Labelled sequent calculi for logics of strict implication

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Abstract

In this paper we study the proof theory of C.I. Lewis' logics of strict conditional **S1**-**S5** and we propose the first modular and uniform presentation of C.I. Lewis' systems. In particular, for each logic **Sn** we present a labelled sequent calculus **G3Sn** and we discuss its structural properties: every rule is height-preserving invertible and the structural rules of weakening, contraction and cut are admissible. Completeness of **G3Sn** is established both indirectly via the embedding in the axiomatic system **Sn** and directly via the extraction of a countermodel out of a failed proof search. Finally, the sequent calculus **G3S1** is employed to obtain a syntactic proof of decidability of **S1**.

Keywords: Strict implication, non-normal modalities, S1, sequent calculi cut elimination.

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1 Introduction

Clarence Irving Lewis proposed the first axiomatic systems of propositional modal logic. In particular, due to his dissatisfaction towards the material conception of classical implication, he devised a new logical operator, namely strict implication. He introduced five systems from **S1** to **S5**. **S4** and **S5** have been intensively studied, whereas **S1**, **S2** and **S3** did not receive much attention.

It can be argue that this depended on the fact that the latter are non-normal modal logics, because the rule of necessitation does not hold unrestrictedly. The semantics of the systems **S2** and **S3** was obtained via a slight modification of the standard Kripke semantics, by considering models with non-normal (or queer) worlds.

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On the contrary, Chellas proposed a semantics for **S1** which combines features of neighborhood and relational models. Due to the rather complex formulation of the semantics **S1** was long considered as an uninteresting system. In our opinion, this position is not justified, insofar as the system exhibits some interesting metalogical properties such as decidability.

In the present work we shall focus on the proof theory of these systems. In a previous paper by one of the two authors labelled sequent calculi were introduced for the logics **S2**, **S3** and some related systems. However, a modular treatment is still lacking, due to the impossibility to encompass the system **S1**.

We propose the first modular and analytic approach to the proof theory of the original systems by C. I. Lewis. The sequent calculi are obtained by converting the truth conditions for the logical operators in corresponding rules. The calculi satisfy good structural properties, namely admissibility of the rules of weakening, contraction and cut.

Completeness is first established by showing the embedding of the axiomatic calculus into the corresponding labelled sequent calculus. The admissibility of the rule of substitution of strict equivalents requires to prove a non trivial lemma. We then establish strong completeness via the extraction of a countermodel out of a failed proof search.

Our proof-theoretic approach enables us to investigate the system **S1** by purely syntactic means which are uniform with respect to the ones traditionally employed for **S2** − **S5**. In particular, we exploit the calculus **G3S1** to obtain the first purely syntactic proof of decidability of the logic **S1** via terminating proof search. Also, the peculiar formulation of the rules of the calculus can be used to show completeness with respect to a bineighborhood semantics for **S1**.

2 Logics of strict implication

Language

The language of strict implications is defined by the following grammar:

$$A \ ::= \ p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid A \mathrel{-\!\!3} A \qquad (\mathcal{L}^{-\!3})$$

where $p \in \mathcal{P}$ for a denumerable set of sentential variables $\mathcal{P}$.

Parentheses are used as customary ($\mathrel{-\!\!3}$ binds lighter than other operators). Capital roman letters will be used for arbitrary formulas and lower-case ones for sentential variables. The symbol $\top$ is a short for $\bot \supset \bot$ and $\neg A$ is short for $A \supset \bot$. The unary modalities $\Box$ and $\Diamond$ can be defined as:

$$\Box A \ \equiv \ \top \mathrel{-\!\!3} A \qquad \text{and} \qquad \Diamond A \ \equiv \neg(\top \mathrel{-\!\!3} \neg A) \qquad (\text{Def}_{\Box})$$

We use $\mathcal{L}^{\Box}$ to denote the standard modal language—i.e., $\mathcal{L}^{-\!3}$ with $\Box$ and $\Diamond$ in place of $\mathrel{-\!\!3}$. The formula $A \mathrel{-\!\!3} B$ can be defined in $\mathcal{L}^{\Box}$ either as $\Box(A \supset B)$ or as $\neg\Diamond(A \wedge \neg B)$. Observe that languages $\mathcal{L}^{-\!3}$ and language $\mathcal{L}^{\Box}$ are not minimal since we have the usual classical and modal interdefinabilities—e.g., Lewis [**?**] considered a language with only $\neg$, $\wedge$ and $\Diamond$ as primitives.

We will use $A[B/p]$ for the formula obtained from A by replacing each occurrence of p with an occurrence of B and $A[B//C]$ for the formula obtained from A by replacing some occurrences of C with occurrences of B.

Axiomatic systems

We present here Lewis [**?**] axiomatisation of the logics **S1** and **S5**. As already anticipated Lewis considered a language with only $\neg$, $\wedge$ and $\Diamond$ as primitives. For simplicity in the following we assume to have the definition of the other symbols as implicit axioms. We simplify Lewis' axiomatisation of **S1** by dropping the redundant axiom $A \strictif \neg\neg A$—see [**?**]—and by considering axiom schemes instead of having as primitive a rule of uniform substitution of material equivalents.

Definition 2.1 [Lewis' axiomatisation of **S1**]

- Axioms:
  - (i) $(A \wedge B) \strictif (B \wedge A)$
  - (ii) $(A \wedge B) \strictif A$
  - (iii) $A \strictif (B \wedge C)$
  - (iv) $((A \wedge B) \wedge C) \strictif (A \wedge (B \wedge C))$
  - (v) $((A \strictif B) \wedge (B \strictif C)) \strictif (A \strictif C)$
  - (vi) $(A \wedge (A \strictif B)) \strictif B$
- Rules:
  - (i) $\dfrac{A \quad (B \strictif C) \wedge (C \strictif B)}{A[B//C]}\ SSE$
  - (ii) $\dfrac{A \quad B}{A \wedge B}\ Adj$
  - (iii) $\dfrac{A \strictif B \quad A}{B}\ MP_{\strictif}$

Definition 2.2 [Axiomatisation of **S2**–**S5**] $\mathbf{S2} = \mathbf{S1} \oplus \Diamond(A \wedge B) \strictif \Diamond A$; $\mathbf{S3} = \mathbf{S2} \oplus (A \strictif B) \strictif (\Box A \strictif \Box B)$; $\mathbf{S4} = \mathbf{S1} \oplus \Box A \strictif \Box\Box A$; $\mathbf{S5} = \mathbf{S4} \oplus A \strictif \Box\Diamond A$.

add Lemmon's axiom.?

Semantics

As it is well-known, standard relational semantics can be used for the normal conditional logics **S4** and **S5**. A modification thereof has been used by Kripke [**?**] to give a semantics for the non-normal conditional logics **S2** and **S3**: we must add so-called *queer* (or *non-normal*) *worlds* where $\Diamond A$ is always true and $\Box A$ is always false. Finally, a semantics for **S1** has been introduced by Cresswell in [**?**] and generalised to logics weaker than **S1** in [**?**]. This semantics is interesting is that it needs both an accessibility relation and a neighbourhood functions to define strict implication (as well- as modalities): in normal worlds we must use the accessibility relation and in queer ones we must use the neighbourhood one. [3]

Formally an **S1**-*frame* is quadruple $\mathcal{F} = \langle \mathcal{W}, \mathcal{N}, \mathcal{R}, \mathcal{I} \rangle$ where: (i) $\mathcal{W}$ is a non-empty set of *worlds*; (ii) $\mathcal{N}$ is a subset of $\mathcal{W}$, of so-called *normal worlds* (worlds in $\mathcal{W}/\mathcal{N}$ will be called *queer worlds*); (iii) $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is a reflexive *accessibility relation* on $\mathcal{W}$; (iv) $\mathcal{I} : \mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ is a *neighbourhood functions* mapping worlds to sets of sets of worlds with the side conditions that if $\alpha \in \mathcal{I}(w)$ then $w \in \alpha$—i.e., $\mathcal{I}$ is reflexive— and that if $X, Y \in I(w)$ then $X \cup Y \neq \mathcal{W}$.

