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Corrigendum to "Unramified sections of the Legendre scheme and modular forms" [J. Geom. Phys. 166 (2021) 104266]

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In this *Corrigendum* we wish to point out an inaccurate assertion made in our paper *Unramified sections of the Legendre scheme and modular forms*, J. Geom. Phys. 166 (2021), concerning a certain (counter)example. (The inaccuracy was pointed out by Yves André, whom we thank.)

In fact, after a more careful analysis we verified that the assertion and the example still hold (and fulfill the sought purpose for insertion in the paper) provided only certain exceptional cases are excluded. We are going to explain all of this in detail.

We avail ourselves of this opportunity and, after discussing this issue, we also point out some typos (we thank Luca Mauri for detecting most of them), and add a precision concerning a reference.

Turning back to our main point, we are referring to the example presented in the paper in question, at page 5, around formula (2.6). We repeat here the essentials, introducing a minimum of (new) notation for convenience. Let $A := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ be the standard (unipotent) generators for the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ denoted (usually) by Γ_2. This is well known to be a free group on these generators.

Consider also the subgroup G of SL_3 made up of matrices of the shape $(g, v) := \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix}$, where $g \in SL_2$ and $v \in \mathbb{G}_a^2$ is a column vector. It is isomorphic to the group of affine automorphism of $\mathbb{A}^2$ given by $x \mapsto gx + v$.

Finally, for a choice w_A, w_B of vectors in $\mathbb{G}_a^2$, consider the subgroup $\Gamma = \Gamma(w_A, w_B)$ of G generated by the matrices

$$A^* := (A, w_A) = \begin{pmatrix} A & w_A \\ 0 & 1 \end{pmatrix}, \qquad B^* := (B, w_B) = \begin{pmatrix} B & w_B \\ 0 & 1 \end{pmatrix}.$$

Using for convenience the present notation, in the paper in question (see p. 5) it was stated that:

[...] and w_A, w_B are a basis for $\mathbb{Z}^2$. It can be easily shown that the Zariski closure of Γ is the full group G, whereas the kernel of the natural map to $\mathrm{SL}_2(\mathbb{Z})$ is trivial. [...]

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Now, this assertion is not true in this generality. In fact, if there exists a $w_0 \in \mathbb{G}_a^2$ such that $w_A = (A - I)w_0$ and $w_B = (B - I)w_0$, it is easily seen that the group Γ consists of the matrices $(g, (g - I)w_0) = \begin{pmatrix} g & (g-I)w_0 \\ 0 & 1 \end{pmatrix}$, where $g \in \Gamma_2$, so its Zariski closure consists of the same kind of matrices but allowing g to be arbitrary in SL_2, and this is obviously smaller than G.

We observe that the existence of w_0 corresponds to a common fixed point for the affine maps corresponding to A^*, B^*. Note that such a common fixed point needs not to exist for arbitrary choices of w_A, w_B; actually, if for instance w_A is chosen outside the image of $A - I$ (which is singular), the affine map corresponding to A^* has no fixed points at all.

It is our purpose here to state (and prove) the correct form of the above assertion, namely to prove the following

Claim. *The Zariski closure of Γ is the full group G, unless there exists $w_0 \in \mathbb{G}_a^2$ such that $w_A = (A - I)w_0$ and $w_B = (B - I)w_0$.*
On the other hand, the kernel of the natural map from Γ to $\mathrm{SL}_2(\mathbb{Z})$ is trivial.

Here we tacitly identify a variety with the set of its points over a(ny) algebraically closed field of characteristic 0.

Note that this Claim entails, in particular, that taking the Zariski closure *does not commute* with taking the kernel in the present situation. Indeed, whereas the kernel from Γ is trivial, the kernel from its Zariski closure has dimension 2, apart from the said special choices for w_A, w_B.

This fact is quite relevant concerning Theorem 2.2 of our paper. Indeed, we stress that the present Claim serves the same purposes as the former (inaccurate) one, concerning the rest of that paper. Namely, it yields a counterexample to a (false but *a priori* plausible) assertion, which would affect the main results of our paper concerning its generality. More precisely, it shows that our Theorem 2.2 does not follow from the theorem of Bertrand quoted in the paper (concerning the differential Galois group) via purely group-theoretic consideration.

We shall now give a sketch of proof of this statement, relying on well-known facts about SL_2.

We start by observing that the natural homomorphism $\pi : (g, v) \mapsto g$ to SL_2, restricted to Γ, has trivial kernel, since the image $\pi(\Gamma)$ equals Γ_2, and since Γ is generated by A^*, B^*, whose images $\pi(A^*) = A$, $\pi(B^*) = B$ are free generators of Γ_2. This also takes care of the last assertion in the Claim.

Let now G_0 be the Zariski closure of Γ (in G), where we have to prove that $G_0 = G$. Observe that since Γ_2 is Zariski dense in SL_2, the image $\pi(G_0)$ is the full SL_2.

