BOUND STATES OF NONLINEAR SCHRÖDINGER EQUATIONS WITH POTENTIALS VANISHING AT INFINITY

By

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1 Introduction

We consider a class of nonlinear Schrödinger equations with potentials

$$(1) \qquad -\varepsilon^2 \Delta u + V(x) u = K(x) u^p, \quad x \in \mathbb{R}^n, \quad u > 0.$$

The physically relevant solutions of (1) are those with finite energy, namely, those belonging to the Sobolev space $W^{1,2}(\mathbb{R}^n)$. These solutions are called *bound states*.

A great deal of work has been devoted to finding bound states of (1), both in the case when ε is arbitrary and when $\varepsilon \sim 0$. Regarding the former case, critical point theory has been used to prove the existence of bound states u_ε of (1) under suitable assumptions on V and K. For example, in [13], it is assumed that $K \equiv 1$, $V > 0$ and $\lim_{|x| \to \infty} V(x) = +\infty$; and in [6], the case in which $V \equiv 1$, $K > 0$, $\lim_{|x| \to \infty} K(x) = k_0 > 0$ and K tends exponentially to k_0 is handled. For other results, see the book [7], which also contains an extensive bibliography. In any case, it is worth pointing out that the common assumptions are

$$(V0) \qquad \inf_{x \in \mathbb{R}^n} V(x) > 0,$$

$$(K0) \qquad \exists\, \kappa > 0 \; : 0 < K(x) \le \kappa, \qquad \forall\, x \in \mathbb{R}^n.$$

Bound states of (1) when $\varepsilon \ll 1$ are called *semiclassical states* and are relevant for the links between classical and quantum mechanics. An important feature of semiclassical states u_ε is that they *concentrate* as $\varepsilon \to 0$. By this we mean, roughly,

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that outside of a neighborhood of a set S, u_ε tends uniformly to zero as $\varepsilon \to 0$. For example, by concentration at a point $x_0 \in \mathbb{R}^n$ we mean that

$$\forall\, \delta > 0,\ \exists\, \varepsilon_0 > 0,\ R > 0 \text{ s.t. } \quad u_\varepsilon(x) \le \delta,\ \forall\, |x - x_0| \ge R,\ \forall\, \varepsilon < \varepsilon_0.$$

The existence of semiclassical states holds true under much weaker assumptions on V and K (although one always supposes that (V0)–(K0) hold). Denote by Q the function

$$\text{(Q)} \qquad Q(x) = [V(x)]^{\theta}[K(x)]^{-2/(p-1)}, \qquad \theta = \frac{p+1}{p-1} - \frac{n}{2}.$$

For Q smooth, we say that x_0 is an isolated *stable* stationary point of Q if the Leray–Schauder index $ind(Q', x_0, 0)$ is different from zero. The index $ind(Q', x_0, 0)$ of Q' with respect to x_0 and 0 is defined as $\lim_{r \to 0} deg(Q', B_r(x_0), 0)$. Here deg denotes the topological degree and $B_r(x_0)$ is the ball in $\mathbb{R}^n$ of radius r, centered at x_0. It is easy to see that local isolated maxima and minima as well as non-degenerate stationary points are stable.

The following is a typical result dealing with the existence of semiclassical states; see, e.g., [5] and Chapter 8 in [2] and the references cited there.

Theorem A. *Suppose that* $1 < p < (n+2)/(n-2)$ *and let* V *and* K *be smooth and satisfy* (V0) *and* (K0), *respectively.*

(i) *If* u_ε *concentrates at a point* x_0, *then* $Q'(x_0) = 0$.

(ii) *Conversely, if* x_0 *is an isolated stable stationary point of* Q, *then for all* $\varepsilon \ll 1$, *there exists a semiclassical state of* (1) *concentrating at* x_0.

We also mention that solutions concentrating on spheres have been found in the radial case, [4].

In a recent paper [3] (see also some previous partial results in [11, 14]), the new case in which V may decay to zero as $|x| \to \infty$ has been addressed. More precisely, it is assumed that the potentials V, K are smooth and satisfy

$$\text{(V1)} \qquad \exists\, A_0, A_1 > 0 : \ \frac{A_0}{1 + |x|^{\alpha}} \le V(x) \le A_1, \qquad 0 \le \alpha \le 2,$$

$$\text{(K1)} \qquad \exists\, \beta, k > 0 : \quad 0 < K(x) \le \frac{k}{1 + |x|^{\beta}}\,.$$

For $n \ge 3$ and

$$\sigma = \begin{cases} \dfrac{n+2}{n-2} - \dfrac{4\beta}{\alpha(n-2)}, & \text{if } 0 < \beta < \alpha, \\ 1 & \text{otherwise,} \end{cases}$$

the following result has been proved by using critical point theory in weighted Sobolev spaces.

Theorem B ([3]). *Let V and K satisfy* (V1) *and* (K1) *respectively, and suppose that* $0 \le \alpha < 2$ *and p satisfies*

$$(\sigma) \qquad \sigma < p < (n+2)\,/\,(n-2)\,.$$

Then, for every $\varepsilon > 0$, (1) *has a solution* u_ε *which is a ground state, i.e., it is a mountain pass solution with minimal energy. Furthermore, as* $\varepsilon \to 0$, u_ε *concentrates at a global minimum of* Q.

Let us point out the following facts.

(B.1) If $\alpha > 0$ and $\beta = 0$, one has $\sigma = (n+2)\,/\,(n-2)$; hence the previous result does not apply. On the other hand, it is possible to show that if $V(x) \sim (1+|x|)^{-\alpha}$, $K(x) \sim (1+|x|)^{-\beta}$ and (σ) is violated, then there are no ground states at all; see [3, Prop. 15].

(B.2) Under the assumptions of Theorem B, the auxiliary potential Q has indeed a minimum on $\mathbb{R}^n$.

The main purpose of the present paper is to show that there exist bound states of (1) for all p satisfying $1 < p < (n+2)\,/\,(n-2)$, provided ε is sufficiently small. We further require that V and K are smooth and satisfy

$$(\text{V2}) \qquad \exists\, V_1 > 0 \,:\, |V'(x)| \le V_1, \quad \forall\, x \in \mathbb{R}^n;$$

$$(\text{K2}) \qquad \exists\, \kappa_1 > 0 \,:\, |K'(x)| \le \kappa_1, \quad \forall\, x \in \mathbb{R}^n.$$

Our main result is the following.

Theorem 1. *Let* $1 < p < (n+2)\,/\,(n-2)$ *(if* $n \ge 3$*; otherwise, any* $p > 1$ *is allowed) and suppose that V and K are smooth and satisfy* (V1) – (V2) *and* (K0) – (K2), *respectively. Moreover, let* x_0 *be an isolated stable stationary point of* Q. *Then for* $\varepsilon \ll 1$, *equation* (1) *has a bound state which concentrates at* x_0.

We anticipate that the case in which Q has a compact set of non-isolated critical points can also be handled; see Theorems 12 and 13.

Of course, if $\alpha = 0$, i.e., when V is bounded away from zero, we recover Theorem A. A comparison with Theorem B is also in order. In Theorem 1,

- K does not need to decay to zero at infinity; in particular, we can deal with the case in which K is constant.

- The full range $1 < p < (n+2)/(n-2)$ is allowed.

- The case $\alpha = 2$ can be handled: we can deal with potentials V such that

$$\frac{A_0}{1+|x|^2} \leq V(x) \leq A_1.$$

On the other hand,

- we prove the existence of solutions only for ε small enough.

- We do not find ground states (which may not exist; see remark (B.1) above), but merely bound states.

The proof of Theorem 1 relies on arguments different from those used in [3]. Specifically, we use a perturbation method, variational in nature (see [1] and [2, Chapter 2]) which we outline in the next section. Although the general procedure is similar to the one used in [4, 5], several changes and different estimates are required here, due to the fact that the potential V decays to zero at infinity.

The paper consists of seven sections. In Sections 2–6, we carry out the proof of Theorem 1. Section 7 contains some further existence results.

General Remarks and Notation

- $B(y, R)$ denotes the ball $\{x \in \mathbb{R}^n : |x - y| \leq R\}$.

- If J is a functional, J' denotes its gradient.

- $W^{1,2}(\mathbb{R}^n)$ denotes the usual Sobolev space.

- In the sequel, we always take $n \geq 3$. The case $n = 1, 2$ requires minor changes.

- Without loss of generality, we assume that $Q'(0) = 0$. Hence we seek solutions concentrating at $x_0 = 0$.

- $c_1, c_2, \ldots, C_1, C_2, \ldots$ denote positive, possibly different, constants.

- $o_\varepsilon(1)$ denotes a quantity such that $\lim_{\varepsilon \to 0} o_\varepsilon(1) = 0$.

2 Outline of the abstract procedure and other preliminaries

After a change of variables, we are led to study the problem

$$-\Delta u + V(\varepsilon x)u = K(\varepsilon x)u^p, \quad u \in W^{1,2}(\mathbb{R}^n), \quad u > 0. \tag{2}$$

If u is a solution of (2), then $u(x/\varepsilon)$ solves (1).

Let us introduce the space $E = E_\varepsilon$,

$$E = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(\varepsilon x)u^2(x)dx < \infty\}.$$

E is a Hilbert space with scalar product and norm given, respectively, by

$$(u, v) = \int_{\mathbb{R}^n} [\nabla u(x) \cdot \nabla v(x) + V(\varepsilon x)u(x)v(x)]\, dx, \qquad \|u\|^2 = (u, u). \tag{3}$$

Since the functions in E might not belong to $L^{p+1}(\mathbb{R}^n)$, we need to introduce a truncated nonlinearity. If $\mu(s)$ is a real C^∞ function such that $0 \le \mu(s) \le 1$, $\mu(s) \equiv 1$ for $s < 1$, $\mu(s) \equiv 0$ for $s > 2$, we set $\Upsilon(x, u) = \mu\left(\bar{c}^{-1}|u|(1 + \varepsilon|x|)^\vartheta\right)$ and define

$$F_\varepsilon(x, u) = \Upsilon(x, u)\frac{|u_+|^{p+1}}{p+1} + (1 - \Upsilon(x, u))\, \bar{c}(1 + \varepsilon|x|)^{-\vartheta(p+1)}, \tag{4}$$

where $\bar{c}$ and ϑ are to be chosen later. For the moment, we simply take $\vartheta(p-1) > n$, which, in particular, implies that $F_\varepsilon(x, u(x)) \in L^1(\mathbb{R}^n)$ for $u \in E$.

For $u \in E$ we set

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}^n} K(\varepsilon x)F_\varepsilon(x, u(x))dx. \tag{5}$$

Clearly, any critical point u of I_ε such that $|u(x)| < \bar{c}(1 + |\varepsilon x|)^{-\vartheta}$ gives rise to a solution of (2). Moreover, since $\vartheta(p-1) > n$, it follows that $I_\varepsilon \in C^2(E, \mathbb{R})$.

Next, denote by U the unique radial positive function satisfying (see [9])

$$-\Delta U + U = U^p, \qquad U \in W^{1,2}(\mathbb{R}^n).$$

Setting

$$z_{\varepsilon,\xi}(x) = \sigma U(\lambda(x - \xi)), \qquad \sigma = \left[\frac{V(\varepsilon\xi)}{K(\varepsilon\xi)}\right]^{1/(p-1)}, \quad \lambda = [V(\varepsilon\xi)]^{1/2},$$

one checks that $z = z_{\varepsilon,\xi}$ satisfies

$$-\Delta z + V(\varepsilon\xi)z = K(\varepsilon\xi)z^p. \tag{6}$$

We are now ready to describe the finite dimensional procedure used below. We introduce the manifold

$$Z_\varepsilon = Z = \{z_{\varepsilon,\xi}(x) : \xi \in \mathbb{R}, |\varepsilon\xi| < 1\},$$

and let $P = P_\xi$ denote the orthogonal projection onto $(T_{z_{\varepsilon,\xi}} Z)^\perp$, the orthogonal complement (with respect to the scalar product in (3)) to the tangent space to Z at $z_{\varepsilon,\xi}$. Critical points of I_ε in the form $u = z_{\varepsilon,\xi} + w$, with $z_{\varepsilon,\xi} \in Z$, will be found by using a finite dimensional reduction which takes into account the variational nature of the problem. For a broader discussion of this abstract perturbation method in critical point theory, as well as for several different applications, including nonlinear Schrödinger equations, we refer to the aforementioned monograph [2].