[3] A semantics for **S1** based on Rantala models has been given in [**?**], and a relational semantics has been given in [**?**].

or $\{\emptyset\}$?

By adding conditions on $\mathcal{N}, \mathcal{R}$, and $\mathcal{I}$ we can define a class of frames for the other Lewis' systems. In particular: (i) an **S2**-*frame* is an **S1**-frame where $\mathcal{I}$ is such that it maps each world to $\emptyset$; (ii) an **S3**-*frame* is a transitive **S2**-frame—i.e., if $w\mathcal{R}v$ and $v\mathcal{R}u$ then $w\mathcal{R}u$; (iii) an **S4**-*frame* is an **S3**-frame where $\mathcal{N} = \mathcal{W}$; (iv) an **S5**-*frame* is a symmetric **S4**-frame—i.e., if $w\mathcal{R}v$ then $v\mathcal{R}w$. Some observations are in order. **S2**-frames can be equivalently defined by simply dropping $\mathcal{I}$ from **S1** fames, thus obtaining Kripke semantics for non-normal logics [**?**]. **S4** can be defined by dropping $\mathcal{N}$ and $\mathcal{I}$ from **S1**-frames, thus obtaining standard relational semantics for normal modalities.

A *model* $\mathcal{M}$ is a frame augmented with a *valuation function* $\mathcal{V} : \mathcal{W} \longrightarrow \mathcal{P}(\mathcal{W})$ mapping each sentential variables to the set of worlds where it holds/is true. We say that $\mathcal{M}$ is an **Sn**-*model* if its underlying frame is an **Sn**-frame.

We are now ready to define *truth* of a formula A at a world w of a model $\mathcal{M}$, $\models^{\mathcal{M}}_w A$ or simply $\models_w A$ when $\mathcal{M}$ is clear from the context. The definition is standard for sentential variables and for the extensional operators—e.g., $\models_w p$ iff $w \in \mathcal{V}(p)$ and $\models_w A \wedge B$ iff $\models_w A$ and $\models_w B$. The only interesting case is that of strict implication where we have:

$$\models_w A \dashv B \qquad \text{iff} \qquad \begin{cases} \forall v \in \mathcal{W}, w\mathcal{R}v \text{ and } \models_v A \text{ imply } \models_w B, & \text{if } w \in \mathcal{N} \\ [\![A \supset B]\!]_{\mathcal{M}} \in \mathcal{I}(w), & \text{else} \end{cases}$$

where $[\![A]\!]_{\mathcal{M}}$ is the truth set of A in $\mathcal{M}$: $[\![A]\!]_{\mathcal{M}} = \{w : \models^{\mathcal{M}}_w A\}$. Equivalently, we have that $\models_w A$ for $w \in \mathcal{W}/\mathcal{N}$ iff $\exists \alpha \in I(w)$ such that, for all $v \in \mathcal{W}$, ($\not\models_v A$ or $\models_w B$) if and only if $v \in \alpha$.

Observe that for **S2**- and **S3**-models the clause for queer worlds says that $A \dashv B$ cannot be true therein, and for **S4**- and **S5**-models it can be dropped.

A formula A is said to be: (i) *True in a model* $\mathcal{M}$, $\models^{\mathcal{M}} A$, if it true in every normal point of that model; (ii) **Sn**-*valid*, **Sn** $\models A$, if it is true in all **Sn**-models; (iii) *An* **Sn**-*consequence* of a set of formulas X, $X \models_{\mathbf{Sn}} A$, if A is true in all normal world of each **Sn**-model where all formulas in X are true.

Theorem 2.3 (Characterisation, [?]) *The axiomatic calculus* **Sn** *is sound and compete for validity w.r.t. the class of all* **Sn**-*frames.*

3 Labelled sequent calculi

We are now going to introduce labelled sequent calculi for the logics of strict implication **S1**-**S5**. Labelled calculi for normal modal logics have been introduced in [**?**] and for the non-normal ones in [**?**]. Labelled calculi for the non-normal logics **S2** and **S3**, as well as for some of their extensions, have been studied in [**?**]. The main novelties w.r.t. [**?**] is that here we consider also **S1** and we have a language with $\dashv$ instead of $\Box$ as primitive.

In order to define the language of sequent calculi we consider two denumerable and disjoint sets of labels: a set $\mathbb{W}$ of *world labels*, for which we use the metavariables $w, v, u, \ldots$, and a set $\mathbb{I}$ of *neighbours label*, to be denoted by $\alpha, \beta, \gamma, \ldots$. Moreover, we add the following atomic predicates $R, N, Q, \in$, and $\notin$ that are syntactic counterparts of the elements of **S1**-frames. he formulas of the labelled language $\mathcal{L}^{ll}$ are the following (where $w, v \in \mathbb{W}$, $\alpha \in \mathbb{I}$ and $A \in \mathcal{L}^{\dashv}$): (i) *relational atoms* wRv; (ii) *normality atoms* Nw; (iii) *queer atoms* Qw; (iv) *neighbour atoms* $\alpha \in Iw$; (v) *inclusion atoms* $w \in \alpha$; (vi) *exclusion atoms* $w \notin \alpha$; (vii) *labelled formulas* $w : A$; (viii) *forcing formulas* $\alpha \Vdash A$; and (ix) *covering formulas* $\alpha \lhd A$.

Definition 3.1 The *label* of a formula E in $\mathcal{L}^{ll}$ of form $u : A$ (resp. $\alpha \Vdash^{\forall} A$ or $\alpha \lhd A$) is u (resp. α) and is denoted by $l(E)$. The *pure part* of a labelled formula E is obtained removing from E the label and the forcing and is denoted by $p(E)$. The notion of *weight* is defined for labels and pure parts of formulas. For every u and for every a, $\mathtt{w}(u) = 0$ and $\mathtt{w}(\alpha) = 1$. The weight of a pure formula A, $\mathtt{w}(A)$ is defined as follows: $\mathtt{w}(p) = \mathtt{w}(\bot) = 1$, $\mathtt{w}(A \circ B) = max(\{\mathtt{w}(A), \mathtt{w}(B)\}) + 1$, where $\circ \in \{\wedge, \vee, \supset\}$, $\mathtt{w}(A \dashv B) = max(\{\mathtt{w}(A), \mathtt{w}(B)\}) + 2$. The *degree* of a labelled formula E is an ordered pair $deg(E) = (\mathtt{w}(p(E)), \mathtt{w}(l(E)))$. For relational formulas we stipulate $deg(u \in \alpha) = deg(\alpha \in I(u)) = deg(N(u)) = (0, 1)$. *Degrees* of labelled formulas are ordered lexicographically.

A *sequent* is an expression $\Gamma \Rightarrow \Delta$ where Γ is a finite multiset of $\mathcal{L}^{ll}$-formulas and Δ is a finite multiset of labelled, forcing, and covering formulas only. Substitutions of labels in an $\mathcal{L}^{ll}$-formula E, $E[v/u]$ and $E[\alpha/\beta]$, are defined as expected and it is extended to multisets by applying it componentwise.

