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Shape Optimization Problems and Regularity of the Free Boundaries

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Contents

Preface

In this preliminary chapter, we go over the document's structure and illustrate the main contributions obtained during the Ph.D. studies, some of which are not a part of this thesis.

Thesis overview and main contributions

In Chapter 1, we give a brief overview of shape optimization theory, discussing some notions and results that play an essential role in this thesis. It is organized as follows:

- In Subsection 1.1.2, we recall the spectral decomposition theorem and apply it to elliptic operators to obtain a sequence of eigenvalues (Theorem 1.3)

$$0 < \lambda_1(L, \Omega) \le \lambda_2(L, \Omega) < \dots \qquad \text{and} \qquad \lambda_n(L, \Omega) \searrow +\infty.$$

  In particular, we focus on the connection between the domain's topological properties and the first eigenvalue's multiplicity (Theorem 1.4).
- In Subsection 1.1.4, we show that eigenvalues can be characterized as solutions to specific min-max problems and, in particular, the first eigenvalue of the Laplacian is given by

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega v^2 \, dx}.$$

- In Section 1.2, we introduce the *Schwarz rearrangement* and the *Steiner symmetrization*, focusing on their role in shape optimization theory (Theorem 1.11).
- In Section 1.3, we examine the notion of Γ-convergence for functionals and study some properties, for example, the convergence of the corresponding minima (Theorem 1.12).
- In Section 1.4, we discuss the **relaxation of functionals**, which is a crucial notion to deal with optimization problems of the form

$$\min \{ J(u) \, : \, u \in \mathcal{X} \},$$

  when F is not lower semicontinuous on $\mathcal{X}$. In particular, we present a method to determine whether or not a functional is the relaxation of another (Proposition 1.4).
  This notion is extremely helpful in the direct methods for optimization problems that lack compactness. Indeed, roughly speaking, we can find minimizers of a relaxed functional (usually more manageable) and, through regularity theory, prove that they are also solutions to the initial problem. This is done, for example, in Chapter 2, Chapter 3 and Chapter 4.
- In Section 1.5, we introduce functions of bounded variation, and we focus, in Subsection 1.5.1, on compactness and lower semicontinuity theorems (mainly due to Ambrosio [6, 7]) since they will play a crucial role in proving existence of a minimizer in Chapter 3.

- Finally, in Section 1.6, we give a brief overview of the problem of minimizing the perimeter for a fixed volume $m > 0$. For this, we first show that

$$\operatorname{Per} E = \mathcal{H}^{d-1}(\partial^* E)$$

is a *good* definition of *perimeter* for irregular sets (since it coincides with the intuitive one if E is regular enoguh), where $\partial^* E$ denotes the reduced boundary (Definition 1.17). Next, we prove the **Faber-Krahn isoperimetric inequality** (Theorem 1.16).

In Chapter 2, we discuss the results obtained in [35], in which we study a shape optimization problem arising from the reinforcement of a membrane Ω with one-dimensional stiffeners, namely

$$\max_S \lambda_1(S),$$

where S ranges into two classes of admissible choices, one of one-dimensional rectifiable sets with prescribed length and another where the constraint of being connected is added, and

$$\lambda_1(S) := \inf_{u \in C_c(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx + m \int_S |\nabla_\tau u|^2 \, d\mathcal{H}^1}{\int_\Omega |u|^2 \, dx}.$$

The maximization in the first class has already been considered in [40] in the framework of reducing traffic congestion; therefore, they consider the energy in place of the eigenvalue

$$\max_{\vartheta \in \mathcal{A}_m} \left\{ -\frac{1}{2} \int_D f u_\vartheta \, dx \right\},$$

where f is given and u_ϑ is the weak solution to

$$\begin{cases} -\operatorname{div}(1+\vartheta)\nabla u_\vartheta = f & \text{in } D, \\ u_\vartheta \in H_0^1(D). \end{cases}$$

Our main result (Theorem 2.1) shows that the existence and the regularity properties obtained in [40] hold if we consider the principal eigenvalue in place of the energy.
The maximization of $\lambda_1(S)$ with the additional constraint connected, on the other hand, was studied in [2] for the energy problem. Our main result (Theorem 2.2) is relatively weak compared to the one obtained in the other class; nonetheless, in Section 2.6 we discuss some possible improvements, based on the remarkable paper [53], which are unfortunately quite difficult to implement.

In Chapter 3, we consider shape optimization problems for general integral functionals of the calculus of variations that may contain a boundary term, namely

$$\inf \left\{ \mathcal{J}(\Omega) \ : \ \Omega \subset D \text{ and } \Omega \text{ Lipschitz} \right\},$$

where the shape functional $\mathcal{J}$ is defined by setting

$$\mathcal{J}(\Omega) := \min_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega j(x, u, \nabla u) \, dx + \int_{\partial\Omega} g(x, u) \, d\mathcal{H}^{d-1} \right\}.$$

In particular, we discuss the main results obtained in [36]. The prototype for our class of integral functionals is obtained by solving the energy PDE with Robin boundary conditions (3.1). A similar shape optimization problem, with $f = 0$ and Dirichlet boundary condition

$$u = u_0 \qquad \text{on some } D_0 \subset D,$$

was studied by Bucur-Giacomini [24]. The main difference is that we have $f \neq 0$, so there is a linear term ($-f(x)u$ in the model) that raises several technical problems. Nevertheless, we prove that a solution to the problem exists (Theorem 3.3) by studying the relaxation to SBV and, using regularity theory, deducing that optimal shapes are Lipschitz.

In Chapter 4, we consider a shape optimization problem in control form. More precisely, if $D \subset \mathbb{R}^d$ is bounded, for every $\Omega \subset D$, we denote by u_Ω the solution of

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \overline{D} \setminus \Omega, \end{cases}$$

where $p > 1$ is given, $f \in W^{-1,p'}(D)$. We consider minimization problems of the form

$$\min \left\{ J(u_\Omega) \;:\; \Omega \text{ open},\ \Omega \subset D \right\},$$

for cost functionals of the form

$$J(u_\Omega) = -\int_D g(x) u_\Omega(x)\, dx + \lambda \left| \{u_\Omega > 0\} \right|,$$

where g is given and satisfies certain assumptions. In [38], we prove that, if $\lambda > 0$, $g \in L^r(D)$ for some $r > 1$ is a non-negative measurable function, f is non-negative and

$$f \in L^q(D) \qquad \text{with } q > \frac{d}{p} \text{ and } q \geq 1,$$

then, if there is a constant $C > 0$ such that

$$f(x) \leq C g(x) \qquad \text{for every } x \in D,$$

the problem (4.2) admits a solution (Theorem 4.1). We distinguish the case $p > d$ (Section 4.2), in which we can exploit the continuity of the embedding

$$W^{1,p}(D) \hookrightarrow C^0(\overline{D}),$$

from the case $p \leq d$, which requires a completely different approach. Indeed, we consider the relaxation of the problem to the class of p-capacitary measures (Subsection 4.3.2), deduce the existence of a p-quasi-open set (Subsection 4.3.3) and, finally, conclude that it is open if more assumptions are satisfied (Subsection 4.3.4).

In Chapter 5, we discuss the results obtained in [124]. More precisely, we give a short and self-contained proof of the boundary Harnack principle (Definition 5.1) for a class of domains satisfying some geometric conditions given in terms of a state function that behaves as the distance function to the boundary, is subharmonic inside the domain and satisfies some suitable estimates on the measure of its level sets. This summarizes the statement of our main result, Theorem 5.1, which will be deduced as an easy consequence of a boundary Harnack inequality (Theorem 5.2).

In Chapter 6, we consider a free boundary system arising in the study of a class of shape optimization problems; more precisely, we let $u, v \in C^0(B_1)$ be solutions to

$$-\Delta u = f \quad \text{and} \quad -\Delta v = g \quad \text{in } \Omega = \{u > 0\} = \{v > 0\}, \qquad \frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = 1 \quad \text{on } \partial\Omega \cap B_1.$$

The boundary condition is intended in a weak sense (see, for example, Definition 6.1) because Ω may be an irregular domain. The main result of [123] is an epsilon-regularity theorem for viscosity solutions of this free boundary system (Theorem 6.1), which is obtained as follows:

- We prove that a viscosity result (Lemma 6.1) as a consequence of the fact that the auxiliary functions

$$\sqrt{uv} \qquad \text{and} \qquad (u+v)/2$$

  are, respectively, a subsolution and a supersolution of the one-phase problem.
- We study the improvement of flatness of the auxiliary functions above in order to trap the boundary $\partial\Omega$ between nearby translations of a half-space (Lemma 6.2).
- We deduce a partial Harnack inequality (Theorem 6.4) for solutions to (6.13).
- We obtain an improvement-of-flatness result (Theorem 6.5), which allows to conclude that flatness (Definition 6.2) implies the $C^{1,\alpha}$-regularity.

In Chapter 7, we give a brief overview of [37], which aims to obtain some regularity properties of the problem studied in Chapter 4 for $p = 2$.

Theorem A. *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a solution to (7.2). Suppose that the following conditions hold:*

(a) $f, g \in C^2_c(B_1)$*;*

(b) $f \geq 0$ *in* $\overline{B}_1$*, and* $u_\Omega > 0$ *in* Ω*;*

(c) there are constants $C_1, C_2 > 0$ *such that*

$$C_1 g \leq f \leq C_2 g \qquad \text{in } \overline{B}_1.$$

Then there is a closed set $S \subseteq \partial\Omega \cap B_1$ such that:

(i) $\partial\Omega \cap B_1 \setminus S \in C^{1,\alpha}$ *for some* $\alpha \in (0,1]$*;*

(ii) S *is empty if* $d \leq 4$ *and* $\dim_{\mathcal{H}}(S) \leq d-5$*, if* $d \geq 5$*.*

The work is still ongoing (although almost complete), so we omit the most technical proofs and divide the chapter as follows:

- In Section 7.2, we discuss the existence of minimizers for the problem (7.2) and, more precisely, we give two optimality conditions (for optimal shapes) that are crucial in studying local properties of the optimal domain in Section 7.3.
- In Subsection 7.2.2, we prove stationary and stability condition under inner variations.

- ► In Section 7.4, we study the compactness of blow-up sequences and the main properties of blow-up limits.
- ► In Subsection 7.4.2, we prove that we can construct blow-up sequences such that the corresponding blow-up limits are one-homogeneous and proportional. This is achieved as a consequence of the boundary Harnack principle obtained in Chapter 5.
- ► In Section 7.5, we prove (i) of Theorem A. Indeed, we show that the state variables corresponding to an optimal shape Ω are viscosity solutions and ϵ-flat, so we can apply the theory developed in Chapter 6 (in particular, Theorem 6.1) to obtain the $C^{1,\alpha}$-regularity.
- ► Finally, in Section 7.6, we use a well-known dimension-reduction argument and exploit the fact that blow-up limits are one-homogeneous and proportional to prove Theorem A (ii).

Other contributions

To conclude this preface, we will now briefly go over some other contributions obtained during the Ph.D. studies that are not a part of this thesis.

Characterization of k-rectifiability in Carnot groups [101]

The notion of rectifiability is crucial in geometric measure theory and many other mathematics areas, as we will see in this thesis. First, we recall the definition [84, Section 3.2.14]:

Definition A. *Let (X, d) be a metric space. We say that $E \subset X$ is $\mathcal{H}^k$-rectifiable if it can be covered, up to a set of $\mathcal{H}^k$-measure zero, by countably many Lipschitz images of subsets of the Euclidean space $\mathbb{R}^k$.*

There is a rich literature on the characterization and basic properties of k-rectifiable sets in metric spaces; see, for example, [9, 64, 83, 127] and the references therein. However, this notion of rectifiability does not always work in all metric spaces; for instance, in [9] it was proved that the three-dimensional Heisenberg group is purely k-unrectifiable for $k = 2, \dots, 4$. Therefore, a natural notion of *intrinsic rectifiability* in codimension one was proposed in [88], where a De Giorgi rectifiability theorem for finite perimeter sets in all Heisenberg groups was established. Moreover, in [86], the following alternative definition of rectifiability was proposed:

Definition B. *Let $\mathbb{G}$ be a stratified group. We say that $E \subset \mathbb{G}$ is $\mathbb{G}$-rectifiable if it is, up to a negligible set, a countable union of level sets of functions with non-vanishing continuous horizontal gradient.*

In the remarkable work [133], Merlo showed that in Heisenberg groups, $\mathbb{G}$-rectifiability could be characterized in terms of densities and intrinsically flat tangents, obtaining an analogous of Preiss' theorem. That said, $\mathbb{G}$-rectifiable sets can be very different from k-rectifiable ones [113], but one might still ask whether they can be covered by countable unions of Lipschitz images of homogeneous groups instead of $\mathbb{R}^k$. We refer to [135] and [10] for a negative answer; that said, their equivalence in Heisenberg groups is still an interesting open question.
In [15, 87], the authors proposed another notion of rectifiability, consisting in countable unions of intrinsic Lipschitz graphs. It was proved that this corresponds to the $\mathbb{G}$-rectifiability as soon as one has a De Giorgi rectifiability theorem (see [85, 87]). In Heisenberg groups, this question has been settled by Vittone [146], using the theory of currents, leading to a Rademacher-type theorem.

