

Quadratic Quasi-Normal Modes of a Schwarzschild Black Hole

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Quadratic quasi-normal modes, generated at second order in black hole perturbation theory, are a promising target for testing gravity in the nonlinear regime with next-generation gravitational wave detectors. While their frequencies have long been known, their amplitudes remain poorly studied. We introduce regular variables and compute amplitudes for Schwarzschild black holes with the Leaver algorithm. We find a nonlinear ratio $\mathcal{R} \simeq 0.154e^{-0.068i}$ for the most excited $\ell = 4$ mode, matching results from Numerical Relativity. We also predict new low-frequency $\ell = 2$ quadratic modes.

Introduction Observations of binary black hole (BH) mergers hold promise for testing General Relativity (GR) in the strong gravity regime with unprecedented precision and perhaps even finding evidence of novel phenomena in the gravitational domain. While the precise moment of the actual merger can only be described using Numerical Relativity (NR), analytic perturbation theory plays a fundamental role at earlier and later times. In this letter, we will focus on the final phase of the merger process, known as the ringdown, specifically examining the scenario where the black hole formed in the process has zero angular momentum.

Linear perturbations around the spacetime geometry resulting from the merger are described at intermediate times by a superposition of a discrete set of damped sinusoids, known as quasinormal modes (QNMs) [1]. They are characterized by complex frequencies determined in GR solely by the mass and angular momentum of the black hole. At linear order and for zero BH spin, the dynamics of the two physical gravitational wave polarizations can be described by two “master scalar” variables $\Psi_{\pm}^{(1)}$ that obey independent wave equations in a potential V_{\pm} :

$$\left(\square - V_{\pm}\right)\Psi_{\pm}^{(1)} = 0. \quad (1)$$

The equations are decoupled because the two perturbations transform oppositely under parity (hence the subscript \pm), which is a symmetry of the Schwarzschild geometry. The resulting frequencies $\omega_{\ell mn}$, labeled by two angular harmonic numbers (ℓ, m) and an overtone number n , can be utilized for accurate gravity testing [2–27]. On the other hand, the amplitudes of these modes depend on the initial conditions of the merger and should be fitted against data or Numerical Relativity simulations [7, 22–24, 28–32].

Since the perturbations decay exponentially, it is not surprising that linear Black Hole Perturbation Theory (BHPT) [33, 34] can effectively model the ringdown after a time roughly equivalent to the QNM decay timescale. However, during the initial phase of the ringdown, just after the peak of the strain, it becomes essential to account for nonlinear effects. This is particularly relevant because the first few cycles encompass most of the observable signal, providing the best opportunities for precise measurements. Indeed, recent studies have shown that second-order BHPT may be necessary to accurately describe the signals obtained from NR [32, 35–41].

To second order in perturbation theory, the dynamics is governed by the same wave equation for the second order master scalars $\Psi_{\pm}^{(2)}$, but now with a source term $S^{(2)}$ quadratic in the linear perturbations,

$$\left(\square - V_{\pm}\right)\Psi_{\pm}^{(2)} = S^{(2)}. \quad (2)$$

As a consequence, there exists a set of quadratic QNMs (QQNMs) generated by the product of two linear modes. For each (ℓ, m) the waveform contains additional modes whose frequencies can be trivially obtained from the product of two monochromatic first-order waves as $\omega^{\text{Q}} = \omega_{\ell_1 m_1 n_1} + \omega_{\ell_2 m_2 n_2}$ or $\omega^{\text{Q}} = \omega_{\ell_1 m_1 n_1} - (\omega_{\ell_2 m_2 n_2})^*$ [42], where $*$ denotes complex conjugation. A crucial feature of QQNMs is that, unlike the linear ones, their *amplitude* is also entirely determined by GR. This essential property arises from the fact that these modes solve a non-homogeneous equation (2). Despite this fact, to date, only a few specific second-order amplitudes have been computed, either by fitting NR simulations [32, 35–40] or by integrating BHPT equations [41, 43–47]. Since at least the loudest QQNM will be observable by next-generation interferometers [48], there is an increased need for precise computation of these amplitudes.