The rules of the calculi **G3S1**–**G3S5** are given in Table 1: **G3S1** contains all initial sequent and all propositional, conditional, and relational rules. **G3S2** = **G3S1** plus rule $S2$. **G3S3** = **G3S2** plus rule *Trans*. **G3S4** = **G3S3** plus rule *Norm*. **G3S5** = **G3S4** plus rule *Sym*. Observe that the calculus **G3S2** (**G3S4**) is equivalent to the simpler calculus obtained by dropping rules $L/R \dashv_Q$ (and removing normality atoms from rule $L/R \dashv_N$) and all relational rules but Ref_R from the calculus **G3S1** (**G3S3**).

A **G3Sn**-*derivation* of a sequent $\Gamma \Rightarrow \Delta$ is a tree of sequents, whose leaves are initial sequents, whose root is $\Gamma \Rightarrow \Delta$, and which grows according to the rules of **G3Sn**. The *height* of a **G3Sn**-derivation is the number of nodes of its longest branch. We say that $\Gamma \Rightarrow \Delta$ is **G3Sn**-derivable (with height n), and we write **G3Sn** $\vdash^{(n)} \Gamma \Rightarrow \Delta$, if there is a **G3Sn**-derivation (of height at most n) of $\Gamma \Rightarrow \Delta$. A rule is said to be *(height-preserving) admissible* in **G3Sn**, if, whenever its premisses are **G3Sn**-derivable (with height at most n), also its conclusion is **G3Sn**-derivable (with height at most n). In each rule depicted in Table 1, Γ and Δ are called *contexts*, the formulas occurring in the conclusion are called *principal*, and those occurring in the premisses only are called *active*.

Lemma 3.2 *The following generalised initial sequents are* **G3Sn**-*derivable:*

$$E, \Gamma \Rightarrow \Delta, E$$

Proof By induction on the degree of the formula E: the rules are applied root-first since in each branch we reach a sequent with a formula occurring both in the antecedent and in the succedent and having lesser degree than E.$\square$

Table 1
Rules of the calculi **G3S1**–**G3S5**

Initial Sequents

$$\frac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\,{\scriptstyle Ax} \qquad \frac{}{w : \bot, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\bot}$$

$$\frac{}{Nw, Qw, \Gamma \Rightarrow \Delta}\,{\scriptstyle Ax_N} \qquad \frac{}{w \in \alpha, w \notin \alpha, \Gamma \Rightarrow \Delta}\,{\scriptstyle Ax_\in}$$

Propositional Rules

$$\frac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\wedge} \qquad \frac{\Gamma \Rightarrow \Delta, w : A \qquad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\,{\scriptstyle R\wedge} \qquad \frac{w : A, \Gamma \Rightarrow \Delta \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\,{\scriptstyle R\vee} \qquad \frac{\Gamma \Rightarrow \Delta, w : A \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \supset B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\supset} \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \supset B}\,{\scriptstyle R\supset}$$

Conditional Rules

$$\frac{Nw, wRv, w : A \mathrel{-\!\!\!3} B, \Gamma \Rightarrow \Delta, v : A \qquad v : B, Nw, wRv, w : A \mathrel{-\!\!\!3} B, \Gamma \Rightarrow \Delta}{Nw, wRv, w : A \mathrel{-\!\!\!3} B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L \mathrel{-\!\!\!3}_N}$$

$$\frac{u : A, wRu, Nw, \Gamma \Rightarrow \Delta, u : B}{Nw, \Gamma \Rightarrow \Delta, w : A \mathrel{-\!\!\!3} B}\,{\scriptstyle R \mathrel{-\!\!\!3}_N,\ u \text{ fresh}} \qquad \frac{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, \Gamma \Rightarrow \Delta}{Qw, w : A \mathrel{-\!\!\!3} B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L \mathrel{-\!\!\!3}_Q,\ \alpha \text{ fresh}}$$

$$\frac{Qw, \alpha \in Iw, \Gamma \Rightarrow \Delta, w : A \mathrel{-\!\!\!3} B, \alpha \Vdash A \supset B \qquad Qw, \alpha \in Iw, \Gamma \Rightarrow \Delta, w : A \mathrel{-\!\!\!3} B, \alpha \lhd A \supset B}{Qw, \alpha \in Iw, \Gamma \Rightarrow \Delta, w : A \mathrel{-\!\!\!3} B}\,{\scriptstyle R \mathrel{-\!\!\!3}_Q}$$

Relational rules

$$\frac{v : A, v \in \alpha, \alpha \Vdash A, \Gamma \Rightarrow \Delta}{v \in \alpha, \alpha \Vdash A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L \Vdash} \qquad \frac{u \in \alpha, \Gamma \Rightarrow \Delta, u : A}{\Gamma \Rightarrow \Delta, \alpha \Vdash A}\,{\scriptstyle R \Vdash,\ u \text{ fresh}}$$

$$\frac{v \notin \alpha, \alpha \lhd A, \Gamma \Rightarrow \Delta, v : A}{v \notin \alpha, \alpha \lhd A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L \lhd} \qquad \frac{u \notin \alpha, u : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha \lhd A}\,{\scriptstyle R \lhd,\ u \text{ fresh}} \qquad \frac{wRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref_R}$$

$$\frac{u \notin \alpha, u \notin \beta, \alpha \in Iw, \beta \in Iw, \Gamma \Rightarrow \Delta}{a \in Iw, b \in Iw, \Gamma \Rightarrow \Delta}\,{\scriptstyle S1,\ u \text{ fresh}} \qquad \frac{w \in a, a \in Iw, \Gamma \Rightarrow \Delta}{a \in Iw, \Gamma \Rightarrow \Delta}\,{\scriptstyle Ref_I}$$

$$\frac{Nw, \Gamma \Rightarrow \Delta \qquad Qw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Norm}$$

Additional rules

$$\frac{}{\alpha \in Iw, \Gamma \Rightarrow \Delta}\,{\scriptstyle S2} \qquad \frac{wRu, wRv, vRu\Gamma \Rightarrow \Delta}{wRv, vRu\Gamma \Rightarrow \Delta}\,{\scriptstyle Trans}$$

$$\frac{Nw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Norm} \qquad \frac{uRw, wRu, \Gamma \Rightarrow \Delta}{wRu, \Gamma \Rightarrow \Delta}\,{\scriptstyle Sym}$$

4 Structural properties

Lemma 4.1 (Substitution) **G3Sn** $\vdash^n \Gamma \Rightarrow \Delta$ *implies* **G3Sn** $\vdash^n \Gamma[v/u] \Rightarrow \Delta[v/u]$ *and* **G3Sn** $\vdash^n \Gamma[\alpha/\beta] \Rightarrow \Delta[\alpha/\beta]$.

Proof A standard induction on the height of the derivation $\mathcal{D}$ of the sequent $\Gamma \Rightarrow \Delta$. We apply to $\mathcal{D}$ the inductive hypothesis either twice or once— depending on whether the last rule instance *Rule* in $\mathcal{D}$ has a variable condition that clashes with the substitution or not— and then we conclude by applying an instance of *Rule*. □

Theorem 4.2 (Weakening) *Let* Σ *contains only labelled, forcing and covering formulas.* **G3Sn** $\vdash^n \Gamma \Rightarrow \Delta$ *implies* **G3Sn** $\vdash^n \Pi, \Gamma \Rightarrow \Delta, \Sigma$.

