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# Asymptotic properties of an optimal principal Dirichlet eigenvalue arising in population dynamics

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We consider a shape optimization problem related to the persistence threshold for a biological species, the unknown shape corresponding to the zone of the habitat which is favorable to the population. Analytically, this translates in the minimization of a weighted eigenvalue of the Dirichlet Laplacian, with respect to a bang-bang indefinite weight. For such problem, we provide a full description of the singularly perturbed regime in which the volume of the favorable zone vanishes, with particular attention to the interplay between its location and shape.

First, we show that the optimal favorable zone shrinks to a connected, nearly spherical set, in $C^{1,1}$ sense, which aims at maximizing its distance from the lethal boundary. Secondly, we show that the spherical asymmetry of the optimal favorable zone decays exponentially, with respect to a negative power of its volume, in the $C^{1,\alpha}$ sense, for every $\alpha < 1$. This latter property is based on sharp quantitative asymmetry estimates for the optimization of a weighted eigenvalue problem on the full space, of independent interest.



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**Contents**



## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ denote a bounded domain (open and connected) with regular boundary, for instance of class $C^1$ (even though such assumption can be relaxed, see below). Let $m \in L^\infty(\Omega)$ denote a (sign-changing) *indefinite weight*. A *principal eigenvalue* of the weighted Dirichlet problem

$$\begin{cases} -\Delta u = \lambda m u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

is a real number $\lambda$ admitting a positive eigenfunction $u \in H_0^1(\Omega)$. If both the positive and the negative parts $m^\pm$ of the weight are nontrivial, (1.1) admits two principal eigenvalues, and we denote with $\lambda^1(m,\Omega) > 0$ the unique positive one:

$$\lambda^1(m,\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2}{\int_\Omega m u^2} : u \in H_0^1(\Omega),\ \int_\Omega m u^2 > 0 \right\} \tag{1.2}$$

(in particular, $m \le 0$ a.e. in $\Omega$ implies $\lambda^1(m,\Omega) = +\infty$).

Among other applications, such eigenvalue is of interest in population dynamics: when modeling the spatial dispersal of a population in a heterogeneous environment through a reaction-diffusion equation of logistic type [36,6,2,7], one is lead to consider the following evolutive problem for the population density $u = u(x,t)$:

$$\begin{cases} u_t - d\Delta u = mu - u^2 & x \in \Omega,\ t > 0, \\ u = u_0 \geq 0 & x \in \Omega,\ t = 0, \\ u = 0 & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.3}$$

Here $m$ describes the heterogeneous habitat, whose favorable and hostile zones for the species correspond to positivity and negativity sets of $m$, respectively, and Dirichlet conditions encode a lethal boundary. It is well known that solutions to (1.3) persist (i.e. they do not extinguish) if and only if $d\lambda^1(m, \Omega) < 1$; accordingly, the smaller $\lambda^1(m, \Omega)$ is, the more chances of survival the population has. For this reason, the minimization of $\lambda^1(m, \Omega)$ with respect to the weight, or to other relevant parameters of the model, has been widely investigated. The literature is quite extensive, and we refer to the recent papers [24,1,35,27,16] and references therein, both for a more detailed description of the problem, also with different boundary conditions, and for various interesting phenomena, ranging from fragmentation to different nonlocal effects. We also mention [8,9], where related problems arising from the study of composite membranes are investigated.

In this paper we deal with the so-called *optimal design problem for the survival threshold*, i.e. the minimization of $\lambda^1(m, \Omega)$ with respect to the weight $m$. For fixed positive constants $\underline{m}$, $\overline{m}$, and $0 < \varepsilon < \mathcal{L}(\Omega)$ (where $\mathcal{L}$ denotes the Lebesgue measure) let us consider the class of indefinite weights:

$$\mathcal{M}_\varepsilon := \left\{ m \in L^\infty(\Omega) : -\underline{m} \leq m \leq \overline{m},\ \int_\Omega m = M_\varepsilon \right\}, \tag{1.4}$$

where $M_\varepsilon = \overline{m}\varepsilon - (\mathcal{L}(\Omega) - \varepsilon)\underline{m}$ (see below), and the class of *bang-bang weights*:

$$\mathcal{BB}_\varepsilon := \left\{ m \in L^\infty(\Omega) : m = \overline{m}\chi_E - \underline{m}\chi_{\Omega\setminus E},\ E \subset \Omega,\ \mathcal{L}(E) = \varepsilon \right\},$$

where $\chi_A$ denotes the indicatrix function of the set $A$. By definition of $M_\varepsilon$, it is a direct check that $\mathcal{BB}_\varepsilon \subset \mathcal{M}_\varepsilon$. Moreover, in the seminal paper [6], Cantrell and Cosner proved that the minimization of $\lambda^1(m, \Omega)$ in the wider class is achieved by an element of the more narrow one:

$$\lambda_\varepsilon(\Omega) := \inf_{m \in \mathcal{M}_\varepsilon} \lambda^1(m, \Omega) = \inf_{m \in \mathcal{BB}_\varepsilon} \lambda^1(m, \Omega) = \inf_{E \subset \Omega,\ \mathcal{L}(E) = \varepsilon} \lambda^1(E, \Omega), \tag{1.5}$$

where, with some abuse of notation, for weights in $\mathcal{BB}_\varepsilon$ we write

$$\lambda^1(E, \Omega) := \lambda^1(\overline{m}\chi_E - \underline{m}\chi_{\Omega\setminus E}, \Omega).$$

In particular, the optimal design problem for the survival threshold is turned into a *shape optimization problem* involving the favorable zone $E$. In the following, when no confusion arises, referring to the set $\Omega$, we will simply write

$$\lambda^1(E) = \lambda^1(E, \Omega), \qquad \lambda_\varepsilon = \lambda_\varepsilon(\Omega).$$

According to [6], it follows that $\lambda_\varepsilon$ is achieved by a set $E_\varepsilon$, and the first order optimality condition translates in the following crucial fact: any optimal set $E_\varepsilon$ is the *super-level set* of the associated positive eigenfunction $u_\varepsilon \in H_0^1(\Omega)$. In particular, $E_\varepsilon$ contains the maximum points of $u_\varepsilon$.

Both for modeling reasons, and from the mathematical point of view, natural questions arise about the *location* and the *shape* of the optimal shape $E_\varepsilon$. This issue is mostly open, and it is completely understood only when the domain $\Omega$ is a ball: in such case, by symmetrization arguments it is proved in [6] that $E_\varepsilon$ is a concentric ball of measure $\varepsilon$, for every $0 < \varepsilon < \mathcal{L}(\Omega)$. On the converse, by [24] if $E_\varepsilon$ is a ball then $\Omega$ itself must be a concentric (larger) ball.

In this paper we address the above questions, in the singularly perturbed, small volume regime $\varepsilon \to 0$. Our aim is to show that, for $\varepsilon$ small, the optimal favorable region $E_\varepsilon$ shrinks to a connected, asymptotically spherical shape, which locates at maximal distance from the boundary. Moreover, we are going to provide sharp estimates on the optimal eigenvalue $\lambda_\varepsilon$, which will reflect on quantitative properties of the shape of the optimal set $E_\varepsilon$. On the one hand, the investigation of concentration phenomena to detect the limit position and the qualitative asymptotic sphericity of the optimal shape is a classical theme in the analysis of variational problems, see e.g. the book [20] and references therein. On the other hand, the use of quantitative estimates to provide explicit bounds to the spherical asymmetry of the optimal shape is more recent: we refer for instance to [19], where this kind of results were obtained for the shape of liquid drops and crystals, in the small mass regime, also in the anisotropic setting (where a suitable Wulff shape replaces the sphere). It is worth mentioning that, in our case, no perimeter term is present: the main feature of the model consists in the interplay between the location and the geometry of the optimal favorable set, as the quantitative estimates about the optimal shape are triggered by a repulsion effect of the lethal boundary of the box. Incidentally, we also mention that a first investigation of an anisotropic version of our problem is contained in the recent paper [34].

A crucial role in our description is played by the rescaled limit problem on $\mathbb{R}^N$, which has already been studied in [29, Section 2] (see also [18, Proposition 2.1]): consistently with the previous notation, for $\mathcal{A} \subset \mathbb{R}^N$ (measurable and) bounded let us define the principal eigenvalue

$$\lambda^1(\mathcal{A}, \mathbb{R}^N) := \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\overline{m} \int_{\mathcal{A}} v^2 - \underline{m} \int_{\mathbb{R}^N \setminus \mathcal{A}} v^2} : u \in H^1(\mathbb{R}^N),\ \overline{m} \int\limits_{\mathcal{A}} v^2 > \underline{m} \int\limits_{\mathbb{R}^N \setminus \mathcal{A}} v^2 \right\}.$$

Then, using symmetrization techniques, one can show that the minimization of such eigenvalue, among sets $\mathcal{A}$ having prescribed measure, is achieved by a ball, with an associated radial and radially decreasing eigenfunction. Precisely, we have that

$$\tilde{\lambda}_0 \coloneqq \inf_{\mathcal{L}(\mathcal{A})=1} \lambda^1(\mathcal{A}, \mathbb{R}^N) = \lambda^1(B, \mathbb{R}^N), \tag{1.6}$$

where $B \subset \mathbb{R}^N$ denotes the ball of unitary Lebesgue measure, centered at the origin, and $r_0$ denotes its radius:

$$B = B_{r_0}, \qquad \mathcal{L}(B_{r_0}) = 1. \tag{1.7}$$

Moreover such minimizer is unique up to translations, and in turn $\lambda^1(B, \mathbb{R}^N)$ is achieved by $w \in H^1(\mathbb{R}^N)$, solution of

$$-\Delta w = \tilde{\lambda}_0 \tilde{m}_0 w \quad \text{in } \mathbb{R}^N, \qquad \text{where } \tilde{m}_0 \coloneqq \overline{m}\chi_B - \underline{m}\chi_{\mathbb{R}^N \setminus B}; \tag{1.8}$$

here $w > 0$, radially symmetric, radially decreasing and normalized in $L^2(\mathbb{R}^N)$, is uniquely determined. As a matter of fact, $w$ is explicit in terms of Bessel functions, and it decays exponentially at infinity:

$$w(\mathbf{x}) \sim C|\mathbf{x}|^{-(N-1)/2} e^{-\sqrt{\tilde{\lambda}_0 \underline{m}}|\mathbf{x}|} \qquad \text{as } |\mathbf{x}| \to +\infty. \tag{1.9}$$

Our main results can be divided in two parts. In the first part of the paper we analyze the qualitative properties of the optimal favorable set $E_\varepsilon$, as well as the asymptotics of the optimal eigenvalue $\lambda_\varepsilon$, leaning on a blow-up procedure. In this respect, our main results can be summarized as follows.

**Theorem 1.1.** *There exists $\bar{\varepsilon}$ such that, for every $0 < \varepsilon < \bar{\varepsilon}$, for every optimal eigenfunction $u_\varepsilon > 0$ and favorable set $E_\varepsilon$, with $\mathcal{L}(E_\varepsilon) = \varepsilon$, associated to $\lambda_\varepsilon = \lambda_\varepsilon(\Omega) = \lambda^1(E_\varepsilon, \Omega)$:*

1. *$u_\varepsilon$ has a unique local maximum point $\mathbf{x}_\varepsilon$, and in particular $E_\varepsilon \ni \mathbf{x}_\varepsilon$ is connected;*
2. *$E_\varepsilon$ is radially diffemorphic to a ball centered at $\mathbf{x}_\varepsilon$, i.e. there exists a function $\varphi_\varepsilon \in C^{1,1}(\mathbb{S}^{N-1})$ such that ($\mathcal{L}(B_{r_0}) = 1$)*

$$E_\varepsilon = \left\{ \mathbf{x} : |\mathbf{x} - \mathbf{x}_\varepsilon| < \varepsilon^{1/N} \left( r_0 + \varphi_\varepsilon \left( \frac{\mathbf{x} - \mathbf{x}_\varepsilon}{|\mathbf{x} - \mathbf{x}_\varepsilon|} \right) \right) \right\}.$$

*Moreover, as $\varepsilon \to 0^+$,*

3. $\operatorname{dist}(\mathbf{x}_\varepsilon, \partial\Omega) \to \max_{\mathbf{p}\in\Omega} \operatorname{dist}(\mathbf{p}, \partial\Omega) =: d^*$*;*
4. $\lambda_\varepsilon = \varepsilon^{-2/N} \left( \tilde{\lambda}_0 + e^{-2\sqrt{\tilde{\lambda}_0 \underline{m}}(1+o(1))d^* \cdot \varepsilon^{-1/N}} \right)$*;*
5. $\|\varphi_\varepsilon\|_{C^{1,1}(\mathbb{S}^{N-1})} \to 0$.

*Finally, properties 2, 3 and 5 remain true if the maximum point $\mathbf{x}_\varepsilon$ is replaced with the barycenter $\operatorname{bar}(E_\varepsilon)$ of the optimal set $E_\varepsilon$, and $\varphi_\varepsilon$ with the corresponding polar parametrization of $\partial E_\varepsilon$ centered at $\operatorname{bar}(E_\varepsilon)$.*

In particular, we obtain an answer to the natural questions raised above: when the volume of the favorable zone is small, then such zone is connected, asymptotically spherical, and it concentrates at a point whose location tends to maximize its distance to the lethal boundary.

Notice that, in general, the uniqueness of the pair $u_\varepsilon, E_\varepsilon$ is not guaranteed: to disprove it, in view of the theorem, it is sufficient to consider a domain $\Omega$, symmetric with respect to a hyperplane, such that points achieving the inradius do not belong to such hyperplane. In other words, the properties in Theorem 1.1 hold true for every choice of $u_\varepsilon, E_\varepsilon$. On the other hand, in case the set of points realizing $\max_{\mathbf{p}\in\Omega} \operatorname{dist}(\mathbf{p}, \partial\Omega)$ is not reduced to a single point, we do not know how to further select the possible accumulation limit: this would probably require to provide an expansion of the optimal eigenvalue to an higher order.

Moreover, it will be clear from the proof that the regularity assumptions on $\partial\Omega$ can be substantially weakened, e.g. requiring it to be Lipschitz: indeed, in such case, one can approximate $\Omega$ both from inside and from outside, with smooth domains having arbitrarily close inradii; in this way one obtains the expansion of $\lambda_\varepsilon$, which is the key to obtain the other properties.

As we mentioned, the proof of Theorem 1.1 is based on a blow-up procedure, and it exploits some analogies with the study of singularly perturbed semilinear elliptic equations. Indeed, it is well known in the literature (see [30–32,15,3]) that least action solutions of equations like

$$-\varepsilon^2 \Delta u + u = f(u) \qquad \text{in } \Omega,$$

complemented with suitable boundary conditions, exhibit concentration phenomena as $\varepsilon \to 0$; moreover, a sharp asymptotic expansion of the least action level $c_\varepsilon$ allows to locate the points where concentration may happen. Now, with respect to such problem, the optimal design problem for the survival threshold in population dynamics exhibits several different aspects: first of all, the driving parameter is not explicit inside the equation, but it is the prescribed volume of the optimal favorable set; moreover, the underlying equation is linear, with a discontinuous and non-homogeneous weight, and degenerate on the relevant solutions (the eigenfunctions). On the other hand, the two problems present also relevant analogies, triggered by the fact that the unknown shape of the optimal favorable set is a superlevel set of the associated eigenfunction, and thus the vanishing-volume regime suggests that some concentration for the eigenfunction may happen. Such analogies have already been pointed out and exploited in [28,29], for the corresponding problems with Neumann boundary conditions.

More specifically, properties 1, 3 and 4 in Theorem 1.1 follow by a repeated application of a blow-up argument. In particular, we obtain the uniqueness of the maximum points, which reflects on the connectedness of $E_\varepsilon$, by a nontrivial adaptation of the arguments in [32]; on the other hand, we infer the location of such maxima and the sharp expansion of the eigenvalue exploiting a simplified argument inspired by [15]. Incidentally, the blow-up procedure provides information not only on the asymptotic behavior of $E_\varepsilon$ and $\lambda_\varepsilon$, but also on that of the optimal eigenfunction $u_\varepsilon$. In particular, we show that a suitable rescaling of $u_\varepsilon$ converges, strongly in $H^1(\mathbb{R}^N)$ and in $C^{1,\alpha}_{\mathrm{loc}}$, $\alpha < 1$, to the eigenfunction $w$ of the limit problem (1.8) (see Lemmas 2.2, 2.3, 3.2 and Corollary 3.11 ahead for more details).

Concerning the properties of the polar parameterization $\varphi_\varepsilon$ of $\partial E_\varepsilon$, it is worth noticing that, the weight $m_\varepsilon$ being merely $L^\infty$, the blow-up procedure naturally provides just a $C^{1,\alpha}$-regularity, for $\alpha$ strictly smaller than 1. To recover the $C^{1,1}$ regularity obtained in Theorem 1.1, which is optimal at the level of the eigenfunction $u_\varepsilon$, we observe that the partial derivatives of the optimal eigenfunction $u_\varepsilon$ satisfy, after a suitable change of variable, a system which can be seen as a transmission problem; this crucial remark allows to apply recent regularity results for such kind of problems, by Caffarelli, Soria-Carro and Stinga [5] and Dong [17], thus concluding the proof of Theorem 1.1. We remark that such further regularity is not only interesting by itself, but it also allows to simplify some arguments that we use in the second part of the paper.

Properties 2, 5 in Theorem 1.1 assert that, from a qualitative point of view, the shape of the optimal favorable set $E_\varepsilon$ is asymptotically a shrinking sphere, in a $C^{1,1}$ sense. Then a natural question, that we address in the second part of the paper, is to which extent such assertion can be made quantitative. The answer is contained in the following result.

**Theorem 1.2.** *Under the assumptions and notation of Theorem 1.1, let $0 < \varepsilon < \bar{\varepsilon}$ and let $\varphi_\varepsilon$ denote the polar parametrization of $\partial E_\varepsilon$, as in property 2 of such theorem, centered at* $\mathrm{bar}(E_\varepsilon)$.

*Then, for a possibly smaller value of $\bar{\varepsilon}$ and for every $0 < \alpha < 1$ there exist positive constants $M, C$ such that*

$$\|\varphi_\varepsilon\|_{C^{1,\alpha}(\mathbb{S}^{N-1})} \le C e^{-M\varepsilon^{-1/N}},$$

*for every $0 < \varepsilon < \bar{\varepsilon}$.*

**Remark 1.3.** In the previous theorem it is not possible to use the maximum point $\mathbf{x}_\varepsilon$ instead of $\mathrm{bar}(E_\varepsilon)$: this would require further information on $|\mathbf{x}_\varepsilon - \mathrm{bar}(E_\varepsilon)|$, which is not available at the moment.

The proof of Theorem 1.2 is based on a sharp quantitative form of the inequality provided by the limit problem (1.6), (1.8), which is of independent interest. Such result is expressed in terms of a quantification of the spherical asymmetry of the favorable set, based on the following definition.

**Definition 1.4.** A (bounded) set $\mathcal{A} \subset \mathbb{R}^N$ is *nearly spherical of class $C^{1,\alpha}$* if there exists $\varphi \in C^{1,\alpha}(\mathbb{S}^{N-1})$, with $\|\varphi\|_{L^\infty} \le r_0/2$, such that $\partial\mathcal{A}$ is represented as

$$\partial\mathcal{A} = \left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = (r_0 + \varphi(\theta))\theta \text{ for } \theta \in \mathbb{S}^{N-1}\right\}$$

(recall that $r_0$ is the radius of the ball of unit measure in $\mathbb{R}^N$: $B = B_{r_0}(\mathbf{0})$), and we denote $\varphi_{\mathcal{A}}$ such $\varphi$.

Under the above definition, and denoting with $\mathrm{bar}(\mathcal{A})$ the barycenter of any bounded measurable set in $\mathbb{R}^N$, we obtain the following sharp quantitative version of (1.6).

**Theorem 1.5.** *There exist constants $C > 0$, $\delta > 0$ such that, for any $C^{1,1}$ nearly spherical set $\mathcal{A} \subset \mathbb{R}^N$ satisfying*

*(i)* $\mathrm{bar}(\mathcal{A}) = \mathbf{0}$,
*(ii)* $\mathcal{L}(\mathcal{A}) = 1$,
*(iii)* $\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})} \leq \delta$,

*it holds*

$$\lambda^1(\mathcal{A}, \mathbb{R}^N) - \lambda^1(B, \mathbb{R}^N) \geq C\|\varphi_{\mathcal{A}}\|^2_{L^2(\mathbb{S}^{N-1})}.$$

In the above theorem, the constant $C > 0$ is obtained by a constructive method and in principle it is computable. As we mentioned, the above estimate is sharp, in the sense that we prove also the existence of a larger constant $C' > C$ such that, under the same assumptions, also the reverse inequality holds:

$$\lambda^1(\mathcal{A}, \mathbb{R}^N) - \lambda^1(B, \mathbb{R}^N) \leq C'\|\varphi_{\mathcal{A}}\|^2_{L^2(\mathbb{S}^{N-1})} \tag{1.10}$$

(see Lemma 6.1 ahead). We also remark that the constraint on the barycenter of the nearly spherical set is necessary to obtain that $B$ is locally a *strict* minimum shape, or equivalently that $C > 0$ strictly. Indeed, since problem (1.6) is settled in the whole $\mathbb{R}^N$, translations of $B$ are still global minimizers.

Theorem 1.5 can be easily improved, at the regularity level, using the Gagliardo-Nirenberg inequality on the sphere $\mathbb{S}^{N-1}$ (and in turn, such improvement is crucial to deduce Theorem 1.2 from property 4 of Theorem 1.1).