Contribution. In collaboration with Idu and Magnani, in [101], we studied the classical notion of k-rectifiability in homogeneous groups. Indeed, motivated by the results obtained in [128], where the authors characterize k-rectifiable sets in Heisenberg groups $\mathbb{H}^n$, $k \leq n$, with the a.e.-existence of *approximate tangent groups* in suitable Grassmannians, we decided to investigate whether this result could be extended to a more general framework.

Remark A. The notion of *approximate tangent group* was introduced in [128], where the projections defining intrinsic cones are given by the semidirect factorization of the Heisenberg group as

$$\mathbb{H}^n = \mathbb{V} \rtimes \mathbb{V}^\perp.$$

The reason is that, as noticed in [135], using Euclidean projections to define cones constitutes an obstacle in reaching a characterization of rectifiability through approximate tangent cones, so a stronger notion is required.

Our goal in [101] is to extend the characterization [128] to homogeneous groups. Before we describe our main result, we give some necessary definitions to entirely understand the notations appearing in the statement. Indeed, let

$$\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$$

be a connected and simply connected graded nilpotent Lie group, and recall that it is *homogeneous* if it is equipped with a one-parameter family of intrinsic dilations

$$\delta_r : \mathbb{G} \to \mathbb{G},$$

which are linear mappings such that $\delta_r(p) = r^i p$ for each $p \in H^i$, $r > 0$ and $i = 1, \ldots, \iota$.

Definition C. *A linear subspace $S \subset \mathbb{G}$ that satisfies $\delta_r(S) \subset S$ for all $r > 0$ and is a Lie subgroup of $\mathbb{G}$, is called **homogeneous subgroup**. Moreover, if we let*

$$S = S^1 \oplus \cdots \oplus S^\iota$$

be the decomposition induced by the graded structure of $\mathbb{G}$, we say that

- *S is horizontal if $S = S^1$;*
- *S is vertical if $S^i = H^i$ for every $i = 2, \ldots, \iota$.*

Definition D. *Let $\mathbb{V}$ be a k-homogeneous subgroup of $\mathbb{G}$. We say that $\mathbb{V}$ belongs to the horizontal Grassmannian of k-dimensional subspaces $\mathscr{G}(\mathbb{G}, k)$ if it is a k-dimensional horizontal subgroup.*

Definition E. *To every homogeneous group $\mathbb{G}$, we can associate a positive integer $\upsilon \leq \dim(\mathbb{G})$, which is the maximal linear dimension among all horizontal subgroups of $\mathbb{G}$.*

Definition F. *Let μ be a Radon measure on $\mathbb{G}$ and let $p \in G$. A Radon measure ν satisfying $\nu(\mathbb{G}) > 0$ is a **tangent measure** of μ at p (and we say that it belongs to $\operatorname{Tan}(\mu, p)$), if there are sequences of positive numbers c_i and $r_i \to 0$ such that the following weak convergence of measures holds:*

$$c_i (T_{p,r_i})_\# \mu \rightharpoonup \nu,$$

where $T_{p,r}$ is the magnification map defined by $T_{p,r}(q) := \delta_{1/r}(p^{-1}q)$.

Definition G. *Let $E \subset \mathbb{G}$ be an $\mathscr{H}^k$-measurable set. A homogeneous subgroup $\mathbb{T}_p$ of dimension k is a $(k, \mathbb{G})$-approximate tangent group to E at p if the following properties hold:*

- $\Theta^{*k}(E, p) > 0$;
- *for all $s \in (0, 1)$ we have*

$$\lim_{r \to 0} \frac{1}{r^k} \mathscr{H}^k \llcorner E(B(p, r) \setminus X(p, \mathbb{T}_p, s)) = 0,$$

where X denotes the ***intrinsic cone*** *of vertex p, axis $\mathbb{T}$ and opening s, given by*

$$X(p, \mathbb{T}, s) = \left\{ q \in \mathbb{G} \,:\, d(p^{-1}q, \mathbb{T}) < s d(p, q) \right\}.$$

We denote by $\operatorname{apTan}_k^{\mathbb{G}}(E, p)$ the set of all $(k, \mathbb{G})$-approximate tangent groups to E at p (and $\mathbb{T}_p$ if unique).

Theorem B. ([101]) *Let $\mathbb{G}$ be a homogeneous group and fix $k \in [1, v]$. If $E \subset \mathbb{G}$ is a Borel set and $\mathscr{H}^k \llcorner E$ is locally finite, then the following conditions are all equivalent:*

(i) *The set E is k-rectifiable.*

(ii) *For $\mathscr{H}^k$-a.e. $p \in E$, there exists $\mathbb{T}_p \in \mathscr{G}(\mathbb{G}, k)$ such that*

$$\frac{1}{r^k} (T_{p,r})_\# \mathscr{H}^k \llcorner E \rightharpoonup \mathscr{H}^k \llcorner \mathbb{T}_p \qquad \text{as } r \to 0^+.$$

(iii) *For $\mathscr{H}^k$-a.e. $p \in E$, there exists $\mathbb{T}_p \in \mathscr{G}(\mathbb{G}, k)$ such that*

$$\operatorname{Tan}(\mathscr{H}^k \llcorner E, p) = \{\lambda\, \mathscr{H}^k \llcorner \mathbb{T}_p \,:\, 0 < \lambda < \infty\}.$$

(iv) *For $\mathscr{H}^k$-a.e. $p \in E$, there exists $\mathbb{T}_p \in \mathscr{G}(\mathbb{G}, k)$ such that*

$$\operatorname{apTan}_{\mathbb{G}}^k(E, p) = \mathbb{T}_p.$$

The first implication is an immediate consequence of the a.e.-differentiability of Lipschitz mappings taking values in homogeneous groups [122, Theorem 1.1] and the associated area formula [122, Theorem 1.2]. On the other hand, the implications from (ii) to (iii) and from (iii) to (iv) are easily deduced from the definitions above. However, the most noteworthy implication is from (iv) to (i), namely the existence a.e. of the $(k, \mathbb{G})$-approximate tangent group implies k-rectifiability. In [128], this is established by proving a lower density theorem; nevertheless, in [101] we followed a different strategy, which is based on the following result:

Theorem C. *Let $E \subset \mathbb{G}$ be purely k-unrectifiable set, $V \in \mathscr{H}(\mathbb{G}, k)$, $0 < s < c_{\mathbb{G}}^3$, $\lambda > 0$ and $\delta > 0$, where $c_{\mathbb{G}}$ is a constant that depends only on $\mathbb{G}$. If for all $p \in E$ and $0 < r \leq \delta$ we have*

$$\mathscr{H}^k \llcorner E \left(X(p, V^\perp, s) \cap B(p, r) \right) \leq \lambda r^k s^k,$$

then, for all $w \in \mathbb{G}$, the following density upper bound holds:

$$\Theta^{*k}(\mathcal{H}^k \llcorner E, w) \leq 2\lambda \left({}^{84}/_{c_{\mathbb{G}}^2} \right)^k .$$

This result is well-known in the Euclidean setting (see, for instance, [84, Lemma 3.3.6], and [127, Theorem 15.19]), but, surprisingly, it is possible to prove it in non-commutative homogeneous groups following the same strategy. Several technical difficulties arise; however, we can get around them thanks to the estimates obtained in [101, Theorem 2.13].

$C^{1,\alpha}$-rectifiability in low codimension in Heisenberg groups [102]

As mentioned, interest in rectifiable sets arises mainly for their geometric, measure-theoretic, and analytic properties. For example, a notion of approximate tangent spaces defined almost everywhere, a version of the area and coarea formulas [9, 112], and a framework for studying the boundedness of a class of singular integral operators [54, 60, 61].
We already pointed out that many definitions of rectifiability diverge along several, not necessarily equivalent, directions. However, a missing piece in the study of rectifiability in metric spaces is the natural notion of higher-order rectifiability, which can be defined as composing a set with countably many objects of higher-order regularity. So, motivated by the seminal work [11] in the Euclidean setting, our goal in [102] is to initiate progress along this line in Heisenberg groups.

Contribution. We introduce a notion of $C^{1,\alpha}$-rectifiability, for $\alpha \in (0, 1]$, which consists of composing a set with countably many $(\alpha, \mathbb{H})$-regular surfaces. Our main result, which surprisingly is the analogous geometric criterion of *approximate tangent paraboloids* as in the Euclidean setting [76, 141], applies to low-codimensional sets of the Heisenberg groups $\mathbb{H}^n$. However, before we can state it, we give a few necessary definitions and notations.

Remark B. The notion of $(\alpha, \mathbb{H})$-regular surface is entirely equivalent to that of $\mathbb{H}$-regular surface introduced in [128, Section 2.3], with the horizontal gradient in $C^{0,\alpha}$ instead of C^0.

Definition H. *Let $\alpha \in (0, 1]$ and $\lambda > 0$. The α-paraboloid, centered at $x \in \mathbb{H}^n$ with base $S \in \mathscr{G}(\mathbb{H}^n, k)$ and parameter λ, is defined by setting*

$$Q_\alpha(x, S, \lambda) = \left\{ y \in \mathbb{H}^n \ : \ d(x^{-1}y, S) \leq \lambda d(x, y)^{1+\alpha} \right\} .$$

Definition I. *A measurable set $E \subset \mathbb{H}^n$ is $C^{1,\alpha}$-rectifiable if there are k-dimensional $(\alpha, \mathbb{H})$-regular surfaces S_i, with $i \in \mathbb{N}$, such that*

$$\mathcal{H}^{k_m} \left(E \setminus \bigcup_{i \in \mathbb{N}} S_i \right) = 0,$$

where k_m is the metric dimension (i.e., k if $k \leq n$, and $k + 1$ otherwise).

The main result of [102] is the following:

Theorem D. *Fix $\alpha \in (0,1]$ and $n < k \leq 2n$. Let $E \subset \mathbb{H}^n$ be a $\mathscr{H}^{k_m}$-measurable set with $\mathscr{H}^{k_m}(E) < \infty$ and assume that, for $\mathscr{H}^{k_m}$-almost every $p \in E$, there are $\mathbb{V}_p \in \mathscr{G}(\mathbb{H}^n, k)$ and $\lambda > 0$ such that*

$$\lim_{r \to 0^+} \frac{1}{r^{k_m}} \mathscr{H}^{k_m} \left(E \cap B(p,r) \setminus Q_\alpha(p, V_p, \lambda) \right) = 0.$$

If $\Theta_*^{k_m}(E,p) > 0$ for $\mathscr{H}^{k_m}$-almost every $p \in E$, then E is $C^{1,\alpha}$-rectifiable.

To conclude the preface, we make a few comments on the proof of this result and suggest some possible improvements to extend it, for example, to Carnot groups:

- To recover the α-Hölder regularity of the distribution of vertical subgroups, we prove an analogous of [76, Lemma 3.5] in the Heisenberg setting. However, the positive lower density condition is essential and, unlike the Euclidean case [76], it is unclear whether this assumption can be derived from the approximate tangent paraboloid condition.
- The structure of $\mathbb{H}^n$ plays a role in the Hölder-continuity mentioned above, but the other technical results hold in any Carnot group. Therefore, we show that our main result can be extended to any Carnot group in codimension one, and we briefly discuss what is missing in the case ≥ 2.

Preliminaries

Introduction to Shape Optimization Theory 1

1.1 Eigenvalues of elliptic operators

In this section, we recall the preliminary results of the elliptic partial differential equations theory. The prototype of elliptic operators is the *Laplacian*, given by

$$\Delta u := \partial^2_{x_1} u + \cdots + \partial^2_{x_d} u,$$

but the results presented below are valid for general linear elliptic operators.

We will mainly follow [98, Chapter 1], but the reader interested in partial differential equations and operator theory can refer to the books [17], [57] and [77].

1.1.1 Partial differential equations

Let $a_{ij}(x)$ be bounded function defined on $\Omega \subset \mathbb{R}^d$ for $i, j = 1, \dots, d$ and suppose that they satisfy the *ellipticity assumption*, namely there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i x i_j \geq \alpha|\xi|^2 \qquad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^d, \tag{1.1}$$

where $|\cdot|$ denotes the standard Euclidean norm. For simplicity, we will also require the coefficients a_{ij} to be *symmetric*, which means that

$$a_{ij}(x) = a_{ji}(x) \qquad \text{for all } x \in \Omega \text{ and } i, j = 1, \dots, d.$$

Let $a_0(x)$ be a bounded function on Ω. We introduce the linear elliptic operator L, defined on the Sobolev space $H^1(\Omega)$, by setting

$$(Lu)(x) := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u(x)\right) + a_0(x)u. \tag{1.2}$$

If u is not regular enough, the partial derivatives are intended in the sense of distributions; more precisely, for a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ we define

$$\langle \partial_x u, \varphi \rangle := -\langle u, \partial_x \varphi \rangle.$$

The prototype operator, the *Laplacian*, is obtained by setting $a_{ij} = \delta_{ij}$, which is equal to one if $i = j$ and zero otherwise, in (1.2) and we denote it as follows:

$$-\Delta u := -\sum_{i=1}^{d} \partial^2_{x_i} u.$$

Definition 1.1 (Dirichlet) *Let $f \in L^2(\Omega)$. We say that u is a **weak solution** of the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

if $u \in H^1_0(\Omega)$ *is the unique solution of the variational problem*

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij}(x)(\partial_{x_i} u)(\partial_{x_j} v)\,dx + \int_\Omega a_0(x) u v\,dx = \int_\Omega f(x) v\,dx \qquad \text{for all } v \in H^1_0(\Omega).$$

The existence and uniqueness of a solution for this problem follows from the ellipticity assumption (1.1), the classical Poincaré inequality (4.12) and the **Lax-Milgram theorem**, which we now recall.

