1 is an upset, $min(X) \subseteq V_1$ and thus also $min(X) \subseteq V_2$. Hence $\mathfrak{G}$ is also a generated subframe of $\mathfrak{F}_{\mathbf{Int}}(n)/\psi$ of finite depth. Furthermore, by the induction hypothesis, $\mathfrak{F}_{\mathbf{Int}}(n)/\psi$ is injective in $\boldsymbol{\mathcal{DF}}$ and thus, by Proposition 3.39, there exists a point $k \in V_2$ such that $k \preceq X$. But then $k \in V$ too and thus there exists a point in V covered by X. By the *consequentia mirabilis*, we can actually conclude that there are points $k \in V$ such that $k \preceq X$. Consequently $\mathfrak{F}_{\mathbf{Int}}(n)/\varphi \to \psi$ is injective in $\boldsymbol{\mathcal{DF}}$ by Proposition 3.39. ☐

We are now in a position to give a different proof of the following theorem by Minari and Wroński:

Theorem 3.46 (Minari, Wroński [124]). *Let L be an intermediate logic. Then, if $\varphi \to (\eta \vee \delta) \in L$ and $\varphi \in \mathcal{H}$, then $(\varphi \to \eta) \vee (\varphi \to \delta) \in L$.*

Proof. Let $\mathbf{Var}\varphi \cup \mathbf{Var}\eta \cup \mathbf{Var}\delta \subseteq \mathbf{Var}\mathcal{L}_n$ and assume that the hypothesis of the theorems hold. Consider the finitely copresented frames

$$\mathfrak{F}_L(n)/\varphi = \langle V_1, R_L \restriction_{V_1}, \mathcal{P}_L \cap V_1 \rangle,$$
$$\mathfrak{F}_L(n)/\eta = \langle V_2, R_L \restriction_{V_2}, \mathcal{P}_L \cap V_2 \rangle,$$
$$\mathfrak{F}_L(n)/\delta = \langle V_3, R_L \restriction_{V_3}, \mathcal{P}_L \cap V_3 \rangle.$$

Since $\varphi \to (\eta \vee \delta) \in L$ it follows that

$$V_1 \supset (V_2 \cup V_3) = W_L \setminus (V_1 \setminus (V_2 \cup V_3)){\downarrow} = W_L,$$

and, consequently,

$$(V_1 \setminus (V_2 \cup V_3))\!\downarrow = \emptyset. \tag{1}$$

Now, in order to show that $(\varphi \to \eta) \vee (\varphi \to \delta) \in L$, we need to show that the equation $(V_1 \supset V_2) \cup (V_1 \supset V_3) = W_L$ holds or, equivalently, that

$$(V_1 \setminus V_2)\!\downarrow \cap\, (V_1 \setminus V_3)\!\downarrow = \emptyset. \tag{2}$$

First notice that if φ is inconsistent, that is, if $V_1 = \emptyset$, then we are done. Thus we can assume that φ is a consistent $\mathcal{L}_n$-formula. For reductio, let us assume that $w \in (V_1 \setminus V_2)\!\downarrow \cap\, (V_1 \setminus V_3)\!\downarrow$ for some $w \in W_L$. Then there are points $y \in V_1 \setminus V_2$ and $z \in V_1 \setminus V_3$ such that wR_Ly and wR_Lz. Notice that if $y \notin V_3$, then $y \in V_1 \setminus (V_2 \cup V_3)$ and, consequently, $w \in (V_1 \setminus (V_2 \cup V_3))\!\downarrow$, contrary to (1). Therefore, $y \in V_3$ and, analogously, $z \in V_2$. This implies that y and z are incomparable elements of $\mathfrak{F}_L(n)$ and w is a proper predecessor of both y and z. Furthermore, since φ is a consistent Harrop formula, it follows from Proposition 3.45 that $\mathfrak{F}_L(n)/\varphi$ is a regular injective descriptive frame in $\boldsymbol{\mathcal{DF}_L}$. So, let $f \colon W_L \to V_1$ be the p-morphism from $\mathfrak{F}_L(n)$ onto $\mathfrak{F}_L(n)/\varphi$ which is the retract of the inclusion $V_1 \subseteq W_L$. Thus $f(w)$ is a point in V_1 such that $f(w)R_Ly$ and $f(w)R_Lz$, but then $f(w) \notin V_2 \cup V_3$ and $f(w) \in (V_1 \setminus (Q \cup R))\!\downarrow$, contradiction. We can thus conclude that (2) holds. □

Finally notice that, since any negated formula is an Harrop formula, the previous result strengthens Prucnal's Theorem 1.32 stating that the Kreisel-Putnam rule

$$\frac{\neg p \to (q \vee r)}{(\neg p \to q) \vee (\neg p \to r)}$$

is admissible in every intermediate logic.

Chapter 4

Structural completeness

In §1.3.1 we introduced the notion of structural completeness for intermediate propositional logics. We are now going to look deeper into this notion in the more general context of propositional logics both from a logical and an algebraic point of view. The main sources for this chapter are [30] for the logical part and [7] and [38] for the algebraic one.

4.1 The Logical Setting

Given a propositional language $\mathcal{L}$, that is a finite set of connectives with their specified finite arity, recall that an *axiomatic system* AS on $\mathcal{L}$ is given by a pair $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$, where $\mathcal{A}x \subseteq \mathbf{For}\mathcal{L}$ is the non-empty set of axioms and $\mathcal{R} \subseteq \mathcal{P}^{<\omega}(\mathbf{For}\mathcal{L}) \times \mathbf{For}\mathcal{L}$ is the set non-empty of inference rules. The notion of an AS-*derivation* of a formula φ from the set of assumption Γ is defined analogously as for intermediate logics and we write $\Gamma \vdash_{AS} \varphi$. The relation $\vdash_{AS} \subseteq \mathcal{P}(\mathbf{For}\mathcal{L}) \times \mathbf{For}\mathcal{L}$ is called the *derivability relation* of AS[1]. Recall moreover that we call a formula φ a *theorem* of AS if $\vdash_{AS} \varphi$ and the *logic of the axiomatic system* AS, denoted by $\mathbf{AS}$, is defined to be the set of theorems of AS. We will also write $Th(\vdash)$ to denote the set of theorems of the consequence relation $\vdash$, so that $\mathbf{AS} = Th(\vdash_{AS})$.

We have also seen that, by the Łoś-Suzko Theorem, $\vdash_{AS}$ is a finitary consequence relation (cfr. Definition 1.5) and every finitary consequence relation $\vdash$ can be obtained as the derivability relation $\vdash_{AS}$ of an axiomatic system AS. In this situation, we say that $\vdash$ is *axiomatized* by AS. Therefore in what follows we will be often making a systematical confusion between a given axiomatic system AS and its derivability relation $\vdash_{AS}$. Moreover, when speaking of a consequence relation $\vdash$ we will always mean a finitary structural consequence relation.

Definition 4.1. Let $\vdash$ and $\vdash'$ be consequence relations. We say that $\vdash'$ is an *extension* of $\vdash$ and we write $\vdash \leq \vdash'$ if $\vdash \subseteq \vdash'$. Moreover, $\vdash'$ is called a *proper*

[1] In §1.1.1 we required that the rule of substitution (Subst) had to be included in the set of inference rules $\mathcal{R}$ of a given axiomatic system $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$ and defined the derivability relation $\vdash_{AS}$ by omitting the reference to substitution instances. We have opted to change the previous definition in order to conform with the common literature on the topic, cfr., for instance, [142], [30] or [127]. Anyway, it is clear that the two definitions give rise to the same derivability relation, which is what really matters.

extension of $\vdash$, written $\vdash < \vdash'$, if $\vdash \subsetneq \vdash'$. We say that $\vdash'$ is an *axiomatic extension* of $\vdash$ if $\vdash'$ can be axiomatized by adding only axioms to some axiomatization of $\vdash$.

Definition 4.2. Let AS_1 and AS_2 be two axiomatic systems. We say that AS_1 and AS_2 are *logically equal* (notation: $AS_1 \sim_0 AS_2$) if $\mathbf{AS}_1 = \mathbf{AS}_2$, i.e. if their logics are the same, and we say that AS_1 and AS_2 are *deductively equal* (notation: $AS_1 \sim_1 AS_2$) if $\vdash_{AS_1} = \vdash_{AS_2}$.

It follows directly from the previous definitions that both $\sim_0$ and $\sim_1$ are equivalence relations on the class of the axiomatic systems and that, for any two axiomatic systems AS_1 and AS_2, $AS_1 \sim_1 AS_2 \Longrightarrow AS_1 \sim_0 AS_2$.

Example 8. In the present context, $Int = \langle \mathcal{A}x_{Int}, \{\mathrm{MP}, (Subst)\}\rangle$, where $\mathcal{A}x_{Int}$ is the set of axioms from Definition 1.4. Furthermore $\vdash_{Int} < \vdash_{Cl}$ and, in particular, $\vdash_{Cl}$ is an axiomatic extension of $\vdash_{Int}$, since $Cl = \langle \mathcal{A}x_{Int} \cup \{p \vee \neg p\}, \mathrm{MP}\rangle$.

The following simple lemma will be useful in the sequel.

Lemma 4.1. *Let $AS = \langle \mathcal{A}x, \mathcal{R}\rangle$ be an axiomatic system. Then*

$$AS \sim_1 \langle \mathbf{AS}, \mathcal{R}\rangle.$$

Proof. Just notice that extending the set of axioms of AS by the set of theorems of AS does not change the definition of AS-consequence. ☐

4.1.1 Admissible and Derivable Rules

Definition 4.3. Let $AS = \langle \mathcal{A}x, \mathcal{R}\rangle$ be an axiomatic system. An inference rule $r : \Gamma/\varphi$ is said to be *admissible* in AS (or in $\vdash_{AS}$), if the logic $\mathbf{AS}$ is closed under r, that is, if $AS \sim_0 \langle \mathcal{A}x, \mathcal{R} \cup \{r\}\rangle$.

The previous notion has been introduced by Paul Lorenzen in [100] in the context of intuitionistic propositional logic[2]. The following lemma shows that the previous definition is equivalent to the definition of admissible rule given in §1.3.1.

Lemma 4.2. *Let $AS = \langle \mathcal{A}x, \mathcal{R}\rangle$ be an axiomatic system and Γ/φ an inference rule. Then Γ/φ is admissible in AS iff, for every substitution $\sigma\colon \mathbf{Var}\mathcal{L} \to \mathbf{For}\mathcal{L}$,*

$$\sigma(\Gamma) \subseteq \mathbf{AS} \Longrightarrow \sigma(\varphi) \in \mathbf{AS}.$$

For an axiomatic system $AS = \langle \mathcal{A}x, \mathcal{R}\rangle$, we denote by $\mathcal{R}^a$ the set of all admissible rules of AS and by $\mathrel{|\!\!\sim}_{AS}$ the consequence relation defined by the axiomatic system $\langle \mathcal{A}x, \mathcal{R}^a\rangle$. Directly from the definition of admissible rule, it follows that $AS \sim_0 \langle \mathcal{A}x, \mathcal{R}^a\rangle$ and in particular that $\mathrel{|\!\!\sim}_{AS}$ is the greatest consequence relation having $\mathbf{AS}$ as its set of theorems. Moreover, since admissibility depends only on the logic of the axiomatic systems, given two logically equal axiomatic systems $AS_1 = \langle \mathcal{A}x_1, \mathcal{R}_1\rangle$ and $AS_2 = \langle \mathcal{A}x_2, \mathcal{R}_2\rangle$, then a rule Γ/φ is admissible in AS_1 iff it is admissible in AS_2, or, equivalently, $\mathrel{|\!\!\sim}_{AS_1} = \mathrel{|\!\!\sim}_{AS_2}$.

[2] Lorenzen's "operative interpretation" which stands behind the definition of admissible rule is that a rule r is admissible if every application of r can be eliminated from the extended calculus. Cfr. [144].

Definition 4.4. Let $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$ be an axiomatic system. An inference rule Γ/φ is said to be *derivable* in AS (or in $\vdash_{AS}$), if $\Gamma \vdash_{AS} \varphi$.

For an axiomatic system $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$, we denote by $\mathcal{R}^d$ the set of all derivables rules of AS. It is easily seen that $AS \sim_1 AS' = \langle \mathcal{A}x, \mathcal{R}^d \rangle$.

4.1.2 Structural Completeness and related notions

We will now recall the definition of structural completeness and introduce some other variants which will be of interest in the rest of this thesis. The notion of structural completeness was first introduced in [134] by Pogorzelski and has been since then an interesting theme of investigation[3].

Structural Completeness

Definition 4.5. An axiomatic system $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$ is said to be *structurally complete* if every admissible rule in AS is derivable in AS, that is, if $\vdash\!\sim_{AS} \subseteq \vdash_{AS}$.

Notice that, since every derivable inference rule in AS is admissible in AS, for a structurally complete axiomatic system AS we have $\vdash\!\sim_{AS} = \vdash_{AS}$. The following lemma provides an intrinsic characterization of structural completeness, which is often used as a definition.

Lemma 4.3 (Makinson, [104])**.** *Let $\vdash$ be a consequence relation. Then $\vdash$ is structurally complete iff every proper extension $\vdash'$ contains new theorems, that is*

$$\vdash \, < \, \vdash' \Longrightarrow Th(\vdash) \subsetneq Th(\vdash'). \tag{sc}$$

Proof. ($\Longrightarrow$) Suppose that $\vdash$ is structurally complete. We show (sc) by contraposition. If $Th(\vdash) = Th(\vdash')$, then, being $\vdash\!\sim$ the greatest consequence relation having the same set of theorems as $Th(\vdash)$, we have $\vdash' \subseteq \vdash\!\sim$. Since $\vdash$ is structurally complete, we have $\vdash\!\sim \subseteq \vdash$. Therefore $\vdash' \leq \vdash$.
($\Longleftarrow$) Assume conversely that (sc) holds. Let $\mathcal{R}$ be the set of rules derivable in $\vdash$. Then, for $AS = \langle Th(\vdash), \mathcal{R} \rangle$, we have $\vdash \, = \, \vdash_{AS}$. Suppose for contradiction that $\vdash_{AS}$ is not structurally complete, that is, suppose that there exists a rule r which is admissible in $\vdash_{AS}$ but not derivable. Then consider the axiomatic system $AS' = \langle Th(\vdash), \mathcal{R} \cup \{r\} \rangle$: we have $\vdash_{AS} < \vdash_{AS'}$, but since r is admissible $Th(\vdash_{AS}) = Th(\vdash_{AS'})$, contradicting (sc). □

For any consequence relation $\vdash$, $\vdash\!\sim$ is also called the *structural completion* of $\vdash$. By factoring out the set of all axiomatic systems by the equivalence relation $\sim_0$, it follows that the structural complete axiomatic systems are the

[3] Immediately after Pogorzelski's 1971 paper, Prucnal [135, 136] showed the structural completeness of some classes of purely implicational axiomatic systems, while Dzik and Wroński [39] proved that the intermediate logics **LC** and $\mathbf{BD}_n$ are structurally complete. Makinson [104] gave a different characterization of structural completeness in the language of consequences operators and Prucnal and Wroński [139] an algebraic reformulation of the same notion (cfr. also [7]). Citkin [27, 28] studied the notion of structural completeness in the lattice of si-logics, while Rybakov [142] investigated structural completeness in the lattice of extension of modal logic $K4$ and Moraschini [125] in the lattice of extensions of the positive fragment of $K4$. For many-valued logics, including Łukasiewicz logics, cfr. the papers of Wojtilak [162, 163, 164]. For fuzzy logics, see Cintula and Metcalfe [25], while the reader interested in structural completeness in the context of substructural logics is referred to Olson et al. [127].

$\leq$-maximal members of the $\sim_0$-equivalence classes. Therefore in this sense we can see structural completeness as a maximality condition[4].

Hereditarily Structural Completeness

Definition 4.6. An axiomatic system $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$ is said to be *hereditarily structurally complete* (*hsc*, for short) if every (not necessarily proper) extension of $\vdash_{AS}$ is structurally complete.

Clearly a hsc axiomatic system is structural complete. One may wonder if the converse also hold, that is if the notion of hereditarily structural completeness and structural completeness are equivalent. That is not the case, since, for instance, in [27] it is shown that Medvedev's logic **ML**, despite being structural complete, it is not hereditarily structural complete[5]. Notice moreover that any extension of a hsc consequence relation is hsc.

The following lemma gives another characterization of hsc axiomatic systems.

Lemma 4.4. *Let $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$ be an axiomatic system. Then AS is hsc iff, for every consequence $\vdash \geq \vdash_{AS}$, $\vdash = \vdash_{AS'}$ where $AS' = \langle \mathcal{A}x \cup Th(\vdash), \mathcal{R} \rangle$.*

Proof. ($\Longrightarrow$) Assume $\vdash_{AS}$ is hsc and let $\vdash$ be a consequence relation such that $\vdash_{AS} \leq \vdash$. Now let AS' be the axiomatic system $\langle \mathcal{A}x \cup Th(\vdash), \mathcal{R} \rangle$. Since $\mathcal{A}x \subseteq Th(\vdash)$ and $\mathcal{R} \subseteq \vdash$, we have $\vdash_{AS} \leq \vdash_{AS'} \leq \vdash$. Moreover, $Th(\vdash_{AS'}) = Th(\vdash)$ and since $\vdash_{AS}$ is hsc, we have that $\vdash_{AS'}$ is structurally complete and thus $\vdash_{AS'} = \vdash$ by Lemma 4.3.
($\Longleftarrow$) Suppose $\vdash_{AS} \leq \vdash_1 < \vdash_2$ and consider $AS_1 = \langle \mathcal{A}x \cup Th(\vdash_1), \mathcal{R} \rangle$ and $AS_2 = \langle \mathcal{A}x \cup Th(\vdash_2), \mathcal{R} \rangle$. Therefore, by the condition of the lemma, $\vdash_{AS_1} = \vdash_1$ and $\vdash_{AS_2} = \vdash_2$. Since $\vdash_1$ propertly extend $\vdash_2$, we must have $Th(\vdash_1) \subsetneq Th(\vdash_2)$, that is $\vdash_1$ is structurally complete by Lemma 4.3. □

Notice that the previous lemma allows us to interpret hereditarily structural completeness in the following way: given a hsc axiomatic system $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$, every new (non derivable) rule of inference r can be replaced by a set of axioms. Indeed, if $AS_1 = \langle \mathcal{A}x, \mathcal{R} \cup \{r\} \rangle$, then by the previous proposition we have $AS_1 \sim_1 AS' = \langle \mathcal{A}x \cup Th(\vdash_{AS_1}), \mathcal{R} \rangle$.

Active and Passive Structural Completeness

It could be the case that, given an axiomatic system AS and an inference rule r, r is admissible in AS only because the condition defining admissibility is trivially satisfied. Such rules have been first isolated by Rybakov et al. in [143] and are called *passive*.

Definition 4.7. Let $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$ be an axiomatic system and let $r : \Gamma/\varphi$ an inference rule. The inference rule r is said to be *active* if there exists a

[4]In logic it is very often the case that the notion of completeness is connected to some kind of maximality and the case of structural completeness provides an example of such a phenomenon. For the study of maximality/completeness conditions such as structural completeness, Post-completeness and definitional completeness on axiomatic systems, cfr. [151].

[5]For the study of hereditarily structural complete axiomatic systems with a particular focus on intermediate logics, cfr. [30].

substitution $\sigma\colon \mathbf{Var}\mathcal{L} \to \mathbf{For}\mathcal{L}$ such that

$$\sigma(\Gamma) \subseteq \mathbf{AS},$$

and r is said to be *passive* otherwise, that is, if, for every substitution σ,

$$\sigma(\Gamma) \nsubseteq \mathbf{AS}.$$

Passive inference rules have been called like this because they are really "passive": they can never be used for any real derivation. Nevertheless, passive rules are still of interest: when taking into consideration passive rules one can get different notions of structural completeness.

Definition 4.8. Let $AS = \langle \mathcal{A}x, \mathcal{R} \rangle$ be an axiomatic system:

(i) we say that AS is *passively structurally complete* if every passive inference rule of AS is derivable in AS;

(ii) we say that AS is *actively structurally complete* if every admissible in AS active inference rule of AS is derivable in AS.

It is clear that an axiomatic system AS is structurally complete iff it is actively and passively structurally complete. The notion of passive structural completeness has been introduced by Wronski [168] under the name of *non-overflow completeness* and has been investigated also by [54] and [25][6], while the notion of active structural completeness has been first introduced by Dzik in [36] and then studied from an algebraic perspective in [38].

It can be easily seen that passive structural completeness is preserved upwards, that is, if a consequence relation $\vdash$ is passively structurally complete, then all its extensions are passively structurally complete too. Moreover, the following observation is of interest.

Proposition 4.5 (Wroński)**.** *The intuitionistic propositional calculus Int is passively structurally complete.*

Proof. Indeed, suppose that the inference rule Γ/φ is not derivable in Int. Then also $\Gamma/\bot$ is not derivable in Int, which is equivalent by the Deduction Theorem to $\bigwedge\Gamma \to \bot \notin \mathbf{Int}$. By Glivenko's theorem, $\bigwedge\Gamma \to \bot \notin \mathbf{Cl}$ and thus, for some model $\mathfrak{M}$ based on the frame $\imath$, we have $\mathfrak{M} \not\models \neg\Gamma$. Therefore $\mathfrak{M} \models \Gamma$ and we can define a substitution σ by

$$\sigma(p_i) = \begin{cases} \top & \text{if } \mathfrak{M} \models p_i \\ \bot & \text{otherwise.} \end{cases}$$

for all $p_i \in \mathbf{Var} \bigwedge\Gamma$. Then for all $\psi \in \Gamma$, $\sigma(\psi) \in \mathbf{Cl}$ and since $\mathbf{Int}$ is equal to $\mathbf{Cl}$ with respect to variables free formulas, we have $\sigma(\psi) \in \mathbf{Int}$ for all $\psi \in \Gamma$. Hence $\sigma(\Gamma) \subseteq \mathbf{Int}$, that is, Γ/φ is not passive. ☐

It follows that every intermediate logic L is passively structural complete and that structural completeness and active structural completeness are equivalent notions in the context of intermediate logics.

[6] An inference rule Γ/p is said to be an *overflow rule* if $p \notin \mathbf{Var}\Gamma$ and an axiomatic system AS is called *overflow complete* if every admissible in AS overflow rule is derivable in AS. Notice that AS is overflow complete iff, for every inference rule Γ/φ, if Γ/φ is not derivable in AS, then there exists a substitution σ such that $\sigma(\Gamma) \subseteq \mathbf{AS}$ and this last condition is equivalent by contraposition to AS being passively structurally complete.

4.2 The Algebraic Setting

We have seen in §1.5 that the algebraic semantics for intuitionistic propositional logic is given by the class of all Heyting algebras $\mathcal{HA}$, while, for any intermediate logic L, the algebraic semantics is given by a specific subclass $\mathcal{V}_L$ of Heyting algebras. All these classes of algebras are actually varieties and, in general, most of the axiomatic systems throughly studied by logicians are algebraizable by a variety of algebras of the appropriate similarity type. However, when dealing with the notion of structural completeness or, more generally, with inference rules, the algebraic language of quasivariety theory is the most suited[7]. Therefore, let us start with some useful facts from the theory of quasivarieties[8].

Recall that a quasivariety is a class $\mathcal{Q}$ of algebras of the same similarity type which is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_U$, or, equivalently, a class $\mathcal{Q}$ satisfying a given set of quasi-identity. Since a quasivariety $\mathcal{Q}$ is not closed under $\mathbb{H}$, we should be careful when dealing with congruences of algebras from $\mathcal{Q}$. Furthermore, let us denote the algebra presented by $\langle Y|S\rangle$ by $\mathbf{F}_{\mathcal{Q}}(Y,S)$ (or simply $\mathbf{F}(Y,S)$ when the class to which it belongs is clear). Notice that if S is empty, then $\mathbf{F}_{\mathcal{Q}}(Y,S)$ is just $\mathbf{F}_{\mathcal{Q}}(Y)$, that is, the $\mathcal{Q}$-free algebra over Y.

The following theorem establishes a connection between quasi-identities and finitely presented algebras in a given quasivariety $\mathcal{Q}$ of Ω-algebras.

Theorem 4.6. *Let $\mathcal{Q}$ be a quasivariety of Ω-algebras and let φ be the following quasi-identity over $\Omega(X)$:*

$$p_1 \approx q_1 \;\&\; \ldots \;\&\; p_n \approx q_n \Rightarrow p \approx q.$$

Denote by ψ the first-order formula $\&_{i=1}^{n} p_i \approx q_i$ and let $S_\psi \subsetneq Id(X)$ be the following set of identities: $S_\psi = \{p_i \approx q_i \mid i \in \{1,\ldots,n\}\}$. Moreover let $\pi\colon \mathbf{F}_{\mathcal{Q}}(\overline{X}) \to \mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi)$ and $\nu\colon \mathbf{Tm}(X) \to \mathbf{F}_{\mathcal{Q}}(\overline{X})$ be the natural maps. Then the two following series of equivalences holds:

(i) $\mathbf{F}_{\mathcal{Q}}(\overline{X}) \models \psi \Longleftrightarrow \mathbf{F}_{\mathcal{Q}}(\overline{X}) \models S_\psi \Longleftrightarrow \mathbf{F}_{\mathcal{Q}}(\overline{X}) \cong \mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi)$

(ii) $\mathcal{Q} \models \varphi \Longleftrightarrow \mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi) \models p \approx q\ [\pi \circ \nu]$.

Proof. (i) The first equivalence is trivial. Now, consider the surjective homomorphism $\pi \circ \nu\colon \mathbf{Tm}(X) \to \mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi)$ given by the composition of the two natural projections ν and π. Notice that, by the Homomorphism's Theorem, $\mathbf{Tm}(X)/\ker(\pi \circ \nu)$ is isomorphic to $\mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi)$ and $\ker(\pi \circ \nu)$ is the smallest congruence on $\mathbf{Tm}(X)$ containing S_ψ. Thus if $\mathbf{F}_{\mathcal{Q}}(\overline{X}) \cong \mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi)$, then $\theta_{\mathcal{Q}}(X) = \ker(\pi \circ \nu)$, thus $S_\psi \subseteq \theta_{\mathcal{Q}}(X)$ and $\mathbf{F}_{\mathcal{Q}}(\overline{X}) \models S_\psi$. Conversely, if $\mathbf{F}_{\mathcal{Q}}(\overline{X}) \models S_\psi$, then $S_\psi \subseteq \theta_{\mathcal{Q}}(X)$, whence $\ker(\pi \circ \nu) \subseteq \theta_{\mathcal{Q}}(X)$. Since $\mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi) \in \mathcal{Q}$, we have $\ker(\pi \circ \nu) = \theta_{\mathcal{Q}}(X)$ by the definition of $\theta_{\mathcal{Q}}(X)$. Therefore $\mathbf{F}_{\mathcal{Q}}(\overline{X}) \cong \mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi)$.
(ii) If $\mathcal{Q} \models \varphi$, then since $\mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi) \in \mathcal{Q}$, we have $\mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi) \models \varphi$. By definition

[7]For every axiomatic system AS, its finitary structural consequence relation $\vdash_{AS}$ can be associated with an algebraic semantics given by a quasivariety. The relation $\vdash_{AS}$ is thus said to be *BP-algebraizable*, that is, algebraizable in the sense of Blok and Pigozzi. Cfr. [51] for issues related to algebraizability of arbitrary structural consequence relations and, more generally, for an introduction to the field of *abstract algebraic logic*. Cfr. also [54] for the study of admissible rules for such consequence relations in the framework of abstract algebraic logic. Cfr. also the footnote 9 below.

[8]The reader interested in the theory of quasivarieties should consult Gorbunov's book [67].

of $\models$, it follows that if $\mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi) \models \psi\ [\pi \circ \nu]$, then $\mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi) \models p \approx q\ [\pi \circ \nu]$. But the antecedent holds by the definition of $\mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi)$, thus $\mathbf{F}_{\mathcal{Q}}(\overline{X}, S_\psi) \models p \approx q\ [\pi \circ \nu]$ and $\langle p, q\rangle \in \ker(\pi \circ \nu)$. Finally, let $\mathfrak{A} \in \mathcal{Q}$, $g\colon \mathbf{Tm}(X) \to \mathfrak{A}$ a valuation in $\mathfrak{A}$ and suppose that $g(\psi)$ holds in $\mathfrak{A}$. Then $S_\psi \subseteq \ker(g)$, whence $\ker(\pi \circ \nu) \subseteq \ker(g)$. Since $\langle p, q\rangle \in \ker(\pi \circ \nu)$, it follows that $g(p) = g(q)$, that is, $\mathfrak{A} \models \varphi$. Since $\mathfrak{A}$ was arbitrary, we conclude that $\mathcal{Q} \models \varphi$. ☐

The following lemma will also be useful.

Lemma 4.7. *Let $\mathcal{Q}$ be a quasivariety and $\mathfrak{A} \cong \mathbf{F}_{\mathcal{Q}}(Y, S)$ be a $\mathcal{Q}$-finitely presented algebra. Consider the quasi-identity*

$$\varphi(y_1, \dots, y_k)\colon p_1 \approx q_1\ \&\ \dots\ \&\ p_n \approx q_n \Rightarrow p \approx q,$$

where $S = \{p_i \approx q_i \mid i \in \{1, \dots, n\}\}$. Then, for any $\mathfrak{B} \in \mathcal{Q}$, $\mathfrak{B} \not\models \varphi \iff$ there exists a homomorphism $h\colon \mathfrak{A} \to \mathfrak{B}$ such that $\mathfrak{B} \not\models p(h([\vec{y}\,])) = q(h([\vec{y}\,]))$.

Proof. ($\Longrightarrow$) If $\mathfrak{B} \in \mathcal{Q}$ is such that $\mathfrak{B} \not\models \varphi(y_1, \dots, y_k)$, then there are elements $\vec{b} = b_1, \dots, b_k \in B$ such that we have $p_i(\vec{b}\,) = q_i(\vec{b}\,)$ for all $i \in \{1, \dots, n\}$ but $p(\vec{b}\,) \neq q(\vec{b}\,)$ in $\mathfrak{B}$. Then let $g\colon Y \to B$ be the function associating to each y_j the corresponding b_j as in the above equalities. Then, by the universal property for finitely presented algebras, there exists a homomorphism $\hat{h}\colon \mathfrak{A} \to \mathfrak{B}$ such that $\hat{h} \circ i_{\mathfrak{A}} = g$. Therefore we have $\mathfrak{B} \not\models p(\hat{h}([\vec{y}\,])) = q(\hat{h}([\vec{y}\,]))$.
($\Longleftarrow$) Follows from the fact that $\mathfrak{A} \models \&_{i=1}^{n} p_i \approx q_i$ and positive formulas are preserved by homomorphism. ☐

4.2.1 Structural Completeness and related notions in Quasivarieties

The following notions probably would not have been taken into consideration if the correspondence between axiomatic systems and quasivarieties mentioned above had not been discovered.

Definition 4.9. Let $\mathcal{Q}$ be a quasivariety. Then we say that

- $\mathcal{Q}$ is *structurally complete* if, for every quasivariety $\mathcal{Q}'$,

  $$\mathcal{Q}' \subsetneq \mathcal{Q} \Longrightarrow \mathbb{V}(\mathcal{Q}') \subsetneq \mathbb{V}(\mathcal{Q}),$$

  that is, if every proper subquasivariety $\mathcal{Q}'$ generates a proper subvariety;

- $\mathcal{Q}$ is *primitive* if $\mathcal{Q}$ is hereditarily structurally complete. Equivalently stated, for all quasivarieties $\mathcal{Q}_1, \mathcal{Q}_2$,

  $$\mathcal{Q}_2 \subsetneq \mathcal{Q}_1 \subseteq \mathcal{Q} \Longrightarrow \mathbb{V}(\mathcal{Q}_2) \subsetneq \mathbb{V}(\mathcal{Q}_1).$$

Since the correspondence between axiomatic systems and quasivarieties is effective, the definitions and the results obtained in §4.1.2 and §4.1.2 for consequence relations can be operationally translated in terms of the corresponding quasivarieties. For instance, consider a consequence relation $\vdash$ and its corresponding quasivariety $\mathcal{Q}^{\vdash}$ and let $\boldsymbol{\tau}$ be the map transforming formulas to (set of) equations in such a way that $\varphi \in Th(\vdash)$ implies that $\mathcal{Q}^{\vdash} \models \boldsymbol{\tau}(\varphi)$. Then an inference rule Γ/φ is admissible in $\vdash$ implies that the $\mathcal{Q}$-free algebra on ω generators satisfies the quasi-equation $\boldsymbol{\tau}(\Gamma) \Rightarrow \boldsymbol{\tau}(\varphi)$, since substitutions can be seen as valuations in the $\mathcal{Q}$-free algebra[9]. Therefore, the translation of lemmas 4.3 and 4.4 reads as follows:

Lemma 4.8. *A quasivariety $\mathcal{Q}$ is structurally complete if, for every quasi-identity q, if $\mathbf{F}_{\mathcal{Q}}(\omega) \models q$, then $\mathcal{Q} \models q$, that is, if*

$$\mathcal{Q} = \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega)).$$

Lemma 4.9. *A quasivariety $\mathcal{Q}$ is primitive iff, for every quasivariety $\mathcal{Q}'$,*

$$\mathcal{Q}' \subseteq \mathcal{Q} \Longrightarrow \mathcal{Q}' = \mathcal{Q} \cap \mathbb{V}(\mathcal{Q}').$$

The notion of structural completion also makes sense in the algebraic framework. Indeed, for a quasivariety $\mathcal{Q}$, the *structural completion* of $\mathcal{Q}$ is the quasivariety $\widetilde{\mathcal{Q}} = \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$. The next lemma shows that $\widetilde{\mathcal{Q}}$ is indeed well defined. See [7, Proposition 2.3] for the proof.

Lemma 4.10. *Let $\mathcal{Q}$ be a quasivariety. Then $\widetilde{\mathcal{Q}}$ is the unique subquasivariety of $\mathcal{Q}$ which is structurally complete and $\mathbb{V}(\mathcal{Q}) = \mathbb{V}(\widetilde{\mathcal{Q}})$.*

Since varieties of algebras are quasivarieties, let us see what happens when applying the previous definitions to an arbitrary variety.

Lemma 4.11. *Let $\mathcal{V}$ be a variety.*

(1) *$\mathcal{V}$ is structurally complete iff every proper subquasivariety $\mathcal{Q}'$ generates a proper subvariety;*

(2) *$\mathcal{V}$ is primitive iff every subquasivariety is a variety.*

Proof. (1) is simply the definition of structural completeness. As to (2), suppose $\mathcal{V}$ is primitive and let $\mathcal{Q}$ be a subquasivariety. Then by Lemma 4.9, it follows that $\mathcal{Q} = \mathcal{V} \cap \mathbb{V}(\mathcal{Q})$ and, since $\mathcal{V}$ is a variety, $\mathcal{Q}$ is a variety as well. Conversely, if every subquasivariety is a variety, then, for every $\mathcal{Q}_1, \mathcal{Q}_2 \subseteq \mathcal{V}$, if $\mathbb{V}(\mathcal{Q}_2) = \mathbb{V}(\mathcal{Q}_1)$, then $\mathcal{Q}_1 = \mathcal{Q}_2$ by assumption and thus $\mathcal{V}$ is primitive. ☐

Active and Passive Structural Completeness in Quasivarieties

Let Ω be a similarity type and X a countable set of variables. Given a quasi-identity over $\Omega(X)$

$$\varphi : \qquad p_1 \approx q_1 \ \& \ \ldots \ \& \ p_n \approx q_n \Rightarrow p \approx q,$$

we denote by φ^a and φ^c respectively the first-order formulas over $\Omega(X)$

$$(p_1 \approx q_1 \ \& \ \ldots \ \& \ p_n \approx q_n) \qquad \text{and} \qquad p \approx q$$

and we denote by $\varphi^{\neg a}$ the first-order formula over $\Omega(X)$

$$\neg(p_1 \approx q_1 \ \& \ \ldots \ \& \ p_n \approx q_n).$$

[9] In abstract algebraic logic, the map $\boldsymbol{\tau}$ is called a (*finitary structural*) *transformer*. When a consequence relation $\vdash$ is algebraizable in the sense of Blok and Pigozzi, the map $\boldsymbol{\tau}$ comes in pair with another structural transformer $\boldsymbol{\rho}$ from equations to (set of) formulas and the pair $\langle \boldsymbol{\tau}, \boldsymbol{\rho} \rangle$ satisfies a few basic conditions that establish the (deductive) equivalence of $\vdash$ with the corresponding quasivariety $\mathcal{Q}^{\vdash}$. Cfr. [51, Definition 3.1.1]. The treatment of admissibility of Rybakov's [142] is actually based on the correspondence between axiomatic systems and quasivarieties in the framework of abstract algebraic logic.

Definition 4.10. Let $\mathcal{Q}$ be a quasivariety of Ω-algebras and φ be a quasi-identity over $\Omega(X)$. We say that φ is $\mathcal{Q}$*-active* if $\mathbf{F}_{\mathcal{Q}}(\omega) \not\models \varphi^{\neg a}$, while φ is said to be $\mathcal{Q}$*-passive* if $\mathbf{F}_{\mathcal{Q}}(\omega) \models \varphi^{\neg a}$.

Having defined an algebraic analogue of the notion of active and passive rule of inference, we can give the following

Definition 4.11. Let $\mathcal{Q}$ be a quasivariety. We say that $\mathcal{Q}$ is

- *passively structurally complete* (*psc* for short) if, for every $\mathcal{Q}$-passive quasi-identity φ,
$$\mathcal{Q} \models \varphi;$$
- *actively structurally complete* (*asc* for short) if, for every $\mathcal{Q}$-active quasi-identity φ,
$$\mathbf{F}_{\mathcal{Q}}(\omega) \models \varphi \Longrightarrow \mathcal{Q} \models \varphi.$$

Let us first characterize passively structurally complete quasivarieties. First recall that, given a similarity type Ω and a set of variables X, by a *positive, existential* sentence φ over $\Omega(X)$ we mean a fist-order formula of the following form
$$\exists \vec{y} \bigvee_{i \in I} \&_{j \in J_i} \psi_{ij},$$
where I and the J_i's are finite non-empty sets, the ψ_{ij}'s are atomic formulas, i.e. identities of type Ω over X, and $\vec{y}$ is a (possibly empty) finite tuple of variables including all $\mathbf{Var}\varphi$.

Now, for a given class of algebras $\mathcal{Q}$, let us denote by $\mathcal{Q}^-$ the class of all non trivial members of $\mathcal{Q}$. The following theorem is basically due to Wroński [168].

Theorem 4.12. *Let $\mathcal{Q}$ be a quasivariety, Then $\mathcal{Q}$ is passively structurally complete iff, for every positive existential sentence φ, either $\mathcal{Q} \models \varphi$ or $\mathcal{Q}^- \models \neg\varphi$.*

Proof. ($\Longrightarrow$) Assume $\mathcal{Q}$ is passively structurally complete and consider a positive existential sentence $\varphi := \exists \vec{y} \bigvee_{i \in I} \&_{j \in J_i} \psi_{ij}$. Now, if $\mathcal{Q}^- \not\models \neg\varphi$, there exists a nontrivial $\mathfrak{A} \in \mathcal{Q}^-$ such that $\mathfrak{A} \not\models \neg\varphi$. So $\mathfrak{A} \models \exists \vec{x} \&_{j \in J_i} \psi_{ij}$ for some $i \in I$. Let us denote $\exists \vec{x} \&_{j \in J_i} \psi_{ij}$ by δ. In order to show that $\mathcal{Q} \models \varphi$, it suffice to show that $\mathcal{Q} \models \delta$. Consider the quasi-identity ξ: $\&_{j \in J_i} \psi_{ij} \Rightarrow y = z$, where $y, z \notin \mathbf{Var}\psi_{ij}$ for all $j \in J_i$. Clearly $\mathcal{Q} \not\models \xi$, since $\mathfrak{A} \not\models \xi$. Therefore ξ is not $\mathcal{Q}$-passive and thus $\mathbf{F}_{\mathcal{Q}}(\omega) \not\models \xi^{\neg a}$, whence $\mathbf{F}_{\mathcal{Q}}(\omega) \models \delta$. Being δ an existential positive sentence, it is preserved by homomorphism and since for every $\mathfrak{B} \in \mathcal{Q}$ we can find an homomorphism $h \colon \mathbf{F}_{\mathcal{Q}}(\omega) \to \mathfrak{B}$, we have $\mathfrak{B} \models \delta$. So, $\mathcal{Q} \models \varphi$.
($\Longleftarrow$) By contraposition, suppose $\mathcal{Q}$ is not passively structurally complete and let φ: $\psi \Rightarrow \eta$ be a $\mathcal{Q}$-passive quasi-identity such that $\mathcal{Q} \not\models \varphi$. Then there exists a nontrivial algebra $\mathfrak{A} \in \mathcal{Q}^-$ such that $\mathfrak{A} \not\models \varphi$. In particular, by letting $\delta := \exists \vec{x} \psi(\vec{x})$, $\mathfrak{A} \models \delta$. Therefore $\mathcal{Q}^- \not\models \neg\delta$. Moreover, since φ is $\mathcal{Q}$-passive, $\mathbf{F}_{\mathcal{Q}}(\omega) \models \neg\delta$ and thus $\mathcal{Q} \not\models \delta$. Hence δ is a positive existential sentence such that both $\mathcal{Q} \not\models \delta$ and $\mathcal{Q}^- \not\models \neg\delta$. □

Now, let us turn the attention to actively structurally complete quasivarieties. The following theorem, except for a few minor changes, is due to Dzik and Stronkowski [38, Theorem 3.1].

Theorem 4.13. *Let $\mathcal{Q}$ be a quasivariety. Then the following are equivalent:*

(1) *$\mathcal{Q}$ is actively structurally complete;*

(2) *for every $\mathfrak{A} \in \mathcal{Q}$ and $n \leq \omega$, $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$;*

(3) *for every $\mathfrak{A} \in \mathcal{Q}_{SI}$ and $n \leq \omega$, $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$;*

(4) *for every $\mathfrak{A} \in \mathcal{Q}$ and $n \leq \omega$, if there exists a morphism $h\colon \mathfrak{A} \to \mathbf{F}_{\mathcal{Q}}(n)$, then $\mathfrak{A} \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$;*

(5) *for every $\mathfrak{A} \in \mathcal{Q}_{FP}$ and $n \leq \omega$, if there exists a morphism $h\colon \mathfrak{A} \to \mathbf{F}_{\mathcal{Q}}(n)$, then $\mathfrak{A} \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$.*

Proof. (1)$\Longrightarrow$(2) Let $\mathfrak{A}$ be an algebra in $\mathcal{Q}$ and fix some arbitrary $n \leq \omega$. Choose $\varphi(y_1, \ldots, y_n)$ to be a quasi-identity such that $\mathbf{F}_{\mathcal{Q}}(n) \models \varphi$. Moreover, notice that $\mathbf{F}_{\mathcal{Q}}(\omega) \models \varphi$. Now, if $\mathcal{Q} \models \varphi$, then, since $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \in \mathcal{Q}$, $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \models \varphi$ and thus $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$, since $\mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n))$ is a quasi-equational class. If $\mathcal{Q} \not\models \varphi$, then, since $\mathcal{Q}$ is asc, $\mathbf{F}_{\mathcal{Q}}(\omega) \models \varphi^{\neg a}$. Now if $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \not\models \varphi^{\neg a}$, then $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \models \exists \vec{x} \varphi^a$ and since existential positive sentences are preserved by homomorphisms we would have $\mathbf{F}_{\mathcal{Q}}(\omega) \not\models \varphi^{\neg a}$, contradiction. Thus $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \models \varphi^{\neg a}$ and therefore $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \models \varphi$. Hence $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n) \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$.
(2)$\Longrightarrow$(3) Trivial.
(3)$\Longrightarrow$(4) Let $\mathfrak{A} \in \mathcal{Q}$ and $h\colon \mathfrak{A} \to \mathbf{F}_{\mathcal{Q}}(n)$ a homomorphism. It can be shown that $\mathfrak{A}$ is isomorphic to a subdirect product of a family $\{\mathfrak{B}_i\}_{i \in I}$ of $\mathcal{Q}$-subdirectly irreducible algebras. If $I = \emptyset$, then $\mathfrak{A}$ is trivial and $\mathfrak{A} \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$. So suppose $I \neq \emptyset$, then, for each $i \in I$, $\mathfrak{B}_i \times \mathbf{F}_{\mathcal{Q}}(n) \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n))$ by (3). Hence $\prod_{i \in I}(\mathfrak{B}_i \times \mathbf{F}_{\mathcal{Q}}(n)) \cong \prod_{i \in I} \mathfrak{B}_i \times \mathbf{F}_{\mathcal{Q}}(n)^I \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n))$. Then, since $\mathbf{F}_{\mathcal{Q}}(n)$ is isomorphic with the diagonal of $\mathbf{F}_{\mathcal{Q}}(n)^I$, $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n)$ is isomorphic to a subalgebra of $\prod_{i \in I} \mathfrak{B}_i \times \mathbf{F}_{\mathcal{Q}}(n)^I$ and therefore it belongs to $\mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n))$. But the subalgebra of $\mathfrak{A} \times \mathbf{F}_{\mathcal{Q}}(n)$ with universe the set $\{\langle a, h(a) \rangle \mid a \in A\}$ is isomorphic to $\mathfrak{A}$. We thus conclude that $\mathfrak{A} \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$.
(4)$\Longrightarrow$(5) Trivial.
(5)$\Longrightarrow$(1) Let φ be a $\mathcal{Q}$-active quasi-identity such that $\mathbf{F}_{\mathcal{Q}}(\omega) \models \varphi$ and consider the $\mathcal{Q}$-finitely presented algebra $\mathbf{F}_{\mathcal{Q}}(\kappa, \varphi^a)$ with presentation given by $\langle \mathbf{Var}\varphi^a | \{p_i(x_1, \ldots x_k) \approx q_i(x_1, \ldots x_k)\}_{i \in I} \rangle$ where each $p_i \approx q_i$ is a conjuct of φ^a. Now, since φ is $\mathcal{Q}$-active, $\mathbf{F}_{\mathcal{Q}}(\omega) \not\models \varphi^{\neg a}$ and therefore $\mathbf{F}_{\mathcal{Q}}(\omega) \models \varphi^a(a_1, \ldots, a_k)$ for some choice $a_1, \ldots, a_k$ of elements of $\mathbf{F}_{\mathcal{Q}}(\omega)$. Since $k < \omega$, we can find a finite bound on the number of generators needed to construct the terms $a_1, \ldots, a_k$ and thus regard such terms as elements of $\mathbf{F}_{\mathcal{Q}}(n)$ for some $n < \omega$. Then, by letting $h\colon \kappa \to \{a_1, \ldots, a_k\}$ be the function respecting the previous choice of elements, by the universal property of fp-algebras, there exists a unique homomorphism $\hat{h}\colon \mathbf{F}_{\mathcal{Q}}(\kappa, \varphi^a) \to \mathbf{F}_{\mathcal{Q}}(n)$ such that $\hat{h} \restriction_{\kappa} = h$. Then by (5), $\mathbf{F}_{\mathcal{Q}}(\kappa, \varphi^a) \in \mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(n))$. Therefore, since $\mathbf{F}_{\mathcal{Q}}(\omega) \models \varphi$ implies $\mathbf{F}_{\mathcal{Q}}(n) \models \varphi$, we have $\mathbf{F}_{\mathcal{Q}}(\kappa, \varphi^a) \models \varphi$ and, in particular, $\mathbf{F}_{\mathcal{Q}}(\kappa, \varphi^a) \models \varphi^c[\nu]$ where $\nu\colon \mathbf{Tm}(\kappa) \to \mathbf{F}_{\mathcal{Q}}(\kappa, \varphi^a)$ is the natural map. Hence $\mathcal{Q} \models \varphi$ by Theorem 4.6 (ii) and $\mathcal{Q}$ is asc. ☐

Notice that we can further simplify condition (5). Indeed we have the following

Corollary 4.14. *Let $\mathcal{Q}$ be a quasivariety. Then $\mathcal{Q}$ is actively structurally complete iff the following condition holds:*

(5') *for every* $\mathfrak{A} \in \mathcal{Q}_{FP}$ *and* $n \leq \omega$, *if there exists a morphism* $h\colon \mathfrak{A} \to \mathbf{F}_{\mathcal{Q}}(n)$, *then* $\mathfrak{A} \in \mathbb{SP}(\mathbf{F}_{\mathcal{Q}}(n))$.