**Corollary 1.6.** *Under the assumptions of Theorem 1.5, for every $0 < \alpha < 1$ there exists a constant $C > 0$, depending also on $\delta$, such that*

$$\lambda^1(\mathcal{A}, \mathbb{R}^N) - \lambda^1(B, \mathbb{R}^N) \geq C\|\varphi_{\mathcal{A}}\|^{(4+N)/(1-\alpha)}_{C^{1,\alpha}(\mathbb{S}^{N-1})}.$$

The proof of Theorem 1.5 is carried out expanding the map

$$\mathcal{A} \mapsto \lambda^1(\mathcal{A}, \mathbb{R}^N)$$

in Taylor series, centered at $B$, up to the second order. After the seminal paper [21], this is a by now classical argument in shape optimization problems, see for instance [13,10,23,11], also to deduce sharp quantitative estimates for volumetric geometric-functional inequalities as in [4,14]. In particular, we advise the paper [26], where Mazari considers the optimization of the first eigenvalue of a Schrödinger operator in the ball, with respect to a bang-bang potential, obtaining results related with ours. In particular, as remarked in [26, Sec. 2.5.2], in shape optimization problems where the shape is a subdomain of a fixed domain, the natural norm to prove quantitative asymmetry estimates for nearly spherical sets seems to be the $L^2(\mathbb{S}^{N-1})$ norm of the deformation, rather than the $H^{1/2}(\mathbb{S}^{N-1})$ norm as it is more usual in the literature (see e.g. [12,4,14,10]). In this respect, our results corroborate this observation. On the other hand, with respect to [26], in our proof we can take advantage of the $C^{1,1}$ regularity of the shape to simplify part of the argument. In particular, on the lines of [26], we think that Theorem 1.5 may be proved also starting from a weaker $C^{1,\alpha}$ regularity assumption, even though we do not pursue this issue here.

The paper is structured as follows: after a description of the blow-up procedure contained in Section 2, Section 3 includes the proof of property 1 of Theorem 1.1, as well as a version in $C^{1,\alpha}$, $\alpha < 1$, of properties 2 and 5; in Section 4 we prove properties 3 and 4 of Theorem 1.1; in Section 5 we complete the proof of Theorem 1.1; finally, Section 6 contains the proof of Theorems 1.5 and 1.2.

*Notation*

- $\mathcal{L}(A)$ denotes the $N$-dimensional Lebesgue measure of the set $A \subset \mathbb{R}^N$.
- $B \subset \mathbb{R}^N$ denotes the ball centered at the origin, with $\mathcal{L}(B) = 1$; consequently, for every $t > 0$, $\mathcal{L}(tB) = t^N$. Moreover, $r_0$ is defined in such a way that

$$B = B_{r_0}(\mathbf{0}).$$

- $C$, $C'$ and so on denote positive constants, whose values may change from line to line.

## 2. Blow-up procedure

In this section we begin to study the qualitative asympotics for $\lambda_\varepsilon$, the first generalized eigenvalue with indefinite weight. We proceed by means of a blow-up argument, similarly to [29, Section 4] and [18, Section 3], to which we refer for further details.

As recalled in the introduction, we denote with $u_\varepsilon$ a family of positive eigenfunctions associated to $\lambda_\varepsilon$, normalized in $L^2(\Omega)$, having associated optimal weight $m_\varepsilon \in L^\infty(\Omega)$ and optimal favorable $E_\varepsilon \subset \Omega$, with $\mathcal{L}(E_\varepsilon) = \varepsilon$. Moreover, we extend the eigenfunctions $u_\varepsilon$ to be 0 outside $\Omega$. Although at first the blow-up procedure will be centered at $\mathbf{x}_\varepsilon \in \Omega$, a global maximum point of $u_\varepsilon$, later we will need to choose different centers. For this reason it is convenient to illustrate the procedure for an arbitrary choice of the blow-up centers $\mathbf{p}_\varepsilon$.

Let

$$\mathbf{p}_\varepsilon \in \Omega, \qquad k_\varepsilon := \varepsilon^{1/N}. \tag{2.1}$$

We define the blow-up functions

$$\tilde{u}_\varepsilon(\mathbf{x}) := k_\varepsilon^{N/2}\, u_\varepsilon(\mathbf{p}_\varepsilon + k_\varepsilon \mathbf{x}), \qquad \tilde{m}_\varepsilon(\mathbf{x}) := m_\varepsilon(\mathbf{p}_\varepsilon + k_\varepsilon \mathbf{x}), \tag{2.2}$$

and (superlevel) sets

$$\begin{aligned} \tilde{\Omega}_\varepsilon &:= \left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{p}_\varepsilon + k_\varepsilon \mathbf{x} \in \Omega\right\} = \left\{\mathbf{x} \in \mathbb{R}^N : \tilde{u}_\varepsilon(\mathbf{x}) > 0\right\}, \\ \tilde{E}_\varepsilon &:= \left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{p}_\varepsilon + k_\varepsilon \mathbf{x} \in E_\varepsilon\right\} = \left\{\mathbf{x} \in \mathbb{R}^N : \tilde{u}_\varepsilon(\mathbf{x}) > \alpha_\varepsilon\right\}, \end{aligned} \tag{2.3}$$

where $\alpha_\varepsilon = k_\varepsilon^{N/2}\, u_\varepsilon|_{\partial E_\varepsilon} > 0$. The normalizations are chosen in such a way that, for every $\varepsilon > 0$,

$$\int_{\tilde{\Omega}_\varepsilon} \tilde{u}_\varepsilon^2 = \mathcal{L}\left(\tilde{E}_\varepsilon\right) = 1.$$

Moreover, the functions $\tilde{u}_\varepsilon$ solve, in $H_0^1(\tilde{\Omega}_\varepsilon)$,

$$\begin{cases} -\Delta \tilde{u}_\varepsilon = \tilde{\lambda}_\varepsilon \tilde{m}_\varepsilon \tilde{u}_\varepsilon & \text{in } \tilde{\Omega}_\varepsilon, \\ \tilde{u}_\varepsilon = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon, \end{cases} \tag{2.4}$$

where

$$\tilde{\lambda}_\varepsilon = \tilde{\lambda}_\varepsilon(\tilde{\Omega}_\varepsilon) := k_\varepsilon^2 \lambda_\varepsilon(\Omega) \tag{2.5}$$

satisfies

$$\tilde{\lambda}_\varepsilon = \inf_{A \subset \tilde{\Omega}_\varepsilon : \mathcal{L}(A) = 1} \lambda^1(A, \tilde{\Omega}_\varepsilon) = \lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon).$$

Since $\Omega$ is bounded, from any sequence $\varepsilon_n \to 0$ we can extract a subsequence, still denoted by $\varepsilon_n$, such that $\mathbf{p}_{\varepsilon_n} \to \mathbf{p}_\infty \in \overline{\Omega}$ for $n \to +\infty$. For easier notation we write $\tilde{\lambda}_n = \tilde{\lambda}_{\varepsilon_n}$, $\tilde{u}_n = \tilde{u}_{\varepsilon_n}$, $\tilde{m}_n = \tilde{m}_{\varepsilon_n}$, and so on. In the following we are going to prove convergence of such sequences, possibly up to further subsequences. The blow-up limits will be related to the corresponding counterpart of the limit problem (1.6), (1.8), namely $\tilde{\lambda}_0$, $w$, $\tilde{m}_0$. We remark that, in case the limit of the subsequence is independent of the starting sequence, then convergence of the whole family follows, as $\varepsilon \to 0$.

We start identifying the limit of the blow-up sequences. To this aim, we notice that, by trivial extension, we can assume that $\tilde{u}_n \in H^1(\mathbb{R}^N)$. On the other hand, it is convenient to extend $\tilde{m}_n$ outside $\tilde{\Omega}_n$ as $-\underline{m}$. In this way we obtain that $\tilde{m}_n + \underline{m} = (\overline{m} + \underline{m})\chi_{\tilde{E}_n}$ so that

$$\tilde{m}_n \in \mathcal{M}' := \left\{ m \in L^\infty(\mathbb{R}^N) : -\underline{m} \le m \le \overline{m}, \quad \int_{\mathbb{R}^N} (m + \underline{m}) \le (\underline{m} + \overline{m}) \right\}. \tag{2.6}$$

**Lemma 2.1.** *For any choice of the blow-up centers $(\mathbf{p}_n)_n$ we have, as $n \to +\infty$,*

1. $\tilde{\lambda}_n \to \tilde{\lambda}_0$;
2. *up to subsequences,*

$$\tilde{\Omega}_n \to \Omega_\infty = \begin{cases} \mathbb{R}^N & \text{if } d(\mathbf{p}_n, \partial\Omega)/k_n \to +\infty \\ \text{an open half-space } H & \text{if } d(\mathbf{p}_n, \partial\Omega)/k_n \le C, \end{cases}$$

*in the sense that, for every $K \subset\subset \Omega_\infty$ (resp. $\mathbb{R}^N \setminus \Omega_\infty$), we have $K \subset\subset \tilde{\Omega}_n$ (resp. $\mathbb{R}^N \setminus \tilde{\Omega}_n$) for $n$ sufficiently large.*

3. *There exists $m_\infty \in \mathcal{M}'$ such that, up to subsequences, $\tilde{m}_n \to m_\infty$ weakly-$*$ in $L^\infty(\mathbb{R}^N)$.*
4. *There exists $u_\infty \in H^1(\mathbb{R}^N) \cap H^1_0(\Omega_\infty)$ such that, up to subsequences, $\tilde{u}_n \to u_\infty$ both weakly in $H^1(\mathbb{R}^N)$ and $C^{1,\alpha}_{loc}(\Omega_\infty)$, for every $0 < \alpha < 1$, and*

$$\int_{\mathbb{R}^N} \nabla u_\infty \cdot \nabla v = \tilde{\lambda}_0 \int_{\mathbb{R}^N} m_\infty u_\infty v \qquad \forall v \in H^1_0(\Omega_\infty). \tag{2.7}$$

*Moreover, $u \ge 0$ in $\mathbb{R}^N$.*

**Proof.** The proof of 1 is analogous to that of [18, Lemma 3.1] (notice, in particular, that $\tilde{\lambda}_n \ge \tilde{\lambda}_0$ is bounded above by the corresponding generalized eigenvalue with spherical weight).

Claim 2 follows by regularity of $\partial\Omega$: in particular, $H$ depends in an elementary way on $\mathbf{p}_\infty$, on the tangent plane to $\partial\Omega$ at $\mathbf{p}_\infty$ and on the asymptotic direction $\lim_n \frac{\mathbf{p}_n - \mathbf{p}_\infty}{|\mathbf{p}_n - \mathbf{p}_\infty|}$.

To show 3 it is sufficient to notice that $(\tilde{m}_n)_n$ is bounded in $L^\infty(\mathbb{R}^N)$, and that $\mathcal{M}'$, defined in (2.6), is closed with respect to the weak-$*$ topology (see [29, Lemma 4.1, Step 3] for further details).

To prove 4 one can proceed similarly as in [18, Proposition 3.3]: first, using the equation and point 1, we show the boundedness of $(\tilde{u}_n)_n$, obtaining weak $H^1(\mathbb{R}^N)$, strong $L^2_{\text{loc}}(\mathbb{R}^N)$ and a.e. convergence to $u_\infty \in H^1(\mathbb{R}^N)$, $u_\infty \ge 0$ (recall that $\tilde{u}_n$ is positive and $L^2$-normalized in $\tilde{\Omega}_n$). Next, by point 2 and definition of $\tilde{u}_n$, we have that $u_\infty = 0$ a.e. in any $K \subset\subset \mathbb{R}^N \setminus \overline{\Omega}_\infty$, implying that (the restriction to $\Omega_\infty$ of) $u_\infty$ belongs to $H^1_0(\Omega_\infty)$.

On the other hand, choose an arbitrary $K \subset\subset \Omega_\infty$ and take any $v \in C_0^\infty(K)$. Then, thanks to 2, for $n$ large enough $\mathrm{supp}(v) \subset \tilde{\Omega}_n$. Thus $\tilde{u}_n$ satisfies:

$$\int_{\mathbb{R}^N} \nabla \tilde{u}_n \cdot \nabla v = \tilde{\lambda}_n \int_{\mathbb{R}^N} \tilde{m}_n \tilde{u}_n v.$$

Using 1, 3, and as $C_0^\infty(\Omega_\infty)$ is dense in $H_0^1(\Omega_\infty)$ we infer (2.7). Finally, compactness in $C^{1,\alpha}(K)$, $0 < \alpha < 1$, follows by the weak equation above, using elliptic regularity and a standard bootstrap argument (see e.g. [29, Lemma 4.1, Step 1]). □

In the following we focus on a subsequence, still denoted with $\varepsilon_n$, such that all the claims in Lemma 2.1 hold true.

**Lemma 2.2.** *For any choice of the blow-up centers $(\mathbf{p}_n)_n$ we have that one of the following two alternatives hold:*

1. *either $u_\infty \equiv 0$ in $\mathbb{R}^N$,*
2. *or $\Omega_\infty = \mathbb{R}^N$ and, up to (the same) translation,*

$$m_\infty = \tilde{m}_0, \qquad u_\infty = Aw,$$

*for some positive constant $A$. In particular, $\mathrm{dist}(\mathbf{p}_n, \partial\Omega)/k^{1/N} \to +\infty$ as $n \to +\infty$.*

**Proof.** Assume that $u_\infty$ is not the null function. Then, taking $v = u_\infty$ in (2.7) we obtain

$$0 < \int_{\mathbb{R}^N} |\nabla u_\infty|^2 = \tilde{\lambda}_0 \int_{\mathbb{R}^N} m_\infty u_\infty^2, \qquad \text{i.e. } \tilde{\lambda}_0 = \frac{\int_{\mathbb{R}^N} |\nabla u_\infty|^2}{\int_{\mathbb{R}^N} m_\infty u_\infty^2}.$$

Since $u \in H^1(\mathbb{R}^N)$, $m \in \mathcal{M}'$ and $\int_{\mathbb{R}^N} m_\infty u_\infty^2 > 0$, the lemma follows by the variational characterization of $\tilde{\lambda}_0$, see [29, Thm. 2.2]. □

To conclude we recall that, in view of [18], one can obtain strong $H^1$ convergence of the blow-up sequence under the further assumption that the optimal favorable sets $\tilde{E}_n$ are uniformly bounded.

**Lemma 2.3.** *Under the above notations, assume that $\tilde{u}_n(\mathbf{0}) \geq C > 0$ and that $\tilde{E}_n \subset K$, for a suitable constant $C$ and a suitable compact set $K$. Then*

$$\tilde{u}_n \to w \qquad \text{strongly in } H^1(\mathbb{R}^N) \text{ and in } C^{1,\alpha}_{loc}$$

*up to a translation.*

**Proof.** By Lemmas 2.1 and 2.2 we only need to prove the strong $H^1$ convergence, which follows proceeding exactly as in [18, Section 4]. □

## 3. Qualitative properties of the favorable region

In this section we prove some qualitative geometric properties of the favorable region $E_\varepsilon$, and of its counterpart in the blow-up scale $\tilde{E}_\varepsilon$, in the asymptotic limit $\varepsilon \to 0^+$. In particular, we will show that it is connected and that it converges in a suitable sense to a ball. The former property will come from the uniqueness of local maximum points for $u_\varepsilon$. Moreover, we will show property 1 of Theorem 1.1, as well as a version in $C^{1,\alpha}$, $\alpha < 1$, of properties 2 and 5.

The main arguments are based on the refinement of the blow-up analysis introduced in the previous section, when centered at maximum points for $u_\varepsilon$. In particular, throughout this section we take $\varepsilon_n \to 0$,

$$\mathbf{p}_n = \mathbf{x}_n = \mathbf{x}_{\varepsilon_n} \text{ a global maximum point of } u_{\varepsilon_n},$$

and we denote with $\tilde{u}_n$ the corresponding blow-up sequence as in (2.2). We are going to show that, for $n$ large enough, any relative maximum $\mathbf{x}'_n$ coincides with $\mathbf{x}_n$.

**Remark 3.1.** In principle, since $\tilde{E}_n$ is a (positive) superlevel set, and homogeneous Dirichlet boundary conditions are required, in general, the number of connected components of $\tilde{E}_n$ is less or equal than the number of local maximum points. Moreover, since $\tilde{u}_n$ is subharmonic in $\tilde{\Omega}_n \setminus \tilde{E}_n$, no local maxima can be achieved outside $\tilde{E}_n$.

**Lemma 3.2.** *Choosing $\mathbf{p}_n = \mathbf{x}_n$ we obtain that the second alternative in Lemma 2.2 occurs, and no translation is needed.*

**Proof.** Taking into account Lemmas 2.1 and 2.2 we have to show that

$$\tilde{u}_n(\mathbf{0}) \geq C > 0. \tag{3.1}$$

Indeed, this will imply convergence of $\tilde{u}_n$ to $Aw$, $A > 0$, up to a translation. However, by uniform convergence (point 4 of Lemma 2.1) and since by construction $\tilde{u}_n(\mathbf{0})$ is a maximum, necessarily the translation must be $\mathbf{0}$ (recall that $w$ is radially symmetric and decreasing).

On the other hand, (3.1) easily follows by the $L^2$ normalization, as in [18, Proposition 3.3 (iv)]. Indeed, since

$$0 < \overline{m} \int_{\tilde{E}_n} \tilde{u}_n^2 - \underline{m} \int_{\mathbb{R}^N \setminus \tilde{E}_n} \tilde{u}_n^2 = (\overline{m} + \underline{m}) \int_{\tilde{E}_n} \tilde{u}_n^2 - \underline{m},$$

$\mathbf{0}$ is a global maximum point and $\mathcal{L}(\tilde{E}_n) = 1$, we infer

$$\tilde{u}_n^2(\mathbf{0}) \geq \int_{\tilde{E}_n} \tilde{u}_n^2 \geq \frac{\underline{m}}{\overline{m} + \underline{m}} > 0. \qquad \square$$

**Remark 3.3.** We notice that the above lemma (and all its consequences throughout this section) hold true by choosing different centers for the blow-up procedure, say $\mathbf{p}_n = \mathbf{y}_n$, as long as $|\mathbf{x}_n - \mathbf{y}_n|/k_n \to 0$ as $n \to +\infty$. Indeed, let us denote with $\tilde{v}_n$ the blow-up sequence of eigenfunctions, centered at $\mathbf{y}_n$; then,

$$\tilde{v}_n(\mathbf{x}) := k_n^{N/2}\, u_n(\mathbf{y}_n + k_n\mathbf{x}) = \tilde{u}_n(\mathbf{x} - (\mathbf{x}_n - \mathbf{y}_n)/k_n),$$

so that the limits of $(\tilde{v}_n)_n$ and $(\tilde{u}_n)_n$ coincides.

The main properties of $\tilde{E}_n$ will descend by the fact that it is a superlevel set. Recall that, as defined in (2.3), $\alpha_n$ is the value of $\tilde{u}_n$ at the boundary of the favorable set. Accordingly, let us denote with $\alpha_0$ the positive real value such that

$$\mathcal{L}\left(\{u_\infty > \alpha_0\}\right) = 1,$$

where $u_\infty = Aw$ is as in Lemma 2.2. We have the following key fact.

**Lemma 3.4.** *We have $\alpha_n \to \alpha_0$ as $n \to +\infty$.*

**Proof.** First notice that $0 < \alpha_n \le \tilde{u}_n(\mathbf{0}) \to u_\infty(\mathbf{0})$. As a consequence, we can extract a subsequence, still denoted with $\alpha_n$, such that $\alpha_n \to \bar{\alpha}$ for $n \to +\infty$. To conclude, we show by contradiction that both $\bar{\alpha} \ge \alpha_0$ and $\bar{\alpha} \le \alpha_0$.

Suppose $\bar{\alpha} < \alpha_0$. Then, since as already remarked $u_\infty$ is radially strictly decreasing, we have that $\mathcal{L}(\{u_\infty > \bar{\alpha}\}) > 1$. Thus, by local uniform convergence on compact sets, for $n$ large enough it also holds that $\mathcal{L}(\{\tilde{u}_n > \alpha_n\}) > 1$, which is a contradiction with the normalization $\mathcal{L}(\{\tilde{u}_n > \alpha_n\}) = 1$. Thus $\bar{\alpha} \ge \alpha_0$.

Now suppose that $\bar{\alpha} > \alpha_0$. Consider the function $\chi_{\{u_\infty > \alpha_0\}} \in L^1(\mathbb{R}^N)$. By Lemma 2.1 we have that

$$\int\limits_{\mathbb{R}^N} \tilde{m}_n \chi_{\{u_\infty > \alpha_0\}} \to \int\limits_{\mathbb{R}^N} \tilde{m}_0 \chi_{\{u_\infty > \alpha_0\}} = \overline{m} \qquad \text{for } n \to +\infty.$$

On the other hand, by uniform convergence on compact sets we have, for $n \to +\infty$,

$$\begin{aligned}\int\limits_{\mathbb{R}^N} \tilde{m}_n \chi_{\{u_\infty > \alpha_0\}} &= \overline{m}\mathcal{L}(\{\tilde{u}_n > \alpha_n\} \cap \{u_\infty > \alpha_0\}) - \underline{m}\mathcal{L}(\{\tilde{u}_n \le \alpha_n\} \cap \{u_\infty > \alpha_0\}) \\ &\le \overline{m}\mathcal{L}(\{\tilde{u}_n > \alpha_n\} \cap \{u_\infty > \alpha_0\}) \le \overline{m}\mathcal{L}(\{u_\infty > \bar{\alpha}\}) + o(1) < \overline{m},\end{aligned}$$

which leads to a contradiction.

Thus $\alpha_n \to \bar{\alpha} = \alpha_0$ for $n \to +\infty$ and the proof is concluded. □

To prove the uniqueness of the local maximum point, whence the connectedness of the favorable region, for $\varepsilon$ small we proceed by contradiction, assuming that $\tilde{u}_n$ admits a second (possibly relative) maximum point, that we denote with $\mathbf{x}'_{\varepsilon_n} = \mathbf{x}'_n$, such that

$$\mathbf{x}_n' \neq \mathbf{x}_n.$$

The first step to obtain a contradiction consists in performing the blow-up analysis, centered at $\mathbf{x}_n'$.