Theorem 1.1 (Lax-Milgram) *Let* H *be a Hilbert space, denote by* $\|\cdot\|_H$ *the corresponding norm and let* $a : H \times H \longrightarrow \mathbb{R}$ *be a function satisfying the following properties:*

- *the map* $v \mapsto a(\cdot, v)$ *is linear and* $a(0, v) = 0$ *for all* $v \in H$*;*
- *for any* $u_1, u_2 \in H$ *and* $v \in H$*, it turns out that*

$$|a(u_1, v) - a(u_2, v)| \le M\|u_1 - u_2\|_H \|v\|_H;$$

- *there exists* $\nu > 0$ *such that*

$$a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2) \ge \nu\|u_1 - u_2\|^2_H$$

  holds for all $u_1, u_2 \in H$.

Then, for any $F : H \longrightarrow \mathbb{R}$ *linear functional, there exists a unique* $u_F \in H$ *such that*

$$a(u_F, v) = F(v) \qquad \text{for all } v \in H.$$

1.1.2 Eigenvalues of elliptic operators

The existence of a sequence of eigenvalues (and the corresponding eigenfunctions) for general elliptic operators follows from a well-known result in spectral theory.

Definition 1.2 *Let* H *be a Hilbert space endowed with a scalar product* $\langle\cdot,\cdot\rangle_H$ *and let* $T : H \longrightarrow H$ *be an operator (namely, a linear and continuous map). We say that*

- T *is positive if* $\langle Tx, x\rangle \ge 0$ *for all* $x \in H$*;*
- T *is self-adjoint if* $(Tx, y) = (x, Ty)$ *for all* $x, y \in H$*;*
- T *is compact if the image of any bounded set is relatively compact.*

Theorem 1.2 (Spectral decomposition) *Let* H *be a separable* Hilbert space and* T *a positive, self-adjoint and compact operator. Then there exist a sequence of real numbers* $(\nu)_{n\in\mathbb{N}}$ *such that*

$$\nu > 0 \qquad \text{and} \qquad \nu \searrow 0,$$

and a sequence of eigenvectors $(x_n)_{n\in\mathbb{N}} \subset H$*, defining a Hilbert base of* H*, satisfying*

$$Tx_n = \nu x_n \qquad \text{for all } n \in \mathbb{N}.$$

If we denote by T^Ω_L the operator defined on $L^2(\Omega)$ given by

$$T^\Omega_L(f) := u_f,$$

* We say that H is *separable* if it contains a countable and dense subset with respect to the topology induced by $\|\cdot\|_H$.

where u_f is the unique solution of (1.3), then it is easy to verify that T_L^Ω is positive, self-adjoint and compact. Thus, there are $(u_n)_{n\in\mathbb{N}}$ Hilbert basis of $L^2(\Omega)$ and $\nu_n \geq 0$ converging to zero such that

$$T_L^\Omega(u_n) = \nu_n u_n \qquad \text{for every } n \in \mathbb{N}.$$

If we plug this result back into (1.3), we get

$$Lu_n = \frac{1}{\nu_n} u_n,$$

which means that, setting $\lambda_n = \frac{1}{\nu_n}$, we have proved the following theorem:

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Then there are a sequence of positive eigenvalues*

$$0 < \lambda_1(L,\Omega) \leq \lambda_2(L,\Omega) \leq \dots$$

with $\lambda_n(L,\Omega) \nearrow +\infty$, and a sequence of corresponding eigenfunctions u_n^Ω, defining a Hilbert basis of $L^2(\Omega)$, that satisfy the following PDE:

$$\begin{cases} Lu_n^\Omega = \lambda_n(L,\Omega)u_n^\Omega & \text{in } \Omega, \\ u_n^\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

When $L = -\Delta$ is the Laplacian, we will simply denote the eigenvalues by $\lambda_n(\Omega)$ and the corresponding eigenfunctions by u_n. Since they are defined up to a constant, we put the additional constraint

$$\int_\Omega u_n(x)^2\, dx = 1.$$

Notice that if the domain is symmetric, some eigenvalues may be multiple. In this case, we count them with their multiplicity.

Remark 1.1 When Ω is not connected, for example if it has two connected components Ω_1 and Ω_2, then the eigenvalues are obtained by reordering the ones of each component:

$$\begin{aligned} \lambda_1(\Omega) &= \min\{\lambda_1(\Omega_1),\ \lambda_1(\Omega_2)\}, \\ \lambda_2(\Omega) &= \min\left\{\max\{\lambda_1(\Omega_1), \lambda_1(\Omega_2)\}, \lambda_2(\Omega_1), \lambda_2(\Omega,2)\right\}, \end{aligned}$$

and so on. As a consequence, we can choose every eigenfunction to vanish on all but one of the connected components of Ω to have some kind of uniqueness.

The *principal eigenvalue* (i.e., the first) may have multiplicity > 1 (for example, if $\Omega_1 = \Omega_2$ in the remark above), but this is not the case if the domain Ω is connected.

Theorem 1.4 *Let Ω be a connected open set with LIpschitz boundary. Then λ_1 is simple (i.e., it has multiplicity equal to one) and u_1 has a constant sign on Ω.*

This result plays a fundamental role in the problem discussed in Chapter 2; indeed, in Theorem 2.1 we assume Ω connected to have a unique eigenfunction u_1 with a constant (positive) sign on Ω.

1.1.3 Properties and regularity of the eigenfunctions

The Laplace operator is invariant for translation and rotations, so we have

$$\lambda_n(\tau_x(\Omega)) = \lambda_n(\Omega),$$

for every translation $\tau_x(y) := y - x$, and

$$\lambda_n\left(R(\Omega)\right) = \lambda_n(\Omega)$$

for every isometry R. That said, one of the most crucial properties for our purposes is the behavior of the eigenvalues under rescaling. More precisely, for $k > 0$ let

$$H_k(x) := kx$$

and, given a function v on Ω, let $H_k v$ be defined on $H_k(\Omega)$ by setting

$$H_k v(x) := v(x/k).$$

Since $H_k \circ \Delta = k^2\, \Delta \circ H_k$, we immediately deduce that

$$\lambda_n(H_k(\Omega)) = \frac{\lambda_n(\Omega)}{k^2}\,.$$

As a simple application, notice that if $\Omega = B_R$ is the ball of radius R and B_1 the unit ball, then

$$\lambda_1(B_R) = \frac{1}{R^2}\lambda_1(B_1),$$

which means that the function

$$(0, +\infty) \ni R \longmapsto \lambda_1(B_R) \in (0, +\infty)$$

is strictly decreasing and goes to zero as $R \to +\infty$. This scaling property will be discussed in more details and applied in the proof of Lemma 3.4.

To conclude this section, we recall some well-known results concerning the interior boundary regularity of the corresponding eigenfunctions.

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^d$ be an open set. Then the eigenfunctions of the Laplacian are analytic in Ω.*

This result is proved, for example, in [59] and [57] and it is a consequence of the hypo-ellipticity of the Laplacian. For more general elliptic operators L it depends on the regularity of the coefficients and we refer the reader to [89] for a detailed discussion.

The regularity up to the boundary, on the other hand, requires some assumptions on the domain. The following (standard) results can be found, for example, in [89] and [94]:

Theorem 1.6 *Let Ω be either $C^{1,1}$ or convex and assume that $a_{ij} \in C^0$ and $a_0 \in L^\infty$. Then the eigenfunctions of L belong to $H^2(\Omega)$.*

Theorem 1.7 *Let $\Omega \in C^{2,\alpha}$ and assume $a_{ij} \in C^{1,\alpha}$ and $a_0 \in C^{0,\alpha}$. Then the eigenfunctions belong to $C^{2,\alpha}(\bar{\Omega})$.*

1.1.4 Min-max principles

The eigenvalues can also be characterized via a variational problem. More precisely, define the *Rayleigh quotient* of the elliptic operator L by setting

$$R_L[v] := \frac{\sum_{i,j=1}^d \int_\Omega a_{ij}(x)\partial_{x_i} v \partial_{x_j} v\, dx + \int_\Omega a_0(x) v^2(x)\, dx}{\int_\Omega v^2(x)\, dx}\,.$$

Then we have

$$\lambda_n(L, \Omega) = \min_{E_k \subset H_0^1(\Omega)} \max_{v \in E_k \setminus \{0\}} R_L[v],$$

where $E_k \subset H_0^1(\Omega)$ ranges among all subspaces of dimension k. For example, the n-th eigenvalue of the Laplace operator is given by the variation problem

$$\lambda_n(\Omega) = \min_{E_k \subset H_0^1(\Omega)} \left\{ \max_{v \in E_k \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega v^2 \, dx} \right\},$$

and, in particular, the principal eigenvalue is given by

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega v^2 \, dx}.$$

For an account of recent results in critical point theory by min-max methods see [134], [137], [138] and the references therein.

1.2 Rearrangements in shape optimization theory

This section aims to introduce rearrangements and discuss the properties that play an essential role in shape optimization. We follow [98] closely for most of this section.

1.2.1 Schwarz rearrangement

The Schwarz rearrangement is an important tool in shape optimization theory and was used by Faber and Krahn to prove the *isoperimetric inequality* (see Theorem 1.16). The reader interested in a more detailed discussion of rearrangements (with applications to other branches of mathematics) may start, for example, from the books [14], [96], [106] and [136].

Definition 1.3 *Let $\Omega \subset \mathbb{R}^d$ be a measurable set. We denote by Ω^* the ball of same volume, namely*

$$\Omega^* = \left\{ x \in \mathbb{R}^d \;:\; |x| \leq \left[|\Omega| \frac{\Gamma(d/2+1)}{\pi^{d/2}} \right]^{\frac{1}{d}} \right\}.$$

If $u : \Omega \to \mathbb{R}$ is a measurable non-negative function vanishing on the boundary, we denote the level sets by

$$\Omega(t) := \{ x \in \Omega \;:\; u(x) \geq t \} \qquad \textit{for all } t > 0.$$

*The **Schwarz rearrangement** of u, denoted by u^*, is the function defined on Ω^* as follows:*

$$u^*(x) := \sup \{ t > 0 \;:\; x \in \Omega(t)^* \}.$$

The general idea is that, starting from u, the function u^* is obtained by rearranging the level sets $\Omega(t)$ in balls of the same volume.

Remark 1.2 The following properties hold by construction:

- the function u^* is radially symmetric and non-increasing;
- the level sets of u and u^* have the same measure;
- $\sup_\Omega u = \sup_{\Omega^*} u^*$.

Theorem 1.8 *Let Ω be a measurable set and u be as above. If $\psi : [0, +\infty) \to \mathbb{R}$ is measurable, then*

$$\int_\Omega \psi(u(x)) \, dx = \int_{\Omega^*} \psi(u^*(x)) \, dx.$$

We now state an important result, which is known as *Pòlya inequality*, which gives a connection between the integrals of ∇u and ∇u^*.

Theorem 1.9 (Pòlya inequality) *Let Ω be an open set and let $u \in H_0^1(\Omega)$ be a non-negative function. Then*

$$u^* \in H_0^1(\Omega^*) \qquad \textit{and} \qquad \int_\Omega |\nabla u|^2\,dx \geq \int_{\Omega^*} |\nabla u^*|^2\,dx.$$

A proof of this result based on the classical isoperimetric inequality can be found in [106]. We also refer the reader to [55] for a discussion on the cases in which equality holds.

To conclude this section, we recall an inequality due to Hardy and Littlewood (see [119, Chapter 3]):

Theorem 1.10 *Let u and v be two functions as above defined on Ω. Then*

$$\int_\Omega u(x)v(x)\,dx \leq \int_{\Omega^*} u^*(x)v^*(x)\,dx.$$

1.2.2 Steiner symmetrization

We now introduce another rearrangement, the *Steiner symmetrization*, which is used in Section 2.4 to prove that radial data lead to a radially symmetric solution for a problem of the form

$$\max_S \lambda_1(S),$$

where $\lambda_1(S)$ denotes the principal eigenvalue when we add a stiffener S (modeled as a one-dimensional rectifiable set) to a membrane Ω - see Chapter 2 for more details -.