4.3 Canonical formulas

In this section we are going to present the machinery of *canonical formulas*, introduced and developed by Zakharyaschev in a series of papers for Ext**Int** as well for transitive modal logics. The motivation behind such an introduction is the attempt to give a characterization of the geometry of frames $\mathfrak{F}$ which refute a given $\mathcal{L}$-formula φ.

Definition 4.12. Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ be frames. A partial map f from W onto V is called a *subreduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if, for all $x, y \in W$ and $v \in V$,

(i) $xRy \;\;\&\;\; x, y \in \mathrm{dom} f \Longrightarrow f(x)Sf(y)$;

(ii) $f(x)Sv \Longrightarrow \exists y \in \mathrm{dom} f\,(xRy \;\;\&\;\; f(y) = v)$;

(iii) $\forall X \in \overline{\mathcal{Q}}\, f^{-1}(X){\downarrow} \in \overline{\mathcal{P}}$,

where $\overline{\mathcal{Q}} = \{V \setminus X \mid X \in \mathcal{Q}\}$ and $\overline{\mathcal{P}} = \{W \setminus X \mid X \in \mathcal{P}\}$. If $\mathfrak{G}$ is a finite Kripke frame, then (iii) is equivalent to

(iv) $W \setminus f^{-1}(v){\downarrow} \in \mathcal{P}$.

A set $X \subseteq W$ is said to be *cofinal* in $\mathfrak{F}$ if $X{\uparrow} \subseteq X{\downarrow}$. We then say that f is a *cofinal* subreduction if $\mathrm{dom} f$ is cofinal in $\mathfrak{F}$ and that f is *globally cofinal* if $W = \mathrm{dom} f{\uparrow}$. Furthermore, $\mathfrak{G}$ is said to be a (*cofinal*) *subframe* of $\mathfrak{F}$ if $\mathfrak{G} \subseteq \boldsymbol{\kappa}\mathfrak{F}$ and the identity map on V is a (cofinal) subreduction of $\mathfrak{F}$ onto $\mathfrak{G}$. Finally, let $\mathfrak{D}$ be a (possibly empty) set of antichains in $\mathfrak{G}$. We say that f satisfies the *closed domain condition* for $\mathfrak{D}$ if

$$\neg\exists x \in \mathrm{dom} f{\uparrow} \setminus \mathrm{dom} f\, \exists \mathfrak{d} \in \mathfrak{D}\; f(x{\uparrow}) = \mathfrak{d}{\uparrow} \tag{CDC}$$

We denote by $\mathfrak{D}^\natural$ the set of all antichains of $\mathfrak{G}$ and if $\mathfrak{K} = \langle K, T\rangle$ is a generated subframe of $\mathfrak{G}$, then we let

$$\mathfrak{D}\restriction_{\mathfrak{K}} := \{\mathfrak{d} \in \mathfrak{D} \mid \mathfrak{d} \subseteq K\}.$$

Definition 4.13. Let $\mathfrak{F} = \langle W, R\rangle$ be a finite frame and let $a_0, \ldots, a_n$ be its points. Assume also that $\mathfrak{D}$ is a (possibly empty) set of antichains in $\mathfrak{F}$. The *intuitionistic canonical formula* $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ is the $\mathcal{L}$-formula

$$\beta(\mathfrak{F}, \mathfrak{D}, \bot) := \bigwedge_{a_i R a_j} \psi_{i,j} \wedge \bigwedge_{\mathfrak{d} \in \mathfrak{D}} \psi_{\mathfrak{d}} \wedge \psi_\bot \to p_0,$$

where

$$\psi_{i,j} = (\bigwedge_{\neg a_j R a_k} p_k \to p_j) \to p_i,$$

$$\psi_{\mathfrak{d}} = \bigwedge_{a_i \in W \setminus \mathfrak{d}{\uparrow}} (\bigwedge_{\neg a_i R a_k} p_k \to p_i) \to \bigvee_{a_j \in \mathfrak{d}} p_j,$$

$$\psi_\bot = \bigwedge_{i=0}^{n} (\bigwedge_{\neg a_i R a_k} p_k \to p_i) \to \bot.$$

The $\mathcal{L}$-formulas $\beta(\mathfrak{F},\mathfrak{D})$ is obtained from $\beta(\mathfrak{F},\mathfrak{D},\bot)$ by deleting the conjunct $\psi_\bot$ and it is called the *intuitionistic negation free canonical formula*; the $\mathcal{L}$-formulas $\beta(\mathfrak{F},\emptyset)$ and $\beta(\mathfrak{F},\emptyset,\bot)$ are called the *subframe* and *cofinal subframe* formulas for $\mathfrak{F}$ respectively. Finally, the $\mathcal{L}$-formula $\beta(\mathfrak{F},\mathfrak{D}^\natural,\bot)$ is said to be the *frame* formula for $\mathfrak{F}$.

The following result is one of the two cornerstones that provides the characterization mentioned at the beginning of this section. The *refutability criterion for canonical formulas* is the following

Theorem 4.15. *Let $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ be any frame, $\mathfrak{F} = \langle W, R\rangle$ a finite rooted frame and let $\mathfrak{D}$ be a (possibly empty) set of antichains in $\mathfrak{F}$. Then the following holds:*
$\mathfrak{G} \not\models \beta(\mathfrak{F},\mathfrak{D},\bot)$ iff there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.

Proof. ($\Longrightarrow$) Suppose that $\beta(\mathfrak{F},\mathfrak{D},\bot)$ is refuted in a model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$. Denote by $\psi^{\mathfrak{F}^{\mathfrak{D}}}$ the premises of $\beta(\mathfrak{F},\mathfrak{D},\bot)$ and define a partial map f from V onto W as follows:

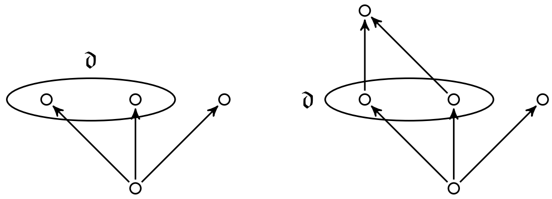
$$f(w) = \begin{cases} a_i & \text{if } w \models \psi^{\mathfrak{F}^{\mathfrak{D}}},\ w \models \bigwedge_{\neg a_i R a_j} p_j \text{ and } w \not\models p_i, \\ undefined & \text{otherwise.} \end{cases}$$

Let us shot that f satisfies (CDC) for $\mathfrak{D}$, leaving to the reader the task of showing that f is actually a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$. So, let $\mathfrak{d} \in \mathfrak{D}$ and suppose, for reductio, that $f(w\uparrow) = \mathfrak{d}\uparrow$ for some $w \in \mathrm{dom} f\uparrow \setminus \mathrm{dom} f$. Then $w \models \psi_{\mathfrak{d}}$ and, since $w \in \bigcap_{a_j \in \mathfrak{d}} f^{-1}(a_j)\downarrow$, $w \not\models \bigwedge_{a_i \in W \setminus \mathfrak{d}\uparrow}(\bigwedge_{\neg a_i R a_k} p_k \to p_i)$. Hence, for some $a_i \in W \setminus \mathfrak{d}\uparrow$, there exists $v \in w\uparrow$ such that $v \models \bigwedge_{\neg a_i R a_k} p_k$ and $v \not\models p_i$. Thus $f(v) = a_i \in f(w\uparrow)$, contrary to our assumption.
($\Longleftarrow$) Conversely, assume that f is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Define a valuation $\mathfrak{U}$ in $\mathfrak{G}$ by letting

$$\mathfrak{U}(p_i) := V \setminus f^{-1}(a_i)\downarrow.$$

Again we leave to the reader the task of proving that $\langle \mathfrak{G}, \mathfrak{U}\rangle \not\models \beta(\mathfrak{F},\mathfrak{D},\bot)$ by showing that $w \models \psi^{\mathfrak{F}^{\mathfrak{D}}}$, for each $w \in f^{-1}(a_0)$. $\square$

Example 9. A frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ refutes the Kreisel-Putnam axiom ***kp*** iff $\mathfrak{F}$ is cofinally subreducible to one of the two following frame, with (CDC) for $\mathfrak{D} = \{\mathfrak{d}\}$ being satisfied,



The next theorem represents a sort of convers of the previous one. For the proof, cfr. [23, Theorem 9.36].

Theorem 4.16. *There is an algorithm which, given a $\mathcal{L}$-formula φ, returns a finite number of finite rooted frames $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ and sets $\mathfrak{D}_1, \dots, \mathfrak{D}_n$ of antichains in them such that, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff there is a cofinal subreduction of $\mathfrak{F}$ onto $\mathfrak{F}_i$, for some $i \in \{1, \dots, n\}$, satisfying (CDC) for $\mathfrak{D}_i$.*

The apparatus of canonical formulas, which is an extension of the Jankov's approach to characteristic formulas, has been proven particularly fruitful in the study of intermediate logics, since every si-logic can be axiomatized by canonical formulas. Indeed, as an immediate consequence of Theorems 4.15 and 4.16, we get the following

Theorem 4.17. *There exists an algorithm which, given a $\mathcal{L}$-formula φ, returns canonical formulas $\beta(\mathfrak{F}_1, \mathfrak{D}_1, \bot), \dots, \beta(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$ such that*

$$\mathbf{Int} + \varphi = \mathbf{Int} + \beta(\mathfrak{F}_1, \mathfrak{D}_1, \bot) + \dots + \beta(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

So the set of intuitionistic canonical formulas is complete for Ext**Int**.

Let us continue with a few useful lemmas on subreductions.

Lemma 4.18. *Let $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ be a frame, $\mathfrak{F} = \langle W, R \rangle$ a finite Kripke frame and $\mathfrak{D}$ a set of antichains in $\mathfrak{F}$. If f is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and $w \in domf$, then the restriction $g := f \upharpoonright_{w\uparrow}$ of f to $w\uparrow$ is a globally cofinal subreduction of $\mathfrak{G}_w$ to $\mathfrak{F}_{f(w)}$ satisfying (CDC) for $\mathfrak{D} \upharpoonright_{\mathfrak{F}_{f(w)}}$.*

Proof. It is clear that g is a subreduction of $\mathfrak{G}_w$ to $\mathfrak{F}_{f(w)}$. Furthermore, if $x \in \mathrm{dom} g\uparrow$, then $x \in y\uparrow$ for some $y \in \mathrm{dom} g = \mathrm{dom} f \cap w\uparrow$. Consequently, since $\mathrm{dom} f$ is cofinal in $\mathfrak{F}$, there exists $z \in \mathrm{dom} f$ such that $x \in z\downarrow$. Hence $z \in \mathrm{dom} g$ and $x \in \mathrm{dom} g\downarrow$. So, g is globally cofinal. Finally, since $\mathrm{dom} g\uparrow \setminus \mathrm{dom} g \subseteq \mathrm{dom} f\uparrow \setminus \mathrm{dom} f$ and each antichain $\mathfrak{d} \in \mathfrak{D} \upharpoonright_{\mathfrak{F}_{f(w)}}$ is also an antichain in $\mathfrak{D}$, it follows that g satisfies (CDC) for $\mathfrak{D} \upharpoonright_{\mathfrak{F}_{f(w)}}$. □

Lemma 4.19. *Let $\mathfrak{F}_i = \langle W_i, R_i, \mathcal{P}_i \rangle$ for $i \in \{1, 2, 3\}$ be frames, f_1 a cofinal subreduction of $\mathfrak{F}_1$ onto $\mathfrak{F}_2$ and f_2 a cofinal subreduction of $\mathfrak{F}_2$ onto $\mathfrak{F}_3$. Then the composition $f_2 \circ f_1$ is a cofinal subreduction of $\mathfrak{F}_1$ onto $\mathfrak{F}_3$.*

Lemma 4.20. *Let $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$ be finite Kripke frames with $\mathfrak{D}$ and $\mathfrak{E}$ sets of antichains for $\mathfrak{F}$ and $\mathfrak{G}$ respectively. If f is a cofinal subreduction of $\mathfrak{G}$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ such that*

- *an antichain $\mathfrak{e} \subseteq domf\uparrow$ is in $\mathfrak{E}$ whenever $f(\mathfrak{e}\uparrow) = \mathfrak{d}\uparrow$ for some $\mathfrak{d} \in \mathfrak{D}$,*

*then $\beta(\mathfrak{G}, \mathfrak{E}, \bot) \in$ **Int** $+ \beta(\mathfrak{F}, \mathfrak{D}, \bot)$.*

Proof. Let $\mathfrak{H}$ be a frame such that $\mathfrak{H} \not\models \beta(\mathfrak{G}, \mathfrak{E}, \bot)$. By Theorem 4.15, there exists a cofinal subreduction g from $\mathfrak{H}$ onto $\mathfrak{G}$ satisfying (CDC) for $\mathfrak{E}$. Since $h = f \circ g$ is a cofinal subreduction of $\mathfrak{H}$ onto $\mathfrak{F}$ by Lemma 4.19, we only need to prove that h satisfies (CDC) for $\mathfrak{D}$. So suppose that there is a point $w \in \mathrm{dom} h\uparrow$ and an antichain $\mathfrak{d} \in \mathfrak{D}$ such that $h(w\uparrow) = \mathfrak{d}\uparrow$. Consider the antichain $\mathfrak{e} \subseteq V$ such that $g(w\uparrow) = \mathfrak{e}\uparrow$. It follows that $\mathfrak{e} \subseteq \mathrm{dom} f\uparrow$ and $f(\mathfrak{e}\uparrow) = \mathfrak{d}\uparrow$, consequently $\mathfrak{e} \in \mathfrak{E}$. Since g satisfies (CDC) for $\mathfrak{E}$, we have $w \in \mathrm{dom} g$. But then $g(w) \in \mathrm{dom} f\uparrow$ and, since $g(w\uparrow) = g(w)\uparrow$, the following equality holds: $f(g(w)\uparrow) = \mathfrak{d}\uparrow$. Therefore, since f satisfies (CDC) for $\mathfrak{D}$, $g(w) \in \mathrm{dom} f$ and, consequently, $w \in \mathrm{dom} h$. We conclude that h also satisfies (CDC) for $\mathfrak{D}$. Hence $\mathfrak{H} \not\models \beta(\mathfrak{F}, \mathfrak{D}, \bot)$ by the refutability criterion. □

We conclude this section by showing the power of the apparatus of canonical formulas in proving the following well-known result concerning Jankov's logic **KC**. Denote by $\mathcal{P}$ the set of positive $\mathcal{L}$-formulas, that is, $\mathcal{L}$-formulas that do not contain any occurrence of $\neg$.

Theorem 4.21 (Jankov, [87]). *Let L be an intermediate logic. The following equivalence holds:*

$$L \subseteq \mathbf{KC} \Longleftrightarrow \mathbf{Int} \cap \mathcal{P} = L \cap \mathcal{P}.$$

Proof. ($\Longrightarrow$) Suppose L is included in **KC**. Since $\mathbf{Int} \subseteq L$, it suffice to show $L \cap \mathcal{P} \subseteq \mathbf{Int}$. By Theorem 4.17, we can consider a negation free canonical formulas $\beta(\mathfrak{F}, \mathfrak{D}) \notin \mathbf{Int}$. So, there exists a finite frame $\mathfrak{G}$ such that $\mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D})$. Then also $\mathbf{1} \oplus \mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D})$ and, since $\mathbf{1} \oplus \mathfrak{G}$ is a **KC**-frame, we get $\beta(\mathfrak{F}, \mathfrak{D}) \notin \mathbf{KC}$. Hence $\beta(\mathfrak{F}, \mathfrak{D}) \notin L$.
($\Longleftarrow$) First notice that if a negation free canonical formulas $\beta(\mathfrak{F}, \mathfrak{D}) \notin \mathbf{KC}$, then $\beta(\mathfrak{F}, \mathfrak{D}) \notin \mathbf{Int}$ and thus $\beta(\mathfrak{F}, \mathfrak{D}) \notin L$ by our assumption. So, suppose that $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin \mathbf{KC}$. By completeness of **KC** with respect to finite frames with a top element, there exists a finite frame $\mathfrak{G}$ such that $\mathbf{1} \oplus \mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D}, \bot)$. By Theorem 4.15, let $f \colon \mathbf{1} \oplus \mathfrak{G} \to \mathfrak{F}$ be a cofinal subreduction satisfying (CDC) for $\mathfrak{D}$. Then it must be the case that $\mathfrak{F} \cong \mathbf{1} \oplus \mathfrak{K}$ for some finite frame $\mathfrak{K}$. Let us denote by $a_{\mathfrak{F}}$ the unique maximal point of $\mathfrak{F}$. Furthermore, we also have $\beta(\mathfrak{F}, \mathfrak{D}) \notin \mathbf{KC}$. Consequently, $\beta(\mathfrak{F}, \mathfrak{D}) \notin L$ and therefore there is a L-frame $\mathfrak{H}$ and a subreduction $g \colon \mathfrak{H} \to \mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Now, extend g by letting $g^*(x) = a_{\mathfrak{F}}$ for every $x \in max(\mathfrak{H})$. Then g^* is a cofinal subreduction from $\mathfrak{H}$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$, hence $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$. ☐

4.3.1 Partial Esakia equivalences

Let us first translate the notions of subreduction in terms of Esakia spaces. Given Esakia spaces $\mathcal{X} = \langle X, \tau, R \rangle$ and $\mathcal{Y} = \langle Y, \gamma, S \rangle$, a partial map f from X onto Y is said to be a *partial Esakia morphism* from $\mathcal{X}$ to $\mathcal{Y}$ if the following hold:

(i) $\mathrm{dom} f$ is a closed subset of X;

(ii) $f \restriction_{\mathrm{dom} f}$ is a p-morphism;

(iii) for every clopen subset U of Y, $f^{-1}(U){\downarrow}$ is a clopen subset of X.

.

It is worth mentioning that the theory of such morphisms was developed in [9], where it is also shown that such a notion sharpens the notion of subreduction as it is defined in Definition 4.12. This comes not as a surprise, since Zakharyaschev's definition deals with general frames and not only with the descriptive ones. Nevertheless, as pointed out in [9], in order to develop a suitable duality theory one has to take into consideration the notion of partial Esakia morphisms and so we do for the purpose of introducing the notion of *partial Esakia equivalence*[10].

[10] Obviously, one can also elaborate on the issue by introducing further specification on the notion of partial Esakia equivalence in order to get an appropriate duality with respect to the notions of *well partial Esakia morphism*, *strong partial Esakia morphism* and partial Esakia morphism satisfying the *Closed Domain Condition* as they are defined in [9].

Recall that a *partial equivalence relation* $\sim$ on a given set X is a symmetric and transitive relation and if we restrict $\sim$ to the set

$$D_\sim := \{x \in X \mid \exists y \in X(x \sim y)\},$$

which is called the *domain* of $\sim$, we get an equivalence relation[11]. Now, given any partial function $h\colon X \to Y$, the relation $\sim_h$, defined, for all $x, y \in X$, as follows,

$$x \sim_h y \iff x, y \in \mathrm{dom} f \;\;\&\;\; h(x) = h(y),$$

is a partial equivalence relation on X. So, given Esakia spaces $\mathcal{X} = \langle X, \tau, R\rangle$ and $\mathcal{Q} = \langle Q, \gamma, S\rangle$, any partial Esakia morphism $f\colon X \to Q$ from $\mathcal{X}$ to $\mathcal{Q}$ induces a partial equivalence relation $\sim_f$. Furthermore, we have the following

Lemma 4.22. *Let f be a partial Esakia morphism from the the Esakia space $\mathcal{X} = \langle X, \tau, R\rangle$ to the Esakia space $\mathcal{Q} = \langle Q, \gamma, S\rangle$. Then the quotient $[\mathcal{D}_f]$ of the ordered topological subspace $\mathcal{D}_f = \langle \mathrm{dom} f, \tau_{domf}, R\restriction_{\mathrm{dom} f}\rangle$, under the equivalence relation $\sim_f$, is an Esakia space order-homeomorphic to $\mathcal{Q}$ and such that, for every clopen subset $[U]$ of $[\mathrm{dom} f]$, $\bigcup[U]{\downarrow}$ is a clopen downset of X.*

Proof. Consider the map $f\restriction_{\mathrm{dom} f}$. By definition, it is a p-morphism. Let us show that $f\restriction_{\mathrm{dom} f}$ is continuous. It suffice to show that, for each clopen upset U of Q, both $f^{-1}(U)$ and $f^{-1}(Q \setminus U)$ belongs to $\tau_{\mathrm{dom} f}$. Notice that, since f is a subreduction, the set $f^{-1}(Q \setminus U){\downarrow}$ is a clopen of X and

$$f^{-1}(Q \setminus U){\downarrow} \cap \mathrm{dom} f = f^{-1}(Q \setminus U{\downarrow}) = f^{-1}(Q \setminus U),$$

since $Q \setminus U$ is a downset. Moreover, it is easy to check that

$$X \setminus f^{-1}(Q \setminus U){\downarrow} \cap \mathrm{dom} f = f^{-1}(U).$$

Consequently, the map $f\restriction_{\mathrm{dom} f}$ is a continuous p-morphisms. Furthermore, since $\mathrm{dom} f$ is closed in X and X is compact, $\mathrm{dom} f$ is also compact. So, being f a continuous function from a compact space to a Hausdorff space, it is closed, hence a quotient map. So, the quotient $[\mathcal{D}_f]$ of $\mathcal{D}_f$ under the equivalence $\sim_f$ is order-homeomorphic to $\mathcal{Q}$ with the induced map

$$\begin{aligned} \tilde{f}\colon [\mathrm{dom} f] &\to Q \\ [x] &\mapsto f(x), \end{aligned}$$

being a homeomorphism. Finally, let $[U]$ be a clopen of $[\mathrm{dom} f]$. Then we have $\bigcup[U] = f^{-1}(f(U))$ and, since $\tilde{f}([U]) = f(U)$ is clopen in Q, we have that $\bigcup[U]{\downarrow}$ is a clopen downset of X, being f a a partial Esakia morphism. □

Corollary 4.23. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and $\mathcal{Q} = \langle Q, \gamma, S\rangle$ an ordered topological space. The following equivalence holds: $\mathcal{Q}$ is a subframe of $\mathcal{X}$ if and only if $\mathcal{Q}$ is a closed subspace of $\mathcal{X}$ and, for every clopen subset U of Q, $U{\downarrow}$ is a clopen downset of X.*

[11] Indeed, if $x \in D_\sim$, then $x \sim y$ for some $y \in X$; consequently, $y \sim x$ by symmetry and $x \sim x$ by reflexivity.

Proof. ($\Longrightarrow$) Suppose that $\mathcal{Q}$ is a subframe of $\mathcal{X}$. Thus the partial identity map $i\colon X \to Q$ is a partial Esakia morphism. In particular, we have that $\operatorname{dom} i = Q$ is a closed subspace of $\langle X, \tau\rangle$. Now, notice that, for any subset $U \subseteq Q$, $i^{-1}(U) = U$ and, since the quotient of $\mathcal{Q}$ under the identity relation is $\mathcal{Q}$ itself, we have that $U{\downarrow}$ is a clopen downset of X, for each clopen subset U of Q by the previous Lemma.
($\Longleftarrow$) Since $\langle Q, \gamma\rangle$ is a closed subspace of the compact space $\langle X, \tau\rangle$, $\langle Q, \gamma\rangle$ is compact as well. Suppose that $\neg(xSy)$. Since $S = R\restriction_Q$, we have $\neg(xRy)$ and by the Priestley separation axiom, there exists a clopen set U of X such that $x \in U$ and $y \notin U$. But then $U \cap Q$ is a clopen upset in Q separating x and y. Hence $\mathcal{Q}$ is a Priestley space. Now, let us consider a clopen subset U of Q. Notice that $U{\downarrow_S} = U{\downarrow} \cap Q$. By hypothesis, $U{\downarrow}$ is a clopen of X and thus $U{\downarrow_S}$ is clopen in Q. Thus S is a clopen relation and $\mathcal{Q}$ an Esakia space. Finally, the partial identity function $i\colon X \to Q$ on Q is a partial Esakia morphism from $\mathcal{X}$ to $\mathcal{Q}$. ☐

Definition 4.14. Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space. A partial equivalence relation $\sim$ on X is called a *partial Esakia equivalence* if the following conditions hold:

1. $D_\sim$ is a closed subset of X;
2. For every $w, v \in D_\sim$, wRv implies $[w] \subseteq [v]{\downarrow}$;
3. For every $w, v \in D_\sim$, if $\neg(w \sim v)$, then w and v are separated by an $\sim$-*saturated* clopen upset of $D_\sim$;
4. $\bigcup[U]{\downarrow}$ is clopen in X, for every clopen subset $[U]$ of $[D_\sim]$.

We can thus consider the quotient $[\mathcal{D}_\sim]$ induced on the ordered topological subspace $\mathcal{D}_\sim = \langle D_\sim, \tau_{D_\sim}, R\restriction_{D_\sim}\rangle$ by $\sim$ and we call such a space the *Esakia prequotient space* of $\mathcal{X}$ given by the partial Esakia equivalence $\sim$.

Lemma 4.24. *Let $\mathcal{X} = \langle X, \tau, R\rangle$ be an Esakia space and $\sim$ a partial Esakia equivalence on X. Then the Esakia prequotient space of $\mathcal{X}$ is an Esakia space and the partial map $f_\sim\colon \mathcal{X} \to [\mathcal{X}]$ from the Esakia space $\mathcal{X}$ onto the Esakia quotient space $[\mathcal{X}]$ given by*

$$f_\sim(x) = [x],$$

for each $x \in D_\sim$, is a partial Esakia morphism.

Proof. Suppose that the relation $\sim$ is a partial Esakia equivalence on $\mathcal{X}$. Being $D_\sim$a closed subset of the Hausdorff space X, $D_\sim$ is compact. Therefore, $[\mathcal{D}_\sim]$ is also compact. Moreover, one can show as in Lemma 2.36 that $[\mathcal{D}_\sim]$ satisfies the Priestley separation axiom and thus $[\mathcal{D}_\sim]$ is a Priestley space. Now, let $[U]$ be a clopen of $[D_\sim]$. By hypothesis, $\bigcup[U]{\downarrow}$ is clopen in X and, consequently, $\bigcup[U]{\downarrow_{D_\sim}} = \bigcup[U]{\downarrow}\cap D_\sim$ is clopen in $D_\sim$. But, $(\bigcup[U]){\downarrow_{D_\sim}} = \bigcup([U][{\downarrow_{D_\sim}}])$, hence $[U][{\downarrow_{D_\sim}}]$ is clopen in $[D_\sim]$. So, $[R\restriction_{D_\sim}]$ is a clopen relation and we conclude that $[\mathcal{D}_\sim]$ is an Esakia space. Finally one can show that $f_\sim$ is a partial Esakia morphism as in Lemma 2.36 and Proposition 2.37. ☐

Proposition 4.25. *Let $\mathcal{X}$ be an Esakia space. There exists a one-to-one correspondence between the partial Esakia equivalence on $\mathcal{X}$ and the partial Esakia morphism from $\mathcal{X}$.*

4.4 Structural completeness and canonical formulas

Given a finite rooted Kripe frame $\mathfrak{F}$ and a set $\mathfrak{D}$ of antichains in it, we denote by $\rho(\mathfrak{F}, \mathfrak{D}, \bot)$ the inference rule whose premises and conclusion are the premises and the conclusion of $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$, that is

$$\rho(\mathfrak{F}, \mathfrak{D}, \bot) := \frac{\{\psi_{i,j} \mid a_i R a_j\},\ \{\psi_{\mathfrak{d}} \mid \mathfrak{d} \in \mathfrak{D}\},\ \psi_\bot}{p_0}.$$

Furthermore, recall that a rule $r : \Gamma/\psi$ is not satisfied in a frame $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ (we write: $\mathfrak{G} \not\models r$) if there exists a descriptive valuation $\mathfrak{V}$ on $\mathfrak{G}$ such that $\mathfrak{V}(\bigwedge \Gamma) = V$ and $\mathfrak{V}(\psi) \neq V$.

Theorem 4.26. *Let $\mathfrak{F}$ be a finite rooted Kripe frame, $\mathfrak{D}$ a set of antichains in $\mathfrak{F}$, and let $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ be a descriptive frame. Then $\mathfrak{G} \not\models \rho(\mathfrak{F}, \mathfrak{D}, \bot)$ if and only if there exists a globally cofinal subreduction from $\mathfrak{G}$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. In particular, if $\mathfrak{D} = \mathfrak{D}^\natural$, then $\mathfrak{G} \not\models \rho(\mathfrak{F}, \mathfrak{D}^\natural, \bot)$ if and only if $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$.*

Proof. ($\Longleftarrow$) Let $h\colon \mathfrak{G} \to \mathfrak{F}$ be a globally cofinal subreduction from $\mathfrak{G}$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. From the proof of Theorem 4.15, it follows that $w \models \bigwedge_{a_i R a_j} \psi_{i,j} \wedge \bigwedge_{\mathfrak{d} \in \mathfrak{D}} \psi_{\mathfrak{d}} \wedge \psi_\bot$ and $w \not\models p_0$, for each $w \in h^{-1}(a_0)$. But, since f is globally cofinal, we have that $min(\mathfrak{G}) \subseteq h^{-1}(a_0)$ and, consequently, $\mathfrak{G} \not\models \rho(\mathfrak{F}, \mathfrak{D}, \bot)$.
($\Longrightarrow$) Suppose that $\mathfrak{G} \not\models \rho(\mathfrak{F}, \mathfrak{D}, \bot)$. Then, by the refutability criterion for canonical formulas, there is a globally cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Furthermore, if $\mathfrak{D} = \mathfrak{D}^\natural$, we are going to show that we can extend f to a plain function. Suppose there exists a point $x \in V \setminus \mathrm{dom} f$. Then, by (CDC), $f(x\uparrow) = a_x\uparrow$ for some $a_x \in W$. Then by defining $g\colon V \to \mathfrak{F}$ as

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom} f, \\ a_x & x \notin \mathrm{dom} f, \end{cases}$$

we get a p-morphism from $\mathfrak{G}$ to $\mathfrak{F}$. □

Proposition 4.27. *Let L be a structurally complete intermediate logic, $\mathfrak{F}$ a finite rooted frame and let $\mathfrak{D}$ be a set of antichains in $\mathfrak{F}$. If $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$, then there exists a globally cofinal subreduction from $\mathfrak{F}_L(n)$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$, for some $n < \omega$. In particular, if $\mathfrak{D} = \mathfrak{D}^\natural$, then $\mathfrak{F}$ is a p-morphic image of the n-canonical frame $\mathfrak{F}_L(n)$, for some $n < \omega$.*

Proof. Assume that $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$ and suppose, for reductio, that there are no globally cofinal subreduction from $\mathfrak{F}_L(n)$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$, for every $n < \omega$. Then, by Theorem 4.26, it follows that, for each $n < \omega$, $\mathfrak{F}_L(n) \models \rho(\mathfrak{F}, \mathfrak{D}, \bot)$ and thus also $\mathfrak{F}_L(\omega) \models \rho(\mathfrak{F}, \mathfrak{D}, \bot)$. Therefore, $\rho(\mathfrak{F}, \mathfrak{D}, \bot)$ is an admissible rule of L and, since L is structurally complete, the $\mathcal{L}$-formula $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \in L$. However, this contradicts our assumption. □

Lemma 4.28. *Let $\mathcal{V}_L$ be a finitely approximable variety of Heyting algebras and let $\mathfrak{A}$ be a subdirectly irreducible finitely presented algebra in $\mathcal{V}_L$. Then $\mathfrak{A}$ is finite.*

Proof. Consider the dual descriptive L-frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ of $\mathfrak{A}$. Then, by Proposition 2.42, $W = w\uparrow$ for some $w \in W_{\text{iso}}$. Furthermore, by Lemma 5.23, W_{fin} is dense in W and thus $W_{\text{iso}} = W_{\text{fin}}$. Indeed, since $\mathfrak{F}$ is finitely generated, $W_{\text{fin}} \subseteq W_{\text{iso}}$ and thus it suffice to show that $W_{\text{iso}} \subseteq W_{\text{fin}}$. If $x \notin W_{\text{fin}}$, then $W_{\text{fin}} \setminus \{x\} = W_{\text{fin}}$ and thus $x \in \overline{W_{\text{fin}} \setminus \{x\}} = W$, since W_{fin} is dense in W. Consequently, x is a limit point of W_{fin}. Since $W_{\text{fin}} \subseteq W$, then x is also a limit point of W and thus $x \notin W_{\text{iso}}$. Therefore $w \in W_{\text{fin}}$, that is, $w\uparrow = W$ is finite. □

By the previous lemma, for any finitely approximable variety of Heyting algebras $\mathcal{V}_L$, the subdirectly irreducible finitely presented algebras of $\mathcal{V}_L$ are exactly the subdirectly irreducible finite algebras of $\mathcal{V}_L$. Furthermore, we have the following

Theorem 4.29. *Let L be a finitely approximable intermediate logic. Then L is structurally complete if and only if every finite rooted frame $\mathfrak{F} \in \boldsymbol{\mathcal{DF}_L}$ is a p-morphic image of the n-canonical L-frame $\mathfrak{F}_L(n)$, for some $n < \omega$.*

Proof. ($\Longrightarrow$) This follows immediately by Proposition 4.27.
($\Longleftarrow$) Suppose that L is not structurally complete and let $r : \Gamma/\varphi$ be an admissible rule that is not derivable in L. Since L is finitely approximable, there exists a finite frame $\mathfrak{G}$ such that $\mathfrak{G} \not\models \bigwedge\Gamma \to \varphi$. In particular, there exists a point $w \in \mathfrak{G}$ such that $w \models \bigwedge\Gamma$ and $w \not\models \varphi$. Hence $\mathfrak{F} = w\uparrow$ is a finite rooted frame in $\boldsymbol{\mathcal{DF}_L}$ such that $\mathfrak{F} \not\models r$. Now, if $\mathfrak{F}$ is a p-morphic image of the n-canonical L-frame $\mathfrak{F}_L(n)$, for some $n < \omega$, then $\mathfrak{F}$ is also a p-morphic image of $\mathfrak{F}_L(\omega)$. But then we have that $\mathfrak{F}_L(\omega) \not\models r$ and thus that r is not admissible in L, contradicting our assumption. □

Chapter 5

Unification in intermediate logics

5.1 Preliminary of Unification Theory

Broadly speaking, unification can be described as the attempt to identify two given symbolic expressions by replacing certain sub-expressions in them by other expressions. More concretely, consider a similarity type Ω and a set of variables X and let t and s be Ω-terms over X. The *unification problem* for the terms t and s is then as follows: is it possible to replace the variables in t and s by some other Ω-terms over X in a way that the resulting Ω-terms are syntactically equal?

A unification problem for two given Ω-terms t and s thus asks for the existence of a substitution $\sigma\colon X \to \mathbf{Tm}(X)$ such that $\sigma(t) = \sigma(s)$. Such a substitution is called a *unifier* of t and s. In general, there can be infinitely many unifiers for a given unification problem, therefore one should be interested in finding the *most general unifier*, that is, a unifier such that every other unifier can be obtained by *instantiation*.

Unification theory is the abstract theory of unification in the sense that it provides the formal definition for the most important notions involved in an abstract unification process, investigates the related properties of these notions and studies general unification algorithms applicable to a wide range of contexts[1].

5.1.1 Symbolic E-Unification

The unification process described above is also called *syntactic*, since it is required that the unified terms turn out to be syntactically equal[2]. By replacing syntactical equality by equality modulo a given equational theory E, we get a

[1]For the history of unification theory, an explanation of the concepts involved as well as an overview of the most interesting topics and results concerned with unification theory, cfr. the survey paper [4].

[2]Furthermore, such kind of unification is called *first-order*, because of the fact that the terms involved in the unification process do not contain higher-order variables, that is, variables ranging over functions symbols. In what follows we will deal only with unification problems of this type.

much harder kind of unification called *E-unification*. Indeed, contrary to syntactic unification, E-unification can be undecidable and, even when it is not, one may not find a most general unifier for a given E-unification problem. Let us define E-unification in a precise formal way.

Let a similarity type Ω and a countable set of variables $X = \{x_0, x_1, x_2, \ldots\}$ be fixed. A *substitution* is any mapping $\sigma\colon X \to \mathbf{Fm}(X)$. Most of the times, we will only consider substitutions σ which are constant on a cofinite subset of X, that is, such that the set

$$\mathcal{D}\mathrm{om}(\sigma) = \{x \in X \mid \sigma(x) \neq x\},$$

called the *domain* of σ, is finite. The *range* of σ is the following set

$$\mathcal{R}\mathrm{an}(\sigma) = \{\sigma(x) \mid x \in \mathcal{D}\mathrm{om}(\sigma)\}$$

and we denote the set of variable occurring in the range of σ as $\mathbf{Var}\mathcal{R}\mathrm{an}(\sigma)$. By the universal mapping property of $\mathbf{Tm}(X)$, substitutions can be extended in a unique way to endomorphisms of $\mathbf{Fm}(X)$ and compose in the standard way.

An *equational theory* over Ω, or an *equational Ω-theory*, is a set

$$E = \{p_i \approx q_i \mid i \in I\}$$

of identities of type Ω over X, that is, E a set of pairs $\langle p_i, q_i\rangle \in \mathbf{Tm}(X) \times \mathbf{Tm}(X)$. Clearly E axiomatizes the variety of Ω-algebras $\mathcal{V}_E$ consisting of those Ω-algebras $\mathfrak{A}$ such that $\mathfrak{A} \models E$.

Notice that any substitution can be “extended” to an endomorphism of $\mathbf{F}_{\mathcal{V}_E}(X)$. Indeed, given a substitution $\tau\colon X \to \mathbf{Tm}(X)$, since the set of variables X is in one-to-one correspondence with the set $[X] = \{[x] \mid x \in X\}$ of the free generators of $\mathbf{F}_{\mathcal{V}_E}(X)$, first consider the map $[\tau]\colon [X] \to \mathbf{Tm}(X)$ given by $[\tau]([x]) = \tau(x)$ for all $[x] \in [X]$; then let $\pi_{\mathcal{V}_E}\colon \mathbf{Tm}(X) \to \mathbf{F}_{\mathcal{V}_E}(X)$ be the natural projection and finally consider the composition $\pi_{\mathcal{V}_E} \circ [\tau]\colon [X] \to \mathbf{F}_{\mathcal{V}_E}(X)$. Now, by the universal mapping property of $\mathbf{F}_{\mathcal{V}_E}(X)$, we get a (unique) endomorphism $\overline{\tau}\colon \mathbf{F}_{\mathcal{V}_E}(X) \to \mathbf{F}_{\mathcal{V}_E}(X)$ such that $\overline{\tau}\restriction_{[X]} = \pi_{\mathcal{V}_E} \circ [\tau]$. It is immediately seen that

$$\tau(q) = p \Longrightarrow \overline{\tau}([q]) = [p].$$

Conversely, given any endomorphism $h\colon \mathbf{F}_{\mathcal{V}_E}(X) \to \mathbf{F}_{\mathcal{V}_E}(X)$, we can recover a substitution $\sigma_h\colon X \to \mathbf{Tm}(X)$ such that $\overline{\sigma_h} = h$ just by letting $\sigma_h(x) = p$ iff $[p] = h([x])$, for all $x \in X$.

Furthermore, by a similar reasoning, we can also show that substitutions are closely related to valuations in $\mathbf{F}_{\mathcal{V}_E}(X)$. In this case, given a substitution $\tau\colon X \to \mathbf{Tm}(X)$, we first extend τ to an endomorphism of $\mathbf{Tm}(X)$ and then we get a valuation $v_\tau\colon \mathbf{Tm}(X) \to \mathbf{F}_{\mathcal{V}_E}(X)$ simply by taking the composition $\pi_{\mathcal{V}_E} \circ \tau$. Conversely, with each valuation $h\colon \mathbf{Tm}(X) \to \mathbf{F}_{\mathcal{V}_E}(X)$, we can associate the substitution $\gamma^h\colon X \to \mathbf{Tm}(X)$, defined by $\gamma^h(x) = p$ iff $[p] = h(x)$, for all $x \in X$. It is evident that $v_{\gamma^h} = \pi_{\mathcal{V}_E} \circ \gamma^h = h$.

Since these facts will be frequently used in what follows, we will formally state them in the following

Remark 15. Let E be an equational theory over Ω and $\tau\colon X \to \mathbf{Tm}(X)$ a substitution. The *τ-endomorphism* of $\mathbf{F}_{\mathcal{V}_E}(X)$ and the *τ-valuation* in $\mathbf{F}_{\mathcal{V}_E}(X)$ are the homomorphism $\overline{\tau}\colon \mathbf{F}_{\mathcal{V}_E}(X) \to \mathbf{F}_{\mathcal{V}_E}(X)$ and $v_\tau\colon \mathbf{Tm}(X) \to \mathbf{F}_{\mathcal{V}_E}(X)$

defined as above and for which it holds that $\tau(x) = p$ implies $\overline{\tau}([x]) = [p]$ and $v_\tau(x) = [p]$. Furthermore, every endomorphism of $\mathbf{F}_{\mathcal{V}_E}(X)$ and every valuation in $\mathbf{F}_{\mathcal{V}_E}(X)$ is the σ-endomorphism and the σ-valuation for some substitution σ.

Remark 16. Let E be an equational theory over Ω and let $\tau\colon X \to \mathbf{Tm}(X)$ be a substitution. Then, for any identity $p \approx q$ of type Ω over X,

$$\mathcal{V}_E \models \tau(p) \approx \tau(q) \iff \mathbf{F}_{\mathcal{V}_E}(X) \models p \approx q\,[v_\tau].$$

Definition 5.1. A (*symbolic*) *E-unification problem* is a finite set of identities $\Sigma \subsetneq \mathrm{Id}(X)$ of type Ω over X, that is a set

$$\Sigma = \{s_j \approx t_j \mid j \in J\}$$

for some finite index set J. A *unifier* for Σ, or a *solution* for Σ, is a substitution σ such that

$$\mathcal{V}_E \models \sigma(s_j) \approx \sigma(t_j), \quad \text{for all } j \in J.$$

We denote by $U_E(\Sigma)$ the set of unifiers for the E-unification problem Σ and we say that Σ is *unifiable*, or *solvable*, if $U_E(\Sigma) \neq \emptyset$[3].

Given two E-unification problems Σ and Σ', we say that Σ is equivalent Σ' if $U_E(\Sigma) = U_E(\Sigma')$.

Definition 5.2. Let σ and τ be substitutions and let $Y \subseteq X$ be a subsets of variables. We say that σ is *more general* than τ (with respect to E and Y), and we write $\tau \preccurlyeq_E^Y \sigma$, if there exists a substitution θ such that

$$\mathcal{V}_E \models (\theta \circ \sigma)(x) \approx \tau(x), \quad \text{for all } x \in Y.$$

Thus $\tau \preccurlyeq_E^Y \sigma$ means that τ is an *istantiation* of σ up to E-equivalence and only as far as variables in Y are concerned. Notice that the relation $\preccurlyeq_E^Y$ is reflexive and transitive, thus by endowing the set of unifiers $U_E(\Sigma)$ of a given solvable unification problem Σ with the relation $\preccurlyeq_E^Y$, where $Y = \mathbf{Var}\Sigma$ is the set of variables occurring in the equations of Σ, we get a preordered set $\langle U_E(\Sigma), \preccurlyeq_E^Y\rangle$ and $\preccurlyeq_E^Y$ is called the *instantiation preorder of* Σ[4].

Now, the most fundamental and valuable piece of information one would like to have with respect to E in connection with unification issues is the *unification type* of E. Let us introduce this notion in an abstract setting.

Let $\langle P, \preceq\rangle$ be a preorder. Define the following equivalence relation on P by requiring, for all $p, q \in P$,

$$p \sim q \iff p \preceq q \;\&\; q \preceq p.$$

Moreover, we can naturally induce a partial order $\leq$ on the class of $\sim$-equivalence classes $P/\sim$ by

$$[p] \leq [q] \iff p \preceq q.$$

Thus $\langle P/\sim, \leq\rangle$ is a poset and it is called the *canonical quotient* of $\langle P, \preceq\rangle$.

[3] In order to be precise, we have just defined what is usually called an *elementary E*-unification problem, namely finite sets of identities Σ that do not contain function symbols not included in the signature Ω. Cfr. [4, Definition 3.9].

[4] We could also have defined unifiers for a given unification problem Σ as substitutions with a finite domain equal to the set of variables occurring in Σ, consistently with the definition of the instantiation preorder $\preccurlyeq_E^Y$. Cfr. [4, §3.2.1] for a discussion concerning the definition of the instantiation preorder.