This letter aims to provide the amplitude of all low multipole QQNMs for a Schwarzschild BH. One of the main obstacles in solving the second-order equation can be traced back to the redundancy that inevitably arises when describing the two degrees of freedom (DOF) of a massless spin-2 particle in terms of the metric ten-

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sor. Conceptually, the procedure should follow what is done at linear order and appears straightforward: one first fixes the gauge redundancy associated with diffeomorphism invariance. Secondly, one selects two master scalar variables $\Psi_{\pm}^{(2)}$, expressed in terms of the metric components, which capture the dynamics of the physical DOF. Once the equation for each $\Psi_{\pm}^{(2)}$ is solved with the appropriate boundary conditions, it becomes possible to reconstruct the full metric and transition to an outgoing radiation gauge, where the amplitude of the physical gravitational wave observable at infinity can be computed.

Regrettably, as noted in the literature [43, 44, 46, 49–54], when the choice of a second-order master scalar is equivalent to the first-order one, quadratic perturbations exhibit a divergent behavior at infinity or the black hole horizon. Although this divergence will eventually become immaterial, it signifies a poor choice of variables, making the extraction of the subleading physical effect extremely challenging. To circumvent this problem, we will demonstrate how to define two new master scalars for arbitrary (ℓ, m) that are regular everywhere. With these variables, the resulting equations, along with appropriate QNM boundary conditions, can be integrated using the Leaver algorithm [55]. While we explicitly present in this article a set of amplitudes for the first few QQNMs, our complete results are available online [56]. In the following, we will use units in which $G = c = 1$.

Second-order BHPT We expand the spacetime metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}^{(1)} + \varepsilon^2 h_{\mu\nu}^{(2)}$, where ε is a small book-keeping parameter for the perturbation order – physically corresponding to the small magnitude of linear QNMs (LQNMs) – and $\bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$ is the Schwarzschild metric, with $f(r) = 1 - 2M/r$. Up to second order, Einstein’s equations read

$$G_{\mu\nu}^{(1)} [h^{(1)}] = 0, \quad (3)$$

$$G_{\mu\nu}^{(1)} [h^{(2)}] = -G_{\mu\nu}^{(2)} [h^{(1)}, h^{(1)}], \quad (4)$$

where we schematically denoted by $G_{\mu\nu}^{(1)} [\cdot]$ the part of the Einstein tensor linear in perturbations and by $G_{\mu\nu}^{(2)} [\cdot, \cdot]$ its part bilinear in perturbations, suppressing indices on $h_{\mu\nu}$ for clarity. Thus, the LQNM $h^{(1)}$ will provide a source term for the QQNM $h^{(2)}$; notice that the same differential operator acts similarly on $h^{(1)}$ and $h^{(2)}$ on the left-hand side of these equations. Spherical symmetry of the background allows us to split spacetime between a “ $t - r$ plane” \mathcal{M} (indices a, b, \dots and metric $\bar{g}_{ab} = \text{diag}(-f, f^{-1})$) and the 2-sphere \mathcal{S} (indices A, B, \dots and metric $\Omega_{AB} = \text{diag}(1, \sin^2 \theta)$). We decom-

pose metric perturbations according to

$$h_{ab}^{(i)} = \sum_{\omega, \ell, m} e^{-i\omega t} h_{ab, \ell m \omega}^{(i)} Y^{\ell m}, \quad (5)$$

$$h_{aB}^{(i)} = \sum_{\omega, \ell, m} e^{-i\omega t} [h_{a+, \ell m \omega}^{(i)} Y_B^{\ell m} + h_{a-, \ell m \omega}^{(i)} X_B^{\ell m}], \quad (6)$$

$$h_{AB}^{(i)} = \sum_{\omega, \ell, m} e^{-i\omega t} [h_{\circ, \ell m \omega}^{(i)} \Omega_{AB} Y^{\ell m} + h_{+, \ell m \omega}^{(i)} Y_{AB}^{\ell m} + h_{-, \ell m \omega}^{(i)} X_{AB}^{\ell m}], \quad (7)$$