**Lemma 3.5.** *Let us choose $\mathbf{p}_n = \mathbf{x}_n'$ and denote with $\tilde{u}_n'$, $\tilde{m}_n'$ the corresponding blow up sequences. Then, up to subsequences,*

$$\tilde{m}_n' \to \tilde{m}_0 \text{ weakly-}* \text{ in } L^\infty(\mathbb{R}^N), \qquad \tilde{u}_n' \to A'w \text{ weakly in } H^1(\mathbb{R}^N) \text{ and in } C^{1,\alpha}_{loc},$$

*$A' > 0$, as $n \to +\infty$.*

**Proof.** Again, the proof will follow from Lemmas 2.1 and 2.2 once we know that

$$\tilde{u}_n'(\mathbf{0}) \geq C > 0.$$

This follows by Lemma 3.4, since $\mathbf{x}_n' \in E_n$ and thus

$$\tilde{u}_n'(\mathbf{0}) > \alpha_n \to \alpha_0 > 0. \quad \square$$

Now, only three mutually exclusive scenarios can happen for $n \to +\infty$:

1. *(repulsion)* $\dfrac{\operatorname{dist}(\mathbf{x}_n', \mathbf{x}_n)}{k_n} \to +\infty$,
2. *(coexistence)* $c \leq \dfrac{\operatorname{dist}(\mathbf{x}_n', \mathbf{x}_n)}{k_n} \leq C$, for some positive constants $c, C$ independent of $n$,
3. *(collapsing)* $\dfrac{\operatorname{dist}(\mathbf{x}_n', \mathbf{x}_n)}{k_n} \to 0$.

Our aim is to prove that none of the cases above is possible, thus leading to a contradiction.

**Lemma 3.6.** *In the limit $n \to +\infty$ repulsion cannot happen.*

**Proof.** We proceed by contradiction. So, suppose repulsion.

By Lemma 3.5 we have that the blow-up sequence $\tilde{u}_n'$ with respect to the points $\mathbf{x}_n'$ converges uniformly on compact sets to $u_\infty' = A'w$, while as usual $\tilde{u}_n$ converges to $u_\infty = Aw$. We now prove that actually $u_\infty = u_\infty'$.

Let us denote with $\alpha_0'$ the positive real value such that $\mathcal{L}(\{u_\infty' > \alpha_0'\}) = 1$. Then one can repeat the proof of Lemma 3.4 to obtain that $\alpha_n \to \alpha_0'$ for $n \to +\infty$. Thus $\alpha_0 = \alpha_0'$ and thanks to their definitions, and to the strict monotonicity of $w$, this implies $u_\infty' = u_\infty$.

Now the proof follows easily. Indeed, using the uniform convergence on compact sets, for any $\eta > 0$, for $n$ large enough there exists an $R > 0$ independent of $n$ such that

$$\mathcal{L}\left(\{\tilde{u}_n > \alpha_n\} \cap B_R(\mathbf{0})\right) > 1-\eta \qquad \text{and} \qquad \mathcal{L}\left(\{\tilde{u}'_n > \alpha_n\} \cap B_R(\mathbf{0})\right) > 1-\eta.$$

However, due to the repulsion assumption, this implies that $\mathcal{L}(\{\tilde{u}_n > \alpha_n\}) > 2(1-\eta)$, which is a contradiction. □

**Lemma 3.7.** *In the limit $n \to +\infty$ coexistence cannot happen.*

**Proof.** We proceed again by contradiction. Notice that, by assumption,

$$\mathbf{z}_n := \frac{\mathbf{x}'_n - \mathbf{x}_n}{k_n} \in K \subset\subset \mathbb{R}^N, \qquad \mathbf{z}_n \to \mathbf{z}_\infty \neq \mathbf{0},$$

up to subsequences, as $n \to +\infty$.

Moreover, by definition, $\nabla \tilde{u}_n(\mathbf{0}) = \nabla \tilde{u}_n(\mathbf{z}_n) = \mathbf{0}$. Since $\tilde{u}_n$ converges in $C^1(K)$ to $Aw$, $A > 0$ (see Lemma 2.2), we infer that $\nabla w(\mathbf{0}) = \nabla w(\mathbf{z}_\infty) = \mathbf{0}$ with $\mathbf{z}_\infty \neq \mathbf{0}$, contradiction. □

**Lemma 3.8.** *In the limit $n \to +\infty$ collapsing cannot happen.*

**Proof.** We are in a similar situation as in the previous lemma, but now

$$\mathbf{z}_n := \frac{\mathbf{x}'_n - \mathbf{x}_n}{k_n} \neq \mathbf{0}, \qquad \mathbf{z}_n \to \mathbf{0},$$

up to subsequences, as $n \to +\infty$. Then we obtain a contradiction using [30, Lemma 4.2] with $\phi = u_\infty$ and $\psi = \tilde{u}_n$ once one notices that, in Lemma 2.1 due to standard elliptic regularity theory, the convergence in $B_a(\mathbf{0})$ for $a > 0$ small enough is actually in $C^k$ for any $k \geq 0$. □

**Corollary 3.9.** *Any relative maximum point $\mathbf{x}'_n$ satisfies, for $n$ large enough, $\mathbf{x}'_n \equiv \mathbf{x}_n$, so that property 1 of Theorem 1.1 holds true.*

Once we know that, for $n$ sufficiently large, $\tilde{u}_n$ has a unique local maximum point and $\tilde{E}_n$ is connected, we can deduce that the optimal set is asymptotically spherical, in the following sense.

**Lemma 3.10.** *For every $\eta > 0$ sufficiently small there exists $\bar{n}$ such that, for every $n > \bar{n}$,*

$$(1-\eta)B \subset \tilde{E}_n \subset (1+\eta)B.$$

**Proof.** It follows readily from Lemmas 2.3 and 3.4, since they imply that $\partial\tilde{E}_n$ is eventually contained in any neighborhood of $\partial B$ (i.e. of the $\alpha_0$ level set of $u_\infty$). □

Once $\tilde{E}_n$ is uniformly bounded, Lemma 2.3 applies straightforwardly.

**Corollary 3.11.** *Under the above notation, $u_\infty = w$ and the convergence is strong in $H^1(\mathbb{R}^N)$.*

Moreover, exploiting the $C^{1,\alpha}$ regularity of the family $\tilde{u}_n$, we can prove that $\partial\tilde{E}_n$ is nearly spherical, in the sense of Definition 1.4.

**Proposition 3.12.** *For $n$ sufficiently large $\partial\tilde{E}_n$ is nearly spherical of class $C^{1,\alpha}$, for all $0 \leq \alpha < 1$, represented by $\varphi_{\tilde{E}_n}$. Moreover*

$$\|\varphi_{\tilde{E}_n}\|_{C^{1,\alpha}(\partial B)} \to 0, \qquad \text{for } n \to +\infty. \tag{3.2}$$

**Proof.** We use polar coordinates $\mathbf{x} = \rho\theta$, with $\rho > 0$, $\theta \in \mathbb{S}^{N-1}$. Recalling that $\partial\tilde{E}_n$ is the $\alpha_n$-level of the function $\tilde{u}_n$, we obtain that $\varphi_{\tilde{E}_n} = \rho - r_0$ is implicitly defined as

$$\tilde{u}_\varepsilon((r_0 + \varphi_{\tilde{E}_n})\theta) = \alpha_n.$$

Now, consider $n$ large enough so that $\partial\tilde{E}_n \subset B_R \setminus B_r$ and

$$\max_{B_R\setminus B_r} \partial_\rho \tilde{u}_n \leq \frac{1}{2} \max_{B_R\setminus B_r} \partial_\rho w < 0, \tag{3.3}$$

where $r < r_0 < R$ are fixed. This is possible thanks to Lemmas 2.3 and 3.10. Since $\partial\tilde{E}_n \subset B_R \setminus B_r$, the existence of $\varphi_{\tilde{E}_n}$ follows; moreover, the Implicit Function Theorem applies and $\varphi_{\tilde{E}_n}$ is at least of class $C^1$. A direct computation gives

$$\nabla\varphi_{\tilde{E}_n} = -\frac{\big(r_0 + \varphi_{\tilde{E}_n}(\theta)\big)}{\partial_\rho \tilde{u}_n(\mathbf{x})} \nabla_T \tilde{u}_n = -\frac{\big(r_0 + \varphi_{\tilde{E}_n}(\theta)\big)}{\partial_\rho \tilde{u}_n(\mathbf{x})} \left(\nabla\tilde{u}_n - (\nabla\tilde{u}_n \cdot \theta)\theta\right), \tag{3.4}$$

where $\nabla_T$ denotes the tangential component of the gradient. Since $\tilde{u}_n$ converges in $C^{1,\alpha}$ to the radial function $w$, we deduce that $\varphi_{\tilde{E}_n}$ is of class $C^{1,\alpha}$ and that (3.2) holds true, concluding the proof. □

**Remark 3.13.** Denote with $\mathrm{bar}(\tilde{E}_n)$ the barycenter of $\tilde{E}_n$. Thanks to Lemma 3.10 we obtain that

$$\mathrm{bar}(\tilde{E}_n) \to \mathbf{0} \qquad \text{as } n \to +\infty.$$

As a consequence, by Remark 3.3 we have that all the results in this section, and in particular the last proposition, hold true if we center the blow-up procedure at the barycenters of the favorable regions $E_\varepsilon$ (and not at the unique maximum points of $u_\varepsilon$).

To conclude this section, we use the maximum principle to show the exponential decay of $\tilde{u}_n$, uniformly in $n$. This will be crucial for the asymptotic location of the favorable zone, discussed in the next section.

**Lemma 3.14.** *For every $\eta > 0$, if $n$ is sufficiently large then*

$$0 \le \tilde{u}_n(\mathbf{x}) \le (1+\eta)w(\mathbf{x}) \qquad \textit{for } \mathbf{x} \in \mathbb{R}^N \setminus (1+\eta)B.$$

**Proof.** Since $\tilde{u}_n \equiv 0$ outside $\tilde{\Omega}_n$, the proof is an easy application of the maximum principle in $A = \tilde{\Omega}_n \setminus (1+\eta)\overline{B}$: indeed, for every $\eta > 0$ and $n$ sufficiently large we have that

$$-\Delta(\tilde{u}_n - (1+\eta)w) + \tilde{\lambda}_0 \underline{m}(\tilde{u}_n - (1+\eta)w) = -(\tilde{\lambda}_n - \tilde{\lambda}_0)\underline{m}\tilde{u}_n \le 0 \qquad \text{in } A$$

by Lemma 3.10, and $\tilde{u}_n - (1+\eta)w \le 0$ on $\partial A$ by uniform convergence on compact sets. □

## 4. Asymptotic location of the favorable zone

In this section we address the asymptotic position of the favorable zone, i.e. the limit location of the global maximum point $\mathbf{x}_n = \mathbf{x}_{\varepsilon_n}$ of the optimal eigenfunction $u_n = u_{\varepsilon_n}$. As a byproduct of our analysis we will provide also an expansion of the associated eigenvalue. To proceed, we introduce the following notation:

$$d^* := \max_{\mathbf{p}\in\Omega} \operatorname{dist}(\mathbf{p}, \partial\Omega), \qquad d_n := \operatorname{dist}(\mathbf{x}_n, \partial\Omega), \qquad d_\infty := \lim_{n\to+\infty} d_n$$

(up to subsequences). Notice that, in the blow-up scale, all the above distances are divided by $k_n = \varepsilon_n^{1/N}$. In particular, $d_n/k_n \to +\infty$, as $n \to +\infty$, by Lemma 3.2. We will prove the following.

**Proposition 4.1.** *Under the previous notation, we have:*

1. $d_\infty = d^*$;
2. $\tilde{\lambda}_n = \tilde{\lambda}_0 + e^{-2\sqrt{\tilde{\lambda}_0 \underline{m}}(1+o(1))d^*/k_n}$, *as $n \to +\infty$.*

**Corollary 4.2.** *Properties 3 and 4 of Theorem 1.1 hold true.*

To prove the proposition, we follow the strategy proposed in [15] to deal with the analogous problem, in the framework of least energy solutions to semilinear elliptic equations. As in [15], we base our analysis on the sharp asymptotics for the problem settled in radially symmetric domains, i.e. in a ball, instead of $\Omega$. We recall that in such case, by Schwarz symmetrization, the optimal favorable domain is a concentric ball of the appropriate measure (as proved in [6, Remark 3.10]).

**Lemma 4.3.** *Let $B_R$ denote the ball of radius $R$, $R$ large, and $B \subset B_R$ denote the concentric ball of unitary volume. Then*

$$\Lambda(R) = \lambda^1(B, B_R) = \inf_{A\subset B_R:\mathcal{L}(A)=1} \lambda^1(A, B_R)$$

*satisfies*

$$\Lambda(R) = \tilde{\lambda}_0 + e^{-2\sqrt{\tilde{\lambda}_0 \underline{m}}(1+o(1))R} \qquad \text{as } r \to +\infty. \tag{4.1}$$

**Proof.** This result can be proved in several different ways: an overcomplicated (although direct) strategy consists in deducing it as a byproduct of [18, Theorem 1.3]; alternatively, one can use sub/supersolutions, in the spirit of Lemma 3.14 above, as done in [15] for the semilinear case; finally, since our problem is linear, one can evaluate $\Lambda(R)$ as an implicit function defined in terms of Bessel functions. □

Proposition 4.1 will follow by estimating $\tilde{\lambda}_n$ both from above and below, in terms of $\Lambda(R)$ evaluated at values of $R$ suitably depending on $d^*, d_\infty$.

**Lemma 4.4.** *We have*

$$\tilde{\lambda}_n \leq \Lambda\left(\frac{d^*}{k_n}\right).$$

**Proof.** By the definitions of $d^*$ and $\tilde{\Omega}_n$, we have that there exists $\mathbf{x}_n^*$ such that $B_{d^*/k_n}(\mathbf{x}_n^*) \subset \tilde{\Omega}_n$. The lemma follows by the characterization of $\Lambda$. □

**Lemma 4.5.** *For every $\delta > 0$ sufficiently small,*

$$\tilde{\lambda}_n \geq \Lambda\left(\frac{d_\infty + \delta}{k_n}\right) + o\left(e^{-2\sqrt{\tilde{\lambda}_0 \underline{m}}(d_\infty+\delta)/k_n}\right) \qquad \text{as } n \to +\infty.$$

**Proof.** Let $\delta > 0$ be fixed, and $n$ sufficiently large, in such a way that $d_\infty + \delta > d_n$. By definition, this means that $B_{(d_\infty+\delta)/k_n}(\mathbf{0})$ is not completely contained in $\tilde{\Omega}_n$ (and the same is true for any larger radius). On the other hand, since $B_{d_n/k_n}(\mathbf{0}) \subsetneq \tilde{\Omega}_n$ (if $\tilde{\Omega}_n$ is a ball there is nothing to prove, by Lemma 4.3), if $\delta$ is sufficiently small we have that $\mathcal{L}(B_{(d_\infty+\delta)/k_n}(\mathbf{0})) < \mathcal{L}(\tilde{\Omega}_n)$. Thanks to these facts, we can choose $\delta_n' > \delta$ in such a way that the set

$$A_n = B_{(d_\infty+\delta_n')/k_n}(\mathbf{0}) \cap \tilde{\Omega}_n$$

satisfies

$$\mathcal{L}(A_n) = \mathcal{L}(B_{(d_\infty+\delta)/k_n}(\mathbf{0})).$$

Notice that, as $n \to +\infty$, up to subsequences $\mathbf{x}_n \to \mathbf{x}_\infty$ and

$$\delta_n' \to \delta' > \delta, \qquad \text{where} \qquad \mathcal{L}(B_{d_\infty+\delta'}(\mathbf{x}_\infty) \cap \Omega) = \mathcal{L}(B_{d_\infty+\delta}(\mathbf{x}_\infty)).$$

Since $(d_\infty + \delta'_n)/k_n \to +\infty$, as $n \to +\infty$, Lemma 3.10 implies that, for $n$ sufficiently large, $\tilde{E}_n \subset\subset A_n$; by [6, Remark 3.10] we infer that

$$\Lambda\left(\frac{d_\infty+\delta}{k_n}\right) \le \inf_{D\subset A_n:\mathcal{L}(D)=1} \lambda^1(D, A_n) \le \lambda^1(\tilde{E}_n, A_n). \tag{4.2}$$

We now construct an admissible function to estimate $\lambda^1(\tilde{E}_n, A_n)$, by suitably cutting-off $\tilde{u}_n$ on $A_n$. More precisely, let $\eta$ be a smooth cut-off function, such that

$$\eta_n = \begin{cases} 1 & B_{(d_\infty+\delta'_n)/k_n-1}(\mathbf{0}) \\ 0 & \mathbb{R}^N \setminus B_{(d_\infty+\delta'_n)/k_n}(\mathbf{0}) \end{cases}$$

and $|\nabla\eta_n| \le 2$; again, by Lemma 3.10, $\tilde{E}_n \subset\subset B_{(d_\infty+\delta'_n)/k_n-1}(\mathbf{0})$ for $n$ large and

$$\eta_n(\mathbf{x}) \ne 1 \qquad \Longrightarrow \qquad \tilde{m}_n(\mathbf{x}) = -\underline{m} < 0. \tag{4.3}$$

Let us consider the function

$$v_n = \eta_n \tilde{u}_n \in H^1_0(A_n).$$

Using (4.3) we obtain

$$\int_{A_n} \tilde{m}_n v_n^2 = \int_{\tilde{\Omega}_n} \tilde{m}_n \eta_n^2 \tilde{u}_n^2 = \int_{\tilde{\Omega}_n} \tilde{m}_n \tilde{u}_n^2 - \int_{\tilde{\Omega}_n} \tilde{m}_n (1-\eta_n^2)\tilde{u}_n^2 \ge \int_{\tilde{\Omega}_n} \tilde{m}_n \tilde{u}_n^2 > 0.$$

Then $v$ is admissible for $\lambda^1(\tilde{E}_n, A_n)$ and (4.2) yields

$$\begin{aligned} \Lambda\left(\frac{d_\infty+\delta}{k_n}\right) &\le \frac{\int_{A_n} |\nabla v_n|^2}{\int_{A_n} \tilde{m}_n v_n^2} = \frac{\int_{A_n} |\nabla(\eta_n\tilde{u}_n)|^2}{\int_{A_n} \tilde{m}_n \eta_n^2 \tilde{u}_n^2} = \frac{\int_{A_n} -\Delta\tilde{u}_n \cdot \eta_n^2\tilde{u}_n + \int_{A_n} \tilde{u}_n^2 |\nabla\eta_n|^2}{\int_{A_n} \tilde{m}_n \eta_n^2 \tilde{u}_n^2} \\ &= \tilde{\lambda}_n + \frac{\int_{A_n} \tilde{u}_n^2 |\nabla\eta_n|^2}{\int_{A_n} \tilde{m}_n \eta_n^2 \tilde{u}_n^2}. \end{aligned}$$

Now, by strong $H^1$ convergence of $\tilde{u}_n$ and weak-$*$ $L^\infty$ convergence of $\tilde{m}_n$, we have that

$$\int_{A_n} \tilde{m}_n \eta_n^2 \tilde{u}_n^2 \ge \int_{\mathbb{R}^N} \tilde{m}_n \tilde{u}_n^2 \to \int_{\mathbb{R}^N} \tilde{m}_0 w^2 > 0.$$

On the other hand, recalling Lemma 3.14 and the decay of $w$ (see equation (1.9)), we have that, for $|\mathbf{x}|$ large, $\tilde{u}_n^2(\mathbf{x}) \le C|\mathbf{x}|^{-(N-1)} e^{-2\sqrt{\tilde{\lambda}_0 \underline{m}}|\mathbf{x}|}$. Integrating, we conclude that, for $n$ sufficiently large,

$$\Lambda\left(\frac{d_\infty+\delta}{k_n}\right) \le \tilde{\lambda}_n + Ce^{-2\sqrt{\tilde{\lambda}_0\underline{m}}(d_\infty+\delta'_n)/k_n} = \tilde{\lambda}_n + Ce^{-2\sqrt{\tilde{\lambda}_0\underline{m}}(1+o(1))(d_\infty+\delta')/k_n},$$

and the lemma follows since $\delta' > \delta$. $\square$

**Proof of Proposition 4.1.** Taking into account Lemmas 4.3, 4.5 and 4.4 we can write, for every $\delta > 0$ sufficiently small,

$$\tilde{\lambda}_0 + e^{-2\sqrt{\tilde{\lambda}_0 \underline{m}}(1+o(1))(d_\infty+\delta)/k_n} \leq \tilde{\lambda}_n \leq \tilde{\lambda}_0 + e^{-2\sqrt{\tilde{\lambda}_0 \underline{m}}(1+o(1))d^*/k_n},$$

as $n \to +\infty$. In particular, we have $d^* \leq d_\infty + \delta$, for every $\delta > 0$. Since by definition $d_\infty \leq d^*$, we deduce that actually $d_\infty = d^*$. Substituting in the inequality above, since $\delta$ is arbitrary we obtain the asymptotics of $\lambda_n$. $\square$

## 5. $C^{1,1}$ regularity

In this section we provide some sharper results on the regularity of the eigenfunctions and of the boundary of the favourable regions. Such results are of interest by themselves, but they will also guarantee enough regularity to justify all the steps in the differentiation of the eigenvalue with respect to the shape, carried out in Section 6.2.

We begin with a sharper analysis of the regularity of the blow-up family of eigenfunctions. To this aim, we exploit some recent results on regularity theory for transmission problems with $C^{1,\alpha}$ interfaces, proved by Caffarelli, Soria-Carro, and Stinga in [5] for the case of harmonic functions, and generalized by Dong in [17] to the case of elliptic equations in divergence form.