Definition 5.3. Let $\langle P, \preceq\rangle$ be a preorder. A subset $M \subseteq P$ of P is called *complete* if, for all $p \in P$, there exists $m \in M$ such that $p \preceq m$. A complete subset M is said to be a *μ-set* of $\langle P, \preceq\rangle$ if all elements of M are mutually $\preceq$-incomparable.

Lemma 5.1. *Let $\langle P, \preceq\rangle$ be a preorder and let M be a μ-set for $\langle P, \preceq\rangle$. Then $[M] = \{[m] \mid m \in M\}$ is a μ-set for the canonical quotient $\langle P/\sim, \leq\rangle$ and $[M]$ coincide with the set of $\leq$-maximal elements of $P/\sim$. Conversely, if the set Q of $\leq$-maximal elements of $P/\sim$ is complete, then the set $\{m \mid [m] \in Q\}$, obtained by choosing exactly one representative for each maximal equivalence class, is a μ-set for $\langle P, \preceq\rangle$.*

It follows moreover that all μ-sets for $\langle P, \preceq\rangle$, if any, have the same cardinality. This allows us to give the following

Definition 5.4. Let $\langle P, \preceq\rangle$ be a preorder. We say that $\langle P, \preceq\rangle$ (or, simply P, when the preorder is clear from the context) has *type*

1 if it has a μ-set of cardinality 1;

$\boldsymbol{\omega}$ if it has a μ-set of finite (greater that 1) cardinality;

$\boldsymbol{\infty}$ if it has a μ-set of infinite cardinality;

0 if it has no μ-set at all.

It has to be understood that the above list of types is arranged in decreasing order of desirability. We are now ready for giving the main definition of this section.

Definition 5.5. Let E be an equational Ω-theory. The *unification type* of E is

- *unitary* (or 1), if, for every solvable E-unification problem Σ, $U_E(\Sigma)$ has type 1;
- *finitary* (or ω), if, for every solvable E-unification problem Σ, $U_E(\Sigma)$ has type 1 or ω and there exists a solvable E-unification problem Σ such that $U_E(\Sigma)$ has type ω;
- *infinitary* (or ∞) if, for every solvable E-unification problem Σ, $U_E(\Sigma)$ has type 1, ω or ∞ and there exists a solvable E-unification problem Σ such that $U_E(\Sigma)$ has type ∞;
- *nullary* (or 0) if there exists a solvable E-unification problem Σ such that $U_E(\Sigma)$ has type 0.

Thus the unification type of an Ω-theory E is defined to be unitary, finitary, infinitary or nullary according to the worst cases among types of solvable unification problems Σ. When an Ω-theory E has unitary unification type - the best case according to our definition -, then, for every solvable unification problem Σ, $U_E(\Sigma)$ has a μ-set of cardinality 1 and, according to Lemma 5.1, the canonical quotient $\langle U_E(\Sigma)/\sim, \leq_E^Y\rangle$ has a maximum element $[\sigma]$. Every element $\tau \in [\sigma]$ is called a *most general unifier* (briefly *mgu*) for Σ. Since a mgu is unique up to $\sim$-equivalence, one speaks of the mgu for Σ. If it is the case that $[\sigma]$ is maximal, but not a maximum, then any element $\tau \in [\sigma]$ is instead called a *maximally general unifier* for Σ.

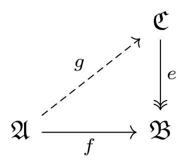
Examples of each kind are presented in [4, §3.4], where it is also stated the solvability and complexity of the related decision problems and references for the unification algorithms are also supplied. Cfr. also [3], for different (non-equivalent) characterizations of unification type 0.

5.1.2 Algebraic E-Unification

We now present an equivalent algebraic approach to E-unification due to Ghilardi [61]. The main feature of this approach is its *categorical* nature: being based only on the notions of *finitely presented* and *regular projective* object, this approach can be introduced in any abstract category and makes unification type a *categorical invariant*[5].

Let us fix an Ω-theory E for some given similarity type Ω and a countable set of variables $X = \{x_0, x_1, x_2, \ldots\}$. Consider moreover the variety $\mathcal{V}_E$ axiomatized by E. Recall that we can consider this variety as the equational category $\boldsymbol{\mathcal{V}_E}$ with Ω-algebras satisfying the equations in E as objects and the related Ω-homomorphism as arrows.

Recall that an algebra $\mathfrak{A}$ in the equational category $\boldsymbol{\mathcal{V}_E}$ is said to be *finitely presented* if $\mathfrak{A} \cong \mathbf{F}_{\mathcal{V}_E}(Y, S)$ for some finite presentation $\langle Y|S\rangle$, that is, if $\mathfrak{A}$ is isomorphic to the algebra $\mathbf{F}_{\mathcal{V}_E}(Y)/\Theta(S)$ where $\mathbf{F}_{\mathcal{V}_E}(Y)$ is the free object in $\boldsymbol{\mathcal{V}_E}$ generated by Y and $\Theta(S)$ is the congruence relation generated by the set S[6]. Moreover, we remind the reader that we call an object $\mathfrak{A} \in \boldsymbol{\mathcal{V}_E}$ regular projective in $\boldsymbol{\mathcal{V}_E}$ if, for any regular epi $e\colon \mathfrak{C} \to \mathfrak{B}$ and any morphism $f\colon \mathfrak{A} \to \mathfrak{B}$, there exists a morphism $g\colon \mathfrak{A} \to \mathfrak{C}$ such that the following diagram commutes



By the way, when we are dealing only with fp algebras, the notion of regular projectivity can be further simplified. Indeed, we have the following

Lemma 5.2. *A fp algebra $\mathfrak{P}$ is regular projective in $\boldsymbol{\mathcal{V}_E}$ iff every regular epi $f\colon \mathfrak{A} \to \mathfrak{P}$ is a split epimorphism, i.e. it has a section $h\colon \mathfrak{P} \to \mathfrak{A}$ such that $f \circ h = 1_{\mathfrak{P}}$.*

Remark 17. It is not difficult to show that a finitely presented algebra $\mathfrak{A}$ in $\boldsymbol{\mathcal{V}_E}$ is regular projective in $\boldsymbol{\mathcal{V}_E}$ iff it is regular projective in $\boldsymbol{\mathcal{V}_E}^{\mathrm{fp}}$, that is, in the full subcategory of $\boldsymbol{\mathcal{V}_E}$ determined by finitely presented algebras. Indeed, coequalizers in $\boldsymbol{\mathcal{V}_E}^{\mathrm{fp}}$ are coequalizers in $\boldsymbol{\mathcal{V}_E}$: given a couple of parallel morphism between fp algebras $f_1, f_2\colon \mathbf{F}(Y, S) \to \mathbf{F}(X, Z)$, the coequalizer of f_1 and f_2, both in $\boldsymbol{\mathcal{V}_E}^{\mathrm{fp}}$ and $\boldsymbol{\mathcal{V}_E}$, is given by the quotient map $f\colon \mathbf{F}(X, Z) \to \mathbf{F}(X, Z \cup Z')$ with $Z' = \{\langle t_1^y, t_2^y\rangle \mid y \in Y\}$ where $f_1([y]) = [t_1^y]$ and $f_2([y]) = [t_2^y]$.

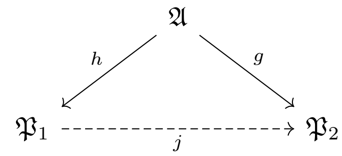
[5] Cfr. [61, §7] for a comparison of Ghilardi's approach to E-unifications through projectivity with other algebraic approaches, for instance the approach presented in [4, §3.3.3].

[6] As we mentioned in the beginning of this section, the notion of finitely presented algebra is a purely categorical notion since it is known that an algebra $\mathfrak{A}$ in $\boldsymbol{\mathcal{V}}$ is finitely presented iff the representable functor of $\mathfrak{A}$, $\mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(A, -)$, preserves filtered colimits. Cfr. [58], in particular §6 and §7.

We are now ready to give the relevant definition of an algebraic E-unification problem.

Definition 5.6. An (*algebraic*) *E-unification problem* is a finitely presented algebra $\mathfrak{A}$ in $\boldsymbol{\mathcal{V}_E}$. An *algebraic unifier* for $\mathfrak{A}$, or a *solution* for $\mathfrak{A}$, is a morphism $h\colon \mathfrak{A} \to \mathfrak{P}$ where $\mathfrak{P}$ is a projective fp algebra in $\boldsymbol{\mathcal{V}_E}$.
We denote the set of algebraic unifiers for the E-unification problem $\mathfrak{A}$ by $U_E(\mathfrak{A})$ and we say that $\mathfrak{A}$ is *algebraically unifiable*, or *solvable*, if $U_E(\mathfrak{A}) \neq \emptyset$.

Definition 5.7. Let $\mathfrak{A}$ be an algebraic E-unification problem and let $h\colon \mathfrak{A} \to \mathfrak{P}_1$ and $g\colon \mathfrak{A} \to \mathfrak{P}_2$ be algebraic unifiers for $\mathfrak{A}$. We say that h is *more general* than g, and we write $g \preccurlyeq h$, if there exists a morphism $j\colon \mathfrak{P}_1 \to \mathfrak{P}_2$ making the following diagram commute



Again, it is easily shown that, for every algebraic E-unification problem $\mathfrak{A}$, $\langle U_E(\mathfrak{A}), \preccurlyeq \rangle$ is a preorder set. Therefore, the definition of the algebraic unification type of $\boldsymbol{\mathcal{V}_E}$, where E is an Ω-equational theory, is the expected one, namely,

Definition 5.8. Let E be an equational Ω-theory. The *algebraic unification type* of $\boldsymbol{\mathcal{V}_E}$ is

- *unitary* (or 1), if, for every solvable E-unification problem $\mathfrak{A}$, $U_E(\mathfrak{A})$ has type 1;
- *finitary* (or ω), if, for every solvable E-unification problem $\mathfrak{A}$, $U_E(\mathfrak{A})$ has type 1 or ω and there exists a solvable E-unification problem $\mathfrak{A}$ such that $U_E(\mathfrak{A})$ has type ω;
- *infinitary* (or ∞) if, for every solvable E-unification problem $\mathfrak{A}$, $U_E(\mathfrak{A})$ has type 1, ω or ∞ and there exists a solvable E-unification problem $\mathfrak{A}$ such that $U_E(\mathfrak{A})$ has type ∞;
- *nullary* (or 0), if there exists a solvable E-unification problem $\mathfrak{A}$ such that $U_E(\mathfrak{A})$ has type 0.

It is clear that, in this algebraic setting, a mgu $h\colon \mathfrak{A} \to \mathfrak{P}$ for a given algebraic unification problem $\mathfrak{A}$ is a morphism through which any other algebraic unifier g for $\mathfrak{A}$ can be factorized.

The following theorem, due to Ghilardi, shows the equivalence of the symbolic and algebraic approach to E-unification[7].

Theorem 5.3 (Ghilardi)**.** *Let E be an Ω-theory for some given similarity type Ω over a countable set of variables $X = \{x_0, x_1, x_2, \ldots\}$ and let $\boldsymbol{\mathcal{V}_E}$ be the equational category associated with the variety $\mathcal{V}_E$ of algebras axiomatized by E. Consider the (symbolic) E-unification problem*

$$\Sigma = \{s_j \approx t_j \mid j \in J\}$$

[7]For the proof of the theorem, cfr. [61, Theorem 4.1].

for some finite index set J and let $Y = \mathbf{Var}(\Sigma)$. Moreover, let $\mathfrak{A}$ be the finitely presented algebra presented by the finite presentation $\langle Y, \Sigma\rangle$. Then the preorder $U_E(\Sigma)$ of unifiers of Σ and the preorder $U_E(\mathfrak{A})$ of algebraic unifiers of $\mathfrak{A}$ are equivalent as categories, that is, the posets of their canonical quotients are isomorphic. In particular, Σ is unifiable iff $\mathfrak{A}$ is algebraically unifiable and the unification type of E and the algebraic unification type of $\boldsymbol{\mathcal{V}_E}$ coincide.

Another advantage of Ghilardi's approach is that the determination of the unification type of locally finite varieties $\mathcal{V}$ over finite algebraic languages becomes concretely simpler. Indeed, for such varieties, the class of fp algebras and the class of finite algebras coincide, as stated in the following

Proposition 5.4. *Let $\mathcal{V}$ be a locally finite variety of Ω-algebras, where $\Omega = \langle \mathcal{F}, ar\rangle$ is a finite similarity type, i.e. $|\mathcal{F}| < \omega$, and let $\mathfrak{A}$ be an Ω-algebra in $\mathcal{V}$. Then $\mathfrak{A}$ is finite iff $\mathfrak{A}$ is finitely presented.*

5.1.3 Other Types of Unification

As we mentioned *en passant* at the end of §5.1.1, there are other different (non-equivalent) characterizations of unification type 0. In this section, we will focus on the opposite side of the unification type list and deal instead with different forms of unitary unification.

Let us consider an equational Ω-theory E and a countable set of variables X. Let σ and τ be substitutions and $Y \subseteq X$ a set of variables. We say that σ *is not seen by* τ (through E and Y) if

$$\mathcal{V}_E \models (\tau \circ \sigma)(x) \approx \tau(x), \quad \text{for all } x \in Y.$$

Now, let Σ be a E-unification problem. A unifier $\sigma \in U_E(\Sigma)$ is said to be *transparent* if $\mathcal{D}\mathrm{om}(\sigma) = \mathbf{Var}\mathcal{R}\mathrm{an}(\sigma)$ and σ is not seen by any $\tau \in U_E(\Sigma)$ through $\mathbf{Var}(\Sigma)$.

Definition 5.9. Let E be an equational Ω-theory. We say that E has *transparent unification* if, for every solvable E-unification problem Σ, there exists a transparent unifier σ in $U_E(\Sigma)$.

Transparent unification has been introduced by Wroński in [167], where it is shown that it is a stronger form of unitary unification. Indeed, we have the following

Lemma 5.5. *Let E be an equational Ω-theory and Σ be a E-unification problem. A unifier $\sigma \in U_E(\Sigma)$ is transparent iff σ is most general and idempotent.*

Proof. ($\Longrightarrow$) Suppose $\sigma \in U_E(\Sigma)$ is transparent. Then σ is clearly a mgu for Σ. Moreover, since σ does not see itself, we have

$$\mathcal{V}_E \models (\sigma \circ \sigma)(x) \approx \sigma(x), \quad \text{for all } x \in \mathbf{Var}(\Sigma),$$

that is, σ is idempotent.
($\Longleftarrow$) Conversely, suppose $\sigma \in U_E(\Sigma)$ is an idempotent mgu for Σ and let $\tau \in U_E(\Sigma)$. Since $\tau \preccurlyeq_E^{\mathbf{Var}(\Sigma)} \sigma$, there exists a substitution θ such that

$$\mathcal{V}_E \models (\theta \circ \sigma)(x) \approx \tau(x), \quad \text{for all } x \in \mathbf{Var}(\Sigma).$$

Thus, by composing with σ on the right,

$$\mathcal{V}_E \models (\theta \circ \sigma \circ \sigma)(x) \approx \tau \circ \sigma(x), \quad \text{for all } x \in \mathbf{Var}(\Sigma);$$

finally, since σ is idempotent, we then get, for all $x \in \mathbf{Var}(\Sigma)$,

$$\mathcal{V}_E \models \tau(x) \approx (\tau \circ \sigma)(x). \qquad \square$$

The following very important class of unifiers has been introduced by Ghilardi [61, 62, 63], mainly for investigations in propositional logics. Consider a E-unification problem $\Sigma = \{s_j \approx t_j \mid j \in J\}$, where $J = \{1, \ldots, j\}$ is a finite index set. A unifier $\sigma \in U_E(\Sigma)$ is said to be *projective* if $\mathcal{D}\mathrm{om}(\sigma) = \mathbf{Var}\mathcal{R}\mathrm{an}(\sigma)$ and $\mathcal{V}_E$ satisfies the following quasi-identities:

$$s_1 \approx t_1 \ \& \ \ldots \ \& \ s_j \approx t_j \Rightarrow \sigma(x) \approx x, \quad \text{for all } x \in \mathbf{Var}(\Sigma).$$

Definition 5.10. Let E be an equational Ω-theory. Then E is said to have *projective unification* if, for every solvable E-unification problem Σ, there exists a projective unifier σ in $U_E(\Sigma)$.

Let us now translate the notion of projective unification in the algebraic context. If a symbolic E-unification problem $\Sigma = \{s_j \approx t_j \mid j \in J\}$, for some finite index set $J = \{1, \ldots, j\}$, and a projective unifier $\sigma \in U_E(\Sigma)$ are given, then consider the corresponding algebraic unification problem $\mathfrak{A} \cong \mathbf{F}_{\mathcal{V}_E}(Y, \Sigma)$ where $Y = \mathbf{Var}\Sigma$ and let $\mathfrak{B}$ be the $\mathcal{V}_E$-free algebra $\mathbf{F}_{\mathcal{V}_E}(Y)$. Define the algebraic unifier $e_\sigma \colon \mathfrak{A} \to \mathfrak{B}$ by letting, for all $t \in \mathbf{Tm}(Y)$,

$$e_\sigma([t]_{\mathfrak{A}}) = [\sigma(t)]_{\mathfrak{B}}.$$

Notice that the previous definition is sound. Indeed, if, for some $t_1, t_2 \in \mathbf{Tm}(Y)$, we have $[t_1]_{\mathfrak{A}} = [t_2]_{\mathfrak{A}}$, then by Theorem 4.6 (ii), $\mathcal{V}_E \models \& \Sigma \Rightarrow t_1 \approx t_2$ and thus $[\sigma(t_1)]_{\mathfrak{B}} = [\sigma(t_2)]_{\mathfrak{B}}$, since $\sigma \in U_E(\Sigma)$. Furthermore, since σ is projective, we have also $\mathcal{V}_E \models s_1 \approx t_1 \ \& \ \ldots \ \& \ s_j \approx t_j \Rightarrow \sigma(x) \approx x$, for all $x \in Y$, and therefore, again by Theorem 4.6, we have $[\sigma(x)]_{\mathfrak{A}} = [x]_{\mathfrak{A}}$. Then, by induction on the construction of terms, we get that $\mathfrak{A}$ validates $[\sigma(t)] = [t]$ for all $t \in \mathbf{Tm}(Y)$. Finally, consider the natural projection $\pi \colon \mathfrak{B} \to \mathfrak{A}$ associating each element $[t]_{\mathfrak{B}}$ of $\mathfrak{B}$ with the corresponding equivalence class $[t]_{\mathfrak{A}}$ in $\mathfrak{A}$. Then it easy to see that $\pi \circ e_\sigma = 1_{\mathfrak{A}}$, that is, $\mathfrak{A}$ is the retract of the $\mathcal{V}_E$-free algebra $\mathfrak{B}$ and it is therefore a regular projective object in $\boldsymbol{\mathcal{V}_E}$.
Conversely, if $\mathfrak{A}$ is regular projective, then $1_{\mathfrak{A}}$ is an algebraic unifier for $\mathfrak{A}$. Since the canonical projection $\pi \colon \mathfrak{B} \to \mathfrak{A}$ is a regular epi, it has a section $s \colon \mathfrak{A} \to \mathfrak{B}$ by Lemma 5.2. It can be easily seen that the substitution $\sigma \colon Y \to \mathbf{Tm}(Y)$ defined, for all the x_i's in Y, by taking $\sigma(x_i)$ to be any term t_i such that $s([x_i]_{\mathfrak{A}}) = [t_i]_{\mathfrak{B}}$ is a unifier for Σ. But $\pi \circ s = 1_{\mathfrak{A}}$ and therefore we have

$$[x_i]_{\mathfrak{A}} = (\pi \circ s)[x_i]_{\mathfrak{A}} = \pi([\sigma(x_i)]_{\mathfrak{B}}) = [\sigma(x_i)]_{\mathfrak{A}},$$

for all $x_i \in Y$. Since $\mathfrak{A}$ is finitely presented by $\langle Y | \Sigma \rangle$, by Theorem 4.6 (ii), we get $\mathcal{V}_E \models \& \Sigma \Rightarrow \sigma(x) \approx x$, for all $x \in Y$, that is, σ is projective.

So, we have just proved the following

Lemma 5.6. *Let E be an equational Ω-theory. Then E has projective unification iff every solvable algebraic E-unification problem $\mathfrak{A}$ in $\boldsymbol{\mathcal{V}_E}$ is a regular projective object in $\boldsymbol{\mathcal{V}_E}$.*

Therefore, having projective unification is a rather strong property. Indeed, we have the following

Lemma 5.7. *Let E be an equational Ω-theory with projective unification. Then $\mathcal{V}_E$ is actively structurally complete.*

Proof. If E has projective unification, then every fp algebra in $\mathcal{V}_E$ is regular projective in $\boldsymbol{\mathcal{V}_E}$. By Theorem 3.34, this means that every fp algebra $\mathfrak{A}$ is the retract of a $\mathcal{V}_E$-free algebra and, in particular, that $\mathfrak{A}$ embeds in $\mathbf{F}_{\mathcal{V}_E}(\omega)$. Therefore, we have $\mathfrak{A} \in \mathbb{Q}(\mathbf{F}_{\mathcal{V}_E}(\omega))$ and $\mathcal{V}_E$ is asc by Theorem 4.13 (5). □

We conclude this section with a lemma which clarifies the relation between projective and transparent unification, but first let us translate also the notion of transparent E-unification in the algebraic setting. Let an equational Ω-theory E be given. An algebraic E-unification problem $\mathfrak{A} = \mathbf{F}_{\mathcal{V}_E}(Y, \Sigma)$ is said to be *transparent* in $\boldsymbol{\mathcal{V}_E}$ if there exists a unifier $h\colon \mathfrak{A} \to \mathbf{F}_{\mathcal{V}_E}(Y)$ in $U_E(\mathfrak{A})$ such that, for any unifier $g \in U_E(\mathfrak{A})$,

$$g \circ \pi \circ h = g,$$

where $\pi\colon \mathbf{F}_{\mathcal{V}_E}(Y) \to \mathbf{F}_{\mathcal{V}_E}(Y, \Sigma)$ is the canonical projection. In such a case, we call h a transparent algebraic unifier for $\mathfrak{A}$.

Lemma 5.8. *Let E be an equational Ω-theory. Then E has transparent unification iff every solvable algebraic E-unification problem $\mathfrak{A}$ is transparent in $\boldsymbol{\mathcal{V}_E}$.*

Proof. ($\Longrightarrow$) Let $\mathfrak{A} \cong \mathbf{F}_{\mathcal{V}_E}(Y, S)$ be a solvable algebraic E-unification problem. Let σ be a transparent unifier for the symbolic E-unification problem S. We claim that the morphism $e_\sigma\colon \mathfrak{A} \to \mathbf{F}_{\mathcal{V}_E}(Y)$, defined as above, is a trasparent algebraic unifier for $\mathfrak{A}$. Indeed, consider an algebraic unifier $g \in U_E(\mathfrak{A})$. Then $g\colon \mathfrak{A} \to \mathfrak{B}$ is a morphism where $\mathfrak{B}$ is a fp projective object in $\boldsymbol{\mathcal{V}_E}$. Thus $\mathfrak{B}$ is the retract of a $\mathcal{V}_E$-free algebra $\mathbf{F}_{\mathcal{V}_E}(Z)$ for some finite generating set Z and, in particular, there exists a monomorphism $s\colon \mathfrak{B} \to \mathbf{F}_{\mathcal{V}_E}(Z)$. Now, define the substitution $\tau\colon Y \to \mathbf{Tm}(Z)$ by taking $\tau(y_i)$ to be any term t_i such that $(s \circ g)[x_i]_{\mathfrak{A}} = [t_i]_{\mathbf{F}_{\mathcal{V}_E}(Z)}$. Therefore, for all the y_i's in Y, we have by definition

$$(s \circ g)[y_i]_{\mathfrak{A}} = [\tau(y_i)]_{\mathbf{F}_{\mathcal{V}_E}(Z)},$$

and thus, by induction on the construction of terms,

$$(s \circ g)[t]_{\mathfrak{A}} = [\tau(t)]_{\mathbf{F}_{\mathcal{V}_E}(Z)},$$

for all $t \in \mathbf{Tm}(Y)$. Therefore, since in $\mathfrak{A}$ all the equalities from S holds, we have that τ is indeed a unifier for S. Furthermore, for all $t \in \mathbf{Tm}(Y)$, we have

$$g([t]_{\mathfrak{A}}) = g([\sigma(t)]_{\mathfrak{A}}).$$

Indeed, suppose otherwise. Then, since s is injective, we would have

$$[\tau(t)]_{\mathbf{F}_{\mathcal{V}_E}(Z)} = (s \circ g)([t]_{\mathfrak{A}}) \neq (s \circ g)([\sigma(t)]_{\mathfrak{A}}) = [\tau \circ \sigma(t)]_{\mathbf{F}_{\mathcal{V}_E}(Z)},$$

contrary to the transparency of σ. Therefore, we have

$$g \circ \pi \circ e_\sigma = g,$$

that is, $\mathfrak{A}$ is transparent.
($\Longleftarrow$) Let Σ be a solvable symbolic E-unification problem and consider the corresponding algebraic E-unification problem $\mathfrak{A} \cong \mathbf{F}_{\mathcal{V}_E}(Y, \Sigma)$, where $Y = \mathbf{Var}\Sigma$. Let $h \in U_E(\mathfrak{A})$ be a transparent algebraic unifier. We claim that the unifier $\sigma_h \colon Y \to \mathbf{Tm}(Y)$ defined as

$$\sigma_h(y_i) = t_i \Longleftrightarrow h([y_i]_{\mathfrak{A}}) = [t_i]_{\mathbf{F}_{\mathcal{V}_E}(Y)},$$

is a transparent unifier for the symbolic E-unification problem Σ. So, consider a unifier $\tau \in U_E(\Sigma)$. Without loss of generality, we can assume that $\mathcal{D}\mathrm{om}(\tau) = Y$. Now consider the morphism $e_\tau \colon \mathfrak{A} \to \mathbf{F}_{\mathcal{V}_E}(Q)$, where $Q = \mathbf{Var}\mathcal{R}\mathrm{an}(\tau)$, defined as above. Since $e_\tau \in U_E(\mathfrak{A})$, we have, for all $y \in Y$,

$$\begin{aligned}
[\tau(y)]_{\mathbf{F}_{\mathcal{V}_E}(Q)} &= e_\tau([y]_{\mathfrak{A}}) \\
&= e_\tau \circ \pi \circ h([y]_{\mathfrak{A}}) && \text{since } h \text{ is transparent} \\
&= e_\tau \circ \pi([\sigma_h(y)]_{\mathbf{F}_{\mathcal{V}_E}(Y)}) \\
&= e_\tau([\sigma_h(y)]_{\mathfrak{A}}) \\
&= [\tau \circ \sigma_h(y)]_{\mathbf{F}_{\mathcal{V}_E}(Q)}
\end{aligned}$$

and thus $\mathcal{V}_E \models (\tau \circ \sigma_h)(y) \approx \tau(y)$, for all $y \in Y$. □

Notice that, given an algebraic E-unification problem $\mathfrak{A} \cong \mathbf{F}_{\mathcal{V}_E}(Y, S)$ and a transparent algebraic unifier h for $\mathfrak{A}$, we also have $h \circ \pi \circ h = h$ and thus the composition $h \circ \pi \colon \mathbf{F}_{\mathcal{V}_E}(Y) \to \mathbf{F}_{\mathcal{V}_E}(Y)$ is an idempotent morphism. We are now ready to make explicit the relation existing between projective and transparent E-unification.

Lemma 5.9. *Let E be an equational Ω-theory. Then E has projective unification iff E has transparent unification and $\mathcal{V}_E$ is actively structurally complete.*

Proof. ($\Longrightarrow$) If E has projective unification, then E has transparent unification, since every projective unifier is transparent. Moreover $\mathcal{V}_E$ is actively structurally complete by Lemma 5.7.
$(\Longleftarrow)_1$ Suppose E has transparent unification and $\mathcal{V}_E$ is actively structurally complete. Consider a solvable unification problem $\Sigma = \{s_j \approx t_j \mid j \in J\}$ and let $\sigma \in U_E(\Sigma)$ be a transparent unifier. Therefore the quasi-identities

$$\varphi_x : \qquad s_1 \approx t_1 \ \& \ \ldots \ \& \ s_j \approx t_j \Rightarrow \sigma(x) \approx x,$$

for all $x \in \mathbf{Var}\Sigma$, are $\mathcal{V}_E$-active. By definition of transparent unifier, we have that σ is not seen by τ for all $\tau \in U_E(\Sigma)$. Equivalently, for every substitution τ, if $\mathcal{V}_E \models \tau(s_1) \approx \tau(t_1) \ \& \ \ldots \ \& \ \tau(s_j) \approx \tau(t_j)$, then $\mathcal{V}_E \models \tau(\sigma(x)) \approx \tau(x)$ for all $x \in \mathbf{Var}\Sigma$. But then, by Remarks 15 and 16, we have $\mathbf{F}_{\mathcal{V}_E}(\omega) \models \varphi_x$ for all $x \in \mathbf{Var}\Sigma$. So, since $\mathcal{V}_E$ is asc, $\mathcal{V}_E \models \varphi_x$ for all $x \in \mathbf{Var}\Sigma$, i.e. σ is a projective unifier for Σ.
$(\Longleftarrow)_2$ Suppose E has transparent unification and $\mathcal{V}_E$ is actively structurally complete. Consider a solvable algebraic unification problem $\mathfrak{A} \cong \mathbf{F}_{\mathcal{V}_E}(Y, S)$. We show that $\mathfrak{A}$ is a regular projective object in $\boldsymbol{\mathcal{V}_E}$. Let $g \colon \mathfrak{A} \to \mathbf{F}_{\mathcal{V}_E}(Y)$ be a transparent unifier for $\mathfrak{A}$. Then by Corollary 4.14 we have $\mathfrak{A} \in \mathbb{SP}(\mathbf{F}_{\mathcal{V}_E}(Y))$, that is, there exists an embedding $h \colon \mathfrak{A} \to \mathbf{F}_{\mathcal{V}_E}(Y)^I$ of $\mathfrak{A}$ into some direct power of $\mathbf{F}_{\mathcal{V}_E}(Y)$ for some index set I. For each $i \in I$, consider the algebraic unifiers

$h_i = \pi_i \circ h\colon \mathfrak{A} \to \mathbf{F}_{\mathcal{V}_E}(Y)$. It is readily seen that $h = \hat{h}$ where $\hat{h}$ is the natural homomorphism obtained from the h_i's. Furthermore, since g is transparent, we have, for all $i \in I$,

$$h_i \circ \pi \circ g = h_i.$$

Now, for every $[p], [q] \in \mathfrak{A}$,

$$\begin{aligned} [p] \neq [q] &\Longrightarrow \hat{h}([p]) \neq \hat{h}([q]) \\ &\Longrightarrow \hat{h}([p])(i) \neq \hat{h}([q])(i) \quad \text{for some } i \in I \\ &\Longrightarrow h_i([p]) \neq h_i([q]) \\ &\Longrightarrow h_i \circ \pi \circ g([p]) \neq h_i \circ \pi \circ g([q]) \\ &\Longrightarrow g([p]) \neq g([q]) \end{aligned}$$

that is, $g\colon \mathfrak{A} \to \mathbf{F}_{\mathcal{V}_E}(Y)$ is an embedding. Therefore, $g(\mathfrak{A}) \cong \mathfrak{A}$ is a subalgebra of $\mathbf{F}_{\mathcal{V}_E}(Y)$ and by letting $f\colon \mathbf{F}_{\mathcal{V}_E}(Y) \to g(\mathfrak{A})$ be the morphism obtained from $g \circ \pi$ by restricting the codomain, we have, by the idempotency of $g \circ \pi$, that f is a retraction of the inclusion map $i\colon g(\mathfrak{A}) \to \mathbf{F}_{\mathcal{V}_E}(Y)$. Thus $g(\mathfrak{A})$ is a retract of a $\mathcal{V}_E$-free algebra and thus $\mathfrak{A}$ is a regular projective object in $\boldsymbol{\mathcal{V}_E}$. □

5.2 Unification in Intermediate Logics

Let us first consider **Int** and its characteristic variety $\mathcal{V}_{\mathbf{Int}}$. We know that $\mathcal{V}_{\mathbf{Int}}$ is the variety of Heyting algebra $\mathcal{HA}$ and we can take any (finite) set $E_{\mathcal{HA}}$ of identities over the similarity type $\langle \wedge, \vee, \to, 0 \rangle$ that axiomatizes $\mathcal{HA}$[8] as the equational theory of Heyting algebras.

Now consider an arbitrary $E_{\mathcal{HA}}$-unification problem for the equational theory of Heyting algebras

$$\Sigma = \{s_j \approx t_j \mid j \in J\}$$

for some finite index set J. It can be easily seen that Σ is equivalent to the $E_{\mathcal{HA}}$-unification problem Σ^* consisting of only the following single equation:

$$\bigwedge_{j \in J} s_j \leftrightarrow t_j \approx 1.$$

Notice moreover that, by the algebraic completeness theorem for **Int**, we have

$$\begin{aligned} \sigma \in U_E(\Sigma^*) &\Longleftrightarrow \mathcal{HA} \models \sigma(\bigwedge_{j \in J} s_j \leftrightarrow t_j \approx 1) \\ &\Longleftrightarrow \mathcal{HA} \models \sigma(\bigwedge_{j \in J} s_j \leftrightarrow t_j) \approx 1 \\ &\Longleftrightarrow \sigma(\bigwedge_{j \in J} s_j \leftrightarrow t_j) \in \mathbf{Int}. \end{aligned}$$

Thus, an $E_{\mathcal{HA}}$-unification problem consists in the problem of making a single $\mathcal{L}$-formula a theorem of the logical calculus Int.

[8] For instance, we can take as $E_{\mathcal{HA}}$ the set of all theorems of **Int** or we can let $E_{\mathcal{HA}}$ be the set of all identities $\varphi \approx 1$ where φ is an axiom of the intuitionistic propositional calculus Int.

We can clearly extend the above considerations to each intermediate logic L. Thus, in the logical context, a *unification problem* for L is just a single $\mathcal{L}$-formula φ and a *L-unifier for* φ is a substitution $\sigma\colon \mathbf{Var}\mathcal{L} \to \mathbf{For}\mathcal{L}$ such that

$$\vdash_L \sigma(\varphi).$$

Also the definition of the instantiation preorder of φ is straightforward. Indeed, given two substitutions σ and τ, we let $Y = \mathbf{Var}\varphi$ and we say that σ is *more general* than τ (with respect to L and Y), and we write $\tau \preccurlyeq_L^Y \sigma$, if there exists a substitution θ such that, for all $p \in Y$,

$$\vdash_L (\theta \circ \sigma)(p) \leftrightarrow \tau(p).$$

Algebraic completeness of Ext**Int** allows us to translate in logical terms the notion of a projective unifier. Indeed, given a unifiable $\mathcal{L}$-formula φ, we say that a substitution $\sigma\colon \mathbf{Var}\mathcal{L} \to \mathbf{For}\mathcal{L}$ is a *projective unifier for* φ if σ is a L-unifier for φ and, moreover,

$$\varphi \vdash_L \sigma(p) \leftrightarrow p, \qquad \text{for all } p \in \mathbf{Var}\varphi.$$

In this case, we say that φ is *projective.* Moreover, notice that by Theorem 1.19 (ii) of replacement of L-equivalents the previous condition is equivalent to

$$\varphi \vdash_L \sigma(\psi) \leftrightarrow \psi,$$

for all $\psi \in \mathbf{For}\mathcal{L}$ such that $\mathbf{Var}\psi \subseteq \mathbf{Var}\varphi$.

Remark 18. Notice that we can also reinterpret the notion of admissible rule by means of unification notions as follows: an inference rule Γ/φ is admissible in an intermediate logic L iff every L-unifier for $\bigwedge\Gamma$ is a L-unifier for φ iff $U_L(\bigwedge\Gamma) \subseteq U_L(\varphi)$.

Investigations on the unification type of intermediate propositional logics have started during the nineties with the works of Wroński and Ghilardi[9], who basically managed to locate, in the lattice of intermediate propositional logics, most of the logics having good unification properties. In particular, in [62] it is shown that **Int** has finitary unification and that any intermediate propositional logic having unitary unification type must be an extension of the Jankov's logic **KC**; in [167], it is stated that any intermediate propositional logic is an extension of the Gödel-Dummet logic **LC** exactly when such a logic enjoys projective unification.

A clear picture of the situation concerning unification type in the lattice of intermediate propositional logics Ext**Int** can be summarized in the following theorem (cfr. [35, Theorem 8]):

Theorem 5.10. *All extensions of* **KC** *coincide with all logics having unitary unification type plus some having nullary unification type. All sublogics of* $\mathrm{Log}\,(\mathtt{1}+\mathtt{1})^{\triangledown}$ *coincide with all logics having finitary unification type, plus some having nullary unification type (plus all sublogics having infinitary unification type, if any).*

[9] However, it would not be wrong to say that such investigations could be traced back to the early 20^{th} century in the works of Löwenheim or even to the beginning of the history of logic itself in Boole's work. Indeed, both logicians gave their contribution to the determination of the unification type of classical logic **Cl**. Cfr. [113].

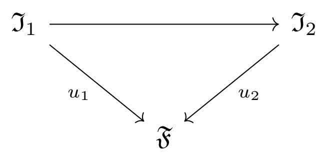
Since the pair $\langle \mathrm{Log}\,(\imath+\imath)^{\triangledown}, \mathbf{KC}\rangle$ is a splitting pair of the lattice Ext**Int**, namely, for every logic $L \in \mathrm{Ext}\mathbf{Int}$, either $\mathbf{KC} \subseteq L$ or $L \subseteq \mathrm{Log}\,(\imath+\imath)^{\triangledown}$, the previous theorem gives indeed an good description of the situation concerning unification types.

The relationship between unification and admissible rules has recently attracted the attention of scholars and has renewed investigations on unification issues. For instance, in [68], it is shown that the logic of bounded branching $\mathbf{T}_n$ has finitary unification type, for every $n \geq 1$. Now, the unification type of many well-known intermediate logics is not known and thus unification issues in intermediate propositional logics are still an open field of research. In particular, the major open problem is whether there exists an intermediate propositional logic having unification type ∞[10].

5.3 Topological approach to unification

We will use the topological framework introduced in the previous chapters in order to investigate unification issues in intermediate logics. Let us first translate the basic notions of unification theory in this new setting.

Let L be an intermediate logic. All the notions concerning unification theory given in §5.1.2 with respect to equational category $\boldsymbol{\mathcal{V}_L}$ corresponding to the variety induced by L can be dualized to the category $\boldsymbol{\mathcal{DF}_L}$. So, in this setting, a *unification problem* is a finitely copresented frame $\mathfrak{F} \in \boldsymbol{\mathcal{DF}_L}$ and a *unifier* for $\mathfrak{F}$ is a p-morphism $u\colon \mathfrak{I} \to \mathfrak{F}$ where $\mathfrak{I}$ is a finitely copresented regular injective frame in $\boldsymbol{\mathcal{DF}_L}$. Furthermore, given two unifiers $u_1\colon \mathfrak{I}_1 \to \mathfrak{F}$ and $u_2\colon \mathfrak{I}_2 \to \mathfrak{F}$ for $\mathfrak{F}$, $u_1 \preceq u_2$, that is, u_2 is *more general* that u_1, if there exists a p-morphism making the following triangle



commute. So, if $\mathfrak{F}$ is regular injective in $\boldsymbol{\mathcal{DF}_L}$, it is immediately seen that the identity morphism $1_{\mathfrak{F}}\colon \mathfrak{F} \to \mathfrak{F}$ is a most general unifier for $\mathfrak{F}$.

As a warm up, let us prove the well-known fact that intuitionistic logic **Int** has finitary unification.

Lemma 5.11. *Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle \subsetneq \mathfrak{F}_{\mathbf{Int}}(n)$, for some $n < \omega$, be a finitely copresented descriptive frame. If $\mathfrak{F}$ is injective, then, for all $k < \omega$, the subframe $\mathfrak{F}{\uparrow}^k \subsetneq \mathfrak{F}_{\mathbf{Int}}(n)$ generated by $W{\uparrow}^k$ is also injective.*

Proof. First notice that $\mathfrak{F}{\uparrow}^k$ is finitely copresented by Lemma 3.21 and thus also finitely approximable by Lemma 5.23. We now prove that $\mathfrak{F}{\uparrow}^k$ has the

[10]During the revision of the present thesis, a very interesting paper, written by Dzik, Kost and Wojtylak, has appeared as a preprint on ArXiv: *Unification types and union splittings in intermediate logics* (https://arxiv.org/abs/2205.10644). The research paper is dense and seems full of very interesting results. In particular, the authors claim that there are exactly four maximal logics with nullary unification, only two minimal logics with hereditary finitary unification and, furthermore, that none of the locally tabular intermediate logics has infinitary unification.

extension property. So, let $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ be a generated subframe of $\mathfrak{F}{\uparrow}^k$ such that $d(\mathfrak{G}) < \omega$. For every point $v \in V$, we can find a point $u_v \in W$ such that $u_v \sim_{k-1} v$. Without loss of generality, we can assume that $d(u_v) < \omega$ (otherwise since $W \cap [u_v]_{k-1}$ is a non-empty clopen set in $W_{\mathbf{Int}}$ and $W^{<\omega}_{\mathbf{Int}}$ is dense, pick a point in $W \cap [u_i]_{k-1} \cap W^{<\omega}_{\mathbf{Int}} \neq \emptyset$). Then, $U = \bigcup_{v \in V} u_v{\uparrow}$ is a generated subframe of $\mathfrak{F}$ of finite depth and consequently, since $\mathfrak{F}$ has the extension property, there exists $z \in W$ such that $z \preceq U$. Now, by the properties of $\mathfrak{F}^{<\infty}_{\mathbf{Int}}(n)$ and by the choice of the u_v's, we can find a point $w \in W_{\mathbf{Int}}$ such that $w \preceq V$ and $col(w) = col(z)$. So, if we show that $z \precsim_k w$ than we are done. By construction, it suffice to show that $z \sim_{k-1} w$. We proceed, by induction on $i \in \{0, \dots, k-1\}$, by showing that $z \sim_i w$.

$(i = 0)$ Just notice that $z \sim_0 w$ iff $col(z) = col(w)$;

$(i = s+1)$ Assume that $z \sim_s w$ holds for all $s \in \{0, \dots, k-2\}$. We prove separately

$$\text{(i)} \quad z \precsim_s w \qquad \text{and} \qquad \text{(ii)} \quad w \precsim_s z.$$

(i) Let $x \in w{\uparrow}$. We need to find $y \in z{\uparrow}$ such that $x \sim_{s-1} y$. So, if $x = w$, then $y := z$ works by induction hypothesis. If $x \neq w$, then $x \in V$, hence, by construction $u_x \sim_{k-1} x$ and, since $k - 1 \geq s - 1$ and $u_x \in z{\uparrow}$, we can take $y := u_x$.

(ii) Let $x \in z{\uparrow}$. We need to find $y \in z{\uparrow}$ such that $x \sim_{s-1} y$. So, if $x = w$, then $y := z$ works again by induction hypothesis. Otherwise, suppose $x \neq z$. Then $x \in u_v{\uparrow}$ for some $v \in V$. Since by construction $u_v \sim_{k-1} v$, there exists $q \in v{\uparrow}$ such that $q \sim_{k-2} x$. But $k-2 \geq s-1$ and, consequently, by letting $y := q$ we are done.

We can conclude that $\mathfrak{F}{\uparrow}^k$ is injective by Proposition 3.39. ☐

Remark 19. Recall that, by the dual of Corollary 3.33, any finitely generated p-morphic image $\mathfrak{F}$ of a finitely copresented frame $\mathfrak{G}$ is also finitely copresented. Furthermore, if $\mathfrak{G}$ is injective, then it is a p-morphic image of a m-canonical frame $\mathfrak{F}_{\mathbf{Int}}(m)$, for some $m < \omega$ and, consequently, $\mathfrak{F}$ is a p-morphic image of $\mathfrak{F}_{\mathbf{Int}}(m)$ as well. Thus, by the dual of Proposition 3.40, $\mathfrak{F}$ is injective too.

Let us consider a solvable unification problem $\mathfrak{F}$ and let $h\colon \mathfrak{I} \to \mathfrak{F}$ be a unifier for $\mathfrak{F}$. By the previous remark, it follows that $h(\mathfrak{I}) \subsetneq \mathfrak{F}$ is a finitely copresented injective descriptive frame. Then, by Lemma 5.11,

$$\mathfrak{F}_h := h(\mathfrak{I}){\uparrow}^{\rho(\mathfrak{F})}$$

is also a finitely copresented injective frame and, since $h(\mathfrak{I}) \subsetneq \mathfrak{F}_h$, it follows that h is also a unifier for $\mathfrak{F}_h$[11]. Furthermore, since $\mathfrak{F}{\uparrow}^{\rho(\mathfrak{F})} = \mathfrak{F}$, we also have $\mathfrak{F}_h \subsetneq \mathfrak{F}$. So, any unifier for $\mathfrak{F}$ is also a unifier for a injective frame of the same rank of $\mathfrak{F}$ which is also a generated subframe of $\mathfrak{F}$. Therefore, it follows that, for any solvable unification problem $\mathfrak{F}$, the finite set of morphisms

$$\{1_{\mathfrak{G}} \mid \mathfrak{G} \subsetneq \mathfrak{F}, \mathfrak{G} \text{ injective } \ \& \ \rho(\mathfrak{G}) \leq \rho(\mathfrak{F})\}$$

is a complete set of unifiers for $\mathfrak{F}$. In particular, we can conclude that

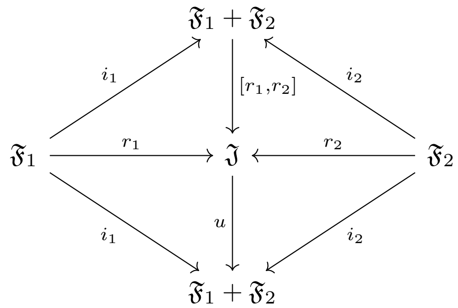
[11] Indeed, if $\mathfrak{A} \subsetneq \mathfrak{B}$, then any unifier $\mathfrak{G} \xrightarrow{g} \mathfrak{A}$ can be extended to $\mathfrak{G} \xrightarrow{g} \mathfrak{A} \xrightarrow{i} \mathfrak{B}$, where i is the obvious inclusion. Alternatively, notice that any substitution σ that unifies the $\mathcal{L}_n$-formula corresponding to $\mathfrak{A}$ also unifies the $\mathcal{L}_n$-formula corresponding to $\mathfrak{B}$ by Lemma 3.23.