where $i = 1, 2$ is the order of perturbation, the sum goes over all QNM frequencies ω and mode numbers ℓ, m , and $Y, Y_A, X_A, Y_{AB}, X_{AB}$ are a set of tensor spherical harmonics defined in app. A. To streamline the discussion, we will omit the indices $\ell m \omega$ from variables in the following. We focus on perturbations with $\ell \geq 2$, as according to the peeling theorem [57–59], these are the only ones carrying radiation to infinity. Following the approach outlined in [60], we define a gauge-invariant variable $\hat{h}_{\mu\nu}$, decomposed in spherical harmonics as in (5)–(7), which coincides with the metric perturbation $h_{\mu\nu}$ in Regge-Wheeler-Zerilli (RWZ) gauge where $h_{\pm} = h_{a+} = 0$. Detailed expressions are provided in app. A. On the other hand, we want to extract QNM amplitudes at large distances and in the physical Transverse-Traceless (TT) gauge, where the (real) \mathfrak{h}_+ and \mathfrak{h}_\times components of the metric perturbation $h_{\mu\nu} = h_{\mu\nu}^{(1)} + \varepsilon h_{\mu\nu}^{(2)}$ are, up to $\mathcal{O}(r^{-2})$:

$$\mathfrak{h}_+ - i\mathfrak{h}_\times = \frac{M}{r} \sum_{\ell m \mathcal{N}} \mathcal{A}_{\ell m \mathcal{N}} e^{i\omega_{\ell \mathcal{N}}(r_* - t)} {}_{-2}Y^{\ell m}(\theta, \phi). \quad (8)$$

In this equation, which applies to both linear and quadratic order, ${}_{-2}Y$ are the spin-weighted spherical harmonics, $r_* = r + 2M \log(r/2M - 1)$ is the tortoise coordinate, $\mathcal{A}_{\ell m \mathcal{N}}$ and $\omega_{\ell \mathcal{N}}$ are the amplitudes and frequencies of QNMs, and \mathcal{N} is an additional mode number defined as follows. For linear modes, $\mathcal{N} = (n, \mathbf{m})$ is composed of the overtone number n and of the mirror modes $\mathbf{m} = \pm$. A mirror mode has a frequency related to regular modes by complex conjugation: $\omega_{\ell n -} = -\omega_{\ell n +}^*$. Note that for Schwarzschild BHs, QNMs frequencies do not depend on the mode number m ; we will use the convention that regular $\mathbf{m} = +$ frequencies have positive real part. For quadratic modes, $\mathcal{N} = (\ell_1, m_1, n_1, \mathbf{m}_1) \times (\ell_2, m_2, n_2, \mathbf{m}_2)$ contains the mode numbers of the product of linear modes which generate them [42]. Finally, we assume (as is common in the literature [29, 30, 48, 61]) that the strain (8) enjoys an additional equatorial symmetry. This implies that mirror modes with $m < 0$ have the same amplitude as regular modes with $m' = -m > 0$ up to a phase. This assumption is typically well satisfied in practise [29]¹, and it fixes the parity of any mode to

¹ Notice that mirror modes are related to but different than retrograde modes introduced in [32, 61], which are modes with $\text{sign}(\text{Re} \omega m) < 0$, and whose amplitude are suppressed in a typical merger [3, 28, 36, 62].