In the present case, we begin by studying, for every $\xi \in \mathbb{R}^n$ with $|\varepsilon\xi| \leq 1$, the auxiliary equation

$$PI'_\varepsilon(z_{\varepsilon,\xi} + w) = 0. \tag{7}$$

Roughly speaking, (7) is first transformed into an equivalent fixed point problem $S_\varepsilon(w) = w$. One selects a subset Γ_ε of E, whose functions satisfy appropriate estimates and have suitable decays. Finally, one shows that, for $\varepsilon \ll 1$, S_ε is a contraction which maps Γ_ε into itself and hence has a unique fixed point $w_{\varepsilon,\xi}$ satisfying (7).

Once (7) is solved, it is a general fact that the manifold $\{u = z_{\varepsilon,\xi} + w_{\varepsilon,\xi}\}$ is a *natural constraint* for I_ε; see [2, Chapter 2]. This means that for finding the critical points of I_ε on E, it suffices to find critical points of the *reduced* (finite dimensional) *functional* $\Phi_\varepsilon(\xi) = I_\varepsilon(z_{\varepsilon,\xi} + w_{\varepsilon,\xi})$.

The existence of solutions of (7) is discussed in Sections 3, 4 and 5. The study of the reduced functional $\Phi_\varepsilon(\xi)$ is carried out in Section 6 and allows us to conclude the proof of Theorem 1.

3 Solving the auxiliary equation, I

In this section, we prove some preliminary results needed to solve the auxiliary equation (7). The following lemma provides a uniform lower bound for $V(\varepsilon x + y)$.

Lemma 2. *Let $\alpha > 0$, suppose that $V(x) > a|x|^{-\alpha}$ for $|x| > 1$, and let $m > 0$ be given. Then there exist $\varepsilon_0 > 0$ and $R > 0$ such that*

$$V(\varepsilon x + y) \geq \frac{m}{|x|^\alpha}, \qquad \forall\, |x| \geq R,\ \varepsilon \leq \varepsilon_0,\ y \in \mathbb{R}^n,\ |y| \leq 1.$$

Proof. Let $\varepsilon_0 > 0$ and $R > 0$ be such that

$$\left(\frac{2a}{3\varepsilon_0}\right)^\alpha = m, \qquad m\, R^{-\alpha} < \min\{V(x) : |x| \le 3\},$$

and take $\varepsilon < \varepsilon_0$, $|x| > R$ and $|y| \le 1$. Then, for $|x| > 2/\varepsilon$, one has $\varepsilon x + y > 1$; and hence

$$\begin{aligned} V(\varepsilon x + y) >& \frac{a}{|\varepsilon x + y|^\alpha} > \frac{a}{\varepsilon^\alpha |x|^\alpha}\left(\frac{|\varepsilon x|}{|\varepsilon x| + 1}\right)^\alpha \\ >& \frac{a}{\varepsilon^\alpha |x|^\alpha}\left(\frac{2}{3}\right)^\alpha = \frac{m}{|x|^\alpha}. \end{aligned}$$

On the other hand, if $R < |x| < 2/\varepsilon$, we have $|\varepsilon x + y| \le 3$; and therefore

$$V(\varepsilon x + y) > \frac{m}{R^\alpha} > \frac{m}{|x|^\alpha}.$$

This concludes the proof. □

Next we estimate the size of $I'_\varepsilon(z_{\varepsilon,\xi})$. The proof is similar to that of Lemma 1 in [5], but we carry out the details since we are using a different functional space with a different norm. Note that we can choose ϑ in the definition (4) of F_ε in such a way that the functional I_ε defined in (5) evaluated on $z_{\varepsilon,\xi}$ takes the form

$$I_\varepsilon(z_{\varepsilon,\xi}) = \tfrac{1}{2}\|z_{\varepsilon,\xi}\|^2 - \int_{\mathbb{R}^n} K(\varepsilon x)|z_{\varepsilon,\xi}(x)|^{p+1}dx.$$

Lemma 3. *There exists $\overline{C} > 0$ such that $\|I'_\varepsilon(z_{\varepsilon,\xi})\| \le \overline{C}\varepsilon$ provided $|\varepsilon\xi| \le 1$.*

Proof. Let us evaluate $I'_\varepsilon(z_{\varepsilon,\xi})[v]$ for an arbitrary $v \in E$. Taking into account that $z_{\varepsilon,\xi}$ satisfies (6), one finds

$$\begin{aligned} |I'_\varepsilon(z_{\varepsilon,\xi})[v]| \le& \left|\int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]z_{\varepsilon,\xi}v dx\right| + \left|\int_{\mathbb{R}^n}[K(\varepsilon x) - K(\varepsilon\xi)]z^p_{\varepsilon,\xi}v dx\right| \\ \le& \left(\int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]^2 z_{\varepsilon,\xi}dx\right)^{1/2}\left(\int_{\mathbb{R}^n} v^2 z_{\varepsilon,\xi}dx\right)^{1/2} \\ &+ \left(\int_{\mathbb{R}^n}[K(\varepsilon x) - K(\varepsilon\xi)]^2 z^p_{\varepsilon,\xi}dx\right)^{1/2}\left(\int_{\mathbb{R}^n} v^2 z^p_{\varepsilon,\xi}dx\right)^{1/2}. \end{aligned}$$

Let us estimate the first integral on the right-hand side. The change of variable $\zeta = x - \xi$ yields

$$\int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]^2 z_{\varepsilon,\xi}dx = \int_{\mathbb{R}^n}[V(\varepsilon\zeta + \varepsilon\xi) - V(\varepsilon\xi)]^2 z_{\varepsilon,\xi}(\zeta + \xi)d\zeta. \tag{8}$$

From the definition of $z_{\varepsilon,\xi}$, it follows that

$$(9)\qquad z_{\varepsilon,\xi}(\zeta+\xi)=\sigma U(\lambda(\zeta)),\qquad \sigma=\left(\frac{V(\varepsilon\xi)}{K(\varepsilon\xi)}\right)^{1/(p-1)},\quad \lambda^2=V(\varepsilon\xi);$$

and this makes it clear that $z_{\varepsilon,\xi}(\cdot+\xi)$ has uniform exponential decay at infinity, provided $|\varepsilon\xi|\le 1$ (in this region, V and K are bounded from above and from below by two fixed positive constants). By the regularity of V, we obtain from Taylor's formula

$$|V(\varepsilon\zeta+\varepsilon\xi)-V(\varepsilon\xi)|^2\le C\varepsilon^2\qquad (C>0),$$

and by the exponential decay of z we infer that

$$\int_{\mathbb{R}^n}[V(\varepsilon\zeta+\varepsilon\xi)-V(\varepsilon\xi)]^2 z_{\varepsilon,\xi}(\zeta+\xi)d\zeta\le C\varepsilon^2\int|\zeta|^2 z_{\varepsilon,\xi}(\zeta+\xi)d\zeta\le C\varepsilon^2.$$

Regarding the integral $\int_{\mathbb{R}^n}v^2 z_{\varepsilon,\xi}$, we claim that there exists $c_1>0$, independent of ε, such that $z_{\varepsilon,\xi}(x)\le c_1\,V(\varepsilon x)$, provided $\varepsilon\ll 1$ and $|\varepsilon\xi|\le 1$. This is equivalent to

$$(10)\qquad z_{\varepsilon,\xi}(\zeta+\xi)\le c_1\,V(\varepsilon\zeta+\varepsilon\xi),\qquad \forall\,\varepsilon\ll 1,\ \zeta\in\mathbb{R}^n,\ |\varepsilon\xi|\le 1.$$

Using Lemma 2 with $m=1$, $x=\zeta$ and $y=\varepsilon\xi$, we deduce that

$$V(\varepsilon\zeta+\varepsilon\xi)\ge\frac{1}{|\zeta|^\alpha},\qquad \forall\,\varepsilon\ll 1,\ |\zeta|>R,\ |\varepsilon\xi|\le 1.$$

Since, as remarked before, $z_{\varepsilon,\xi}(\cdot+\xi)$ has uniform exponential decay at infinity, taking c sufficiently large, we have

$$z_{\varepsilon,\xi}(\zeta+\xi)\le\frac{c}{|\zeta|^\alpha}\le c_1\,V(\varepsilon\zeta+\varepsilon\xi),\qquad \forall\,\varepsilon\ll 1,\ |\zeta|>R,\ |\varepsilon\xi|\le 1.$$

By taking c_1 possibly larger, we can ensure that the preceding inequality holds for $|\zeta|\le R$, too. This shows that (10) holds and proves our claim. From (10) we infer that

$$\int_{\mathbb{R}^n}z_{\varepsilon,\xi}v^2dx\le c_1\int_{\mathbb{R}^n}V(\varepsilon x)v^2dx\le c_2\,||v||^2.$$

Similar estimates hold for the terms involving the function K. This completes the proof. □

We now start the study of $I''_\varepsilon(z_{\varepsilon,\xi})$.

Lemma 4. *The operator $I''_\varepsilon(z_{\varepsilon,\xi}):E\to E$ is a compact perturbation of the identity.*

Proof. The proof follows an argument from [1] but takes into account that the potential V may tend to zero at infinity. One has

$$\begin{aligned} I''_\varepsilon(z_{\varepsilon,\xi})[u,v] &= \int \left[\nabla u\cdot\nabla v + V(\varepsilon x)uv - pz_{\varepsilon,\xi}^{p-1}uv\right]dx \\ &=(u,v) - p\int z_{\varepsilon,\xi}^{p-1}uv\,dx, \end{aligned}$$

where (u,v) is the scalar product in E. We need to prove that the operator $K(u,v) = \int z_{\varepsilon,\xi}^{p-1}uv$ is compact. Take $u_m \rightharpoonup u_0$, $v_m \rightharpoonup v_0$ in E; we claim that $K(u_m,v_m) \to K(u_0,v_0)$.

Clearly, we can restrict ourselves to the case $u_0 = 0$, $v_0 = 0$. Observe that the sequences u_m, v_m must be bounded in E; suppose, for instance, that $\|u_m\| \le 1$, $\|v_m\| \le 1$.

Given $\delta > 0$, because of the exponential decay of $z_{\varepsilon,\xi}$, we can take $R > 0$ such that $(\int_{|x-\xi|>R} z_{\varepsilon,\xi}^{(p-1)n/2})^{2/n} < \delta/2S^2$, where S is the Sobolev constant. Then

$$\begin{aligned} \int z_{\varepsilon,\xi}^{p-1}|u_m v_m|dx &= \int_{|x-\xi|\le R} z_{\varepsilon,\xi}^{p-1}|u_m v_m|dx + \int_{|x-\xi|>R} z_{\varepsilon,\xi}^{p-1}|u_m v_m|dx \\ &\le \int_{|x-\xi|\le R} z_{\varepsilon,\xi}^{p-1}|u_m v_m|dx \\ &\quad + \left[\int_{|x-\xi|>R} z_{\varepsilon,\xi}^{(p-1)n/2}dx\right]^{2/n} \|u_m\|_{L^{2^*}}\|v_m\|_{L^{2^*}} \\ &\le \int_{|x-\xi|\le R} z_{\varepsilon,\xi}^{p-1}|u_m v_m|dx + \delta/2. \end{aligned}$$

Since $V(\varepsilon x) \ge \gamma > 0$ for any x, $|x-\xi| \le R$, u_m, v_m belong to $W^{1,2}(B(\xi,R))$. Moreover, $u_m \rightharpoonup 0$ in $W^{1,2}(B(\xi,R))$. By the compact embedding, $u_m \to 0$ in $L^2(B(0,R))$; the same holds for v_m. So, we just need to take m large enough so that

$$\max\{z_{\varepsilon,\xi}(x)^{p-1} : x \in \mathbb{R}^n\}\int_{B(\xi,R)} |u_m v_m|dx \le \delta/2.$$

This completes the proof. □

Lemma 5. *There exists $C' > 0$ such that if ε is small enough, then $PI''_\varepsilon(z_{\varepsilon,\xi})$ is uniformly invertible for all $\xi \in \mathbb{R}^n$ with $|\varepsilon\xi| \le 1$ and $\|[PI''_\varepsilon(z_{\varepsilon,\xi})]^{-1}\| \le C'$.*

Proof. We only need to verify that there exists $\theta > 0$ such that the interval $(-\theta,\theta)$ does not contain any eigenvalue of $PI''_\varepsilon(z_{\varepsilon,\xi})$, provided ε is small enough and $|\varepsilon\xi| \le 1$.