**Proposition 5.1.** *For any $R$ sufficiently large there exists $\varepsilon_R > 0$ such that, if $\varepsilon \leq \varepsilon_R$, the functions $\tilde{u}_\varepsilon$ belong to $C^{2,\alpha}\left(\overline{\tilde{E}_\varepsilon}\right) \bigcap C^{2,\alpha}\left(\overline{B_R \setminus \tilde{E}_\varepsilon}\right)$.*

**Proof.** The idea is to derive a transmission problem for each component of $\nabla \tilde{u}_\varepsilon$, and then apply the regularity result [17, Theorem 1.2].

Consider any partial derivative $\partial_i \tilde{u}_\varepsilon$ along the vectors of the orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_N$. Applying the Gauss-Green formula and exploiting the $C^{1,\alpha}\left(\overline{\tilde{\Omega}_\varepsilon}\right) \cap H^2\left(\tilde{\Omega}_\varepsilon\right)$ regularity of $\tilde{u}_\varepsilon$ that holds by standard elliptic regularity, for any $\varphi \in C_c^\infty(\tilde{\Omega}_\varepsilon)$ we can write

$$\begin{aligned}
\lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon) \int_{\tilde{\Omega}_\varepsilon} \tilde{m}_\varepsilon \varphi \partial_i \tilde{u}_\varepsilon &= \lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon) \left[ \overline{m} \int_{\tilde{E}_\varepsilon} \varphi \partial_i \tilde{u}_\varepsilon - \underline{m} \int_{\tilde{\Omega}_\varepsilon \setminus \tilde{E}_\varepsilon} \varphi \partial_i \tilde{u}_\varepsilon \right] = \\
&= \lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon)(\overline{m} + \underline{m}) \int_{\partial \tilde{E}_\varepsilon} \tilde{u}_\varepsilon \varphi \nu_i - \lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon) \left[ \overline{m} \int_{\tilde{E}_\varepsilon} \tilde{u}_\varepsilon \partial_i \varphi - \underline{m} \int_{\tilde{\Omega}_\varepsilon \setminus \tilde{E}_\varepsilon} \tilde{u}_\varepsilon \partial_i \varphi \right],
\end{aligned} \tag{5.1}$$

where $\nu_i$ denotes the $i$-th component of the outer unit normal to $\tilde{E}_\varepsilon$. Using the equation of $\tilde{u}_\varepsilon$ we obtain

$$\begin{aligned}\lambda^1(\tilde{E}_\varepsilon,\tilde{\Omega}_\varepsilon)\left[\overline{m}\int_{\tilde{E}_\varepsilon}\tilde{u}_\varepsilon\partial_i\varphi-\underline{m}\int_{\tilde{\Omega}_\varepsilon\setminus\tilde{E}_\varepsilon}\tilde{u}_\varepsilon\partial_i\varphi\right]&=-\int_{\tilde{E}_\varepsilon}\Delta\tilde{u}_\varepsilon\partial_i\varphi-\int_{\tilde{\Omega}_\varepsilon\setminus\tilde{E}_\varepsilon}\Delta\tilde{u}_\varepsilon\partial_i\varphi=\\ =\int_{\tilde{\Omega}_\varepsilon}\nabla\tilde{u}_\varepsilon\partial_i\nabla\varphi=-\int_{\tilde{\Omega}_\varepsilon}\nabla(\partial_i\tilde{u}_\varepsilon)\nabla\varphi.\end{aligned}\tag{5.2}$$

Let us denote $v := \partial_i\tilde{u}_\varepsilon$. Combining (5.1) with (5.2), we get that $v$ solves

$$\begin{cases}-\Delta v=\lambda^1(\tilde{E}_\varepsilon,\tilde{\Omega}_\varepsilon)\tilde{m}_\varepsilon v & \text{in } \tilde{E}_\varepsilon\bigcup\left(\tilde{\Omega}_\varepsilon\setminus\overline{\tilde{E}_\varepsilon}\right),\\ [v]=0,\quad [\partial_\nu v]=-\lambda^1(\tilde{E}_\varepsilon,\tilde{\Omega}_\varepsilon)(\overline{m}+\underline{m})\tilde{u}_\varepsilon\nu_i & \text{on } \partial\tilde{E}_\varepsilon,\\ v=\partial_i\tilde{u}_\varepsilon & \text{on } \partial\tilde{\Omega}_\varepsilon,\end{cases}\tag{5.3}$$

where the symbol $[\cdot]$ denotes the jump operator:

$$[f](\mathbf{x})=\lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in\tilde{E}_\varepsilon}}f(\mathbf{y})-\lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in\tilde{\Omega}_\varepsilon\setminus\overline{\tilde{E}_\varepsilon}}}f(\mathbf{y})\tag{5.4}$$

and the boundary condition is understood in the sense of traces. Now, for $\varepsilon$ sufficiently small we have that $B_R\subset\subset\tilde{\Omega}_\varepsilon$. To get rid of the boundary conditions, we notice that, since $\partial_i\tilde{u}_\varepsilon\in C^\infty(B_{R+\delta})$ for some $\delta$ sufficiently small, depending on $R$ but independent of $\varepsilon$, by smooth cutoff there exists an extension $\bar{v}$ of $\partial_i\tilde{u}_\varepsilon$ in $B_R$ of class $C^\infty(\overline{B_R})$. Hence, we can rewrite problem (5.3) with respect to the function $v-\bar{v}$ in $B_R$, and apply [17, Theorem 1.2]: since $\partial\tilde{E}_\varepsilon$ is $C^{1,\alpha}$, by Proposition 3.12, we infer that $v$ is $C^{1,\alpha}$ on both sides and up to $\partial\tilde{E}_\varepsilon$, and the proposition follows. □

From Proposition 5.1, we can deduce the following result.

**Corollary 5.2.** *For any $R$ sufficiently large there exists $\varepsilon_R>0$ such that, if $\varepsilon\le\varepsilon_R$, the functions $\tilde{u}_\varepsilon$ belong to $C^{1,1}\left(\overline{B_R}\right)$.*

**Proof.** Since $\tilde{u}_\varepsilon\in C^{2,\alpha}\left(\overline{\tilde{E}_\varepsilon}\right)\bigcap C^{2,\alpha}\left(\overline{B_R\setminus\tilde{E}_\varepsilon}\right)$, $\tilde{u}_\varepsilon\in C^{1,\alpha}\left(\overline{B_R}\right)$ and $\partial\tilde{E}_\varepsilon$ is of class $C^{1,\alpha}$, we deduce that $\tilde{u}_\varepsilon\in W^{2,\infty}(B_R)$. In particular, each partial derivative is in $W^{1,\infty}(B_R)$ and thus it is Lipschitz continuous (see e.g. [25, Section 11.4]). □

The next result concerns the convergence, in a certain sense, of $\tilde{u}_\varepsilon$ to $w$ in $C^{1,1}$, as $\varepsilon$ approaches $0^+$.

We remark that, in general, we cannot expect a $C^{1,1}$ convergence in $\overline{B_R}$, since the second derivatives of both $\tilde{u}_\varepsilon$ and $w$ have jumps, and the jump points coincide only in the limit $\varepsilon\to 0^+$ but never coincide for $\varepsilon\neq 0$. Indeed, recall that $\tilde{E}_\varepsilon$ is never exactly a ball, unless $\Omega$ is a ball [24], while $w$ is radially symmetric by [18, Proposition 2.1].

Notice however that, at least for $w$, the jump occurs only for the derivative along the normal to the optimal region $\partial B$, while the tangential derivatives vanish. Hence, we expect that, also for the blow-up solutions $\tilde{u}_\varepsilon$, the tangential second derivatives to spheres vanish uniformly, where they are defined.

To make our ideas rigorous, we exploit once again the aforementioned regularity theory for interface problems. Moreover, for $\varepsilon$ small, we consider the nearly spherical representation $\varphi_{\tilde{E}_\varepsilon}$ of $\partial\tilde{E}_\varepsilon$, which is of class $C^{1,\alpha}$, $\alpha < 1$, by Proposition 3.12. We consider the following family of diffeomorphisms $\mathbf{\Phi}_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$, written in polar coordinates $\mathbf{x} = \rho\,\theta$ as

$$\mathbf{\Phi}_\varepsilon(\rho\,\theta) := \left[\rho^N + h(\rho)\left((r_0 + \varphi_{\tilde{E}_\varepsilon}(\theta))^N - r_0^N\right)\right]^{1/N}\theta \qquad \text{for } \theta \in \mathbb{S}^{N-1},\ \rho \geq 0,$$

where the smooth cut-off function $0 \leq h(\rho) \leq 1$ is such that $h \equiv 1$ for $3r_0/4 \leq \rho \leq 5r_0/4$ and $h \equiv 0$ outside $(r_0/2, 3r_0/2)$. Then one can directly check that $\mathbf{\Phi}_\varepsilon$ satisfies the following properties:

$$\begin{aligned} &\|\mathbf{\Phi}_\varepsilon - \mathbf{I}\|_{C^{1,\alpha}(\mathbb{R}^N)} = o(1) \text{ for some } 0 < \alpha < 1, \\ &\mathbf{\Phi}_\varepsilon = \mathbf{I} \text{ in } \mathbb{R}^N \setminus B_{R/2}, \quad \mathbf{\Phi}_\varepsilon(B) = \tilde{E}_\varepsilon, \quad \mathbf{\Phi}_\varepsilon(\partial B) = \partial\tilde{E}_\varepsilon, \end{aligned} \tag{5.5}$$

where we are assuming $R > 3r_0$, and the quantities $o(1)$ are to be intended for $\varepsilon \to 0^+$. For more details on the properties of $\mathbf{\Phi}_\varepsilon$ we refer to Section 6.1, where a more general class of such maps is treated.

Given a family of orthonormal coordinate axes $\mathbf{e}_1, \ldots, \mathbf{e}_N$, we define the transported functions

$$v_{\varepsilon,i} := (\partial_i \tilde{u}_\varepsilon) \circ \mathbf{\Phi}_\varepsilon, \qquad i = 1, \ldots, N$$

then we can prove the following.

**Proposition 5.3.** *For any $R$ sufficiently large, in the limit $\varepsilon \to 0^+$ it holds that*

$$v_{\varepsilon,i} \to \partial_i w \qquad \text{in } C^{1,\alpha}\left(\overline{B}\right) \cap C^{1,\alpha}\left(\overline{B_R \setminus B}\right)$$

*for any $i = 1, \ldots, N$ and any $0 \leq \alpha < 1$.*

**Proof.** Fix an $i \in \{1, \ldots, N\}$. Since $\partial_i \tilde{u}_\varepsilon$ solves (5.3), the function $v_{\varepsilon,i}$ solves in $H^1(B_R)$

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla v_{\varepsilon,i}) = \lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon)\tilde{m}_0 v_{\varepsilon,i} J_\varepsilon & \text{in } B \bigcup \left(B_R \setminus \overline{B}\right), \\ [v_{\varepsilon,i}] = 0, \quad [\partial_{\mathbf{n}} v_{\varepsilon,i}] = -\lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon)(\overline{m} + \underline{m})(\tilde{u}_\varepsilon \nu_i) \circ \mathbf{\Phi}_\varepsilon J_{\varepsilon,T} & \text{on } \partial B, \\ v_{\varepsilon,i} = (\partial_i \tilde{u}_\varepsilon) \circ \mathbf{\Phi}_\varepsilon & \text{on } \partial B_R, \end{cases} \tag{5.6}$$

where $\mathbf{n}$ and $\nu$ are the unitary outer normal vectors to $\partial B$ and $\partial\tilde{E}_\varepsilon$ respectively, while

$$\begin{aligned} J_\varepsilon := |\det(D\mathbf{\Phi}_\varepsilon)|, \quad A_\varepsilon := J_\varepsilon D\mathbf{\Phi}_\varepsilon^{-1} D\mathbf{\Phi}_\varepsilon^{-T} \qquad &\text{in } \mathbb{R}^N \\ J_{\varepsilon,T} := |D\mathbf{\Phi}_\varepsilon^{-T}\mathbf{n}| J_\varepsilon \qquad &\text{on } \partial B. \end{aligned} \tag{5.7}$$

Taking into account the fact that the function $v_{0,i} := \partial_i w$ solves in $H^1(B_R)$

$$\begin{cases} -\Delta v_{0,i} = \lambda^1(B,\mathbb{R}^N)\tilde{m}_0 v_{0,i} & \text{in } B \bigcup \left(B_R \setminus \overline{B}\right), \\ [v_{0,i}] = 0, \quad [\partial_{\mathbf{n}} v_{0,i}] = -\lambda^1(B,\mathbb{R}^N)(\overline{m}+\underline{m}) w n_i & \text{on } \partial \tilde{E}_\varepsilon, \\ v_{0,i} = \partial_i w & \text{on } \partial B_R, \end{cases} \tag{5.8}$$

combining (5.6) and (5.8) we can write the following problem, in $H^1(B_R)$, for the difference $z_{\varepsilon,i} := v_{\varepsilon,i} - v_{0,i}$:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla z_{\varepsilon,i}) = \operatorname{div}(F_{\varepsilon,i}) + f_{\varepsilon,i} & \text{in } B \bigcup \left(B_R \setminus \overline{B}\right), \\ [z_{\varepsilon,i}] = 0, \quad [\partial_{\mathbf{n}} z_{\varepsilon,i}] = g_{\varepsilon,i} & \text{on } \partial B, \\ z_{\varepsilon,i} = h_{\varepsilon,i} & \text{on } \partial B_R, \end{cases} \tag{5.9}$$

where we have defined

$$\begin{aligned} &F_{\varepsilon,i} := (A_\varepsilon - \mathbf{I})\nabla v_{0,i}, \qquad f_{\varepsilon,i} := \lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon)\tilde{m}_0 v_{\varepsilon,i} J_\varepsilon - \lambda^1(B,\mathbb{R}^N)\tilde{m}_0 v_{0,i}, \\ &g_{\varepsilon,i} := (\overline{m}+\underline{m}) \left[\lambda^1(B,\mathbb{R}^N) w n_i - \lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon)(\tilde{u}_\varepsilon \nu_i) \circ \mathbf{\Phi}_\varepsilon J_{\varepsilon,T}\right], \\ &h_{\varepsilon,i} := \partial_i \tilde{u}_\varepsilon - \partial_i w \text{ on } \partial B_R \end{aligned} \tag{5.10}$$

(recall that $\mathbf{\Phi}$ is the identity near $\partial B_R$). Now we wish to reduce problem (5.9) to the case of homogeneous Dirichlet boundary conditions. To this aim, we recall that, by standard (interior) elliptic regularity, it holds that $h_{\varepsilon,i} = \partial_i \tilde{u}_\varepsilon - \partial_i w \to 0$, in a neighborhood of $\partial B_R$, in $C^k$ for every $k \geq 0$. Using a smooth cutoff, we can easily construct functions $\bar{z}_{\varepsilon,i} \in C^{1,\alpha}\left(\overline{B_R}\right)$ satisfying

$$\bar{z}_{\varepsilon,i} = h_{\varepsilon,i} \text{ on } \partial B_R, \quad \bar{z}_{\varepsilon,i} = 0 \text{ in } B_{(1-\eta)R}, \quad \|\bar{z}_{\varepsilon,i}\|_{C^{1,\alpha}(\overline{B_R})} \to 0 \text{ for } \varepsilon \to 0^+. \tag{5.11}$$

Exploiting $\bar{z}_{\varepsilon,i}$ and introducing the quantity

$$\tilde{F}_{\varepsilon,i} := F_{\varepsilon,i} + A_\varepsilon \nabla \bar{z}_{\varepsilon,i}, \tag{5.12}$$

we can rewrite (5.9) as

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla(z_{\varepsilon,i} - \bar{z}_{\varepsilon,i})) = \operatorname{div}(\tilde{F}_{\varepsilon,i}) + f_{\varepsilon,i} & \text{in } B \bigcup \left(B_R \setminus \overline{B}\right), \\ [z_{\varepsilon,i}] = 0, \quad [\partial_{\mathbf{n}} z_{\varepsilon,i}] = g_{\varepsilon,i} & \text{on } \partial B, \\ z_{\varepsilon,i} - \bar{z}_{\varepsilon,i} = 0 & \text{on } \partial B_R. \end{cases} \tag{5.13}$$

Now we are in the position to use again [17, Theorem 1.2] to deduce that

$$\begin{aligned} &\|z_{\varepsilon,i} - \bar{z}_{\varepsilon,i}\|_{C^{1,\alpha}(\overline{B})} + \|z_{\varepsilon,i} - \bar{z}_{\varepsilon,i}\|_{C^{1,\alpha}\left(\overline{B_R \setminus B}\right)} \leq \\ &\leq C\left(\|g_{\varepsilon,i}\|_{C^{0,\alpha}(\partial B)} + \|\tilde{F}_{\varepsilon,i}\|_{C^{0,\alpha}(B)} + \|\tilde{F}_{\varepsilon,i}\|_{C^{0,\alpha}\left(\overline{B_R \setminus B}\right)} + \|f_{\varepsilon,i}\|_{L^\infty(B_R)}\right), \end{aligned} \tag{5.14}$$

for some positive constant $C$ independent of $\varepsilon$, thanks to (5.5). Thus, thanks to (5.11), to conclude the proof of Proposition 5.3 it is sufficient to establish that the right hand side of (5.14) is infinitesimal for $\varepsilon \to 0^+$.

For the reader's convenience, we split this estimate in Lemmas 5.5, 5.6, 5.7 below, which are preceded by the preliminary Lemma 5.4 dealing with the properties of geometrical quantities. $\square$

**Lemma 5.4.** *Given the definitions in* (5.7)*, it holds that*

*(i)* $\|J_\varepsilon - 1\|_{C^{0,\alpha}(\mathbb{R}^N)} \to 0$ *for* $\varepsilon \to 0^+$,
*(ii)* $\|A_\varepsilon - \mathbf{I}\|_{C^{0,\alpha}(\mathbb{R}^N)} \to 0$ *for* $\varepsilon \to 0^+$,
*(iii)* $\|J_{\varepsilon,T} - 1\|_{C^{0,\alpha}(\partial B)} \to 0$ *for* $\varepsilon \to 0^+$,
*(iv)* $\|\nu \circ \mathbf{\Phi}_\varepsilon - \mathbf{n}\|_{C^{0,\alpha}(\partial B)} \to 0$ *for* $\varepsilon \to 0^+$,

*where* $\nu$ *and* $\mathbf{n}$ *are the unitary outer normal vectors to* $\partial \tilde{E}_\varepsilon$ *and* $\partial B$*, respectively.*

**Proof.** Let us prove point (i). The fact that $\|J_\varepsilon - 1\|_{C^0(\mathbb{R}^N)} \to 0$ for $\varepsilon \to 0^+$ follows directly from (5.5). Moreover, since the map $\det : M^{N\times N} \to \mathbb{R}$ is of class $C^\infty$, by composition and exploiting again (5.5), it follows that $\||\det(D\mathbf{\Phi}_\varepsilon)| - 1\|_{C^{0,\alpha}(\mathbb{R}^N)} = \|\det(D\mathbf{\Phi}_\varepsilon) - 1\|_{C^{0,\alpha}(\mathbb{R}^N)} \to 0$ for $\varepsilon \to 0^+$.

For point (ii) we notice that, thanks to (5.5), it holds that $\|D\mathbf{\Phi}_\varepsilon^{-1} - \mathbf{I}\|_{C^{0,\alpha}(\mathbb{R}^N)} \to 0$ for $\varepsilon \to 0^+$. An analogous property holds for $D\mathbf{\Phi}_\varepsilon^{-T}$. Hence, exploiting point $(i)$ the conclusion follows.

To deal with point (iii), we first notice that, thanks to (5.5), we have $\|D\mathbf{\Phi}_\varepsilon^{-T}\mathbf{n} - \mathbf{n}\|_{C^{0,\alpha}(\partial B)} \to 0$ for $\varepsilon \to 0^+$. In a similar fashion, it also holds that $\||D\mathbf{\Phi}_\varepsilon^{-T}\mathbf{n}| - 1\|_{C^{0,\alpha}(\partial B)} = \||D\mathbf{\Phi}_\varepsilon^{-T}\mathbf{n}| - |\mathbf{n}|\|_{C^{0,\alpha}(\partial B)} \to 0$ for $\varepsilon \to 0^+$. The conclusion follows exploiting point $(i)$.