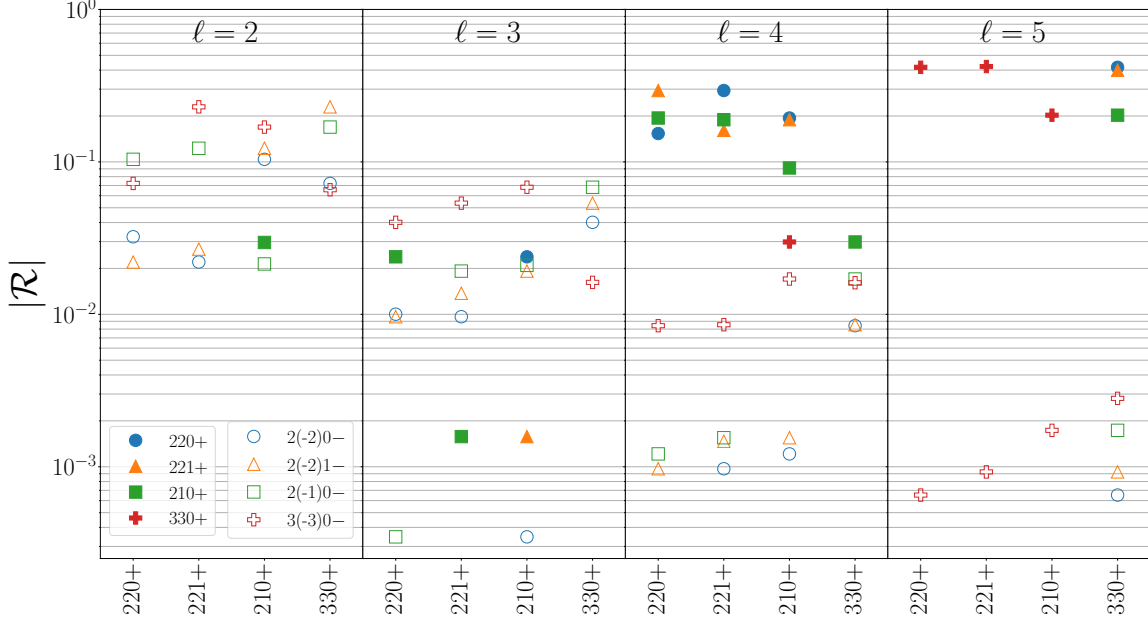


Figure 1: Nonlinear ratio of amplitudes $|\mathcal{R}|$ for different mode numbers. The x axis represents the mode numbers $\ell m m$ of the first linear parent mode, while the second linear parent mode is represented with different markers. Filled symbols represents the combination of two normal modes with $m > 0$ ($\omega^Q = \omega_{\ell_1 n_1 +} + \omega_{\ell_2 n_2 +}$), while empty ones represents a normal mode with $m > 0$ combined with a mirror mode with $m < 0$ ($\omega^Q = \omega_{\ell_1 n_1 +} - (\omega_{\ell_2 n_2 +})^*$). A missing marker means that $|\mathcal{R}| < 2.5 \times 10^{-4}$. Notice that most points in this figure correspond to contributions to various spherical harmonics, as each point corresponds to its own $m = m_1 + m_2$ mode and m_1, m_2 vary.

$(-1)^{\ell+m}$.

By relating \tilde{h} to \mathfrak{h}_+ and \mathfrak{h}_\times , it can be checked that $\tilde{h}_{ab}^{(i)} = \mathcal{O}(r)$, $\tilde{h}_{a-}^{(i)} = \mathcal{O}(r)$, $\tilde{h}_o^{(i)} = \mathcal{O}(r^2)$ for $r \rightarrow \infty$ regardless of the gauge choice (since \tilde{h} is gauge-invariant). The precise relationships between asymptotic amplitudes of the various variables are given in the GitHub package [56] and in our companion paper [63]. We now define modified RWZ variables as

$$\Psi_-^{(i)} = \frac{r}{\lambda} \left[D_r \tilde{h}_{t-}^{(i)} - D_t \tilde{h}_{r-}^{(i)} - \frac{2}{r} \tilde{h}_{t-}^{(i)} \right] + g_-^{(i)}(r), \quad (9)$$

$$\Psi_+^{(i)} = \frac{2r}{\ell(\ell+1)} \left[\frac{\tilde{h}_o^{(i)}}{r^2} + \frac{1}{\lambda + 3M/r} \left(f^2(r) \tilde{h}_{rr}^{(i)} - r f(r) D_r \left(\frac{\tilde{h}_o^{(i)}}{r^2} \right) \right) \right] + g_+^{(i)}(r), \quad (10)$$

where D denotes the covariant derivative on \mathcal{M} , $\lambda = (\ell+2)(\ell-1)/2$, and $g_\pm^{(i)}$ is a *regularizing* term present at second order only (i.e., $g_\pm^{(1)} = 0$) enforcing the correct asymptotic limits. Indeed, one can check that, were that term not present, the RWZ variables would exhibit more divergent behaviors at second order compared to first-order QNMs [43, 49–54]. Our definition ensures that

$$\Psi_\pm^{(i)} \underset{r \rightarrow \infty}{\simeq} \mathcal{B}_\pm^{(i)} e^{i\omega(r_* - t)}, \quad \Psi_\pm^{(i)} \underset{r \rightarrow 2M}{\simeq} \mathcal{C}_\pm^{(i)} e^{-i\omega(r_* + t)}, \quad (11)$$

where $\mathcal{B}_\pm^{(i)}, \mathcal{C}_\pm^{(i)}$ are constants. At second order, a specific QQNM is generated by the product of two LQNMs $\Psi_1^{(1)}$ and $\Psi_2^{(1)}$ (for brevity we include the parity of first-order modes in the set of suppressed indices). We thus take