By direct computation using (6), we have

$$
\begin{aligned}
I''_\varepsilon(z_{\varepsilon,\xi})[z,z] &= \int \left(|\nabla z|^2 + V(\varepsilon x)z^2 - pK(\varepsilon x)z^{p+1}\right) \\
&= \int [V(\varepsilon x) - V(\varepsilon\xi)]z^2 + (1-p)K(\varepsilon\xi)\int_{\mathbb{R}^n} z^{p+1} \\
&\quad + p\int_{\mathbb{R}^n} [K(\varepsilon\xi) - K(\varepsilon x)]z^{p+1} \\
&= o_\varepsilon(1) + (1-p)K(\varepsilon\xi)\int_{\mathbb{R}^n} z^{p+1} < -c\|z\|^2.
\end{aligned}
$$

Next, let

$$X = \langle z_{\varepsilon,\xi}, \partial z_{\varepsilon,\xi}/\partial\xi_1, \ldots, \partial z_{\varepsilon,\xi}/\partial\xi_n\rangle.$$

Observe that $X \subset W^{1,2}(\mathbb{R}^n)$.

We show the following inequality:

$$I''_\varepsilon(z_{\varepsilon,\xi})[v,v] \geq c\|v\|^2, \quad \forall\, v \in X^\perp.$$

Fix $v \in X^\perp$, and suppose that $\|v\| = 1$. We require the following technical result.

Claim 1. *There exists $R \in (\varepsilon^{-1/4}, \varepsilon^{-1/2})$ such that*

$$\int_{R<|x-\xi|<R+1} [|\nabla v|^2 + v^2]dx < C_1\varepsilon^{1/2}\|v\|^2 = C_1\varepsilon^{1/2}.$$

Note that if $|x-\xi| < \varepsilon^{-1/2}+1$, then $|x| < |\xi| + \varepsilon^{-1/2} + 1$. Thus, for ε small, we have $|\varepsilon x| < 2$. Define $0 < m < \min\{V(x) : |x| \leq 2\}$, $m < 1$. Then

$$
\begin{aligned}
\int_{|x-\xi|<\varepsilon^{-1/2}+1} [|\nabla v|^2 + v^2]dx &\leq \int_{|x-\xi|<\varepsilon^{-1/2}+1} \left[|\nabla v|^2 + \frac{V(\varepsilon x)}{m}v^2\right]dx \\
&< \frac{1}{m}\int_{|x-\xi|<\varepsilon^{-1/2}+1} [|\nabla v|^2 + V(\varepsilon x)v^2]dx \leq \frac{1}{m}.
\end{aligned}
$$

Note that the sum

$$\sum_{R\in\mathbb{N}}^{\varepsilon^{-1/4}<R<\varepsilon^{-1/2}} \int_{R<|x-\xi|<R+1} [|\nabla v|^2 + v^2]dx \leq \frac{1}{m}$$

has more than $\varepsilon^{-1/2}/2$ summands (for ε small). Thus it is always possible to choose $R \in \mathbb{N}$, $R \in (\varepsilon^{-1/4}, \varepsilon^{-1/2})$ so that Claim 1 holds.

For any $R > 0$, define $B_R = B(\xi, R)$, $B_R^c = \mathbb{R}^n \setminus B_R$, $C_R = B_{R+1} \setminus B_R$. Now fix R as in the previous claim, and choose $\chi_R : \mathbb{R}^n \to \mathbb{R}$ to be a C^∞ function such that $\chi_R = 1$ in B_R, $\chi_R = 0$ in B^c_{R+1}, $|\nabla \chi_R| < 2$. We decompose v as $v = v_1 + v_2$, where $v_1 = \chi_R v$, $v_2 = (1 - \chi_R)v$.

First, we estimate the norms of v_1 and v_2. We have

$$\begin{aligned}||v_1||^2 &= \int_{\mathbb{R}^n} \left(|\chi_R \nabla v + v\nabla \chi_R|^2 + V(\varepsilon x)\chi_R^2 v^2\right) dx \\ &= \int_{C_R} [v^2|\nabla \chi_R|^2 + 2v\chi_R \nabla v \cdot \nabla \chi_R] dx + \int_{B_{R+1}} [\chi_R^2 |\nabla v|^2 + V(\varepsilon x)\chi_R^2 v^2] dx \\ &= O(\varepsilon^{1/2}) + \int_{B_{R+1}} [|\nabla v|^2 + V(\varepsilon x)v^2] dx.\end{aligned}$$

In the same way, we can show that

$$||v_2||^2 = O(\varepsilon^{1/2}) + \int_{B^c_R} [|\nabla v|^2 + V(\varepsilon x)v^2] dx.$$

We conclude that

$$||v_1||^2 + ||v_2||^2 = 1 + O(\varepsilon^{1/2}). \tag{11}$$

We also get

$$1 = ||v||^2 = ||v_1||^2 + ||v_2||^2 + 2(v_1, v_2) = 1 + O(\varepsilon^{1/2}) + 2(v_1, v_2),$$

which implies $(v_1, v_2) = O(\varepsilon^{1/2})$.

After these preliminaries, we decompose $I''_\varepsilon(z_{\varepsilon,\xi})[v, v]$ as

$$I''_\varepsilon(z_{\varepsilon,\xi})[v_1 + v_2, v_1 + v_2] = I''_\varepsilon(z_{\varepsilon,\xi})[v_1, v_1] + I''_\varepsilon(z_{\varepsilon,\xi})[v_2, v_2] + 2I''_\varepsilon(z_{\varepsilon,\xi})[v_1, v_2]. \tag{12}$$

For the last term, one has

$$|I''_\varepsilon(z_{\varepsilon,\xi})[v_1, v_2]| = o_\varepsilon(1). \tag{13}$$

Indeed, one finds that

$$\begin{aligned}|I''_\varepsilon(z_{\varepsilon,\xi})[v_1, v_2]| &= \left|\int [\nabla v_1 \nabla v_2 + V(\varepsilon x)v_1 v_2 - pK(\varepsilon x)z^{p-1}_{\varepsilon,\xi} v_1 v_2] dx\right| \\ &\le |(v_1, v_2)| + C \left|\int_{C_R} p z^{p-1}_{\varepsilon,\xi} v^2\right| dx \\ &\le C_1 \varepsilon^{1/2} \quad (C_1 > 0).\end{aligned}$$

We apply Lemma 2 to obtain $V(\varepsilon x) - pK(\varepsilon x)z^{p-1}_{\varepsilon,\xi}(x) \ge \frac{1}{2}V(\varepsilon x)$ for any x with $|x - \xi| > R$. Using this inequality, we may estimate the second term in the right-hand side of (12) by

$$I_\varepsilon''(z_{\varepsilon,\xi})[v_2,v_2] = \int_{B_R^c} \left[|\nabla v_2|^2 + [V(\varepsilon x) - pK(\varepsilon x)z_{\varepsilon,\xi}^{p-1}]v_2^2\right] dx$$
$$\geq \int_{B_R^c} \left[|\nabla v_2|^2 + \frac{1}{2}V(\varepsilon x)v_2^2\right] dx.$$

This implies

$$I_\varepsilon''(z_{\varepsilon,\xi})[v_2,v_2] \geq \frac{1}{2}\|v_2\|^2. \tag{14}$$

We now focus our attention to the first term on the right-hand side of (12). Observe that, since v_1 has compact support, it belongs to $W^{1,2}(\mathbb{R}^n)$. Indeed,

$$\int v_1^2 dx = \int_{B_{R+1}} v_1^2 dx \leq \frac{1}{m}\int_{B_{R+1}} V(\varepsilon x)v_1^2 dx \leq \frac{1}{m} + C_2\varepsilon^{1/2},$$

where $0 < m < \min\{V(x) : |x| \leq 2\}$, and $C_2 > 0$.

As mentioned above, we are concerned with the estimate of

$$I_\varepsilon''(z_{\varepsilon,\xi})[v_1,v_1] = \int_{\mathbb{R}^n} \left[|\nabla v_1|^2 + [V(\varepsilon x) - pK(\varepsilon x)z_{\varepsilon,\xi}^{p-1}]v_1^2\right] dx$$
$$= \int_{B_{R+1}} \left[|\nabla v_1|^2 + [V(\varepsilon \xi) - pK(\varepsilon \xi)z_{\varepsilon,\xi}^{p-1}]v_1^2\right] dx$$
$$+ \int_{B_{R+1}} [V(\varepsilon x) - V(\varepsilon\xi)]v_1^2 dx + p\int_{B_{R+1}} [K(\varepsilon\xi) - K(\varepsilon x)]z_{\varepsilon,\xi}^{p-1}v_1^2 dx.$$

We now use the boundedness of V' and K' (see (V2) and (K2)) to conclude that $|V(\varepsilon x) - V(\varepsilon\xi)| \leq M\varepsilon|x-\xi|$ and $|K(\varepsilon x) - K(\varepsilon\xi)| \leq M\varepsilon|x-\xi|$, for some $M > 0$. Since $x \in B_{R+1}$ and $R < \varepsilon^{-1/2}$, we obtain

$$\left|I_\varepsilon''(z_{\varepsilon,\xi})[v_1,v_1] - \int_{\mathbb{R}^n} \left[|\nabla v_1|^2 + [V(\varepsilon \xi) - pK(\varepsilon \xi)z_{\varepsilon,\xi}^{p-1}]v_1^2\right] dx\right| \leq M\varepsilon^{1/2}. \tag{15}$$

Let

$$(u,v)_\xi = \int [\nabla u \cdot \nabla v + V(\varepsilon\xi)uv]\, dx$$

denote a scalar product in $W^{1,2}(\mathbb{R}^n)$ equivalent to the standard one $\int_{\mathbb{R}^n}(\nabla u\cdot\nabla v + uv)$ (uniformly for $|\varepsilon\xi| \leq 1$), and let $\|u\|_\xi^2 = (u,u)_\xi$ be the associated norm, also equivalent to the standard one.

Now write $v_1 = \phi + w$, where $\phi \in X$ and $w \perp_\xi X$, where $\perp_\xi$ stands for orthogonality in the $(\cdot,\cdot)_\xi$ sense. From [12], we have

$$\int_{\mathbb{R}^n} \left[|\nabla w|^2 + (V(\varepsilon\xi) - pK(\varepsilon\xi)z_{\varepsilon,\xi}^{p-1})w^2\right] dx \geq c_1\|w\|_\xi^2. \tag{16}$$

Roughly speaking, our aim is to show that ϕ is small compared to w, so v_1 turns out to be close to w. After that, we estimate $I''_\varepsilon(z_{\varepsilon,\xi})[v_1, v_1]$, taking into account (15) and (16).

Claim 2. *As* $\varepsilon \to 0$, *one has* $\|\phi\|_\xi = o_\varepsilon(1)$.