Let $d \geq 2$ and fix the hyperplane $H := \{x_d = 0\}$. For any measurable set $\Omega \subset \mathbb{R}^d$ we denote by Ω' the projection onto H, namely

$$\Omega' := \left\{x' \in \mathbb{R}^{d-1} \;:\; (x', x_d) \in \Omega \text{ for some } x_d \in \mathbb{R}\right\}.$$

Moreover, for $x' \in \mathbb{R}^{d-1}$ we denote by $\Omega(x')$ the intersection between Ω and all the lines through x', which is given by

$$\Omega(x') = \Omega \cap (\{x'\} \times \mathbb{R}) = \{x_d \in \mathbb{R} \;:\; (x', x_d) \in \Omega\}.$$

Definition 1.4 *Let $\Omega \subset \mathbb{R}^d$ be a measurable set. We say that*

$$\Omega^\star := \left\{x = (x', x_d) \;:\; x' \in \Omega',\; -\frac{1}{2}|\Omega(x')| < x_d < \frac{1}{2}|\Omega(x')|\right\}$$

is the ***Steiner symmetrization*** *of Ω with respect to the hyperplane $\{x_d = 0\}$.*

Consequently, the set $\Omega^\star$ is symmetric with respect to $\{x_d = 0\}$ and concave in the corresponding direction e_d. Moreover, it is easy to verify that

$$\Omega \text{ open} \implies \Omega^\star \text{ open}.$$

Similarly to the Schwarz rearrangement, it is possible to define the Steiner symmetrization of a non-negative measurable function u that vanishes at the boundary.

Definition 1.5 *Let u as above and denote by $\Omega(t)$ the corresponding level sets. The* ***Steiner symmetrization*** *of u is the function $u^\star$ defined on $\Omega^\star$ as follows:*

$$u^\star(x) := \sup\{t > 0 \;:\; x \in \Omega(t)^\star\}.$$

The following result summarizes all the fundamental properties of the Steiner symmetrization of a function. The proof can be found, for example, in the book [78].

Theorem 1.11 *Let $\Omega \subset \mathbb{R}^d$ be a measurable set and u a non-negative measurable function. Then*

(i) the volume of Ω coincides with the volume of $\Omega^\star$;

(ii) if $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a measurable function, then

$$\int_\Omega \psi(u(x))\,dx = \int_{\Omega^\star} \psi(u^\star(x))\,dx\,;$$

(iii) if Ω is open and $u \in W_0^{1,p}(\Omega)$, $1 \le p < \infty$, then $u^\star \in W_0^{1,p}(\Omega^\star)$ and

$$\int_\Omega |\nabla u|^p\,dx \ge \int_{\Omega^\star} |\nabla u^\star|^p\,dx\,;$$

(iv) if Ω is open and $u, v \in L^2(\Omega)$, then

$$\int_\Omega uv\,dx \le \int_{\Omega^\star} u^\star v^\star\,dx.$$

As an immediate consequence of (ii) we get the following result:

Corollary 1.1 *For all $1 \le p < \infty$ we have*

$$\|u\|_{L^p(\mathbb{R}^d)} = \|u^*\|_{L^p(\mathbb{R}^d)}.$$

1.3 Γ-convergence of functionals

The goal of this section is to briefly introduce Γ-convergence and discuss some of the main properties, particularly those that concern the convergence of minima - see Subsection 2.2.3 for a concrete application to a shape optimization problem -. The reader may refer to [126] and [12] for a more thorough introduction, while the original papers on the subject by De Giorgi et all are collected in [62].

1.3.1 The definition of Γ-convergence

Roughly speaking, Gamma-convergence is a notion of convergence for functionals that is designed to have, under some conditions, the convergence of minimizers. Indeed, if $(\mathcal{X}, d)$ is a metric space and

$$F_n : \mathcal{X} \longrightarrow [0, \infty]$$

a sequence of lower semicontinuous functionals, then $F_n \xrightarrow{\Gamma} F$ should imply that *a sequence x_n of minimizers for F_n converges to a minimum point x of F.*

Definition 1.6 (Γ-convergence) *Let $(\mathcal{X}, d)$ be a metric space. We say that $F_n : \mathcal{X} \longrightarrow [0, \infty]$ Γ-converges to some $F : \mathcal{X} \longrightarrow [0, \infty]$ if the following properties hold:*

(Γ-i) For every $x \in \mathcal{X}$ and every sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $x_n \longrightarrow x$, it turns out that

$$\liminf_{n\to\infty} F_n(x_n) \ge F(x). \tag{1.4}$$

(Γ-ii) *For every $x \in \mathcal{X}$ there is a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $x_n \longrightarrow x$ and*

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x). \tag{1.5}$$

*The sequence $(x_n)_{n\in\mathbb{N}}$ is usually referred to in the literature as **recovery sequence**.*

Remark 1.3 The condition (1.4) is often called $\Gamma - \liminf$ inequality, while (1.5) is known as $\Gamma - \limsup$ inequality.

1.3.2 Convergence of minima

Proposition 1.1 *Let $\bar{x}_n$ be a minimizer for F_n and suppose that F is the Γ-limit of F_n. If $\bar{x}_n \longrightarrow \bar{x} \in \mathcal{X}$, then $\bar{x}$ is a minimizer for F.*

Proof. It suffices to prove that, given any $x \in \mathcal{X}$, we have

$$F(x) \geq F(\bar{x}).$$

Let x_n be a recovery sequence for x and notice that by minimality $F_n(\bar{x}_n) \leq F_n(x_n)$, so we have

$$\liminf_{n\to\infty} F_n(\bar{x}_n) \leq \lim_{n\to\infty} F_n(x_n) = F(x),$$

by (1.5). On the other hand, the inequality

$$\liminf_{n\to\infty} F_n(\bar{x}_n) \geq F(\bar{x}),$$

follows immediately from (1.4), and this concludes the proof. □

Example 1.1 Let $\mathcal{X} = \mathbb{R}$ and $F_n(x) = x^2 + \sin(nx)$. We claim that

$$\Gamma - \lim_{n\to\infty} F_n(x) = x^2 - 1.$$

The $\Gamma - \liminf$ inequality is easy to check because -1 is the minimum of $\sin(nx)$. Similarly, given $x \in \mathbb{R}$ the recovery sequence is $x_n \in \mathbb{R}$ such that

$$\sin(nx_n) = -1 \qquad \text{for all } n \in \mathbb{N}.$$

An immediate consequence is that, if $\bar{x}_n$ is a minimizer for $F_n(x)$, then $\bar{x}_n \longrightarrow 0$ as the function $x^2 + 1$ has a unique minimum point which is $\bar{x} = 0$.

Note that Proposition 1.1 requires $\bar{x}_n$ to converge to some $\bar{x} \in \mathcal{X}$ as an assumption. Thus, it makes sense to introduce the notion of *coercivity* for sequences:

Definition 1.7 *A sequence of functionals $F_n : \mathcal{X} \longrightarrow [0, \infty]$ is said to be **equicoercive** if $\{x_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{X}$ whenever*

$$F_n(x_n) \leq C < \infty$$

for some uniform constant C that does not depend on n.

This notion is extremely useful in optimization problems even if we are not using Γ-convergence; for example, in Subsection 3.2.2, we use equicoercivity to obtain the existence of a minimizer.

Theorem 1.12 *Let $\bar{x}_n$ be a minimizer for F_n and suppose that F is the Γ-limit of the sequence F_n. If F_n is equicoercive, then $\{\bar{x}_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{X}$ and each accumulation point is a minimizer for F.*

Remark 1.4 In the applications, we need to choose the "right" topology for the space $\mathcal{X}$. In fact, we need to find a balance between the following two points:

- If the topology is too weak, then the $\Gamma - \liminf$ inequality has to be tested on more sequences and thus it may fail.
- If the topology is too strong, then (1.4) is easier but the equicoerciveness (strictly related to compactness) may fail.

Remark 1.5 The drawback of Theorem 1.12 is that it concerns the convergence of global minima, but it does not give any information on local minima.

Remark 1.6 From a numerical point of view, Γ-convergence is rather useless because it gives no information about the rate of convergence of minimizers (so, given $\bar{x}_n$, we do not know where to stop to find a good approximation of $\bar{x}$).

The result of Theorem 1.12 is also used to extrapolate information on the minimizing sequence, but sometimes it is not possible to do so, as the following example shows:

Example 1.2 Let $\mathcal{X} = \mathbb{R}$ and $F_n(x) = x^2/n + \sin(nx)$. Then we have

$$F_n \xrightarrow{\Gamma} F \equiv -1,$$

but any $x \in \mathbb{R}$ minimizes F so the Γ-convergence gives no information on $\bar{x}_n$. However, if we consider the sequence of functionals

$$G_n(x) := n(F_n(x) + 1) = x^2 + n(\sin(nx) + 1),$$

then it is easy to verify that $\operatorname{argmin}(F_n) = \operatorname{argmin}(G_n)$ and

$$\Gamma - \lim_{n \to \infty} G_n = x^2,$$

which admits a unique minimum point; therefore, any minimizing sequence $\bar{x}_n$ has to be infinitesimal.

1.4 Relaxation of functionals

This section aims to discuss the *relaxation* of functionals, which is a crucial notion to deal with optimization problems of the form

$$\min_{u \in \mathcal{X}} F(u),$$

where $F : \mathcal{X} \to [-\infty, \infty]$ is a functional that is not lower semicontinuous on $\mathcal{X}$. We will only give a brief overview, so the reader interested in further applications can read [13, Chapter 11 and 13].

Definition 1.8 *The relaxation of F on $\mathcal{X}$, denoted by $\bar{F}$, is the lower semicontinuous envelope of F, namely*

$$\bar{F}(u) := \liminf_{v \to u} F(v) = \inf \left\{ \liminf_{n \to \infty} F(v_n) \ : \ v_n \to u \right\}$$

Remark 1.7 By construction (taking the constant sequence $v_n \equiv u$), the relaxation $\bar{F}$ is the larger lower semicontinuous functional on $\mathcal{X}$ which satisfies $\bar{F} \leq F$.

Proposition 1.2 *Let F be a coercive functional on $\mathcal{X}$. Then $\bar{F}$ is also coercive.*

Proof. Notice that for every $M \in \mathbb{R}$ we have

$$\left\{u \in \mathcal{X} \;:\; \bar{F}(u) \leq M\right\} \subset \overline{\{u \in \mathcal{X} \;:\; F(u) \leq M\}},$$

so the level sets of $\bar{F}$ are relatively compact in $\mathcal{X}$, which means that $\bar{F}$ is coercive. □

Remark 1.8 The relaxation $\bar{F}$ is lower semicontinuous by definition so, if F is coercive, then $\bar{F}$ always admits a minimizer on $\mathcal{X}$ even if F does not. More precisely,

- minimizers of $\bar{F}$ give information on the behavior of minimizing sequences for F;
- the functional F has a minimizer if and only if there is $\bar{u} \in \mathcal{X}$ that minimizes $\bar{F}$ and satisfies

$$\bar{F}(\bar{u}) = F(\bar{u}).$$

At this point, one might wonder how to determine whether or not a functional G is the relaxation of F. The following proposition answers this question.

Proposition 1.3 *A functional G is the relaxation of F if the following properties hold:*

(1) G is lower semicontinuous on $\mathcal{X}$ and $G(u) \leq F(u)$ for all $u \in \mathcal{X}$;
(2) for all $u \in \mathcal{X}$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ such that

$$u_n \longrightarrow u \qquad \text{and} \qquad G(u) = \lim_{n\to\infty} F(u_n).$$

The condition (ii) can be replaced by a weaker one which is often easier to verify. For this, we introduce the notion of *dense in energy*:

Definition 1.9 *Let $G : \mathcal{X} \longrightarrow [-\infty, \infty]$ be a functional. A set $\mathcal{D} \subset \mathcal{X}$ is G-dense (or dense in energy for G) if for all $u \in \mathcal{X}$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ such that*

$$u_n \longrightarrow u \qquad \text{and} \qquad G(u) \lim_{n\to\infty} G(u_n).$$

As a consequence, the proposition above can be restated as follows:

Proposition 1.4 *A functional G is the relaxation of F if the following properties hold:*

(1) G is lower semicontinuous on $\mathcal{X}$ and $G(u) \leq F(u)$ for all $u \in \mathcal{X}$;
(2) for all $u \in \mathcal{D}$, where $\mathcal{D}$ is G-dense, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ such that

$$u_n \longrightarrow u \qquad \text{and} \qquad G(u) = \lim_{n\to\infty} F(u_n).$$

We now have all the ingredients to describe the strategy known as *direct method* in the most general case, focusing on the importance of relaxation for the problems studied in this thesis.

Let F be a functional defined on some space $\mathcal{X}'$ of "regular" (depending on the problem) functions and suppose that $\mathcal{X}'$ has no good compactness properties.

- **Step 1.** Find a larger space $\mathcal{X} \supset \mathcal{X}'$ and an extension of F, denoted by F_{ext}, such that F_{ext} is coercive and lower semicontinuous on $\mathcal{X}$.
- **Step 2.** Prove the existence of a function $\bar{u} \in \mathcal{X}$ that solves $\min_{u\in\mathcal{X}} F_{\text{ext}}(u)$.
- **Step 3.** Use regularity theory to conclude that $\bar{u} \in \mathcal{X}'$ so that $F(\bar{u}) = \min_{u\in\mathcal{X}'} F(u)$.