Theorem 5.12. *Intuitionistic logic* **Int** *has finitary unification type.*

5.3.1 Some general results on unification

Proposition 5.13. *Let L be an intermediate logic with unitary unification and let $\mathfrak{F}_1 = \langle X_1, R_1, \mathcal{P}_1\rangle$ and $\mathfrak{F}_2 = \langle X_2, R_2, \mathcal{P}_2\rangle$ be two finitely copresented regular injective frames in $\boldsymbol{\mathcal{DF}_L}$. If the L-frame $\mathfrak{F}_1 + \mathfrak{F}_2$ is finitely copresented, then $\mathfrak{F}_1 + \mathfrak{F}_2$ is also regular injective in $\boldsymbol{\mathcal{DF}_L}$.*

Proof. Notice that the canonical injections $i_i \colon \mathfrak{F}_i \to \mathfrak{F}_1 + \mathfrak{F}_2$, $i = 1, 2$, are unifiers for $\mathfrak{F}_1 + \mathfrak{F}_2$ and therefore, since L has unitary unification, there exist a finitely copresented regular injective frame $\mathfrak{J}$ in $\boldsymbol{\mathcal{DF}_L}$ and a unifier $u \colon \mathfrak{J} \to \mathfrak{F}_1 + \mathfrak{F}_2$ that is more general than both i_1 and i_2. Furthermore, by taking into consideration the fact that $\mathfrak{F}_1 + \mathfrak{F}_2$ is the coproduct of $\mathfrak{F}_1$ and $\mathfrak{F}_2$, we have the following commuting diagram



Consequently, for both $i = 1, 2$, we have $u \circ [r_1, r_2] \circ i_i = u \circ r_i = i_i$ and thus, by the universal property of coproducts, it follows that $u \circ [r_1, r_2] = 1_{\mathfrak{F}_1 + \mathfrak{F}_2}$. So $\mathfrak{F}_1 + \mathfrak{F}_2$ is regular injective in $\boldsymbol{\mathcal{DF}_L}$ being a retract of the regular injective frame $\mathfrak{J}$. □

By taking advantage of the previous result we can prove the following

Proposition 5.14 (Ghilardi)**.** *Let L be an intermediate logic with unitary unification. Then* **KC** $\subseteq L$*.*

Proof. Towards a contradiction, suppose that L has unitary unification but $\boldsymbol{wem} \notin L$. Consider the finitely copresented frames $\mathfrak{F}_L(1)/\neg p = \langle X_1, S_1, \mathcal{Q}_1\rangle$ and $\mathfrak{F}_L(1)/\neg\neg p = \langle X_2, S_2, \mathcal{Q}_2\rangle$. Notice that, by Proposition 3.45, both frames $\mathfrak{F}_L(1)/\neg p$ and $\mathfrak{F}_L(1)/\neg\neg p$ are regular injective frames in $\boldsymbol{\mathcal{DF}_L}$. Furthermore, since $X_1 = \mathfrak{V}_L(\neg p)$ and $X_2 = \mathfrak{V}_L(\neg\neg p)$, the X_i's are disjoint clopen upsets of the 1-canonical frame $\mathfrak{F}_L(1) = \langle W, R, \mathcal{P}\rangle$ and thus

$$\mathfrak{F}_L(1)/\boldsymbol{wem} = \mathfrak{F}_L(1)/\neg p + \mathfrak{F}_L(1)/\neg\neg p$$

is a regular injective finitely copresented frame in $\boldsymbol{\mathcal{DF}_L}$ by Proposition 5.13. Therefore, $\mathfrak{F}_L(1)/\boldsymbol{wem}$ is a retract of $\mathfrak{F}_L(1)$ and so there exits an onto p-morphism $h \colon W \to X_1 \uplus X_2$ such that $h \restriction_{X_1 \uplus X_2} = 1_{\mathfrak{F}_L(1)/\boldsymbol{wem}}$. Furthermore, since $\boldsymbol{wem} \notin L$, $W \setminus X_1 \uplus X_2$ is non-empty and so let $w \in W \setminus X_1 \uplus X_2$. Since $w \notin X_1 \uplus X_2 = \mathfrak{V}_L(\boldsymbol{wem}) = \mathfrak{V}_L(\neg p) \cup \mathfrak{V}_L(\neg\neg p)$, there are incomparable points $v_1 \in \mathfrak{V}_L(p)$ and $v_2 \in \mathfrak{V}_L(\neg p)$ such that wRv_1 and wRv_2. Since $\mathfrak{V}_L(p) \subseteq \mathfrak{V}_L(\neg\neg p)$, both $v_1, v_2 \in X_1 \uplus X_2$ and so, by definition of h, we have $h(w)Sv_1$

and $h(w)Sv_2$. Consequently, since $h(w) \in \mathfrak{V}_L(\boldsymbol{wem})$, either $h(w) \in \mathfrak{V}_L(\neg p)$ or $h(w) \in \mathfrak{V}_L(\neg\neg p)$, but then we would have either $v_1 \in \mathfrak{V}_L(p) \cap \mathfrak{V}_L(\neg p) = \emptyset$ or $v_2 \in \mathfrak{V}_L(\neg p) \cap \mathfrak{V}_L(\neg\neg p) = \emptyset$, respectively. □

In an analogous way, we can also show the following

Proposition 5.15 (Wroński)**.** *Let L be an intermediate logic. Then L has projective unification if and only if* $\mathbf{LC} \subseteq L$*.*

Proof. ($\Longrightarrow$) Aiming for a contradiction, suppose that L has projective unification but $\boldsymbol{da} \notin L$. Consider the finitely copresented frame $\mathfrak{F}_L(2)/\boldsymbol{da} = \langle X, S, \mathcal{Q}\rangle$. By hypothesis, $\mathfrak{F}_L(2)/\boldsymbol{da}$ is regular injective in $\boldsymbol{\mathcal{DF}_L}$ and thus it is the retract of the 2-canonical frame $\mathfrak{F}_L(2) = \langle W, R, \mathcal{P}\rangle$ for L. In particular, there exits an onto p-morphism $h\colon W \to X$ such that $h \restriction_X = 1_{\mathfrak{F}_L(2)/\boldsymbol{da}}$. Furthermore, X is a clopen upset of W and, since $\boldsymbol{da} \notin L$, $W \setminus X$ is non-empty. By Corollary 2.18, $max(W \setminus X) \neq \emptyset$ and so let $w \in max(W \setminus X)$. Since $w \notin X = \mathfrak{V}_L(\boldsymbol{da}) = \mathfrak{V}_L(p \to q) \cup \mathfrak{V}_L(q \to p)$, there are incomparable points $v_1 \in \mathfrak{V}_L(p) \setminus \mathfrak{V}_L(q)$ and $v_2 \in \mathfrak{V}_L(q) \setminus \mathfrak{V}_L(p)$ such that wRv_1 and wRv_2. By maximality of w, both $v_1, v_2 \in X$, but then, by definition of h, we have $h(w)Sv_1$ and $h(w)Sv_2$ and thus $h(w) \notin \mathfrak{V}_L(\boldsymbol{da}) = X$, contradiction.
($\Longleftarrow$) Since projectivity is preserved by proper extensions, it suffice to show that $\mathbf{LC}$ has projective unification. So let $\mathfrak{F}$ be a solvable unification problem. Since $\mathbf{LC}$ is locally tabular, $\mathfrak{F}$ is a finite generated subframe of some n-canonical frame $\mathfrak{F}_{\mathbf{LC}}(n)$ for $\mathbf{LC}$. Now, notice that any generated subframe $\mathfrak{G} \subsetneq \mathfrak{F}$ such that $\mathfrak{G}^{\triangledown} \in \boldsymbol{\mathcal{DF}_{\mathbf{LC}}}$ has to be rooted. Hence $\mathfrak{F}$ is regular injective by Proposition 3.38 and, consequently, $\mathbf{LC}$ has projective unification. □

5.3.2 Negative results on unification

In this section we generalize a result of Ghilardi [64] in order to show that a wide range of intermediate logics do not have finitary unification type. But before proceeding further, recall that the notion of finitely presented algebra is a purely categorical notion: it is a well known fact that an algebra $\mathfrak{A}$ in an equational category $\boldsymbol{\mathcal{V}}$ is finitely presented iff the representable functor of $\mathfrak{A}$, $\mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathfrak{A}, -)\colon \boldsymbol{\mathcal{V}} \to \mathbf{Set}$, preserves filtered colimits, that is, $\mathfrak{A}$ is finitely presented iff, for each filtered diagram D of type $\mathbf{J}$ in $\boldsymbol{\mathcal{V}}$,

$$\mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathfrak{A}, \varinjlim_j D_j) \cong \varinjlim_j \mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathfrak{A}, D_j). \qquad (\star)$$

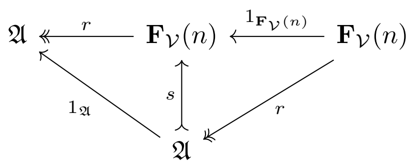
Proposition 5.16. *Let $\mathbf{C}$ be a cocomplete category. A finite colimit of finitely presented objects is finitely presented.*

The previous proposition allows us to prove the following

Lemma 5.17. *Let $\boldsymbol{\mathcal{V}}$ be an equational category. A finitely generated (regular) projective algebra $\mathfrak{A} \in \boldsymbol{\mathcal{V}}$ is finitely presented.*

Proof. Let $\mathfrak{A}$ be a finitely generated (regular) projective object in $\boldsymbol{\mathcal{V}}$. Then, by Theorem 3.34, $\mathfrak{A}$ is a retract of a finitely generated free $\mathcal{V}$-algebra $\mathbf{F}_{\mathcal{V}}(n)$ for some $n < \omega$, that is, there are morphism $s\colon \mathfrak{A} \rightarrowtail$ and $r\colon \mathbf{F}_{\mathcal{V}}(n) \twoheadrightarrow \mathfrak{A}$ such that

$s \circ r = 1_{\mathfrak{A}}$. It is immediate to see from the following diagram



that $r\colon \mathbf{F}_{\mathcal{V}}(n) \twoheadrightarrow \mathfrak{A}$ is the coequalizer of the pair of parallel arrows

$$\begin{aligned} 1_{\mathbf{F}_{\mathcal{V}}(n)}\colon \mathbf{F}_{\mathcal{V}}(n) &\to \mathbf{F}_{\mathcal{V}}(n) \\ s \circ r\colon \mathbf{F}_{\mathcal{V}}(n) &\to \mathbf{F}_{\mathcal{V}}(n). \end{aligned}$$

Being a finite colimit of the finitely presented algebra $\mathbf{F}_{\mathcal{V}}(n)$, $\mathfrak{A}$ is finitely presented as well by Proposition 5.16. $\square$

Let L be a fixed finitely approximable intermediate logic. We are now going to introduce a method to make a finitely generated frame in $\boldsymbol{\mathcal{DF}_L}$ regular injective. Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a n-generated descriptive L-frame. By Lemma 3.9, $\mathfrak{F}$ is isomorphic to a generated subframe of $\mathfrak{F}_L(n)$ and we can thus assume that $\mathfrak{F} \subsetneq \mathfrak{F}_L(n)$. Define inductively the Kripke frames $\iota_n(\mathfrak{F})$ as follows:

- $\iota_0(\mathfrak{F}) = \mathfrak{F}^{<\omega}$;
- $\iota_{n+1}(\mathfrak{F})$ is the frame obtained from $\iota_n(\mathfrak{F})$ by adding a new point x_S whenever S is a generated subframe of $\iota_n(\mathfrak{F})$ such that
  1. $d(S) < \omega$;
  2. $S^{\nabla} \in \boldsymbol{\mathcal{DF}_L}$;
  3. $\neg\exists y \in \iota_n(\mathfrak{F})\,(y \preceq S)$.

  The partial order of $\iota_{n+1}(\mathfrak{F})$ is the reflexive closure of the union of the partial order of $\iota_n(\mathfrak{F})$ together with all the pair $\langle x_S, y\rangle$ where $y \in S$.

Finally, we let $\iota(\mathfrak{F})$ be the double dual of the union of all the $\iota_n(\mathfrak{F})$, that is

$$\iota(\mathfrak{F}) := ((\bigcup_{n=0}^{\omega} \iota_n(\mathfrak{F}))^+)_+.$$

Notice that $\iota(\mathfrak{F})$ is nothing but the closure of the upset $\bigcup_{n=0}^{\omega} \iota_n(\mathfrak{F})$ in $\mathfrak{F}_L(n)$ by Lemma 3.15 and that $\iota(\mathfrak{F})^{<\omega} = \bigcup_{n=0}^{\omega} \iota_n(\mathfrak{F})$. Indeed, if $x \in \iota(\mathfrak{F})^{<\omega} \setminus \bigcup_{n=0}^{\omega} \iota_n(\mathfrak{F})$, then $x \in \overline{\bigcup_{n=0}^{\omega} \iota_n(\mathfrak{F}) \setminus \{x\}}$, that is, x is a limit point of $\bigcup_{n=0}^{\omega} \iota_n(\mathfrak{F})$ and, since $\bigcup_{n=0}^{\omega} \iota_n(\mathfrak{F}) \subseteq \iota(\mathfrak{F})$, x is also a limit point of $\iota(\mathfrak{F})$. But this contradicts the fact that x is isolated in $\iota(\mathfrak{F})$, since $x \in \iota(\mathfrak{F})^{<\omega}$.

Lemma 5.18. *For any finitely generated frame $\mathfrak{F} \in \boldsymbol{\mathcal{DF}_L}$, the following hold:*

1. *$\iota(\mathfrak{F})$ is a regular injective frame in $\boldsymbol{\mathcal{DF}_L}$;*
2. *if $\mathfrak{G}$ is a non-empty generated subframe of $\mathfrak{F}$ such that $d(\mathfrak{G}) < \omega$ and $\mathfrak{G}^{\nabla} \in \boldsymbol{\mathcal{DF}_L}$ and there is no $v \in \mathfrak{F}$ such that $v \preceq \mathfrak{G}$ but there exists $w \in \iota(\mathfrak{F})$ such that $w \preceq \mathfrak{G}$, then $w \in \iota_1(\mathfrak{F})$ and must be $x_{\mathfrak{G}}$.*

Proof. (1) By construction, $\iota(\mathfrak{F})$ is a generated subframe of $\mathfrak{F}_L(n)$ for some $n < \omega$ and thus finitely generated as well. Furthermore, $\iota(\mathfrak{F})$ is finitely approximable, since, by the above remark, $\iota(\mathfrak{F})^{<\omega}$ is dense in $\iota(\mathfrak{F})$. Finally, every generated subframe $\mathfrak{G} \subsetneq \iota(\mathfrak{F})$ of finite depth such that $\mathfrak{G}^\triangledown \in \boldsymbol{\mathcal{DF}_L}$ covers a point $w \in \iota(\mathfrak{F})$. So the conditions of Theorem 3.37 apply and $\iota(\mathfrak{F})$ is regular injective in $\boldsymbol{\mathcal{DF}_L}$. (2) Immediate from the construction of $\iota(\mathfrak{F})$. □

The following Lemma shows that p-morphisms do not increase the depth between points.

Lemma 5.19. *Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$ be descriptive frames and let $f\colon \mathfrak{F} \to \mathfrak{G}$ be a p-morphism. If $\mathfrak{F}$ is Noetherian, then $d(x) \geq d(f(x))$ for every point $x \in W$.*

Proof. Inductively assume that the claim holds for all proper successor of x and suppose for reductio that $d(x) < d(f(x))$. Then let $v \in f(x){\uparrow}$ be such that $d(v) = d(x)$. Since f is a p-morphism, there exists $y \in x{\uparrow}$ such that $f(y) = v$. Moreover, since v is a proper successor of $f(x)$, y must be a proper successor of x. Consequently, $d(y) < d(x) = d(v) = d(f(y))$, contradiction. □

Now let $f\colon \mathfrak{F} \to \mathfrak{G}$ be a p-morphism between the finitely generated descriptive L-frames $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q}\rangle$. Define by induction on $n < \omega$ the maps $\iota_n(f)\colon \iota_n(\mathfrak{F}) \to \iota_n(\mathfrak{G})$ as follows:

- $\iota_0(f) = f\restriction_{W^{<\omega}}$;
- $\iota_{n+1}(f)$ is the extension of $\iota_n(f)$ obtained by letting

$$\iota_{n+1}(f)(x_S) = y \iff y \preceq \iota_n(f)(S) \text{ and } \forall z(z \preceq \iota_n(f)(S) \to col(z) \trianglelefteq col(y)),$$

  where col is the standard colouring of $\mathfrak{F}_L(n)$.

Notice that, for each $n < \omega$, $\iota_n(\mathfrak{F})$ is actually a Noetherian Kripke frame by Remark 6 and the contruction of $\iota_n(\mathfrak{F})$. Therefore, it follows by the very definition of $\iota_n(f)$ and Lemma 1.12 that $\iota_n(f)$ is actually a p-morphism between $\iota_n(\mathfrak{F})$ and $\iota_n(\mathfrak{G})$. However, we also prefer to show it directly with the following

Lemma 5.20. *For every $n < \omega$, the function $\iota_n(f)\colon \iota_n(\mathfrak{F}) \to \iota_n(\mathfrak{G})$ is a well defined p-morphism.*

Proof. Since the base case is covered by Lemma 5.19, suppose for induction hypothesis that the claim holds for n. Consider a point $x_S \in \iota_{n+1}(\mathfrak{F})\backslash\iota_n(\mathfrak{F})$. Then, $\iota_n(f)(S)$ is a generated subframe of $\iota_n(\mathfrak{G}) \subsetneq \mathfrak{F}_L(n)$ such that $d(\iota_n(f)(S)) < \omega$ and $\iota_n(f)(S)^\triangledown \in \boldsymbol{\mathcal{DF}_L}$ since it is a p-morphic image of $S^\triangledown$. So, if there exists no point $v \in \iota_n(\mathfrak{G})$ such that $v \preceq \iota_n(f)(S)$, then there exists a unique point $x_{\iota_n(f)(S)} \in \iota_{n+1}(\mathfrak{G})$ covered by $\iota_n(f)(S)$ and $\iota_{n+1}(f)(x_S) = x_{\iota_n(f)(S)}$; if, otherwise, there are points in $\iota_n(\mathfrak{G})$ covered by $\iota_n(f)(S)$, then there exists also a point y whose colour is maximal with respect to colours $c \trianglelefteq col(u)$ for each $u \in \iota_n(f)(S)$. Since, in this case, $\iota_{n+1}(f)(x_S) = y$ and such a y is uniquely determined by its colour, $\iota_{n+1}(f)$ is well defined. Moreover, the map is clearly monotone. So, in order to check that $\iota_{n+1}(f)$ is a p-morphism, let $\iota_{n+1}(f)(x) < y$ for some $x \in \iota_{n+1}(\mathfrak{F})$ and $y \in \iota_{n+1}(\mathfrak{G})$. We have to find a point $z \in x{\uparrow}$ such that $\iota_{n+1}(f)(z) = y$. If $x \in \iota_n(\mathfrak{F})$, then $\iota_{n+1}(f)(x) = \iota_n(f)(x) \in \iota_n(\mathfrak{G})$ and thus y must be in $\iota_n(\mathfrak{G})$ as well and the back-condition holds by induction hypothesis. If $x \notin \iota_n(\mathfrak{F})$, then $x = x_S$ for some non-empty $S \subsetneq \iota_n(\mathfrak{F})$ satisfying (1)-(3). Then by definition $\iota_{n+1}(f)(x_S) \preceq \iota_n(f)(S)$ and so $y \in \iota_n(f)(S)$, that is, $y = \iota_n(f)(z)$ for some $z \in S$. Since $x_S \leq q$ for all $q \in S$ and $z \in S \subseteq \iota_n(\mathfrak{F})$, $\iota_{n+1}(f)$ is indeed a p-morphism. □

Now let $\iota_\omega(f)$ be the union of all the $\iota_n(f)$. It is immediate to see that $\iota_\omega(f)$ is a p-morphism between $\iota(\mathfrak{F})^{<\omega}$ and $\iota(\mathfrak{G})^{<\omega}$. Then we let $\iota(f)$ be the unique p-morphism between $\iota(\mathfrak{F})$ and $\iota(\mathfrak{G})$ extending $\iota_\omega(f)$ as in Lemma 3.36. Furthermore, notice the following immediate corollary of Lemma 5.17.

Lemma 5.21. *Let $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ be a finitely generated regular injective frame in $\boldsymbol{\mathcal{DF}_L}$. Then $\mathfrak{F}$ is finitely copresented.*

Therefore, for every finitely generated descriptive frame $\mathfrak{F}$ for L, $\iota(\mathfrak{F})$ is finitely copresented. Moreover, since finitely copresented frames are finitely generated as well, our construction makes sense also for every finitely copresented frame.

Remark 20. The ι-construction just presented, despite the fact that applies not only to finite frames but to finitely generated frames as well, is very much alike to the procedure introduced by Ghilardi in [64]. One big difference of Ghilardi's construction with respect to our's is the insertion of a new point x_S covering a generated subframe S even in the case such point already existed. As stated by Ghilardi himself in the footnote 6 of [64], such a condition

> is due to the fact that we want to keep the construction functorial, i.e. operating well on morphisms: if we do not insert such points when it is not needed, then we could be forced to make arbitrary choices for choosing the image of an inserted point which is not inserted anymore in the codomain of a morphism (it goes without saying that such arbitrary choices make functoriality problematic).

And in fact our construction faces such a problematic. Let us make clear the situation with an example. Denote by ${\boldsymbol{\mathcal{DF}_L}}^{fcp}$ the full subcategory of $\boldsymbol{\mathcal{DF}_L}$ of finitely copresented descriptive frame for L. Then $\iota\colon {\boldsymbol{\mathcal{DF}_L}}^{fcp} \to {\boldsymbol{\mathcal{DF}_L}}^{fcp}$ would be a functor if the following conditions were satisfied:

(1) $\iota(1_{\mathfrak{F}}) = 1_{\iota(\mathfrak{F})}$, for any frame $\mathfrak{F}$ in ${\boldsymbol{\mathcal{DF}_L}}^{fcp}$;

(2) $\iota(f) \circ \iota(g) = \iota(f \circ g)$, for any morphism f, g in ${\boldsymbol{\mathcal{DF}_L}}^{fcp}$.

As to (1) our construction still works. However, condition (2) is not satisfied as it is shown in the following picture, where the p-morphisms $g\colon \mathfrak{F} \to \mathfrak{G}$ and $f\colon \mathfrak{G} \to \mathfrak{D}$ are drawn above in between the corresponding frames[12].

[12]Here we are clearly assuming an intermediate logic L such that $\mathfrak{F}^\triangledown$ is a L-frame, such as **Int**. Notice moreover that $\mathfrak{F}$ is nothing but the sum of the three point frame $(\mathbf{1} + \mathbf{1})^\triangledown$, also known as "the fork", with itself.

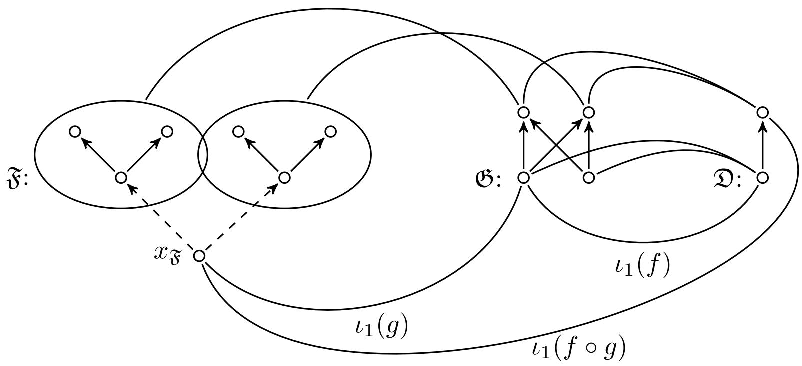
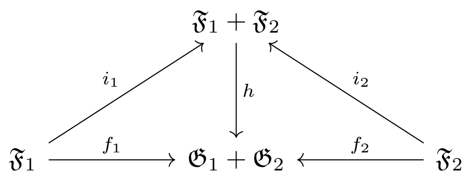


Figure 5.1: *A counterexample to the functoriality of* $\iota(\cdot)$.

Now, Ghilardi's procedure works well in the context of locally finite varieties of Heyting algebras, because the insertion of a new point covered by a generated subframe is not more possible after a certain depth. However, when dealing with finitely approximable varieties, there is no assurance that such a bound exists and thus we might end up with a frame that is not finitely generated anymore. Consequently, since being finitely generated is of greatest importance for our purposes, it seems that we need to give up to the functoriality of $\iota(\cdot)$. Furthermore, another drawback of our construction is that the sufficiency condition of point (i) of Lemma 8 in [64] does not hold anymore, that is, it is not the case that, given any morphism $f\colon \mathfrak{F} \to \mathfrak{G}$ in $\boldsymbol{\mathcal{DF}_L}$, for all $w \in \iota_{n+1}(\mathfrak{F})$, $\iota_{n+1}(f)(w) \in \iota_n(\mathfrak{G})$ implies $w \in \iota_n(\mathfrak{F})$. Since this condition plays a key rôle in the proof of the main result on unification in locally finite varieties, we need to work harder in order to show that such a result still holds in our setting.

Before proceeding further, let us prove some other useful facts which deal, more or less directly, with $\iota(\cdot)$ and, first of all, let us look more closely to what happen when ι is applied to sums of frames. So, suppose we have two p-morphism $f_1\colon \mathfrak{F}_1 \to \mathfrak{G}_1$ and $f_2\colon \mathfrak{F}_2 \to \mathfrak{G}_2$ and let $h\colon \mathfrak{F}_1 + \mathfrak{F}_2 \to \mathfrak{G}_1 + \mathfrak{G}_2$ be $[f_1, f_2]$, the unique arrow making the co-product diagram commute



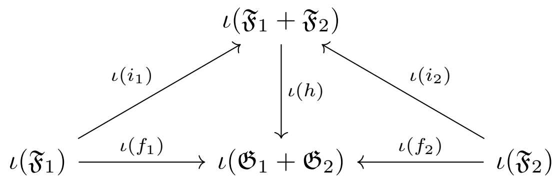
Now let us consider the p-morphism $\iota(h)\colon \iota(\mathfrak{F}_1 + \mathfrak{F}_2) \to \iota(\mathfrak{G}_1 + \mathfrak{G}_2)$. First notice that the set of points of $\iota(\mathfrak{F}_1 + \mathfrak{F}_2)$ can be written as the union of the disjoint underlying sets of $\iota(\mathfrak{F}_1)$ and $\iota(\mathfrak{F}_2)$ with the set of points in the middle part of $\iota(\mathfrak{F}_1 + \mathfrak{F}_2)$, that is

$$\iota(\mathfrak{F}_1 + \mathfrak{F}_2)^* = \{x \in \iota(\mathfrak{F}_1 + \mathfrak{F}_2) \,|\, x \preceq S,\, S \cap \mathfrak{F}_i \neq \emptyset \text{ for } i = 1, 2\}.$$

Notice moreover that $\iota(\mathfrak{F}_1+\mathfrak{F}_2)^* \cap \iota_0(\mathfrak{F}_1+\mathfrak{F}_2)=\emptyset$. Furthermore, in such a situation, we have, for any $x \in \iota^{<\omega}(\mathfrak{F}_1+\mathfrak{F}_2)$,

$$\iota(h)(x) \in \begin{cases} \iota^{<\omega}(\mathfrak{G}_1) & \text{if } x \in \iota^{<\omega}(\mathfrak{F}_1), \\ \iota^{<\omega}(\mathfrak{G}_2) & \text{if } x \in \iota^{<\omega}(\mathfrak{F}_2), \\ \iota(\mathfrak{G}_1+\mathfrak{G}_2)^* & \text{if } x \in \iota(\mathfrak{F}_1+\mathfrak{F}_2)^*. \end{cases}$$

In particular, one can easily show that, when $\iota(\cdot)$ is applied to a coproduct diagram, then the new diagram preserves its commuting properties, that is,



The previous simple remark will be of greatest importance in the proof of Theorem 5.25 below.

Lemma 5.22. *Let $\mathfrak{F}=\langle W,R,\mathcal{P}\rangle$ and $\mathfrak{G}=\langle V,S,\mathcal{Q}\rangle$ be finitely generated descriptive frames for L and let $f\colon \mathfrak{F}\to\mathfrak{G}$ be a p-morphism. Furthermore, suppose that $\mathfrak{F}$ is finitely approximable. Then if there exists $v \in V^{<\omega}$ such that $f(x)=v$ for some point $x \in W$, then there exists also $z \in W^{<\omega}$ such that $f(z)=v$.*

Proof. Since $v \in V^{<\omega}$, v is an isolated point of V and thus $\{v\}$ is clopen in V. Hence $f^{-1}(v)$ is a non-empty clopen set in W and since $W^{<\omega}$ is dense in W, $f^{-1}(v)\cap W^{<\omega}\neq\emptyset$. Therefore there exists $z \in W^{<\omega}$ such that $f(z)=v$. □

Remark 21. Let $\mathfrak{F}=\langle W,R,\mathcal{P}\rangle$ be a descriptive frame, $x \in W$ and $S,Q \subsetneq W$ be two upsets of W such that both $x \notin S$ and $x \notin Q$. If $x \preceq S$ and $x \preceq Q$, then $S=Q$.

Proof. By definition of the relation $\preceq$, we have $S \subseteq x\uparrow$, $Q \subseteq x\uparrow$, $x^{>} \subseteq S$ and $x^{>} \subseteq Q$. Now let $y \in S$. Then xRy and $x \neq y$, since otherwise $x \in S$ as well. So y is a proper successor of x and thus there exists an immediate successor z of x such that zRy. Hence $z \in Q$ and thus $y \in Q$, since Q is upward closed. Thus $S \subseteq Q$ and analogously $Q \subseteq S$. Consequently $S=Q$. □

Lemma 5.23. *Let $\mathfrak{F}=\langle W,R,\mathcal{P}\rangle$ be a finitely copresented descriptive frame for a finitely approximable intermediate logic L. Then $\mathfrak{F}$ is finitely approximable.*

Proof. Without loss of generality we can assume that W is a clopen upset of the Esakia space $\mathcal{E}_L(n)=\langle X,\tau,S\rangle$ dual to the free Heyting algebra $\mathbf{F}_{\mathcal{V}_L}(n)$ for some $n<\omega$. Now, since L is finitely approximable, $X^{<\omega}$ is dense in X and since $W \in \tau$ it follows that $W \cap X^{<\omega}=W^{<\omega}$ is dense in W. Thus $\mathfrak{F}$ is finitely approximable. □

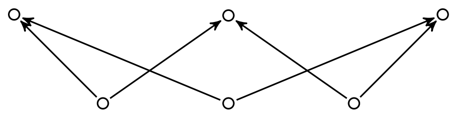
Now consider the sequence of Kripke frames $\mathfrak{G}_n=\langle V_n,S_n\rangle$ (for $n\geq 2$) defined as follows:

$$V_n=\{1,\ldots,n\}\cup\{\langle i,j\rangle \mid 1\leq i<j\leq n\},$$

while the partial ordering S_n is defined by letting

$$xS_ny \iff \begin{array}{l} x = y, \text{ or} \\ x = \langle i,j\rangle \text{ and } y = i, \text{ or} \\ x = \langle i,j\rangle \text{ and } y = j. \end{array}$$

Thus, for instance, $\mathfrak{G}_3$ is the following frame:



The following lemma concerning the injectivization of the frames $\mathfrak{G}_n$ is crucial for our purposes.

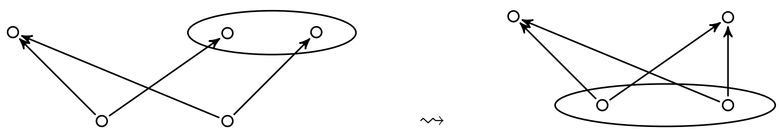
Lemma 5.24. *Let $\mathfrak{Y}$ be a generated subframe of $\iota_k(\mathfrak{G}_n)$ $(k \geq 0)$ such that there exists a point $\langle i,j\rangle$ of V_n in $\mathfrak{Y}$. Then the three point frame $(\mathbf{1}+\mathbf{1})^\nabla$ is a p-morphic image of $\mathfrak{Y}$.*

Proof. We proceed by induction on k. For the base case, let $\mathfrak{Y} \subsetneq \mathfrak{G}_n$ and $\langle i,j\rangle \in \mathfrak{Y}$ for some $1 \leq i < j \leq n$. We distinguish three cases:

(a) $\mathfrak{Y} = \langle i,j\rangle\uparrow$: then clearly $\mathfrak{Y} \cong (\mathbf{1}+\mathbf{1})^\nabla$;

(b) $\mathfrak{Y} = \langle i,j\rangle\uparrow \cup S$, where $S \subseteq max(\mathfrak{G}_n) \setminus \{i,j\}$: then by identifying all the points of S either with i or with j we get a p-morphism from $\mathfrak{Y}$ onto $(\mathbf{1}+\mathbf{1})^\nabla$;

(c) $\mathfrak{Y} = \langle i,j\rangle\uparrow \cup S$, where $S \subsetneq \mathfrak{G}_n$ contains points $\langle k,l\rangle$ different from $\langle i,j\rangle$: first, if there are points in $max(S)$ that are not above any point of the form $\langle k,l\rangle$ in S, then identify such points either with i or with j as in (b). We can thus assume that $\mathfrak{Y}$ is the union of upsets of the form $\langle k,l\rangle\uparrow$. Since there are at most $k(\mathfrak{G}_n) = \frac{n^2-n}{2}$ such upsets, we proceed by induction on the number $m \geq 2$ of such upsets. Consider the case $m = 2$, that is, $\mathfrak{Y} = \langle i,j\rangle\uparrow \cup \langle k,l\rangle\uparrow$. Now, there are two possibilities:
(1) either $i = k$ or $j = l$: then first identify the two other distinct points and then the two points $\langle i,j\rangle$ and $\langle k,l\rangle$ at the bottom as shown in the picture below:



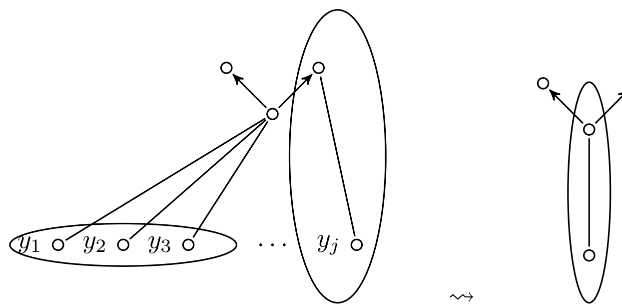
The composition of the two β-reductions is the wanted p-morphism from $\mathfrak{Y}$ onto $(\mathbf{1}+\mathbf{1})^\nabla$.
(2) Both $i \neq k$ and $j \neq l$: then by first identifying either i and k or j and l, we get back to the previous scenario and again we get a p-morphism from $\mathfrak{Y}$ onto $(\mathbf{1}+\mathbf{1})^\nabla$.

Now, for the induction step, let the claim holds for $m < k(\mathfrak{G}_n)$ and suppose that $\mathfrak{Y}$ is the union of $m+1$ different upsets of the form $\langle i,j\rangle\uparrow$. Consider any two such upsets $\langle i,j\rangle\!\uparrow$ and $\langle k,l\rangle\!\uparrow$. If they are disjoint, then reduce them according to the possibility (2) of the base case; otherwise, reduce them according to the possibility (1) of the base case. In either cases, $\mathfrak{Y}$ reduces to a generated subframe $\mathfrak{Y}'$ of $\mathfrak{G}_n$ which is the union of m upsets of the form $\langle i,j\rangle\!\uparrow$. By the inductive hypothesis, $(\imath + \imath)^{\triangledown}$ is a p-morphic image of $\mathfrak{Y}'$ and thus also of $\mathfrak{Y}$.

Now, assume for induction hypothesis that the claim of the Lemma holds for any generated subframe of $\iota_k(\mathfrak{G}_n)$ such that there is a point $\langle i,j\rangle$ belonging to it for some $1 \le i < j \le n$ and let us prove it for $k+1$. So let $\mathfrak{Y} \subsetneq \iota_{k+1}(\mathfrak{G}_n)$ and $\langle i,j\rangle \in \mathfrak{Y}$ for some $1 \le i < j \le n$. Now, if $min(\mathfrak{Y}) \subseteq \iota_k(\mathfrak{G}_n)$, then $\mathfrak{Y} \subsetneq \iota_k(\mathfrak{G}_n)$ and the claim of the Lemma follows by induction hypothesis. Therefore, let $y_1,\dots,y_j \in min(\mathfrak{Y}) \setminus \iota_k(\mathfrak{G}_n)$. By definition of $\iota(\mathfrak{G}_n)$, for each $i \in \{1,\dots,j\}$, there exists $S_i \subsetneq \iota_k(\mathfrak{G}_n)$ such that $y_i \preceq S_i$. Then

$$S = \bigcup_{x \in min(\mathfrak{Y}) \cap \iota_k(\mathfrak{G}_n)} x\!\uparrow \cup \bigcup_{1 \le i \le j} S_i$$

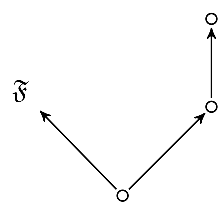
is a generated subframe of $\iota_k(\mathfrak{G}_n)$ containing a point $\langle i,j\rangle$ for some $1 \le i < j \le n$ and thus, by IH, there exists an onto p-morphism $h\colon S \to (\imath+\imath)^{\triangledown}$. Since $S \subsetneq \mathfrak{Y}$, by Lemma 1.13 it follows that $\mathfrak{Y}[S/(\imath+\imath)^{\triangledown}]$ is a p-morphic image of $\mathfrak{Y}$. But then we can further reduce $\mathfrak{Y}[S/(\imath+\imath)^{\triangledown}]$ by a finite sequence of steps of β- and α-reduction, namely, for example,



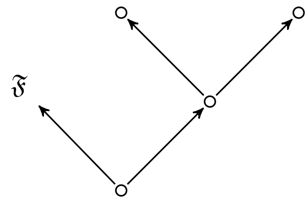
and we can thus conclude that $(\imath + \imath)^{\triangledown}$ is a p-morphic image of $\mathfrak{Y}$. ☐

We are ready for the main result of this section.

Theorem 5.25. *Let $\mathcal{V}_L$ be a finitely approximable variety of Heyting algebras whose unification type is finitary and let $\mathfrak{F}$ be a finite frame. If the finite frame $(\mathfrak{F} + \imath^{\triangledown})^{\triangledown}$*



belongs to $\boldsymbol{\mathcal{DF}_L}$, then the finite frame $(\mathfrak{F} + (\imath + \imath)^{\triangledown})^{\triangledown}$



also belongs to $\boldsymbol{\mathcal{DF}_L}$.

Proof. Aiming for a contradiction, suppose that the unification type of $\mathcal{V}_L$ is ω, $(\mathfrak{F}+\mathtt{1}^\triangledown)^\triangledown \in \boldsymbol{\mathcal{DF}_L}$ but $(\mathfrak{F}+(\mathtt{1}+\mathtt{1})^\triangledown)^\triangledown \notin \boldsymbol{\mathcal{DF}_L}$. Since $\mathtt{1}$ is a p-morphic image of any frame, we have that the frame $(\mathtt{1}+\mathtt{1})^\triangledown$ is a p-morphic image of $(\mathfrak{F}+\mathtt{1}^\triangledown)^\triangledown$ and thus belongs to $\boldsymbol{\mathcal{DF}_L}$ as well. Now consider the sequence $\mathfrak{G}_n = \langle V_n, S_n\rangle$ (for $n \geq 2$) of frames defined above. Notice that $\mathfrak{G}_n \in \boldsymbol{\mathcal{DF}_L}$ for all $n \geq 2$, since all the cones of $\mathfrak{G}_n$ are isomorphic either to $\mathtt{1}$ or to $(\mathtt{1}+\mathtt{1})^\triangledown$. Thus the frames $\mathfrak{F}+\mathfrak{G}_n$'s all belong to $\boldsymbol{\mathcal{DF}_L}$ and the maps $g_n\colon \mathfrak{F}+\mathfrak{G}_n \to \mathfrak{F}+\mathtt{1}^\triangledown$ defined by letting g_n be the identity on $\mathfrak{F}$ and mapping the pair $\langle i,j\rangle$ onto the root of $\mathtt{1}^\triangledown$ and the maximal points of $\mathfrak{G}_n$ onto the maximal point of $\mathtt{1}^\triangledown$ are surjective morphism in $\boldsymbol{\mathcal{DF}_L}$.
Since finite frames are finitely copresented, we can apply the functor ι and get p-morphisms

$$\iota(g_n)\colon \iota(\mathfrak{F}+\mathfrak{G}_n) \to \iota(\mathfrak{F}+\mathtt{1}^\triangledown).$$

Now consider the following subset $Q \subseteq \iota(\mathfrak{F}+\mathtt{1}^\triangledown)$:

$$Q = \{z \in \iota(\mathfrak{F}+\mathtt{1}^\triangledown) \mid \forall y(z \leq y \to y \not\preceq \mathfrak{F}+\mathtt{1}^\triangledown)\}. \qquad (\star)$$

Notice that since $(\mathfrak{F}+\mathtt{1}^\triangledown)^\triangledown \in \boldsymbol{\mathcal{DF}_L}$, by Lemma 5.18 we have that the point $x_{\mathfrak{F}+\mathtt{1}^\triangledown} \in \iota_1(\mathfrak{F}+\mathtt{1}^\triangledown)$ is the unique point covered by $\mathfrak{F}+\mathtt{1}^\triangledown$ and therefore

$$\begin{aligned} Q &= \{z \in \iota(\mathfrak{F}+\mathtt{1}^\triangledown) \mid \forall y(z \leq y \to y \neq x_{\mathfrak{F}+\mathtt{1}^\triangledown})\} \\ &= \{z \in \iota(\mathfrak{F}+\mathtt{1}^\triangledown) \mid z \not\leq x_{\mathfrak{F}+\mathtt{1}^\triangledown}\}, \end{aligned}$$

that is, V is a non-empty upset of $\iota(\mathfrak{F}+\mathtt{1}^\triangledown)$. We let $\mathfrak{D}$ be the generated subframe of $\iota(\mathfrak{F}+\mathtt{1}^\triangledown)$ whose underlying set is Q.
Now, since $\iota(\mathfrak{F}+\mathtt{1}^\triangledown)$ is finitely copresented, it is (isomorphic to) an admissible set of the n-canonical frame $\mathfrak{F}_L(n)$ for L for some $n < \omega$. Thus, by identifying the points, we have that $x_{\mathfrak{F}+\mathtt{1}^\triangledown}$ is a point in $\mathfrak{F}_L(n)^{<\omega}$ and, by Theorem 3.12, the generated subframe $\mathfrak{F}_L(n) \setminus x_{\mathfrak{F}+\mathtt{1}^\triangledown}{\downarrow}$ is admissible. Therefore, the intersection $\iota(\mathfrak{F}+\mathtt{1}^\triangledown) \cap \mathfrak{F}_L(n) \setminus x_{\mathfrak{F}+\mathtt{1}^\triangledown}{\downarrow} = \mathfrak{D}$ is also an admissible set of $\mathfrak{F}_L(n)$, that is, $\mathfrak{D}$ is finitely copresented. So we shall take $\mathfrak{D}$ as an example of a non-finitary unification problem.
First of all, let us prove that, for any $2 \leq n < \omega$, the range of $\iota(g_n)$ is included in $\mathfrak{D}$. Suppose not: there exists $z \in \iota(\mathfrak{F}+\mathfrak{G}_n)$ such that $\iota(g_n)(z) \notin Q$. Thus $\iota(g_n)(z) \leq x_{\mathfrak{F}+\mathtt{1}^\triangledown}$ and, since $\iota(g_n)$ is a p-morphism, there exists $z' \in z{\uparrow}$ such that $\iota(g_n)(z') = x_{\mathfrak{F}+\mathtt{1}^\triangledown}$. Furthermore, since $x_{\mathfrak{F}+\mathtt{1}^\triangledown} \in \iota(\mathfrak{F}+\mathtt{1}^\triangledown)^{<\omega}$, by Lemma 5.22 there exists also a point $v \in \iota(\mathfrak{F}+\mathfrak{G}_n)^{<\omega}$ such that $\iota(g_n)(v) = x_{\mathfrak{F}+\mathtt{1}^\triangledown}$. However, such a point cannot exist as a consequence of the following:

Lemma 5.26. *For every $k \geq 1$, if $v \in \iota_k(\mathfrak{F}+\mathfrak{G}_n)$ is such that $\iota(g_n)(v) = x_{\mathfrak{F}+\mathtt{1}^\triangledown}$, then the frame $(\mathfrak{F}+(\mathtt{1}+\mathtt{1})^\triangledown)^\triangledown$ is a p-morphic image of some generated subframe of $v{\uparrow}$.*

Proof. By induction on k. For the base case, let us assume that $v \in \iota_1(\mathfrak{F}+\mathfrak{G}_n)$. Since $\iota(g_n)(v) \notin \iota_0(\mathfrak{F}+\mathbf{1}^\triangledown)$, it must be the case that $v \notin \iota_0(\mathfrak{F}+\mathfrak{G}_n)$ and thus it is of the form v_S for some upset $S \subsetneq \mathfrak{F}+\mathfrak{G}_n$ such that $S^\triangledown \in \boldsymbol{\mathcal{DF}_L}$. Furthermore, since by definition of $\iota(g_n)$, $\iota_0(g_n)(S) = g_n(S) = \mathfrak{F}+\mathbf{1}^\triangledown$, S must contain $\mathfrak{F}$ and at least one point $\langle i,j\rangle$ of V_n. Then, since $\mathfrak{F}$ and $\mathfrak{G}_n$ are disjoint, $S \setminus \mathfrak{F}$ is a generated subframe of $\iota_0(\mathfrak{G}_n)$ containing a point of the form $\langle i,j\rangle$ and thus, by Lemma 5.24, the fork $(\mathbf{1}+\mathbf{1})^\triangledown$ is a p-morphic image of $S \setminus \mathfrak{F}$. Therefore, $(\mathfrak{F}+(\mathbf{1}+\mathbf{1})^\triangledown)^\triangledown$ is a p-morphic image of $(\mathfrak{F}+S\setminus\mathfrak{F})^\triangledown = S^\triangledown \cong v_S{\uparrow}$.
For the inductive step, let us assume that the statement of the lemma holds for k and suppose that $v \in \iota_{k+1}(\mathfrak{F}+\mathfrak{G}_n)$. Without loss of generality, we can further assume that $v \notin \iota_k(\mathfrak{F}+\mathfrak{G}_n)$ and thus that it is of the form v_S for some upset $S \subsetneq \iota_k(\mathfrak{F}+\mathfrak{G}_n)$ such that $S^\triangledown \in \boldsymbol{\mathcal{DF}_L}$ and such that there is no point in $\iota_k(\mathfrak{F}+\mathfrak{G}_n)$ covered by S. Then, by definition of $\iota(g_n)$, $x_{\mathfrak{F}+\mathbf{1}^\triangledown} \preceq \iota_k(S)$ and we can distinguish two cases:

(a) $x_{\mathfrak{F}+\mathbf{1}^\triangledown} \in \iota_k(S)$: in which case $x_{\mathfrak{F}+\mathbf{1}^\triangledown}$ is the image under $\iota(g_n)$ of a point of S and the lemma follows by induction hypothesis;

(b) $x_{\mathfrak{F}+\mathbf{1}^\triangledown} \notin \iota_k(S)$: then $\mathfrak{F}+\mathbf{1}^\triangledown = \iota_k(S)$ by Remark 21 and thus $\iota_k(S)$ does not contain any point of $\iota_m(\mathfrak{F}+\mathbf{1}^\triangledown)$ for any $m \geq 1$. First notice that, for any $j \geq 1$, $(\iota_j(\mathfrak{F}) \setminus \iota_0(\mathfrak{F})) \cap S = \emptyset$. Indeed, if there exists such a point y in the intersection, then, since $y \in \iota_{s+1}(\mathfrak{F}) \setminus \iota_s(\mathfrak{F})$ for some $0 \leq s < k$ and $\iota(g_n) \restriction_{\iota(\mathfrak{F})}$ is the identity on $\iota(\mathfrak{F})$, $\iota_k(S)$ would contain a point of $\iota_j(\mathfrak{F}+\mathbf{1}^\triangledown)$ for some $j \geq 1$, contrary to our previous remark. Analogously, S does not contain any point of $\iota^*(\mathfrak{F}+\mathfrak{G}_n)$, since, otherwise, $\iota_k(S)$ would again contain a point of $\iota_m(\mathfrak{F}+\mathbf{1}^\triangledown)$ for some $m \geq 1$. Therefore, it must be the case that the set $v^>$ of immediate successors of v contains all the minimal points of $\mathfrak{F}$, at least a point $q \in \iota_k(\mathfrak{G}_n) \setminus \iota_{k-1}(\mathfrak{G}_n)$, other points $q_1,\dots,q_l$ of $\iota_k(\mathfrak{G}_n)$ and nothing else, that is, we let

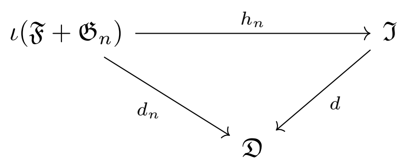
$$min(S) = v^> = min(\mathfrak{F}) \cup \{q_1,\dots,q_l,q\}.$$

Notice, furthermore, that $Q = q{\uparrow} \cup \bigcup_{i=1}^{l} q_i{\uparrow}$ is a generated subframe of $\iota_{k+1}(\mathfrak{G}_n)$ containing a point of V_n of the form $\langle i,j\rangle$ and thus, by Lemma 5.24, the frame $(\mathbf{1}+\mathbf{1})^\triangledown$ is a p-morphic image of Q. Finally, since $Q \cap \mathfrak{F} = \emptyset$, it follows that the frame $(\mathfrak{F}+(\mathbf{1}+\mathbf{1})^\triangledown)^\triangledown$ is a p-morphic image of the frame $(\mathfrak{F}+Q)^\triangledown = S^\triangledown \cong v_S{\uparrow}$. □

We can thus restrict the codomain of the $\iota(g_n)$'s to $\mathfrak{D}$. Let us denote by d_n the restricted maps. Therefore the maps

$$d_n \colon \iota(\mathfrak{F}+\mathfrak{G}_n) \to \mathfrak{D}$$

are unifiers for our unification problem $\mathfrak{D}$. Since $\mathcal{V}_L$ has finitary unification, for cardinality reasons there exists a unifier $d \colon \mathfrak{I} \to \mathfrak{D}$ which is more general than infinitely many d_n's. Hence, for infinitely many n's, the following triangles commute



Now, $\mathfrak{I}$ is a finitely copresented injective frame in $\boldsymbol{\mathcal{DF}_L}$ and thus $max(\mathfrak{I}) \leq 2^k$ for some $k < \omega$. Since p-morphisms map maximal points to maximal points, for some n, there exists distinct points $i_1, i_2 \leq n$ of $V_n \subseteq \iota(\mathfrak{F} + \mathfrak{G}_n)$ such that $h_n(i_1) = h_n(i_2)$. Consider the generated subframe $\mathfrak{S} = \mathfrak{F} + \{\langle i_1, i_2\rangle, i_1, i_2\}$ of $\iota(\mathfrak{F} + \mathfrak{G}_n)$. Notice that the image $h_n(\mathfrak{S})$ is isomorphic to $\mathfrak{F} + \mathbf{1}^\triangledown$, since h_n identifies the maximal points i_1 and i_2 but does not identify anything else, since $d_n \upharpoonright_{\mathfrak{F}+\mathfrak{G}_n} = g_n$ and the above triangle commutes. Then $h_n(\mathfrak{S})$ is a generated subframe of $\mathfrak{I}$ of finite depth and, since $(\mathfrak{F} + \mathbf{1}^\triangledown)^\triangledown \in \boldsymbol{\mathcal{DF}_L}$ and $\mathfrak{I}$ is injective, there exists a point $x \in \mathfrak{I}$ such that $x \preceq h_n(\mathfrak{S})$. But then, since p-morphisms preserve the covering relation and by the commutativity of the above triangle, we have $d(x) \preceq \mathfrak{F} + \mathbf{1}^\triangledown$. However, by the very definition $(\star)$ of $\mathfrak{D}$, there can not be any such point $d(x)$ in $\mathfrak{D}$. □

Corollary 5.27. *The following intermediate logics do not have finitary unification type:*

- *the logics of bounded width* $\mathbf{BW}_k (k \geq 2)$*;*
- *the logics of bounded top width* $\mathbf{BTW}_k (k \geq 2)$*;*
- *the Kreisel-Putnam logic* **KP***;*
- *the weak Kreisel-Putnam logic* **WKP** *and all the logics* $\mathbf{ND}_k (k \geq 3)$[13]*;*
- *the logics of bounded cardinality* $\mathbf{BC}_k (k \geq 4)$*.*

[13]For each $k \geq 1$, the logic $\mathbf{ND}_k$ is the logic $\mathbf{Int} + \boldsymbol{nd_k}$ as defined at the beginning of §6.2.