Since

$$\phi = (v_1, z_{\varepsilon,\xi})_\xi z_{\varepsilon,\xi}\|z_{\varepsilon,\xi}\|_\xi^{-2} + \left(v_1, \frac{\partial z_{\varepsilon,\xi}}{\partial \xi_i}\right)_\xi \frac{\partial z_{\varepsilon,\xi}}{\partial \xi_i}\left\|\frac{\partial z_{\varepsilon,\xi}}{\partial \xi_i}\right\|_\xi^{-2},$$

in order to prove this claim, it suffices to evaluate $(v_1, z_{\varepsilon,\xi})_\xi$ and $(v_1, \partial z_{\varepsilon,\xi}/\partial \xi_i)_\xi$. We first show that $|(v_1, z_{\varepsilon,\xi}) - (v_1, z_{\varepsilon,\xi})_\xi| \le C_1\varepsilon^{1/2}$, for some $C_1 > 0$. Indeed

$$\begin{aligned}
|(v_1, z) - (v_1, z)_\xi| &= \left|\int_{B_{R+1}} [V(\varepsilon x) - V(\varepsilon\xi)] v_1 z_{\varepsilon,\xi} dx\right| \\
&\le \left(\int_{B_{R+1}} [V(\varepsilon x) - V(\varepsilon\xi)]^2 v_1^2 dx\right)^{\frac{1}{2}} \left(\int_{B_{R+1}} z_{\varepsilon,\xi}^2 dx\right)^{\frac{1}{2}} \\
&\le C_1\varepsilon^{1/2}\left(\int_{B_{R+1}} v_1^2 dx\right)^{\frac{1}{2}} \left(\int_{B_{R+1}} z_{\varepsilon,\xi}^2 dx\right)^{\frac{1}{2}} \\
&\le C_2\varepsilon^{1/2}.
\end{aligned}$$

In the last formula, we have used the inequality $|V(\varepsilon x) - V(\varepsilon\xi)| \le M\varepsilon^{1/2}$ whenever $x \in B_{R+1}$.

Since $v = v_1 + v_2$ and $v \perp X$ with respect to the scalar product in E, we have $|(v_1, z_{\varepsilon,\xi})| = |(v_2, z_{\varepsilon,\xi})|$. Then we find

$$\begin{aligned}
|(v_1, z_{\varepsilon,\xi})| &= |(v_2, z_{\varepsilon,\xi})| \\
&\le \int_{B_R^c} [|\nabla v_2 \cdot \nabla z_{\varepsilon,\xi}| + V(\varepsilon x)|v_2| z_{\varepsilon,\xi}]\, dx \\
&\le \left(\int_{B_R^c} |\nabla v_2|^2 dx\right)^{1/2} \left(\int_{B_R^c} |\nabla z_{\varepsilon,\xi}|^2 dx\right)^{1/2} \\
&\quad + C_2 \left(\int_{B_R^c} V(\varepsilon x)|v_2|^2 dx\right)^{1/2} \left(\int_{B_R^c} z_{\varepsilon,\xi}^2 dx\right)^{1/2}.
\end{aligned}$$

Recall that $R > \varepsilon^{-\frac{1}{4}}$. Then, because of the exponential decay of $z_{\varepsilon,\xi}$ and its derivatives, taking into account that $\|v_2\|^2 \le 1 + O(\varepsilon^{1/2})$ (cf. (11)), we obtain $|(v_1, z_{\varepsilon,\xi})| = o_\varepsilon(1)$. This convergence to zero is uniform in v. We conclude that $|(v_1, z_{\varepsilon,\xi})_\xi| \le |(v_1, z_{\varepsilon,\xi})_\xi - (v_1, z_{\varepsilon,\xi})| + |(v_1, z_{\varepsilon,\xi})| = o_\varepsilon(1)$.

In the same way, we can prove that $|(v_1, \partial z_{\varepsilon,\xi}/\partial \xi_i)|_\xi = o_\varepsilon(1)$ for any $i = 1, \dots, n$. Claim 2 is thereby proved.

Taking into account (15), we have

$$I_\varepsilon''(z_{\varepsilon,\xi})[v_1, v_1] = \int_{B_{R+1}} \left[|\nabla v_1|^2 + [V(\varepsilon\xi) - pK(\varepsilon\xi) z_{\varepsilon,\xi}^{p-1}] v_1^2\right] dx + o_\varepsilon(1).$$

Next, using Claim 2, we get

$$\int_{B_{R+1}} \left[|\nabla v_1|^2 + [V(\varepsilon\xi) - pK(\varepsilon\xi) z_{\varepsilon,\xi}^{p-1}] v_1^2\right] dx$$
$$= \int_{B_{R+1}} \left[|\nabla w|^2 + [V(\varepsilon\xi) - pK(\varepsilon\xi) z_{\varepsilon,\xi}^{p-1}] w^2\right] dx + o_\varepsilon(1),$$

and hence

$$I_\varepsilon''(z_{\varepsilon,\xi})[v_1, v_1] = \int_{B_{R+1}} \left[|\nabla w|^2 + [V(\varepsilon\xi) - pK(\varepsilon\xi) z_{\varepsilon,\xi}^{p-1}] w^2\right] dx + o_\varepsilon(1).$$

This equality and (16) imply that

$$I_\varepsilon''(z_{\varepsilon,\xi})[v_1, v_1] \geq c_2 \|w\|_\xi^2.$$

Then, using Claim 2, we deduce

$$I_\varepsilon''(z_{\varepsilon,\xi})[v_1, v_1] \geq c_3 \|v_1\|_\xi^2 + o_\varepsilon(1) \geq c_4 \|v_1\|^2 + o_\varepsilon(1). \tag{17}$$

Finally, (17), (13) and (14) imply that

$$I_\varepsilon''(z_{\varepsilon,\xi})[v, v] \geq c_5 \|v_1\|^2 + c_6 \|v_2\|^2 + o_\varepsilon(1).$$

Since $\|v_1\|^2 + \|v_2\|^2 = 1 + O(\varepsilon^{1/2}) = \|v\| + O(\varepsilon^{1/2})$, we get $I_\varepsilon''(z_{\varepsilon,\xi})[v, v] \geq c_7 \|v\|^2$. This concludes the proof. □

We are now ready to transform the auxiliary equation (7) into a fixed point problem. Specifically, solving the equation $PI_\varepsilon'(z_{\varepsilon,\xi} + w) = 0$ is clearly equivalent to finding fixed points of the map S_ε, defined by setting

$$S_\varepsilon(w) = w - \left[PI_\varepsilon''(z_{\varepsilon,\xi})\right]^{-1} \left(PI_\varepsilon'(z_{\varepsilon,\xi} + w)\right). \tag{18}$$

Note that S_ε is well-defined by virtue of Lemma 5. Fixed points of S_ε will be found in a suitable subset of E consisting of functions satisfying an appropriate decay estimate, by means of Bessel functions. In the next section, we discuss some preliminary material on this topic.

4 Linear equation and decay estimate

Motivated by some comparison arguments we need in the sequel (see, in particular, the proof of Lemma 9), we are interested in the behavior of the radial solutions of the linear problem

$$(L_R) \qquad \begin{cases} -\Delta u + \frac{m}{|x|^\alpha}\, u = f(|x|), & x \in \mathbb{R}^n, \quad |x| > R, \\ u(x) = 1, & |x| = R, \\ u(x) \to 0, & |x| \to \infty, \end{cases}$$

where $R > 0$, $m > 0$, $\alpha \in (0, 2]$ and $f : (R, +\infty) \to \mathbb{R}$ is positive and has a certain decay (the exact hypotheses on f are described later).

If $u(x) = u(|x|)$ is a radial solution of (L_R), then the function $u(r)$ is a solution of the problem

$$(19) \qquad \begin{cases} -u''(r) - (n-1)\frac{u'(r)}{r} + m\frac{u(r)}{r^\alpha} = f(r), & r > R, \\ u(R) = 1, & \\ u(r) \to 0, & r \to +\infty. \end{cases}$$

Since this is a linear problem, we are interested in the solutions of the homogeneous equation

$$(20) \qquad -u''(r) - (n-1)\frac{u'(r)}{r} + m\frac{u(r)}{r^\alpha} = 0.$$

Making the change of variables $v(r) = u(r)r^{(n-1)/2}$, from equations (19), (20), we obtain, respectively,

$$(21) \qquad \begin{cases} -v''(r) + \frac{(n-1)(n-3)}{4r^2}v(r) + \frac{m}{r^\alpha}v(r) = f(r)r^{(n-1)/2}, & r > R, \\ v(R) = R^{(n-1)/2}, & \\ v(r)r^{-(n-1)/2} \to 0, & r \to +\infty; \end{cases}$$

$$(22) \qquad -v''(r) + \frac{(n-1)(n-3)}{4r^2}v(r) + m\frac{v(r)}{r^\alpha} = 0.$$

As mentioned before, we first solve the homogeneous problem. Afterwards we use this information to study the solutions of (21); see Lemma 6 below.

Equations (20), (22) admit a two-dimensional vector space of solutions. We denote by ψ_1 and ψ_2 two generators the space of solutions of (22). Then $u_1(r) = r^{-(n-1)/2}\psi_1(r)$ and $u_2(r) = r^{-(n-1)/2}\psi_2(r)$ span the space of solutions of (20). We choose u_1, u_2 so that they are both positive and $u_1(r) \to 0$, $u_2(r) \to +\infty$ as $r \to +\infty$.

Since equation (22) has the form $v''(r) = q(r)v(r)$, there exists a constant $d_0 \neq 0$ such that the Wronskian $\psi_1(r)\psi_2'(r) - \psi_1'(r)\psi_2(r) = d_0$. We now show that d_0 is a positive constant. Since $u_2/u_1 = \psi_2/\psi_1$ is positive and tends to infinity as $r \to +\infty$, its derivative must be positive at a certain point. But $(\psi_2/\psi_1)' = d_0/\psi_1^2$, and this implies $d_0 > 0$.

The algebraic expressions defining ψ_1 and ψ_2 depend upon the coefficient α, and this forces us to distinguish two cases, $\alpha < 2$ and $\alpha = 2$.

Case 1: $\alpha < 2$.

In this case, the functions ψ_1, ψ_2 can be written by means of Bessel functions. Precisely, let $B_\ell^I(r)$, respectively $B_\ell^K(r)$, denote the modified Bessel function of the first kind, respectively of the second kind. Recall B_ℓ^I and B_ℓ^K are both positive solutions of the modified Bessel equation

$$r^2 y'' + ry' - (r^2 + \ell^2)y = 0.$$

Furthermore, B_ℓ^I is increasing and tends to infinity, while B_ℓ^K is decreasing and tends to zero.

By direct computation, one finds that

$$\psi_1(r) = \sqrt{r} \cdot B_\ell^K \left(\frac{2\sqrt{m}}{2-\alpha} r^{(2-\alpha)/2} \right), \qquad \ell = \frac{n-2}{2-\alpha};$$

$$\psi_2(r) = \sqrt{r} \cdot B_\ell^I \left(\frac{2\sqrt{m}}{2-\alpha} r^{(2-\alpha)/2} \right), \qquad \ell = \frac{n-2}{2-\alpha}.$$

From the asymptotics of the Bessel functions, we have

$$\psi_1(r) \sim r^{\alpha/4} e^{-\frac{2\sqrt{m}}{2-\alpha} r^{\frac{2-\alpha}{2}}}, \qquad \psi_2(r) \sim r^{\alpha/4} e^{\frac{2\sqrt{m}}{2-\alpha} r^{\frac{2-\alpha}{2}}}.$$

Having ψ_1 and ψ_2, we then obtain

$$u_1(r) = r^{-(n-1)/2}\psi_1(r), \qquad u_2(r) = r^{-(n-1)/2}\psi_2(r), \tag{23}$$

the solutions of (20). Observe that $u_1(r) \to 0$, $u_2(r) \to \infty$ for $r \to +\infty$. In particular, from the asymptotic behavior of Bessel functions (see [10, Sections 5.7 and 5.16.4]), one has that

$$u_1(r) \sim r^{\frac{\alpha}{4} - \frac{n-1}{2}} e^{-\frac{2\sqrt{m}}{2-\alpha} r^{\frac{2-\alpha}{2}}}. \tag{24}$$

Case 2: $\alpha = 2$.