Proof By induction on the height of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$, possibly applying an (hp-admissible) instance of substitution if the last rule instance in $\mathcal{D}$ has a variable condition. □

Corollary 4.3 *If* A *is an axiom of the axiomatic system* **Sn** *then the sequent* $Nw \Rightarrow w : A$ *is* **G3Sn**-*derivable.*

Proof The proof is straightforward by a root-first application of the rules of the calculi, possibly using the admissiblity of weakening. We limit ourselves to considering axiom (iv).

$$\dfrac{Nw, Nu, wRu, u : A \mathrel{-\!\!3} B, u : B \mathrel{-\!\!3} C \Rightarrow u : A \mathrel{-\!\!3} C \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[...], v : B, v : A \Rightarrow v : B, v : C, [...]}{[...], v : A \Rightarrow v : B, v : B \supset C, [...]}\,\mathrm{L}\lhd}{[...] \Rightarrow v : A \supset B, v : B \supset C, [...]}\,\mathrm{R}\supset}{[...], v \notin \alpha, v \notin \beta, \alpha \lhd A \supset B, \beta \lhd B \supset C \Rightarrow u : A \mathrel{-\!\!3} C, v : A \supset B}\,\mathrm{L}\lhd}{[...], v \notin \alpha, v \notin \beta, \alpha \in Iu, \alpha \lhd A \supset B, \beta \in Iu, \beta \lhd B \supset C \Rightarrow u : A \mathrel{-\!\!3} C}\,\mathrm{L}\lhd}{[...], \alpha \in Iu, \alpha \lhd A \supset B, \beta \in Iu, \beta \Vdash B \supset C, \beta \lhd B \supset C \Rightarrow u : A \mathrel{-\!\!3} C}\,S1}{[...], Qu, \alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, u : B \mathrel{-\!\!3} C \Rightarrow u : A \mathrel{-\!\!3} C}\,\mathrm{L}\mathord{-\!\!3}_Q}{Qu, u : A \mathrel{-\!\!3} B, u : B \mathrel{-\!\!3} C \Rightarrow u : A \mathrel{-\!\!3} C}\,\mathrm{L}\mathord{-\!\!3}_Q}{\dfrac{\dfrac{Nw, wRu, u : A \mathrel{-\!\!3} B, u : B \mathrel{-\!\!3} C \Rightarrow u : A \mathrel{-\!\!3} C}{Nw, wRu, u : (A \mathrel{-\!\!3} B) \wedge (B \mathrel{-\!\!3} C) \Rightarrow u : A \mathrel{-\!\!3} C}\,\mathrm{L}\wedge}{Nw \Rightarrow w : ((A \mathrel{-\!\!3} B) \wedge (B \mathrel{-\!\!3} C)) \mathrel{-\!\!3} (A \mathrel{-\!\!3} C)}\,\mathrm{R}\mathord{-\!\!3}_N}\,Norm$$

The leftmost top-sequent is provable via applications of rules $\mathrm{R}\mathord{-\!\!3}_N$ and $\mathrm{L}\mathord{-\!\!3}_N$.□

Lemma 4.4 *Each rule of* **G3Sn** *is height-preserving invertible.*

Proof For the rules with repetition of the principal formulas in the premiss hp-invertibility follows from Theorem 4.2. For the other rules, if we are inverting w.r.t. the principal formula of the last rule instance in $\mathcal{D}$ there is nothing to prove. Else we reason by induction on $\mathcal{D}$, possibly applying Lemma 4.1. To illustrate, assume we are inverting rule$R \mathrel{-\!\!3}_N$ and the last rule instance in $\mathcal{D}$ is the following instance of $R \Vdash$:

$$\dfrac{u \in \alpha, \Gamma \Rightarrow \Delta', w : B \mathrel{-\!\!3} C, u : A}{\Gamma \Rightarrow \Delta', \alpha \Vdash A, w : B \mathrel{-\!\!3} C}\; R \Vdash,\ u \text{ fresh}$$

We transform $\mathcal{D}$ into the following derivation having at most the same height:

$$\dfrac{\dfrac{\dfrac{u \in \alpha, \Gamma \Rightarrow \Delta', w : B \mathrel{-\!\!3} C, u : A}{u' \in \alpha, \Gamma \Rightarrow \Delta', w : B \mathrel{-\!\!3} C, u' : A}\,[u'/u]}{w' : B, wRw', u' \in \alpha, \Gamma \Rightarrow \Delta', u' : A, w' : C}\,IH}{w' : B, wRw', \Gamma \Rightarrow \Delta', \alpha \Vdash A, w' : C}$$

where the substitutions are needed if $w' \equiv u$. □

Theorem 4.5 (Contraction) **G3Sn** $\vdash^n \Pi, \Pi, \Gamma \Rightarrow \Delta, \Sigma, \Sigma$ *implies* **G3Sn** $\vdash^n \Pi, \Gamma \Rightarrow \Delta, \Sigma$.

Proof By induction on the height of the derivation $\mathcal{D}$ of $\Pi, \Pi, \Gamma \Rightarrow \Delta, \Sigma, \Sigma$, where one of Π and Σ is a singleton and the other is empty. The theorem follows by induction on the number of formulas in Π, Σ. We consider only the cases where we are contracting an occurrence of $w : A \mathrel{-\!\!3} B$ on the left.

Let's assume the conclusion of $\mathcal{D}$ is $w : A \mathrel{-\!\!3} B, w : A \mathrel{-\!\!3} B, \Gamma' \Rightarrow \Delta$. if no instance of $w : A \mathrel{-\!\!3} B$ is principal in the last rule *Rule* applied in $\mathcal{D}$ then we apply the inductive hypothesis to its premiss and an instance of *Rule*. Else, we have two cases depending on whether *Rule* is an instance of $L \mathrel{-\!\!3}_N$ or of $L \mathrel{-\!\!3} Q$. In the former case we can proceed as when no instance of $w : A \mathrel{-\!\!3} B$ is principal since $L \mathrel{-\!\!3}_N$ is a rule with repetition of the principal formulas. In the latter case we transform

$$\dfrac{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, w : A \mathrel{-\!\!3} B, \Gamma'' \Rightarrow \Delta}{Qw, w : A \mathrel{-\!\!3} B, w : A \mathrel{-\!\!3} B, \Gamma'' \Rightarrow \Delta}\; L \mathrel{-\!\!3}_Q,\ \alpha \text{ fresh}$$

into the following derivation of at most the same height:

$$\dfrac{\dfrac{\dfrac{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, w : A \strictif B, \Gamma'' \Rightarrow \Delta}{\beta \in Iw, \beta \Vdash A \supset B, \beta \lhd A \supset B, \alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, \Gamma'' \Rightarrow \Delta}\ \text{Lem.4.4}}{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, \Gamma'' \Rightarrow \Delta}\ \text{Lem.4.1}}{\dfrac{\alpha \in Iw, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, Qw, \Gamma'' \Rightarrow \Delta}{Qw, w : A \strictif B, \Gamma'' \Rightarrow \Delta}\ L\strictif_Q}\ IH$$

where both α and β do not occur in the conclusion. □

Theorem 4.6 (Cut) *Let E be a relational, forcing, or covering formula. The following rule of cut is admissible in* **G3Sn***:*

$$\dfrac{\Gamma \Rightarrow \Delta, E \qquad E, \Pi \Rightarrow \Sigma}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ Cut$$

Proof We consider an uppermost instance of *Cut* and we proceed by induction on the degree of the cut-formula with a sub-induction on the cut-height of $\mathcal{D}$—i.e., the sum of the heights of the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of the two premisses. The theorem then follows by induction on the number of cuts in the derivation.