Chapter 6

Friedman logics

In his famous *One Hundred and Two Problems in Mathematical Logic* [53], Harvey Friedman asked, in the section devoted to propositional calculi, the following question[1]:

> 41. There is a set of formulae T in the propositional calculus based on $\bot, \&, \vee, \rightarrow$ obeying (i) $\bot \notin T$, (ii) $A \& B \in T$ if and only if $A, B \in T$, (iii) $A \vee B \in T$ if and only if $A \in T$ or $B \in T$, and (iv) $A \rightarrow B \in T$ if and only if every substitution that puts A in T also puts B in T. Furthermore this T is not unique. [53, p. 118]

It is immediate to show that such a set T of $\mathcal{L}$-formulas must be a proper subset of **For**$\mathcal{L}$ closed under modus ponens. Furthermore, it is also assumed that T is closed under uniform substitution and thus it is not difficult to prove that it contains all the axioms of the intuitionistic propositional calculus *Int*[2]. Therefore, such a set T must be an intermediate logic. Since condition (iii) and (iv) for intermediate logics correspond respectively to the disjunction property and to structural completeness, any intermediate logic satisfying these two properties could be a solution for Friedman's problem.

Definition 6.1. An intermediate logic L is said to be *Friedman* if L enjoys the disjunction property and L is structurally complete.

A positive (almost total) solution for Friedman's problem 41 has been provided by Tadeusz Prucnal in [137, 138] where he showed that Medvedev's logic **ML** is Friedman. However, whether there exists a unique Friedman logic or not it is still an open issue.

6.1 Negatively stable logics

Before starting, it is worth mentioning that the main source for this section is [119]. Cfr. also [24] and, particular, §3.2.1, §3.4.2 and §5, for analogous results.

[1]Even if what follows has the form of a statement, Friedman explicitly write that it "should not be viewed as a conjecture" since it can "be at least likely as [its] negation". [53, p. 113]

[2]Indeed, it is possible to show that such a subset $T \subsetneq$ **For**$\mathcal{L}$ is closed under substitution iff **Int** $\subseteq T$ (cfr. [138, Lemma 1]) and Friedman claims that "such a T must properly contain all provable formulae of the intuitionistic propositional calculus".

Given a formula $\varphi \in \mathbf{For}\mathcal{L}$, we say that φ is *negative* if it is of the form $\neg\psi$, for some $\psi \in \mathbf{For}\mathcal{L}$ and we denote by $\mathcal{N}$ the set of negative formulas. Furthermore, we say that a substitution $\sigma\colon \mathbf{Var}\mathcal{L} \to \mathbf{For}\mathcal{L}$ is *negative* if, for each $p \in \mathbf{Var}\mathcal{L}$, $\sigma(p) \in \mathcal{N}$. We will usually indicate negative substitutions by σ_N.

Definition 6.2. A *non-standard intermediate logic* (*nsi-logic*, for short) in the language $\mathcal{L}$ is any consistent set L of $\mathcal{L}$-formulas satisfying the following conditions:

- $\mathbf{Int} \subseteq L$;
- L is closed under modus ponens (MP);
- L is closed under the following rule of negative substitution: $\dfrac{\varphi}{\sigma_N(\varphi)}$, for any negative substitution σ_N.

Thus any intermediate logic is a nsi-logic, since it is closed under uniform substitution, but the converse does not hold.

For any $\mathcal{L}$-formula φ, we denote by φ^n the formula obtained from φ by replacing any occurrence of a propositional variable p with its negation $\neg p$.

Definition 6.3. Let L be an intermediate logic. The *negative variant* of L is the set of $\mathcal{L}$-formulas L^n defined as follows:

$$L^n = \{\varphi \,|\, \varphi^n \in L\}.$$

Notice that L^n can not be be an intermediate logic, unless $L^n = \mathbf{Cl}$. Indeed, L^n will not be closed under uniform substitution, since, for every $p \in \mathbf{Var}\mathcal{L}$, $\neg\neg p \to p \in L^n$. However, we have the following

Lemma 6.1. *Let L be an intermediate logic. Then the negative variant L^n of L is a non-standard intermediate logic including L. In particular, L^n is the smallest set of $\mathcal{L}$-formulas that contains L, all the formulas of the form $\neg\neg p \to p$, where $p \in \mathbf{Var}\mathcal{L}$, and closed under modus ponens.*

We have just seen that $(\cdot)^n$ is an operator from the lattice of intermediate logics to the lattice of non-standard intermediate logic. We are now going to introduce an operator on the lattice of nsi-logics which gives us back an intermediate logic.

Definition 6.4. Let L be a nsi-logic. The *standardization* of L is the set of $\mathcal{L}$-formulas L^s defined as follows:

$$L^s = \{\varphi \,|\, \sigma(\varphi) \in L, \text{ for every substitution } \sigma\}.$$

Since **Int** is included in the standardization of any nsi-logic, it can be readily seen that L^s is the greatest intermediate logic contained in L. Now, the composition $(\cdot)^{ns}$ of the negative-variant operator with the standardization operator gives us an operator defined on the lattice of the intermediate logics, which we denote, following [24], by $(\cdot)^\nu$. In particular, it is not hard to show that $(\cdot)^\nu$ is a closure operator on the complete lattice of intermediate logics[3].

The fixed points of a given closure operator c on a poset $\langle W, \leq \rangle$, that is the elements $u \in W$ such that $u = c(u)$, are commonly called closed elements. With this is mind, we give the following

Definition 6.5. Let L be an intermediate logic. We say that L is *negatively stable* if it is a closed element of the operator $(\cdot)^\nu$.

As a consequence of very known facts about closure operators[4], we have that the set of negatively stable logics forms a complete lattice under set-theoretic inclusion. In particular, given a family $\{L_i\}_{i \in I}$ of negatively stable logics, $\bigcap_{i \in I} L_i$ is the greatest negatively stable logic contained in each L_i and the logic $(\sum_{i \in I} L_i)^\nu$ is the smallest negatively stable logic containing each L_i. We are now going to present a different syntactical characterization of negatively stable logics.

A $\mathcal{L}$-formula φ is said to be *essentially negative* if every occurrence of a variable in it is under the scope of some $\neg$ and we denote by $\mathcal{EN}$ the set of essentially negative formulas. Moreover, we call any substitution σ an *essentially negative substitution* if $\sigma(\mathbf{Var}\mathcal{L}) \subseteq \mathcal{EN}$ and we will usually indicate essentially negative substitutions by $\sigma_{N!}$. Notice that the set $\mathcal{EN}$ of essentially negative formulas can be defined as the smallest set of $\mathcal{L}$-formulas containing the set of negative formulas $\mathcal{N}$ and closed under the following set of connectives: $\{\wedge, \vee, \rightarrow\}$. Equivalently, every essentially negative formula φ is of the form $\psi(\neg\xi_1, \ldots, \neg\xi_n)$ for some $\mathcal{L}$-formulas $\psi(p_1, \ldots, p_n), \xi_1, \ldots, \xi_n$. Therefore, any substitution instance $\sigma(\varphi)$ of an essentially negative formula φ is still essentially negative. Furthermore, notice the following feature of essentially negative formulas:

Lemma 6.2. *Let φ be an $\mathcal{L}$-formula. Then $\neg\varphi \leftrightarrow \neg(\varphi^{nn}) \in \mathbf{Int}$. Therefore, if φ is an essentially negative formula, $\varphi \leftrightarrow \varphi^{nn} \in \mathbf{Int}$.*

Proof. We proceed by induction on φ. Since $\neg\gamma \leftrightarrow \neg\neg\neg\gamma \in \mathbf{Int}$, the claim holds for $p \in \mathbf{Var}\mathcal{L}$ and $\varphi \in \mathbf{For}^{\neg}\mathcal{L}$. So, let $\varphi := \eta \circ \delta$ where $\circ \in \{\wedge, \vee, \rightarrow\}$. Notice that since, for any $\mathcal{L}$-formulas ξ and ψ, $(\xi \rightarrow \neg\psi) \leftrightarrow \neg(\xi \wedge \psi) \in \mathbf{Int}$, it suffice to show that both

$$\neg(\neg(\eta \circ \delta) \wedge (\eta \circ \delta)^{nn}) \in \mathbf{Int} \qquad \text{and} \qquad \neg(\neg((\eta \circ \delta)^{nn}) \wedge (\eta \circ \delta)) \in \mathbf{Int}.$$

By using the induction hypothesis, it can immediately seen that both formulas belong to **Cl** and thus they also belong to **Int** by Glivenko's Theorem. Furthermore, if φ is essentially negative, then φ is of the form $\varphi'(\neg\xi_1, \ldots, \neg\xi_k)$ for some $\mathcal{L}$-formulas $\varphi'(p_1, \ldots, p_k)$ and $\xi_1, \ldots, \xi_k$. Since, for each $i \leq k$, we have $\neg\xi_i \leftrightarrow \neg(\xi_i^{nn}) \in \mathbf{Int}$, the equivalence $\varphi \leftrightarrow \varphi^{nn} \in \mathbf{Int}$ holds by the Replacement Theorem. □

---

[3]Recall that a *closure operator* on a given a partially ordered set $\langle W, \leq \rangle$ is any mapping $c \colon W \to W$ satisfying, for all $u, v \in W$, the following properties:

(i)	$u \leq c(u)$	(Extensivity)
(ii)	$u \leq v \Longrightarrow c(u) \leq c(v)$	(Monotonicity)
(iii)	$c(u) = c(c(u))$	(Idempotency)

[4]Cfr., for instance, [20, Theorem 5.2].

Definition 6.6. Let L be an intermediate logic. We say that L is *essentially negative determined* if the following holds for every $\mathcal{L}$-formula φ:

$$\sigma_{N!}(\varphi) \in L, \text{ for every essentially negative } \sigma_{N!} \Longrightarrow \varphi \in L.$$

It then follows that an essentially negative determined intermediate logic L is a logic whose set of essentially negative theorems "determines" the set of all theorems of L. Now the next theorem (cfr. [119, Theorem 3]) provides the syntactic characterization of negatively stable logics mentioned before.

Theorem 6.3. *Let L be an intermediate logic. Then L is negatively stable iff L is essentially negative determined.*

Proof. ($\Longrightarrow$) Suppose that L is stable and let φ be such that $\sigma_{N!}(\varphi) \in L$ for every $\sigma_{N!}$. Now notice that, since $\neg\neg p \leftrightarrow p \in L^n$, every substitution σ is equivalent in L^n to $\sigma_{N!}$, for some essentially negative substitution $\sigma_{N!}$. Thus, $\sigma(\varphi) \in L^n$ for every substitution σ, and, consequently, $\varphi \in L^{ns} = L^{\nu} = L$.
($\Longleftarrow$) Suppose that L is essentially negative determined and let $\varphi \in L^{\nu}$. So, $\sigma(\varphi) \in L^n$, for every substitution σ, and, in particular, $\sigma_{N!}(\varphi) \in L^n$, for every essentially negative substitution $\sigma_{N!}$. Hence $(\sigma_{N!}(\varphi))^n \in L$ by definition of L^n and $(\sigma_{N!}(\varphi))^{nn} \in L$, since L is closed under substitution. But then, since $\sigma_{N!}(\varphi)$ is essentially negative, we have $\sigma_{N!}(\varphi) \leftrightarrow (\sigma_{N!}(\varphi))^{nn} \in \mathbf{Int}$ by Lemma 6.2. Therefore $\sigma_{N!}(\varphi) \in L$ and, consequently, $\varphi \in L$. ☐

We are now going to introduce a particular class of Kripke frames that, as we shall see, are directly connected with the stability property previously introduced.

Definition 6.7. Let $\mathfrak{F} = \langle W, R\rangle$ be a finite Kripke frame. We say that $\mathfrak{F}$ is *coatomistic* if $\mathfrak{F}$ satisfies the following condition:

$$\neg(xRy) \Longrightarrow max(\mathfrak{F}_y) \nsubseteq max(\mathfrak{F}_x), \tag{c-A}$$

for every points x, y in W.

Since, for every frame $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$, we have $xRy \Longrightarrow max(\mathfrak{F}_y) \subseteq max(\mathfrak{F}_x)$, it follows that in coatomistic frame the position of every point is determined by the set of the maximal points above it[5].

Theorem 6.4. *Let $L = Log\mathcal{K}$ be an intermediate logic and suppose that each $\mathfrak{F} \in \mathcal{K}$ is a finite coatomistic frame. Then L is negatively stable.*

Proof. Suppose $\varphi(p_1, \dots, p_n) \notin L$. Then there exists a finite coatomistic frame $\mathfrak{F} = \langle W, R\rangle \in \mathcal{K}$ refuting φ under some valuation $\mathfrak{V}$. For every point x in $\mathfrak{F}$, let q_x be a new variable and extend the valuation $\mathfrak{V}$ to these new variables by letting

$$\mathfrak{V}^*(q_x) = max(\mathfrak{F}) \cap W \setminus x{\uparrow},$$

[5]The adjective "coatomistic" comes from lattice and order theory. Indeed, in a poset $\langle P, \leq\rangle$ with greatest element 1, a *coatom* is an element covered by 1 and $\langle P, \leq\rangle$ is said to be *coatomistic* if every elements $p \in P$ is the greatest lower bound of a set of coatoms. In general, every poset $\langle P, \leq\rangle$ such that every elements $p \in P$ is the greatest lower bound of the maximal elements above it is coatomistic.

that is $\mathfrak{V}^*(q_x)$ is the set of the maximal points in $\mathfrak{F}$ which are not accessible from x. Now, consider a point y in $\mathfrak{F}$. Then, by definition of $\mathfrak{V}$ and the fact that $\mathfrak{F}$ satisfies (c-A), we have

$$\begin{aligned} y \models \neg q_x \quad &\iff \quad max(\mathfrak{F}_y) \subseteq max(\mathfrak{F}_x) \\ &\iff \quad y \in x\uparrow. \end{aligned}$$

and thus we have, for every $p_i \in \mathbf{Var}\mathcal{L}$,

$$\mathfrak{V}^*(\bigvee_{x \in \mathfrak{V}(p_i)} \neg q_x) = \mathfrak{V}(p_i).$$

Now, define the substitution σ as follows: for each $i \in \{1, \dots, n\}$,

$$\sigma(p_i) = \bigvee_{x \in \mathfrak{V}(p_i)} \neg q_x.$$

Since $\mathfrak{V}(p_i)$ is finite, σ is a well-defined essentially negative substitution such that $\mathfrak{V}^*(\sigma(\varphi)) = \mathfrak{V}(\varphi)$. Consequently, $\sigma(\varphi) \notin L$, L is essentially negative determined and so negatively stable by Theorem 6.3. □

The previous theorem is particular interesting since it allows us to easily exhibit a *pletora* of intermediate negatively stable logics. Indeed, we have the following

Corollary 6.5. *The following intermediate logics are negatively stable:*

- *Intuitionistic logic* **Int***;*
- *the logics of bounded branching* $\mathbf{T}_n (n \geq 1)$*;*
- *Medvedev's logic* **ML***.*

Proof. Just notice that every Jaśkowski's frame $\mathfrak{J}_n$[6], every n-ary tree $(n > 1)$ and each Medvedev frame $\mathfrak{P}_n$ are finite coatomistic frames and the class of such frames respectively characterize **Int**, $\mathbf{T}_n (n \geq 1)$ and **ML**. □

6.1.1 Other negatively stable logics: the logic of rhombuses and their variants

Another interesting example of a stable logic is the *logic of rhombuses* **RH**, introduced by Maksimova in [106], which is defined as the logics of the class of the frames $\mathfrak{R}_n$, for $n < \omega$, defined as follows:
for every $n < \omega$, let us consider the subset $N = \{1, \dots, n+1\} \subseteq \mathbb{N}$ of natural numbers with the restriction of the standard linear ordering on $\mathbb{N}$ and, for every $1 \leq i \leq j \leq n+1$, consider the intervals $[i, j] = \{m \in N \mid i \leq m \leq j\}$; then the Kripke frame $\mathfrak{R}_n = \langle W_n, S_n \rangle$ is the frame such that

$$\begin{aligned} W_n &= \{[i, j] \mid i, j \in N, i \leq j\} \\ [i_1, j_1] S_n [i_2, j_2] \quad &\iff \quad [i_1, j_1] \supseteq [i_2, j_2] \end{aligned}$$

[6]Cfr. the footnote 7 in Chapter 3.

For instance, the **RH**-frames $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_3$ are respectively as follows:

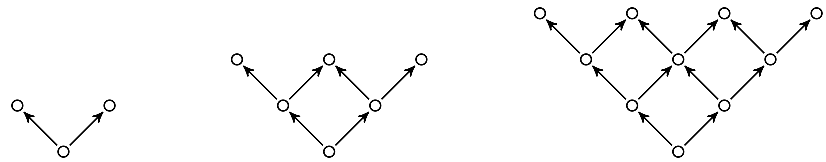


Figure 6.1: *The* **RH**-*frames* $\mathfrak{R}_1$, $\mathfrak{R}_2$ *and* $\mathfrak{R}_3$.

By letting $\mathcal{R} = \{\mathfrak{R}_n \,|\, n < \omega\}$, it is immediate to see that $\mathbf{RH} := \mathrm{Log}\,\mathcal{R}$ is a finitely approximable intermediate logic with the disjunction property and such that $\mathbf{S} + \mathbf{T}_2 \subseteq \mathbf{RH}$. Furthermore, since every $\mathfrak{R}_n$ is a coatomistic frame, it follows immediately from Theorem 6.4 that

Corollary 6.6. **RH** *is negatively stable.*

The logic **RH** has been introduced in order to provide a counterexample to the maximality of $\mathbf{T_2}$ as a constructive logic and it has also been considered as a potential candidate for that rôle (cfr. [22, §2.2]). However, such a conjecture has been falsified by Ferrari and Miglioli who, in [47], showed that **RH** is actually properly contained in another constructive logic.

We are now going to define a sequence of logics that are characterized by frames which arise as a variation on the construction of the frames for **RH**. Let us fix some positive natural numbers $m \geq 1$ and consider, for every $n < \omega$, the subsets $N_n^m = \{1, \ldots, (n \cdot m) + 1\} \subseteq \mathbb{N}$ of natural numbers with the restriction of the standard linear ordering on $\mathbb{N}$. Then we define the frame $\mathfrak{R}_n^m$ to be the pair

$$\mathfrak{R}_n^m := \langle W_n^m, \supseteq \rangle,$$

where

$$W_n^m = \{[i, j] \,|\, i, j \in N_n^m,\ j = i + (m \cdot k) \text{ for some } 0 \leq k \leq n\}.$$

Finally, we let

$$\mathcal{R}^m = \{\mathfrak{R}_n^m \,|\, n < \omega\} \qquad \text{and} \qquad \mathbf{RH}^m = \mathrm{Log}\,\mathcal{R}^m.$$

In what follows, some examples of $\mathbf{RH}^m$-frames for $m \in \{2, 3, 4\}$ are depicted.

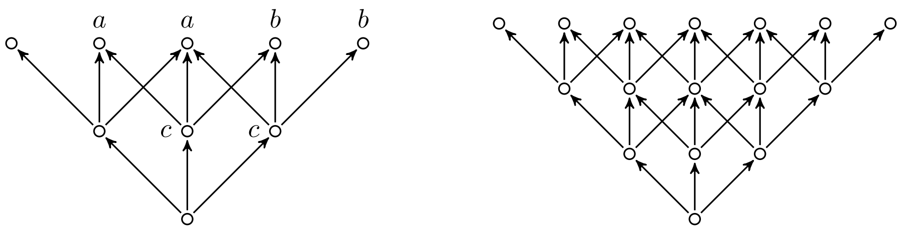


Figure 6.2: *The* $\mathbf{RH}^2$-*frames* $\mathfrak{R}_2^2$ *and* $\mathfrak{R}_3^2$.

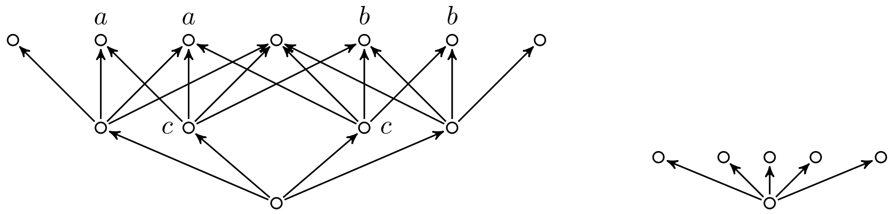


Figure 6.3: *The* $\mathbf{RH}^m$*-frames* $\mathfrak{R}^3_2$ *and* $\mathfrak{R}^4_1$*.*

Notice that, for $m = 1$, the frames in $\mathcal{R}^m$ coincide with the frames in $\mathcal{R}$, that is, we have $\mathbf{RH}^1 = \mathbf{RH}$. Furthermore, each consideration we made concerning $\mathbf{RH}$ can be made identically for every $\mathbf{RH}^m$. In particular, for every $m \geq 1$ and every $n < \omega$, the frame $\mathfrak{R}^m_n \in \mathcal{R}^m$ is coatomistic and since every logic $\mathbf{RH}^m$ is finitely approximable by definition, we have

Proposition 6.7. *For each* $m \geq 1$*, the logic* $\mathbf{RH}^m$ *is a finitely approximable constructive negatively stable intermediate logic containing the logic* $\mathbf{S} + \mathbf{T}_{m+1}$*.*

We are now going to show that the logics $\mathbf{RH}^m$ form a sequence of logics such that, for each $m > 0$, $\mathbf{RH}^{m+1} \subsetneq \mathbf{RH}^m$. In order to show that fact, we prove the following

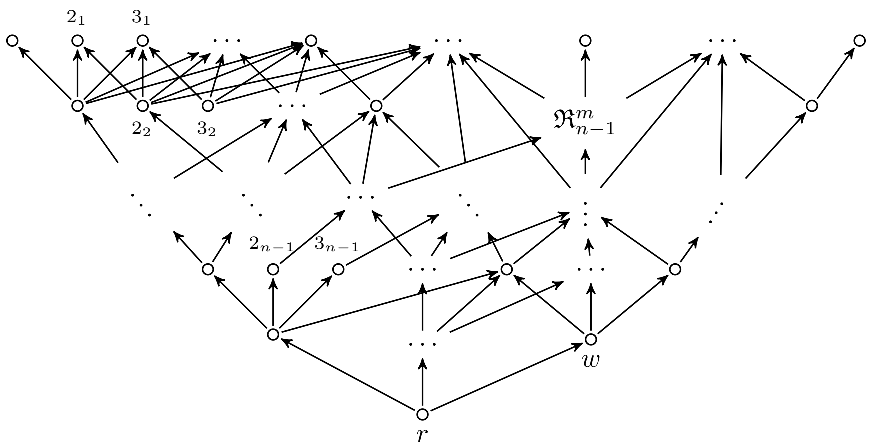
Lemma 6.8. *For each* $m > 0$ *and each* $n < \omega$*, the frame* $\mathfrak{R}^m_n$ *is a p-morphic image of the frame* $\mathfrak{R}^{m+1}_n$*.*

Proof. Let an arbitrary $m > 0$ be fixed. We are going to recursively define a uniform algorithmic procedure that reduces $\mathfrak{R}^{m+1}_n$ to $\mathfrak{R}^m_n$ for every $n > 0$[7] and then prove by induction on n that it is indeed sound.
Consider $\mathfrak{R}^{m+1}_n$ and let r be its root. By construction, we have that

$$\begin{aligned} d(\mathfrak{R}^{m+1}_n) &= n + 1; \\ |max(\mathfrak{R}^{m+1}_n)| = |(\mathfrak{R}^{m+1}_n)^{=1}| &= ((m+1)\cdot n) + 1; \\ |(\mathfrak{R}^{m+1}_n)^{=k+1}| &= |(\mathfrak{R}^{m+1}_n)^{=k}| - (m+1). \end{aligned}$$

So, r has exactly $m+2$ immediate successor and we let $w \in r^{>}$ be the rightmost one. Then $w\uparrow$ is a generated subframe of $\mathfrak{R}^{m+1}_n$ isomorphic to $\mathfrak{R}^{m+1}_{n-1}$ and thus we can apply our reduction procedure to $w\uparrow$. Then $w\uparrow$ reduces to $\mathfrak{R}^m_{n-1}$ and we can consider the frame $\mathfrak{R}^{m+1}_n[w\uparrow/\mathfrak{R}^m_{n-1}]$, which looks like as follows:

[7] Indeed, notice that, for any $k > 0$, the frames $\mathfrak{R}^k_0$ are isomorphic to the one-point frame $\mathfrak{1}$ and thus we can skip the case $n = 0$.



Now, for each layer $k \leq n$ of $\mathfrak{R}^{m+1}_n[w{\uparrow}/\mathfrak{R}^m_{n-1}]$, beginning from $k = 1$ and going down until $k = n$ following the natural progression, identify the elements 2_k and 3_k, that is, the second and the third element of $(\mathfrak{R}^{m+1}_n[w{\uparrow}/\mathfrak{R}^m_{n-1}])^{=k}$ starting from the left. The resulting reduced frame is then isomorphic to $\mathfrak{R}^m_n$.
Let us check, by induction on $n \geq 1$, that our algorithm is sound and that the obtained frame is indeed isomorphic to $\mathfrak{R}^m_n$. For $n = 1$, notice that our algorithm boils down to the identification of the maximal points 2_1 and 3_1 of $\mathfrak{R}^{m+1}_1$. The previous identification is a β-reduction and the reduced frame is nothing but r with $m+1$ immediate successor, which is indeed isomorphic to the frame $\mathfrak{R}^m_1$. For the induction step, assume that our reduction procedure works for $n-1$ and let us prove that it is also sound for n. Consider again the frame $\mathfrak{R}^{m+1}_n[w{\uparrow}/\mathfrak{R}^m_{n-1}]$ depicted above. By the induction step, it is indeed a reduction of the frame $\mathfrak{R}^{m+1}_n$ by Lemma 1.13. Then the identification of the two maximal points 2_1 and 3_1 is again β-reduction and thus this step is sound. Furthermore, notice that the cardinality of the maximal points in this reduced frame, call it $\mathfrak{D}_1$, is

$$\begin{aligned}|max(\mathfrak{D}_1)| &= |max(\mathfrak{R}^{m+1}_n)| - (|max(\mathfrak{R}^{m+1}_{n-1}) - |max(\mathfrak{R}^m_{n-1})|) - 1\\ &= (m+1)n - [(m+1)n - mn - 1]\\ &= |max(\mathfrak{R}^m_n)|,\end{aligned}$$

that is, $\mathfrak{D}^{\leq 1}_1 \cong (\mathfrak{R}^m_n)^{\leq 1}$. So assume that the identification of 2_k and 3_k is sound for some arbitrary $1 < k < n$ and that $\mathfrak{D}^{\leq k}_k \cong (\mathfrak{R}^m_n)^{\leq k}$. Consider the points 2_{k+1} and 3_{k+1}. They are the intervals $[2, 2+m\cdot k]$ and $[3, 3+m\cdot k]$ respectively. Therefore, it follows that in $\mathfrak{R}^{m+1}_n$ we have

$$\begin{aligned}2^{>}_{k+1} =& \{2_k, 2_k+1, \ldots, 2_k+(m+1)\}\\ 3^{>}_{k+1} =& \{3_k, 3_k+1 \ldots, 3_k+(m+1)\},\end{aligned}$$

where $s_k + c$ is the interval $[s+c, s+(m\cdot(k-1))+c]$ for $s \in \{2, 3\}$. Now notice that, since

$$|(\mathfrak{R}^{m+1}_n)^{=k}| = |(\mathfrak{R}^{m+1}_{n-1})^{=k-1}| = (m+1) + |(\mathfrak{R}^{m+1}_{n-1}))^{=k}|,$$

the $m+1^{th}$-element of $(\mathfrak{R}_n^{m+1})^{=k}$ coincide with the 1^{st}-element of the generated subframe $(w\uparrow)^{=k} \cong (\mathfrak{R}_{n-1}^{m+1})^{=k}$, and, in particular, we have that

$$2_k + (m+1) = 2_k^{\mathfrak{R}_{n-1}^{m+1}}$$
$$3_k + (m+1) = 3_k^{\mathfrak{R}_{n-1}^{m+1}}.$$

But then, since the elements $2_k^{\mathfrak{R}_{n-1}^{m+1}}$ and $3_k^{\mathfrak{R}_{n-1}^{m+1}}$ are identified in $\mathfrak{R}_n^{m+1}[w\uparrow/\mathfrak{R}_{n-1}^{m}]$, it follows that $2_{k+1}^{>} = 3_{k+1}^{>}$ in $\mathfrak{D}_k$. Hence the identification of the two points is a β-reduction and thus the obtained frame $\mathfrak{D}_{k+1}$ is a sound reduction of $\mathfrak{D}_k$. Finally, since

$$\begin{aligned} |\mathfrak{D}_{k+1}^{=k+1}| &= |\mathfrak{D}_k^{=k+1}| - 1 \\ &= |\mathfrak{D}_k^{=k}| - (m+1) - 1 = |(\mathfrak{R}_n^m)^{=k}| - m \\ &= |(\mathfrak{R}_n^m)^{=k+1}| \end{aligned}$$

and every element of $\mathfrak{D}_{k+1}^{=k+1}$ has exactly $m+1$ immediate successors, we have $\mathfrak{D}_{k+1}^{\leq k+1} \cong (\mathfrak{R}_n^m)^{\leq k+1}$. We can thus conclude that the frame $\mathfrak{D}_n$ obtained as the final step of our reducing procedure is a sound reduction of $\mathfrak{R}_n^{m+1}$ isomorphic to $\mathfrak{R}_n^m$, as we wanted[8]. □

Since $\boldsymbol{bb}_{n+1} \in \mathbf{RH}^n$ but $\boldsymbol{bb}_{n+1} \notin \mathbf{RH}^{n+1}$, by the previous lemma we can immediately infer that

Proposition 6.9. *For each $m > 0$, $\mathbf{RH}^{m+1} \subsetneq \mathbf{RH}^m$.*

Finally, notice that $\bigcap_{1\leq m} \mathbf{RH}^m$ is still a negatively stable logic and that $\mathbf{S} \subseteq \bigcap_{1\leq m} \mathbf{RH}^m$. In [24], Ciardelli showed that Scott logic $\mathbf{S}$ is negatively stable too. Moreover, both logics enjoy the disjuction property. We thus make the following

Conjecture. $\bigcap_{1\leq m} \mathbf{RH}^m = \mathbf{S}$.

6.2 Friedman logics in ExtInt

We are now going to see where Friedman's logics are situated in the lattice of all the si-logics Ext**Int**. First notice that, by Theorem 1.32, any Friedman logic L include **KP** and thus we have a first lower bound on the family of Friedman logics. As we shall see later on, this bound is not optimal and can be improved, but let us now focus on the upper bound.

Consider the following sequence of $\mathcal{L}$-formulas ($k \geq 1$)

$$\boldsymbol{nd_k} = (\neg p \to \neg q_1 \vee \ldots \vee \neg q_k) \to (\neg p \to q_1) \vee \ldots \vee (\neg p \to \neg q_k)$$

and define the logic **ND** to be the following intermediate logic:

$$\mathbf{ND} = \mathbf{Int} + \{\boldsymbol{nd_k} \mid k \geq 1\}.$$

[8] The reader can also look at Figure 6.2 and 6.3, where the case for $m = 2, 3$ and $n = 2$ is illustrated as follows: the identification of the points labelled by b is the induction step that reduces the cone of the rightmost immediate successor of the root to the frame $\mathfrak{R}_1^2$ and $\mathfrak{R}_1^3$ respectively. Then the two β-reduction of the elements labelled by a and then by c give a frame isomorphic to $\mathfrak{R}_2^1$ and $\mathfrak{R}_2^2$ respectively.

Notice that every occurrence of a variable in $\boldsymbol{nd_k}$ is under the scope of some $\neg$ and thus, for each $k \geq 1$, $\boldsymbol{nd_k} \in \mathcal{EN}$.

The logic **ND** has been introduced by Maksimova in [106] where it is also proved that it has the disjunction property and is not finitely axiomatizable. Thus it is clear that $\mathbf{ND} \subsetneq \mathbf{KP} \subsetneq \mathbf{ML}$. Moreover, the logic **ND** has other interesting properties as well. The following result is well known[9]:

Theorem 6.10. *Let L be a decidable or finitely approximable si-logic and φ an essentially negative $\mathcal{L}$-formula. Then the logic $L + \varphi$ is also decidable or finitely approximable respectively.*

As an immediate corollary of the previous theorem we get that **ND** is a decidable intermediate logic with the finite model property. Furthermore, we have the following *normal form* lemma for $\mathcal{L}$-formulas in $\mathcal{EN}$.

Lemma 6.11. *Let φ be an essentially negative $\mathcal{L}$-formula. Then φ is equivalent in* **ND** *to a $\mathcal{L}$-formula of the form $\neg\xi_1 \vee \ldots \vee \neg\xi_k$ $(k \geq 1)$.*

We are now ready to prove the following result of Maksimova [106] which gives us the desidered upper bound on Friedman logics.

Theorem 6.12 (Maksimova)**.** *Let L be an intermediate logic with the disjunction property and such that $\mathbf{ND} \subseteq L$. Then L is included in Medvedev's logic* **ML**.

Proof. Suppose for reductio that there exists an intermediate logic L with the disjunction property and such that $\mathbf{ND} \subseteq L \nsubseteq \mathbf{ML}$. Then there exists a formula $\varphi \in L \setminus \mathbf{ML}$. Since **ML** is stable by Corollary 6.5, without loss of generality, we can assume that φ is essentially negative. Now, since $\varphi \in \mathcal{EN}$ and $\mathbf{ND} \subseteq L$, by Lemma 6.11 it follows that $\neg\xi_1 \vee \ldots \vee \neg\xi_m \in L \setminus \mathbf{ML}$ for some $m \geq 1$. Since L has the disjunction property, $\neg\xi_j \in L$ for some j but then, by Glivenko's Theorem, $\neg\xi_j \in \mathbf{ML}$ and thus $\neg\xi_1 \vee \ldots \vee \neg\xi_m \in \mathbf{ML}$, contradiction. □

From the previous theorem and the fact that $\mathbf{ND} \subsetneq \mathbf{KP}$, it follows that any Friedman logic is included in **ML**. Since Medvedev's logic itself is Friedman, the upper bound is thus optimal.

Remark 22. Notice that, since the inclusion $\mathbf{ND} \subseteq \mathbf{ML}$ holds, Theorem 6.12 tells us that Medvedev's logic is in fact a maximal intermediate logic with the disjunction property. Whether there exists a finitely axiomatizable maximal intermediate logic with the disjunction property is still an open issue.

The following characterization of Medvedev's logic is due to the Soviet logician L.A. Levin [98]:

Theorem 6.13 (Levin)**.** *Medvedev's logic of finite problems* **ML** *coincides with the set of $\mathcal{L}$-formulas φ such that $\sigma(\varphi) \in \mathbf{KP}$ for every essentially negative substitution σ. Formally,*

$$\mathbf{ML} = \{\varphi \in \mathbf{For}\mathcal{L} \mid \sigma(\varphi) \in \mathbf{KP},\ \textit{for every } \sigma\colon \mathbf{Var}\mathcal{L} \to \mathcal{EN}\}.$$

[9] Cfr. [23, Theorem 11.9 and Theorem 11.10] for a proof.

Proof. Suppose that $\varphi \notin \mathbf{ML}$. Then, by the proof of Theorem 6.12, we can find an essentially negative substitution instance $\sigma(\varphi)$ of φ such that $\sigma(\varphi) \notin \mathbf{ML}$. Since $\mathbf{KP} \subseteq \mathbf{ML}$, it follows that $\sigma(\varphi) \notin \mathbf{KP}$. Conversely, let $\varphi \in \mathbf{ML}$ and consider an essentially negative substitution σ. Since $\sigma(\varphi) \in \mathbf{ML}$ and $\mathbf{ND} \subseteq \mathbf{ML}$, by Lemma 6.11 we have $\sigma(\varphi) \leftrightarrow \bigvee_{j=1}^{m} \neg\xi_j \in \mathbf{ML}$. But then, by Glivenko's theorem, the fact that **ML** has the disjunction property and $\mathbf{ND} \subseteq \mathbf{KP}$, we have that $\sigma(\varphi) \in \mathbf{KP}$. □

Notice that in the proof of the previous theorem the only assumption used with respect to **KP** is the fact that $\mathbf{ND} \subseteq \mathbf{KP}$. Therefore, we can generalize the previous theorem by substituting to **KP** any intermediate logic L such that $\mathbf{ND} \subseteq L \subseteq \mathbf{ML}$. In particular, we have the following

Theorem 6.14. *Let L be an intermediate logic with the disjunction property and such that $\mathbf{ND} \subseteq L$. Then*

$$\mathbf{ML} = \{\varphi \in \mathbf{For}\mathcal{L} \mid \sigma(\varphi) \in L, \text{ for every } \sigma\colon \mathbf{Var}\mathcal{L} \to \mathcal{EN}\}.$$

Moreover, from the previous results, we also have

Corollary 6.15. *Let L be an intermediate logic with the disjunction property and such that $\mathbf{ND} \subseteq L$. Then $L \cap \mathcal{EN} = \mathbf{ML} \cap \mathcal{EN}$.*

A fortiori, any Friedman logic also coincides with Medvedev's logic on essentially negative formulas. Now, let us improve such a coincidence by introducing another set of $\mathcal{L}$-formulas. Let $\mathcal{EN}^*$ be the smallest set of $\mathcal{L}$-formulas extending the set of the essentially negative formulas $\mathcal{EN}$ with the following condition

(*) if $\varphi \in \mathcal{EN}^*$ and $\psi \in \mathbf{For}\mathcal{L}$, then $\psi \to \varphi \in \mathcal{EN}^*$.

We are going to show that **ML** coincide with any Friedman logic on the set $\mathcal{EN}^*$. Indeed, such a claim is an immediate consequence of the the following useful lemma

Lemma 6.16. *Let L be an intermediate logic with the disjunction property and such that $\mathbf{ND} \subseteq L$. If a rule r: $\dfrac{\psi_1, \ldots, \psi_n}{\varphi}$ is admissible in L, then r is derivable in **ML**. Furthermore, if φ is essentially negative, then the converse also holds.*

Proof. By contraposition, suppose that the formula $\bigwedge_i \psi_i \to \varphi \notin \mathbf{ML}$. By the fact that **ML** is structurally complete, it follows that there exists a substitution σ such that $\sigma(\psi_i) \in \mathbf{ML}$ for each $i \in \{1, \ldots, n\}$ and $\sigma(\varphi) \notin \mathbf{ML}$. Since Medvedev's logic is negatively stable, there also is an essentially negative substitution τ such that $(\tau \circ \sigma)(\varphi) \notin \mathbf{ML}$. So, since by Theorem 6.12 L is included in **ML**, we have that $(\tau \circ \sigma)(\varphi) \notin L$. However, for each $i \in \{1, \ldots, n\}$, $(\tau \circ \sigma)(\psi_i) \in L$ by Theorem 6.14 and thus we can conclude that the rule r is not admissible in L.
For the second part of the Lemma, assume that $\varphi \in \mathcal{EN}$ and suppose that the $\mathcal{L}$-formula $\bigwedge_i \psi_i \to \varphi \in \mathbf{ML}$. Consider an arbitrary substitution σ and suppose that $\sigma(\psi_i) \in L$ for each $i \in \{1, \ldots, n\}$. Then both $\sigma(\bigwedge_i \psi_i \to \varphi)$ and the $\sigma(\psi_i)$'s belong to **ML** and consequently we also have $\sigma(\varphi) \in \mathbf{ML}$. But since the set of essentially negative formulas is closed under substitution, it follows that $\sigma(\varphi) \in L$ by Corollary 6.15. Hence, by the arbitrariness of σ, we conclude that r is admissible in L. □

Proposition 6.17. *Let L be a Friedman logic. $L \cap \mathcal{EN}^* = \mathbf{ML} \cap \mathcal{EN}^*$.*

Notice that the Scott axiom $\boldsymbol{sa}$ is a formula in $\mathcal{EN}^*$ which is a theorem of Medvedev's logic and thus, from the previous proposition, it follows that any Friedman logic must be an extension of the logic $\mathbf{SKP} = \mathbf{Int} + \{\boldsymbol{sa}, \boldsymbol{kp}\}$[10].