$$g_\pm^{(2)} = (a_{\pm,2} r^2 + a_{\pm,1} r) \Psi_1^{(1)} \Psi_2^{(1)} \quad (12)$$

where $a_{\pm,1}, a_{\pm,2}$ are chosen to cancel the divergences of the unregulated master scalars. Another possibility would have been to work with “unregularized” variables; however, for numerical accuracy we have chosen here to analytically subtract these spurious terms. We have verified that they do not contribute to the asymptotic waveform in TT gauge. Explicit expressions for $g_\pm^{(2)}$ are given in the companion article [63] and in the GitHub package [56]. Eqs. (3)-(4) then translate in the RWZ equations with a source term

$$\frac{d^2 \Psi_\pm^{(i)}}{dr_*^2} + (\omega^2 - V_\pm) \Psi_\pm^{(i)} = S_\pm^{(i)}, \quad (13)$$

where V_\pm are the usual RWZ potentials given in app. A and the source is zero at first order, $S_\pm^{(1)} = 0$. Our choice of regulator for $\Psi_\pm^{(2)}$ ensures that the left-hand side of eq. (13) is the same at first and second order. On the other hand, the explicit source at second order is

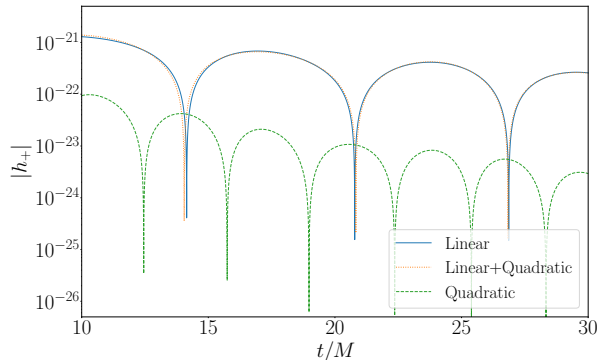


Figure 2: Waveform polarization $|h_+|$ in eq. (8) including only linear modes, only quadratic modes, and both of them for a GW150914-like merger. LQNMs amplitudes are taken from the fits in [32] with progenitor parameters from [64]; we include the (220), (221), (210), (330) and (440) LQNMs. All quadratic modes amplitudes are added using results from this article. The starting time is chosen to be $t_0 = 10M$ after the peak of the waveform to ensure QNM regime.

regulator-dependent and we get

$$S_{\pm}^{(2)} = \mathcal{F}_1(r)\Psi_1^{(1)}\Psi_2^{(1)} + \mathcal{F}_2(r)(\Psi_1^{(1)})'\Psi_2^{(1)} + \mathcal{F}_3(r)\Psi_1^{(1)}(\Psi_2^{(1)})' + \mathcal{F}_4(r)(\Psi_1^{(1)})'(\Psi_2^{(1)})', \quad (14)$$

where $' = d/dr$, and the four functions \mathcal{F} depend on ℓm , the parities of second and first-order modes as well as $\mathcal{N} = (\ell_1, m_1, n_1, \mathbf{m}_1) \times (\ell_2, m_2, n_2, \mathbf{m}_2)$. We give their explicit expression in the companion article [63] and in the GitHub package [56]. We have verified that the asymptotic behavior of LQNMs (11) imposes $S_{\pm}^{(2)} \propto r^{-2}$ for $r \rightarrow \infty$ and $S_{\pm}^{(2)} \propto 1 - 2M/r$ for $r \rightarrow 2M$, which is consistent with eq. (13) at second order once we impose QNM boundary conditions on $\Psi_{\pm}^{(2)}$. This serves as a confirmation of the consistency of our computations.

Checks and results Second-order RWZ equations were investigated in several other works, see e.g. [43–47, 49–54, 60, 65]. However, we cannot directly compare our expression for the source term in (14) to their computations since our definitions of master scalars (9)–(10) differ. What is unequivocally defined, however, is the asymptotic amplitude of quadratic modes in the GW strain (8), which is our main result and that we will now provide.