As usual it is convenient to divide the proof in three exhaustive cases: in case (i) one premiss has a derivation of height 1; in case (ii) the cut-formula is not principal in the last step of at least one of the two premisses; in case (iii) the cut-formula is principal in the last step of both premisses.

The proof of cases (i) and (ii) as well as the sub-cases of case (iii) where the principal operator of the cut-formula is in $\wedge, \vee, \rightarrow$, are standard and can thus be omitted. The proof of the sub-cases of (iii) when the cut-formula has shape $\alpha \vdash A$ or $\alpha \lhd A$ can be found in [**?**]. Hence, we have to consider only the sub-cases of (iii) where the cut-formula has shape $w : B \strictif C$ and either the multiset Nw, wRv or $Qw, \alpha \in Iw$ occurs in Γ.

In the first case suppose $\mathcal{D}$ is a follows (for u not in the conclusion):

$$\dfrac{\dfrac{\begin{matrix}\vdots\mathcal{D}_{11}\\ u : B, wRu, Nw, \Gamma' \Rightarrow \Delta, u : C\end{matrix}}{Nw, \Gamma' \Rightarrow \Delta, w : B \strictif C}\ R\strictif_N \qquad \dfrac{\begin{matrix}\vdots\mathcal{D}_{21}\\ Nw, wRv, w : B \strictif C, \Pi' \Rightarrow \Sigma, v : B\end{matrix} \qquad \begin{matrix}\vdots\mathcal{D}_{22}\\ v : C, Nw, wRv, w : B \strictif C, \Pi' \Rightarrow \Sigma\end{matrix}}{w : B \strictif C, Nw, wRv, \Pi' \Rightarrow \Sigma}\ L\strictif_N}{Nw, Nw, wRv, \Pi', \Gamma' \Rightarrow \Delta', \Sigma}\ Cut$$

We transform it into the following derivation ($[\Gamma]^n$ stands for n copies of Γ, and, for the sake of space, we omit the premisses of dotted inferences):

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\mathcal{D}_1 \quad \vdots\mathcal{D}_{21}\end{matrix}}{[Nw]^2, wRv, \Pi', \Gamma' \Rightarrow \Delta, \Sigma, v : B}\ Cut_j \qquad \dfrac{\vdots\mathcal{D}_{11}}{v : B, wRv, Nw, \Gamma' \Rightarrow \Delta, v : C}\ Lem.4.1}{[Nw]^3, [wRv]^2, \Pi', [\Gamma']^2 \Rightarrow [\Delta]^2, \Sigma, v : C}\ Cut_i \qquad \dfrac{\begin{matrix}\vdots\mathcal{D}_1 \quad \vdots\mathcal{D}_{22}\end{matrix}}{v : C, [Nw]^2, wRv, \Pi', \Gamma' \Rightarrow \Delta, \Sigma}\ Cut_j}{\dfrac{[Nw]^5, [wRv]^3, [\Pi']^2, [\Gamma']^3 \Rightarrow [\Delta]^3, [\Sigma]^2}{Nw, wRv, \Pi, \Gamma \Rightarrow \Delta, \Sigma}\ Lem.4.5}\ Cut_i$$

where instances of *Cut* with subscript i (j) are admissible by the (sub-)induction hypothesis.

Finally, if $\mathcal{D}$ is a follows (for β not in the conclusion):

$$\dfrac{\dfrac{\begin{matrix}\vdots\mathcal{D}_{11}\\ Qw, \beta \in Iw, \Gamma' \Rightarrow \Delta, w : B \strictif C, \beta \Vdash B \supset C\end{matrix} \qquad \begin{matrix}\vdots\mathcal{D}_{12}\\ Qw, \beta \in Iw, \Gamma' \Rightarrow \Delta, w : B \strictif C, \beta \lhd B \supset C\end{matrix}}{Qw, \beta \in Iw, \Gamma' \Rightarrow \Delta, w : B \strictif C}\ R\strictif_Q \qquad \dfrac{\begin{matrix}\vdots\mathcal{D}_{21}\\ \alpha \in Iw, \alpha \Vdash B \supset C, \alpha \lhd B \supset C, Qw, \Pi' \Rightarrow \Sigma\end{matrix}}{Qw, w : B \strictif C, \Pi' \Rightarrow \Sigma}\ L\strictif_Q}{Qw, \alpha \in Iw, \Pi', \Gamma' \Rightarrow \Delta, \Sigma}\ Cut$$

we transform it into the following derivation ($\mathcal{D}_{1i}[\star]$ stands for the derivation $\mathcal{D}_{1i}$ with α in place of β by an instance of Lemma 4.1, and D stands for $B \supset C$):

$$\dfrac{\dfrac{\dfrac{\vdots\mathcal{D}_{12}[\star] \qquad \vdots\mathcal{D}_2}{[Qw]^2, \alpha \in Iw, \Pi', \Gamma' \Rightarrow \Delta, \Sigma, \alpha \lhd D}\;{\scriptstyle Cut_j} \qquad \dfrac{\dfrac{\vdots\mathcal{D}_{11}[\star] \qquad \vdots\mathcal{D}_2}{[Qw]^2, \alpha \in Iw, \Pi', \Gamma' \Rightarrow \Delta, \Sigma, \alpha \Vdash D}\;{\scriptstyle Cut_j} \qquad \begin{matrix}\vdots\mathcal{D}_{21}\\ \alpha \Vdash D, \alpha \lhd D, \alpha \in Iw, Qw, \Pi' \Rightarrow \Sigma\end{matrix}}{\alpha \lhd D, [Qw]^3, [\alpha \in Iw]^2, [\Pi']^2, \Gamma' \Rightarrow \Delta, [\Sigma]^2}\;{\scriptstyle Cut_i}}{[Qw]^5, [\alpha \in Iw]^3, [\Pi']^3, [\Gamma']^2 \Rightarrow [\Delta]^2, [\Sigma]^3}\;{\scriptstyle Cut_i}}{Nw, \alpha \in Iw, \Pi, \Gamma \Rightarrow \Delta, \Sigma}\;{\scriptstyle Thm.4.5}$$

□

Corollary 4.7 *The rule $MP_{⥽}$ is* **G3Sn***-admissible:*

$$\dfrac{Nw \Rightarrow w : A ⥽ B \qquad Nw \Rightarrow w : A}{Nw \Rightarrow w : B}\; Det.$$

Proof By applying Lemma 4.4 to $Nw \Rightarrow w : A ⥽ B$ we obtain the derivability of $u : A, wRu, Nw \Rightarrow u : B$ for some fresh label u. By an instance of Lemma 4.1 this becomes $w : A, wRw, Nw \Rightarrow w : B$ and, by a *Cut* with $Nw \Rightarrow w : A$, we obtain $wRw, Nw, Nw \Rightarrow w : B$. Finally, we apply an instance of Rule Ref_R and one of Theorem 4.5 to conclude that $Nw \Rightarrow w : B$ is derivable. □

Corollary 4.8 **G3Sn***-derivations are analytic—i.e., every label occurring in a derivation either occurs in its conclusion or it is an* eigenvariable. needed?

Proof See [**?**, Lemma 3.17]. □

Lemma 4.9 *For every formula A, B and C, if the sequents $w : A \Rightarrow w : B$ and $w : B \Rightarrow w : A$ are derivable in* **G3Sn***, then the sequents:*

$$w : C \Rightarrow w : C[A//B] \text{ and } w : C[A//B] \Rightarrow w : C$$

are provable in **G3Sn**.