Assume the contrary, i.e. that G_0 is strictly smaller than G; then we contend that the kernel of π is trivial even on G_0. In fact, let (I, v) be in the kernel. Then, for every $(g, w) \in G_0$, $(g, w)(I, v)(g, w)^{-1}$ also lies in the kernel, and is easily computed to be (I, gv). So if $v \neq 0$, we would have $(I, z) \in G_0$ for every $z \in \mathbb{G}_a^2$. But then clearly $G_0 = G$.

Since the kernel of π restricted to G_0 is trivial, and since $\pi : G_0 \to \mathrm{SL}_2$ is surjective, we have a section $g \mapsto (g, v_g)$ from SL_2 to G_0. This defines a 1-cocycle of SL_2 with values in $\mathbb{G}_a^2$, since it is immediately checked that $v_{gh} = v_g + g v_h$.

But $H^1(\mathrm{SL}_2, \mathbb{G}_a^2) = 0$ in characteristic 0: this well-known fact may be proved in several ways. The simplest is probably using the restriction-inflation sequence with H being the central subgroup of SL_2 generated by $e := -I$. Note in fact that e has no (nonzero) fixed points on $\mathbb{G}_a^2$, and it is immediately proved that $H^1(\langle -I \rangle, \mathbb{G}_a^2) = 0$ directly from the definitions. Even more explicitly, the cocycle equations for $v_{eg} = v_{ge}$ yield $v_e - v_g = v_g + g v_e$, whence $2v_g = (1 - g)v_e$, so v_g is the coboundary associated with $-v_e/2$.

Therefore there exists $w_0 \in \mathbb{G}_a^2$ such that $v_g = (g - I)w_0$, as required.

As a final remark, we note that the result originates from the different behavior of $H^1(\cdot, \mathbb{G}_a^2)$ on Γ_2 and SL_2: we have just seen that $H^1(\mathrm{SL}_2, \mathbb{G}_a^2)$ is trivial. On the other hand, $H^1(\Gamma_2, \mathbb{G}_a^2)$ is far from being trivial. Indeed, we may choose w_A, w_B arbitrarily and define inductively a cocycle on Γ_2 by induction on the length of a word in A, B; the fact that A, B are free generators says that this is well defined, and we come out with a coboundary only for the above special choices of w_A, w_B. A reason behind this is that while Γ_2 is free, there are many relations among elements of SL_2, which yield strong restrictions on cocycles; actually, the above argument shows that the relations $(-I) \cdot g = g \cdot (-I)$, for $g \in \mathrm{SL}_2$, suffice to ensure the triviality of $H^1(\mathrm{SL}_2, \mathbb{G}_a^2)$.

For 'several' other discrete subgroups of linear algebraic groups, the corresponding H^1 is trivial, which excludes examples similar to the one discussed here.

Some typos:

- Page 6, line 7: with $\pi_1(\mathcal{E}_{b_0}, \sigma(b_0)) \simeq \mathbb{Z}^2$. The index b_0 was missing. The same occurs in Theorem 2.3 at the subsequent line.
- Page 6, two lines before Theorem 2.4: $\sigma_B \to \mathcal{E}$ should be $\sigma : B \to \mathcal{E}$.
- Page 10, line -5: G should in fact be the operator $\mathcal{G}$.
- Page 13, formula (3.32): $\Im(w)$ should be $\Im(z)$.
- Page 14, fifth line of the proof of Theorem 4.1: 'for a cocyle' → 'form a cocycle'.
- Page 17, penultimate line: $A(f, g)$ should be $A(f_1, f_2)$.

- Page 14, first displayed formula of the proof of Lemma 4.5: some parentheses are missing. The formula should read

$$g \circ \Im(\Phi(f)) \circ g^{-1} = \Im(\Phi(f)).$$

The last term in the following displayed formula should be

$$\int_{\mathcal{H}/C} d\Im(\Phi(f)) \wedge d\Im(\omega_h).$$

- Page 20, third displayed formula. The sum should read

$$\sum_{t \in \rho^{-1}(t')} \sigma(t).$$

- Page 21, first displayed formula: $\tilde{g}$ should be g.
- Page 22, formula (6.54). The sum should appear with a minus sign. In the sequel, this mistake is corrected.
- Page 24, line -13: compilation error: Ee should be $\mathcal{E}$.

About the quotation of [23] appearing in the introduction, page 2: We wrongly asserted that Shioda and Schütt proved Silverman's conjecture in the particular case the elliptic scheme has rank zero and is 'extremal', in the sense that its Picard rank equals the Hodge number $h^{1,1}$. Actually, in a previous version of [23] appearing on the arXiv, they proved that the property of extremality is preserved under unramified extensions. This does not seem to imply what we attributed to them.

Data availability

No data used.