Admissible rules of KP

We are now going to introduce an operation on Kripke frames which has been first taken into consideration by G. C. Meloni in [117].
Let two Kripke frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$ be given. The *connected product* of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ is the Kripke frame $\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \langle W_1 \otimes W_2, R \rangle$ defined as follows:

- $W_1 \otimes W_2 = W_1 \uplus W_2 \uplus (W_1 \times W_2)$;
- for $x, y \in W$, $\quad xRy \iff$
$$\begin{aligned} & (x, y \in W_1 \text{ and } xR_1y) \text{ or} \\ & (x, y \in W_2 \text{ and } xR_2y) \text{ or} \\ & (x = \langle u, v \rangle, y \in W_1 \text{ and } uR_1y) \text{ or} \\ & (x = \langle u, v \rangle, y \in W_2 \text{ and } vR_2y) \text{ or} \\ & (x = \langle u, v \rangle, y = \langle u', v' \rangle \text{ and } uR_1u', vR_2v'). \end{aligned}$$

Clearly, if both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are rooted and w_1, w_2 are the roots of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively, then also $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ is rooted and $\langle w_1, w_2 \rangle$ is its root. For instance, the connected product of the two elements frame $\mathfrak{1}^\triangledown$ with itself is the following frame

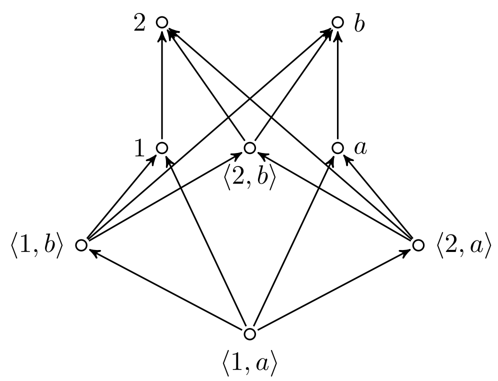


Figure 6.4: *The Kripke frame $\mathfrak{1}^\triangledown \otimes \mathfrak{1}^\triangledown$.*

Notice that, for each $i \in \{1, 2\}$, $\mathfrak{F}_i$ is a generated subframes of $\mathfrak{F}_1 \otimes \mathfrak{F}_2$. The following lemma states that the class $\mathcal{KP}_{\mathrm{fin}}$ of finite rooted frames for $\mathbf{KP}$ is closed under $\otimes$. For the proof, cfr. [117] or [123, Lemma 2.1].

Lemma 6.18. *Let $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$ be two finite rooted Kripke frames in $\mathcal{KP}_{fin}$. Then $\mathfrak{F} \otimes \mathfrak{G}$ also belongs to $\mathcal{KP}_{fin}$.*

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[10] Actually, any Friedman logic must be a proper extension of **SKP**, since such a logic is not structurally complete as shown in [123].

Notice that, as an immediate corollary of the previous lemma, we get another proof of the fact that **KP** has the disjunction property. However, the main reason we have introduced such a construction is to show the admissibility of some interesting rules in **KP**. In particular, we have the following

Lemma 6.19. *The following rules are admissible in* **KP***:*

$$(a) : \frac{(\neg\neg p \to p) \to p \vee \neg p}{\neg p \vee \neg\neg p} \qquad\qquad (b) : \frac{(p \to q) \to (r \vee s)}{((p \to q) \to r) \vee (\neg p \to s)}$$

Proof. (a) Notice that since the conclusion is an essentially negative formula and $\boldsymbol{sa} \in \mathbf{ML}$, such a rule is admissible in **KP** by Lemma 6.16.
(b) Suppose that for some $\mathcal{L}$-formulas φ, ψ, γ and δ,

$$((\varphi \to \psi) \to \gamma) \vee (\neg\varphi \to \delta) \notin \mathbf{KP}.$$

Since $\mathbf{KP} = \mathrm{Log}\,\mathcal{KP}_{\mathrm{fin}}$, there exist finite rooted Kripke frames $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$ in $\mathcal{KP}_{\mathrm{fin}}$ and valuations $\mathfrak{V}_1$ and $\mathfrak{V}_2$ such that, for some points $w \in \mathfrak{F}$, $v \in \mathfrak{G}$,

$$w \models \varphi \to \psi \text{ and } w \not\models \gamma$$
$$v \models \neg\varphi \text{ and } v \not\models \delta.$$

Consider the model $(\mathfrak{F} \otimes \mathfrak{G}, \mathfrak{V})$, given by the valuation $\mathfrak{V}$ defined as follows: for all $p \in \mathbf{Var}\mathcal{L}$, $\mathfrak{V}(p) = \mathfrak{V}_1(p) \cup \mathfrak{V}_2(p)$. Now, consider the point $k = \langle w, v\rangle \in \mathfrak{F} \otimes \mathfrak{G}$. Clearly, we have $k \not\models \gamma \vee \delta$. Furthermore, if $k \not\models \varphi \to \psi$, then there is a point $q \in k{\uparrow}$ such that $q \models \varphi$ and $q \not\models \psi$. Then, if $q \in \mathfrak{F}$, $w \leq q$ and consequently, by the monotonicity of the entailment, $q \models \psi$, which contradicts the assumption. If $q \in \mathfrak{G}$, then $v \leq q$ and thus $q \models \varphi \wedge \neg\varphi$, contradiction. Finally if $q = \langle q_1, q_2\rangle \in \mathfrak{F} \times \mathfrak{G}$, then $q \leq q_2$ and $v \leq q_2$ and consequently we get again $q \models \varphi \wedge \neg\varphi$. Therefore $k \models \varphi \to \psi$ and thus $k \not\models (\varphi \to \psi) \to \gamma \vee \delta$. So, since $\mathfrak{F} \otimes \mathfrak{G} \in \mathcal{KP}_{\mathrm{fin}}$, $(\varphi \to \psi) \to \gamma \vee \delta \notin \mathbf{KP}$ and we conclude that (b) is admissible in **KP**. □

Corollary 6.20. *The Kreisel-Putnam logic* **KP** *is not structurally complete.*

Proof. By the previous lemma, the rule corresponding to the Scott axiom $\boldsymbol{sa}$ is admissible in **KP**, however $\boldsymbol{sa} \notin \mathbf{KP}$. □

6.3 On Medvedev frames

Let us consider a Medvedev frame $\mathfrak{P}_n = \langle \mathcal{P}(n) \setminus \{\emptyset\}, \supseteq\rangle$. Denote by W the universe of $\mathfrak{P}_n$ and, for every point $U \in \mathfrak{P}_n$, let x_U be the principal upset $U{\uparrow}$, so that $x_{\{i\}} = \{\{i\}\}$. The frame $\mathfrak{P}_n$ is a n-generated frame, since the $x_{\{i\}}$'s generate every possible other upset of $\mathfrak{P}_n$. Indeed, since $\mathfrak{P}_n$ is coatomistic, the position of every point in $\mathfrak{P}_n$ is determined by the set of the final points above it and thus we have, for every $U \in \mathfrak{P}_n$,

$$x_U = W \setminus (\bigcup_{i \notin U} x_{\{i\}}){\downarrow}. \qquad (\alpha)$$

Since $\mathfrak{P}_n$ is finite, it follows that every upset V of $\mathfrak{P}_n$ is a finite union of cones and thus, for some subset $\Theta \subseteq W$, it has the form

$$V = \bigcup_{K\in\Theta} x_K = \bigcup_{K\in\Theta} (W \setminus (\bigcup_{i\notin K} x_{\{i\}})\downarrow)^{11}.$$

However, we can actually refine such result. So, let us take a closer look to $\mathfrak{P}_n$.

Lemma 6.21. *For every $n < \omega$, the Medvedev frame $\mathfrak{P}_n$ is $\lceil \log_2 n \rceil$-generated.*

Proof. The set $max(\mathfrak{P}_n)$ of final elements of $\mathfrak{P}_n$ consists of the sigletons $\{i\}$, where $i \in n = \{0, \dots, n-1\}$. Write each $i \in n$ in binary notation, prefixing, if needed, some 0's, in order to have a $\lfloor \log_2 n \rfloor + 1$-bit sequence as follows

$$i = \langle i_{\lfloor \log_2 n \rfloor + 1}, \dots, i_{\lfloor \log_2 i \rfloor + 1} \dots, i_1 \rangle.$$

Now, for $j \in \{1, \dots, \lceil \log_2 n \rceil\}$, define

$$U_j := \{i \in n \,|\, i_j = 1\}$$

and consider the corresponding principal upset x_{U_j} of $\mathfrak{P}_n$. We are going to show that the sets of upsets $\{x_{U_j} \,|\, j \in \{1, \dots, \lceil \log_2 n \rceil\}$ is a generating set for $\mathfrak{P}_n^+$. Let $f\colon \mathfrak{P}_n \to \mathfrak{F}$ be a proper onto p-morphism. Since $\mathfrak{P}_n$ is finite, by Lemma 3.2, f can be written as the composition $f_1 \circ \dots \circ f_m$ of α- or β-reductions. Now, we notice that, by the structure of $\mathfrak{P}_n$, any proper p-morphism must first identify two distinct final points: hence f_m has to be a β-reduction[12]. Now, let $\{i\}, \{k\}$ be distinct points of $max(\mathfrak{P}_n)$ such that $f_m(\{i\}) = f_m(\{k\})$. Since $i \neq k$, we have $i_j \neq k_j$ for some $j \in \{1, \dots, \lceil \log_2 n \rceil\}$ and thus, without loss of generality, we can assume that $i \in U_j$ and $k \notin U_j$. Hence $\{i\} \in x_{U_j}$ and $\{k\} \notin x_{U_j}$ and, consequently, $col(\{i\}) \neq col(\{j\})$. As f was arbitrary, we conclude that $\mathfrak{P}_n^+$ is actually generated by the x_{U_j}'s by the Colouring Theorem 3.3[13]. □

Due to the boolean structure of every Medvedev frame, we have the following interesting property.

[11] In particular, by letting $\mathfrak{V}$ be the valuation on $\mathfrak{P}_n$ defined by $\mathfrak{V}(p_i) = x_{\{i\}}$, we have that V can be defined through $\mathfrak{V}$ by the $\mathcal{L}_n$-formula

$$\varphi_V := \bigvee_{K\in\Theta} \neg(\bigvee_{i\notin K} p_i).$$

Notice that φ_V is an essentially negative formula.

[12] Here we are assuming $n > 2$, since, in such a case, a proper p-morphisms can begin with an α-reduction too.

[13] By the remarks in the previous paragraph the set $\Theta = \{x_{\{i\}} \,|\, i \in n\}$ generates $\mathfrak{P}_n^+$, so, in order to prove that $\{x_{U_j} \,|\, j \in \{1, \dots, \lceil \log_2 n \rceil\}$ generates $\mathfrak{P}_n^+$, it is also possible to show that each element $x_{\{i\}} \in \Theta$ is definable directly from the x_{U_j}'s by means of the set-theoretic operations $\cap, \cup, \supset$. For instance, for $n = 8$, one has

$$U_1 = \{1,3,5,7\}, \qquad U_2 = \{2,3,6,7\}, \qquad U_3 = \{4,5,6,7\},$$

and thus gets the upsets $x_{\{i\}}$ in the following ordered way:

(1)	$x_{\{0\}} = W \setminus (x_{U_1} \cup x_{U_2} \cup x_{U_3})\downarrow;$	(2)	$x_{\{1\}} = W \setminus (x_{U_2} \cup x_{U_3} \cup x_{\{0\}})\downarrow;$
(3)	$x_{\{2\}} = W \setminus (x_{U_1} \cup x_{U_3} \cup x_{\{0\}})\downarrow;$	(4)	$x_{\{3\}} = W \setminus (x_{U_3} \cup \bigcup_{i\in\{1,2,0\}} x_{\{i\}})\downarrow;$
(5)	$x_{\{4\}} = W \setminus (x_{U_1} \cup x_{U_2} \cup x_{\{0\}})\downarrow;$	(6)	$x_{\{5\}} = W \setminus (x_{U_2} \cup \bigcup_{i\in\{1,4,0\}} x_{\{i\}})\downarrow;$
(7)	$x_{\{6\}} = W \setminus (x_{U_1} \cup \bigcup_{i\in\{2,4,0\}} x_{\{i\}})\downarrow;$	(8)	$x_{\{7\}} = x_{U_1} \cap x_{U_2} \cap x_{U_3}.$

Lemma 6.22. *Let $\mathfrak{P}_n$ be a Medvedev frame and let σ be a permutation of the maximal elements of $\mathfrak{P}_n$. Then σ induces an automorphism $j_\sigma \colon \mathfrak{P}_n \to \mathfrak{P}_n$.*

Proof. Let $x \in \mathfrak{P}_n$ and define j_σ by induction on $d(x)$ as follows:

- if $d(x) = 1$, then let $j_\sigma(x) = \sigma(x)$;
- if $d(x) = k + 1$, then, whenever $x \prec U$, we let $j_\sigma(x) = y$ if $y \prec j_\sigma(U)$.

We now show, for each $x, y \in \mathfrak{P}_n$,

$$x \leq y \iff j_\sigma(x) \leq j_\sigma(y).$$

($\Longrightarrow$) If $d(x) = 1$, then $x = y$ and $j_\sigma(x) = \sigma(x) = \sigma(y) = j_\sigma(y)$. Now assume that $d(x) = k + 1$ and suppose that $x < y$. Then $x \lessdot s \leq y$ and the depth of s is k. Thus, by induction hypothesis, $j_\sigma(s) \leq j_\sigma(y)$. Furthermore, since $s \in x^>$ and $x \prec x^>$, by the definition of j_σ we have $j_\sigma(x) \prec j_\sigma(x^>)$. Consequently $j_\sigma(s) \in j_\sigma(x^>)$ and $j_\sigma(x) \lessdot j_\sigma(s) \leq j_\sigma(y)$.

($\Longleftarrow$) Let $j_\sigma(x) \leq j_\sigma(y)$. If $d(x) = 1$, then $\sigma(x) = \sigma(y)$ and thus $x = y$, since σ is a bijection. So suppose that $d(x) = k + 1$. Let s be an immediate successor of $j_\sigma(x)$ such that $j_\sigma(y) \in s\uparrow$. Since $j_\sigma(x) \prec j_\sigma(x^>)$, we have that $s = j_\sigma(q)$ for some $q \in x^>$. Hence $j_\sigma(q) \leq j_\sigma(y)$ and by the induction hypothesis we get $q \leq y$. Hence $x \lessdot q \leq y$. □

Remark 23. Given a transposition (xy) of two maximal elements of $\mathfrak{P}_n$, we also get an automorphism $j_{(xy)} \colon \mathfrak{P}_n \to \mathfrak{P}_n$. Furthermore, since every permutation σ is a product of transpositions, we have $j_\sigma = j_{\tau_m} \circ \ldots \circ j_{\tau_1}$, where $\sigma = \tau_m \ldots \tau_1$.

Proposition 6.23. *For every $n < \omega$, the Medvedev frame $\mathfrak{P}_n$ is regular injective in $\boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$.*

Proof. Since the frame $\mathfrak{P}_n$ is a finite rooted frame in $\boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$ and $\mathbf{ML}$ is structurally complete, there exists an onto p-morphism $h \colon \mathfrak{F}_{\mathbf{ML}}(k) \to \mathfrak{P}_n$ for some $k < \omega$, by Proposition 4.27. Furthermore, since $max(\mathfrak{P}_n) = n$, it must be the case that $k > \lceil \log_2 n \rceil$ and consequently $\mathfrak{P}_n$ is a generated subframe of $\mathfrak{F}_{\mathbf{ML}}(k)$ by Lemma 6.21. Now, we show that, for all $Q \in \mathfrak{P}_n \subsetneq \mathfrak{F}_{\mathbf{ML}}(k)$,

$$h(Q) = Q, \tag{*}$$

by induction on the depth of Q. Using if necessary Lemma 6.22, (*) holds for the final points of $\mathfrak{P}_n$, so let us assume by induction hypothesis that $h(Q) = Q$ for all $Q \in \mathfrak{P}_n$ such that $d(Q) = s < n$. Let A be a point of $\mathfrak{P}_n$ such that $d(A) = s + 1$ and consider the set of immediate successor $A^>$ of A. Then, $A \preceq A^>\uparrow$ and, since p-morphisms preserve coverings, we have $h(A) \preceq h(A^>\uparrow)$. Since, for all $B \in A^>\uparrow$, $d(B) \leq s$, it follows by the induction hypothesis that $h(A) \preceq h(A^>\uparrow) = A^>\uparrow$. Finally, since for every $Q \in \mathfrak{P}_n$ and $\Delta \subseteq \mathfrak{P}_n$, $Q \preceq \Delta$ implies $\bigcup \Delta = Q$, we conclude $h(A) = A$. Being a retract of the k-canonical frame, $\mathfrak{P}_n$ is regular injective in $\boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$. □

As an immediate consequence of the previous lemma, we have the following result, for which we give also a direct proof.

Lemma 6.24. *For every $n \in \omega$, $\mathfrak{P}_n$ is a retract of $\mathfrak{P}_{n+1}$.*

Proof. Clearly $\mathfrak{P}_n \subsetneq \mathfrak{P}_{n+1}$. Define a retraction $r\colon \mathfrak{P}_{n+1} \to \mathfrak{P}_n$ of the inclusion map $i\colon \mathfrak{P}_n \to \mathfrak{P}_{n+1}$ as follows: for each point $U \in \mathfrak{P}_{n+1}$,

$$r(U) = \begin{cases} U, & \text{if } n \notin U; \\ (U \setminus \{n\}) \cup \{n-1\}, & \text{otherwise.} \end{cases}$$

Then it is easily seen that r is a onto p-morphism and $r \circ i = 1_{\mathfrak{P}_n}$. □

The following lemma will also be useful in what follows.

Lemma 6.25. *Let a Medvedev frame $\mathfrak{P}_n$, for some $n \in \omega$, be given and let $k \in \omega$. Then $\biguplus_{i=0}^{k-1} \mathfrak{P}_{n_i} \subsetneq \mathfrak{P}_{k\cdot n}$ and there exists an onto p-morphism $f\colon \mathfrak{P}_{k\cdot n} \to \mathfrak{P}_n$ such that $f \restriction_{\mathfrak{P}_{n_i}} = 1_{\mathfrak{P}_n}$.*

Proof. It is immediately seen that the disjoint union of k copies of $\mathfrak{P}_n$ is a generated subframe of $\mathfrak{P}_{k\cdot n}$. In particular, for each $i \in k$, the root of $\mathfrak{P}_{n_i}$ is taken to be the set $\mathfrak{r}_{\mathfrak{P}_{n_i}} = \{i \cdot n, (i \cdot n) + 1, \dots, (i \cdot n) + (n-1)\}$. Now, let $f\colon \mathfrak{P}_{k\cdot n} \to \mathfrak{P}_n$ be defined as follows: for every $A \in \mathfrak{P}_{k\cdot n}$,

$$f(A) = \mathrm{mod}_n(A),$$

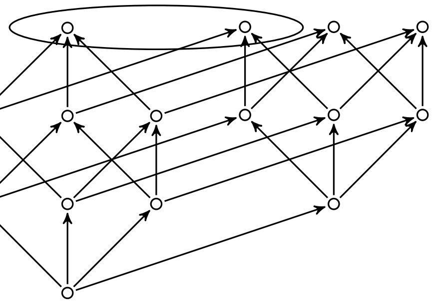
where $\mathrm{mod}_n(A) = \{\mathrm{mod}_n(i) \mid i \in A\}$. Let us show that f is a p-morphism. Let $A, B \in \mathfrak{P}_{k\cdot n}$ and suppose first that $A \supseteq B$. Then clearly $\mathrm{mod}_n(A) \supseteq \mathrm{mod}_n(B)$ and so f is monotone. Now assume that $f(A) \supset Q$ for some $Q \in \mathfrak{P}_n$. Then, for $C = \{i \in A \mid \mathrm{mod}_n(i) \in Q\}$, we have $A \supset C$ and $f(C) = Q$. Consequently, f is a p-morphism and, since $f(\{0, \dots, (k \cdot n) - 1\}) = \{0, \dots, n-1\}$, f is onto. Finally, notice that, for each $i \in k$, $f(\mathfrak{r}_{\mathfrak{P}_{n_i}}) = n$ and thus $f \restriction_{\mathfrak{P}_{n_i}} = 1_{\mathfrak{P}_n}$. □

Let us focus on some p-morphic images of Medvedev frames. The following is a well-known result due to Maksimova, Skvortsov and Shehtman [110].

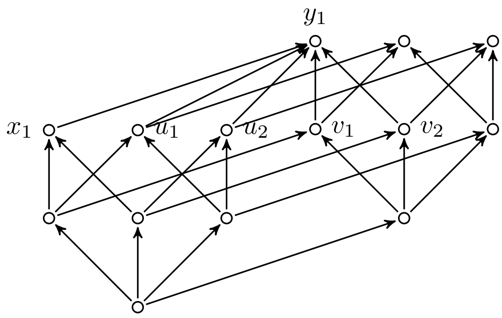
Lemma 6.26. *Let $\mathfrak{F}$ be a finite rooted frame. Then, for some $n < \omega$, there exists a p-morphism from $\mathbb{1} \oplus \mathfrak{P}_n$ onto $\mathbb{1} \oplus \mathfrak{F}$. Consequently, there exists a p-morphisms from $\mathfrak{P}_{n+1}$ onto $\mathbb{1} \oplus \mathfrak{F}$.*

Now consider a Medvedev's frame $\mathfrak{P}_n$ for some arbitrary $n \in \omega$. Let us identify the first two leftmost maximal point of $\mathfrak{P}_n$ and denote by $\mathfrak{F}_0 := [\mathfrak{P}_n]$ the resulting quotient frame and notice that $max(\mathfrak{F}_0) \cong max(\mathfrak{P}_{n-1})$. Now, among the points in $\mathfrak{F}_0^{=2}$, we have exactly one point x with only one immediate successor y and $n-2$ pairs $\langle u_i, v_i \rangle$ of points that have the same immediate successors. So, let $\mathfrak{F}_1 = [\mathfrak{F}_0]$ be the quotient frame obtained by identifying x and y and the points u_i and v_i for all $i \in \{1, \dots n-2\}$. Notice that $\mathfrak{F}_1^{\leq 2} \cong \mathfrak{P}_{n-1}^{\leq 2}$ Now, consider the points in $\mathfrak{F}_1^{=3}$: there are $n-2$ points x_i that have only one immediate successor y_i and $\binom{n-2}{2}$ pairs $\langle u_i, v_i \rangle$ of points that have the same immediate successors. Again identify the x_i's with the y_i's and the u_i's with the v_i's and denote by $\mathfrak{F}_2 = [\mathfrak{F}_1]$ the resulting frame. Again notice that $\mathfrak{F}_2^{\leq 3} \cong \mathfrak{P}_{n-1}^{\leq 3}$. Proceeding in this fashion, we have that $\mathfrak{F}_k$ has $\binom{n-2}{k}$ points of depth $k+2$ with a single immediate successor and $\binom{n-2}{k+1}$ pair of points of depth $k+2$ with the same set of immediate successors and $\mathfrak{F}_k^{\leq k+1} \cong \mathfrak{P}_{n-1}^{\leq k+1}$. We exemplify the previous reduction procedure for the Medvedev's frame $\mathfrak{P}_4$.

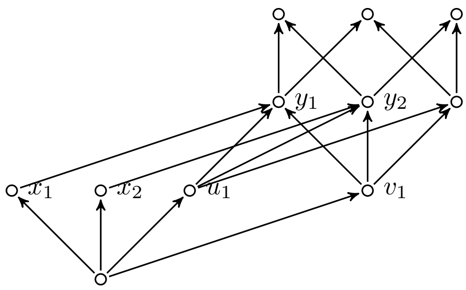
$\mathfrak{P}_4$



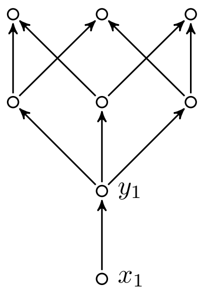
$\mathfrak{F}_0$



$\mathfrak{F}_1$



$\mathfrak{F}_2$



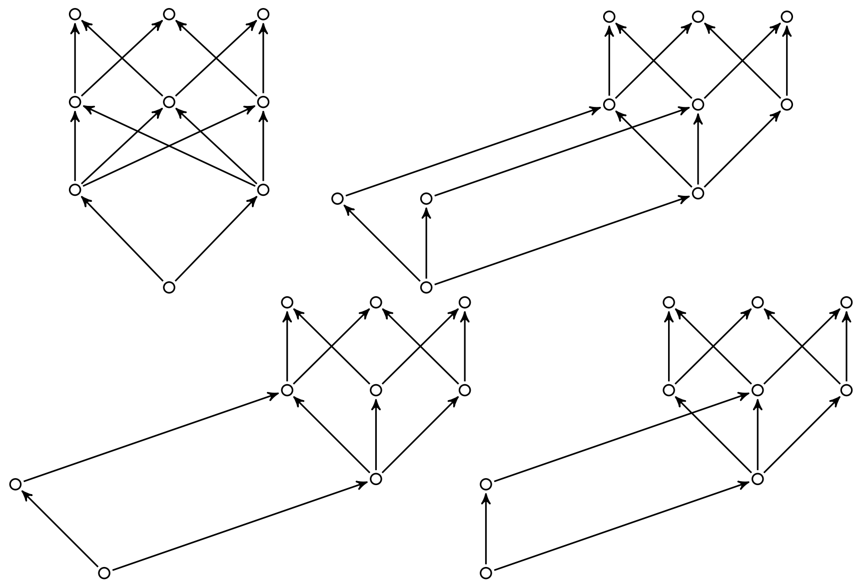
By considering the previous reduction procedure, we notice the following facts. First of all, we have $\mathfrak{F}_{n-2} \cong \mathfrak{P}_{n-1} \oplus \mathbf{1}$. Furthermore, when we get to $\mathfrak{F}_{n-3}$, we can reduce the frame in the following three different ways:

(a) by identifying only the linear points that have only one immediate successors;

(b) by identifying only the points that have strictly more than one immediate successors;

(c) by identifying only $1 \leq k < n-2$ of the $n-2$ points that have exactly one immediate successors and all the points that have strictly more than one immediate successors.

The resulting quotient frames will be respectively isomorphic to the frames $\mathfrak{P}_{n-1}^{\leq n-2} \oplus (\mathbf{1}+\mathbf{1})^{\triangledown}$, $\frac{\mathfrak{P}_{n-1}}{n-2} \oplus \mathbf{1}$ and $\frac{\mathfrak{P}_{n-1}}{k} \oplus \mathbf{1}$, where $\frac{\mathfrak{P}_{n-1}}{k}$ is the frame obtained from $\mathfrak{P}_{n-1}$ by adding k linear points below the points in $\mathfrak{P}_{n-1}^{=n-2}$. For instance, in the case $n = 4$, we get the following four frames



We summarize the previous considerations in the following

Lemma 6.27. *For every $n \in \omega$, $k \leq n-2$,*

(i) $\mathfrak{P}_n \oplus \mathbf{1}$ *is a p-morphic image of* $\mathfrak{P}_{n+1}$*;*

(ii) $\mathfrak{P}_n^{\leq n-1} \oplus (\mathbf{1}+\mathbf{1})^{\triangledown}$ *is a p-morphic image of* $\mathfrak{P}_{n+1}$*;*

(iii) $\frac{\mathfrak{P}_{n-1}}{k} \oplus \mathbf{1}$ *is a p-morphic image of* $\mathfrak{P}_{n+1}$*.*

We remark that point (i) of the previous Lemma is nothing but a special case of the following more encompassing result, first discovered by Grigolia[14].

Lemma 6.28. *Let $\mathfrak{F}$ be a finite rooted frame. Then, $\mathfrak{P}_n \oplus \mathfrak{F} \in \boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$, for every $n \in \omega$.*

Proof. By Lemma 6.26, there exist an onto p-morphism $g\colon \mathbf{1} \oplus \mathfrak{P}_k \to \mathbf{1} \oplus \mathfrak{F}$, for some $k \in \omega$. We notice that, for every point u different from the root of an arbitrary Medvedev frame, $u{\downarrow} \cong \mathbf{1} \oplus \mathfrak{P}_j$, where $j = d(u) + 1$. Now, consider the Medvedev frame $\mathfrak{P}_{(k+1)+n}$ and let w be a point in it such that $d(w) = n$. Then we have $w{\uparrow} \cong \mathfrak{P}_n$ and $w{\downarrow} \cong \mathbf{1} \oplus \mathfrak{P}_k$. Let $\mathfrak{G} \subsetneq \mathfrak{P}_{(k+1)+n}$ be the generated subframe of $\mathfrak{P}_{(k+1)+n}$ with domain $W \backslash w{\downarrow} \cup \{w\}$. Since $w{\uparrow} \subsetneq \mathfrak{G}$ and $w{\uparrow}$ is regular injective in $\boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$ by Proposition 6.23, there exists a retraction $h\colon \mathfrak{G} \to \mathfrak{P}_n$. Now, let $j\colon \mathfrak{P}_{(k+1)+n} \to \mathfrak{P}_n \oplus \mathfrak{F}$ be the function defined as follows:

[14]Private communication.

$$j(u) = \begin{cases} h(u), & \text{if } u \in \mathfrak{G}; \\ g(u) & \text{if } u \in w{\downarrow} \setminus \{w\}. \end{cases}$$

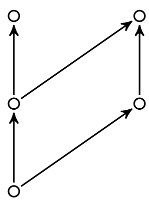
It is not hard to show that j is actually a surjective p-morphism and, consequently, $\mathfrak{P}_n \oplus \mathfrak{F} \in \boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$. □

Notice that every finite rooted frame $\mathfrak{F} \in \boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$ is actually a p-morphic image of some Medvedev frame $\mathfrak{P}_n$. Indeed, if $\mathfrak{F} \in \boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$, then the frame-formula $\beta(\mathfrak{F}, \mathfrak{D}^\natural, \bot) \notin \mathbf{ML}$. Consequently, for some $n \in \omega$, $\mathfrak{P}_n \not\models \beta(\mathfrak{F}, \mathfrak{D}^\natural, \bot)$ and thus there exists an onto p-morphism $f\colon \mathfrak{P}_m \to \mathfrak{F}$ for some $m \leq n$ by Theorem 4.15. So, as an immediate corollary of the previous lemmas, we get

Proposition 6.29. *Let $\mathfrak{F}$ be a finite rooted frame in $\boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$ and $\mathfrak{G}$ a finite rooted frame. Then the frames $\mathfrak{F} \oplus \mathfrak{G}$ and $\mathfrak{F}^{\leq d(\mathfrak{F})-1} \oplus (\mathbf{1} + \mathbf{1})^\triangledown$ both belong to $\boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$. Furthermore, if $|\mathfrak{F}^{=d(\mathfrak{F})-1}| = k$, then the frames $\frac{\mathfrak{F}}{j} \oplus \mathbf{1}$, for $j < k$, also belong to $\boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$.*

Proof. Let us show the last part of the claim. Since $\mathfrak{F} \in \boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$, without loss of generality, $\mathfrak{F}$ is a p-morphic image of a Medvedev frame $\mathfrak{P}_n$ such that each point $x \in \mathfrak{P}_n$ such that $x \neq n$ is mapped to a point different to $\mathfrak{F}$'s root. Therefore, if $|\mathfrak{F}^{=d(\mathfrak{F})-1}| = k$, it must be the case that $n \geq k$. Now, for every choice of $j < k$ points of $\mathfrak{F}^{=d(\mathfrak{F})-1}$, we can pick (possibly using Lemma 6.22) a sequence of immediate successors of $\mathfrak{P}_n$'s root which are mapped bijectively onto the selected points of $\mathfrak{F}$. Then it becomes clear that $\frac{\mathfrak{F}}{j} \oplus \mathbf{1}$ is a p-morphic image of the frame $\frac{\mathfrak{P}_n}{j} \oplus \mathbf{1}$. □

Notice that if $d(\mathfrak{F}) > k$, then we can relax the bound on j by letting $j \leq k$. For instance, by letting $\mathfrak{F}$ be the 1-canonical Medvedev's frame $\mathfrak{F}_{\mathbf{ML}}(1)$ (cfr. the following section §6.4)



we get the following frames:

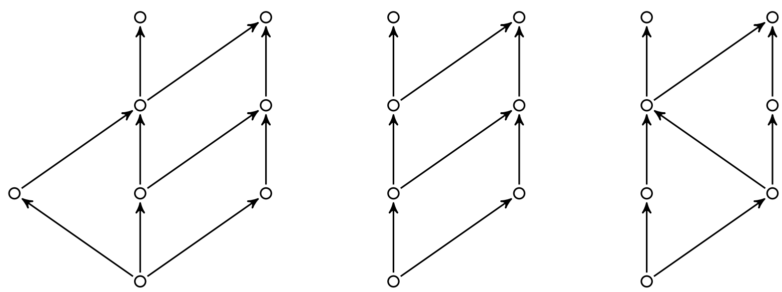


Figure 6.5: *The derivatives of* $\mathfrak{F}_{\mathbf{ML}}(1)$.

However, we cannot always relax the bound, since, for each $n \in \omega$, the frame $\frac{\mathfrak{P}_n}{n} \oplus \mathbf{1}$ does not belong to $\mathcal{DF}_{\mathbf{ML}}$, as shown in the following

Proposition 6.30. *For each $n \in \omega$, $\frac{\mathfrak{P}_n}{n} \oplus \mathbf{1}$ is not a p-morphic image of any Medvedev frame.*

Proof. Suppose on the contrary that $f \colon \mathfrak{P}_j \to \frac{\mathfrak{P}_n}{n} \oplus \mathbf{1}$ is an onto p-morphism. Notice that $j > n$ and, moreover, we can assume, without loss of generality, that the root of $\mathfrak{P}_j$ is the only point which is mapped to the root of $\frac{\mathfrak{P}_n}{n} \oplus \mathbf{1}$. Let $a_1, \ldots, a_n$ be the immediate successors of $\frac{\mathfrak{P}_n}{n} \oplus \mathbf{1}$'s root that have only one immediate successor ordered from left to right and let $U_i \in \mathfrak{P}_j^{=j-1}$ be such that $f(U_i) = a_i$ for each $i \in \{1, \ldots, n\}$. Now, we distinguish two cases according to whether n is even or odd. If n is even, then consider the points in $\mathfrak{P}_j$

$$A = \bigcap_{i \in \{1,\ldots,\frac{n}{2}\}} U_i \quad \text{and} \quad B = \bigcap_{i \in \{\frac{n}{2}+1,\ldots,n\}} U_i;$$

if n is odd, consider instead the following points in $\mathfrak{P}_j$

$$A = \bigcap_{i \in \{1,\ldots,\lfloor\frac{n}{2}\rfloor+1\}} U_i \quad \text{and} \quad B = \bigcap_{i \in \{\lfloor\frac{n}{2}\rfloor+1,\ldots,n\}} U_i.$$

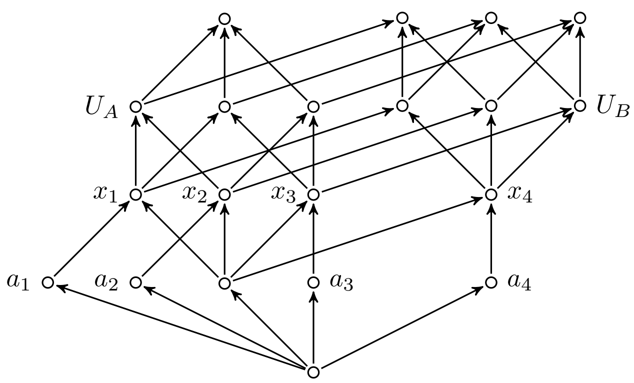
First notice that, by the choice of the U_i's and the fact that $j > n$, both A and B are indeed points of $\mathfrak{P}_j$, that is, both sets are non-empty. Moreover, by the structure of $\frac{\mathfrak{P}_n}{n} \oplus \mathbf{1}$ and the fact that f is monotone, $f(A)$ and $f(B)$ are points of $\mathfrak{P}_n$ which are above or equal to the points

$$U_A = \bigcap_{i \in \{1,\ldots,\frac{n}{2}\}} x_i \quad \text{and} \quad U_B = \bigcap_{i \in \{\frac{n}{2}+1,\ldots n\}} x_i,$$

if n even, or above or equal to the points

$$U_A = \bigcap_{i \in \{1,\ldots,\lfloor\frac{n}{2}\rfloor+1\}} x_i \quad \text{and} \quad U_B = \bigcap_{i \in \{\lfloor\frac{n}{2}\rfloor+1,\ldots n\}} x_i,$$

if n is odd, where each x_i is the point of $\mathfrak{P}_n$ which is above a_i. For instance, the case $n = 4$ is depicted as follows:



Now, notice that $U_A{\uparrow} \cap U_B{\uparrow} = \emptyset$ and, consequently, $f(A){\uparrow} \cap f(B){\uparrow} = \emptyset$. However, again relying on the choice of the U_i's and the fact that $j > n$, we have $A \cap B \neq \emptyset$ and thus $f(A \cap B) \in f(A){\uparrow} \cap f(B){\uparrow}$, which is a contradiction. $\square$

Let us now investigate some connections between Medvedev frames and canonical formulas. First, notice the following

Lemma 6.31. *Let $\mathfrak{F}$ be a finite rooted frame, $f\colon \mathfrak{P}_n \to \mathfrak{F}$ a cofinal subreduction, for some $n \in \omega$, and let $a \in max(\mathfrak{F})$. Then there exists a cofinal subreduction $g\colon \mathfrak{P}_{n+1} \to \mathfrak{F}$ such that $\{n\} \in g^{-1}(a)$.*

Proof. Since f is cofinal, there exists $\{i\} \in max(\mathfrak{P}_n)$ such that $\{i\} \in f^{-1}(a)$. Let $g\colon \mathfrak{P}_{n+1} \to \mathfrak{F}$ be the composition $f \circ j_\tau \circ r$, where $j_\tau\colon \mathfrak{P}_n \to \mathfrak{P}_n$ is the automorphism induced by the transposition $\tau = (\{i\}\{n-1\})$ and $r\colon \mathfrak{P}_{n+1} \to \mathfrak{P}_n$ the retraction of Lemma 6.24. ☐

Lemma 6.32. *Let $n, k \in \mathbb{N}$ such that $0 < k \leq n$. Then $k \cdot 2^{(n-k)+1} \leq 2^{n+1}$.*

Proof. By double induction on n, k.

$(n = 1)$ So $k = n = 1$ and the inequality holds.

$(n = q + 1)$ For $k = 1$, we have that the inequality is true. Now, let $k = s + 1$ and assume that the inequality holds for $k \leq s$. Then we compute

$$\begin{aligned} k \cdot 2^{(n-k)+1} &= (s+1) \cdot 2^{(q-s)+1} \\ &\leq (s+s) \cdot 2^{(q-s)+1} \\ &= s \cdot 2^{(q-s)+1} + s \cdot 2^{(q-s)+1} \\ &\leq 2^{q+1} + 2^{q+1} = 2^{n+1} \end{aligned}$$ ☐

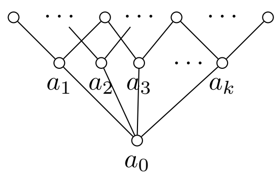
Proposition 6.33. *Let $\mathfrak{F} = \langle W, R\rangle$ be a finite rooted frame. If $\beta(\mathfrak{F}, \bot) \notin$ **ML**, then $\mathfrak{P}_{2^{|\mathfrak{F}|}} \not\models \beta(\mathfrak{F}, \bot)$.*

Proof. By induction on $|\mathfrak{F}|$.

$(|\mathfrak{F}| = 1)$ Then $\mathfrak{P}_{2^{|\mathfrak{F}|}} = \mathfrak{P}_2 \cong (\mathfrak{1} + \mathfrak{1})^\nabla$ is easily seen to be reducible to the one point frame $\mathfrak{1} \cong \mathfrak{F}$. Consequently $\mathfrak{P}_{2^{|\mathfrak{F}|}} \not\models \beta(\mathfrak{F}, \bot)$ by Theorem 4.15.

$(|\mathfrak{F}| = n + 1)$ First notice that $\beta(\mathfrak{F}, \bot) \notin$ **ML** implies $\beta(\mathfrak{F}_a, \bot) \notin$ **ML** for each $a \in W$. Now, if $\mathfrak{F} = \mathfrak{G} \oplus \mathfrak{1}$, then, since $|\mathfrak{G}| = n$, we have $\mathfrak{P}_{2^{|\mathfrak{G}|}} \not\models \beta(\mathfrak{G}, \bot)$ by induction hypothesis and thus $\mathfrak{P}_{2^{|\mathfrak{G}|}} \oplus \mathfrak{1} \not\models \beta(\mathfrak{F}, \bot)$. Consequently, $\mathfrak{P}_{2^{|\mathfrak{G}|+1}} \not\models \beta(\mathfrak{F}, \bot)$ by Lemma 6.27 (i), hence also $\mathfrak{P}_{2^{|\mathfrak{F}|}} \not\models \beta(\mathfrak{F}, \bot)$. So, without loss of generality, we can depict $\mathfrak{F}$ as follows:

$\mathfrak{F}$:



and we notice that k is such that $k \leq n$. Since $|\mathfrak{F}_{a_i}| \leq (n-k)+1$, by the induction hypothesis, $\mathfrak{P}_{2^{(n-k)+1}} \not\models \beta(\mathfrak{F}_{a_i}, \bot)$ for each $i \in \{1, \ldots, k\}$. Consequently, we get cofinal subreductions $f_i\colon \mathfrak{P}_{2^{(n-k)+1}} \to \mathfrak{F}_{a_i}$ for each i by Theorem 4.15 and we can consider each domf_i as a subset of the $i-1^{th}$ Medvedev frame isomorphic to $\mathfrak{P}_{2^{(n-k)+1}}$ in the Medvedev frame

$\mathfrak{P}_{k\cdot 2^{(n-k)}+1}$ as in Lemma 6.25. Then we can define a cofinal subreduction h from the Medvedev frame $\mathfrak{P}_{k\cdot 2^{(n-k)}+1}$ onto $\mathfrak{F}$ by letting

$$h(x) = \begin{cases} f_i, & \text{if } x \in \mathrm{dom} f_i; \\ a_0 & \text{if } x{\uparrow} = \mathfrak{P}_{k\cdot 2^{(n-k)}+1}; \\ undefined & \text{otherwise.} \end{cases}$$

So, we have $\mathfrak{P}_{k\cdot 2^{(n-k)}+1} \not\models \beta(\mathfrak{F}, \bot)$ by Theorem 4.15 and thus also

$$\mathfrak{P}_{2^{n+1}} = \mathfrak{P}_{2^{|\mathfrak{F}|}} \not\models \beta(\mathfrak{F}, \bot)$$

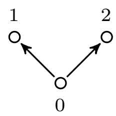
by Lemma 6.32. □

6.4 On the canonical L-frames for Friedman logics

Now let us turn back to Friedman logics and let L be any such intermediate logic. Let us investigate the structure of $\mathfrak{F}_L^{<\infty}(n)$, for some $n < \omega$. Recall that, since L has the disjunction property, by Theorem 3.16 every canonical L-frame $\mathfrak{F}_L(n)$ is rooted. Furthermore, notice that

Lemma 6.34. *The three point frame $(\mathfrak{1} + \mathfrak{1})^{\triangledown}$ belongs to $\boldsymbol{\mathcal{DF}_L}$.*

Proof. Consider the 1-canonical L-frame $\mathfrak{F}_L(1)$, which is rooted by the previous remark. Let A_i, for $i \in \{0, 1, 2\}$, be the set of points of $\mathfrak{F}_L(1)$ which see respectively both the final points and only one of the two point final points of $\mathfrak{F}_L(1)$. Notice that each A_i is non-empty and $\bigcup_{i\in\{0,1,2\}} A_i = W_L(1)$. Then, the map

$h\colon \mathfrak{F}_L(1) \to$ 

defined by letting $h(x) = i \iff x \in A_i$ is a well defined onto p-morphism. □

Proposition 6.35. *Let $\mathfrak{F}$ be a finite rooted frame and $\mathfrak{D}$ a set of antichains in $\mathfrak{F}$. If $\mathfrak{d} = \{a, b\} \subseteq max(\mathfrak{F})$ does not totally cover any point in $\mathfrak{F}$ and $\mathfrak{d} \in \mathfrak{D}$, then $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \in L$.*

Proof. Suppose for contradiction that $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$. Then, by Proposition 4.27, there exists a globally cofinal subreduction $f\colon \mathfrak{F}_L(k) \to \mathfrak{F}$ of the k-canonical frame for L to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ for some $k \in \omega$. Since f is cofinal, let x_a and x_b be maximal point of $\mathfrak{F}_L(k)$ such that $x_u \in f^{-1}(u)$ for $u \in \mathfrak{d}$. By Lemma 6.34, there exists a point $y \in \mathfrak{F}_L(k)$ such that $y \prec \{x_a, x_b\}$. Notice that $y \in \mathrm{dom} f{\uparrow}$ and $\mathfrak{d} \subseteq f(y{\uparrow})$, consequently, y must belong to $\mathrm{dom} f$ by (CDC). But then $\mathfrak{d}$ totally covers $f(y)$, contrary to our hypothesis. □

By the previous results, we have that the finite rooted frames in Figure 6.6 do not belong to $\boldsymbol{\mathcal{DF}_L}$.