Proof The proof runs by induction on the weight of the formula C. We assume that $C \neq A$, otherwise the proof is trivial. If C is a sentential variable p, the claim is trivial. If C is a conjunction, a disjunction or a formula of the shape $D \supset E$, then the proof easily follows by applying the induction hypothesis. We discuss the case in which C is of the form $D ⥽ E$. Since $C \neq A$, we have $(D ⥽ E)[A//B] \equiv D[A//B] ⥽ E[A//B]$.

We first show that $Nw, w : D ⥽ E \Rightarrow w : D[A//B] ⥽ E[A//B]$ is derivable.

$$\dfrac{\dfrac{[...], w : D ⥽ E, u : D[A//B] \Rightarrow u : E[A//B], u : D \qquad [...], w : D ⥽ E, u : D[A//B], u : E \Rightarrow u : E[A//B]}{Nw, wRu, w : D ⥽ E, u : D[A//B] \Rightarrow u : E[A//B]}\;{\scriptstyle \text{L}⥽_N}}{Nw, w : D ⥽ E \Rightarrow w : D[A//B] ⥽ E[A//B]}\;{\scriptstyle \text{R}⥽_N}$$

The derivability of the topmost sequents follows from the induction hypothesis and weakening. The sequent $Qw, w : D ⥽ E \Rightarrow w : D[A//B] ⥽ E[A//B]$ is derivable too.

$$\dfrac{\dfrac{\dfrac{\dfrac{[...], o \in \alpha, o : D \supset E \Rightarrow o : D[A//B] \supset E[A//B]}{[...], o \in \alpha, \alpha \Vdash D \supset E \Rightarrow o : D[A//B] \supset E[A//B]}\;{\scriptstyle \text{L}\Vdash}}{[...], \alpha \Vdash D \supset E \Rightarrow \alpha \Vdash D[A//B] \supset E[A//B]}\;{\scriptstyle \text{R}\Vdash} \qquad \dfrac{\dfrac{[...], u : D[A//B] \supset E[A//B], \alpha \lhd D \supset E \Rightarrow u : D \supset E}{[...], u \notin \alpha, u : D[A//B] \supset E[A//B], \alpha \lhd D \supset E \Rightarrow}\;{\scriptstyle \text{L}\lhd}}{[...], \alpha \lhd D \supset E \Rightarrow \alpha \lhd D[A//B] \supset E[A//B]}\;{\scriptstyle \text{R}\lhd}}{Qw, \alpha \in Iw, \alpha \lhd D \supset E, \alpha \Vdash D \supset E \Rightarrow w : D[A//B] ⥽ E[A//B]}\;{\scriptstyle \text{R}⥽_Q}}{Qw, w : D ⥽ E \Rightarrow w : D[A//B] ⥽ E[A//B]}\;{\scriptstyle \text{L}⥽_Q}$$

The derivability of the topmost sequents follows from application of the rules L⊃, R⊃, the induction hypothesis and weakening. The desired conclusion follows from an application of $Norm$.

We now discuss the other part of the claim, i.e. $w : D[A//B] \strictif E[A//B] \Rightarrow w : D \strictif E$. We first show that $Nw, w : D[A//B] \strictif E[A//B] \Rightarrow w : D \strictif E$:

$$\dfrac{\dfrac{[...], u : D \Rightarrow u : D[A//B], u : E \qquad [...], , u : E[A//B], u : D \Rightarrow u : E}{[...], w : D[A//B] \strictif E[A//B], u : D \Rightarrow u : E}\,\mathrm{L}{\strictif}_N}{Nw, w : D[A//B] \strictif E[A//B] \Rightarrow w : D \strictif E}\,\mathrm{R}{\strictif}_N$$

Again, the derivability of the topmost sequents follows from the induction hypothesis and weakening. For the other direction we proceed as follows (we omit to display redundant repetition of formulas):

$$\dfrac{\dfrac{\dfrac{[...], o : D[A//B] \supset E[A//B] \Rightarrow o : D \supset E, [...]}{[...], o \in \alpha, \alpha \Vdash D[A//B] \supset E[A//B] \Rightarrow \alpha \Vdash D \supset E, [...]}\,\mathrm{L}{\Vdash} \qquad \dfrac{\dfrac{[...], u : D \supset E \Rightarrow u : D[A//B] \supset E[A//B], [...]}{[...], u \notin \alpha, u : D \supset E, \alpha \lhd D[A//B] \supset E[A//B] \Rightarrow [...]}\,\mathrm{L}{\lhd}}{[...], \alpha \lhd D[A//B] \supset E[A//B] \Rightarrow \alpha \lhd D \supset E, [...]}\,\mathrm{R}{\lhd}}{Qw, \alpha \in Iw, \alpha \Vdash D[A//B] \supset E[A//B], \alpha \lhd D[A//B] \supset E[A//B] \Rightarrow w : D \strictif E}\,\mathrm{R}{\strictif}_Q}{Qw, w : D[A//B] \strictif E[A//B] \Rightarrow w : D \strictif E}\,\mathrm{L}{\strictif}_Q$$

The topmost sequents are derivable via applications of the rules R⊃, L⊃, the induction hypothesis and admissibility of weakening.

□

We shall now prove the admissibility of the rule of substitution of strict equivalents.

Corollary 4.10 *The rule of substitution of strict equivalents is* **G3Sn***-admissible:*

$$\dfrac{Nw \Rightarrow w : A \qquad Nw \Rightarrow w : B \strictif C \qquad Nw \Rightarrow w : C \strictif B}{Nw \Rightarrow w : A[B//C]}\,SSE$$

Proof We assume that we have a proof of the sequents $Nw \Rightarrow w : A$ and $Nw \Rightarrow w : (B \strictif C) \wedge (C \strictif B)$. By invertibility of the rule R∧ we get the derivations of $Nw \Rightarrow w : B \strictif C$ and $Nw \Rightarrow w : C \strictif B$.

We apply again the invertibility of the rule R$\strictif$ we get $Nw, wRu, u : B \Rightarrow u : C$ and $Nw, wRu, u : C \Rightarrow u : B$. We observe that the normality atoms and the relational atoms are never active in a derivation, so we can remove them.

So the sequents $u : C \Rightarrow u : B$ and $u : B \Rightarrow u : C$ are derivable and we can apply Lemma 4.9 which yields $w : A \Rightarrow w : A[B//C]$. Finally, a cut gives the desired result. □

We are now in the position to state and prove the embedding of the axiomatic calculi **Sn** into **G3Sn**.

Theorem 4.11 *If* **Sn** $\vdash A$*, then* **G3Sn** $\vdash Nw \Rightarrow w : A$.

Proof The proof runs by induction on the height of the derivation in the axiomatic calculi **Sn**. The axioms are derivable by Lemma **??**. The rule *Adj* is admissible by rule R∧. The admissibility of $MP_{\strictif}$ is a consequence of the Corollary 4.7, and that of *SSE* follows from Theorem 4.10. □

5 Characterisation

We will now propose an alternative and more direct form of completeness which is obtained by extracting a countermodel out of a failed proof search. We start by defining the notion of validity of labelled sequents [**?**].