Remark 1.9 This strategy looks simple, but there are a couple of issues regarding the last step that we need to consider. Indeed, regularity theory is usually challenging, and (as we will see in Chapter 3) it can happen[†] that the minimizer $\bar{u}$ of F_{ext} is not an element of $\mathcal{X}'$.

[†] In which case, putting additional assumptions on the functional may help overcome the issue.

The relaxation plays a fundamental role in the direct method described here because we proved that if we start with a coercive functional F, then $\bar{F}$ is coercive and lower semicontinuous. Therefore, to conclude this section, we give a definition of relaxation which is more suitable for our goals:

Definition 1.10 *Let $F : \mathcal{X}' \subset \mathcal{X} \to [-\infty, \infty]$ be a functional. The relaxation of F to $\mathcal{X}$ is defined as the relaxation possibly extended to ∞. More precisely, we have*

$$\bar{F}(u) = \begin{cases} \inf \{\liminf_{n\to\infty} F(u_n) \ : \ (u_n)_{n\in\mathbb{N}} \subset \mathcal{X}' \text{ and } u_n \to u\} & \text{if } u \in \overline{\mathcal{X}'} \\ +\infty & \text{otherwise.} \end{cases}$$

This strategy based on finding minimizers of a relaxed functional and, through regularity theory, proving that they are solutions to the initial problem, will be employed in Chapter 2, Chapter 3 and Chapter 4 to deal with some shape optimization problems.

1.5 Compactness in BV-spaces

In this section, we develop the theory of (special) bounded variation spaces since it is the natural framework of the work [36], which is the main topic of Chapter 3.

We will mainly discuss the properties needed in our shape optimization problem, so we refer the reader to [6] and the references therein for a complete overview of BV-spaces.

Definition 1.11 *Let $A \subset \mathbb{R}^d$ be an open set. The **total variation** of $u \in L^1(A)$ is defined as*

$$\int_A |Du| := \sup\left\{\int_A u \ \mathrm{div}(\phi)\, dx \ : \ \phi \in C_c^1(A, \mathbb{R}^d),\ \|\phi\|_\infty \le 1\right\}.$$

The space $\mathrm{BV}(A)$ *consists of all functions with bounded total variation in A, namely*

$$\mathrm{BV}(A) := \left\{u \in L^1(A) \ : \ \int_A |Du| < +\infty\right\}.$$

In other words, a function $u \in L^1(A)$ belongs to $\mathrm{BV}(A)$ *if and only if its distributional derivative Du belongs to the space of finite vector-valued Radon measures.*

Given a function $u : \mathbb{R}^d \to \mathbb{R}$ the *precise representative* of u, which belongs to the same equivalence class in $L^1(\mathbb{R}^d)$, is defined by setting

$$\mathrm{u}(x) := \lim_{r\to 0} \fint_{B_r(x)} u(y)\, dy. \tag{1.6}$$

We now introduce the notion p-capacity as it will be used multiple times throughout this thesis.

Definition 1.12 *Let K be a compact subset of $\mathbb{R}^d$, $d \ge 2$, and let $p \in [1, d)$. The p-capacity of K is*

$$\mathrm{cap}_p(K) := \inf\left\{\int_{\mathbb{R}^d} |\nabla f|^p\, dx \ : \ f \in C_c^\infty(\mathbb{R}^d),\ f(x) \ge 1 \text{ for all } x \in K\right\}.$$

In Section 4.3 we expand this definition by introducing p-capacitary measures and the notion of γ_p-convergence to prove existence for a shape optimization problem.

Remark 1.10 If $u \in W^{1,p}(\mathbb{R}^d)$, $1 < p < \infty$, the limit (1.6) exists up to a set of p-capacity zero.

Definition 1.13 *Let $u \in \mathrm{BV}(A)$. The asymptotic values (or traces) of u near a point of discontinuity $x \in A$ are defined, respectively, as follows:*

$$u^-(x) = \sup\left\{t \in \mathbb{R} \,:\, \lim_{r\to 0} \frac{|B_r(x) \cap \{u < t\}|}{|B_r(x)|} = 0\right\},$$
$$u^+(x) = \inf\left\{t \in \mathbb{R} \,:\, \lim_{r\to 0} \frac{|B_r(x) \cap \{u > t\}|}{|B_r(x)|} = 0\right\}.$$

Moreover, we denote by J_u the set of all jump points, which are all points of discontinuity $x \in A$ for which the traces are well-defined and $u^+(x) \neq u^-(x)$.

We now have all the ingredients needed to define the class of *special functions of bounded variation*, which De Giorgi and Ambrosio first introduced in [63].

Definition 1.14 *Let $A \subseteq \mathbb{R}^d$ be an open set. The class $\mathrm{SBV}(A)$ of special functions with bounded variation on A consists of all $u \in \mathrm{BV}(A)$ such that the singular part Du^s of the measure Du is concentrated on the set*

$$\left\{x \in A \,:\, u(x) \neq \frac{u(x^+) + u(x^-)}{2}\right\}.$$

In other words, the singular part of the measure Du is concentrated on the set of points where the precise representative of u is not defined.

1.5.1 Lower semicontinuity and compactness in $\mathrm{SBV}(\mathbb{R}^d)$

As mentioned above, the space $\mathrm{SBV}(\mathbb{R}^d)$ is the natural framework to solve the optimization problem presented in Chapter 3, but the free discontinuity functional

$$\mathcal{F}(u) = \int_{\mathbb{R}^d \cap \{u\neq 0\}} j(x, u, \nabla u)\, dx + \int_{J_u} \left[g(x, u^+) + g(x, u^-)\right] d\mathcal{H}^{d-1}$$

is not coercive because pathological behaviors are, in principle, possible. Thus, following [23] and considering that we do not have a natural constraint on the L^∞-norm of u, we introduce

$$\mathcal{X}_D := \left\{u \,:\, u \vee \epsilon,\ u \wedge (-\epsilon) \in \mathrm{GSBV}(\mathbb{R}^d) \text{ for all } \epsilon > 0,\ u = 0 \text{ on } \mathbb{R}^d \setminus D\right\}.$$

The reason is that we want to avoid functions that either oscillate a lot when sufficiently close to zero or are unbounded. Indeed, the space GSBV is defined as follows:

Definition 1.15 *Let $A \subseteq \mathbb{R}^d$ be open. We say that $u \in \mathrm{GSBV}(A)$ if for every $M > 0$ we have*

$$u \wedge M \text{ and } u \vee (-M) \text{ are in } \mathrm{SBV}(A).$$

In other words, a function belongs to $\mathrm{GSBV}(A)$ if and only if any truncation that makes the L^∞-norm finite is an element of $\mathrm{SBV}(A)$.

Going back to our framework, we consider $\mathrm{GSBV}(\mathbb{R}^d)$ equipped with the weak topology, so the following characterization in $\mathrm{SBV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ can be useful:

Remark 1.11 Let $A \subseteq \mathbb{R}^d$ be an open set. A sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to u weakly in $\mathrm{SBV}(A) \cap L^\infty(A)$ if the following properties are satisfied:

(1) $u_n(x) \to u(x)$ at a.e. $x \in A$;
(2) $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^1(A)$;
(3) both $\|u_n\|_\infty$ and $\mathcal{H}^{d-1}(J_{u_n})$ are uniformly bounded.

This also explains why in [36] we cannot work directly in $\mathrm{SBV}(\mathbb{R}^d)$. Indeed, if we have a positive constant such that $\mathcal{F}(u_n) \leq C$, then the jump set satisfy

$$\mathcal{H}^{d-1}(J_{u_n}) \leq c_n,$$

where c_n is a constant that may depend on n and thus (3) is not verified (see Remark 3.4).

The lower semicontinuity of functionals defined in $\mathrm{SBV}(A)$ was first studied by Ambrosio in [7] and will play a crucial role in Chapter 3.

Theorem 1.13 *Let $\varphi(x,s,p)$ be a Carathéodory function on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ and let $\psi(x,a,b)$ be a continuous function on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$. Suppose that*

(i) the function $\varphi(x,s,\cdot)$ is convex;

(ii) there is $r > 1$ such that the estimate

$$\varphi(x,s,p) \geq |p|^r$$

holds for all $p \in \mathbb{R}^d$, all $s \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^d$;

(iii) the function ψ is non-negative and satisfies the triangular inequality

$$\psi(x,a,b) \leq \psi(x,a,c) + \psi(x,c,b).$$

Then, for every open set $A \subset \mathbb{R}^d$, the functional

$$F(u) = \int_A \varphi(x,u,\nabla u)\,dx + \int_{J_u} \psi(x,u^+,u^-)\,d\mathcal{H}^{d-1}$$

is lower semicontinuous in $\mathrm{SBV}(A)$ with respect to the $L^1_{\mathrm{loc}}(A)$ topology.

Remark 1.12 In [7] this result is proved under milder assumptions. However, in Chapter 3 we have

$$\beta_1\left[g(u^+) + g(u^-)\right] \leq \psi(x,u^+,u^-) \leq \beta_2\left[g(u^+) + g(u^-)\right],$$

with $\beta_1, \beta_2 > 0$ and g a non-negative function; hence, the triangular inequality is automatically satisfied in view of the assumptions in Subsection 3.1.2.

Remark 1.13 Notice that, if $\varphi(x,\cdot,\cdot)$ is lower semicontinuous then, using an approximation argument and Beppo-Levi's lemma, we easily deduce that

$$(s,p) \longmapsto \varphi(x,s,p) \text{ is continuous for a.e. } x \in \mathbb{R}^d.$$

As a consequence, the assumption of Theorem 1.13 can be weakened in such a way that this result can be applied in the framework of Chapter 3.

Remark 1.14 Note that this result also applies to sequences u_n in $\mathrm{GSBV}(A)$. Indeed, using the same notations, we can write the functional as

$$F(u) = \int_{A\cap\{u<M\}} \varphi(x,u,\nabla u)\,dx + \int_{J_u\cap\{u<M\}} \psi(x,u^+,u^-)\,d\mathcal{H}^{d-1} + o(1) =: F_1(u) + o(1)$$

for $M \to \infty$, which means that F_1 is lower semicontinuous in $\mathrm{SBV}(A)$ with respect to the $L^1_{\mathrm{loc}}(A)$ topology by Theorem 1.13. In particular, we have

$$\liminf_n F_1(u_n) \geq F_1(u)$$

so, taking the limit as $M \to \infty$, proves that the same is true with F in place of F_1.

To conclude this section, we present a result proved by Ambrosio in [6] that gives the existence of a converging subsequence under certain conditions.

Theorem 1.14 *Let A be an open bounded set in $\mathbb{R}^d$, let $\phi : [0,\infty) \to [0,\infty]$ be a convex non-decreasing function satisfying the condition*

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty,$$

and let $\Theta : [0,\infty] \to [0,\infty]$ be a concave non-decreasing function such that

$$\lim_{t\to 0^+} \frac{\Theta(t)}{t} = \infty.$$

Let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{SBV}(A) \cap L^\infty(A)$ be a sequence such that $\|u_n\|_{L^\infty} \le C$ for a suitable constant, and

$$\sup_{n\in\mathbb{N}} \left\{ \int_A \phi(|\nabla u_n|)\,dx + \int_{J_{u_n}} \Theta(|u_n^+ - u_n^-|)\,d\mathcal{H}^{d-1} \right\} < \infty. \tag{1.7}$$

Then there exists a subsequence converging in measure to a function $u \in \mathrm{SBV}(A) \cap L^\infty(A)$ such that

$$\nabla u_{n_k} \rightharpoonup \nabla u \qquad \text{weakly in } L^1(A).$$

Remark 1.15 If we take the concave non-decreasing function

$$\Theta(t) := \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0, \end{cases}$$

and the convex non-decreasing function

$$\phi(t) = |t|^q,$$

then condition (1.7) can be rewritten as

$$\sup_{n\in\mathbb{N}} \left\{ \int_A |\nabla u_n|^q\,dx + \mathcal{H}^{d-1}(J_{u_n}) \right\} < \infty. \tag{1.8}$$

The functional appearing in (1.8) is usually known as *Mumford-Shah functional* and it will play a fundamental role in Section 3.4 to prove that any optimal shape is open.

Lemma 1.1 *Let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{GSBV}(A)$ be a sequence and assume that the following properties hold:*

(1) There are $c_1, c_2 > 0$ and $q > 1$ such that $\|u_n\|_{L^q(A)} \le c_1$ and $\|\nabla u_n\|_{L^q(A)} \le c_2$.
(2) There is $c_3 > 0$ such that $\int_{J_{u_n}} |u_n|^q\,dx \le c_3$.
(3) There is $c_4 > 0$ such that $\mathcal{H}^{d-1}(J_{u_n}) \le c_4$.

Then u_n converges, up to subsequences, strongly in $L^q(A)$ to a function $u \in \mathrm{GSBV}(A)$ that satisfies the properties (1)–(3) with the same constants.