In this case, the functions ψ_1, ψ_2 are given by

$$\psi_1(r) = r^{\frac{1-\sqrt{(n-2)^2+4m}}{2}}, \qquad \psi_2(r) = r^{\frac{1+\sqrt{(n-2)^2+4m}}{2}}.$$

Observe that here we still obtain the asymptotics $\psi_1(r) \to 0$, $\psi_2(r) \to \infty$ for $r \to +\infty$.

The functions u_1 and u_2 are given by

$$u_1(r) = r^{\frac{2-n-\sqrt{(n-2)^2+4m}}{2}}, \quad u_2(r) = r^{\frac{2-n+\sqrt{(n-2)^2+4m}}{2}}. \tag{25}$$

Clearly, we still have that $u_1(r) \to 0$, $u_2(r) \to +\infty$ as $r \to \infty$.

We can now state and prove the main result of the section.

Lemma 6. *Let u_1, u_2 be defined as above, and let $\overline{\varphi}$ be a solution of problem (L_R), where $f : (R, +\infty) \to \mathbb{R}$ is a positive continuous function satisfying the integrability condition*

$$\int_R^{+\infty} r^{n-1} f(r) u_2(r)\, dr < +\infty. \tag{26}$$

Then there exists $\gamma(R) > 0$ such that $\overline{\varphi}(r) \leq \gamma(R) u_1(r)$ for all $r \in (R, +\infty)$.

Proof. We argue by passing to problem (21), so we write $v(r) = r^{(n-1)/2}\overline{\varphi}(r)$, $\widetilde{f}(r) = r^{(n-1)/2} f(r)$. In terms of ψ_2 and $\widetilde{f}$, condition (26) can be rewritten as

$$\int_R^{+\infty} \widetilde{f}(r)\psi_2(r)\, dr < +\infty. \tag{27}$$

Variation of parameters shows that $v(r)$ is of the form

$$v(r) = \frac{1}{d_0}\left[\psi_1(r)\int_R^r \psi_2(s)\widetilde{f}(s)\, ds - \psi_2(r)\int_R^r \psi_1(s)\widetilde{f}(s)\, ds\right] + a_1\psi_1(r) + a_2\psi_2(r),$$

where $d_0 = \psi_1\psi_2' - \psi_1'\psi_2 > 0$ is the constant given by the Wronskian and a_1, a_2 are suitable constants.

Observe that

$$0 = \lim_{r\to+\infty} r^{-(n-1)/2} v(r) = \left[\frac{-1}{d_0}\int_R^r \psi_1(s)\widetilde{f}(s)\, ds + a_2\right]\psi_2(r) r^{-(n-1)/2}.$$

Recall that in both cases 1 and 2, we have $u_2(r) = \psi_2(r) r^{-(n-1)/2} \to +\infty$ as $r \to +\infty$. This implies that a_2 must coincide with $-\frac{1}{d_0}\int_R^{+\infty} \psi_1(s)\widetilde{f}(s)\, ds$. The constant a_1 can be found by taking into account the boundary condition $v(R) = R^{(n-1)/2}$.

We now compute the limit of the quotient $v(r)/\psi_1(r)$. We have

$$\lim_{r\to\infty}\frac{v(r)}{\psi_1(r)} = \lim_{r\to+\infty}\frac{1}{d_0}\left[\int_R^r \psi_2(s)\widetilde{f}(s)\,ds + \frac{\psi_2(r)}{\psi_1(r)}\int_r^{+\infty}\psi_1(s)\widetilde{f}(s)\,ds\right] + a_1$$
$$= \frac{1}{d_0}\int_R^{+\infty}\psi_2(s)\widetilde{f}(s)\,ds + a_1 + \frac{1}{d_0}\lim_{r\to+\infty}\frac{\psi_2(r)}{\psi_1(r)}\int_r^{+\infty}\psi_1(s)\widetilde{f}(s)\,ds.$$

In order to compute the last limit, we first note that ψ_1/ψ_2 is a positive decreasing function ($(\psi_1/\psi_2)' = -d_0/\psi_2^2 < 0$). Thus, we can write

$$0 \le \lim_{r\to+\infty}\frac{\psi_2(r)}{\psi_1(r)}\int_r^{+\infty}\psi_1(s)\widetilde{f}(s)\,ds = \lim_{r\to+\infty}\frac{\psi_2(r)}{\psi_1(r)}\int_r^{+\infty}\frac{\psi_1(s)}{\psi_2(s)}\psi_2(s)\widetilde{f}(s)\,ds$$
$$\le \lim_{r\to+\infty}\frac{\psi_2(r)}{\psi_1(r)}\frac{\psi_1(r)}{\psi_2(r)}\int_r^{+\infty}\psi_2(s)\widetilde{f}(s)\,ds = \lim_{r\to+\infty}\int_r^{+\infty}\psi_2(s)\widetilde{f}(s)\,ds = 0.$$

Hence, we obtain

$$\lim_{r\to\infty}\frac{v(r)}{\psi_1(r)} = \frac{1}{d_0}\int_R^{+\infty}\psi_2(s)\widetilde{f}(s)\,ds + a_1.$$

To complete the proof, it suffices to note that $v(r)/\psi_1(r) = \overline{\varphi}(r)/u_1(r)$. □

5 Solving the auxiliary equation, II

Using the analysis carried out in the previous section, we can now choose the set where we shall find the fixed points of the map S_ε defined in (18).

As before, it is always understood that $|\varepsilon\xi| \le 1$.

Let us introduce the set $\mathcal{W}_\varepsilon(R)$ of the functions $w \in E$ such that

$$(28)\qquad |w(x+\xi)| \le \begin{cases} \gamma(R)\,\sqrt{\varepsilon}\,u_1(|x|), & \text{if } |x| \ge R, \\ \sqrt{\varepsilon}, & \text{if } |x| \le R, \end{cases}$$

where $u_1(r)$ is defined in (23) (respectively, in (25)) if $0 \le \alpha < 2$ (respectively, if $\alpha = 2$), and $\gamma(R)$ is the constant found in Lemma 6. Next, for $|\varepsilon\xi| \le 1$ we set

$$\Gamma_\varepsilon(R) = \{w \in E : \|w\| \le c_0\varepsilon, \quad w \in \mathcal{W}_\varepsilon(R) \cap (T_{z_{\varepsilon,\xi}})^\perp\},$$

where c_0 is a fixed positive constant to be chosen later (see (31)). Clearly, we can choose $\bar{c}$ and ϑ in equation (4) defining F_ε, in such way that $|z_{\varepsilon,\xi}(x) + w(x)| < \bar{c}(1+|\varepsilon x|)^{-\vartheta}$ for any $w \in \Gamma_\varepsilon(R)$. Thus we have

$$I_\varepsilon(z_{\varepsilon,\xi} + w) = \tfrac{1}{2}\|z_{\varepsilon,\xi} + w\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} K(\varepsilon x)|z_{\varepsilon,\xi} + w|^{p+1}dx,$$

and any critical point $u = z_{\varepsilon,\xi} + w$ of I_ε, with $w \in \Gamma_\varepsilon(R)$, gives rise to a solution of (2).

By Lemma 2, given any $m > 0$, we can find $R_1 > 0$ such that

$$V(\varepsilon x + \varepsilon\xi) \geq \frac{m}{|x|^\alpha}, \qquad \forall\, |x| \geq R_1. \tag{29}$$

The choice of m depends upon the fact that $\alpha < 2$ or $\alpha = 2$. In the former case, we can take, say, $m = 2$. In the latter, we have to choose m sufficiently large (see below). Furthermore, let $z_0(x) = z_{\varepsilon,\xi}(x + \xi)$. We note that z_0 depends on ε, ξ, but has uniform decay at infinity because $|\varepsilon\xi| \leq 1$. Hence there exists $R_2 > 0$ such that

$$p z_0^{p-1}(x) \leq 1/|x|^\alpha, \qquad \forall\, |x| \geq R_2. \tag{30}$$

Set $\rho = \max\{R_1, R_2\}$. Note that this choice is independent of ε.

Proposition 7. *$S_\varepsilon(\Gamma_\varepsilon(\rho)) \subset \Gamma_\varepsilon(\rho)$ and is a contraction provided ε is sufficiently small.*

Proposition 7 is an immediate consequence of the following two lemmas.

Lemma 8. *For c_0 large enough and for ε sufficiently small, one has $\|S_\varepsilon(w)\| \leq c_0\varepsilon$, for all $w \in \Gamma_\varepsilon(\rho)$; and S_ε is a contraction in $\Gamma_\varepsilon(\rho)$.*

Lemma 9. *For all ε sufficiently small, one has that $S_\varepsilon(\Gamma_\varepsilon(\rho)) \subset \mathcal{W}_\varepsilon(\rho)$ for every $w \in \Gamma_\varepsilon(\rho)$.*

Proof of Lemma 8. Let $\overline{C}$ be given by Lemma 3. Observe that by Lemma 5, $\|[PI''_\varepsilon(z_{\varepsilon,\xi})]^{-1}\| \leq C'$ for some $C' > 0$. Choose

$$c_0 = 2C'\overline{C} \tag{31}$$

in the definition of Γ_ε. We first compute $S'_\varepsilon(w)$ for $w \in \Gamma_\varepsilon$. We have

$$S'_\varepsilon(w)[v] = v - [PI''_\varepsilon(z_{\varepsilon,\xi})]^{-1}(PI''_\varepsilon(z_{\varepsilon,\xi} + w)[v]).$$

We apply $PI''_\varepsilon(z_{\varepsilon,\xi})$ and obtain

$$\|PI''_\varepsilon(z_{\varepsilon,\xi})\,[S'_\varepsilon(w)[v]]\,\| = \|PI''_\varepsilon(z_{\varepsilon,\xi})[v] - PI''_\varepsilon(z_{\varepsilon,\xi} + w)[v])\|. \tag{32}$$

In the next Lemma, we estimate the above quantity.

Lemma 10. *There exist $C_1 > 0$ and $\delta > 0$ such that for all $w \in \Gamma_\varepsilon$, $\|PI''_\varepsilon(z_{\varepsilon,\xi}) - PI''_\varepsilon(z_{\varepsilon,\xi} + w)\| \leq C_1\|w\|^\delta$, provided ε is small enough.*

Proof. Let $w_1,\ w_2 \in E$. Since $|K| \le \kappa$, we have by direct computation

$$\begin{aligned}|I_\varepsilon''(z_{\varepsilon,\xi})[w_1,w_2] - I_\varepsilon''(z_{\varepsilon,\xi}+w)[w_1,w_2]| \le & \kappa\left|\int_{\mathbb{R}^n}[(z_{\varepsilon,\xi}+w)^{p-1} - z_{\varepsilon,\xi}^{p-1}]w_1w_2dx\right| \\ \le & C_1\int_{\mathbb{R}^n}(|w|+|w|^{p-1})w_1w_2dx \\ \le & C_2\left[\int_{\mathbb{R}^n}\left(|w|+|w|^{p-1}\right)^{n/2}dx\right]^{2/n} \\ & \times \|w_1\|_{L^{2^*}}\|w_2\|_{L^{2^*}}.\end{aligned}$$

We now estimate the term $\int_{\mathbb{R}^n}|w|^{(p-1)n/2}dx$. Let $q>1$ and $q'=q/(q-1)$. Then

$$\int_{\mathbb{R}^n}|w|^{(p-1)n/2} \le \left(\int_{\mathbb{R}^n}|w|^{q(p-1)n/4}\right)^{1/q}\left(\int_{\mathbb{R}^n}|w|^{q'(p-1)n/4}\right)^{1/q'}.$$