Definition 5.1 Given a set S of world labels w and a set L of neighborhood labels a, and an **Sn** model $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{R}, \mathcal{I}, v \rangle$, an SN realisation (ρ, σ) is a pair of functions mapping each $w \in S$ into $\rho(x) \in W$ and mapping each $a \in NL$ into $\sigma(a) \in Nw$ for some $w \in W$. We introduce the notion $\mathcal{M}$ *satisfies a formula E under an SN-realisation* (σ, ρ) and denote it by $\mathcal{M} \vDash_{\rho,\sigma} E$, where we assume that the labels in E occur in S, NL. The definition extends by cases on the form of E, we give some examples:

- $\mathcal{M} \vDash_{\rho,\sigma} w \in \alpha$ if $\rho(w) \in \sigma(\alpha)$.
- $\mathcal{M} \vDash_{\rho,\sigma} w : A$ if $\vDash_{\rho(w)} A$
- $\mathcal{M} \vDash_{\rho,\sigma} \alpha \lhd A$ if for all u s. t. $\mathcal{M} \vDash_{\rho,\sigma} u : A$, $\rho(u) \in \sigma(\alpha)$.
- $\mathcal{M} \vDash_{\rho,\sigma} w : A \multimap B$ if either $\rho(w) \in \mathcal{N}$ and for every $u \in \mathcal{W}$ such that $\mathcal{M} \vDash_{\rho,\sigma} w\mathcal{R}u$, then $\mathcal{M} \vDash_{\rho,\sigma} w : A \supset B$, or $\rho(w) \notin \mathcal{N}$ and for some α, $\sigma(\alpha) \in \mathcal{I}(\rho(w))$, $\mathcal{M} \vDash_{\rho,\sigma} \alpha \lhd A \supset B$ and $\mathcal{M} \vDash_{\rho,\sigma} \alpha \Vdash A \supset B$.

Given a sequent $\Gamma \Rightarrow \Delta$, let S, NL be the sets of worlds and neighborhood labels occurring in $\Gamma \cup \Delta$, and let (ρ, σ) be an SN-realisation; we define $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ to hold if whenever $\mathcal{M} \vDash_{\rho,\sigma} E$ for all formulas $E \in \Gamma$ then $\mathcal{M} \vDash_{\rho,\sigma} \psi$ for some formula $\psi \in \Delta$. We further define $\mathcal{M}$-validity by:

$$\mathcal{M} \vDash \Gamma \Rightarrow \Delta \text{ iff } \mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta \text{ for every } SN\text{-realisation } (\rho, \sigma).$$

We finally say that a sequent $\Gamma \Rightarrow \Delta$ is valid in a **Sn** frame if $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ for every model based on it.

Theorem 5.2 (Soundness) *If* **G3Sn** $\vdash \Gamma \Rightarrow \Delta$*, then* $\Gamma \Rightarrow \Delta$ *is* **Sn***-valid.*

Proof By induction on the height of the derivations in the calculus **G3Sn**. □

We introduce the notion of saturated sequent in a derivation. For every branch in a derivation we write $\downarrow \Gamma$ ($\downarrow \Delta$) to denote the union of the antecedents (succedents) in the branch from the endsequent up to the sequent $\Gamma \Rightarrow \Delta$.

Definition 5.3 A branch in a proof search in the system **G3S1** from the endsequent up to the sequent $\Gamma \Rightarrow \Delta$ is *saturated* if, for every rule R, if the principal formulas of R occur in the branch, the formulas introduced by one of the premises of R also occur in the branch. In detail, a saturated branch up to $\Gamma \Rightarrow \Delta$ has to satisfy the following conditions (we omit some of them): (Ax) There is no sentential variable p such that $w : p \in \Gamma \cap \Delta$. (Ax$_C$) There are no α, w such that $w \in \alpha, w \in \alpha, \in \Gamma$. (Ax$_N$) There is no w such that $Nw, Qw, \in \Gamma$. (L$\bot$) It is not the case that $w : \bot \in \Gamma$ for every w. (L-$\multimap_Q$) If Qw and $w : A \multimap B \in \downarrow \Gamma$, then for some α $\alpha \in Iw$, $\alpha \lhd A \supset B$ and $\alpha \Vdash A \supset B$ are in $\downarrow \Gamma$. (R-$\multimap_Q$) If Qw, $\alpha \in Iw$ are in $\downarrow \Gamma$ and $w : A \multimap B \in \downarrow \Delta$, then $\alpha \lhd A \supset B \in \downarrow \Delta$ or $\alpha \Vdash A \supset B \in \downarrow \Delta$. The notion of saturated sequent is extended to the systems **G3Sn** by adding conditions relative to the additional rules.

Given a sequent $\Gamma \Rightarrow \Delta$ we build a *proof search tree* by applying all possible rules of the calculus. To avoid repetitions, we fix a counter. At stage 1 we apply rule L$\wedge$, at stage 2 the rule R$\wedge$ and so forth. There are 20 + m different stages (where m is the number of relational rules depending on the system). At stage 20 + m + 1 we start again. If the construction ends we obtain a derivation or a finite tree in which a branch is saturated, otherwise we obtain an infinite tree. By König's Lemma there is an infinite branch which is saturated from which we can extract a countermodel.

Theorem 5.4 *Given a saturated branch $\mathcal{B}$ in a proof search tree for the sequent $\Gamma \Rightarrow \Delta$ built according to the rules of system* **G3Sn***, we can extract a countermodel $\mathcal{M}$ to $\Gamma \Rightarrow \Delta$ based on a* **Sn***-frame.*

Proof Given a saturated branch $\mathcal{B}$ in a proof search tree we define the following countermodel: $\langle \mathcal{W}, \mathcal{R}, \mathcal{I}, \mathcal{V} \rangle$ such that:

- $\mathcal{W}$ is the set of all world labels occurring in Γ.
- $w\mathcal{R}u$ if and only if wRu occurs in Γ.
- $\mathcal{I}(w)$ is the set of all the neighbours α such that $\alpha \in Iw$ occurs in Γ and every α consists of all the worlds w such that $w \in \alpha$ occur in Γ.
- $\mathcal{V}(p)$ is the set of all worlds w such that $w : p$ occurs in Γ.

Notice that $\mathcal{V}$ is well defined by condition *Init*. For every system **G3Sn**, the frame $\langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle$ satisfies the properties of **Sn**-frames by the saturation conditions regarding relational and additional rules. We define the realization (ρ, σ) such that $\rho(w) \equiv w$ and $\sigma(\alpha) \equiv \alpha$. We claim that:

(i) If $w : A$ is in Γ, then $\mathcal{M} \vDash_{\rho,\sigma} w : A$.

(ii) If $w : A$ is in Δ, then $\mathcal{M} \nvDash_{\rho,\sigma} w : A$.

The proof is by simultaneous induction on the degree of A. We focus on the case of strict implication.

(a) If $w : A \multimap B$ is in Γ, then by the saturation condition there is either Qw or Nw in Γ. In the first case, again by the saturation condition, there are $\alpha \in Iw$, $\alpha \lhd A \supset B$ and $\alpha \Vdash A \supset B$ in Γ. By definition of $\mathcal{M}$ and induction hypothesis we have $\alpha \in \mathcal{I}(w)$, $\mathcal{M} \vDash_{\rho,\sigma} \alpha \lhd A \supset B$ and $\mathcal{M} \vDash_{\rho,\sigma} \alpha \Vdash A \supset B$, therefore $\mathcal{M} \vDash_{\rho,\sigma} w : A \multimap B$. In the second case, we distinguish two subcases. If there is no label u such that wRu occurs in Γ, then the claim trivially follows. Otherwise for every u sucht that wRu occurs in Γ, by the saturation condition either $u : A$ is in Δ or $u : B$ is in Γ. By induction hypothesis we get $\mathcal{M} \nvDash_{\rho,\sigma} u : A$ or $\mathcal{M} \vDash_{\rho,\sigma} u : B$. Therefore we get $\mathcal{M} \vDash_{\rho,\sigma} w : A \multimap B$.