Proof. Using (1) we deduce that u_n converges (up to subsequences) to some u weakly in $L^q(A)$ and

$$\|u\|_{L^q(A)} \le \liminf_{n\to\infty} \|u_n\|_{L^q(A)} \le c_1,$$

so, to prove that u_n converges strongly to u in $L^q(A)$, it is enough to show that

$$\int_A |u_n|^q\,dx \xrightarrow{n\to\infty} \int_A |u|^q\,dx.$$

Now notice that the weak derivative of u_n can be written as the sum of the absolutely continuous part and the singular one; more precisely, we have

$$Du_n = \nabla u_n \, dx + u_n \, \mathcal{H}^{d-1} \llcorner J_{u_n}.$$

If we let $w_n := u_n^q$, then a simple computation shows that its weak derivative is given by

$$Dw_n = u_n^{q-1} \nabla u_n \, dx + u_n^q \;\; \mathcal{H}^{d-1} \llcorner J_{u_n},$$

so that its total variation as a measure is given by

$$\begin{aligned}
\int_A |Dw_n| &= \int_A u_n^{q-1} \nabla u_n \, dx + \int_{J_{u_n}} |u_n|^q \, d\mathcal{H}^{d-1} \\
&\le \int_A (|u_n|^q + |\nabla u_n|^q) \, dx + \int_{J_{u_n}} |u_n|^q \, d\mathcal{H}^{d-1} \\
&\le c_1^q + c_2^q + c_3^q =: C.
\end{aligned}$$

Consequently, the sequence w_n converges to some w in $\mathrm{BV}(A)$ and, in particular, this convergence is strong with respect to the $L^1(A)$ topology. Finally, we notice that

$$\begin{aligned}
\int_A |Du_n| &= \int_A \nabla u_n \, dx + \int_{J_{u_n}} u_n \, d\mathcal{H}^{d-1} \\
&\le \int_A (1 + |\nabla u_n|^q) \, dx + \int_{J_{u_n}} (1 + |u_n|^q) \, d\mathcal{H}^{d-1} \\
&\le |A| + \mathcal{H}^{d-1}(J_{u_n}) + \int_A |\nabla u_n|^q \, dx + \int_{J_{u_n}} |u_n|^q \, d\mathcal{H}^{d-1} \\
&\le |A| + c_4^q + c_2^q + c_3^q =: \tilde{C},
\end{aligned}$$

which means that u_n converges strongly to u in $L^1(A)$ and weakly in $L^q(A)$. Therefore, since

$$\int_A |u_n|^q \, dx = \int_A |w_n| \, dx \xrightarrow{n \to \infty} \int_A |w| \, dx,$$

we easily deduce that $u = w$ and conclude the proof. □

1.6 Perimeter and isoperimetrical inequality

The problem of minimizing the perimeter for a fixed volume $m > 0$ is extremely important in the theory of shape optimization and can be written as

$$J(m) := \inf_{|E|=m} \operatorname{Per} E,$$

where E ranges among all subsets of $\mathbb{R}^d$. A priori, there is no reason a solution should exist in such a large class of admissible sets, but, as we will see, any ball of volume m is a minimizer.

That said, we first need to establish what *perimeter* of a set exactly means. If E is regular enough (e.g., with Lipschitz boundary), then it makes sense to define

$$\operatorname{Per} E := \mathcal{H}^{d-1}(\partial E),$$

where ∂E is the topological boundary of E. However, this definition is not good enough if E is an irregular set as the next example shows:

Example 1.3 Let E be the unit disk. If we remove the diameter, we obtain a new set $\tilde{E} \subset \mathbb{R}^N$ such that

$$\mathcal{H}^{d-1}(\partial E) < \mathcal{H}^{d-1}(\partial \widetilde{E})$$

since the crack inside contributes to the $\mathcal{H}^{d-1}$-measure. However, as far as the perimeter is concerned, a different definition is required since we still would like to have

$$\operatorname{Per} E = \operatorname{Per} \widetilde{E}.$$

Definition 1.16 (Finite perimeter) *A set $E \subset \mathbb{R}^d$ has finite perimeter if $\mathbb{1}_E \in \mathrm{BV}(\mathbb{R}^d)$, where*

$$\mathbb{1}_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, the perimeter of E is given by the total variation of the distributional derivative:

$$\operatorname{Per} E := |D\mathbb{1}_E| \tag{1.9}$$

Remark 1.16 If E has finite perimeter and Lipschitz boundary, then we have

$$D\mathbb{1}_E = -\nu_E \, \mathcal{H}^{d-1}(\partial E)$$

in sense of distributions. As a consequence, the total variation is given by

$$|D\mathbb{1}_E| = \mathcal{H}^{d-1}(\partial E),$$

which means that (1.9) coincides with the intuitive notion of perimeter given above for regular sets.

A simple consequence of the definition (1.9) is that the topological boundary is not reasonable in our framework, and therefore, we introduce the notion of *reduced boundary*:

Definition 1.17 (Reduced boundary) *Let E be a set of finite perimeter. We say that x belongs to the reduced boundary of E, and we write $x \in \partial^* E$, if there exists $\nu \in \mathbb{S}^{d-1}$ such that*

$$\lim_{\epsilon \to 0^+} \frac{|(B^+(x,\epsilon) \cap E)|}{|B^+(x,\epsilon)|} = 0 \qquad \text{and} \qquad \lim_{\epsilon \to 0^+} \frac{|(B^-(x,\epsilon) \cap E)|}{|B^-(x,\epsilon)|} = 1,$$

where B^+ and B^- are respectively defined as

$$B^+(x,\epsilon) = \{y \in B(x,\epsilon) \, : \, (y-x)\cdot \nu > 0\},$$
$$B^-(x,\epsilon) = \{y \in B(x,\epsilon) \, : \, (y-x)\cdot \nu < 0\}.$$

In this case, the unit vector ν is unique and referred to in the literature as outer normal to $\partial^* E$ at x.

Theorem 1.15 *Let E be a set of finite perimeter. Then*

$$\operatorname{Per} E = \mathcal{H}^{d-1}(\partial^* E).$$

For the proof of this result and a complete discussion of finite perimeter sets we encourage the reader to consult the book [92] and the references therein.

Definition 1.18 *The d-dimensional density of E at x is defined by*

$$\Theta_d(E,x) := \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|}$$

wherever this limit exists. We denote by E^k the set $\{p \in E \,:\, \Theta_d(E,p) = k\}$.

Remark 1.17 If E has finite perimeter, then the function $\mathbb{1}_E$ belongs to $L^1(\mathbb{R}^d)$ and

$$p \in E^0 \text{ or } p \in E^1 \qquad \text{for } \mathcal{H}^d - \text{a.e. } p \in E.$$

In particular, if $p \in \partial^* E$, then $p \in E^{1/2}$. Moreover, we have the inclusions

$$\partial^* E \subseteq E^{1/2} \subseteq \mathbb{R}^d \setminus \left(E^0 \cup E^1\right).$$

1.6.1 Isoperimetrical inequality

We can now take a step back to look at the minimization problem

$$J(m) := \inf_{|E|=m} \operatorname{Per} E.$$

Notice that for every $x \in \mathbb{R}^d$ and $\lambda > 0$ we have

$$|E + x| = |E| \qquad \text{and} \qquad |\lambda E + x| = \lambda^d |E|,$$

so the perimeter is translation-invariant and scales with a factor; namely, we have

$$\operatorname{Per}(\lambda E + x) = \lambda^{d-1} \operatorname{Per} E.$$

Therefore, it is enough to solve the isoperimetric problem for a fixed volume (say $m = 1$) and deduce the value of the minimum for every $m > 0$ using the formula

$$J(m) = m^{\frac{d-1}{d}} J(1).$$

The goal of this section is to prove that the set with minimal perimeter for a fixed volume is the ball, but first we need a technical result:

Lemma 1.2 *Let E be a minimizer for J which is symmetric with respect to the origin. Then E is a ball.*

Theorem 1.16 (Faber-Krahn) *Let E be an isoperimetrical set. Then E is a ball. In particular, for any $S \subset \mathbb{R}^d$ we have the* ***isoperimetric inequality***

$$\operatorname{Per} S \geq d|S|^{(d-1)/d}|B_1|^{1/d}, \tag{1.10}$$

where B_1 is a unit sphere in $\mathbb{R}^d$ and equality holds if and only if S is a ball.

This result was first a conjecture by Rayleigh and was later proven (independently) by Faber [81] and Krahn [114].

Proof. Up to translations we can always assume that

$$\left|E \cap \{x_1 > 0, \dots, x_j > 0\}\right| = 2^{-j}|E| \qquad \text{for any } 1 \leq j \leq d.$$

Now E^1 is isoperimetric because $\{x_1 = 0\}$ bisects E, where E^1 is the Steiner symmetrization of the right part, that is,

$$E^1 = \left\{x \in \mathbb{R}^d \,:\, (|x_1|, x_2, \dots, x_d) \in E\right\}.$$

Similarly, the hyperplane $\{x_2 = 0\}$ bisects E_1 (although it may not bisect E), and hence

$$E^2 = \left\{x \in \mathbb{R}^d \,:\, (|x_1|, |x_2|, x_3, \dots, x_d) \in E\right\}$$

is also isoperimetric. We can iterate the argument to find that

$$E^d = \{x \in \mathbb{R}^d \;:\; (|x_1|, \dots, |x_d|) \in E\}$$

is isoperimetric. Furthermore, by construction it is symmetric with respect to the origin, so we can use Lemma 1.2 to infer that E^d is a ball. Then the set

$$E \cap \{x_1 > 0, \dots, x_d > 0\}$$

is also a ball and, since we can do the same for all possible quadrants of the space $\mathbb{R}^d$, we immediately conclude that E must be a ball and (1.10) holds. □

Shape optimization problems for integral functionals

Optimal one-dimensional structures for the principal eigenvalue 2

As mentioned in the previous chapter, the problem of finding the vibration modes of an elastic membrane $\Omega \subset \mathbb{R}^2$ fixed at its boundary $\partial\Omega$ is equivalent to solving the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, we mentioned that the principal eigenvalue $\lambda_1(\Omega)$ can be characterized as the solution of a variational problem; more precisely, we have

$$\lambda_1(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} \ : \ u \in H_0^1(\Omega), \ u \neq 0 \right\}.$$

This chapter aims to present the results obtained in [35], which investigates how $\lambda_1(\Omega)$ is modified if we attach to the membrane a one-dimensional stiffener S (modeled as a one-dimensional rectifiable set $S \subset \Omega$). In this case, the principal eigenvalue depends on S and is given by

$$\lambda_1(S) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx + m \int_S |\nabla_\tau u|^2 \, d\mathcal{H}^1}{\int_\Omega |u|^2 \, dx} \ : \ u \in C_0^\infty(\Omega), \ u \neq 0 \right\}, \tag{2.1}$$

where the parameter m is the stiffness coefficient of S (and depends on the material), ∇_τ the tangential derivative along S and $\mathcal{H}^1$ the one-dimensional Hausdorff measure.

The following example shows how this problem naturally arises in applications and why it is important to study the regularity properties of optimal stiffeners.

Example 2.1 Let Ω be a two-dimensional heat conductor with an initial temperature u_0 (zero at the boundary). We are interested in cooling it as fast as possible, so if we add one-dimensional conducting wires S, the corresponding heat problem reads as follows:

$$\begin{cases} \partial_t u(t,x) + \mathcal{A}_S u(t,x) = 0 & \text{if } (t,x) \in (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{if } x \in \Omega, \end{cases}$$

where $\mathcal{A}_S$ is a second-order operator that depends on S and is given in weak form by

$$\langle \mathcal{A}_S u, \phi \rangle := \int_\Omega \nabla u \nabla \phi \, dx + m \int_S (\nabla_\tau u)(\nabla_\tau \phi) \, d\mathcal{H}^1, \qquad \text{for } u, \phi \in H_0^1(\Omega) \cap H^1(S).$$

Using Fourier analysis we can write the solution as

$$u(t,x) = \sum_{n \geq 1} c_n u_n(x) e^{-t\lambda_n(S)}, \qquad \text{with } c_n = \int_\Omega u_0 u_n \, dx,$$

where $\lambda_n(S)$ are the eigenvalues of the operator $\mathcal{A}_S$ and u_n the corresponding (normalized) eigenfunctions. Since by Theorem 1.3 we have

$$0 < \lambda_1(S) \leq \lambda_2(S) \leq \ldots \qquad \text{and} \qquad \lambda_n(S) \nearrow +\infty,$$

then the problem of cooling Ω as fast as possible reduces to finding the structure S, in a suitable class of admissible sets, that maximizes the principal eigenvalue $\lambda_1(S)$.

2.1 Formulation of the problem and main results

In [35] we consider the shape optimization problem corresponding to the functional (2.1) the following two classes of admissible choices for the stiffener:

$$\mathcal{A}_L := \{S \subset \Omega \; : \; S \text{ rectifiable and } \mathcal{L}(S) \leq L\} \,,$$
$$\mathcal{A}_L^c := \{S \subset \Omega \; : \; S \text{ rectifiable, connected and } \mathcal{L}(S) \leq L\} \,,$$

where $L > 0$ is fixed and $\mathcal{L}(S)$ is the length of S. This constraint is natural since, in applications, there are costs associated with the material and other factors (e.g., weight) to consider.