Now fix q so that $\tau = q(p-1)n/4 > 2$. Since $w \in \Gamma_\varepsilon$, the expression $(\int_{\mathbb{R}^n}|w|^{q'(p-1)n/4})^{1/q'}$ is finite. From Lemma 2, we deduce that there exists $C_3>0$ so that $|w(x)|^{\tau-2} < C_3V(\varepsilon x)$ (recall again that $w \in \Gamma_\varepsilon$). Therefore, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|w|^{q(p-1)n/4}\right)^{1/q} \le & C_4\left(\int|w|^2|w|^{\tau-2}\right)^{1/q} \\ \le & C_5\left(\int|w|^2V(\varepsilon x)\right)^{1/q} \le C_6\|w\|^{1/q}.\end{aligned}$$

The estimate of the term $\int|w|^{n/2}$ can be carried out in the same way. □

Proof of Lemma 8 completed. Using Lemma 10 and (32), we get

$$\|PI_\varepsilon''(z_{\varepsilon,\xi})\,[S_\varepsilon'(w)[v]]\,\| \le C_1\|w\|^\delta\|v\|.$$

Then, for any $w_1,\,w_2 \in \Gamma_\varepsilon$, we have

$$\begin{aligned}\|S_\varepsilon(w_1)-S_\varepsilon(w_2)\| \le & \|[PI_\varepsilon''(z_{\varepsilon,\xi})]^{-1}\|\,\|PI_\varepsilon''(z_{\varepsilon,\xi})(S_\varepsilon(w_1)-S_\varepsilon(w_2))\| \\ \le & C'\int_0^1\|PI_\varepsilon''(z)\,(S_\varepsilon'(w_2+s(w_1-w_2))[w_1-w_2])\,\|ds.\end{aligned}$$

Thus we obtain

$$\|S_\varepsilon(w_1)-S_\varepsilon(w_2)\| \le C''\left(\max_{s\in[0,1]}\|w_2+s(w_1-w_2)\|\right)^\delta\|w_1-w_2\|$$

for some $C'',\delta>0$. Since both w_1 and w_2 belong to $\Gamma_\varepsilon(\rho)$, we easily find that

$$\|S_\varepsilon(w_1)-S_\varepsilon(w_2)\| = o_\varepsilon(1)\|w_1-w_2\|. \tag{33}$$

Equation (33) yields the contraction property for S_ε.

Next, we show that $\|S_\varepsilon(w)\| \le c_0\varepsilon$ for any $w \in \Gamma_\varepsilon$. Using (33) with $w_1 = w$ and $w_2 = 0$, we obtain

$$\|S_\varepsilon(w) - S_\varepsilon(0)\| = o_\varepsilon(1)\|w\|.$$

On the other hand, by using Lemma 3 and Lemma 5, we obtain

$$\|S_\varepsilon(0)\| = \|[PI_\varepsilon''(z_{\varepsilon,\xi})]^{-1}(PI_\varepsilon'(z_{\varepsilon,\xi}))\| \le C'\|PI_\varepsilon'(z_{\varepsilon,\xi})\| \le C'\overline{C}\varepsilon.$$

Hence, we finally deduce

$$\begin{aligned}\|S_\varepsilon(w)\| \le& \|S_\varepsilon(w) - S_\varepsilon(0)\| + \|S_\varepsilon(0)\|\\ \le& o_\varepsilon(1)\|w\| + C'\overline{C}\varepsilon = o_\varepsilon(1)\|w\| + \tfrac{1}{2}c_0\varepsilon.\end{aligned}$$

Since $w \in \Gamma_\varepsilon(\rho)$, $\|w\| \le c_0\varepsilon$; hence we get

$$\|S_\varepsilon(w)\| \le o_\varepsilon(1)c_0\varepsilon + \tfrac{1}{2}c_0\varepsilon \le c_0\varepsilon,$$

provided ε is sufficiently small. This concludes the proof. □

Proof of Lemma 9. First, let us introduce some notation. We set

$$\begin{aligned}\widetilde{w} =& S_\varepsilon(w),\\ L[v] =& -\Delta v + V(\varepsilon x)v - pK(\varepsilon x)z_{\varepsilon,\xi}^{p-1}v,\\ \dot{z}_{\varepsilon,\xi} =& D_\xi z_{\varepsilon,\xi},\\ \eta =& \|\dot{z}_{\varepsilon,\xi}\|^{-2}\left(I_\varepsilon''(z_{\varepsilon,\xi})[\widetilde{w} - w] + I_\varepsilon'(z_{\varepsilon,\xi} + w), \dot{z}_{\varepsilon,\xi}\right),\\ g(v) =& K(\varepsilon x)[(z_{\varepsilon,\xi} + v)^p - z_{\varepsilon,\xi}^p - pz_{\varepsilon,\xi}^{p-1}v] + \eta\left[-\Delta\dot{z}_{\varepsilon,\xi} + V(\varepsilon x)\dot{z}_{\varepsilon,\xi}\right]\\ &- \left[-\Delta z_{\varepsilon,\xi} + V(\varepsilon x)z_{\varepsilon,\xi} - K(\varepsilon x)z_{\varepsilon,\xi}^p\right].\end{aligned}$$

Here, for brevity, the symbol $D_\xi z_{\varepsilon,\xi}$ stands for a linear combination of the derivatives $D_{\xi_1}z_{\varepsilon,\xi}, \ldots, D_{\xi_n}z_{\varepsilon,\xi}$ (related to the projection of the equation $\widetilde{w} = S_\varepsilon(w)$ onto $(T_zZ)^\perp$). With all this notation, using integration by parts and the definitions of $I_\varepsilon'(z+w), I_\varepsilon''(z)$, one finds that the function $\widetilde{w}$ satisfies $L[\widetilde{w}] = g(w)$. Moreover, if we set $z_0(x) = z_{\varepsilon,\xi}(x+\xi)$ (as at the beginning of this section),

$$w_0(x) = w(x+\xi), \qquad \widetilde{w}_0(x) = \widetilde{w}(x+\xi),$$

$$L_0 = -\Delta + V(\varepsilon x + \varepsilon\xi) - pK(\varepsilon\xi + \varepsilon x)z_0^{p-1}$$

and

$$\begin{aligned}g_1(v) =& K(\varepsilon x + \varepsilon\xi)[(z_0 + v)^p - z_0^p - pz_0^{p-1}v],\\ g_2 =& \eta\left[-\Delta\dot{z}_0 + V(\varepsilon x + \varepsilon\xi)\dot{z}_0\right],\\ g_3 =& -\Delta z_0 + V(\varepsilon x + \varepsilon\xi)z_0 - K(\varepsilon x + \varepsilon\xi)z_0^p;\end{aligned}$$

$\widetilde{w}_0$ satisfies

$$L_0[\widetilde{w}_0] = g_0(w_0) := g_1(w_0) + g_2 - g_3. \tag{34}$$

Below, for $q > n$, we need the estimates

$$\|g_0(w_0)\|_{L^{q/2}(B_{2\rho})} \leq o_\varepsilon(1)\sqrt{\varepsilon}; \tag{35}$$

$$|g_0(w_0)(x)| \leq o_\varepsilon(1)\, \sqrt{\varepsilon}\, u_1^{2\wedge p}, \quad \forall\, |x| \geq \rho. \tag{36}$$

The proof of (35) and (36) are postponed to the end of this section.

Now we prove separately that for $\varepsilon \ll 1$,

$$|\widetilde{w}_0(x)| \leq \sqrt{\varepsilon}, \qquad \text{if } |x| \leq \rho, \tag{37}$$

and

$$|\widetilde{w}_0(x)| \leq \gamma(\rho)\, \sqrt{\varepsilon}\, u_1(|x|), \qquad \text{if } |x| \geq \rho. \tag{38}$$

Concerning the former estimate, we apply Theorem 8.24 of [8] to (34) to infer that

$$\|\widetilde{w}_0\|_{L^\infty(B_\rho)} \leq c_1\|\widetilde{w}_0\|_{L^2(B_{2\rho})} + c_2\|g_0(w_0)\|_{L^{q/2}(B_{2\rho})} \quad (q > n). \tag{39}$$

Using Lemma 8 and recalling that $\|z_{\varepsilon,\xi}\| \leq const.$, we get

$$\|\widetilde{w}_0\|_{L^2(B_{2\rho})} \leq c_3\varepsilon. \tag{40}$$

Inserting (35) and (40) into (39), we find that $\|\widetilde{w}_0\|_{L^\infty(B_\rho)} \leq c_4\varepsilon + c_2 o_\varepsilon(1)\sqrt{\varepsilon}$; (37) follows, provided that ε is sufficiently small.

Let us now prove (38). It is convenient to consider first the case $\alpha < 2$.

From (34) and (37), it follows that $\widetilde{w}_0$ verifies the equation

$$L_0[\widetilde{w}_0] = g_0(w_0), \qquad |x| > \rho, \tag{41}$$

together with the boundary condition

$$|\widetilde{w}_0(\rho)| \leq \sqrt{\varepsilon}. \tag{42}$$

Let φ be the solution of the linear problem

$$\begin{cases} -\Delta\varphi + \frac{1}{|x|^\alpha}\, \varphi = \sqrt{\varepsilon}\, u_1^{2\wedge p}, & |x| > \rho; \\ \varphi(x) = \sqrt{\varepsilon}, & |x| = \rho. \end{cases} \tag{43}$$

Obviously, $\varphi = \sqrt{\varepsilon}\overline{\varphi}$, where $\overline{\varphi}$ is the solution of (L_R) with $R = \rho$ and $f(r) = u_1^{2\wedge p}(r)$. Recalling the discussion carried out in Section 4, in particular (24), we infer that

$f(r) = u_1^{2\wedge p}(r)$ satisfies the integrability condition (26). Hence Lemma 6 applies, yielding

(44) $$\varphi(r) \leq \gamma(\rho)\ \sqrt{\varepsilon}\, u_1(r) \qquad (r > \rho).$$

Using (29) (with $m = 2$) and (30), we infer that

$$V(\varepsilon x + \varepsilon\xi) - pK(\varepsilon x + \varepsilon\xi) z_0^{p-1}(x) \geq (1)\,/|x|^\alpha \qquad (|x| > \rho).$$

From (36), we clearly deduce that

$$g_0(w_0) \leq \sqrt{\varepsilon}\, u_1^{2\wedge p} \qquad \text{for } |x| > \rho.$$

This allows us to compare (41) – (42) with (43), yielding

$$|\widetilde{w}_0(r)| \leq \varphi(r) = \sqrt{\varepsilon}\, \overline{\varphi}(r) \qquad (r > \rho).$$

Finally, using (44), we get $|\widetilde{w}_0(r)| \leq \gamma(\rho)\ \sqrt{\varepsilon}\, u_1(r)$, for $r > \rho$, proving (38) in the case $\alpha < 2$.

To complete the proof, we have to treat the case $\alpha = 2$. We indicate the changes required in the above arguments. First, we observe that the solution u_1 of (L_R) has polynomial decay that depends on m; see (25). Therefore, if we take m sufficiently large, the function $f(r) = u_1^{2\wedge p}(r)$ still satisfies the integrability condition (26). Substituting the comparison problem (43) with

$$\begin{cases} -\Delta\varphi + \frac{(m-1)}{|x|^\alpha}\,\varphi = \sqrt{\varepsilon}\, u_1^{2\wedge p}, & |x| > \rho, \\ \varphi(x) = \sqrt{\varepsilon}, & |x| = \rho, \end{cases}$$

and noticing that $V(\varepsilon x + \varepsilon\xi) - pK(\varepsilon x + \varepsilon\xi) z_0^{p-1}(x) \geq (m-1)\,/|x|^\alpha$, for $|x| > \rho$, we can repeat the preceding arguments to obtain that (38) holds.

It remains to carry out the proofs of (35) and (36).