Now we study point (iv). We recall that

$$\nu \circ \mathbf{\Phi}_\varepsilon = \frac{D\mathbf{\Phi}_\varepsilon^{-T}\mathbf{n}}{|D\mathbf{\Phi}_\varepsilon^{-T}\mathbf{n}|},$$

so that the proof immediately follows from the properties introduced in the analysis of point (iii). $\square$

**Lemma 5.5.** *Let* $\tilde{F}_{\varepsilon,i}$ *be defined as in* (5.12)*. Then*

$$\|\tilde{F}_{\varepsilon,i}\|_{C^{0,\alpha}(B_R)} \to 0 \qquad \text{for } \varepsilon \to 0^+.$$

**Proof.** Since by definition $\tilde{F}_{\varepsilon,i} = F_{\varepsilon,i} + A_\varepsilon \nabla \bar{z}_{\varepsilon,i}$, where $F_{\varepsilon,i}$ is defined in (5.10), thanks to (5.11) and Lemma 5.4 (ii) it is sufficient to prove that $\|F_{\varepsilon,i}\|_{C^{0,\alpha}(B)} \to 0$ for $\varepsilon \to 0^+$. This follows immediately from Lemma 5.4 (ii). $\square$

**Lemma 5.6.** *Let $f_{\varepsilon,i}$ be defined as in (5.10). Then*

$$\|f_{\varepsilon,i}\|_{L^\infty(B_R)} \to 0 \qquad \text{for } \varepsilon \to 0^+.$$

**Proof.** Let us decompose $f_{\varepsilon,i} = f_1 + f_2 + f_3$, where

$$f_1 := \left[\lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon) - \lambda^1(B, \mathbb{R}^N)\right]\tilde{m}_0 v_{\varepsilon,i} J_\varepsilon, \qquad f_2 := \lambda^1(B, \mathbb{R}^N)\tilde{m}_0 v_{\varepsilon,i}\left[J_\varepsilon - 1\right],$$
$$f_3 := \lambda^1(B, \mathbb{R}^N)\tilde{m}_0\left[v_{\varepsilon,i} - v_{0,i}\right].$$

Now, for $\varepsilon \to 0^+$, $\|f_1\|_{L^\infty(B_R)} \to 0$ by Lemma 2.1, while $\|f_2\|_{L^\infty(B_R)} \to 0$ by Lemma 5.4 (i). For $f_3$ we proceed as follows. We write

$$v_{\varepsilon,i} - v_{0,i} = (\partial_i \tilde{u}_\varepsilon) \circ \mathbf{\Phi}_\varepsilon - \partial_i w = (\partial_i \tilde{u}_\varepsilon - \partial_i w) \circ \mathbf{\Phi}_\varepsilon + ((\partial_i w) \circ \mathbf{\Phi}_\varepsilon - \partial_i w),$$

and notice that $\|(\partial_i \tilde{u}_\varepsilon - \partial_i w) \circ \mathbf{\Phi}_\varepsilon\|_{L^\infty(B_R)} \to 0$ for $\varepsilon \to 0^+$ thanks to Lemma 2.3, while $\|(\partial_i w) \circ \mathbf{\Phi}_\varepsilon - \partial_i w\|_{L^\infty(B_R)} \to 0$ for $\varepsilon \to 0^+$ thanks to the $C^{1,\alpha}\left(\overline{B_R}\right)$ regularity of $w$ and (5.5). □

**Lemma 5.7.** *Let $g_{\varepsilon,i}$ be defined as in (5.10). Then*

$$\|g_{\varepsilon,i}\|_{C^{0,\alpha}(\partial B)} \to 0 \qquad \text{for } \varepsilon \to 0^+.$$

**Proof.** We can decompose $g_{\varepsilon,i} = g_1 + g_2 + g_3 + g_4$, where

$$\begin{aligned}
g_1 &:= \left[\lambda^1(B, \mathbb{R}^N) - \lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon)\right](\overline{m} + \underline{m})(\tilde{u}_\varepsilon \nu_i) \circ \mathbf{\Phi}_\varepsilon J_{\varepsilon,T},\\
g_2 &:= \lambda^1(B, \mathbb{R}^N)(\overline{m} + \underline{m})(\tilde{u}_\varepsilon \nu_i) \circ \mathbf{\Phi}_\varepsilon\left[1 - J_{\varepsilon,T}\right],\\
g_3 &:= \lambda^1(B, \mathbb{R}^N)(\overline{m} + \underline{m})\tilde{u}_\varepsilon \circ \mathbf{\Phi}_\varepsilon\left[n_i - \nu_i \circ \mathbf{\Phi}_\varepsilon\right],\\
g_3 &:= \lambda^1(B, \mathbb{R}^N)(\overline{m} + \underline{m})n_i\left[w - \tilde{u}_\varepsilon \circ \mathbf{\Phi}_\varepsilon\right].
\end{aligned}$$

The facts that $\|g_1\|_{C^{0,\alpha}}(\partial B) \to 0$, $\|g_2\|_{C^{0,\alpha}}(\partial B) \to 0$ and $\|g_3\|_{C^{0,\alpha}}(\partial B) \to 0$ for $\varepsilon \to 0^+$ follow readily from Lemma 2.1, Lemma 5.4 (iii) and (iv), respectively. For what concerns $g_4$, we notice that

$$\|\tilde{u}_\varepsilon \circ \mathbf{\Phi}_\varepsilon - w\|_{C^{0,\alpha}(\overline{B_R})} \leq \|\tilde{u}_\varepsilon \circ \mathbf{\Phi}_\varepsilon - w \circ \mathbf{\Phi}_\varepsilon\|_{C^{0,\alpha}(\overline{B_R})} + \|w \circ \mathbf{\Phi}_\varepsilon - w\|_{C^{0,\alpha}(\overline{B_R})}$$

and the right hand side vanishes for $\varepsilon \to +\infty$, thanks to Lemma 2.3 and (5.5). □

Once Proposition 5.3 is established, we draw the following direct consequence.

**Corollary 5.8.** *For any $r < R$, with $B_r \subset\subset B \subset\subset B_R$, in the limit $\varepsilon \to 0^+$ it holds that*

$$\|\nabla \tilde{u}_\varepsilon - (\nabla \tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}\|_{C^{1,\alpha}\left(\overline{\tilde{E}_\varepsilon \setminus B_r}\right)} + \|\nabla \tilde{u}_\varepsilon - (\nabla \tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}\|_{C^{1,\alpha}\left(\overline{B_R \setminus \tilde{E}_\varepsilon}\right)} \to 0,$$

*where* $\mathbf{n} = \dfrac{\mathbf{x}}{|\mathbf{x}|}$ *denotes the unitary outer normal vector to* $B$*, extended by radial prolongation over* $\mathbb{R}^N \setminus \{\mathbf{0}\}$*.*

**Proof.** First, we claim that

$$\| (\nabla\tilde{u}_\varepsilon - (\nabla\tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}) \circ \mathbf{\Phi}_\varepsilon\|_{C^{1,\alpha}\left(\overline{B\setminus B_r}\right)} + \| (\nabla\tilde{u}_\varepsilon - (\nabla\tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}) \circ \mathbf{\Phi}_\varepsilon\|_{C^{1,\alpha}\left(\overline{B_R\setminus B}\right)} \to 0.$$

Indeed, it is sufficient to notice that, by Proposition 5.3 it holds that

$$\| (\nabla\tilde{u}_\varepsilon - (\nabla\tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}) \circ \mathbf{\Phi}_\varepsilon - (\nabla w - (\nabla w \cdot \mathbf{n})\mathbf{n}) \|_{C^{1,\alpha}\left(\overline{B\setminus B_r}\right)} +$$
$$+ \| (\nabla\tilde{u}_\varepsilon - (\nabla\tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}) \circ \mathbf{\Phi}_\varepsilon - (\nabla w - (\nabla w \cdot \mathbf{n})\mathbf{n}) \|_{C^{1,\alpha}\left(\overline{B_R\setminus B}\right)} \to 0$$

for $\varepsilon \to 0^+$, and that, since $w$ is radially symmetric, it also holds that

$$\nabla w - (\nabla w \cdot \mathbf{n})\mathbf{n} \equiv \mathbf{0} \quad \text{in } \overline{B_R \setminus B_r}.$$

Once the claim is proved, to conclude just notice that

$$\nabla\tilde{u}_\varepsilon - (\nabla\tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n} = [(\nabla\tilde{u}_\varepsilon - (\nabla\tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}) \circ \mathbf{\Phi}_\varepsilon] \circ \mathbf{\Phi}_\varepsilon^{-1},$$

that $\|\mathbf{\Phi}_\varepsilon^{-1} - \mathbf{I}\|_{C^{1,\alpha}} \to 0$ for $\varepsilon \to 0^+$ and that $\mathbf{\Phi}_\varepsilon(\partial B) = \partial\tilde{E}_\varepsilon$ (see (5.5)). □

Once the $C^{1,1}$ regularity of the eigenfunctions is established, we turn to the regularity of the free boundary, i.e. of its nearly spherical representation $\varphi_{\tilde{E}_\varepsilon}$.

Given any $\theta_1$, $\theta_2 \in \mathbb{S}^{N-1}$ let us give the following definitions:

$$\mathbf{x}_i \coloneqq (r_0 + \varphi_{\tilde{E}_\varepsilon}(\theta_i))\theta_i, \qquad \text{for } i = 1, 2,$$

$\gamma_{\theta_1,\theta_2} : [0, L(\gamma_{\theta_1,\theta_2})] \to \mathbb{S}^{N-1}$ denotes a geodesic on $\mathbb{S}^{N-1}$ connecting $\theta_1$ with $\theta_2$, with velocity 1, and $\Gamma_{\theta_1,\theta_2} : [0, L(\gamma_{\theta_1,\theta_2})] \to \partial\tilde{E}_\varepsilon$ is defined as

$$\Gamma_{\theta_1,\theta_2} \coloneqq (r_0 + \varphi_{\tilde{E}_\varepsilon} \circ \gamma_{\theta_1,\theta_2})\gamma_{\theta_1,\theta_2}$$

(here $L$ denotes the length of $C^1$-parametric curves).

As a consequence of Proposition 3.12, we can prove the following

**Lemma 5.9.** *There exists* $\bar{\varepsilon} > 0$ *such that, for any* $0 < \varepsilon \le \bar{\varepsilon}$ *and any* $\theta_1, \theta_2 \in \mathbb{S}^{N-1}$*,*

*(i)* $L(\Gamma_{\theta_1,\theta_2}) = (1 + o(1))L(\gamma_{\theta_1,\theta_2})$,
*(ii)* $|\mathbf{x}_1 - \mathbf{x}_2| = (1 + o(1))|\theta_1 - \theta_2|$,

*where all* $o(1)$ *terms are intended for* $\varepsilon \to 0^+$ *and uniformly in* $\theta_1, \theta_2$*.*

**Proof.** We begin with point (i). First of all, it holds that

$$\dot{\Gamma}_{\theta_1,\theta_2} = (r_0 + \varphi_{\tilde{E}_\varepsilon} \circ \gamma_{\theta_1,\theta_2})\dot{\gamma}_{\theta_1,\theta_2} + (\nabla_T \varphi_{\tilde{E}_\varepsilon} \cdot \dot{\gamma}_{\theta_1,\theta_2})\gamma_{\theta_1,\theta_2},$$

so that

$$|r_0 + \varphi_{\tilde{E}_\varepsilon} \circ \gamma_{\theta_1,\theta_2}| - |\nabla_T \varphi_{\tilde{E}_\varepsilon} \cdot \dot{\gamma}_{\theta_1,\theta_2}| \le |\dot{\Gamma}_{\theta_1,\theta_2}| \le |r_0 + \varphi_{\tilde{E}_\varepsilon} \circ \gamma_{\theta_1,\theta_2}| + |\nabla_T \varphi_{\tilde{E}_\varepsilon} \cdot \dot{\gamma}_{\theta_1,\theta_2}|$$

and thanks to Proposition 3.12 we can write $|\dot{\Gamma}_{\theta_1,\theta_2}| = 1 + o(1)$. Point (i) follows immediately.

Now we turn to point (ii). We have

$$|\theta_1 - \theta_2| - |\varphi_{\tilde{E}_\varepsilon}(\theta_1)\theta_1 - \varphi_{\tilde{E}_\varepsilon}(\theta_2)\theta_2| \le |\mathbf{x}_1 - \mathbf{x}_2| \le |\theta_1 - \theta_2| + |\varphi_{\tilde{E}_\varepsilon}(\theta_1)\theta_1 - \varphi_{\tilde{E}_\varepsilon}(\theta_2)\theta_2|$$

and the result follows again from Proposition 3.12. □

Now, exploiting the regularity results previously obtained for $\tilde{u}_\varepsilon$, we can extend Proposition 3.12 to the case $\alpha = 1$.

**Proposition 5.10.** *There exists $\bar{\varepsilon} > 0$ such that, for any $0 < \varepsilon \le \bar{\varepsilon}$, it holds that $\partial\tilde{E}_\varepsilon$ is nearly spherical of class $C^{1,1}$ and*

$$\|\varphi_{\tilde{E}_\varepsilon}\|_{C^{1,1}} \to 0, \qquad \text{for } \varepsilon \to 0^+.$$

**Proof.** Given the $C^{1,1}$ regularity of $\tilde{u}_\varepsilon$ stated in Corollary 5.2, exactly as in Proposition 3.12 we can prove that $\partial\tilde{E}_\varepsilon$ is nearly spherical of class $C^{1,1}$ and $\|\varphi_{\tilde{E}_\varepsilon}\|_{C^{1,\alpha}} \to 0$ for $\varepsilon \to 0^+$, for any $0 \le \alpha < 1$. Hence, the only thing left to prove is that

$$\|\nabla_T \varphi_{\tilde{E}_\varepsilon}\|_{C^{0,1}} \to 0 \qquad \text{for } \varepsilon \to 0^+. \tag{5.15}$$

To prove (5.15), the fundamental tool will be Corollary 5.8. Fix $r < r_0 < R$. To begin with, we remark that since $\partial_n \tilde{u}_\varepsilon = \nabla \tilde{u}_\varepsilon \cdot \mathbf{n}$, thanks to Proposition 5.1 and Corollary 5.8, for $\varepsilon \to 0^+$ it holds that, for $\alpha < 1$,

$$\begin{aligned} &\left\| \frac{1}{\partial_n \tilde{u}_\varepsilon} \left(\nabla \tilde{u}_\varepsilon - (\nabla \tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}\right) \right\|_{C^{1,\alpha}\left(\overline{\tilde{E}_\varepsilon \setminus B_r}\right)} \\ &+ \left\| \frac{1}{\partial_n \tilde{u}_\varepsilon} \left(\nabla \tilde{u}_\varepsilon - (\nabla \tilde{u}_\varepsilon \cdot \mathbf{n})\mathbf{n}\right) \right\|_{C^{1,\alpha}\left(\overline{B_R \setminus \tilde{E}_\varepsilon}\right)} \to 0. \end{aligned} \tag{5.16}$$

Let $\mathbf{x}_i := (r_0 + \varphi_{\tilde{E}_\varepsilon}(\theta_i))\theta_i$, $i = 1, 2$. Then, using Lemma 5.9, we have

$$\frac{|\nabla_T \varphi_{\tilde{E}_\varepsilon}(\theta_2) - \nabla_T \varphi_{\tilde{E}_\varepsilon}(\theta_1)|}{|\theta_2 - \theta_1|} \le 2\frac{|\nabla_T \varphi_{\tilde{E}_\varepsilon}(\theta_2) - \nabla_T \varphi_{\tilde{E}_\varepsilon}(\theta_1)|}{|\mathbf{x}_2 - \mathbf{x}_1|}.$$

Recalling equation (3.4) and using (5.16) one can easily conclude. □

**Remark 5.11.** The nearly spherical representation $\varphi_{\tilde{E}_\varepsilon}$ depends of course on the choice of the centers of the blow-up procedure. In view of Remark 3.13 we have that Proposition 5.10 holds true both in case such centers are taken at the unique maximum point of $u_\varepsilon$, and when they are identified with the barycenter of the optimal favorable region $E_\varepsilon$.

**End of the proof of Theorem 1.1.** In view of Corollaries 3.9 and 4.2 we are left to prove that properties 2 and 5 in the theorem hold true, regardless of the blow-up procedure being centered at the maximum points of $u_\varepsilon$ or at the barycenters of $E_\varepsilon$. In turn, taking into account the definition of $\tilde{E}_\varepsilon$ (2.3) and the blow-up scaling (2.1), (2.3), such properties are direct consequences of Proposition 5.10 and Remark 5.11 above, by simply writing

$$\varphi_\varepsilon = \varphi_{\tilde{E}_\varepsilon}. \quad \square$$

## 6. Quantitative asymmetry estimates for the problem in $\mathbb{R}^N$

This section is mainly devoted to the proof of Theorem 1.5. Before dealing with it, we first notice that such result readily implies both Corollary 1.6 and Theorem 1.2.

**Proof of Corollary 1.6.** Using Theorem 1.5 and the Gagliardo-Nirenberg inequality we obtain, for every $0 < \alpha < 1$,

$$\begin{aligned}\|\varphi_{\mathcal{A}}\|_{C^{1,\alpha}(\mathbb{S}^{N-1})} &\leq C_1\|\varphi_{\mathcal{A}}\|_{W^{2,\infty}(\mathbb{S}^{N-1})}^{(2+N+2\alpha)/(4+N)}\|\varphi_{\mathcal{A}}\|_{L^2(\mathbb{S}^{N-1})}^{(2-2\alpha)/(4+N)} + C_2\|\varphi_{\mathcal{A}}\|_{L^2(\mathbb{S}^{N-1})} \\ &\leq C_3\delta^{(2+N+2\alpha)/(4+N)}\|\varphi_{\mathcal{A}}\|_{L^2(\mathbb{S}^{N-1})}^{(2-2\alpha)/(4+N)},\end{aligned}$$

where $C_1, C_2, C_3$ only depend on $N$ and $\alpha$, see e.g. [33, Thm. 1] (actually, such result is stated on bounded domains of $\mathbb{R}^N$, but it can be extended to $\mathbb{S}^{N-1}$ in a standard way, by reasoning as in e.g. [22, Sec. 2.6]). □

**Proof of Theorem 1.2.** Using the same notation of the previous sections, let $\tilde{E}_\varepsilon$ denote the optimal favorable set in the blow-up scale, centered at $\mathrm{bar}(E_\varepsilon)$. In particular, if $\varphi_\varepsilon$ is defined as in Theorem 1.1 (with $\mathrm{bar}(E_\varepsilon)$ instead of $\mathbf{x}_\varepsilon$), we have that

$$\varphi_\varepsilon = \varphi_{\tilde{E}_\varepsilon}.$$

Since $\tilde{\Omega}_\varepsilon \subset \mathbb{R}^N$, we infer

$$\lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon) - \lambda^1(B, \mathbb{R}^N) \geq \lambda^1(\tilde{E}_\varepsilon, \mathbb{R}^N) - \lambda^1(B, \mathbb{R}^N).$$

Now, by assumption $\mathrm{bar}(\tilde{E}_\varepsilon) = \mathbf{0}$, while $\|\varphi_{\tilde{E}_\varepsilon}\|_{C^{1,1}(\mathbb{S}^{N-1})}$ can be made arbitrarily small using Proposition 5.10 (recall Remark 5.11). Then we can apply Corollary 1.6 and Proposition 4.1, obtaining

$$\begin{aligned}\|\varphi_{\tilde{E}_\varepsilon}\|_{C^{1,\alpha}(\mathbb{S}^{N-1})} &\leq C\left[\lambda^1(\tilde{E}_\varepsilon, \tilde{\Omega}_\varepsilon) - \lambda^1(B, \mathbb{R}^N)\right]^{(1-\alpha)/(4+N)} = C\left[\tilde{\lambda}_\varepsilon - \tilde{\lambda}_0\right]^{(1-\alpha)/(4+N)} \\ &\leq C\exp\left(-2\sqrt{\tilde{\lambda}_0 \underline{m}}\cdot 2d^* \cdot \varepsilon^{-1/N}\cdot\frac{1-\alpha}{4+N}\right),\end{aligned}$$

for every $0 < \varepsilon < \bar{\varepsilon}$, and the conclusion follows. □

The proof of Theorem 1.5 proceeds in several steps, that for the reader's convenience are briefly and conceptually listed as follows, each corresponding to a section below:

**Step 0:** we set the main notation and the basic properties of deformation paths (Section 6.1);
**Step 1:** we prove differentiability of eigenfunction and eigenvalue via the implicit function theorem (Section 6.2);
**Step 2:** for $C^{1,1}$-nearly spherical domains, we carry out the expansion of the eigenvalue up to the second order (Section 6.3);
**Step 3:** for $C^{1,1}$-nearly spherical domains, we prove coercivity of the second order shape derivative of the eigenvalue, computed at the origin, with respect to the $L^2(\mathbb{S}^{N-1})$ norm of the deformation (Section 6.4); this is the point where the constraint on the barycenter plays a crucial role;
**Step 4:** for $C^{1,1}$-nearly spherical domains, we prove continuity of the remainder with respect to the $L^2(\mathbb{S}^{N-1})$ norm of the deformation and a modulus of continuity depending on the $C^{1,1}(\mathbb{S}^{N-1})$ error of the deformation (Section 6.5).

Once the above steps are performed, the proof of Theorem 1.5 is straightforward and it is concluded at the end of Section 6.5.

As a final remark, we notice that Theorem 1.5 is sharp, in the sense of equation (1.10).

**Lemma 6.1.** *Under the assumptions of Theorem 1.5 there exists a constant $C' > 0$ such that*

$$\lambda^1(\mathcal{A}, \mathbb{R}^N) - \lambda^1(B, \mathbb{R}^N) \leq C'\|\varphi_{\mathcal{A}}\|^2_{L^2(\mathbb{S}^{N-1})}$$

**Proof.** The proof is a direct computation, bounding $\lambda^1(\mathcal{A}, \mathbb{R}^N)$ from above with the associated Rayleigh quotient of $w$.

First, since $\mathcal{L}(\mathcal{A}) = \mathcal{L}(B) = 1$, we have

$$\begin{aligned}0 = \int_B d\mathbf{x} - \int_{\mathcal{A}} d\mathbf{x} &= \int_{\mathbb{S}^{N-1}} \int_{r_0+\varphi_{\mathcal{A}}(\theta)}^{r_0} r^{N-1} dr\, d\sigma \\ &= \frac{1}{N}\int_{\mathbb{S}^{N-1}} \left[r_0^N - (r_0 + \varphi_{\mathcal{A}}(\theta))^N\right] d\sigma \qquad (6.1)\end{aligned}$$

which yields

$$\int_{\mathbb{S}^{N-1}} \varphi_{\mathcal{A}}(\theta) = -\frac{N-1}{2r_0}\|\varphi_{\mathcal{A}}\|^2_{L^2(\mathbb{S}^{N-1})}\left(1+R(\varphi_{\mathcal{A}})\right), \qquad \text{where } |R(\varphi_{\mathcal{A}})| \le C_\delta \tag{6.2}$$

(indeed, $R(\varphi_{\mathcal{A}}) = \sum_{n=3}^{N} \frac{2(N-2)!}{n!(N-n)!r_0^{n-2}} \int_{\mathbb{S}^{N-1}} \varphi_{\mathcal{A}}^{n-2} d\sigma$ and, for instance, $C_\delta = \sum_{n=3}^{N} \frac{2N(N-2)!\delta^{n-2}}{n!(N-n)!r_0^{n-2}}$).