We numerically solve eq. (13) with QNM boundary conditions (11). We employ the adaptation of the Leaver algorithm to QQNMs described in [43], which provides the ratio of QQNMs to LQNMs amplitudes at infinity $\mathcal{B}_{\pm}^{(2)}/(\mathcal{B}_1^{(1)}\mathcal{B}_2^{(1)})$ with negligible numerical error. Subsequently, we invert eqs. (9)–(10) to reconstruct the gauge-invariant variable \tilde{h} from the RWZ scalars, which finally provides us the physical amplitude of QNMs in the TT gauge as per eq. (8). Rotational symmetry imposes that the only dependence of $\mathcal{A}_{\ell m \mathcal{N}}^{(2)}$ on m, m_1, m_2 is contained

in the 3j-symbol $\begin{pmatrix} \ell & \ell_1 & \ell_2 \\ -m & m_1 & m_2 \end{pmatrix}$ which we factor out from our numerical computations. We give in the companion file [56] the ratios $\mathcal{R} = \mathcal{A}_{\ell m \mathcal{N}}^{(2)}/(\mathcal{A}_{\ell_1 m_1 \mathcal{N}_1}^{(1)}\mathcal{A}_{\ell_2 m_2 \mathcal{N}_2}^{(1)})$ for all $2 \leq \ell, \ell_1, \ell_2 \leq 7$ respecting the Clebsch-Gordan rules $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$. We also include the cases where one of the linear amplitudes is an overtone $n = 1$ or a mirror mode $\mathbf{m} = -$.

Our results for some of the low-lying modes are displayed in fig. 1. In particular, we find $\mathcal{R} \simeq 0.154e^{-0.068i}$ for $(220+) \times (220+) \rightarrow (44)$ which is the most excited nonlinear mode in NR simulations and BHPT fits [32, 35–38, 40, 41]. This is $\sim 10\%$ higher than the semi-analytical calculation on a Kerr background given in [46]. However, our result is in perfect agreement with the fits to NR simulations provided in [40] at zero spin². Our phase seem also to be in good agreement with [40]. Another QQNM of interest is $(330+) \times (220+) \rightarrow (55)$ for which we find $\mathcal{R} \simeq 0.417e^{-0.081i}$, which is a $\sim 7\%$ difference in amplitude from the fits provided in [32], extrapolated to zero spin. The phase is however half of the value provided in [32]: this difference may be due to the extrapolation. Finally we find $\mathcal{R} \simeq 0.194e^{-0.084i}$ for $(220+) \times (210+) \rightarrow (43)$, a QQNM not yet mentioned in the literature.

Let us now describe other peculiarities of our results. First, one has $|\mathcal{R}| < 1$ for all of the ratios displayed in fig. 1; this was not granted since perturbation theory only requires $\mathcal{A}^{(2)} \ll \mathcal{A}^{(1)}$. In addition, it appears that combinations of a regular $\mathbf{m} = +$ and $m > 0$ with a mirror $\mathbf{m} = -$ and $m < 0$ linear modes give a non-negligible ratio in the $\ell = 2$ sector; we find for example $\mathcal{R} \simeq 0.169e^{-0.194i}$ for $(330+) \times (2(-1)0-) \rightarrow (22)$ and $\mathcal{R} \simeq 0.104e^{-1.58i}$ for $(220+) \times (2(-1)0-) \rightarrow (21)$. These QQNMs were not mentioned in previous works using fits to waveforms [32, 35–38, 40, 41]. This may be due to the fact that: a) They have a small or even vanishing real part of the frequency due to the relation $\omega^{\text{Q}} = \omega_{\ell_1 n_1 +} - (\omega_{\ell_2 n_2 +})^*$, making them potentially more challenging to incorporate into fitting algorithms. b) Their ratio \mathcal{R} is non-negligible mostly for odd m_1 or m_2 , whose parent modes amplitudes are suppressed in equal-mass mergers [32, 66]. Lastly, the combination of fundamental $n = 0$ and overtone $n = 1$ can also present non-negligible ratios \mathcal{R} , although these modes are expected to be more challenging to detect due to their shorter decay time.