Proof of (35). We estimate $g_1(w_0)$, g_2 and g_3 separately. Since $w \in \mathcal{W}_\varepsilon(\rho)$ and u_1 is decreasing, w_0 can be estimated by

$$\begin{cases} |w_0(x)| \leq \sqrt{\varepsilon} & \text{if } |x| < \rho, \\ |w_0(x)| \leq \gamma(\rho)\ \sqrt{\varepsilon}\, u_1(\rho) & \text{if } \rho < |x| < 2\,\rho. \end{cases}$$

Moreover, one has

(45) $$|g_1(w_0)| \leq c_1 |w_0(x)|^{2\wedge p}.$$

From these estimates, we readily obtain

$$\|g_1(w_0)\|_{L^{q/2}(B_{2\rho})} \le c_3 \left(\sqrt{\varepsilon}\right)^{2\wedge p}.$$

Since $2 \wedge p > 1$, we get

$$\|g_1(w_0)\|_{L^{q/2}(B_{2\rho})} \le o_\varepsilon(1)\, \sqrt{\varepsilon}, \qquad \text{where } o_\varepsilon(1) \to 0 \text{ as } \varepsilon \to 0. \tag{46}$$

Let us now estimate g_2. From the definition of $z_{\varepsilon,\xi}$ (see Section 2), its exponential decay, and from the boundedness of V' and K', one finds

$$\|\dot{z}_{\varepsilon,\xi}\| \le C; \qquad \| - \Delta \dot{z}_0 + V(\varepsilon x + \varepsilon\xi)\dot{z}_0\|_{L^{q/2}(B_{2\rho})} \le C.$$

Moreover, from Lemma 3 and Lemma 10, we have

$$\begin{aligned}
\|I'_\varepsilon(z_{\varepsilon,\xi} + w)\| \le& \|I'_\varepsilon(z_{\varepsilon,\xi})\| + \int_0^1 \|I''_\varepsilon(z_{\varepsilon,\xi} + sw)[w]\| ds \\
\le& \|I'_\varepsilon(z_{\varepsilon,\xi})\| + \|I''_\varepsilon(z_{\varepsilon,\xi})[w]\| + \int_0^1 \|(I''_\varepsilon(z_{\varepsilon,\xi} + sw) - I''_\varepsilon(z_{\varepsilon,\xi}))[w]\| ds \\
\le& C_1\varepsilon + C_2\varepsilon + C_3\varepsilon^\delta\varepsilon \le o_\varepsilon(1)\sqrt{\varepsilon}.
\end{aligned}$$

In addition, since $\|\widetilde{w}\| \le c_0\varepsilon$ (see Lemma 8) and $w \in \Gamma_\varepsilon(\rho)$, we easily infer that

$$\|I''_\varepsilon(z_{\varepsilon,\xi})[\widetilde{w} - w]\| \le C_4\varepsilon.$$

Therefore, we find that $|\eta| \le o_\varepsilon(1)\sqrt{\varepsilon}$ and obtain

$$\|g_2\|_{L^{q/2}(B_{2\rho})} = o_\varepsilon(1)\, \sqrt{\varepsilon}. \tag{47}$$

We finally turn to g_3. From the fact that $-\Delta z_0 + V(\varepsilon\xi) z_0 = K(\varepsilon\xi) z_0^p$, we get $g_3 = [V(\varepsilon x + \varepsilon\xi) - V(\varepsilon\xi)]\, z_0(x) + [K(\varepsilon\xi) - K(\varepsilon x)] z_0^p(x)$. Using the assumption (V2), we deduce that $|g_3| \le V_1\ \varepsilon\ |x|\ |z_0(x)| + K_0\ \varepsilon\ |x|\ |z_0(x)|^p$. Since z_0 has exponential decay, it follows that

$$\|g_3\|_{L^{q/2}(B_{2\rho})} = o_\varepsilon(1)\, \sqrt{\varepsilon}. \tag{48}$$

Putting together (46), (47) and (48), we find that (35) holds. □

Proof of (36). By (45) and the fact $w \in \mathcal{W}_\varepsilon(\rho)$, we have

$$|g_1(w_0)(x)| \le c_1 \cdot \left[\gamma(\rho)\, \sqrt{\varepsilon}\, u_1(x)\right]^{2\wedge p} = o_\varepsilon(1)\sqrt{\varepsilon}\, (u_1(x))^{2\wedge p} \quad (|x| > \rho).$$

Furthermore, since z_0 (and its derivatives, even multiplied by polynomials in x) decays faster than $u_1^{2\wedge p}$, repeating the arguments carried out above, we get

$$|g_i| = o_\varepsilon(1)\sqrt{\varepsilon}\, u_1^{2\wedge p} \quad (|x| > \rho), \quad i = 1, 2.$$

This completes the proof of (36). □

As a consequence of the above arguments, we obtain the existence of a function w satisfying $S_\varepsilon(w) = w$. We summarize the existence result, collecting some properties about the dependence on ξ in the next proposition.

Proposition 11. *Under the assumptions of Theorem 1, there exists a unique $w \in \Gamma(\rho)$ satisfying $S_\varepsilon(w) = w$. Moreover, w is differentiable with respect to ξ; and there exist $C_1 > 0$ and $\delta > 0$ such that*

$$\|w\| \le C_1\varepsilon, \qquad \|\nabla_\xi w\| \le C_1\varepsilon^\delta.$$

Proof. The estimate on the norm of w has already been proved. We turn now to the dependence of w on ξ. The equation $S_\varepsilon(w) = w$ is equivalent to $H(\xi, w, \mu) = 0$, where $\mu \in \mathbb{R}^n$ and $H : \mathbb{R}^n \times E \times \mathbb{R}^n \to E \times \mathbb{R}^n$ is given by

$$H(\xi, w, \mu) = \begin{pmatrix} I'_\varepsilon(z_{\varepsilon,\xi} + w) - \sum_{i=1}^n \mu_i \partial z_{\varepsilon,\xi}/\partial\xi_i \\ (w, \partial_{\xi_1} z_{\varepsilon,\xi}), \ldots, (w, \partial_{\xi_n} z_{\varepsilon,\xi}) \end{pmatrix}.$$

Fix $\xi \in \mathbb{R}^n$ with $|\varepsilon\xi| \le 1$. We know that for ε sufficiently small there exists a (locally unique) solution of $H(\xi, w, \mu) = 0$ which coincides with the one found by Proposition 7. Observe that the function H is of class C^1 in ξ, w and μ. Moreover, we have

$$\frac{\partial H}{\partial(w,\mu)}[v,\nu] = \begin{pmatrix} I''_\varepsilon(z_{\varepsilon,\xi} + w)[v] - \sum_{i=1}^n \nu_i \partial z_{\varepsilon,\xi}/\partial\xi_i \\ (v, \partial_{\xi_1} z_{\varepsilon,\xi}), \ldots, (v, \partial_{\xi_n} z_{\varepsilon,\xi}) \end{pmatrix}.$$

Using Lemma 5 and Lemma 10 and arguing as in [1], one can prove that $\partial H/\partial(w_\xi, \mu)$ is uniformly invertible for $|\varepsilon\xi| \le 1$. As a byproduct of this fact, one obtains an estimate in norm for μ similar to that for w.

Then, by the local uniqueness of the function w, applying the implicit function theorem we obtain

$$\|(w,\mu)\| \le c_1 \left\| \frac{\partial H}{\partial\xi}(\xi, \mu, w) \right\|,$$

where c_1 is independent of ξ for $|\varepsilon\xi| \le 1$. Without loss of generality, we can consider the derivative with respect to ξ_1, which gives

$$\frac{\partial H}{\partial\xi_1} = \begin{pmatrix} I''_\varepsilon(z + w)[\partial_{\xi_1} z_{\varepsilon,\xi}] - \sum_{i=1}^n \mu_i \partial^2 z_{\varepsilon,\xi}/\partial\xi_i\partial\xi_1 \\ (w, \partial^2_{\xi_1^2} z_{\varepsilon,\xi}), \ldots, (w, \partial^2_{\xi_1\xi_n} z_{\varepsilon,\xi}) \end{pmatrix}.$$

Since (w,μ) is bounded by $c_2\varepsilon$ (see also the estimate of η before (47)), we immediately find that

$$\left\| \frac{\partial H}{\partial\xi_1} \right\| \le c_3\left(\varepsilon + \|I''_\varepsilon(z_{\varepsilon,\xi} + w)[\partial_{\xi_1} z]\|\right).$$

By Lemma 10, we can write

$$\|I_\varepsilon''(z_{\varepsilon,\xi} + w)[\partial_{\xi_1} z_{\varepsilon,\xi}]\| \leq \|I_\varepsilon''(z_{\varepsilon,\xi})[\partial_{\xi_1} z_{\varepsilon,\xi}]\| + c_4 \varepsilon^\delta,$$

where δ is some fixed positive constant. Therefore, at this point, it is sufficient to estimate the norm $\|I_\varepsilon''(z_{\varepsilon,\xi})[\partial_{\xi_1} z_{\varepsilon,\xi}]\|$. Arguing as in [5], formula (6), one finds that

$$\partial_{\xi_1} z_{\varepsilon,\xi} = -\partial_{x_1} z_{\varepsilon,\xi} + O(\varepsilon) \qquad \text{in } E.$$

It follows that

$$\|I_\varepsilon''(z_{\varepsilon,\xi})[\partial_{\xi_1} z_{\varepsilon,\xi}]\| \leq c_5 \varepsilon + \|I_\varepsilon''(z_{\varepsilon,\xi})[\partial_{x_1} z_{\varepsilon,\xi}]\|.$$

Since $\partial_{x_1} z_{\varepsilon,\xi}$ belongs to the kernel of the linearization of (6), for any function $v \in E$, we have

$$I_\varepsilon''(z_{\varepsilon,\xi})[\partial_{x_1} z_{\varepsilon,\xi}, v] = \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)](\partial_{x_1} z_{\varepsilon,\xi}) v.$$

Reasoning as for the proof of Lemma 3, one finds

$$\left| \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)](\partial_{x_1} z) v \right| \leq c_6 \varepsilon \|v\|.$$

Therefore, since v is arbitrary, we deduce from the last formulas that

$$\|\partial_{\xi_1} w\| \leq c_7 \varepsilon^\delta$$

for some $\delta > 0$. This concludes the proof. □

6 Proof of Theorem 1

We now complete the proof of Theorem 1 by showing that the reduced functional Φ_ε has a critical point $\xi_\varepsilon/\varepsilon$, with $\xi_\varepsilon \sim 0$, provided ε is small enough. According to the discussion carried out in Section 2, this implies that (2) has a solution u_ε; and therefore $u_\varepsilon(x/\varepsilon)$ solves (1). Moreover, since $u_\varepsilon \sim z_{\varepsilon,\xi}$,

$$u_\varepsilon(x/\varepsilon) \sim z_{\varepsilon,\xi_\varepsilon}\left(\frac{x - \xi_\varepsilon}{\varepsilon}\right);$$

and hence such a solution concentrates at $x_0 = 0$.