Using the properties of $w$ and writing $m_{\mathcal{A}} = \overline{m}\chi_{\mathcal{A}} - \underline{m}\chi_{\mathbb{R}^N\setminus\mathcal{A}}$ (and the same with $B$ instead of $\mathcal{A}$) we have

$$\begin{aligned}\lambda^1(\mathcal{A},\mathbb{R}^N) - \lambda^1(B,\mathbb{R}^N) &\le \frac{\int_{\mathbb{R}^N}|\nabla w|^2}{\int_{\mathbb{R}^N} m_{\mathcal{A}}w^2} - \frac{\int_{\mathbb{R}^N}|\nabla w|^2}{\int_{\mathbb{R}^N} m_B w^2} \\ &= \frac{\lambda^1(B,\mathbb{R}^N)}{\int_{\mathbb{R}^N} m_{\mathcal{A}}w^2}\int_{\mathbb{R}^N}(m_B - m_{\mathcal{A}})w^2.\end{aligned}$$

Now, $\int_{\mathbb{R}^N} m_{\mathcal{A}}w^2 \ge \int_{\mathbb{R}^N} m_{B_{r_0-\delta}}w^2$, which is a positive constant independent of $\mathcal{A}$; on the other hand, since $w^2(r) \le A_\delta - B_\delta r$ for $r_0 - \delta < r < r_0 + \delta$, with $A_\delta, B_\delta$ positive,

$$\begin{aligned}\int_{\mathbb{R}^N}(m_B - m_{\mathcal{A}})w^2 &= (\overline{m}+\underline{m})\int_{\mathbb{S}^{N-1}}\int_{r_0+\varphi_{\mathcal{A}}(\theta)}^{r_0} r^{N-1}w^2(r)dr\,d\sigma \\ &\le -B_\delta(\overline{m}+\underline{m})\int_{\mathbb{S}^{N-1}}\int_{r_0+\varphi_{\mathcal{A}}(\theta)}^{r_0} r^N dr\,d\sigma\end{aligned}$$

where the term with $A_\delta$ cancels because of (6.1). Finally, we can evaluate the last integral and use (6.2) to write

$$\begin{aligned}&\int_{\mathbb{S}^{N-1}}\int_{r_0+\varphi_{\mathcal{A}}(\theta)}^{r_0} r^N dr\,d\sigma \\ &= (N+1)r_0^{N+1}\left(-\int_{\mathbb{S}^{N-1}}\varphi_{\mathcal{A}}(\theta) - \frac{N}{2r_0}\|\varphi_{\mathcal{A}}\|^2_{L^2(\mathbb{S}^{N-1})}\left(1+R'(\varphi_{\mathcal{A}})\right)\right) \\ &= -\frac{(N+1)r_0^N}{2}\|\varphi_{\mathcal{A}}\|^2_{L^2(\mathbb{S}^{N-1})}\left(1+R''(\varphi_{\mathcal{A}})\right) \ge -C\|\varphi_{\mathcal{A}}\|^2_{L^2(\mathbb{S}^{N-1})},\end{aligned}$$

with $C > 0$ independent on $\mathcal{A}$, and the lemma follows. □

### *6.1. Deformation paths and their properties*

Before proceeding with the proof, we need to specify in which sense we intend the differentiation with respect to the shape, for instance of the eigenvalue. Analogous considerations hold for the associated eigenfunction, and for all the other quantities depending on a shape.

A standard procedure is to reduce to a one-dimensional problem, in a way we briefly recall. Consider a reference shape $\mathcal{A}_0 \in \mathbb{R}^N$, and a (small) perturbation $\mathcal{A} \in \mathbb{R}^N$. Suppose the existence of a one-parameter family of smooth displacement fields $\Phi(t, \mathbf{x}) : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ which are diffeomorphisms for any fixed $t$, satisfying

$$\Phi(0, \mathbf{x}) = \mathbf{I} \quad \text{and} \quad \Phi(1, \mathcal{A}_0) = \mathcal{A}. \tag{6.3}$$

At this point, one can introduce the parametric family

$$\lambda(t) := \lambda^1(\Phi(t, \mathcal{A}_0), \mathbb{R}^N).$$

Such quantity, once the reference shape has been fixed, depends only on the parameter $t$. Hence, one defines all the desired quantities exactly as in the case of functions of one variable. For instance, by first derivative of the eigenvalue, we mean the quantity

$$\dot{\lambda}(t) := \frac{d}{dt}\lambda(t).$$

Of course, one needs to check in advance that the desired quantities are well defined. In our case, this will be the subject of Section 6.2.

The existence of a family of diffeomorphisms satisfying (6.3) together with additional properties, for a general class of problems, has been the object of many works in the literature. An example is [10], where the vector fields $\partial_t \Phi(t, \mathbf{x})$ are required to be divergence free in a neighborhood of $\partial \mathcal{A}_0$.

As remarked in [4, Appendix A], when the reference configuration $\mathcal{A}_0$ coincides with a ball and when nearly spherical perturbations are considered, as for our problem, the construction of a family $\Phi(t, \mathbf{x})$ can be made explicit. In radial coordinates, it is sufficient to consider

$$\begin{gathered}\Phi_{\mathcal{A}}(t, \rho\,\theta) := \left[\rho^N + t\left((r_0 + \varphi_{\mathcal{A}}(\theta))^N - r_0^N\right)\right]^{1/N} \theta \\ \text{for } \theta \in \mathbb{S}^{N-1},\ 3\frac{r_0}{4} < \rho < 5\frac{r_0}{4}.\end{gathered} \tag{6.4}$$

Then, one can extend (6.4) to the whole $\mathbb{R}^N$, so that it also satisfies some additional properties. More precisely, similarly as in [4, Appendix A], we have the following.

**Lemma 6.2.** *There exists $\delta > 0$ sufficiently small and a modulus of continuity $\eta$ such that, for any set $\mathcal{A} \subset \mathbb{R}^N$ nearly spherical of class $C^{1,1}$ with*

$$\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})} < \delta \quad \text{and} \quad \mathcal{L}(\mathcal{A}) = \mathcal{L}(B) = 1, \tag{6.5}$$

*there exists a family $\Phi_{\mathcal{A}}(t, \rho\,\theta)$ extending (6.4) to the whole $\mathbb{R}^N$, such that*

*(i)* *The map* $t \in [0,1] \mapsto \Phi_{\mathcal{A}}(t, \mathbf{x}) - \mathbf{I} \in W^{2,\infty}(\mathbb{R}^N)$ *is of class* $C^\infty$.
*(ii)* *For* $3r_0/4 < |\mathbf{x}| < 5r_0/4$,

$$\partial_t \Phi_{\mathcal{A}}(t, \mathbf{x}) = \mathbf{X}_{\mathcal{A}}(\Phi_{\mathcal{A}}(t, \mathbf{x})), \qquad \operatorname{div}(\mathbf{X}_{\mathcal{A}}) = 0,$$

*where*

$$\mathbf{X}_{\mathcal{A}}(\rho, \theta) \coloneqq \frac{(r_0 + \varphi_{\mathcal{A}}(\theta))^N - r_0^N}{N\,\rho^{N-1}}\,\theta. \tag{6.6}$$

*(iii)* $\mathcal{L}(\Phi_{\mathcal{A}}(t, B)) = 1$ *for all* $t \in [0,1]$.
*(iv)* $\|\Phi_{\mathcal{A}} - \mathbf{I}\|_{C^{1,1}(\mathbb{R}^N)} \le \eta\left(\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})}\right)$ *uniformly for* $t \in [0,1]$.
*(v)* $\mathbf{X}_{\mathcal{A}} \circ \Phi_{\mathcal{A}} - \mathbf{X}_{\mathcal{A}} = O\left(\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})}\right)\mathbf{X}_{\mathcal{A}}$ *on* $\partial B$, *uniformly for* $t \in [0,1]$.
*(vi)* $\left\|\mathbf{X}_{\mathcal{A}}(r_0, \cdot) \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - \varphi_{\mathcal{A}}\right\|_{L^2(\mathbb{S}^{N-1})} \le \eta\left(\|\varphi_{\mathcal{A}}\|_{L^\infty(\mathbb{S}^{N-1})}\right)\|\varphi_{\mathcal{A}}\|_{L^2(\mathbb{S}^{N-1})}$.

**Remark 6.3.** In Lemma 6.2 (i) the derivatives as $t = 0$ and $t = 1$ are understood from the right and from the left, respectively. Moreover, in Lemma 6.2 (iv) and (v) analogous estimates continue to hold if the $C^{1,1}(\partial B)$ norm is substituted by the $C^{1,\alpha}(\partial B)$ norm, for any $0 < \alpha < 1$.

**Proof.** Consider a smooth cutoff function $h$, such that $h \equiv 1$ for $3r_0/4 \le \rho \le 5r_0/4$ and $h \equiv 0$ for $r_0/2 \le \rho \le 3r_0/2$. We define

$$\Phi_{\mathcal{A}}(t, \rho\,\theta) \coloneqq \left[\rho^N + t\,h(\rho)\left((r_0 + \varphi_{\mathcal{A}}(\theta))^N - r_0^N\right)\right]^{1/N}\theta \tag{6.7}$$

in the whole $\mathbb{R}^N$. Clearly, (6.7) is an extension of (6.4).

Let us prove point $(i)$. Thanks to the restriction on $\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})}$ required in (6.5), for $\delta > 0$ sufficiently small it holds that the quantities $\Phi_{\mathcal{A}}(t, \mathbf{x}) - \mathbf{I}$, $D_{\mathbf{x}}\Phi_{\mathcal{A}}(t, \mathbf{x}) - \mathbf{I}$ and $D^2_{\mathbf{x}}\Phi_{\mathcal{A}}(t, \mathbf{x})$ are $C^\infty$ in time in the classical sense, for almost every $\mathbf{x} = (\rho, \theta) \in \mathbb{R}^N$ fixed. By a simple computation, all time derivatives are bounded in $L^\infty(\mathbb{R}^N)$, uniformly for $t \in [0,1]$. This is sufficient to prove point $(i)$.

Point $(ii)$ follows immediately by direct computation.

Point $(iii)$ follows from the fact that, by [23, Corollaire 5.2.8], it holds that

$$\frac{d}{dt}\mathcal{L}(\Phi_{\mathcal{A}}(t, B)) = \frac{d}{dt}\int\limits_{\Phi_{\mathcal{A}}(t,B)} 1 = \int\limits_{\partial\Phi_{\mathcal{A}}(t,B)} \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}, \tag{6.8}$$

where in the last passage we have used points $(i)$ and $(ii)$. Now, using (6.6), the last integral in (6.8) vanishes thanks to the condition $\mathcal{L}(\mathcal{A}) = \mathcal{L}(B)$ in (6.5). Hence, point $(iii)$ follows.

Point $(iv)$ follows easily by direct computation.

Point $(v)$ also follows by direct computation, using point $(iv)$.

Point $(vi)$ follows expanding the expression (6.6) for $\mathbf{X}_{\mathcal{A}}$ in terms of $\varphi_{\mathcal{A}}$. Notice that, since $\|\varphi_{\mathcal{A}}\|_{L^2(\mathbb{S}^{N-1})} \leq 2\|\mathbf{X}_{\mathcal{A}}(r_0,\cdot)\cdot\frac{\mathbf{x}}{|\mathbf{x}|}\|_{L^2(\mathbb{S}^{N-1})}$ for $\delta$ sufficiently small, up to redefining $\eta$ it also holds that

$$\begin{aligned}&\left\|\mathbf{X}_{\mathcal{A}}(r_0,\cdot)\cdot\frac{\mathbf{x}}{|\mathbf{x}|}-\varphi_{\mathcal{A}}\right\|_{L^2(\mathbb{S}^{N-1})}\\&\leq \eta\left(\|\varphi_{\mathcal{A}}\|_{L^\infty(\mathbb{S}^{N-1})}\right)\left\|\mathbf{X}_{\mathcal{A}}(r_0,\cdot)\cdot\frac{\mathbf{x}}{|\mathbf{x}|}\right\|_{L^2(\mathbb{S}^{N-1})}. \quad \square \end{aligned} \tag{6.9}$$

**Remark 6.4** *(on notation).* In the following, we will denote with $\Phi_{\mathcal{A}}(t,\rho\theta)$ or equivalently $\Phi_{\mathcal{A}}(t,\mathbf{x})$ the family of diffeomorphic deformations of the form (6.4) already extended to the whole $\mathbb{R}^N$, in order to satisfy the conditions of Lemma 6.2.

### *6.2. Differentiability*

Consider $\Theta \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)$, with $\|\Theta\|_{W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)}$ small enough so that $\mathbf{I}+\Theta$ is a diffeomorphism of $\mathbb{R}^N$ in itself. We introduce the domain $D_\Theta$ defined as the image of $B$ via $\mathbf{I}+\Theta$, the correspondent weight

$$\tilde{m}_\Theta := \overline{m}\chi_{D_\Theta} - \underline{m}\chi_{\mathbb{R}^N\setminus D_\Theta}$$

and the eigenfunction $u_\Theta$, solution in $H^1(\mathbb{R}^N)$ of

$$-\Delta u_\Theta = \lambda^1(D_\Theta,\mathbb{R}^N)\tilde{m}_\Theta u_\Theta \quad \text{in } \mathbb{R}^N, \tag{6.10}$$

with the normalization condition

$$\int_{\mathbb{R}^N} \tilde{m}_\Theta\, u_\Theta^2 = 1.$$

A common practice to study the differentiability of $u_\Theta$ with respect to $\Theta$, is to bring back the problems for different vector fields $\Theta$ on the same domain $B$ and study first the properties of such problem, which are usually easier to tackle. Then, one tries to transfer the obtained information to the function $u_\Theta$. This can be done introducing the functions

$$v_\Theta := u_\Theta \circ (\mathbf{I}+\Theta).$$

We remark that, thanks to the regularity required to $\Theta$, it holds that $v_\Theta \in H^1(\mathbb{R}^N)$. Moreover, performing a change of variables in the weak formulation of (6.10), it is easy to see that the functions $v_\Theta$ solve, in $H^1(\mathbb{R}^N)$

$$-\operatorname{div}(A(\Theta)v_\Theta) = \lambda^1(D_\Theta,\mathbb{R}^N)J(\Theta)\tilde{m}_0 v_\Theta \quad \text{in } \mathbb{R}^N, \tag{6.11}$$

where $J(\Theta)$ denotes the jacobian of the diffeomorphism $\mathbf{I} + \Theta$, i.e.

$$J(\Theta) := \det(\mathbf{I} + D\Theta)$$

while $A(\Theta)$ is defined as

$$A(\Theta) := J(\Theta)(I + D\Theta)^{-1}(I + D\Theta)^{-T}.$$

The main result of this section concerns a local differentiability of the function $v_\Theta$, and of the corresponding eigenvalue, with respect to $\Theta$. To prove such result we need a preliminary lemma.

**Lemma 6.5.** *For every $h \in L^2(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} wh = 0$ there exists a unique $y \in H^2(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \tilde{m}_0 wy = 0$, such that*

$$-\Delta z - \tilde{\lambda}_0 \tilde{m}_0 z = h \qquad \Longleftrightarrow \qquad z = y + cw, \ c \in \mathbb{R}.$$

*On the other hand, if $\int_{\mathbb{R}^N} wh \neq 0$ the above equation has no solution.*

**Proof.** The necessary condition on $h$ for the existence of solutions follows by testing the equation with $w$ and using the corresponding equation.

To show existence, let us introduce the bilinear form on $H^1(\mathbb{R}^N)$:

$$a(y, v) := \int_{\mathbb{R}^N} \nabla y \cdot \nabla v - \tilde{\lambda}_0 \tilde{m}_0 y v.$$

By the variational characterization of $\tilde{\lambda}_0 > 0$ and $w$ we have that

$$a(v, v) \leq 0 \qquad \Longleftrightarrow \qquad v = cw, \ c \in \mathbb{R} \tag{6.12}$$

(and hence $a(v, v) = 0$). Moreover, considering the Hilbert space

$$V := \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \tilde{m}_0 w v = 0 \right\}, \qquad \|v\|_V^2 := \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \tilde{\lambda}_0 \underline{m} \|v\|_{L^2(\mathbb{R}^N)}^2,$$

we have that

$$a(v, v) = \|v\|_V^2 - \tilde{\lambda}_0 (\overline{m} + \underline{m}) \|v\|_{L^2(B)}^2; \tag{6.13}$$

Then we can apply the Lax-Milgram theorem to the variational problem:

$$\text{find } y \in V \text{ such that} \qquad a(y, v) = \int_{\mathbb{R}^N} hv \qquad \text{for every } v \in V. \tag{6.14}$$

Indeed, (bi)linearity and continuity are straightforward, while the coercivity of $a$ follows by contradiction: assume the existence of a sequence $(v_n)_n \subset V$ such that

$$\|v_n\|_V = 1, \qquad a(v_n, v_n) \leq \frac{1}{n}.$$

Then, up to subsequences, $v_n$ converges to some $\bar{v}$, weakly in $V$ and strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$, as $n \to +\infty$. In particular, exploiting the lower weak semicontinuity of the norm in (6.13) we infer $a(\bar{v}, \bar{v}) \leq 0$, and, since $\bar{v} \in V$, (6.12) yields $\bar{v} \equiv 0$; on the other hand, again by (6.13),

$$\begin{aligned}\tilde{\lambda}_0(\overline{m} + \underline{m})\|\bar{v}\|^2_{L^2(B)} &= \tilde{\lambda}_0(\overline{m} + \underline{m})\|v_n\|^2_{L^2(B)} + o(1) \\ &= \|v_n\|^2_V - a(v_n, v_n) + o(1) \geq 1 - \frac{1}{n} + o(1)\end{aligned}$$

for $n$ large, contradiction.

Then Lax-Milgram theorem applies, yielding the existence and uniqueness of $y \in V$ satisfying (6.14). Taking $v \in H^1(\mathbb{R}^N)$ we have that $v - t_v w \in V$, where $t_v = \int_{\mathbb{R}^N} \tilde{m}_0 wv / \int_{\mathbb{R}^N} \tilde{m}_0 w^2$. Substituting in (6.14) we have

$$a(y, v) = \int\limits_{\mathbb{R}^N} hv - t_v \left[a(y, w) - \int\limits_{\mathbb{R}^N} hw\right] = \int\limits_{\mathbb{R}^N} hv \qquad \text{for every } v \in H^1(\mathbb{R}^N),$$

where we used the equation of $w$ and the fact that $\int_{\mathbb{R}^N} wh = 0$ by assumption. Then by elliptic regularity $y \in H^2(\mathbb{R}^N)$, and $z = y + cw$, $c \in \mathbb{R}$, satisfies the equation

$$-\Delta z - \tilde{\lambda}_0 \tilde{m}_0 z = h.$$

Finally, consider $z_1, z_2 \in H^1(\mathbb{R}^N)$ solutions of the above equation. Then subtracting we have $a(z_1 - z_2, z_1 - z_2) = 0$ and we can conclude the proof using again (6.12). □

**Proposition 6.6.** *The function*

$$\Theta \in W^{2,\infty}(\mathbb{R}^N, \mathbb{R}^N) \mapsto (\lambda^1(D_\Theta, \mathbb{R}^N), v_\Theta) \in \mathbb{R} \times H^2(\mathbb{R}^N)$$

*is $C^\infty$ in a neighborhood of $\Theta = \mathbf{0}$.*

**Proof.** The proof follows from the implicit function theorem, similarly as in [11].

Let us define the map $F : W^{2,\infty}(\mathbb{R}^N, \mathbb{R}^N) \times \mathbb{R} \times H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \times \mathbb{R}$ as follows:

$$F(\Theta, \lambda, v) := \left( -\operatorname{div}(A(\Theta)\nabla v) - \lambda \tilde{m}_0 v J(\Theta), \int\limits_{\mathbb{R}^N} \tilde{m}_0\, v^2\, J(\Theta) - 1 \right).$$

Then there exists a neighborhood $\mathcal{U} \subset W^{2,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ of the origin such that

$$F(\Theta, \lambda, v) \in C^\infty(\mathcal{U} \times \mathbb{R} \times H^2(\mathbb{R}^N)). \tag{6.15}$$

Indeed, it is a standard result (reasoning for instance as in [23, Théorème 5.3.2] and its proof) that the maps

$$(\Theta, v) \in \mathcal{U} \times H^2(\mathbb{R}^N) \mapsto \operatorname{div}(A(\Theta)\nabla v) \in L^2(\mathbb{R}^N),$$
$$\Theta \in W^{2,\infty}(\mathbb{R}^N, \mathbb{R}^N) \mapsto J(\Theta) \in \mathbb{R}$$

are of class $C^\infty$, in an appropriate $\mathcal{U}$. Thanks to the regularity of $J(\Theta)$, it is immediate to deduce the $C^\infty$ regularity property also for

$$(\Theta, \lambda, v) \in \mathcal{U} \times \mathbb{R} \times H^2(\mathbb{R}^N) \mapsto \lambda \tilde{m}_0 v J(\Theta) \in L^2(\mathbb{R}^N),$$
$$(\Theta, v) \in \mathcal{U} \times H^2(\mathbb{R}^N) \mapsto \int\limits_{\mathbb{R}^N} \tilde{m}_0\, v^2\, J(\Theta) \in \mathbb{R}.$$

Hence, (6.15) is proved.

Next, recalling that $J(\mathbf{0}) = 1$ and $A(\mathbf{0}) = \mathbf{I}$, we easily infer that, for any $\mu \in \mathbb{R}$ and $z \in H^2(\mathbb{R}^N)$,

$$D_{\lambda,v}(\mathbf{0}, \tilde{\lambda}_0, w)[\mu, z] = \left( -\Delta z - \mu \tilde{m}_0 w - \tilde{\lambda}_0 \tilde{m}_0 z, \int\limits_{\mathbb{R}^N} 2\tilde{m}_0\, z\, w \right).$$

In order to apply the implicit function theorem, we are left to prove that $D_{\lambda,v}(\mathbf{0}, \tilde{\lambda}_0, w)$ is a diffeomorphism. Actually, proving its invertibility is sufficient since it is continuous and linear, hence the continuity of its inverse would follow from the inverse mapping theorem in Banach spaces.