The class $\mathcal{A}_L^c$ can be easily generalized to admit an upper bound on the number of connected components, so we also consider

$$\mathcal{A}_L^{c,N} := \left\{S \subset \Omega \; : \; S \text{ rectifiable, } \leq N \text{ connected components and } \mathcal{L}(S) \leq L\right\}.$$

No essential differences exist between $N > 1$ and $N = 1$ concerning the proof of existence and regularity of optimal structures. Nonetheless, in Section 2.6 we discuss the problem of finding the optimal number of connected components when $N > 1$.

2.1.1 Setting of the problem for rectifiable sets

Let $\Omega \subset \mathbb{R}^2$ be a bounded set with Lipschitz boundary. In the first part of this chapter, we discuss the maximization problem associated to the first class, namely

$$\max \{\lambda_1(S) \; : \; S \in \mathcal{A}_L\} \,, \tag{2.2}$$

where $\lambda_1(S)$ is given by (2.1). Due to the lack of compactness in $\mathcal{A}_L$ we cannot solve the problem directly, so the first step is to find a suitable relaxation (see Section 1.4 for more details).

Let $(S_n)_{n\in\mathbb{N}} \subset \mathcal{A}_L$ be a maximizing sequence and consider the auxiliary measures given by

$$\mu_n := \mathcal{H}^1 \llcorner S_n.$$

By definition, the total variation of μ_n is uniformly bounded by the constant L, so we can find a subsequence μ_{n_k} weakly-$\star$ converging to a suitable measure μ. If we now set

$$\lambda_1(\nu) := \inf\left\{\frac{\int_\Omega |\nabla u|^2 \, dx + m \int |\nabla u|^2 \, d\nu}{\int_\Omega |u|^2 \, dx} \; : \; u \in C_c^\infty(\Omega), \; u \neq 0\right\},$$

it is easy to verify that, in general, the minimum is not attained in C_c^∞ and minimizing sequences converge strongly in L^2 and weakly in H^1_ν to solutions of the relaxed problem

$$\min\left\{\frac{\int_\Omega |\nabla u|^2 \, dx + m \int |\nabla_\nu u|^2 \, d\nu}{\int_\Omega |u|^2 \, dx} \; : \; u \in H^1_0(\Omega) \cap H^1_\nu, \; u \neq 0\right\}.$$

Here ∇_ν is a kind of tangential gradient that was defined in [16] for every measure ν. Moreover, the Sobolev space H^1_ν is defined as the space of all functions u such that

$$\int |\nabla_\nu u|^2 \, d\nu < \infty.$$

Remark 2.1 If $\nu = \mathcal{H}^1 \llcorner S$, then ∇_ν coincides with the tangential gradient on S, which means that

$$\nu = \mathcal{H}^1 \llcorner S \implies \lambda_1(\nu) = \lambda_1(S).$$

In particular, the functional $\lambda_1(\nu)$ extends (2.1) to any measure ν defined on Ω. Thus, the relaxation of the optimization problem (2.2) is given by

$$\max \left\{ \lambda_1(\mu) \ : \ \mu \in \mathfrak{A}_L \right\}, \tag{2.3}$$

where $\mathfrak{A}_L$ is the class of all non-negative measures on Ω with total variation bounded by L, i.e.,

$$\mathfrak{A}_L = \left\{ \mu \ : \ \mu \text{ measure on } \Omega, \ |\mu| \leq L \right\}.$$

Proposition 2.1 *The optimization problem (2.3) admits a solution $\bar{\mu} \in \mathfrak{A}_L$ that satisfies $|\bar{\mu}| = L$.*

Proof. For every $u \in C_c^\infty(\Omega)$, the functional

$$\mathfrak{A}_L \ni \mu \longmapsto \frac{\int_\Omega |\nabla u|^2 \, dx + m \int |\nabla u|^2 \, d\mu}{\int_\Omega |u|^2 \, dx} \in \mathbb{R}$$

is weakly-$\star$ continuous, so $\lambda_1(\mu)$ is weakly-$\star$ upper semicontinuous as it is given by the infimum of such functionals. The existence now follows from the fact that

$$|\mu| \leq L \text{ for all } \mu \in \mathfrak{A}_L \implies \mathfrak{A}_L \text{ is weakly-}\star \text{ compact.}$$

Finally, the saturation of the constraint ($|\bar{\mu}| = L$) follows from the fact that the functional above is monotone increasing with respect to μ. □

Remark 2.2 The equivalent expression for the principal eigenvalue

$$\lambda_1(\mu) = \min \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx + m \int |\nabla_\mu u|^2 \, d\mu}{\int_\Omega |u|^2 \, dx} \ : \ u \in H_0^1(\Omega) \cap H_\mu^1, \ u \neq 0 \right\}$$

requires more refined tools (such as the tangential gradient ∇_μ and the Sobolev space H^1_μ), and for the precise definitions, we refer to [16]. However, we will see that in our case, any optimal measure $\bar{\mu}$ obtained in Proposition 2.1 can be written as

$$\bar{\mu} = \bar{\vartheta} \, dx \qquad \text{for some } \bar{\vartheta} \in L^p(\Omega),$$

which, in turn, implies that there is no need to employ such refined notions since

$$\nabla_{\bar{\mu}} u = \nabla_\tau u \qquad \text{and} \qquad H_0^1(\Omega) \cap H^1_{\bar{\mu}} = H_0^1(\Omega).$$

Before stating our main result for (2.2), we introduce a technical assumption necessary to have a bound on the L^∞-norm of the gradient on the boundary $\partial\Omega$.

Definition 2.1 (External Ball Condition) *A set $\Omega \subset \mathbb{R}^d$ satisfies the external ball condition at $x_0 \in \partial\Omega$ with radius $\rho > 0$ if there exists $y_0 \in \mathbb{R}^d$ such that*

$$B(y_0, \rho) \subset \mathbb{R}^d \setminus \Omega \qquad \text{and} \qquad x_0 \in \partial B(y_0, \rho).$$

*Moreover, we say that Ω satisfies the **uniform external ball condition** if there is $\rho > 0$ such that the external ball condition holds at all $x_0 \in \partial\Omega$ with the same radius ρ.*

Theorem 2.1 *Let Ω be a connected* subset of $\mathbb{R}^2$ with Lipschitz boundary satisfying the uniform external ball*

* This assumption is not necessary, but it is used to make the presentation more clear since we can work with a unique eigenfunction (of fixed L^2 norm) which is positive on all Ω.

condition. Then the relaxed problem (2.3) admits a solution of the form

$$\bar{\mu} = \bar{\vartheta}\, dx,$$

where $\bar{\vartheta}$ belongs $L^p(\Omega)$ for all $p < \infty$, is equal to zero almost everywhere on the set

$$\{x \in \Omega \;:\; |\nabla \bar{u}_\vartheta|(x) < \|\nabla \bar{u}_\vartheta\|_\infty\}$$

and the auxiliary function $\bar{u}_\vartheta$ is obtained as a solution of the following problem:

$$\lambda_1(\bar{\mu}) = \max_{\mu \in \mathfrak{A}_L} \lambda_1(\mu) = \min_{u \in H^1_0(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla u|^2\, dx + mL\|\nabla u\|^2_\infty}{\int_\Omega |u|^2\, dx} = \frac{\int_\Omega |\nabla \bar{u}_\vartheta|^2\, dx + mL\|\nabla \bar{u}_\vartheta\|^2_\infty}{\int_\Omega |\bar{u}_\vartheta|^2\, dx}.$$

Moreover, the following additional regularity properties hold:

(1) if Ω is convex, then $\bar{\vartheta}$ belongs to $L^\infty(\Omega)$;

(2) if $\partial\Omega \in C^{2,\alpha}$, then there exists $\beta \in (0,1)$ that depends on α, such that $\bar{\vartheta} \in C^{1,\beta}(\bar{\Omega})$.

We prove this result in Section 2.2 except for the convex case, discussed in Section 2.3. It requires an entirely different strategy due to some technical limitations (see Remark 2.7).

Moreover, Section 2.4 discusses the case in which the initial data are radially symmetric. We show that it is possible to find an explicit formula for the optimal density $\bar{\vartheta}$.

2.1.2 Setting of the problem for rectifiable and N-connected sets

In the second part of this chapter we consider the connected case, namely

$$\max\left\{\lambda_1(S) \;:\; S \in \mathcal{A}^c_L\right\}. \tag{2.4}$$

As above, it is necessary to find a suitable relaxation so let $(S_n)_{n\in\mathbb{N}} \subset \mathcal{A}^c_L$ be a maximizing sequence and consider the corresponding auxiliary measures

$$\mu_n := \mathcal{H}^1 \llcorner S_n.$$

The total variation is uniformly bounded by L so there exists a subsequence μ_{n_k} that converges in the weak-$star$ topology to a measure μ that satisfies the following property

$$\operatorname{spt}\mu = S,$$

where S is a closed connected set as it is obtained as the limit of the sequence S_n, which is compact with respect to the Hausdorff convergence. Moreover, by Gołąb's theorem [93] we have

$$\mu \geq \mathcal{H}^1 \llcorner S,$$

which implies $\mathcal{L}(S) \leq \mu(S) \leq L$, and therefore $S \in \mathcal{A}^c_L$ is an admissible competitor. Consequently, the relaxation of (2.4) is given by the optimization problem

$$\max\{\lambda_1(\mu) \;:\; \mu \in \mathfrak{A}^c_L\}, \tag{2.5}$$

where $\mathfrak{A}^c_L$ is the class of measures defined as follows:

$$\mathfrak{A}^c_L := \left\{\mu \;:\; \mu \text{ measure on } \Omega,\; \operatorname{spt}\mu = S \text{ closed, connected and } \mu \geq \mathcal{H}^1 \llcorner S\right\}.$$

Proposition 2.2 *The optimization problem (2.5) admits a solution $\bar{\mu} \in \mathfrak{A}^c_L$.*

Proof. The same argument used in Proposition 2.1 works since the weak-$\star$ compactness of $\mathfrak{A}_L^c$ follows from the discussion above. $\square$

We are now ready to state the main result in $\mathcal{A}_L^c$, which is a weaker version of Theorem 2.1 due to the additional constraint "connected". Nonetheless, in Section 2.6 we discuss possible improvements of this theorem by looking at what is known for the energy problem.

Theorem 2.2 *The optimization problem* (2.5) *admits a solution of the form*

$$\bar{\mu} = \bar{\vartheta}\,\mathcal{H}^1 \llcorner S,$$

where $S \subset \bar{\Omega}$ is closed and connected, $\mathcal{L}(S) \leq L$, $\bar{\vartheta} \in L^1(S)$ and $\bar{\vartheta} \geq 1$ on S.

If we allow N connected components, $N > 1$, then the relaxation of (2.4) is given by

$$\max\left\{\lambda_1(\mu) \,:\, \mu \in \mathfrak{A}_L^{c,N}\right\}, \tag{2.6}$$

where $\mathfrak{A}_L^{c,N}$ is the class of measures defined as follows:

$$\mathfrak{A}_L^{c,N} := \left\{\mu \,:\, \mu \text{ measure on } \Omega,\ \operatorname{spt}\mu = S \text{ closed and } N\text{-connected and } \mu \geq \mathcal{H}^1 \llcorner S\right\}.$$

The same proof given for Theorem 2.2 works, with minimal changes, for every $N > 1$; as a consequence, we get for free the following result:

Theorem 2.3 *The optimization problem* (2.6) *admits a solution of the form*

$$\bar{\mu}_N = \sum_{j=1}^{\ell} \bar{\vartheta}_j \mathcal{H}^1 \llcorner S_j, \qquad \textit{with } \ell \leq N,$$

where each $S_j \subset \bar{\Omega}$ is closed and connected, the total length is $\leq L$, $\bar{\vartheta}_j \in L^1(S_j)$ and $\bar{\vartheta}_j \geq 1$ on S_j.

Remark 2.3 This theorem does not give the optimal number of connected components; we will discuss this issue in more detail in Section 2.6.

2.2 Proof of Theorem 2.1 for non-convex domains

In this section, we prove that if Ω satisfies the uniform external ball condition, any optimal measure $\bar{\mu}$ given by Proposition 2.1 can be written as $\bar{\vartheta}\,dx$, where $\bar{\vartheta}$ solves the optimization problem

$$\max\{\lambda_1(\vartheta) \,:\, \vartheta \in \mathfrak{A}_L\} \tag{2.7}$$

and $\lambda_1(\vartheta)$ is defined by

$$\lambda_1(\vartheta) := \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (1 + m\vartheta)|\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx}.$$

Furthermore, we will see that the optimal densities solving (2.7) satisfy some higher-integrability properties such as $\bar{\vartheta} \in L^p(\Omega)$ for all $p < \infty$.