First of all, let us expand the reduced functional

$$\Phi_\varepsilon(\xi) = I_\varepsilon(z_\xi + w_{\varepsilon,\xi})$$

in the variable ξ. To simplify notation, we write z instead of $z_{\varepsilon,\xi}$ and w instead of $w_{\varepsilon,\xi}$. As in [5, Subsection 4.2], we find

$$\begin{aligned}\Phi_\varepsilon(\xi) &= C_1[V(\varepsilon\xi)]^\theta[K(\varepsilon\xi)]^{-2/(p-1)} + I'_\varepsilon(z)[w] + \Lambda_\varepsilon(\xi) + \Psi_\varepsilon(\xi)\\ &= C_1 Q(\varepsilon\xi) + I'_\varepsilon(z)[w] + \Lambda_\varepsilon(\xi) + \Psi_\varepsilon(\xi),\end{aligned}$$

where $\theta = \frac{p+1}{p-1} - \frac{n}{2}$, C_1 is a positive constant depending only on n and p, and

$$\Lambda_\varepsilon(\xi) = \frac{1}{2}\int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]z^2 - \frac{1}{p+1}\int_{R^n}[K(\varepsilon x) - K(\varepsilon\xi)]z^{p+1},$$

$$\Psi_\varepsilon(\xi) = \frac{1}{2}\|w\|^2 - \int_{\mathbb{R}^n} K(\varepsilon x)\left[|z+w|^{p-1} - z^{p+1} - (p+1)z^p w\right].$$

First of all, observe that, by Lemma 3 and Proposition 11,

$$I'_\varepsilon(z)[w] \le \overline{C}\varepsilon\|w\| \le C_2\varepsilon^2.$$

The Taylor expansions of V and K give

$$V(\varepsilon x) - V(\varepsilon\xi) = \varepsilon V'(\varepsilon\xi)(x-\xi) + O(\varepsilon^2|x-\xi|^2);$$

$$K(\varepsilon x) - K(\varepsilon\xi) = \varepsilon K'(\varepsilon\xi)(x-\xi) + O(\varepsilon^2|x-\xi|^2).$$

Therefore, from elementary estimates involving the evenness of $z^2_{\varepsilon,\xi}(x-\xi)$ and the oddness of $V'(\varepsilon\xi)x$,

$$|\Lambda_\varepsilon(\xi)| = o(\varepsilon).$$

Furthermore, arguing as in the proof of Lemma 10, one finds

$$|\Psi_\varepsilon(\xi)| = o(\|w\|) = o(\varepsilon).$$

Hence it follows that

$$\Phi_\varepsilon(\xi) = C_1 Q(\varepsilon\xi) + o(\varepsilon). \tag{49}$$

We remark that (49) would suffice to prove Theorem 1 in the case that x_0 is an isolated local minimum or maximum. Indeed, setting $\widetilde{\Phi}_\varepsilon(\xi) = \Phi_\varepsilon(\xi/\varepsilon)$, one finds

$$\widetilde{\Phi}_\varepsilon(\xi) = C_1 Q(\xi) + o(\varepsilon).$$

From this one readily deduces that $\widetilde{\Phi}_\varepsilon(\xi)$ possesses a critical point $\xi_\varepsilon \sim 0$; hence Φ_ε has a critical point $\xi_\varepsilon/\varepsilon$ with $\xi_\varepsilon \sim 0$, yielding a solution

$$u_\varepsilon(x/\varepsilon) \sim z_{\varepsilon,\xi_\varepsilon}\left(\frac{x-\xi_\varepsilon}{\varepsilon}\right)$$

concentrating at $x_0 = 0$.

In order to handle the more general case in which x_0 is an isolated stable critical point of Q, we need to estimate the derivatives of Φ_ε with respect to ξ. We write z' for $\partial_\xi z$ and w' for $\partial_\xi w$. One has

$$\partial_\xi(I'_\varepsilon(z)[w]) = I''_\varepsilon(z)[z', w] + I'_\varepsilon(z)[w'].$$

As for Lemma 3, one can prove that $\|I''_\varepsilon(z)[z']\| = o_\varepsilon(1)$; hence, from Proposition 11 and the fact that $\|I'_\varepsilon(z)\| \leq C\varepsilon$ (see Lemma 3), we obtain

$$|\partial_\xi(I'_\varepsilon(z)[w])| = \varepsilon o_\varepsilon(1), \qquad \text{as } \varepsilon \to 0.$$

Regarding the function Λ_ε, using a change of variables, we can write

$$\Lambda_\varepsilon(\xi) = \frac{1}{2}\int_{\mathbb{R}^n}[V(\varepsilon x + \varepsilon\xi) - V(\varepsilon\xi)]z_0^2 - \frac{1}{p+1}\int_{R^n}[K(\varepsilon x + \varepsilon\xi) - K(\varepsilon\xi)]z_0^{p+1},$$

where z_0, as before, stands for the function $z_0(x) = \sigma U(\lambda x)$, with

$$\lambda^2 = V(\varepsilon\xi) \quad \text{and} \quad \sigma = \left(\frac{V(\varepsilon\xi)}{K(\varepsilon\xi)}\right)^{\frac{1}{p-1}}.$$

Hence

$$\begin{aligned}\Lambda'_\varepsilon(\xi) =& \frac{\varepsilon}{2}\int_{\mathbb{R}^n}[V'(\varepsilon\xi + \varepsilon x) - V'(\varepsilon\xi)]z_0^2 + \int_{\mathbb{R}^n}[V(\varepsilon x + \varepsilon\xi) - V(\varepsilon\xi)]z_0 z_0' \\ &- \frac{\varepsilon}{p+1}\int_{\mathbb{R}^n}[K'(\varepsilon\xi + \varepsilon x) - K'(\varepsilon\xi)]z_0^{p+1} + \int_{\mathbb{R}^n}[K(\varepsilon x + \varepsilon\xi) - K(\varepsilon\xi)]z_0^p z_0'.\end{aligned}$$

Using the Dominated Convergence Theorem, we obtain

$$|\Lambda'_\varepsilon(\xi)| = o(\varepsilon).$$

We have also

$$\Psi'_\varepsilon(\xi) = (w, w') + (p+1)\int_{\mathbb{R}^n} K(\varepsilon x)G(z, w)dx,$$

where

$$G(z, w) = |z + w|^{p-1}(z + w)(z' + w') - z^p z' - p z^{p-1} z' w - z^p w'.$$

From (K2), we infer

$$|\Psi'_\varepsilon(\xi)| \leq \|w\|\|w'\| + C_7\int_{\mathbb{R}^n} G(z, w)dx;$$

hence, using Proposition 11, we find

$$|\Psi_\varepsilon'(\xi)| \le C_7 \int_{\mathbb{R}^n} G(z,w)dx + o(\varepsilon). \tag{50}$$

Next, let us write $G(z,w) = G_1(z,w)z' + G_2(z,w)w'$ with

$$G_1(z,w) = |z+w|^{p-1}(z+w) - z^p - pz^{p-1}w,$$
$$G_2(z,w) = |z+w|^{p-1}(z+w) - z^p,$$

and estimate separately $\int_{\mathbb{R}^n} G_1(z,w)z'dx$ and $\int_{\mathbb{R}^n} G_2(z,w)w'dx$. As for the former, since $G_1(z,w) = |z+w|^{p-1}(z+w) - z^p - pz^{p-1}w = O(|w|^{2\wedge p})$, we have

$$\int_{\mathbb{R}^n} G_1(z,w)z'dx \le \|w\|_\infty^{1\wedge(p-1)} \int_{\mathbb{R}^n} \frac{wz'}{\sqrt{V(\varepsilon x)}} \sqrt{V(\varepsilon x)}dx.$$

Using Hölder's inequality, we get

$$\int_{\mathbb{R}^n} G_1(z,w)z'dx \le \|w\|_\infty^{1\wedge(p-1)}\|w\| \left(\int_{\mathbb{R}^n} \frac{z'^2}{V(\varepsilon x)}dx\right)^{\frac{1}{2}}.$$

Taking into account the exponential decay of z' as well as the fact that $\|w\|_\infty = O(\varepsilon^{1/2})$ and $\|w\| \le \varepsilon$ (see the definition of Γ_ε), we obtain

$$\int_{\mathbb{R}^n} G_1(z,w)z'dx = o(\varepsilon). \tag{51}$$

In a quite similar way, using once more that $\|w'\| = o_\varepsilon(1)$, we find

$$\int_{\mathbb{R}^n} G_2(z,w)w'dx \le O(\varepsilon)\|w'\| = o(\varepsilon). \tag{52}$$

In conclusion, from (50), (51) and (52), we have

$$|\Psi_\varepsilon'(\xi)| = o(\varepsilon).$$

From these estimates, it follows that

$$\Phi_\varepsilon'(\xi) = C_1\, \varepsilon\, Q'(\varepsilon\xi) + o(\varepsilon), \qquad |\varepsilon\xi| \le 1. \tag{53}$$

As before, we set $\widetilde{\Phi}_\varepsilon(\xi) = \Phi_\varepsilon(\xi\,/\,\varepsilon)$. Then (53) implies that, for $\varepsilon \ll 1$,

$$ind(\widetilde{\Phi}_\varepsilon', 0, 0) = ind(Q', 0, 0) \ne 0.$$

Hence $\widetilde{\Phi}_\varepsilon$ has a critical point $\xi_\varepsilon \sim 0$. As a consequence, the reduced functional Φ_ε possesses a critical point $\xi_\varepsilon/\varepsilon$ with $\xi_\varepsilon \sim 0$, and the conclusion follows.

7 Further results

In this final section, we discuss some extensions of Theorem 1.

Let $\mathcal{X}_0$ be a compact set of critical points of Q. We say that $\mathcal{X}_0$ is a stable critical set of Q if the topological degree $deg(Q', \mathcal{X}_{0,\delta}, 0) \neq 0$, for all small $\delta > 0$, where $\mathcal{X}_{0,\delta}$ is a δ-neighborhood of $\mathcal{X}_0$. Then the same arguments carried out in the previous sections lead to the following result.

Theorem 12. *Let* $1 < p < (n+2)/(n-2)$ *and suppose that* V *and* K *are smooth and satisfy* (V1) – (V2) *and* (K0) – (K2)*, respectively. Moreover, let* $\mathcal{X}_0$ *be a compact, stable critical set of* Q*. Then for* $\varepsilon \ll 1$*,* (1) *has a bound state that concentrates at some point of* $\mathcal{X}_0$*.*

Of course, if $\mathcal{X}_0 = \{x_0\}$, Theorem 12 is nothing but Theorem 1. However, in the more general case covered by Theorem 12, we cannot establish a priori at which point of $\mathcal{X}_0$ the concentration occurs.

In certain circumstances, one can also find a multiplicity result. Let Σ be a smooth compact manifold of critical points of Q. We say that Σ is non-degenerate if every $x \in \Sigma$ is a non-degenerate critical point of $Q_{|\Sigma^\perp}$. Combining the arguments used in [5] and those carried over in the present paper, one can prove the existence of multiple solutions of (1) concentrating at points of Σ.

Theorem 13. *Let* $1 < p < (n+2)/(n-2)$ *and suppose that* V *and* K *are smooth and satisfy* (V1) – (V2) *and* (K0) – (K2)*, respectively. Moreover, let* Σ *be either a non-degenerate compact manifold of critical points of* Q *or a compact set of minima/maxima of* Q*. Then for* $\varepsilon > 0$ *small,* (1) *has at least* $l(\Sigma)$*, respectively,* $cat(\Sigma, \Sigma_\delta)$*, solutions concentrating near points of* Σ*.*

Above, $l(\Sigma)$ is the *cup length* of Σ, defined by

$$l(\Sigma) = 1 + \sup\{k \in \mathbb{N} : \exists\, \alpha_1, \dots, \alpha_k \in \check{H}^*(\Sigma) \setminus 1,\ \alpha_1 \cup \cdots \cup \alpha_k \neq 0\}.$$

If no such class exists, we set $l(\Sigma) = 1$. Here $\check{H}^*(\Sigma)$ is the Alexander cohomology of Σ with real coefficients and $\cup$ denotes the cup product. Moreover, $cat(\Sigma, \Sigma_\delta)$ denotes the Lusternik–Schnirelman category of Σ with respect to the δ-neighborhood Σ_δ of Σ, namely, the least number k such that $\Sigma \subseteq \bigcup_1^k T_i$, with T_i closed and contractible in Σ_δ. In general, one has $l(\Sigma) \leq cat(\Sigma, \Sigma_\delta)$.

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Added in proofs. After this paper was completed, we became aware of the work, P. Souplet and Qi S. Zhang, *Stability for semilinear parabolic equations with decaying potentials in* $\mathbb{R}^n$ *and dynamical approach to the existence of ground states*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 683–703, where the Schrödinger equation with a decaying potential V is studied. However, they do not deal with semiclassical states and do not study spikes. Moreover, they only consider radial potentials V satisfying (V1) with $0 \leq \alpha < 2(n-1)(p-1)/(p+3)$, which is strictly smaller than 2.

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