Additionally, in fig. 2 we illustrate the impact of quadratic modes on a typical ringdown waveform for a GW150914-like merger. From the figure, it is evident that the quadratic oscillations occur at twice the

² We refer here to their results based on full NR simulations, which is still $\sim 10\%$ lower than their time evolution of second-order perturbation equations. Based on this observation we conjecture that the fits to second-order perturbation theory in [40] contains some source of systematic uncertainties, as was suggested in that article.

frequency of the linear modes, and the fact that the quadratic amplitude is proportional to the square of the linear one. Notice that, for the purpose of this figure, we have extrapolated our results to the spinning case by simply neglecting the spin dependence of the nonlinear ratio \mathcal{R} . This assumption is approximately supported by NR simulations [32, 40].

Conclusions In this letter we have provided the first *ab initio* computation of all low multipole QQNMs amplitudes using perturbation theory on a Schwarzschild BH background. Our findings should facilitate a straightforward integration of second-order nonlinearities into ring-down models, as QQNMs frequencies and amplitudes are now fully determined by their parent LQNMs. In addition to providing more accurate models for fitting to data or NR, our work enables to test GR by measuring the nonlinear ratio \mathcal{R} in data and comparing it with our prediction, provided that \mathcal{R} has a weak dependence on spin effects and deviations to reflection symmetry, as suggested by recent works [32, 41, 67]. We leave these aspects, as well as the extension of our results to Kerr backgrounds, to future investigations.

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Note added Our results are compatible with the recent related article [67], uploaded to the Arxiv just after ours, by setting $C_{\ell m}^- = 0$ in their work, which corresponds to our assumption of equatorial symmetry. While we computed the amplitude of several QQNMs, they concentrated on the most excited QQNM and studied devi-

ations from equatorial symmetry. Their approach could explain the discrepancies we mention when comparing our results to the literature.

Appendix A: Technical reference

Spherical harmonics. Following the literature, we define tensor spherical harmonics as

$$Y_A^{\ell m} = D_A Y^{\ell m}, \quad (\text{A1})$$

$$X_A^{\ell m} = -\epsilon_A^C D_C Y^{\ell m}, \quad (\text{A2})$$

$$Y_{AB}^{\ell m} = \left(D_A D_B + \frac{1}{2} \ell(\ell+1) \Omega_{AB} \right) Y^{\ell m}, \quad (\text{A3})$$

$$X_{AB}^{\ell m} = -\epsilon_{(A}^C D_{B)} D_C Y^{\ell m}, \quad (\text{A4})$$

where $Y^{\ell m}$ are the standard spherical harmonics, D_A and ϵ_{AB} are the covariant derivative and the Levi-Civita symbol on the unit sphere respectively.

Gauge-invariant perturbations. Following [60], we define

$$\tilde{h}_{ab}^{(i)} = h_{ab}^{(i)} + H_{ab}^{(i)} + 2D_{(a} \zeta_{b)}^{(i)}, \quad (\text{A5})$$

$$\tilde{h}_{a-}^{(i)} = h_{a-}^{(i)} + H_{a-}^{(i)} + r^2 D_a Z_-^{(i)}, \quad (\text{A6})$$

$$\tilde{h}_o^{(i)} = h_o^{(i)} + H_o^{(i)} + 2rf(r)\zeta_r^{(i)} - \ell(\ell+1)r^2 Z_+^{(i)}, \quad (\text{A7})$$

where $Z_{\pm}^{(i)} = -(h_{\pm}^{(i)} + H_{\pm}^{(i)})/(2r^2)$ and $\zeta_a^{(i)} = -h_{a+}^{(i)} - H_{a+}^{(i)} - r^2 D_a Z_+^{(i)}$. The various $H^{(i)}$ components can be found by decomposing a tensor $H_{\mu\nu}^{(i)}$ in spherical harmonics in the same way as eqs (5)-(7), where $H_{\mu\nu}^{(1)} = 0$ and $H_{\mu\nu}^{(2)} = \mathcal{L}_{\xi}(h_{\mu\nu}^{(1)} + \mathcal{L}_{\xi}\bar{g}_{\mu\nu}/2)$, where \mathcal{L} is the Lie derivative, $\xi_{\mu} = (\zeta_a^{(1)}, Z_A^{(1)})$ and $Z_A^{(1)} = Z_+^{(1)} Y_A + Z_-^{(1)} X_A$.

RWZ potentials. They are given by

$$V_- = f(r) \left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3} \right),$$

$$V_+ = f(r) \frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 M r^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r + 3M)^2}, \quad (\text{A8})$$

where $\lambda = (\ell+2)(\ell-1)/2$.

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