Lemma 6.36. *Let L be an intermediate logic. Then $\mathbf{SL} \subseteq L \iff \mathfrak{S} \notin \boldsymbol{\mathcal{DF}_L}$.*

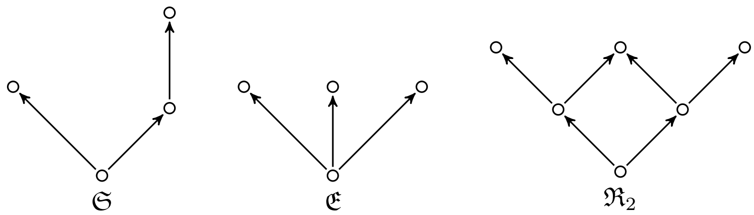


Figure 6.6: *Three frames that are not in $\boldsymbol{\mathcal{DF}_L}$.*

Proof. It is well known that $\mathbf{SL} = \mathbf{Int} + \chi_{\mathfrak{S}}$, where $\chi_{\mathfrak{S}}$ is the frame formula for $\mathfrak{S}$. Therefore the following equivalence holds: $\mathbf{SL} \subseteq L \iff \chi_{\mathfrak{S}} \in L$. Now, if $\mathbf{SL} \nsubseteq L$, then $\chi_{\mathfrak{S}} \notin L$ and thus there exits a frame $\mathfrak{F} \in \boldsymbol{\mathcal{DF}_L}$ such that $\mathfrak{F} \not\models \chi_{\mathfrak{S}}$. By the refutability criterion for canonical formulas it follows that $\mathfrak{S}$ is a p-morphic image of a generated subframe of $\mathfrak{F}$ and thus $\mathfrak{S} \in \boldsymbol{\mathcal{DF}_L}$. Conversely, if $\mathfrak{S} \in \boldsymbol{\mathcal{DF}_L}$, then $L \subseteq \operatorname{Log}\mathfrak{S}$. Since $\mathfrak{S} \not\models \chi_{\mathfrak{S}}$, $\chi_{\mathfrak{S}} \notin L$ and, consequently $\mathbf{SL} \nsubseteq L$. □

Corollary 6.37. *Let L be a Friedman logic. Then* $\mathbf{SL} \subseteq L$.

Let us partially investigate some points at finite depth of the n-canonical frame of a Friedman logic L. Clearly, $\mathfrak{F}_L(n)$ is a generated subframe of $\mathfrak{F}_{\mathbf{Int}}(n)$, for each $n < \omega$. In particular, we have $\mathfrak{F}_L(1) \subsetneq \mathfrak{F}_{\mathbf{Int}}(1)$. Furthermore, since the Scott frame $\mathfrak{S}$ does not belong to $\boldsymbol{\mathcal{DF}_L}$ and $\mathfrak{F}_L(1)$ is rooted, it follows that $\mathfrak{F}_L(1)$ must be a rooted generated subframe of the following frame

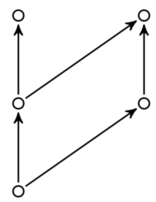
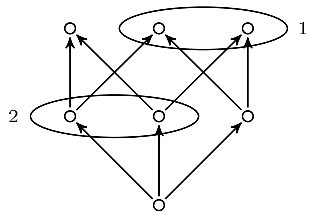


Figure 6.7: *The frame $\mathfrak{D}$.*

Moreover, we already know that $L \subseteq \mathbf{ML}$ and thus, for each $n < \omega$, the Medvedev frame $\mathfrak{P}_n \in \boldsymbol{\mathcal{DF}_L}$. Now, notice that the frame $\mathfrak{P}_3$ can be reduced to the frame $\mathfrak{D}$ by the following two steps of β-reduction



and, thus, it follows that $\mathfrak{D} \in \boldsymbol{\mathcal{DF}_L}$ too. Hence, we have the following

Proposition 6.38. *Let L be a Friedman logic. The* 1*-canonical frame* $\mathfrak{F}_L(1)$ *is isomorphic to the frame $\mathfrak{D}$ of Figure 6.7.*

For $n \geq 2$, we obtain $\mathfrak{F}_L^{\leq 2}(n)$ from $\mathfrak{F}_{\mathbf{Int}}^{\leq 2}(n)$ by removing each point of depth 2 that has strictly more than two immediate successors, since, otherwise, the frame $\mathfrak{C}$ would belong to $\boldsymbol{\mathcal{DF}_L}$. So, for instance, we have

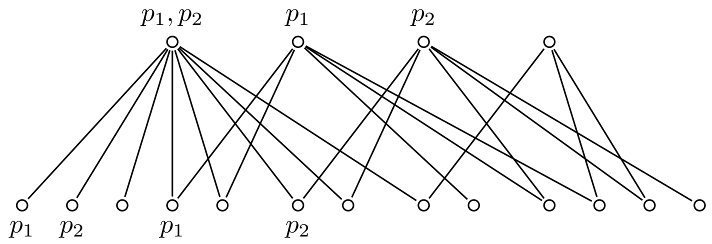


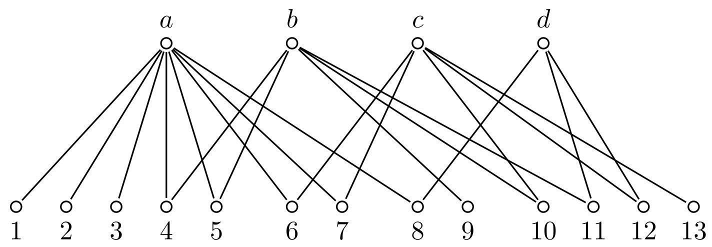
Figure 6.8: *The Friedman submodel* $\mathfrak{M}_L^{\leq 2}(2)$.

Furthermore, by Lemma 6.21, we get the following

Proposition 6.39. *Let L be a Friedman logic. For every $n < \omega$, the Medvedev frame $\mathfrak{P}_{2^n}$ is a generated subframe of the n-canonical frame $\mathfrak{F}_L(n)$ for L such that*

$$max(\mathfrak{P}_{2^n}) = max(\mathfrak{F}_L(n)).$$

Now, let us continue our inquiry on the points at finite depth of n-canonical L-frames. Consider again the frame $\mathfrak{F}_L^{\leq 2}(2)$ with the following labelling of the points



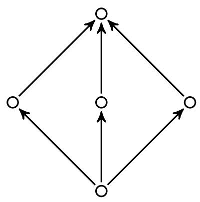
and let us focus on the points at depth 3. First of all, notice that there can not be any points $x_A \in W_L^{=3}(2)$ such that $x_A \preceq A$, where $A \subseteq W_L^{\leq 2}(2)$ is an antichain such that $A \cap W_L^{=1}(2) \neq \emptyset$, that is, containing points of both depth 1 and 2. Indeed, suppose otherwise and let A and x_A be as claimed. Let $A^{=i} = A \cap W_L^{=i}(2)$ be the set of immediate successor of x_A at depth $i \in \{1, 2\}$. Then it follows that $B = A^{=2}\uparrow \cap\, max(\mathfrak{F}_L(2))$ is disjoint from $A^{=1}$ and thus, by identifying all the points in the sets $A^{=1}, A^{=2}$ and B, we reduce the frame $x_A\uparrow$ to the Scott frame $\mathfrak{S}$, which, by Proposition 6.35, does not belong to $\boldsymbol{\mathcal{DF}_L}$. So, we can limit ourselves to considering only antichains $X \subseteq W_L^{=2}(2)$ of points of depth 2. Moreover, observe that there cannot be points of depth 3 covered by an antichain $X \subseteq \{1, 2, 3, 9, 10, 11, 12, 13\}$ such that $\{n, m\} \subseteq X$, where $1 \leq n \leq 3$ and $9 \leq m \leq 13$, because it is easily seen that such an upset would again be reducible to the frame $\mathfrak{S}$. That being said, let us first consider the antichains given by two points.

There are $\binom{13}{2}$ possible combinations of two elements from $W_L^{=2}(2)$. Among them, the combinations allowing the covering of an element are exactly those contained in the following sets:

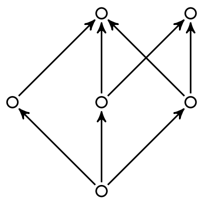
$$\begin{aligned}
\Theta_1 &= \{\{1,2\},\{1,3\},\{2,3\}\},\\
\Theta_2 &= \{\{4,5\},\{6,7\}\},\\
\Theta_3 &= \{\{n,m\} \mid n \in \{1,2,3\}, m \in \{4,5,6,7,8\}\}\\
&\quad \cup \{\{9,m\} \mid m \in \{4,5,10,11\}\}\\
&\quad \cup \{\{13,m\} \mid m \in \{6,7,10,12\}\}.
\end{aligned}$$

In fact, there can not be point at depth 3 covered by a two-point antichain which is not in $\bigcup_{i=1}^{3} \Theta_i$, since it can be shown that the principal upset generated by such a point would be reducible to a frame that does not belong to $\boldsymbol{\mathcal{DF}_L}$. For instance, take the antichain $\{6,11\}$ and suppose there exists a point $x_{\{6,11\}}$ such that $x_{\{6,11\}} \preceq \{6,11\}$. Then, by identifying the final points $b, c \in x_{\{6,11\}}\!\uparrow$, we get the frame $\mathfrak{R}_2$, which, as we have already seen, does not belong to $\boldsymbol{\mathcal{DF}_L}$. So, for each $A \in \Theta_1$, we have a point $x_A \in W_L^{=3}(2)$ such that $x_A \preceq A$ and $x_A\!\uparrow \cong \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1}$. For each $A \in \Theta_2$ there exists a point $x_A \in W_L^{=3}(2)$ such that $x_A \preceq A$ and $x_A\!\uparrow \cong \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1}$[15] and, finally, for each $A \in \Theta_3$ there exists a point $x_A \in W_L^{=3}(2)$ such that $x_A \preceq A$ and $x_A\!\uparrow \cong \mathfrak{F}_L(1)$.

Now, regarding the points of $W_L^{=3}(2)$ covered by a three-point antichain, first notice that the frames



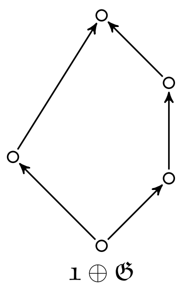
$\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{1}$



$\mathfrak{P}_3^*$

are actually Friedman-frames, since they can be obtained from $\mathfrak{P}_3$ by identifying all the final points and only two final points respectively.

Remark 24. A beautiful and well-known result of Citkin states that an intermediate logic L is *hereditarily structurally complete* if and only if the finite rooted frames of Figure 6.6 as well as the frame $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{1}$ and the following *penthagon* frame



$\mathbf{1} \oplus \mathfrak{G}$

[15] Notice that the frames $\mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1}$ and $\mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1}$ belong to $\boldsymbol{\mathcal{DF}_L}$. Indeed, just identify the final points of the L-frames $\mathfrak{F}_L(1)$ in order to get $\mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1}$, while by identifying all the points in the principal upset generated by the rightmost point of depth 2 of $\mathfrak{P}_3$ one gets $\mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1}$.

do not belong to $\boldsymbol{\mathcal{DF}_L}$[16]. Therefore, since $\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{1}$ is actually a Friedman frame, it follows that any Friedman logic is not hereditarily structurally complete. In particular, the Medvedev's logic **ML** is not hereditarily structurally complete. Notice moreover that also the frame $\mathbf{1}\oplus\mathfrak{G}$ belongs to $\boldsymbol{\mathcal{DF}_L}$ for L Friedman by Lemma 6.26.

Now, consider the following frame, called the *butterfly*,

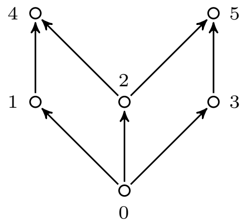


Figure 6.9: *The* butterly *frame* $\mathfrak{B}$*.*

Notice that $\mathfrak{B}\cong\frac{\mathfrak{P}_2}{2}\oplus\mathbf{1}$ and, consequently, $\mathfrak{B}$ does not belong to $\boldsymbol{\mathcal{DF}_{\mathbf{ML}}}$. Furthermore, we also have (cfr. Assertion 4.12 of [69])

Proposition 6.40. *The butterfly frame* $\mathfrak{B}$ *does not belong to* $\boldsymbol{\mathcal{DF}_L}$*.*

Proof. Suppose for reductio that $\mathfrak{B}\in\boldsymbol{\mathcal{DF}_L}$. Since $\mathfrak{B}$ is rooted and finite, it follows by Proposition 4.27 that there exists a p-morphism $h\colon\mathfrak{F}_L(n)\to\mathfrak{B}$ from the n-canonical frame $\mathfrak{F}_L(n)=\langle W_L,R_L,\mathcal{P}_L\rangle$ for L onto $\mathfrak{B}$, for some $n<\omega$. Let us consider the final point $a\in max(\mathfrak{F}_L(n))$ such that $col(a)=\langle 0,\dots,0\rangle$. Then h maps a onto a final point of $\mathfrak{B}$, say $h(a)=5$. Consider the set

$$A:=(h^{-1}(5)\cap max(\mathfrak{F}_L(n)))\setminus\{a\}$$

and, for each $i\in\mathfrak{B}$, define the sets U_i as follows:

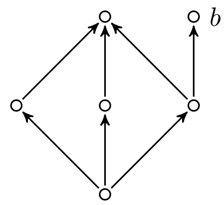
$$U_i:=h^{-1}(i)\setminus A{\downarrow}.$$

Then, each U_i is clopen in W_L and, for $j=0,3$, we have $U_j=\emptyset$. Indeed, if U_3 is non-empty, let $w\in U_3$ be maximal by Corollary 2.18. Then $max(w)\subseteq h^{-1}(5)\cap max(\mathfrak{F}_L(n))$. Since $w\notin A{\downarrow}$, it must be the case that $max(w)=\{a\}$, that is, a is the only maximal point seen by w. So, there exists also a point $v\in w{\uparrow}$ such that $v\lessdot a$. But then, by construction of $\mathfrak{F}_L(n)$, we have $col(v)\lhd col(a)$, which is a contradiction. Now, let $b\in max(\mathfrak{F}_L(n))$ be a point such that $h(b)=4$. Then, by the structure of $\mathfrak{F}_L(n)$, there exists a point $c\in W_L^{=2}$ such that $c\prec\{a,b\}$ and $h(c)=2$. Consequently, we have $c\in U_2$ and thus $U_j\neq\emptyset$ for $j=1,2$.
Now, consider the generated subframe $\mathfrak{F}_L(n)\setminus A{\downarrow}$ of $\mathfrak{F}_L(n)$. For each $i\in\mathfrak{B}$, U_i is also a clopen subset of $W_L\setminus A{\downarrow}$ and thus, by Corollary 2.18, we can choose two minimal points $u_j\in min(U_j)$ for $j=1,2$. Clearly, u_1 and u_2 are incomparable; moreover, there are no points in $\mathfrak{F}_L(n)\setminus A{\downarrow}$ which are below both u_1 and u_2.

[16]In Citkin's original paper [27], such a result is stated as a theorem without proof. In [142], Citkin's theorem is obtained as a corollary of an analogous result concerning hereditarily structurally complete modal logics extending **K4**. A direct, self-contained proof of Citkin's theorem, based on Esakia duality and the method of subframe formulas, can be found in [16].

Indeed, if $w \in W_L \setminus A{\downarrow}$ is such that wR_Lu_j for $j = 1, 2$, then $h(w) = 0$, hence $w \in U_0$, contrary to the fact that U_0 is empty. Thus, *a fortiori*, the frame $\mathfrak{F}_L(n) \setminus A{\downarrow}$ can not be rooted. However, since $A \subsetneq max(\mathfrak{F}_L(n))$, by Corollary 3.42 and Lemma 3.44, $\mathfrak{F}_L(n) \setminus A{\downarrow}$ is regular injective in $\boldsymbol{\mathcal{DF}_L}$ and so, being the retract of the rooted frame $\mathfrak{F}_L(n)$, $\mathfrak{F}_L(n) \setminus A{\downarrow}$ is also rooted. □

Finally, notice that the frame



does not belong to $\boldsymbol{\mathcal{DF}_L}$, since $a{\uparrow} \setminus b{\downarrow}$ has two minimums and thus it is not a **KP**-frame. Therefore, there are $\binom{13}{3}$ possible three elements antichains of elements from $W_L^{=2}(2)$ and those covering a point in $W_L^{=3}(2)$ are exactly the ones that belong to the following sets:

$$\begin{aligned}
\Psi_1 &= \{\{1,2,3\}\},\\
\Psi_2 &= \{\{a,4,5\} \mid a \in \{1,2,3,9\}\}\\
&\quad \cup \{\{a,6,7\} \mid a \in \{1,2,3,13\}\}\\
\Psi_3 &= \{\{a,b,10\} \mid a \in \{4,5\}, b \in \{6,7\}\}\\
&\quad \cup \{\{a,8,11\} \mid a \in \{4,5\}\}\\
&\quad \cup \{\{a,8,12\} \mid a \in \{6,7\}\}\\
&\quad \cup \{\{10,11,12\}\}.
\end{aligned}$$

Again, there can not be any point at depth 3 covered by a three-point antichain which is not in $\bigcup_{i=1}^{3} \Psi_i$, since it can be shown that the principal upset generated by such a point would be (reducible to) a frame that does not belong to $\boldsymbol{\mathcal{DF}_L}$. For each $A \in \Psi_1$, we have a point $x_A \in W_L^{=3}(2)$ such that $x_A \preceq A$ and $x_A{\uparrow} \cong \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{1}$; for each $A \in \Psi_2$, there is a point $x_A \in W_L^{=3}(2)$ such that $x_A \preceq A$ and $x_A{\uparrow} \cong \mathfrak{P}_3^*$ and, finally, for each $A \in \Psi_3$, there exists $x_A \in W_L^{=3}(2)$ such that $x_A \preceq A$ and $x_A{\uparrow} \cong \mathfrak{P}_3$.

Lemma 6.41. *For each $w \in \mathfrak{F}_L^{=3}(2)$, w has at most 3 immediate successors.*

Proof. Let $w \in \mathfrak{F}_L^{=3}(2)$ and suppose that $|w^{>}| = k > 3$. Let us distinguish various cases depending on $m = |max(w)|$. First notice that $m > 1$, since there are at most three distinct points of depth 2 with the same unique successor.

($m = 2$) Then $w^{>}$ could be either a subset of $\{1,2,3,4,5,9\}$ or a subset of $\{1,2,3,6,7,13\}$. Indeed, $w^{>} = \{1,2,3,8\}$ is not allowed, because $w{\uparrow} \setminus d{\downarrow}$ has three minimums and thus w refutes $\boldsymbol{kp}$. Without loss of generality, we can assume that $w^{>} \subseteq \{1,2,3,4,5,9\} := A$, since the resulting frame would be isomorphic. Now notice that, for each $\{m,8\} \subseteq U \subseteq A$ such that $|U| \geq 4$, where $m \in \{1,2,3\}$, it must be the case that $n \in U$, where $n \in \{4,5\}$, otherwise the resulting generated subframe would not be prefinally connected and thus would refute the Scott axiom $\boldsymbol{sa}$. But then, it can be easily seen that $w{\uparrow}$ would be reducible to the butterfly frame and, consequently, $w{\uparrow} \notin \boldsymbol{\mathcal{DF}_L}$, contrary to $w \in \mathfrak{F}_L^{=3}(2)$. So, the only possible choice for $w^>$ containing at least 4 points is as a subset of $\{1,2,3,4,5\}$ and thus it must contain at least two element of $\{1,2,3\}$ along with either 4 or 5. But then the resulting frame $w{\uparrow}$ refutes $\boldsymbol{kp}$, since $w{\uparrow} \setminus b{\downarrow}$ has at least two minimums.

($m = 3$) There are 3 distinct pair of final points of $w{\uparrow}$ and thus, by Proposition 4.27, each such pair A_i, $(i = 1,2,3)$ has to cover a point $x_{A_i} \in w^>$. Consequently, $w^>$ must contain at least another point u distinct from the x_{A_i}'s. Since u can see at most 2 points, there is a final point $v \in max(w)$ which is not seen by u. Moreover, for some i, $v \notin A_i$ and thus it follows that $\{u, x_{A_i}\} \subseteq w{\uparrow} \setminus v{\downarrow}$, that is, $w{\uparrow} \setminus v{\downarrow}$ has at least two minimums and therefore $w \not\models \boldsymbol{kp}$, contrary to the fact that $w{\uparrow}$ is frame for $L \supseteq \mathbf{KP}$.

($m = 4$) In this case, due to Proposition 4.27, we have $\{n, m, 8, 10, 11, 12\} \subseteq w^>$, for $n \in \{4,5\}$ and $m \in \{6,7\}$. But then we have $|min(w{\uparrow} \setminus d{\downarrow})| \geq 3$ and consequently $w \not\models \boldsymbol{kp}$, contrary to the fact that $w{\uparrow}$ is frame for $L \supseteq \mathbf{KP}$.

Therefore we conclude that there is no point $w \in \mathfrak{F}_L(2)$ of depth 3 with strictly more than 3 immediate successors. □

By the previous lemma, we have completely determined the points at depth 3 of the 2-canonical Friedman frame $\mathfrak{F}_L(2)$, whose principal upsets are depicted in Figure 6.10. In particular, one can show that there are exactly 48 points in $\mathfrak{F}_L^{=3}(2)$.

Someone could wonder whether Lemma 6.41 can be somehow generalized in order to cover the case of points at arbitrary depth and of arbitrary n-canonical frames. There seems to be no method for doing this task. For instance, notice that there are points of depth 3 in the n-canonical frame $\mathfrak{F}_L(n)$ that can have $2^n - 1$ immediate successors, since, by Lemma 6.26, the frame $\mathfrak{1} \oplus (\mathfrak{2}^{\mathfrak{n}} - \mathfrak{1}) \oplus \mathfrak{1}$ belongs to $\boldsymbol{\mathcal{DF}}_{\mathbf{ML}} \subseteq \boldsymbol{\mathcal{DF}_L}$. Still, we can bound the cardinality of the set of the immediate successors of some particular points using a technique of [110].

A frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ is said to be *everywhere branching* if every point $w \in W \setminus max(W)$ has at least two immediate successors, or, equivalently, if no point in W has only one immediate successor.

Lemma 6.42. *Let $\mathfrak{F} = \langle W, R \rangle$ be a finite rooted everywhere branching frame and assume that $h \colon \mathfrak{P}_j \to \mathfrak{F}$ is a p-morphism from the Medvedev frames $\mathfrak{P}_j$ onto $\mathfrak{F}$, for some $j \in \omega$. Then, for each $v \in W$, there exists $V \in \mathfrak{P}_j$ such that $h(V) = v$ and $|V| \leq 2^{d(v)-1}$.*

Proof. Consider an arbitrary point $v \in W$. We proceed by induction on $d(v)$.

($d(v) = 1$) Since h is onto, there exists $U \in \mathfrak{P}_i$ such that $h(U) = v$. Then, for any $V \in max(U)$, we have $h(V) = v$ and $|V| = 1 \leq 2^{d(u)-1}$.

($d(v) = n+1$) Let $U \in \mathfrak{P}_i$ be such that $h(U) = v$. Then the restriction $h\restriction_{U\uparrow} \colon \mathfrak{P}_k \to \mathfrak{F}_v$ of h to $U{\uparrow}$ is a p-morphism from a Medvedev's frame $\mathfrak{P}_k$ onto the finite rooted everywhere branching frame $\mathfrak{F}_v$, for some $k \leq j$. Let v_1 and v_2 be immediate successors of v. By induction hypothesis, there

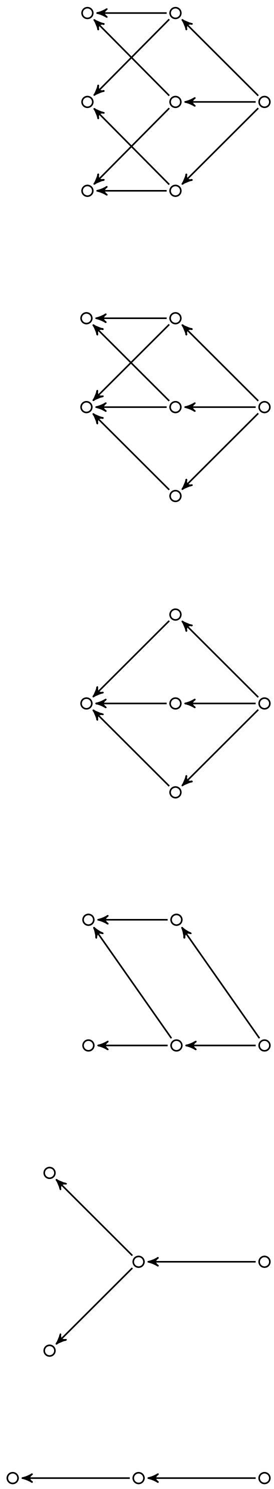


Figure 6.10: *The frames of depth 3 of the 2-canonical Friedman frame* $\mathfrak{F}_L(2)$.

exist $U_1, U_2 \in \mathfrak{P}_k$ such that $h(U_i) = v_i$ and $|U_i| \leq 2^{n-1}$ for $i \in \{1,2\}$. Then, for $V = U_1 \cup U_2 \subseteq U$, we have, for $i \in \{1,2\}$,

$$v = h(U)R\,h(V)R\,h(U_i) = v_i,$$

and, since the v_i's are immediate successors of v, it must be the case that $h(V) = v$ and $|V| \leq |U_1| + |U_2| \leq 2^n = 2^{d(v)-1}$. □

Corollary 6.43. *Let the finite rooted everywhere branching frame $\mathfrak{F} = \langle W, R\rangle$ be a p-morphic image of a Medvedev frames $\mathfrak{P}_j$ for some $j \in \omega$. Then, for all $v \in W$, $|v^>| \leq 2^{d(v)-1}$.*

Proof. Suppose not, that is, there exists $v \in W$ such that $|v^>| = k > 2^{d(v)-1}$. By the previous lemma, v is the image (under some p-morphisms h) of a point V in $\mathfrak{P}_j$ such that $|V| \leq 2^{d(v)-1}$. Since h is a p-morphism, V must have at least k immediate successors, but there are only $\binom{|V|}{|V|-1} = |V| < k$ of them. □

Now, for any $n < \omega$, consider the n-canonical frame $\mathfrak{F}_L(n)$ for L Friedman. We say that a point $w \in \mathfrak{F}_L(n)$ is *everywhere branching* if the generated subframe $w\!\uparrow\, \subsetneq \mathfrak{F}_L(n)$ is everywhere branching. Then, relying on Corollary 6.47 of the following Section 6.4.1, we can prove the following

Proposition 6.44. *Let $w \in \mathfrak{F}_L^{<\infty}(n)$ be an everywhere branching point at finite depth of the n-canonical frame $\mathfrak{F}_L(n)$ $(n < \omega)$ for a Friedman logic L. Then $|w^>| \leq 2^{d(w)-1}$.*

Proof. Since $w\!\uparrow\, \in \boldsymbol{\mathcal{DF}_{\mathbf{ML}}}$ by Corollary 6.47, the frame formula $\beta(w\!\uparrow, \mathfrak{D}^\natural, \bot)$ does not belong to **ML** and, consequently, $\mathfrak{P}_j \not\models \beta(w\!\uparrow, \mathfrak{D}^\natural, \bot)$ for some $j \in \omega$. Thus $w\!\uparrow$ is a p-morphic image of some Medvedev frame $\mathfrak{P}_k$ for some $k \leq j$. Since $w\!\uparrow$ is a finite rooted everywhere branching frame, the result follows from Corollary 6.43. □

Finally, notice that the proof of Proposition 6.23 can be repeated words by words in order to prove that

Proposition 6.45. *For every $n < \omega$, the Medvedev frame $\mathfrak{P}_n$ is regular injective in $\boldsymbol{\mathcal{DF}_L}$.*

6.4.1 Is Medvedev's logic the only Friedman logic?

In the previous sections we have seen that any Friedman logic L coincides with Medvedev's logic **ML** on a large class of formulas. Furthermore, the analysis on the canonical L-frames has not revealed any reason to think that L has to be different from **ML** and that's why we align ourselves with Grigolia [69] in making the following

Conjecture. Medvedev's logic **ML** is the only Friedman logic.

There are many ways one could possibly show that $\mathbf{ML} \subseteq L$, for L Friedman. For instance, one could prove that L has to be negatively stable. But that is easier said than done, since there is actually no clue on how to define an essentially negative substitution σ such that $\sigma(\varphi) \notin L$ for any $\mathcal{L}$-formula $\varphi \notin L$. We think that the most straightforward way to prove the conjecture is to rely on the apparatus of canonical formulas. Indeed, by Theorem 4.17, if we show that, for every finite rooted frame $\mathfrak{F}$ and set of antichains $\mathfrak{D}$ in $\mathfrak{F}$,

$$\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin L \Longrightarrow \beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin \mathbf{ML}, \qquad (\star)$$

then we are done. Let us see what are the intricacies of this approach.

Suppose that $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$. Without loss of generality, we shall assume that

(i) $\mathfrak{D}$ has to be non-empty (since **ML** enjoys the disjunction property);

(ii) $|max(\mathfrak{F})| > 1$ (by Lemma 6.26);

(iii) $\mathfrak{F}$ is not of the form $\mathfrak{G} \oplus \mathfrak{K}$ for some finite rooted frames $\mathfrak{G}, \mathfrak{K}$ (by Proposition 6.29).

Furthermore, we can also assume that $(\star)$ holds for all finite frames $\mathfrak{G}$ and set of antichains $\mathfrak{E}$ in $\mathfrak{G}$ such that $d(\mathfrak{G}) < d(\mathfrak{F})$ as inductive hypothesis.

Now, from the assumption, it follows that $\beta(\mathfrak{F}_{a_i}, \mathfrak{D} \restriction_{\mathfrak{F}_{a_i}}, \bot) \notin L$ for each $i \in \{1, \ldots, k\}$, where $\{a_i\}_{i \in \{1,\ldots,k\}}$, $k \geq 2$, is the set of immediate successor of $\mathfrak{F}$'s root. Then, by induction hypothesis, it follows that the canonical formula $\beta(\mathfrak{F}_{a_i}, \mathfrak{D} \restriction_{\mathfrak{F}_{a_i}}, \bot) \notin \mathbf{ML}$ for each $i \in \{1, \ldots, k\}$ and, in particular, there are globally cofinal subreductions $f_{a_i} \colon \mathfrak{P}_{j_i} \to \mathfrak{F}_{a_i}$ satisfying (CDC) for $\mathfrak{D} \restriction_{\mathfrak{F}_{a_i}}$ for some Medvedev frame $\mathfrak{P}_{j_i}$. Moreover, there is also a globally cofinal subreduction $h \colon \mathfrak{F}_L(k) \to \mathfrak{F}$ of the k-canonical frame for L to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ for some $k \in \omega$. Now, the task should be that of constructing a cofinal subreduction f of some Medvedev frame $\mathfrak{P}_m$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ using the f_{a_i}'s as building blocks. Actually one can show that there is a cofinal subreduction $j \colon \mathfrak{P}_m \to \mathfrak{F}$ satisfying (CDC) for $\bigcup_{i \in \{1,\ldots,k\}} \mathfrak{D} \restriction_{\mathfrak{F}_{a_i}}$ but there is no clue on how to deal with closed domains $\mathfrak{d} \in \mathfrak{D} \setminus \bigcup_{i \in \{1,\ldots,k\}} \mathfrak{D} \restriction_{\mathfrak{F}_{a_i}}$. Of course one should look at h and try to get some informations on it but there seems to be no way to make any progress. The problem is that the f_{a_i}'s are completely unrelated to h. Actually one can show that f and h coincide on the maximal elements (which implies that closed domains $\mathfrak{d} \subseteq max(\mathfrak{F})$ are taken care of), but nothing more.

Of course, the situation would be different, if the globally cofinal subreduction h were induced by a projective unifier σ of the antecedent of $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$: in such a scenario, the inductive hypothesis would ensure the existence of points $x_{a_i} \in \mathfrak{F}_{\mathbf{ML}}(|\mathfrak{F}|)$ such that $x_{a_i} \in h^{-1}(a_i)$ for each $i \in \{1, \ldots, k\}$ and, consequently, any point $x \in \bigcap_{i \in \{1,\ldots,k\}} x_{a_i}{\downarrow} \cap \mathfrak{F}_{\mathbf{ML}}(|\mathfrak{F}|)$, which exists since $\mathfrak{F}_{\mathbf{ML}}(|\mathfrak{F}|)$ is rooted, would be mapped to the root of $\mathfrak{F}$ because of the fact that h satisfies (CDC) for $\mathfrak{D}$ and that $\{a_i\}_{i \in \{1,\ldots,k\}}$ can be assumed to belong to $\mathfrak{D}$.

It seems that the key to make progress is to find a way to relate the subreductions in **ML** with that of L. For instance, notice that the following condition would be sufficient in order to prove the conjecture:

(¿) for every globally cofinal subreduction $h \colon \mathfrak{F}_L(k) \to \mathfrak{F}$ of the k-canonical frame for L to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ for some $k \in \omega$, the restriction of h to $\mathfrak{F}_{\mathbf{ML}}(k)$ is still a cofinal subreduction onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.

Let us try to prove (¿). Let $h \colon \mathfrak{F}_L(k) \to \mathfrak{F}$ be a globally cofinal subreduction satisfying (CDC) for $\mathfrak{D}$ for some $k \in \omega$. Define h' to be the restriction of h to $\operatorname{dom} h \setminus \bigcup_{b\in\mathfrak{F}\setminus a\uparrow} h^{-1}(b)$, where a is an immediate successor of $\mathfrak{F}$'s root. Then $h'\colon \mathfrak{F}_L(k) \to \mathfrak{F}_a$ is a cofinal subreduction and, by the refutability criterion for canonical formulas, we have $\mathfrak{F}_L(k) \not\models \beta(\mathfrak{F}_a, \mathfrak{D}\restriction_{\mathfrak{F}_a}, \bot)$, under the valuation $\mathfrak{V}(p_j) = W_L \setminus h'^{-1}(a_j)$. Since $\mathfrak{V}(p_j) = \mathfrak{V}_L(\sigma(p_j))$, where $\mathfrak{V}_L$ is the canonical valuation and σ the substitution associated to h, it follows that the substitution instance $\sigma(\beta(\mathfrak{F}_a, \mathfrak{D}\restriction_{\mathfrak{F}_a}, \bot)) \notin L$. Being L structurally complete, there exists a substitution τ such that $\tau(\sigma(\psi)) \in L$ and $\tau(\sigma(p_a)) \notin L$, where ψ is the antecedent of the canonical formula $\beta(\mathfrak{F}_a, \mathfrak{D}\restriction_{\mathfrak{F}_a}, \bot)$. In particular, by considering the p-morphism $h_\tau\colon \mathfrak{F}_L(s) \to \mathfrak{F}_L(k)$ induced by τ, we have that $h\circ h_\tau\colon \mathfrak{F}_L(s) \to \mathfrak{F}_a$ is a globally cofinal subreduction satisfying (CDC) for $\mathfrak{D}\restriction_{\mathfrak{F}_a}$. By induction hypothesis, the restriction of $h \circ h_\tau$ to $\mathfrak{F}_{\mathbf{ML}}(s)$ is again a cofinal subreduction onto $\mathfrak{F}_a$ satisfying (CDC) $\mathfrak{D}\restriction_{\mathfrak{F}_a}$. Hence, by denoting by $r_{\mathbf{ML}}$ the root of $\mathfrak{F}_{\mathbf{ML}}(s)$, it follows that $h_\tau(r_{\mathbf{ML}})$ is a point in $\mathfrak{F}_{\mathbf{ML}}(k)$ such that $h_\tau(r_{\mathbf{ML}}) \in h^{-1}(a)$. But now we face some problems.

Firstly, what if a is the unique successor of $\mathfrak{F}$'s root? In such a case we can assume without loss of generality (recall that **ML** is finitely approximable) that there exists a point at finite depth $x \in \mathfrak{F}_{\mathbf{ML}}^{<\infty}$ such that $x \prec y$ and $y \in h^{-1}(a)$. Now, if x is not mapped by h to the root of $\mathfrak{F}$, we can extend h to a globally cofinal subreduction h' that maps x to $\mathfrak{F}$'s root, being $\{x\}$ clopen. So we still get a globally cofinal subreduction from $\mathfrak{F}_{\mathbf{ML}}(k)$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ (hence $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin \mathbf{ML}$), but the claim (¿) is not proved, since h' is different from h.

Secondly, even if $\mathfrak{F}$'s root a_0 has more that one immediate successor, say $a_0^{>} = \{a_1, \ldots, a_n\}$, it could be the case that $a_0^{>} \notin \mathfrak{D}$. In such a case, there is no guarantee that a point $x \in \mathfrak{F}_{\mathbf{ML}}(k) \cap \bigcap_{i=1}^{n} h^{-1}(a_i){\downarrow}$ (which exists, since $\mathfrak{F}_{\mathbf{ML}}(k)$ is rooted) belongs also to $h^{-1}(a_0)$. Consequently, again, we can still get a globally cofinal subreduction from $\mathfrak{F}_{\mathbf{ML}}(k)$ onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ (hence $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \notin \mathbf{ML}$), but we can not claim to have proved (¿).

So there seems to be no easy way to prove (¿) unless some further conditions on the globally cofinal subreduction h are imposed. Nevertheless, taking into consideration the previous remarks, we can prove the following

Proposition 6.46. *Let $\mathfrak{F}$ be a finite rooted everywhere branching frame and let h be a globally cofinal subreduction from the k-canonical frame $\mathfrak{F}_L(k)$ for L, for some $k \in \omega$, onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ where $\{a^{>} \mid a \in \mathfrak{F}\} \subseteq \mathfrak{D}$. Then the restriction of h to $\mathfrak{F}_{\mathbf{ML}}(k)$ is still a cofinal subreduction onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.*

Proof. Let us denote by a_0 the root of $\mathfrak{F}$ and proceed by induction on $d(\mathfrak{F})$.

($d(\mathfrak{F}) = 2$) Notice that if $|max(\mathfrak{F})| > 2$, then $\mathfrak{D} = \{a_0^{>}\}$. Otherwise, let $\mathfrak{D} \ni \mathfrak{d} \subset max(\mathfrak{F})$ be different from $a_0^{>}$ and, for each $d \in \mathfrak{d}$, pick a maximal point $x_d \in max(\mathfrak{F}_L(k))$ such that $x_d \in h^{-1}(d)$. Since the Medvedev frame $x\uparrow$ whose maximal points are the x_d's is a generated subframe of $\mathfrak{F}_L(k)$, we have $h(x\uparrow) = \mathfrak{d}\uparrow$, contrary to the fact that h satisfies (CDC) for $\mathfrak{D}$. So, the only closed domain is $a_0^{>}$ and, reasoning as above, there exists a point $x \in \mathfrak{F}_{\mathbf{ML}}(k)$ such that $h(x) = a_0$. In particular, the restriction of h to $\mathfrak{F}_{\mathbf{ML}}(k)$ is still a cofinal subreduction onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.

($d(\mathfrak{F}) = n+1$) Let $a_0^{>} = \{b_1, \ldots, b_n\}$, for $n > 1$. Reasoning as in the case of (¿) above, we can find points $x_{b_i} \in \mathfrak{F}_{\mathbf{ML}}(k) \cap h^{-1}(b_i)$ for each $i \in \{1, \ldots, n\}$.

Since $\mathfrak{F}_{\mathbf{ML}}(k)$ is rooted, there exists a point $x \in \mathfrak{F}_{\mathbf{ML}}(k)$ such that

$$x \in \bigcap_{i=1}^{n} b_i{\downarrow}.$$

But $x \in \mathrm{dom} h{\uparrow}$ and $h(x{\uparrow}) \supseteq a_0^{>}{\uparrow}$, consequently $x \in h^{-1}(a_0)$ by (CDC). Thus the restriction of h to $\mathfrak{F}_{\mathbf{ML}}(k)$ is a cofinal subreduction onto $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. □

Corollary 6.47. *Let L be a Friedman logic. For every $k \in \omega$, the everywhere branching points of finite depth of $\mathfrak{F}_L(k)$ coincide with those of $\mathfrak{F}_{\mathbf{ML}}(k)$, that is:*

$$\mathcal{B}(\mathfrak{F}_L^{<\infty}(k)) = \mathcal{B}(\mathfrak{F}_{\mathbf{ML}}^{<\infty}(k)),$$

where $\mathcal{B}(A) = \{x \in A \mid x \text{ is everywhere branching}\}$.

Proof. Since $\mathfrak{F}_{\mathbf{ML}}(k) \subsetneq \mathfrak{F}_L(k)$, the inclusion $\supseteq$ is clear. So, let $x \in \mathcal{B}(\mathfrak{F}_L^{<\infty}(k))$. Then $x{\uparrow} \in \boldsymbol{\mathcal{DF}_L}$ and, consequently, the frame formula $\beta(x{\uparrow}, \mathfrak{D}^{\natural}, \bot) \notin L$. Then, by Proposition 4.27 , there exists a globally cofinal subreduction $h\colon \mathfrak{F}_L(s) \to x{\uparrow}$ satisfying (CDC) for $\mathfrak{D}^{\natural}$. Since $x{\uparrow}$ is a finite rooted everywhere branching frame and $\{a^{>} \mid a \in x{\uparrow}\} \subseteq \mathfrak{D}^{\natural}$, by the previous Proposition the restriction of h to $\mathfrak{F}_{\mathbf{ML}}(s)$ is still a cofinal subreduction onto $x{\uparrow}$ satisfying (CDC) for $\mathfrak{D}^{\natural}$. Hence, by the refutability criterion for canonical formulas, $\beta(x{\uparrow}, \mathfrak{D}^{\natural}, \bot) \notin \mathbf{ML}$ and $x{\uparrow} \in \boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$. Therefore, we conclude that $x \in \mathcal{B}(\mathfrak{F}_{\mathbf{ML}}^{<\infty}(k))$. □

6.5 Some remarks on the decidability of the n-letter fragment of Medvedev's logic

The issue of the decidability of Medvedev's logic **ML** is a major open problem in the field of intermediate logics. The only direct try to settle the question seems to be the one made by Dov Gabbay in [55]: in §6 he outlined a decidability proof for **ML**, which, however, has been shown to be wrong by Skvortsov in [147]. Since **ML** is finitely approximable by definition, to provide a recursive axiomatization of **ML** would imply the decidability of **ML**. Still, very little is known on the matter and the only significative result is the one by Maksimova, Shetman and Skvortsov in [110], namely that **ML** is not finitely axiomatizable[17].

Recently, Ciardelli [24], rediscovering some interesting results of Miglioli *et al.* [119], has proposed a new line of investigation on the issue of the decidability of **ML** which relies on Levin's characterization of Medvedev's logic. Let us investigate the issue more deeply.

Definition 6.8. An essentially negative substitution $\sigma\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$ is said to be *m-simple* ($m \in \omega$) if

$$\sigma(p_i) = \bigvee_{j \in m} \neg p_j^i, \qquad \text{for every } p_i \in \mathbf{Var}\mathcal{L}_n,$$

[17] Interestingly enough, Skvortsov [147] showed that the so-called *logic of infinite problems* $\mathbf{ML}^{\infty} = \mathrm{Log}\, \mathfrak{P}_\omega$, where $\mathfrak{P}_\omega = \langle \mathcal{P}(\omega) \setminus \{\emptyset\}, \supseteq \rangle$, is a recursively axiomatizable intermediate logic included in **ML**. Therefore, one could try to prove the equality $\mathbf{ML}^{\infty} = \mathbf{ML}$ in order to prove the decidability of Medvedev's logic.

and $p^i_k \neq p^j_l$ for every $k, l \in m$ and $i \neq j$. Moreover, an essentially negative substitution $\sigma\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$ is said to be *simple* if σ is m-simple for some $m \in \omega$.

A simple $\mathcal{EN}$-substitution is thus completely determined by the number m of distinct propositional variables occurring in each disjunct and clearly, for $i \leq j$, a j-simple substitution $\sigma\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$ is more general than a i-simple substitution $\tau\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$. Furthermore, notice the following easy

Lemma 6.48. *For every $\mathcal{EN}$-substitution $\tau\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$, there exists a simple $\mathcal{EN}$-substitution σ that is more general than τ modulo* **ND***-equivalence.*

Proof. For every $p_i \in \mathbf{Var}\mathcal{L}_n$, $\tau(p_i)$ is equivalent to some disjuctive formula $\bigvee_{j\in k_i} \neg\xi^i_j$, for some finite $k_i \in \omega$, by Lemma 6.11. Let $k = max\{k_i\}_{i\in n}$ and consider a k-simple substitution $\sigma\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$. Let $\theta\colon \mathbf{Var}\mathcal{L} \to \mathbf{For}\mathcal{L}$ be the substitution defined as follows:

$$\theta(p^i_j) = \begin{cases} \xi^i_j & \text{if } j \in k_i, \\ \top & \text{otherwise.} \end{cases}$$

Then $\theta \circ \sigma(p_i) \leftrightarrow \tau(p_i) \in \mathbf{ND}$, for every $p_i \in \mathbf{Var}\mathcal{L}_n$, that is $\tau \preceq_{\mathbf{ND}} \sigma$. □

As an immediate consequence of the previous lemma, Levin's characterization of Medvedev's logic can be refined as follows:

$$\mathbf{ML} = \{\varphi(p_0, \ldots, p_{n-1}) \in \mathbf{For}\mathcal{L} \mid \sigma(\varphi) \in \mathbf{KP}, \quad \forall \text{ simple } \sigma\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}\}.$$

Hence, the following question comes to mind:

(?) is it possible to find, for every $n \in \omega$, a bound $k(n)$ such that, given any formula $\varphi \in \mathbf{For}\mathcal{L}_n$, $\varphi \in \mathbf{ML}$ if and only if $\sigma(\varphi) \in \mathbf{KP}$ for a $k(n)$-simple substitution $\sigma\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$?

Of course a positive answer to the previous question would provide a decision algorithm for Medvedev's logic based on the decidability of **KP** and this is in fact the line of investigation proposed by Ciardelli. Furthermore, both [24] and [119] proved that for $n = 1$ such a bound exists and it is equal to 2. However, a 2-simple substitution is not enough to deal with formulas with two variables, since, as shown in [119, p. 556], there exists a $\mathcal{L}$-formula $\varphi(p_0, p_1) \notin \mathbf{ML}$ such that $\varphi(\neg p^0_0 \vee \neg p^0_1, \neg p^1_0 \vee \neg p^1_1) \in \mathbf{KP}$.

In order to understand what is going on, let us see in detail why a 2-simple substitution $\sigma\colon \mathbf{Var}\mathcal{L}_1 \to \mathcal{EN}$ suffices to deal with formulas in $\mathbf{For}\mathcal{L}_1$, while a 1-simple substitution is not enough. First, notice that (?) can be equivalently restated in topological terms as follows:

(?*) is it possible to find, for every $n \in \omega$, a bound $k(n)$ such that the continuous p-morphism $h_\sigma\colon \mathfrak{F}_{\mathbf{ML}}(k(n) \cdot n) \to \mathfrak{F}_{\mathbf{ML}}(n)$, induced by a $k(n)$-simple substitution $\sigma\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$, is onto?