(b) If $w : A \multimap B$ is in Δ, then by the saturation condition there is either Qw or Nw in Γ. In the first case, by the saturation condition, for every $\alpha \in \mathcal{I}(w)$, there is $\alpha \lhd A \supset B$ or $\alpha \Vdash A \supset B$ in Δ. In both cases by induction hypothesis it follows $\mathcal{M} \nvDash_{\rho,\sigma} w : A \multimap B$. In the second case, by saturation there are $wRu, u : A \in \Gamma$ and $u : B$ in Δ. By induction hypothesis we get $\mathcal{M} \vDash_{\rho,\sigma} u : A$ and $\mathcal{M} \nvDash_{\rho,\sigma} u : B$, which yields $\mathcal{M} \vDash_{\rho,\sigma} w : A \dashv B$.

□

Corollary 5.5 (Completeness) *For every formula A:*

$$\mathbf{Sn} \vDash A \text{ if and only if } \mathbf{G3Sn} \vdash Nw \Rightarrow w : A$$

Proof The direction from right to left is the content of the soundness theorem. For the other direction we prove the contrapositive. Suppose that $\mathbf{G3Sn} \nvdash Nw \Rightarrow w : A$, hence there is a saturated branch and we can extract a **Sn**-countermodel for $Nw \Rightarrow w : A$, which gives $\mathbf{Sn} \nvDash A$. □

6 Decidability

Per il momento questa sezione è soltanto abbozzata, ma intanto ho buttato giù qualche idea. Given an endsequent $Nw \Rightarrow w : A$, we build a branch by backward applications of the rules. The branch is a sequence $Nw \Rightarrow w : A \equiv \Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, \ldots$ where $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is obtained from $\Gamma_i \Rightarrow \Delta_i$ with an application of a rule R.

To establish decidability we need to show that the search for a derivation can be interrupted at a certain point and that we can extract a finite countermodel.

First we prove some preliminary lemmata which are easily proved via height-preserving admissibility of the rule of contraction.

Lemma 6.1 *The rules Ref_R, Ref_I, $S1$, $Norm$, $L\Vdash$, $L\lhd$, $L\text{-}\!\!\dashv_N$ and $R\text{-}\!\!\dashv_Q$ need not be instantiated more than once on the same label in every branch in a proof search.*

The dynamic rules are R$\Vdash$, $S1$, R$\lhd$, L$\dashv_Q$ or R$\dashv_N$. We now introduce some definitions which allow us to check the relations between world labels and neighbourhood labels.

Definition 6.2 In a branch $\mathcal{B}$ of a proof search tree of the sequent $Nw \Rightarrow w : A$ we define the relation $\rightarrow_{\mathcal{B}}$ of *immediate successor* (for $w, v \in \mathbb{W}$ and $\alpha \in \mathbb{I}$): (i) $w \rightarrow_{\mathcal{B}} \alpha$ if $\alpha \in Iw$ occurs in $\mathcal{B}$; (ii) $\alpha \rightarrow_{\mathcal{B}} w$ if in $\mathcal{B}$ there is either $w \in \alpha$ or $w \notin \alpha$, and there is not $\alpha \in Iw$; (iii) $w \rightarrow_{\mathcal{B}} u$ if some α is such that $w \rightarrow_{\mathcal{B}} \alpha \rightarrow_{\mathcal{B}} u$ **or** wRu is in $\mathcal{B}$.

Fact. The transitive closure of $\rightarrow_{\mathcal{B}}$ defines a tree which, as it is easy to check, does not contain cycles (except for the reflexive relations).

Fact. The immediate successors of a world label in an open brach of a proof search tree are either all neighbourhood labels or world labels, but not both. This depends on the fact that every world label w such that is either Nw or Qw occurs in a branch $\mathcal{B}$. eliminable?

Theorem 6.3 *Each label in an branch $\mathcal{B}$ of a proof search tree of an endsequent $Nw \Rightarrow w : A$ has only a finite number of immediate successors.*

Proof We observe that immediate successors of a label can be introduced only by applications of the dynamic rules R$\Vdash$, $S1$, R$\lhd$, L$\dashv_Q$ or R$\dashv_N$. The is Cor.4.8 useful?

subformulas of the formula A are finite, therefore if there were infinite immediate successors there would be more than one application of one of the above mentioned rules to the same principal labelled formulas.

We show that every derivation can be transformed in a proof search in which every branch contains at most one application of such rules to the same principal labelled formulas. We detail the case of L-$\prec_Q$ as an example.

$$\dfrac{Nu, \beta \in Iu, \beta \Vdash A \supset B, \beta \lhd A \supset B, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}{Nu, u : A \prec B, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}\ \text{L-}\prec_Q$$

$$\vdots\ \mathcal{D}$$

$$\dfrac{Nu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}{Nu, u : A \prec B, \Gamma \Rightarrow \Delta}\ \text{L-}\prec_Q$$

We transform the derivation as follows:

$$\dfrac{\dfrac{\dfrac{Nu, \beta \in Iu, \beta \Vdash A \supset B, \beta \lhd A \supset B, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}{Nu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}\ \text{Lem.4.1}}{Nu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}\ \text{Thm.4.5}}{Nu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, u : A \prec B, \Gamma \Rightarrow \Delta}\ \text{Thm.4.2}$$

$$\vdots\ \mathcal{D}$$

$$\dfrac{Nu, \alpha \in Iu, \alpha \Vdash A \supset B, \alpha \lhd A \supset B, \Gamma \Rightarrow \Delta}{Nu, u : A \prec B, \Gamma \Rightarrow \Delta}\ \text{L-}\prec_Q$$

The application of the hp-admissible rules of substitution, contraction and weakening does not introduce new applications of L-$\prec$ (this can be easily checked). □

As a consequence, the tree defined by $\rightarrow_{\mathcal{B}}$ is finitely branching. The second part of the proof of termination consists in showing that in every branch the length of a chain of labels is finite. The proof for **S1** and **S2** is relatively simple. The key point is that the relation defined by $\rightarrow_{\mathcal{B}}$ is not transitive. In particular, this means that a label *sees* only its immediate successors and itself (by reflexivity).

Theorem 6.4 *Every chain of labels in a branch in a proof search for the sequent $Nw \Rightarrow w : A$ is finite.*

Proof Given a chain of labels and a label w in the chain, every successor of w in the chain labels a formula whose weight is strictly lower than those labelled by w. Since the degree of each formula is finite, the chain is clearly finite. □

Theorem 6.5 *The proof search for a sequent $Nw \Rightarrow w : A$ in the system* **G3S1** *terminates.*

Proof The proof is immediate because in every branch the number of labels generated is finite. □

was 5.8,typo?

By Theorem 5.4 we can extract a countermodel out of a saturated branch, therefore we obtain the finite model property and the decidability of the system.

Corollary 6.6 *If* $\mathbf{X}$ *is either* **G3S1** *or* **G3S2***, the relation* $\mathbf{X} \vdash Nw \Rightarrow w : A$ *is decidable.*

This is the first purely syntactic proof of the decidability of the system **S1**.

7 Conclusion

Appendix

Here starts the appendix. If you don't wish an appendix, please remove the `\Appendix` command from the LaTeX file.