To prove invertibility, consider a couple $(k, p) \in L^2(\mathbb{R}^N) \times \mathbb{R}$. We have to show existence and uniqueness of the pair $(\mu, z) \in \mathbb{R} \times H^2(\mathbb{R}^N)$ solving

$$\begin{cases} -\Delta z - \tilde{\lambda}_0 \tilde{m}_0 z = k + \mu \tilde{m}_0 w, \\ \int_{\mathbb{R}^N} 2\tilde{m}_0\, z\, w = p. \end{cases}$$

The first equation is solvable if and only if the condition on the right hand side appearing in Lemma 6.5 is satisfied, i.e.

$$\mu = -\frac{\int_{\mathbb{R}^N} kw}{\int_{\mathbb{R}^N} \tilde{m}_0 w^2},$$

with infinitely many solutions given by

$$z = y + cw, \qquad c \in \mathbb{R},$$

where $y$ is uniquely determined in such a way that $\int_{\mathbb{R}^N} \tilde{m}_0 w y = 0$. Substituting in the second equation we have that $c$ is uniquely determined by

$$c = \frac{p}{2\int_{\mathbb{R}^N} \tilde{m}_0 w^2},$$

concluding the proof. $\square$

Our aim now is to exploit the differentiability of $v_\Theta$ to prove a corresponding result for $u_\Theta$. Actually, since as recalled in Section 6.1 we will exploit differentiation along paths, we derive the result directly for $u_\Theta$ restricted on such paths.

Let $\mathcal{A}$ be a nearly spherical set of class $C^{1,1}$, with $\|\varphi_\mathcal{A}\|_{C^{1,1}(\mathbb{S}^{N-1})}$ sufficiently small. Consider the path $\Phi_\mathcal{A}(t, \mathbf{x})$ associated to $\mathcal{A}$ (see also Remark 6.4) and introduce the parametric family $\mathcal{A}_t := \Phi_\mathcal{A}(t, B)$, the weights $m_t := \overline{m}\chi_{\mathcal{A}_t} - \underline{m}\chi_{\mathbb{R}^N\setminus\mathcal{A}_t}$ and the functions $u_t \in H^1_0(\mathbb{R}^N)$ for any fixed $t \in [0, 1]$, solutions of

$$-\Delta u_t = \lambda^1(\mathcal{A}_t, \mathbb{R}^N) m_t u_t \qquad \text{in } \mathbb{R}^N \tag{6.16}$$

As usual, the principal eigenvalue $\lambda^1(\mathcal{A}_t, \mathbb{R}^N)$ in (6.16) has a unique eigenfunction $u_t$, up to normalization, see e.g. [29, Lemma 2.6].

Then, from Proposition 6.6 we can deduce the following.

**Lemma 6.7.** *There exists $\delta > 0$ sufficiently small such that, for any nearly spherical set $\mathcal{A}$ of class $C^{1,1}$ with $\|\varphi_\mathcal{A}\|_{C^{1,1}(\mathbb{S}^{N-1})} < \delta$, the map $t \in [0, 1] \mapsto u_t \in H^1(\mathbb{R}^N)$ is of class $C^1$.*

**Proof.** Let us denote $v_t$ the map $v_t := v_{\Phi_\mathcal{A}(t)-\mathbf{I}}$. Thanks to Lemma 6.2 and Proposition 6.6, by composition it holds that $t \in [0, 1] \mapsto v_t \in H^2(\mathbb{R}^N)$ is of class $C^\infty$.

Now, the function $u_t$ can be written as $u_t = v_t \circ \Phi_\mathcal{A}(t, \mathbf{x})^{-1}$. Since the map $\Theta \in W^{2,\infty} \mapsto (\mathbf{I} + \Theta)^{-1} \in W^{1,\infty}$ is of class $C^1$ in a neighborhood of $\Theta = \mathbf{0}$, by composition the map $t \mapsto \Phi_\mathcal{A}(t, \mathbf{x})^{-1} \in W^{1,\infty}$ is $C^1$ in $[0, 1]$ (see e.g. [23, Sec. 5.2.2]). Summing up, we deduce that

$$t \mapsto (v_t, \Phi_\mathcal{A}(t, \mathbf{x})^{-1} - \mathbf{I}) \in H^2(\mathbb{R}^N) \times W^{1,\infty} \qquad \text{is of class } C^1. \tag{6.17}$$

On the other hand, by [23, Lemme 5.3.9], in a sufficiently small neighborhood $\mathcal{U} \subset W^{1,\infty}$ of the origin, the map

$$(g, \Theta) \in H^2(\mathbb{R}^N) \times \mathcal{U} \mapsto g \circ (\mathbf{I} + \Theta) \in H^1(\mathbb{R}^N) \qquad \text{is of class } C^1. \tag{6.18}$$

Hence, using Lemma 6.2 (iv), for $\delta$ sufficiently small we can conclude the proof by composition of the maps in (6.17) and (6.18). $\square$

*6.3. The expansion*

In this section we perform the expansion of the eigenvalue up to the second order. We will use extensively the Hadamard formula, see e.g. [23, Corollaire 5.2.8]. In particular, let

$$f(x,t) \in C^1\left([0,T); L^1(\mathbb{R}^N)\right) \cap C^0\left([0,T); W^{1,1}(\mathbb{R}^N)\right),$$

then, recalling the definition and properties of $\mathbf{X}_{\mathcal{A}}$ (see Lemma 6.2),

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{A}_t} f(x,t) &= \int_{\mathcal{A}_t} \partial_t f(x,t) + \int_{\mathcal{A}_t} \operatorname{div}\left(f(x,t)\mathbf{X}_{\mathcal{A}}(x)\right),\\
\frac{d}{dt}\int_{\mathbb{R}^N\setminus\mathcal{A}_t} f(x,t) &= \int_{\mathbb{R}^N\setminus\mathcal{A}_t} \partial_t f(x,t) + \int_{\mathbb{R}^N\setminus\mathcal{A}_t} \operatorname{div}\left(f(x,t)\mathbf{X}_{\mathcal{A}}(x)\right).
\end{aligned}$$

In the following we denote with $\lambda_t := \lambda^1(\mathcal{A}_t, \mathbb{R}^N)$, which is of class $C^\infty$ in time by Proposition 6.6, and with $u_t$ the corresponding eigenfunction as in (6.16) which is $C^1$ in time, with values in $H^1$, by Lemma 6.7. Moreover, we denote with $\dot{\lambda}_t$, $\dot{u}_t$ the time derivatives of $\lambda_t$ and of $u_t$, respectively.

We start with the first order expansion.

**Lemma 6.8.** *There exists $\delta > 0$ sufficiently small such that, for any nearly spherical set $\mathcal{A}$ of class $C^{1,1}$ with $\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})} < \delta$*

$$\dot{\lambda}_t = -\lambda_t(\overline{m} + \underline{m}) \int_{\partial\mathcal{A}_t} u_t^2\, \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_t, \tag{6.19}$$

*where $\mathbf{n}_t$ denotes the unitary outer normal to $\partial\mathcal{A}_t$.*

**Proof.** Choosing the normalization $\int_{\mathbb{R}^N} m_t u_t^2 = 1$ we have that

$$\dot{\lambda}_t = \frac{d}{dt}\int_{\mathbb{R}^N} |\nabla u_t|^2 = 2\int_{\mathbb{R}^N} \nabla \dot{u}_t \cdot \nabla u_t. \tag{6.20}$$

Our aim is to rewrite the previous formula only in terms of $u_t$. Exploiting the chosen normalization and using the Hadamard formula, we have that

$$\begin{aligned}
0 = \frac{d}{dt}\int_{\mathbb{R}^N} m_t u_t^2 &= \frac{d}{dt}\left[\overline{m}\int_{\mathcal{A}_t} u_t^2 - \underline{m}\int_{\mathcal{A}_t^c} u_t^2\right] = \\
&= 2\int_{\mathbb{R}^N} m_t \dot{u}_t u_t + (\overline{m} + \underline{m}) \int_{\partial\mathcal{A}_t} u_t^2\, \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_t.
\end{aligned} \tag{6.21}$$

Testing (6.16) with $\dot{u}_t$, we get

$$\int_{\mathbb{R}^N} \nabla \dot{u}_t \cdot \nabla u_t = \lambda_t \int_{\mathbb{R}^N} m_t \dot{u}_t u_t. \tag{6.22}$$

Substituting (6.22) in (6.20) and the using (6.21), we can conclude. □

Now we differentiate the eigenvalue once again, to obtain the following.

**Lemma 6.9.** *There exists $\delta > 0$ sufficiently small such that, for any nearly spherical set $\mathcal{A}$ of class $C^{1,1}$ with $\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})} < \delta$*

$$\begin{aligned} \ddot{\lambda}_t = \lambda_t &\left[ (\overline{m} + \underline{m}) \int_{\mathcal{A}_t} \operatorname{div}(u_t^2 \mathbf{X}_{\mathcal{A}}) \right]^2 + \\ &- 2\lambda_t (\overline{m} + \underline{m}) \left[ \int_{\partial \mathcal{A}_t} u_t \left(\nabla u_t \cdot \mathbf{X}_{\mathcal{A}}\right) \left(\mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_t\right) + \int_{\partial \mathcal{A}_t} \dot{u}_t u_t \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_t \right], \end{aligned} \tag{6.23}$$

*where $\mathbf{n}_t$ denotes the unitary outer normal to $\partial \mathcal{A}_t$. In particular,*

$$\ddot{\lambda}_t|_{t=0} = 2\tilde{\lambda}_0 (\overline{m} + \underline{m}) w|_{\partial B} \left[ \left| \frac{\partial w}{\partial \rho} \right|_{\partial B} \int_{\partial B} |\mathbf{X}_{\mathcal{A}}|^2 - \int_{\partial B} \dot{w} \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_0 \right]. \tag{6.24}$$

**Proof.** We rewrite (6.19) as

$$\dot{\lambda}_t = -\lambda_t (\overline{m} + \underline{m}) \int_{\mathcal{A}_t} \operatorname{div}\left(u_t^2 \, \mathbf{X}_{\mathcal{A}}\right),$$

and differentiating with respect to $t$ using the Hadamard formula, we get

$$\begin{aligned} \ddot{\lambda}_t = &-\dot{\lambda}_t (\overline{m} + \underline{m}) \int_{\mathcal{A}_t} \operatorname{div}\left(u_t^2 \, \mathbf{X}_{\mathcal{A}}\right) + \\ &- 2\lambda_t (\overline{m} + \underline{m}) \int_{\mathcal{A}_t} \operatorname{div}\left(\dot{u}_t u_t \, \mathbf{X}_{\mathcal{A}}\right) - \lambda_t (\overline{m} + \underline{m}) \int_{\partial \mathcal{A}_t} \operatorname{div}\left(u_t^2 \, \mathbf{X}_{\mathcal{A}}\right) \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_t = \\ = &-\dot{\lambda}_t (\overline{m} + \underline{m}) \int_{\mathcal{A}_t} \operatorname{div}\left(u_t^2 \, \mathbf{X}_{\mathcal{A}}\right) - 2\lambda_t (\overline{m} + \underline{m}) \int_{\mathcal{A}_t} \operatorname{div}\left(\dot{u}_t u_t \, \mathbf{X}_{\mathcal{A}}\right) + \\ &- 2\lambda_t (\overline{m} + \underline{m}) \int_{\partial \mathcal{A}_t} u_t \left(\nabla u_t \cdot \mathbf{X}_{\mathcal{A}}\right) \left(\mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_t\right) - \lambda_t (\overline{m} + \underline{m}) \int_{\partial \mathcal{A}_t} u_t^2 \operatorname{div}\left(\mathbf{X}_{\mathcal{A}}\right) \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_t. \end{aligned}$$

Substituting $\dot{\lambda}_t$ with (6.19) and noticing that the last term vanishes thanks to Lemma 6.2 (ii), gives (6.23).

To obtain (6.24) it is sufficient to notice that $u_0 = w$, $\mathbf{n}_0 = \mathbf{x}/|\mathbf{x}|$ and that the first term vanishes since $\partial B$ is a level set for $w$ and thanks to Lemma 6.2 (ii). □

To conclude this section we derive an equation for $\dot{u}_t$.

**Lemma 6.10.** *For any $t \in [0,1]$, the function $\dot{u}_t$ solves, in $H^1(\mathbb{R}^N)$, the following problem:*

$$\begin{cases} -\Delta \dot{u}_t = \dot{\lambda}_t m_t u_t + \lambda_t m_t \dot{u}_t & \text{in } \mathbb{R}^N \setminus \partial\mathcal{A}_t, \\ [\dot{u}_t] = 0, \quad [\nabla \dot{u}_t \cdot \mathbf{n}_t] = \lambda_t(\overline{m} + \underline{m}) u_t \, \mathbf{X}_\mathcal{A} \cdot \mathbf{n}_t & \text{on } \partial\mathcal{A}_t \\ 2\int_{\mathbb{R}^N} m_t \dot{u}_t u_t = -(\overline{m} + \underline{m}) \int_{\partial\mathcal{A}_t} u_t^2 \, \mathbf{X}_\mathcal{A} \cdot \mathbf{n}_t, \end{cases} \tag{6.25}$$

*and the jump is defined as in (5.4).*

*In particular, for $t = 0$ it holds that $u_0 = w$, $\mathbf{n}_0 = \mathbf{x}/|\mathbf{x}|$ and*

$$\begin{cases} -\Delta \dot{w} = \tilde{\lambda}_0 \tilde{m}_0 \dot{w} & \text{in } \mathbb{R}^N \setminus \partial B, \\ [\dot{w}] = 0, \quad [\partial_\rho \dot{w}] = \tilde{\lambda}_0(\overline{m} + \underline{m}) w \, \mathbf{X}_\mathcal{A} \cdot \mathbf{n}_0 & \text{on } \partial B \\ 2\int_{\mathbb{R}^N} \tilde{m}_0 \dot{w} w = -(\overline{m} + \underline{m}) \int_{\partial B} w^2 \, \mathbf{X}_\mathcal{A} \cdot \mathbf{n}_0 = 0, \end{cases} \tag{6.26}$$

*with $[\partial_\rho \dot{w}](r_0\theta) = \partial_\rho \dot{w}(r_0^-\theta) - \partial_\rho \dot{w}(r_0^+\theta)$.*

**Proof.** The proof is straightforward, writing the weak formulation of (6.16) for any test function $\psi \in C_0^\infty(\mathbb{R}^N)$, and differentiating the formulation using the Hadamard formula. □

## 6.4. Coercivity

In this section we prove the following result.

**Proposition 6.11.** *There exist $\delta > 0$ sufficiently small and a positive constant $C > 0$ depending only on $\overline{m}$, $\underline{m}$ and the dimension $N$ such that, for any nearly spherical set $\mathcal{A}$ of class $C^{1,1}$ with $\|\varphi_\mathcal{A}\|_{C^{1,1}(\mathbb{S}^{N-1})} < \delta$, $\mathcal{L}(\mathcal{A}) = 1$ and $\mathrm{bar}(\mathcal{A}) = \mathbf{0}$, it holds*

$$\ddot{\lambda}_t\big|_{t=0} \geq C \, \|\mathbf{X}_\mathcal{A}(r_0, \cdot) \cdot \mathbf{n}_0\|^2_{L^2(\mathbb{S}^{N-1})}, \tag{6.27}$$

*where $\mathbf{n}_0 = \mathbf{x}/|\mathbf{x}|$.*

To prove coercivity, i.e. equation (6.27), it is a standard technique (see e.g. [11,10,26]) to expand

$$\mathbf{X}_\mathcal{A}(r_0, \theta) \cdot \mathbf{n}_0 = \frac{(r_0 + \varphi_\mathcal{A}(\theta))^N - r_0^N}{N\,\rho^{N-1}} = \frac{r_0}{N}\left(\left(1 + \frac{\varphi_\mathcal{A}(\theta)}{r_0}\right)^N - 1\right) \tag{6.28}$$

in spherical harmonics: let us denote with $(S_k)_k$ the orthonormal basis of $L^2(\mathbb{S}^{N-1})$ constituted by the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$, with eigenvalues $(\sigma_k)_k$:

$$-\Delta_{\mathbb{S}^{N-1}} S_k = \sigma_k S_k. \tag{6.29}$$

Then it is well known that $\sigma_0 = 0$, with constant eigenfunction, while

$$\sigma_1 = \sigma_2 = \cdots = \sigma_N = N - 1,$$

and the associated eigenspace is spanned by the orthogonal family of homogeneous harmonic polynomials of degree 1 composed by the $N$ cartesian coordinates $x_1, \dots, x_N$; finally, $\sigma_k \geq \sigma_{N+1} > N - 1$ for $k \geq N + 1$.

A first crucial comment is that, since $\mathcal{L}(\mathcal{A}) = 1$ and $\mathrm{bar}(\mathcal{A}) = 0$, in the expansion of $\mathbf{X}_{\mathcal{A}}(r_0, \cdot) \cdot \mathbf{n}_0$ the terms corresponding to $S_k$, $k \leq N$, are negligible.

**Lemma 6.12.** *Under the assumptions of Proposition 6.11, let us write*

$$\mathbf{X}_{\mathcal{A}}(r_0, \theta) \cdot \mathbf{n}_0 = \sum_{k=0}^{+\infty} c_k S_k(\theta), \qquad \text{where } c_k := (\mathbf{X}_{\mathcal{A}}(r_0, \cdot) \cdot \mathbf{n}_0, S_k)_{L^2(\mathbb{S}^{N-1})}. \tag{6.30}$$

*Then $c_0 = 0$ and*

$$\sum_{k=N+1}^{+\infty} c_k^2 = \|\mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_0\|^2_{L^2(\partial B)} \left(1 + \eta\left(\|\varphi_{\mathcal{A}}\|_{L^\infty(\mathbb{S}^{N-1})}\right)\right), \tag{6.31}$$

*for some modulus of continuity $\eta$.*

**Proof.** First of all, $c_0 = 0$ by Lemma 6.2 (iii). Next, recalling (6.28), let us consider the coefficients

$$c_k = \int_{\mathbb{S}^{N-1}} \theta_k \, \mathbf{X}_{\mathcal{A}}(r_0, \theta) \cdot \mathbf{n}_0 = \frac{r_0}{N} \int_{\mathbb{S}^{N-1}} \theta_k \left[(1 + \varphi_{\mathcal{A}}(\theta)/r_0)^N - 1\right],$$
$$k = 1, \dots, N. \tag{6.32}$$

Since the barycenter of $\mathcal{A}$ coincides with $\mathbf{0}$ by assumption, we can write

$$0 = \int_{\mathcal{A}} x_k - \int_B x_k = \frac{r_0^{N+1}}{N+1} \int_{\mathbb{S}^{N-1}} \theta_k \left[(1 + \varphi_{\mathcal{A}}(\theta)/r_0)^{N+1} - 1\right]. \tag{6.33}$$

Now, a direct computation gives

$$\left|\frac{r_0}{N}\int_{\mathbb{S}^{N-1}}\theta_k\left[(1+\varphi_{\mathcal{A}}(\theta)/r_0)^N-1\right]-\frac{r_0}{N+1}\int_{\mathbb{S}^{N-1}}\theta_k\left[(1+\varphi_{\mathcal{A}}(\theta)/r_0)^{N+1}-1\right]\right|\le$$
$$\le\eta\left(\|\varphi_{\mathcal{A}}\|_{L^\infty(\mathbb{S}^{N-1})}\right)\|\varphi_{\mathcal{A}}\|_{L^2(\mathbb{S}^{N-1})}\le\eta\left(\|\varphi_{\mathcal{A}}\|_{L^\infty(\mathbb{S}^{N-1})}\right)\|\mathbf{X}_{\mathcal{A}}\cdot\mathbf{n}_0\|_{L^2(\mathbb{S}^{N-1})},\tag{6.34}$$

where in the last passage we have used (6.9). Hence, combining (6.32), (6.33) and (6.34) we can deduce (6.31). □

To prove Proposition 6.11 we will plug the expansion (6.30) into (6.24). To clarify the role of $\dot{w}$ in the latter, it is convenient to expand in spherical harmonics also the associated differential problem (6.26).

**Lemma 6.13.** *For every integer $k\ge 1$ the problem*

$$\begin{cases}-\Delta z_k=\tilde{\lambda}_0\tilde{m}_0 z_k & \text{in } \mathbb{R}^N\setminus\partial B,\\ [z_k]=0,\quad [\partial_\rho z_k](\mathbf{x})=jS_k(\mathbf{x}/|\mathbf{x}|) & \text{on } \partial B\\ \int_{\mathbb{R}^N}\tilde{m}_0 zw=0,\end{cases}\tag{6.35}$$

*where*

$$[\partial_\rho z_k](r_0\theta)=\partial_\rho z_k(r_0^-\theta)-\partial_\rho z_k(r_0^+\theta)\qquad\text{and}\qquad j=\tilde{\lambda}_0(\overline{m}+\underline{m})w(r_0)>0,$$

*has a unique solution $z_k\in H^1(\mathbb{R}^N)$. Moreover:*

1. *$z_k(r\theta)=g_k(r)S_k(\theta)$, where*

$$\begin{cases}(r^{N-1}g_k')'+m(r)g_k=r^{N-3}\sigma_k g_k & \text{for } r>0,\ r\ne r_0,\\ g_k(0)=g_k(+\infty)=g_k(r_0^-)-g_k(r_0^+)=0,\\ g_k'(r_0^-)-g_k'(r_0^+)=j,\end{cases}$$

   *$\sigma_k$ is defined in (6.29), and $m(r)=\tilde{\lambda}_0\tilde{m}_0(r\mathbf{e})$, for every $|\mathbf{e}|=1$;*
2. *if $k=1,\dots,N$ then $z_k=\partial_{x_k}w$ and $g_k(r)=-w'(r)$;*
3. *for every $k\ge 1$, $g_k\ge 0$ on $(0,+\infty)$;*
4. *$g_k(r_0)$ is strictly monotone decreasing with respect to $\sigma_k$; in particular,*

$$g_k(r_0)\le g_{N+1}(r_0)<g_N(r_0)=-w'(r_0)=-\partial_\rho w|_{\partial B},\qquad \text{for every } k\ge N+1.$$

Actually, $g_k$ can be written explicitly in terms of Bessel's functions, yielding their further properties. Nonetheless, for the reader's convenience we provide a self-contained proof.