Consider the 1-simple substitution

$$\sigma_1\colon \mathbf{Var}\mathcal{L}_1 \to \mathcal{EN}$$
$$p_1 \mapsto \neg p_1$$

and let $h_{\sigma_1}\colon \mathfrak{F}_{\mathbf{ML}}(1) \to \mathfrak{F}_{\mathbf{ML}}(1)$ be the induced p-morphism. As shown in the following Figure 6.11, h_{σ_1} is not surjective, since the point d cannot be in the range of h_{σ_1}. Indeed, if $z \in h_{\sigma_1}^{-1}(d)$, then $z \in 2{\downarrow}$, since $a \notin \mathfrak{V}(p_1) = \{b\}$; moreover, since h_{σ_1} is a p-morphism, there exists $k \in z{\uparrow}$ such that $k \in h_{\sigma_1}^{-1}(b)$ and, consequently, $k \notin 2{\downarrow}$. Thus, $z \in \{3, 5\}$ and, by the monotonicity of h_{σ_1}, we have $d \leq a$, which is a contradiction.

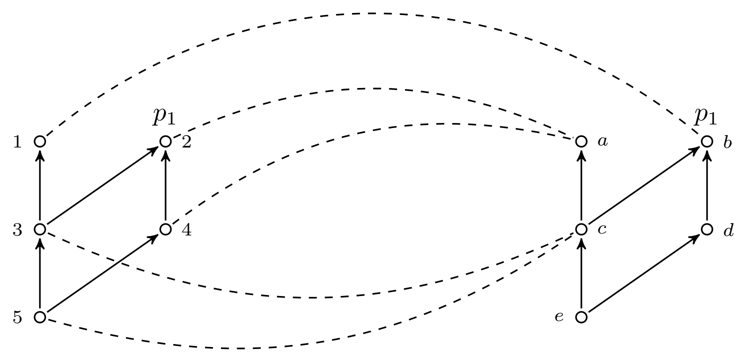


Figure 6.11: *The dashed lines represent the p-morphism* $h_{\sigma_1}\colon \mathfrak{F}_{\mathbf{ML}}(1) \to \mathfrak{F}_{\mathbf{ML}}(1)$.

Now, consider instead the 2-simple substitution

$$\begin{aligned}\sigma_2\colon \mathbf{Var}\mathcal{L}_1 &\to \mathcal{EN}\\ p_1 &\mapsto \neg p_1 \vee \neg p_2\end{aligned}$$

and let $h_{\sigma_2}\colon \mathfrak{F}_{\mathbf{ML}}(2) \to \mathfrak{F}_{\mathbf{ML}}(1)$ be the induced p-morphism. In particular, notice that we have

$$\begin{aligned}h_{\sigma_2}^{-1}(\mathfrak{V}(p_1)) &= \mathfrak{V}(\sigma(p_1))\\ &= W_{\mathbf{ML}}(2) \setminus (\mathfrak{V}(p_1){\downarrow} \cap \mathfrak{V}(p_2){\downarrow}).\end{aligned}$$

By the finiteness of $\mathfrak{F}_{\mathbf{ML}}(1)$ it is not hard to prove that h_{σ_2} is surjective, as shown in Figure 6.12. In this scenario, the point d can indeed be reached by h_{σ_2}: since $7 \prec \{2, 3\}$, we have that $7 \in \mathfrak{V}(p_1){\downarrow} \cap \mathfrak{V}(p_2){\downarrow}$ and $h_{\sigma_2}(7) \preceq \{h_{\sigma_2}(2), h_{\sigma_2}(3)\} = \{a\}$. Furthermore, $h_{\sigma_2}(7) \neq a$, since otherwise, $7 \in h_{\sigma_2}^{-1}(a) = h_{\sigma_2}^{-1}(\mathfrak{V}(p_1))$ and, consequently, $7 \notin \mathfrak{V}(p_1){\downarrow} \cap \mathfrak{V}(p_2){\downarrow}$, which is a contradiction. Therefore, the only point that 7 can be mapped to is d.

What we have discovered from the previous analysis is the fact that difficulties arise when points like d are concerned, that is, points that have a single immediate successor. Let us call such points *linear points.* Moreover, when we are dealing with the n-canonical $\mathfrak{F}_{\mathbf{ML}}(n)$ for $n \geq 2$, there are other particular points that need specific consideration, that is, points that have the same immediate successors, like 6 and 6∗ in the 2-canonical frame $\mathfrak{F}_{\mathbf{ML}}(2)$ of Figure 6.12. We call such points *twin points.* Finally we say that a point is *critical* if it is either a linear point or a twin point.

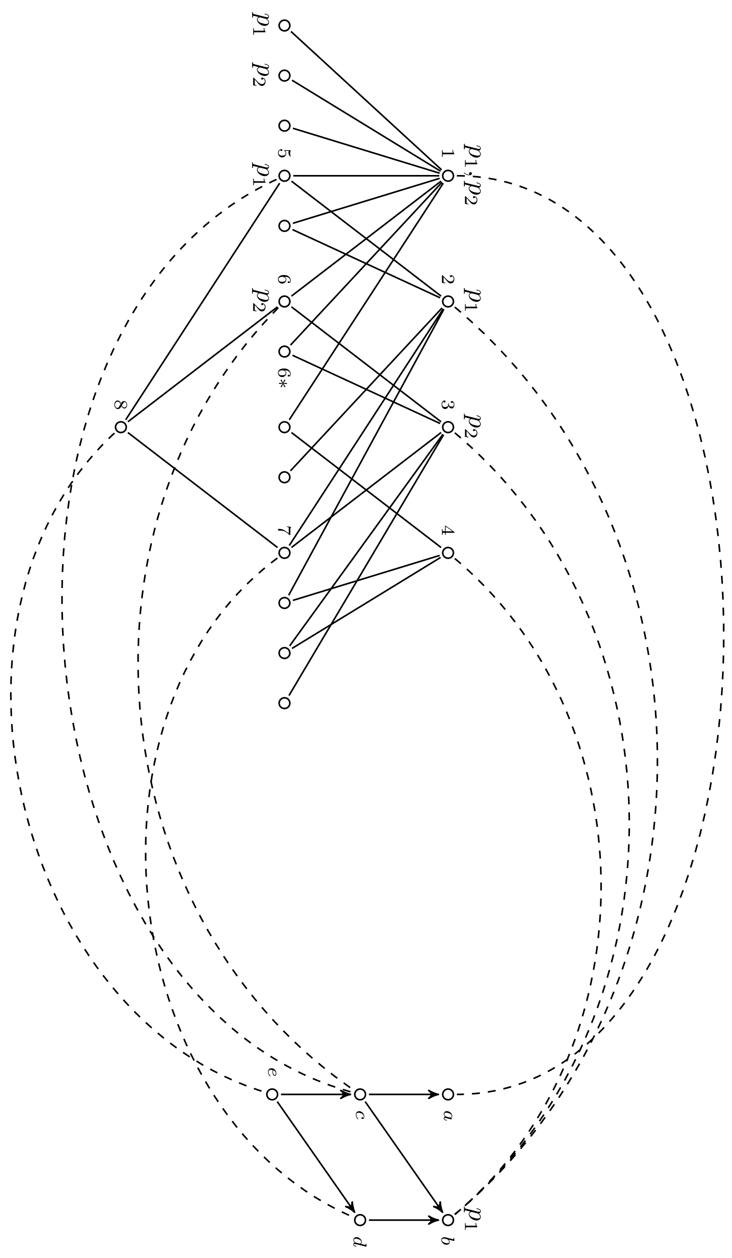


Figure 6.12: *A partial representation of the onto p-morphism* $h_{\sigma_2} : \mathfrak{F}_{\mathbf{ML}}(2) \to \mathfrak{F}_{\mathbf{ML}}(1)$.

Now consider the 2-canonical frame $\mathfrak{F}_{\mathbf{ML}}(2)$ for Medvedev's logic. Following the analysis of the n-canonical Friedman frames of §6.4, we have seen that the maximal depth of a linear point $x \in \mathfrak{F}_{\mathbf{ML}}(2)$ is 4, namely, $x\!\uparrow\, \cong \mathfrak{F}_{\mathbf{ML}}(1) \oplus \mathbf{1}$, while the maximal depth of a twin point is 3. Since the maximal depth of a linear point in $\mathfrak{F}_{\mathbf{ML}}(1)$ is 2 and a 2-simple substitution was enough to cover it, let us see if a 4-simple substitution σ induces an onto p-morphism h_σ between n-canonical frames for **ML**.

Let $\sigma\colon \mathbf{Var}\mathcal{L}_2 \to \mathcal{EN}$ be the 4-simple substitution defined as follows

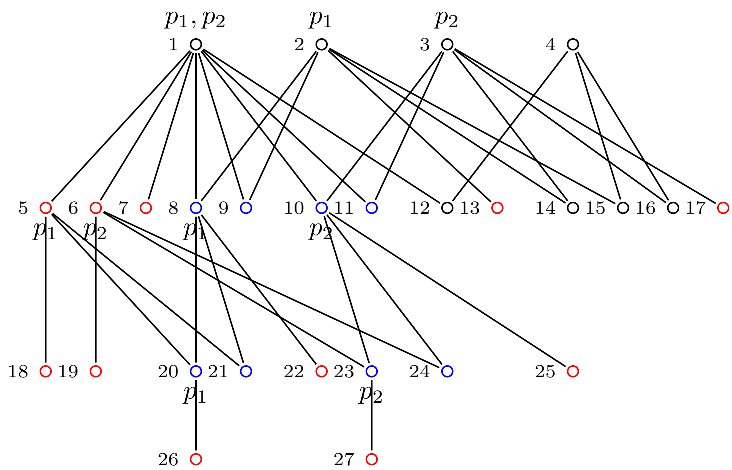
$$\sigma(p_1) = \bigvee_{i\in\{1,3,5,7\}} \neg p_i, \qquad\qquad \sigma(p_1) = \bigvee_{i\in\{2,4,6,8\}} \neg p_i,$$

and let $h_\sigma\colon \mathfrak{F}_{\mathbf{ML}}(8) \to \mathfrak{F}_{\mathbf{ML}}(2)$ be the induced p-morphism for which the following equalities hold:

$$h_\sigma^{-1}(\mathfrak{V}(p_1)) = W_{\mathbf{ML}}(8) \setminus \bigcap_{i\in\{1,3,5,7\}} \mathfrak{V}(p_i)\!\downarrow \tag{1}$$

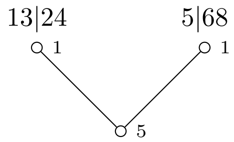
$$h_\sigma^{-1}(\mathfrak{V}(p_2)) = W_{\mathbf{ML}}(8) \setminus \bigcap_{i\in\{2,4,6,8\}} \mathfrak{V}(p_i)\!\downarrow. \tag{2}$$

Consider the following partial picture of $\mathfrak{F}_{\mathbf{ML}}(2)$ where all the critical points are depicted (the linear points are red and the twin points blue).

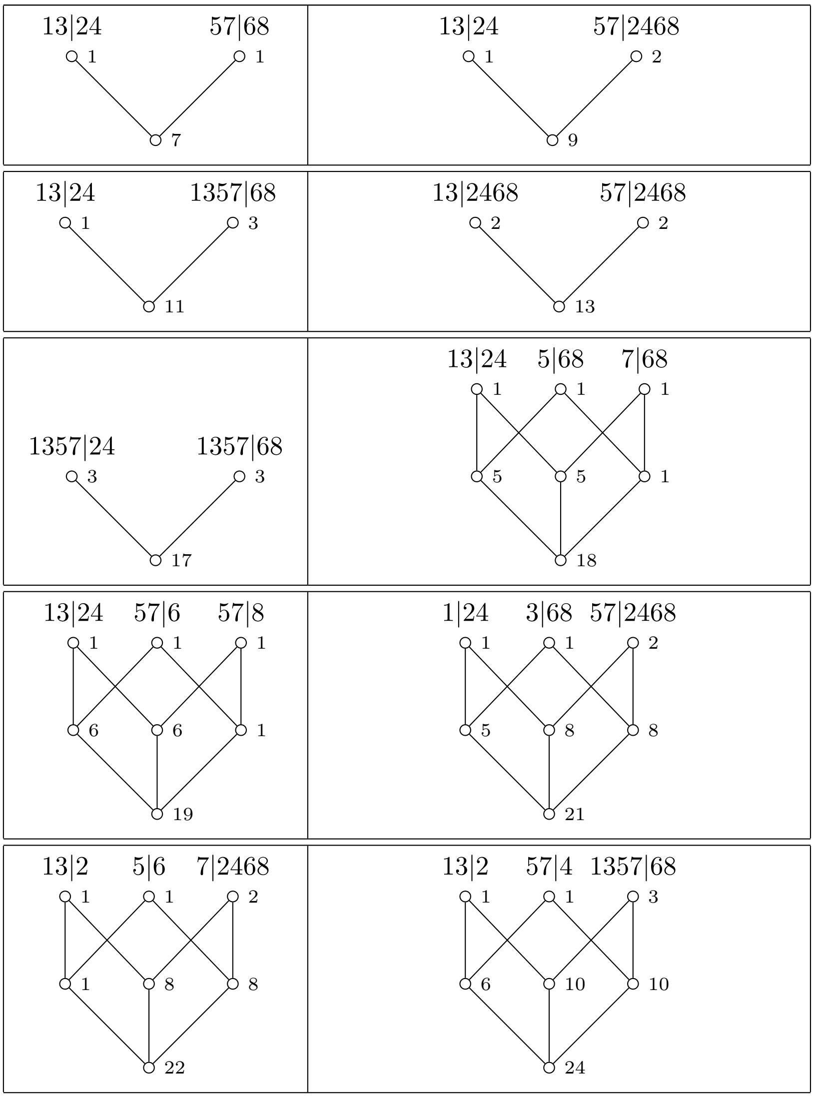


We are now going to prove that every critical point of $\mathfrak{F}_{\mathbf{ML}}(2)$ is in the range of h_σ. In particular, we show that, for every critical point $x \in \mathfrak{F}_{\mathbf{ML}}(2)$, there exists a Medvedev frame $\mathfrak{P}_j \subsetneq \mathfrak{F}_{\mathbf{ML}}(8)$ such that $h_\sigma(\mathfrak{P}_j) = x\!\uparrow$.

We proceed in the following manner. For every critical point $x \in \mathfrak{F}_{\mathbf{ML}}(2)$, we draw the corresponding Medvedev frame $\mathfrak{P}_j^x$ that is purported to show the claim: the label at the right of each point $U \in \mathfrak{P}_j^x$ expresses the point in $\mathfrak{F}_{\mathbf{ML}}(2)$ on which U is mapped to by h_σ, while the label above each point $U \in max(\mathfrak{P}_j^x)$ expresses the set $J \subseteq \{1,\ldots,8\}$ such that $U \in \mathfrak{V}(p_j)$ for each $j \in J$. For instance, the case $x = 5$ can be shown simply like this:



Indeed, the root of the above Medvedev frame (denote it by r) belongs to the set $\bigcap_{i\in\{2,4,6,8\}} \mathfrak{V}(p_i)\!\downarrow$ and thus $h_\sigma(r) \notin \mathfrak{V}(p_2)$ by (2); however r does not belong to the set $\bigcap_{i\in\{1,3,5,7\}} \mathfrak{V}(p_i)\!\downarrow$ and, consequently, $h_\sigma(r) \in \mathfrak{V}(p_1)$ by (1). Since the points above r are mapped by h_σ onto 1, it follows that $h_\sigma(r)$ must be 5. Furthermore, notice that, since h_σ is a p-morphism, we need not to take into consideration the critical points in the following set $\{5, 6, 8, 10, 20, 23\}$. Here is a possible mapping of the critical points of $\mathfrak{F}_{\mathbf{ML}}(2)$.



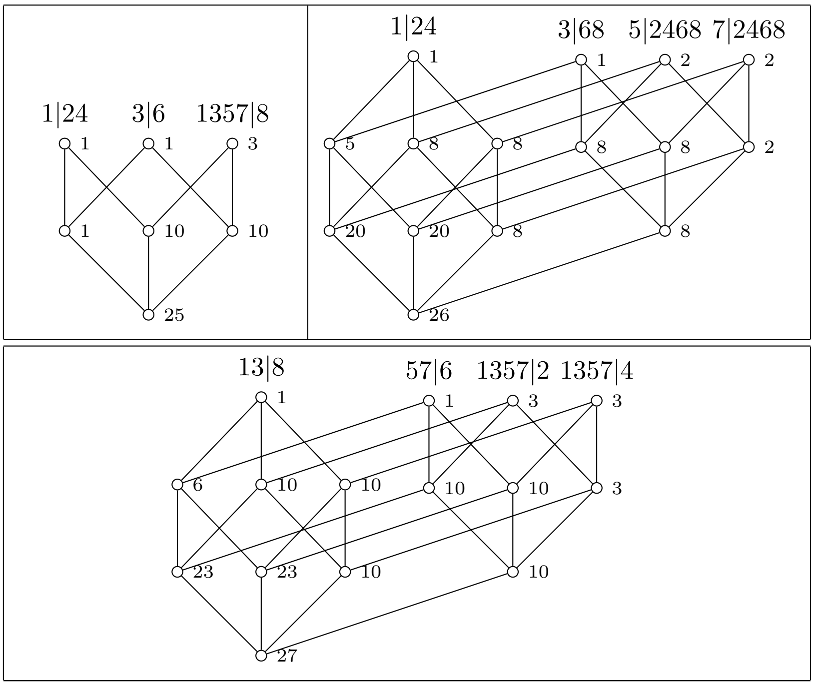


Table 6.1: *The mapping of the critical points of* $\mathfrak{F}_{\mathbf{ML}}(2)$.

As a consequence of the previous considerations, we can prove the following

Proposition 6.49. *The continuous p-morphisms* $h_\sigma \colon \mathfrak{F}_{\mathbf{ML}}(8) \to \mathfrak{F}_{\mathbf{ML}}(2)$ *induced by the* 4*-simple substitution* $\sigma \colon \mathbf{Var}\mathcal{L}_2 \to \mathcal{EN}$ *is surjective.*

Proof. Let us first show that every point of $x \in \mathfrak{F}^{<\infty}_{\mathbf{ML}}(2)$ is in the range of h_σ by induction on $d(x)$. By the previous considerations we can assume, without loss of generality, that x is not a critical point. Of course $max(\mathfrak{F}_{\mathbf{ML}}(2)) \subseteq h_\sigma(\mathfrak{F}_{\mathbf{ML}}(8))$, so let $d(x)$ be $n+1$. Since $x \prec \{y_1, \ldots, y_k\}$, $(k > 1)$, by the induction hypothesis there are points $z_i \in \mathfrak{F}_{\mathbf{ML}}(8)$ such that $h_\sigma(z_i) = y_i$ for every $i \in \{1, \ldots, k\}$. Since $\mathfrak{F}_{\mathbf{ML}}(8)$ is rooted, we can pick a point $z \in \bigcap_{i \in \{1,\ldots,k\}} z_i{\downarrow}$. Hence, $h_\sigma(z) < y_i$ for each $i \in \{1, \ldots, k\}$ and, consequently, $h_\sigma(z) \leq x$, because x is not a twin point. Therefore, as h_σ is a p-morphisms, there exists $z' \in z{\uparrow}$ such that $h_\sigma(z') = x$. So, we have proved that $\mathfrak{F}^{<\infty}_{\mathbf{ML}}(2) \subseteq h_\sigma(\mathfrak{F}_{\mathbf{ML}}(8))$. But then $\mathfrak{F}_{\mathbf{ML}}(2) = h_\sigma(\mathfrak{F}_{\mathbf{ML}}(8))$, since $\mathfrak{F}^{<\infty}_{\mathbf{ML}}(2)$ is dense in $\mathfrak{F}_{\mathbf{ML}}(2)$ and $h_\sigma(\mathfrak{F}_{\mathbf{ML}}(8))$ is closed in $\mathfrak{F}_{\mathbf{ML}}(2)$. ☐

Corollary 6.50. *The* 2*-letter fragment of Medvedev's logic* **ML** *is decidable.*

It is well known that, for the negatively stable logic **Int**, a 2-simple substitution $\sigma \colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$ induces an onto p-morphims $h_\sigma \colon \mathfrak{F}_{\mathbf{Int}}(2n) \to \mathfrak{F}_{\mathbf{Int}}(n)$ for every $n \in \omega$. This is possible because, for every finite frame $\mathfrak{F}$, the frame $(\mathfrak{F} + \mathfrak{F})^\triangledown \in \boldsymbol{\mathcal{DF}}_{\mathbf{Int}}$ and thus any linear point $x \in \mathfrak{F}_{\mathbf{Int}}(n)$ can be shown to be in the range of h_σ as follows: choose carefully two isomorphic disjoint rooted generated subframe $\mathfrak{G}_i \subsetneq \mathfrak{F}_{\mathbf{Int}}(2n)$, $i \in \{1, 2\}$, such that $h_\sigma(\mathfrak{G}_i) = y{\uparrow}$, where $x \prec y$; then by the choice of the $\mathfrak{G}_i$'s, one has that $h_\sigma((\mathfrak{G}_1 + \mathfrak{G}_2)^\triangledown) = x{\uparrow}$.

Of course the previous technique is not allowed in **ML**, since every frame $\mathfrak{F} \in \boldsymbol{\mathcal{DF}}_{\mathbf{ML}}$ has to be prefinally connected. Furthermore, due to the connected structure of $\mathfrak{F}_{\mathbf{ML}}(n)$, it seems plausible to suppose that, in order to cover, with a simple substitution $\sigma\colon \mathbf{Var}\mathcal{L}_n \to \mathcal{EN}$, a linear point $x \in \mathfrak{F}^{<\infty}_{\mathbf{ML}}(n)$, at least $d(x) \cdot 2$ different maximal points are needed in the domain of h_σ. This in fact worked out well with $\mathfrak{F}_{\mathbf{ML}}(2)$, since the maximal depth of a linear point was 4. However, we shall not expect that this works out with every $\mathfrak{F}_{\mathbf{ML}}(n)$ for $n \geq 3$ for the following reason: Medvedev's logic is finitely approximable but it is not locally tabular. Consequently, for some $n \geq 2$, the n-canonical frame $\mathfrak{F}_{\mathbf{ML}}(n)$ will be infinite and thus, in $\mathfrak{F}_{\mathbf{ML}}(n+1)$, the upset $\mathfrak{V}_{\mathbf{ML}}(p_i)$ will also be infinite, since it is isomorphic to $\mathfrak{F}_{\mathbf{ML}}(n)$. Therefore, for every point $x \in \mathfrak{V}_{\mathbf{ML}}(p_i)$ at finite depth k, there will also be a linear point $y \prec x$ at depth $k+1$. So $\mathfrak{F}_{\mathbf{ML}}(n+1)$ will contain linear points at every possible depth and, if our previous supposition is sound, there won't be any simple substitution which induces a p-morphism onto $\mathfrak{F}_{\mathbf{ML}}(n+1)$.

Appendix A

Is Medvedev's logic the logic of knowledge?

In [120] Miglioli and Usberti provide a philosophical analysis of the notion of knowledge to arrive at a so-called "paradigm of logical validity" which, in the end, turns out to be different from the classical and intuitionist ones. Let us briefly recall their main ideas.

A.1 Miglioli and Usberti's analysis of knowledge

The authors first start by stating two different and largely shareable theses which they take for granted:

> *i/* If logic is understood as the theory of abstract laws of knowledge, then a law of logic is a proposition that can be recognized as true only by virtue of the meaning of the logical constants it contains.
>
> *ii/* In turn, the meaning of the logical constants must be characterized in terms of the notion of knowledge by explaining how the knowledge of the truth of a complex proposition depends on the knowledge of the truth of the propositions of which it is composed. [120, p. 112]

Therefore, they conclude that the paradigm of logical validity they sought should be extracted from a logical analysis of the notion of knowledge. Furthermore, in order for such an analysis not to be circular, Miglioli and Usberti carefully decide to involve a new notion, conceptually prior to that of knowledge. In particular, the authors maintain that such a notion, say A, must share a common feature with the notion of truth, namely its being independent of any cognitive domain:

> We call " modal " such a role of the notions of truth and falsehood which consists in marking that invariant feature enjoyed by all sentences used to express knowledge or share information, to whatever cognitive domain they belong to. [120, p. 113]

Afterwards, Miglioli and Usberti realize that the notion of evidence, as it is commonly understood and initially considered as a potential candidate for A, is not suitable for that rôle, since it is sensitive to the context in which it is used: the fact that having evidence for an empirical proposition φ does not imply the truth of φ, whereas this usually holds when φ is a mathematical proposition, means that the notion of evidence is indeed not invariant with respect to different cognitive domains.

The authors thus distinguish two different notions of evidence that attain to two different cognitive modalities which can be experienced towards the objects of knowledge: given any proposition φ,

(A) a *justification*, or a *possible evidence*, for φ is anything that entitles a *fallible subject* to assert φ;

(B) an *evidence* for φ is anything that entitles an *omniscient observer* to assert φ.

The notion of justification, being fallible, allows one to fulfill the modal rôle of the notion of truth and therefore to consider knowledge from a unitary point of view, despite of the cognitive domain in which one is situated; the notion of evidence, being infallible, can instead replace the notion of truth as a key notion of the theory[1].

Bearing in mind the traditional platonistic account of knowledge as "justified true belief", Miglioli and Usberti propose the following as a first intuitive conceptual analysis of the sentence "s knows that φ":

s knows that φ	$\iff$	(i) s believes that φ;
		(ii) s has justifications for (believing that) φ;
		(iii) s has evidences for (believing that) φ.

Since the notion of evidence introduced by Miglioli and Usberti has the same properties of the notion of truth, the previous account of knowledge faces the same problems of the traditional ones and, in particular, it is challenged by the classical *Gettier's counterexamples.* However, by taking into consideration the Gettier cases, the authors pinpoint as a possible solution to such difficulties the necessity of an inductive definition of the notion of knowledge:

> It seems to us that Gettier's counterexample highlights a deep flaw of the traditional analysis, namely the fact that it tries to give a definition of the notion "s knows that A" *without analyzing the internal structure of* A. [...]
>
> It seems to us that between the characterization given [as justified true belief] and a desired adequate definition there must be the same relationship as between the aristotelian definition [of truth] and the one given by Tarski. [120, pp. 119, 120]

[1]Miglioli and Usberti clearly realize that, by making reference to the cognitive modality of the omniscient observer, they reintroduced the classical notion of truth in their setting; however, they explicitly state that such a "notion of truth does not play the rôle of the key notion of the theory, as it does in the classical theory of the logical constant, but it has only an explanatory function; explanatory of the concept of evidence." [120, p. 117]

So, Miglioli and Usberti continue their enquiry by giving a formal, inductive characterizations of the relation (ii) and (iii) of the intuitive definition of knowledge[2].

Definition A.1. An *assignment* is any function $\mathbf{a}\colon \mathbf{Var}\mathcal{L} \to \mathcal{P}(\mathcal{O})$ associating to each propositional variable p a finite non-empty set $\mathbf{a}(p)$ of (arbitrary) objects.

Given an assignment $\mathbf{a}$, we can uniquely extend it to a function $f_{\mathbf{a}}$ from the set $\mathbf{For}\mathcal{L}$ of well-formed formulas to $\mathcal{P}(\mathcal{O})$ as follows:

- $f_{\mathbf{a}}(\bot) = \{\bot\}$;
- $f_{\mathbf{a}}(p) = \mathbf{a}(p)$, for any $p \in \mathbf{Var}\mathcal{L}$;
- $f_{\mathbf{a}}(\varphi \wedge \psi) = f_{\mathbf{a}}(\varphi) \times f_{\mathbf{a}}(\psi)$;
- $f_{\mathbf{a}}(\varphi \vee \psi) = (f_{\mathbf{a}}(\varphi) \times 0) \cup (f_{\mathbf{a}}(\psi) \times 1)$;
- $f_{\mathbf{a}}(\varphi \to \psi) = f_{\mathbf{a}}(\psi)^{f_{\mathbf{a}}(\varphi)}$.

We can think of the set $f_{\mathbf{a}}(\varphi)$ as the set of justification for φ and we denote by $\widehat{\varphi}$ any element of $f_{\mathbf{a}}(\varphi)$. The requirement that $\mathbf{a}(p)$ is finite is due to the fact that the class of the possible evidences for any given proposition has to be controllable by an individual with finite capacities. Furthermore, notice the constructive character of such a definition: for instance, a justification for a disjunctive proposition $\varphi \vee \psi$ is defined as the disjoint union of the set of justification for φ with that of ψ and is thus given by either a justification for φ or a justification for ψ; a justification for an implicative sentence $\varphi \to \psi$ is nothing but a function from the set of possible evidence for φ to the set of possible evidence for ψ and embodies the idea of having a method to transform every justification for φ into a justification for ψ.

The reader could be puzzled as to the case of negated sentences $\neg\varphi$, which, as usual, are defined as $\varphi \to \bot$. In particular, one could wonder why the class of possible evidences for $\bot$ is its own singleton. The reason for this choice is purely technical and seems to be driven only to justify the following consequence: there can be only one justification for a negated sentence $\neg\varphi$, namely the constant function with value $\bot$. Since the philosophical motivation given by the authors is not completely coherent, let us examine the issue more deeply.

Miglioli and Usberti argue that the contradiction can either be characterized as that thing for which there cannot be justifications, or as that thing for which there cannot be evidence. Now, if the latter, then the set of possible evidence for $\bot$ must be non-empty (otherwise the two alternatives coincide). So, given any assignment $\mathbf{a}$, every $\widehat{\bot} \in f_{\mathbf{a}}(\bot)$ must fail to be an evidence and therefore the individual subject must *always* be able to say that this is in fact the case. However, there is no a priori reason to think that a fallible subject can always tell whether a justification is an evidence or a pseudo-justification and, consequently,

> no subject would ever be able to assert a negated formula [$\varphi \to \bot$], since he would not be able to recognize a justification as a justification *of the contradiction.* [120, p. 124]

[2]The relation (i) is taken as primitive since its study attain more to the domain of psychology than to that of logic.

Therefore, the only viable option is the first alternative, that is, the set of justification for $\bot$ is the empty set:

> [t]his treatment has two notable consequences. Firstly, the class of justification for a negated formula $[\neg\varphi]$ will contain exactly one element: *the* constant function with value $\emptyset$ defined on the class of justifications for $[\varphi]$. Secondly, a subject would never be wrong on the fact that $\bot$ express a contradiction: since there are no justifications for $\bot$, a fortiori there can not be evidences for $\bot$. [120, p. 124]

Finally, Miglioli and Usberti reckon that these two consequences could also be obtained by defining the class of possible evidences for $\bot$ as $\{\bot\}$, "provided that this will always turn out to be a pseudo-justification (never an evidence)" [120, p. 124].

Now, leaving aside the fact that this final choice is exactly the second rejected alternative, the careful reader will immediately recognize that the first alternative has to be rejected as well: since there are no function with empty codomain, there can not be justifications for negated sentences! So, a philosophical motivation for the case of negated sentences is clearly missing.

One possible way to motivate the case of negated sentences is to claim that any two distinct individual subjects s_1 and s_2 should share the same set of justification for $\bot$, since any two such justifications for $\bot$ must ultimately be the same. Usually, contradiction is taken to consist in an incongruity between two or more propositions. However, in order to avoid any circularity, $\bot$ has to be taken as an explicit contradiction of the simplest syntactical form (for instance, we can not have $p \wedge \neg p$ as a definition of $\bot$). Now, it can be the case that a subject s has a pseudo-justification $\widehat{\varphi}$ for a contingent proposition φ that is sound in itself: $\widehat{\varphi}$ could be logically deduced from another justification $\widehat{\psi}$ for ψ. However, any justification $\widehat{\bot}$ for $\bot$ an individual subject s can have not only has to be a pseudo-justification, but it has to be unsound in itself: it must contain some errors, perhaps it boils down to a sort of paralogism. Consequently, it is safely to suppose that s could be in principle be able to recognize the mistake. Therefore, one could argue that justifications for $\bot$ share this characteristic feature and thus can be identified with one another: the set of justification for $\bot$ would thus be a singleton. Leaving a question mark on this issue, let us move on with the formal definition of evidence.

Definition A.2. Given an assignment $\mathbf{a}$, a *discrimination* (relative to $\mathbf{a}$) is a function $\chi_{\mathbf{a}} \colon \bigcup_{p \in \mathbf{Var}\mathcal{L}} \mathbf{a}(p) \to \mathbf{2}$ associating with each element $\widehat{p}$ of every $\mathbf{a}(p)$ one of the two possible values 0 or 1.

A discrimination $\chi_{\mathbf{a}}$ can uniquely be extended to a function from the set $\bigcup_{\varphi \in \mathbf{For}\mathcal{L}} f_{\mathbf{a}}(\varphi)$ of the justifications of any given formula as follows:

- $\chi_{\mathbf{a}}(\bot) = 0$;
- $\chi_{\mathbf{a}}(\langle \widehat{\varphi}, \widehat{\psi} \rangle) = 1 \iff \chi_{\mathbf{a}}(\widehat{\varphi}) = 1 \ \ \& \ \ \chi_{\mathbf{a}}(\widehat{\psi}) = 1$;
- $\chi_{\mathbf{a}}(\langle \widehat{\varphi}, 0 \rangle) = 1 \iff \chi_{\mathbf{a}}(\widehat{\varphi}) = 1$;
- $\chi_{\mathbf{a}}(\langle \widehat{\psi}, 1 \rangle) = 1 \iff \chi_{\mathbf{a}}(\widehat{\psi}) = 1$;

- $\chi_{\mathbf{a}}(\widehat{\varphi \to \psi}) = 1 \Longleftrightarrow \forall \widehat{\varphi} \in f_{\mathbf{a}}(\varphi)(\chi_{\mathbf{a}}(\,\widehat{\varphi}\,) = 1 \Longrightarrow \chi_{\mathbf{a}}(\widehat{\varphi \to \psi}(\widehat{\varphi})) = 1)$.

In this setting, a justification $\widehat{\varphi}$ for φ is an *evidence* (relatively to $\chi_{\mathbf{a}}$) for φ if $\chi_{\mathbf{a}}(\,\widehat{\varphi}\,) = 1$. Notice that a discrimination is nothing but the characteristic function of the set of all possible given justification (relative to an assignment) and thus formalize the idea that

> it is not a intersubjective feature which makes justifications evidences, but the objective fact that things really are as those justifications entitle one to believe. [120, p.125]

Finally, Miglioli and Usberti reckon that any explicit reference to the subject s in the relation "s knows that φ" can be dispensed with, since one can assume that a given class of justification $f_{\mathbf{a}}(\varphi)$ canonically represents a class of subjects:

> speaking of a justification for a given sentence is thus making implicit reference to the subject(s) that use(s) it. [120, p.125]

Here is thus the final definition proposed by the authors in order to conceptually describe the relation of knowing something:

s knows that φ	$\Longleftrightarrow$	(i) s believes that φ;
(relative to $\mathbf{a}$ and $\chi_{\mathbf{a}}$)		(ii) there exists a justification $\widehat{\varphi}$ for φ;
		(iii) $\widehat{\varphi}$ is an evidence for φ.

A.2 Medvedev's logic as a paradigm of logical validity

Having completed the task of giving an adequate definition of knowledge, Miglioli and Usberti continue their investigation on the paradigm of logical validity. In particular, alongside with the distinction between justification and evidence, they introduce the distinction between the *epistemic content* and the *meaning* of a sentence.

> Very roughly, the epistemic content of a sentence can be conceived as what it means for an epistemically limited subject, the meaning as what it means for the omniscient observer. [...] it is natural to identify the epistemic content of a sentence with the class of its justifications, its meaning with the class of the evidences for it. [120, p.127]

In the formal setting, by fixing an assigment $\mathbf{a}$ one thus assigns epistemic contents to formulas, while by fixing a discrimination $\chi_{\mathbf{a}}$ one can specify their meanings. Therefore, as a consequence of (*i/*), a logical law is a law which can be perceived as true by only virtue of the epistemic content of the logical constants it contains. So, in order to capture the notion of a logical law, Miglioli and Usberti propose the following

Definition A.3. Let an assignment $\mathbf{a}$ be fixed. A formula φ is said to be *cognitively evident* (relatively to $\mathbf{a}$) if and only if there exists a justification $\widehat{\varphi}$ which is an evidence for φ relatively to every discrimination $\chi_{\mathbf{a}}$. Formally, by letting $\mathcal{E}_{\mathbf{a}} := \{\varphi \mid \varphi \text{ is } \mathbf{a}\text{-cognitively evident}\}$, we have

$$\varphi \in \mathcal{E}_{\mathbf{a}} \Longleftrightarrow \exists \widehat{\varphi} \in f_{\mathbf{a}}(\varphi)\, \forall \chi_{\mathbf{a}}\, (\chi_{\mathbf{a}}(\,\widehat{\varphi}\,) = 1). \tag{A.1}$$

Definition A.4. A formula φ is said to be *constructively logically valid* if and only if it is cognitively evident relative to every possible assignment. Formally, by letting $\mathcal{C} := \{\varphi \,|\, \varphi \text{ is constructively logically valid}\}$, we have

$$\varphi \in \mathcal{C} \Longleftrightarrow \forall \mathbf{a}\, \exists \widehat{\varphi} \in f_{\mathbf{a}}(\varphi)\, \forall \chi_{\mathbf{a}}\, (\chi_{\mathbf{a}}(\,\widehat{\varphi}\,) = 1). \tag{A.2}$$

Notice the two universal quantifiers in (A.2). Starting from the right, we have a first generalization on the class of discriminations (given an arbitrary assignment). As was said above, when we fix an assignment **a**, we assign a specific epistemic content to formulas. Consequently, if there exists a justification $\widehat{\varphi}$ for a formula φ which turns out to be an evidence no matter how things in the universe are, then such a formula φ would express a truth in the specific cognitive domain pictured by **a**. Here we have thus a first level of validity:

> it is clear that the knowledge we have about [a specific field] is neither a logical truth nor is it dependent on a single discrimination. It seems reasonable to understand it as related to the epistemic content assigned to the constant of [this field]. Cognitive evidence can thus be proposed as an alternative to the classical notion of validity in a theory. [120, p.128]

Finally, the second universal quantifier represents a generalization about the class of assignments. Thus, by abstracting from any possible specific epistemic content, a second level of validity is reached which would coincide with the logical one.

The reader familiar with the work of the Soviet logician Juri T. Medvedev will immediately recognize that, given an assignment **a**, the set of justifications $f_{\mathbf{a}}(\varphi)$ for φ is nothing but the set *admissible possibilties* for the finite problem φ in Medvedev's formalization of Kolmogorov's interpretation of intuitionistic logic as a calculus of problem. Furthermore, by considering the restriction of a discrimination $\chi_{\mathbf{a}}$ to the set of justification $f_{\mathbf{a}}(\varphi)$, we get a characteristic function for $f_{\mathbf{a}}(\varphi)$ which identifies the set of justifications for φ which are evidences: the set $\{\widehat{\varphi} \,|\, \chi_{\mathbf{a}}(\,\widehat{\varphi}\,) = 1\}$ can therefore be identified with Medvedev's set of *solution* for the finite problem φ. Finally, Medvedev's *identically solvable* formulas coincide with constructively logically valid formulas, so Miglioli and Usberti's approach toward knowledge turns out to be equivalent to Medvedev's account for specifying a calculus of problem and we have that the class $\mathcal{C}$ coincides with Medvedev's logic **ML**.

A.3 Minimal adequate conditions for the logic of knowledge

On the ground of their analysis of the notion of knowledge, Miglioli and Usberti identify the logic of knowledge with Medvedev's logic of finite problems **ML**. However, it might be the case that a different characterization $\mathcal{K}$ of the relation "s knows that φ" yields a different logic $L_{\mathcal{K}}$. If we assume that all the axioms of intuitionistic logic are reasonable principle of knowledge, then every such logic would be both an extension of intuitionistic logic **Int** and a proper sublogic of classical logic **Cl** and thus it would be an intermediate logic. Is it possible to pinpoint any specific feature that such a logic must possess? Is there a set of minimal adequate conditions that a logic should have to be called the logic of knowledge? And if it is so, does such a set identify a unique logic?

A.3.1 The disjunction property

On the basis of Miglioli and Usberti's account of knowledge, we have the following equivalence:

$$s \text{ knows that } \varphi \vee \psi \Longleftrightarrow s \text{ knows that } \varphi \text{ or } s \text{ knows that } \psi. \qquad \text{(D)}$$

The previous equivalence entails on the logical level that the logic $\mathcal{C}$ enjoys the disjunction property and it is straightforward to see that if a given account of knowledge $\mathcal{K}$ enjoys (D), then the corresponding logic $L_{\mathcal{K}}$ enjoys the disjunction property.

Now, (D) seems to be a natural requirement as to an explication of the disjunction and it is fairly plausible to assume that any characterization of knowledge would satisfy (D). Indeed, since the direction ($\Longleftarrow$) is obvious, any possible counterexample to (D) must provide a situation in which a subject s knows a disjunctive fact $\varphi \vee \psi$ without knowing either φ or ψ. Moreover, such a counterexample can not rely on the principle of excluded middle (or on any principle classically equivalent to it) and it is thus really hard to formulate it. Consequently, we can assume that the logic of knowledge enjoys the disjunction property.

Is the disjunction property a sufficient criterion in order to identify *the* logic of knowledge? It is a well known fact that there exist a continuum of intermediate logics with the disjunction property, however, as Miglioli and Usberti notice,

> from this standpoint, it is clear that any constructive logic (that is, with the disjunction property) that contains any other constructive logic would be *the* logic we sought. [120, p.129]

As Miglioli and Usberti sadly reckon, such a maximum constructive logic does not exist. However, among the continuum-many maximal intermediate logics with the disjunction property, maybe it is still possible to identify the logic of knowledge. In particular, Theorem 6.12 tells us that Medvedev's logic is maximal with respect to the intermediate logics with the disjunction property containing the logic **ND**. Consequently, if, for instance, one can manage to show that ***kp*** is a valid principle of knowledge, then Miglioli and Usberti's logic $\mathcal{C}$ (and thus **ML**) can indeed be considered the logic of knowledge.

The authors are confident about this point. In fact, they notice the following fact:

> Intuitionistically [***kp***] is invalid because it is possible to have a procedure to transform every proof of [$\neg\varphi$] into a proof of [$\psi \vee \xi$], but only a few proofs of [$\neg\varphi$] are transformed by the procedure into proofs of [ψ] and only some others are transformed into proofs of [ξ]. However, this argument does not apply to the notion of justification we provided since, as we saw, the justification for a negated formula could be only one. [120, p.129]

Therefore, they go on to say that a philosophical motivation is needed for this feature of justification and that perhaps it can be achieved by providing an argument in favor of an extensional interpretation of the notion of procedure as a set-theoretic function.

A.3.2 Structural completeness

We have seen that (D) is not in general a sufficient condition for determining the logic of knowledge. Can we find another analogous equivalence to boil down the set of possible logics of knowledge? Now, (D) is taken to be an explication of the logical constant of disjunction, so we can try to formulate an appropriate condition as an explication of a different logical constant. Since the only connective with a constructive character is the implication, our choice is forced. First notice that, in Miglioli and Usberti's framework, the following equivalence holds:

$$s \text{ knows that } \varphi \to \psi \iff \text{If } s \text{ knows that } \varphi,\ \text{then } s \text{ knows that } \psi. \qquad \text{(I)}$$

Indeed, ($\Longrightarrow$) is obvious. Conversely, assume that the right-hand side of (I) holds, that is, suppose that whenever s knows that φ, then s also knows that ψ. Now, consider both the sets of justifications s has for φ and ψ, which, by our hypothesis, are the non-empty sets $f_{\mathbf{a}}(\varphi)$ and $f_{\mathbf{a}}(\psi)$, for some $\mathbf{a}$-assignment correlated to s. Let us consider a justification $\widehat{\varphi} \in f_{\mathbf{a}}(\varphi)$. If $\widehat{\varphi}$ is an evidence, that is, $\chi_{\mathbf{a}}(\widehat{\varphi}) = 1$, then s knows that φ; so, by (I), it follows that s also knows that ψ and, consequently, there exists a justification $\widehat{\psi}$ of ψ which is an evidence. Thus, we let $f(\widehat{\varphi}) = \widehat{\psi}$. If $\widehat{\varphi}$ is not an evidence, then we let $f(\widehat{\varphi}) = \widehat{\psi}$ for some arbitrary $\widehat{\psi} \in f_{\mathbf{a}}(\psi)$. Then $f \in f_{\mathbf{a}}(\psi)^{f_{\mathbf{a}}(\varphi)}$, that is, f is a justification for $\varphi \to \psi$ and, moreover, f is an evidence by construction. Therefore, we have that s knows that $\varphi \to \psi$.

Intuitively, the right-hand side of (I) can be thought as an explanation of the implication in the following sense: it embodies, at a "meta-theoretical" level, a method to transform knowledge of a given proposition φ into knowledge of a proposition ψ. Since this feature is exactly what a constructive explanation of the implication should be, we may assume that any characterization of knowledge must actually satisfy (I).

We can now ask ourselves which could be the logical counterpart of (I). Here the situation is clearly more involved that in the case of (D). In fact, for the case of disjunction, we immediately get a well defined logical property simply by substituting "$\varphi \in L$" for "s knows that φ" in (D). However, such a move does not work for (I), since the equivalence

$$\varphi \to \psi \in L \iff \text{ if } \varphi \in L,\ \text{then } \psi \in L$$

makes no sense. So, let's try to look at the issue from a different perspective. As we noticed, the right-hand side of (I) represents a rule of knowledge according to which, whenever s knows that φ, then s also knows that ψ. Such a rule allows s to gain knowledge of ψ in the case s already has knowledge of φ and its use is always permitted in the cognitive domain of s. Thus we may consider the admissibility of the inference rule $\dfrac{\varphi}{\psi}$ as the logical counterpart of the right-hand side of (I). Furthermore, since $\varphi \to \psi \in L$ is equivalent to the derivability of the rule $\dfrac{\varphi}{\psi}$, it follows that the logical counterpart of (I) could be the property of structural completeness.

Structural completeness seems to be a necessary feature of that logic which is purported to give a cognitive, or constructive, explanation of the logical connectives. Indeed, along with valid principles of knowledge, one should also consider admissible rules of inference. So we should ask ourselves when a law of inference φ/ψ is valid, that is, whether its use is justified as a way of gaining knowledge of ψ whenever one has knowledge of φ. Now, the only possible answer seems to be the following: the rule φ/ψ is valid because the implication $\varphi \to \psi$ is actually a valid principle of knowledge.

Now, if that is the case, the logic of knowledge would be a Friedman logic. But then, on the basis of Conjecture of §6.4.1, such a logic would be completely determined, since there should exist a unique such logic, namely Medvedev's logic of finite problems **ML**. Thus, we can not say whether Miglioli and Usberti's account of knowledge is completely adequate, however what it can be said is that they might have at least found *the* logic of knowledge.

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