**Proof.** The well-posedness of the problem for $z_k$ follows by the Lax-Milgram theorem, arguing as in Lemma 6.5: indeed, notice that the jump condition on $\partial B$ can be inserted in the weak formulation as an $H^{-1}$ term $h$ in the right hand side, as it was obtained in Lemma 6.10; in particular, the condition $\langle h, \psi\rangle = 0$ for every $\psi \in H^1(\mathbb{R}^N)$ requires the restriction $k \neq 0$.

Once the solution is unique, property 1 easily follows by separation of variables, using (6.29) and the well-posedness of the problem for $g_k$, and it can be checked by direct computation. Uniqueness also implies property 2, since

$$\partial_{x_k} w = w'(|\mathbf{x}|) \cdot \frac{x_k}{|\mathbf{x}|} = w'(r) S_k(\theta), \qquad k = 1, \dots, N$$

(with the abuse of notation $w(\mathbf{x}) = w(|\mathbf{x}|) = w(r)$) and, in particular, using the equation for $w$,

$$[w''] = [-\tilde{\lambda}_0 \tilde{m}_0 w] = -\tilde{\lambda}_0(\overline{m} + \underline{m}) w(r_0) \qquad \text{on } \partial B,$$

so that $g_k = -w'(r)$ and $z_k = \partial_{x_k} w$, $1 \le k \le N$. Since $w'(r) < 0$ for $r > 0$, we also infer property 3 for $k \le N$.

In order to prove properties 3 and 4, we first need a Sturm-Picone-type comparison result for solutions with derivative jumps: let $0 \le r_1 < r_2 \le +\infty$, $h, k \ge 1$, and let us test the equation for $g_k$ with $r^{N-1} g_h$ on $(r_1, r_2)$. Then, in case $r_0 \notin (r_1, r_2)$, we have

$$\int_{r_1}^{r_2} r^{N-1} \left(g_h' g_k' - m(r) g_h g_k\right) dr = \left[r^{N-1} g_h g_k'\right]_{r_1}^{r_2} - \sigma_k \int_{r_1}^{r_2} r^{N-3} g_h g_k \, dr,$$

while if $r_0 \in (r_1, r_2)$ then we must integrate separately on the two subinterval and add the term $[r^{N-1} g_h g_k']_{r_0^+}^{r_0^-}$ to the right hand side. Exchanging the role of $h$ and $k$, and assuming $g_k(r_1) = g_k(r_2) = 0$, we finally obtain

$$(\sigma_k - \sigma_h) \int_{r_1}^{r_2} r^{N-3} g_h g_k \, dr + j(g_k(r_0) - g_h(r_0)) \chi_{(r_1, r_2)}(r_0) = \left[r^{N-1} g_h g_k'\right]_{r_1}^{r_2}. \tag{6.36}$$

Now, to show property 1 for $k > N$ let us assume by contradiction that $g_k$, $k > N$, is negative somewhere, and denote with $(r_1, r_2)$ a maximal interval of negativity: $g_k(r_1) = g_k(r_2) = 0$, $g_k(r) < 0$ for $r_1 < r < r_2$. Then $g_k'(r_1) \le 0 \le g_k'(r_2)$ and, recalling that $g_N = -w' > 0$ on $\mathbb{R}_+$, we infer

$$(\sigma_k - \sigma_N) \int_{r_1}^{r_2} r^{N-3} g_N g_k \, dr < 0, \quad j(g_k(r_0) - g_N(r_0)) \chi_{(r_1, r_2)}(r_0) \le 0,$$

$$\left[r^{N-1} g_N g_k'\right]_{r_1}^{r_2} \ge 0,$$

in contradiction with (6.36) (with $h = N$), thus property 3 follows.

Finally, take any $h < k$ such that $\sigma_h < \sigma_k$. Thanks to property 3 we obtain

$$(\sigma_k - \sigma_h) \int_0^{+\infty} r^{N-3} g_h g_k \, dr > 0, \qquad \left[r^{N-1} g_h g_k'\right]_0^{+\infty} = 0,$$

and (6.36) (with $r_1 = 0$, $r_2 = +\infty$) implies

$$j(g_k(r_0) - g_h(r_0)) < 0,$$

yielding property 4. □

**Proof of Proposition 6.11.** As a consequence of (6.30), (6.31), the proof of (6.27) can be reduced to the following statement: there exists $C = C(\overline{m}, \underline{m}, N) > 0$ such that, for any nearly spherical set $\mathcal{A}$ of class $C^{1,1}$ with $\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})} < \delta$, we have

$$\ddot{\lambda}_t\Big|_{t=0} \geq C \sum_{k=N+1}^{+\infty} c_k^2. \tag{6.37}$$

To prove (6.37) we insert (6.30) into (6.24). On the one hand, also recalling Lemma 6.13, 2,

$$\left|\frac{\partial w}{\partial \rho}\right|_{\partial B} \int_{\partial B} |\mathbf{X}_{\mathcal{A}}|^2 = g_N(r_0) \int_{\partial B} |\mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_0|^2 = g_N(r_0) \sum_{k=1}^{+\infty} c_k^2. \tag{6.38}$$

On the other hand, Lemmas 6.10 and 6.13, together with the expansion (6.30), yield

$$\dot{w}(r\theta) = \sum_{k=1}^{+\infty} c_k g_k(r) S_k(\theta),$$

whence

$$\int_{\partial B} \dot{w} \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_0 = \sum_{k=1}^{+\infty} g_k(r_0) c_k^2. \tag{6.39}$$

Plugging (6.38), (6.38) into (6.24), and using Lemma 6.13, 2, we finally obtain

$$\begin{aligned}
\ddot{\lambda}_t|_{t=0} &= 2\tilde{\lambda}_0(\overline{m} + \underline{m}) w(r_0) \sum_{k=1}^{+\infty} \left(g_N(r_0) - g_k(r_0)\right) c_k^2 \\
&= 2\tilde{\lambda}_0(\overline{m} + \underline{m}) w(r_0) \sum_{k=N+1}^{+\infty} \left(g_N(r_0) - g_k(r_0)\right) c_k^2,
\end{aligned}$$

and (6.37) follows with

$$C = 2\tilde{\lambda}_0(\overline{m} + \underline{m})w(r_0)\left[g_N(r_0) - g_{N+1}(r_0)\right],$$

which is strictly positive (and universal) by Lemma 6.13, 4. □

### 6.5. Continuity of the remainder

The aim of this section is to prove the following.

**Proposition 6.14.** *There exist $\delta > 0$ sufficiently small and a modulus of continuity $\eta$ such that, for any nearly spherical set $\mathcal{A}$ of class $C^{1,1}$ with $\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})} < \delta$ it holds that*

$$|\ddot{\lambda}_t - \ddot{\lambda}_0| \leq \eta\left(\|\varphi_{\mathcal{A}}\|_{C^{1,1}(\mathbb{S}^{N-1})}\right)\|\mathbf{X}_{\mathcal{A}}(r_0,\cdot)\cdot\mathbf{n}\|^2_{L^2(\mathbb{S}^{N-1})} \tag{6.40}$$

*uniformly for $t \in [0, 1]$.*

**Remark 6.15.** Notice that, as it will be clear from the following proofs and as remarked in [26], Proposition 6.14 holds even for nearly spherical sets $\mathcal{A}$ of class $C^{1,\alpha}$, under the only constraint that $\|\varphi_{\mathcal{A}}\|_{C^{1,\alpha}(\mathbb{S}^{N-1})}$ is small enough, for some $0 < \alpha < 1$. Of course, in such a case the $C^{1,1}$ norm in (6.40) has to be substituted by the $C^{1,\alpha}$ norm.

Before giving the proof, it is useful to introduce two lemmas concerning the $C^{1,\alpha}$ convergence of $u_t$ to $w$ and the $C^{0,\alpha}$ convergence of $\dot{u}_t$ to $\dot{w}$, respectively. Since we obtain such results by contradiction, exploiting the convergence induced by elliptic regularity, it is convenient to work in the case $\alpha < 1$.

For any nearly spherical set $\mathcal{A}$ of class $C^{1,\alpha}$, with $0 \leq \alpha < 1$, we denote with $u_{\mathcal{A}}$ the eigenfunction over the whole $\mathbb{R}^N$ associated to the eigenvalue $\lambda^1(\mathcal{A}, \mathbb{R}^N)$. As already remarked, such function exists and is uniquely defined, up to a normalization that we choose for instance to be in $L^2(\mathbb{R}^N)$. Notice that one could equivalently choose $u_{\mathcal{A}}$ normalized such that $\int_{\mathbb{R}^N} m_{\mathcal{A}}u_{\mathcal{A}}^2 = 1$, and the following discussion would remain unchanged. Then, we proceed with the following

**Lemma 6.16.** *Fix $0 \leq \alpha < 1$, $R > 0$. There exists a modulus of continuity $\eta$ such that, for any nearly spherical set $\mathcal{A}$ of class $C^{1,\alpha}$, it holds that*

$$\|u_{\mathcal{A}} - w\|_{C^{1,\alpha}(\overline{B_R})} \leq \eta\left(\|\varphi_{\mathcal{A}}\|_{C^{1,\alpha}(\mathbb{S}^{N-1})}\right).$$

**Proof.** We proceed by contradiction. Suppose the existence of a sequence of nearly spherical set $\mathcal{A}_k$ of class $C^{1,\alpha}(\partial B)$, satisfying $\|\varphi_{\mathcal{A}}\|_{C^{1,\alpha}(\mathbb{S}^{N-1})} \to 0$ for $k \to +\infty$, but such that $\|u_{\mathcal{A}_k} - w\|_{C^{1,\alpha}(\overline{B_R})} \geq \delta$, for any $k \in \mathbb{N}$ and for some $\delta > 0$.

However, proceeding as in Section 2, and in particular in Lemma 2.3, one gets convergence in $C^{1,\alpha}\left(\overline{B_R}\right)$ of $u_{\mathcal{A}_k}$ to $w$, both with the same normalization. Hence we have reached a contradiction and the proof is concluded. □

Now we turn to $\dot{u}_t$. For any nearly spherical set $\mathcal{A}$ consider its flow $\mathcal{A}_t := \Phi_{\mathcal{A}}(t, B)$. Denoting with $u_{\mathcal{A}_t}$ the eigenfunction associated to $\mathcal{A}_t$, we can prove the following

**Lemma 6.17.** *Fix $0 \le \alpha < 1$, $R > 0$. There exists a modulus of continuity $\eta$ such that, for any nearly spherical set $\mathcal{A}$ of class $C^{1,\alpha}$, it holds that*

$$\|\dot{u}_{\mathcal{A}_t} - \dot{w}\|_{C^{0,\alpha}(\overline{B_R})} \le \eta\left(\|\varphi_{\mathcal{A}_t}\|_{C^{1,\alpha}(\mathbb{S}^{N-1})}\right),$$

*where $\eta$ is independent of $t \in [0,1]$.*

**Proof.** Without loss of generality we can assume $B \subset\subset B_R$. First, we notice that $\dot{u}_{\mathcal{A}_t}$ actually belongs to $C^{0,\alpha}\left(\overline{B_R}\right)$ for any fixed $R > 0$ and any $t \in [0,1]$. To this aim, we consider the function $z_{\mathcal{A}_t} := \dot{u}_{\mathcal{A}_t} + \nabla u_{\mathcal{A}_t} \cdot \mathbf{X}_{\mathcal{A}}$.

Since the function $\nabla u_{\mathcal{A}_t}$ solves a problem analogous to (5.3), using (6.10) one can prove that the function $z_{\mathcal{A}_t}$ solves

$$\begin{cases} -\Delta z_{\mathcal{A}_t} = \dot{\lambda}_t m_t u_t + \lambda_t m_t z_{\mathcal{A}_t} - \operatorname{div}(\nabla(\nabla u_{\mathcal{A}_t} \cdot \mathbf{X}_{\mathcal{A}})) - \lambda_t m_t \nabla u_{\mathcal{A}_t} \cdot \mathbf{X}_{\mathcal{A}} & \text{in } \mathbb{R}^N, \\ [z_{\mathcal{A}_t}] = 0, \quad [\partial_{\mathbf{n}} z_{\mathcal{A}_t}] = 0 & \text{on } \partial\mathcal{A}_t \end{cases} \tag{6.41}$$

Since $\nabla u_{\mathcal{A}_t} \in W^{1,p}(B_R)$ for any $1 \le p < +\infty$ (this follows by elliptic regularity in $L^p$ spaces for $u_{\mathcal{A}_t}$), standard elliptic regularity for (6.41) tells us that $z_{\mathcal{A}_t} \in C^{0,\alpha}\left(\overline{B_R}\right)$ for any $\alpha \in [0,1)$. Moreover, by bootstrap, the bound is uniform with respect to $t \in [0,1]$ and $\mathcal{A}$, since $\|u_{\mathcal{A}_t}\|_{H^1(B_R)}$ is bounded uniformly with respect to $t \in [0,1]$ and $\mathcal{A}$, when $\|\varphi_{\mathcal{A}_t}\|_{C^{1,\alpha}(\mathbb{S}^{N-1})}$ is sufficiently small. The last assertion follows from Lemma 6.16. Consequently, the same regularity property holds for $\dot{u}_{\mathcal{A}_t}$.

Now we proceed by contradiction. Suppose the existence of a sequence $\mathcal{A}_{t_k}$ of nearly spherical sets of class $C^{1,\alpha}$ with $\|\varphi_{\mathcal{A}_t}\|_{C^{1,\alpha}(\mathbb{S}^{N-1})} \to 0$ for $k \to +\infty$, but $\|\dot{u}_{\mathcal{A}_t} - \dot{w}\|_{C^{0,\alpha}(\overline{B_R})} \ge \delta$ for some constant $\delta > 0$. We remark that the notation $\mathcal{A}_{t_k}$ is to be intended in the sense that both $\mathcal{A}$ and $t$ can change with $k$. To this sequence we associate the functions $\dot{u}_{t_k}$ and $\dot{w}_k$.

Notice that, since for any $0 < \alpha < 1$ the quantity $\|\mathbf{X}_{\mathcal{A}_k}\|_{C^{1,\alpha}(B_R)}$ is bounded uniformly in $k$, up to a subsequence there exists a limit $\mathbf{X} \in C^{1,\alpha}(B_R)$.

The first thing we do is proving an $H^1(\mathbb{R}^N)$ bound for $\dot{u}_{t_k}$ uniformly in $k$. This can be done supposing that $\|\dot{u}_{t_k}\|_{H^1(\mathbb{R}^N)}$ diverges, and then reaching a contradiction. Since this is a method that we have already used many times, we will be short.

Rescaling (6.25) by $\|\dot{u}_{t_k}\|_{H^1(\mathbb{R}^N)}$, up to a subsequence the functions $\dot{u}_{t_k}$ converge weakly in $H^1(\mathbb{R}^N)$ and in $C^{0,\alpha}\left(\overline{B_R}\right)$ (thanks to the uniform bound found above) to a solution $s \in H^1(\mathbb{R}^N)$ of

$$\begin{cases} -\Delta s = \tilde{\lambda}_0 \tilde{m}_0 s & \text{in } \mathbb{R}^N, \\ [s] = 0, \quad [\partial_{\mathbf{n}} s] = 0 & \text{on } \partial\mathcal{A}_t \\ 2\int_{\mathbb{R}^N} \tilde{m}_0 s w = 0. \end{cases}$$

Hence $s$ is a multiple of $w$, but the integral condition implies that $s \equiv 0$. Hence, $\dot{u}_{t_k} \to 0$ in $C^{0,\alpha}\left(\overline{B_R}\right)$. However, from the rescaled version of (6.25), one can notice that for $k$ sufficiently large it holds that $\|\dot{u}_{t_k}\|_{H^1(\mathbb{R}^n)} \le C\|\dot{u}_{t_k}\|_{L^2(B_R)} + o(1)$ for some universal constant $C > 0$, which leads to a contradiction.

We have just proved that $\|\dot{u}_{t_k}\|_{H^1(\mathbb{R}^n)}$ is uniformly bounded with respect to $k$. Of course, analogous properties also hold for $\dot{w}_k$.

To conclude, since $\dot{\lambda}_{t_k} \to 0$ for $k \to +\infty$ thanks to Lemma 6.16, and since $u_{t_k} \to w$ strongly in $H^1(\mathbb{R}^N)$ (proceeding as in Sections 2 and 3), passing to the limit in (6.25) and (6.26) it can be noticed that both the sequences $\dot{u}_{t_k}$ and $\dot{w}_k$ converge up to a subsequence, weakly in $H^1(\mathbb{R}^N)$ and in $C^{0,\alpha}\left(\overline{B_R}\right)$ to a solution in $H^1(\mathbb{R}^N)$ of the problem

$$\begin{cases} -\Delta v = \tilde{\lambda}_0 \tilde{m}_0 v & \text{in } \mathbb{R}^N, \\ [v] = 0, \quad [\partial_{\mathbf{n}} v] = \tilde{\lambda}_0(\overline{m} + \underline{m}) w\, \mathbf{X}\cdot\mathbf{n} & \text{on } \partial B \\ 2\int_{\mathbb{R}^N} \tilde{m}_0 v w = -(\overline{m}+\underline{m})\int_{\partial B} w^2\, \mathbf{X}\cdot\mathbf{n}. \end{cases} \tag{6.42}$$

However, as already remarked, there exist at most one solution in $H^1(\mathbb{R}^N)$ to problem (6.42), so that $\dot{u}_{t_k}$ and $\dot{w}_k$ converge in $C^{0,\alpha}\left(\overline{B_R}\right)$ to the same function, and we have reached a contradiction. □

Now we are ready to prove Proposition 6.14.

**Proof of Proposition 6.14.** We write $\ddot{\lambda}_t - \ddot{\lambda}_0 := A - B - C$, with

$$A := \lambda_t \left[ (\overline{m}+\underline{m}) \int_{\mathcal{A}_t} \operatorname{div}(u_t^2 \mathbf{X}_{\mathcal{A}}) \right]^2,$$

$$B := 2(\overline{m}+\underline{m}) \left[ \lambda_t \int_{\partial\mathcal{A}_t} u_t \nabla u_t \cdot\, \mathbf{X}_{\mathcal{A}}\, \mathbf{X}_{\mathcal{A}}\cdot \mathbf{n}_t - \lambda_0 \int_{\partial B} w \nabla w \cdot \mathbf{n} |\mathbf{X}_{\mathcal{A}}|^2 \right],$$

$$C := 2(\overline{m}+\underline{m}) \left[ \lambda_t \int_{\partial\mathcal{A}_t} \dot{u}_t u_t \mathbf{X}_{\mathcal{A}}\cdot\mathbf{n}_t - \lambda_0 \int_{\partial B} \dot{w} w \mathbf{X}_{\mathcal{A}}\cdot\mathbf{n} \right].$$

Rewriting $A$ as a surface integral on $\partial\mathcal{A}_t$, transporting all the integrals on $\partial B$ and denoting with $J_{T,t}$ the tangential jacobian associated with $\partial\mathcal{A}_t$, the above terms can be rewritten as

$$A := \lambda_t \left[ (\overline{m}+\underline{m}) \int_{\partial B} (u_t^2 \mathbf{X}_{\mathcal{A}}\cdot\mathbf{n}_t)\circ\Phi_{\mathcal{A}} J_{T,t} \right]^2,$$

$$B := 2(\overline{m}+\underline{m}) \left[ \lambda_t \int_{\partial B} (u_t \nabla u_t \cdot\, \mathbf{X}_{\mathcal{A}}\, \mathbf{X}_{\mathcal{A}}\cdot\mathbf{n}_t)\circ\Phi_{\mathcal{A}} J_{T,t} - \lambda_0 \int_{\partial B} w\nabla w\cdot\mathbf{n}|\mathbf{X}_{\mathcal{A}}|^2 \right],$$

$$C := 2(\overline{m} + \underline{m}) \left[ \lambda_t \int\limits_{\partial B} (\dot{u}_t u_t \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n}_t) \circ \Phi_{\mathcal{A}} J_{T,t} - \lambda_0 \int\limits_{\partial B} \dot{w} w \mathbf{X}_{\mathcal{A}} \cdot \mathbf{n} \right].$$

Taking into account that the $C^{1,\alpha}$ norm of $\varphi_{\mathcal{A}}$ is controlled by the $C^{1,1}$ norm, the proof can be concluded using Lemma 6.16 to treat $u_t$, Lemma 6.17 to treat $\dot{u}_t$, Lemma 6.2 (iv), (v) to deal with the composition by $\Phi_{\mathcal{A}}$, Lemma 5.4 (iii), (iv) (notice that Lemma 5.4 can be easily rewritten in terms of a modulus of continuity) to deal with $\mathbf{n}_t$ and $J_{T,t}$ and, to conclude, the continuity of $\lambda_t$ stated in Proposition 6.6. □

**End of the proof of Theorem 1.5.** Expanding $\lambda_t$ in Taylor series up to the second order, we obtain

$$\lambda^1(\mathcal{A}, \mathbb{R}^N) - \lambda^1(B, \mathbb{R}^N) = \frac{1}{2}\ddot{\lambda}_0 + \int\limits_0^1 (1-t)(\ddot{\lambda}_t - \ddot{\lambda}_0).$$

The proof can be concluded using Propositions 6.14, 6.11 and Lemma 6.2 (vi) above. □

## Data availability

No data was used for the research described in the article.

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