2.2.1 Outline of the section

Let $\bar{\mu}$ be any optimal measure given by Proposition 2.1. The strategy used to prove its existence does not give any information on its regularity, and thus, to show that

$$\bar{\mu} = \bar{\vartheta}\, dx \qquad \text{with } \bar{\vartheta} \in L^p(\Omega),$$

we first study the optimization problem (2.3) under an additional constraint. Namely, for every $p > 1$ we consider $\max_\mu \lambda_1(\mu)$ among all $\mu \in \mathfrak{A}_L$ satisfying the following properties:

$$\vartheta = \mu\, dx \qquad \text{and} \qquad \left[\int_\Omega \vartheta^p(x)\, dx\right]^{1/p} \leq L.$$

In Subsection 2.2.2, we prove that each max-min problem (indexed by $p > 1$) yields an optimal density ϑ_p, which can be written explicitly in terms of an auxiliary function u_p. If we introduce

$$F_p(u) := \frac{\int_\Omega |\nabla u|^2\, dx + mL \| |\nabla u|^2 \|_{L^{p'}(\Omega)}}{\int_\Omega |u|^2\, dx},$$

then u_p can be characterized as the unique positive solution with norm in $L^2(\Omega)$ equal to one of the associated minimization problem:

$$\min_{u \in H_0^1(\Omega)\setminus\{0\}} F_p(u).$$

In Subsection 2.2.3 we show that a suitable extension of the sequence $(F_p)_{p>1}$ Γ-converges to a functional F_1 and, as a consequence, we obtain the same for the corresponding minimizers:

$$u_p \xrightarrow{p \to 1^+} u_1$$

with respect to a suitable topology (see Proposition 2.3). Next, in Subsection 2.2.4 we prove that there is a constant $c := c(d, \operatorname{diam}\Omega, \rho) > 0$ such that

$$\frac{1 + C_p \|\nabla u_p\|_{L^\infty(\partial\Omega)}^{2(p'-1)}}{\|u_p\|_{L^\infty(\Omega)}} \leq c$$

for all $p > 1$ (see Lemma 2.5 and Lemma 2.6), where $\rho > 0$ is the radius for which Ω satisfies the uniform external ball condition and C_p is a constant given by (2.12). Once this almost-uniform estimate of ∇u_p is established, we show that the L^r-norm, $r > 1$, of ϑ_p is bounded by

$$\|\vartheta_p\|_{L^r(\Omega)} \leq c_1(r) \qquad \text{for all } p \in (1, 1+\delta)$$

for some $\delta > 0$. This is achieved in Subsection 2.2.5 by extending De Pascale-Evans-Pratelli a priori estimate to the eigenvalue problem (Lemma 2.7) and exploiting it to prove Proposition 2.4.

Finally, in Subsection 2.2.6, we put everything together to prove Theorem 2.1, except for of the L^∞-regularity for convex domains since an entirely different strategy involving Monge-Kantorovich regularity in optimal transport is required; see Section 2.3 for more details.

2.2.2 Optimization in a compact class

We consider the optimization problem (2.7) with an additional constraint, namely

$$\max\left\{\lambda_1(\vartheta) \;:\; \vartheta \in \mathfrak{A}_{L,p}\right\} \tag{2.8}$$

where $\mathfrak{A}_{L,p}$ is the class of measures defined follows:

$$\mathfrak{A}_{L,p} = \left\{ \mu = \vartheta \, dx \; : \; \vartheta \geq 0, \; \int_\Omega \vartheta^p \, dx \leq L^p \right\}.$$

In this case, the principal eigenvalue can be rewritten as

$$\lambda_1(\vartheta) = \min_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega (1 + m\vartheta) |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx},$$

which means that if we can somehow exchange max and min, then we can also find an explicit expression for the optimal density, as the following result shows:

Lemma 2.1 *For every $p > 1$ the optimization problem (2.8) admits a unique solution ϑ_p given by*

$$\vartheta_p(x) = L \frac{|\nabla u_p|^{2/(p-1)}(x)}{\| |\nabla u_p|^{2/(p-1)} \|_{L^p(\Omega)}}, \tag{2.9}$$

where u_p is the unique positive solution with $\|u_p\|_{L^2(\Omega)} = 1$ of the auxiliary problem[†]

$$\min_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx + mL \| |\nabla u|^2 \|_{L^{p'}(\Omega)}}{\int_\Omega |u|^2 \, dx}.$$

Moreover, the function u_p belongs to $L^\infty(\Omega) \cap W_0^{1,2p'}(\Omega)$ and, if $\partial\Omega \in C^{2,\alpha}$, then there exists $\beta := \beta(\alpha) \in (0,1)$ such that $u_p \in C^{2,\beta}(\bar{\Omega})$, which means that

$$\vartheta_p \in C^{1,\beta}(\bar{\Omega}).$$

Proof. To simplify the notations, let

$$E(\vartheta, u) = \frac{\int_\Omega (1 + m\vartheta) |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx},$$

so that the optimization problem (2.8) can be rewritten as

$$\max \left\{ \min_{u \in H_0^1(\Omega) \setminus \{0\}} E(\vartheta, u) \; : \; \vartheta \in \mathfrak{A}_{L,p} \right\}.$$

At this point, we would like to exchange the position of max and min, but the functional is not concave with respect to u so we only have the trivial inequality

$$\max_{\vartheta \in \mathfrak{A}_{L,p}} \left\{ \min_{u \in H_0^1(\Omega) \setminus \{0\}} E(\vartheta, u) \right\} \leq \min_{u \in H_0^1(\Omega) \setminus \{0\}} \left\{ \max_{\vartheta \in \mathfrak{A}_{L,p}} E(\vartheta, u) \right\}. \tag{2.10}$$

To obtain the opposite inequality, we solve the maximization problem on the right-hand side; indeed, for any $u \in H_0^1(\Omega)$ fixed, the maximum is achieved at

$$\vartheta_u(x) = L \frac{|\nabla u|^{2/(p-1)}(x)}{\| |\nabla u|^{2/(p-1)} \|_{L^p(\Omega)}}.$$

[†] Throughout this thesis, we denote by p' the conjugate of p, i.e. the unique positive real number such that $1/p + 1/p' = 1$.

Therefore, the right-hand side reduces to the simple minimization problem

$$\min_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + mL\||\nabla u|^2\|_{L^{p'}(\Omega)}}{\int_\Omega |u|^2\,dx},$$

which admits a unique positive solution u_p with $\|u_p\|_{L^2(\Omega)} = 1$ as a consequence of the direct methods[‡] in the calculus of variations. If we now set

$$\vartheta_p(x) := L\frac{|\nabla u_p|^{2/(p-1)}(x)}{\||\nabla u_p|^{2/(p-1)}\|_{L^p(\Omega)}},$$

and we plug it into (2.10), then we get

$$\min_{u\in H_0^1(\Omega)\setminus\{0\}} E(\vartheta_p,u) \le \max_{\vartheta\in\mathfrak{A}_{L,p}}\left\{\min_{u\in H_0^1(\Omega)\setminus\{0\}} E(\vartheta,u)\right\} \le \mathscr{E}(u_p),$$

where

$$\mathscr{E}(u) := \frac{\int_\Omega |\nabla u|^2\,dx + mL\||\nabla u|^2\|_{L^{p'}(\Omega)}}{\int_\Omega |u|^2\,dx}.$$

The minimum problem on the left-hand side has u_p as a solution (it suffices to compute the Euler-Lagrange equation), and substituting, we quickly verify that

$$E(\vartheta_p,u_p) = \mathscr{E}(u_p),$$

which, in turn, implies that

$$\lambda(\vartheta_p) = \max_{\vartheta\in\mathfrak{A}_{L,p}} \lambda_1(\vartheta).$$

To prove the regularity, we notice u_p can be characterized as the unique positive solution of

$$-\operatorname{div}\left((1+m\vartheta_p)\nabla u_p\right) = \lambda_1(\vartheta_p)u_p$$

satisfying $\|u_p\|_{L^2(\Omega)} = 1$, so $u_p \in L^\infty(\Omega)$ follows from a standard argument (see Remark 2.4). Finally, if $\partial\Omega \in C^{2,\alpha}$, then [120] shows that there exists $\beta \in (0,1)$, depending on α only, such that

$$u_p \in C^{1,\beta}(\bar{\Omega}).$$

A straightforward application of [98, Theorem 1.2.12] gives $u_p \in C^{2,\beta}(\bar{\Omega})$ since the coefficient $1+m\vartheta_p$ of the equation above is β-Hölder continuous as we have

$$u_p \in C^{1,\beta}(\bar{\Omega}) \quad + \quad (2.9) \implies \vartheta_p \in C^{0,\beta}(\bar{\Omega}).$$

□

To conclude this section, we show that $\lambda_1(\vartheta_p)$ can be bounded from above uniformly with respect to p so that it does not lead to any issue when we take the limit for $p \to 1^+$.

Lemma 2.2 *There exists a positive constant $\tilde{c}$ such that*

$$\lambda_1(\vartheta_p) \le \tilde{c} \qquad \textit{for all } p \ge 1.$$

Proof. Since Ω has finite measure, we can estimate the $L^{p'}$-norm of $|\nabla u|^2$ using Hölder's inequality;

[‡] Choosing the topology induced on $H_0^1(\Omega)$ by the standard norm for the lower semicontinuity is sufficient.

more precisely, we have

$$\||\nabla u|^2\|_{L^{p'}(\Omega)} \le |\Omega|^{1/p'}\||\nabla u|^2\|_{L^\infty(\Omega)} \le \max\{1,|\Omega|\}\|\nabla u\|^2_{L^\infty(\Omega)}$$

for all $u \in W^{1,2p'}(\Omega) \cap W^{1,\infty}(\Omega)$. It follows that

$$\begin{aligned}\lambda_1(\vartheta_p) &= \min_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + mL\||\nabla u|^2\|_{L^{p'}(\Omega)}}{\int_\Omega |u|^2\,dx} \\ &\le \min_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + \max\{1,|\Omega|\}\|\nabla u\|^2_{L^\infty(\Omega)}}{\int_\Omega |u|^2\,dx},\end{aligned}$$

and the right-hand side does not depend on p; moreover, a straightforward application of the direct method shows that the minimum is achieved at some $\tilde{u} \in H^1_0(\Omega)$, so

$$\lambda_1(\vartheta_p) \le \frac{\int_\Omega |\nabla \tilde{u}|^2\,dx + \max\{1,|\Omega|\}\|\nabla \tilde{u}\|^2_{L^\infty(\Omega)}}{\int_\Omega |\tilde{u}|^2\,dx} := \tilde{c}, \qquad \text{for all } p \ge 1,$$

and this concludes the proof. □

2.2.3 Γ-convergence as $p \to 1^+$

Let $p > 1$ and consider the family of functionals

$$F_p(u) := \begin{cases} \frac{\int_\Omega |\nabla u|^2\,dx + mL\||\nabla u|^2\|_{L^{p'}(\Omega)}}{\int_\Omega |u|^2\,dx} & \text{if } u \in W_0^{1,2p'}(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus W_0^{1,2p'}(\Omega). \end{cases}$$

In Lemma 2.1 we proved that each F_p has a unique minimizer u_p with $\|u_p\|_{L^2(\Omega)} = 1$. Our goal is to prove that this sequence Γ-converges to the functional

$$F_1(u) := \begin{cases} \frac{\int_\Omega |\nabla u|^2\,dx + mL\||\nabla u|^2\|_{L^\infty(\Omega)}}{\int_\Omega |u|^2\,dx} & \text{if } u \in W_0^{1,\infty}(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus W_0^{1,\infty}(\Omega), \end{cases}$$

and deduce that the same is true for $(u_p)_{p>1}$ in a suitable topology. For more details on the definition and properties of the Γ-convergence we refer the reader to Section 1.3.

Proposition 2.3 *If $\Omega \subset \mathbb{R}^2$ is a bounded set with finite measure, then*

$$F_p \xrightarrow{\Gamma} F_1 \qquad \text{in } L^2(\Omega).$$

Moreover, the sequence of minima $(u_p)_{p>1}$ converges strongly in $H^1(\Omega)$ to u_1 and

$$\lim_{p\to1^+} \|\nabla u_p\|_{L^{2p'}(\Omega)} = \|\nabla u_1\|_{L^\infty(\Omega)}.$$

Proof. The same proof given in [40, Proposition 3.3] works with $f := \lambda_1(\vartheta_p)u_p$. Indeed, the L^2-norm of this term is bounded above since

$$\|\lambda_1(\vartheta_p)u_p\|_{L^2(\Omega)} = \lambda_1(\vartheta_p)\underbrace{\|u_p\|_{L^2(\Omega)}}_{=1} \le \tilde{c},$$

where $\tilde{c}$ is the constant given in Lemma 2.2. Moreover, we know that u_p is the unique positive function with unitary L^2-norm such that

$$\lambda_1(\vartheta_p) = \int_\Omega |\nabla u_p|^2\,dx + mL\||\nabla u_p|^2\|_{L^{p'}(\Omega)},$$

so we can also estimate the norm of the gradient by

$$\int_\Omega |\nabla u_p|^2\,dx = \lambda_1(\vartheta_p) - mL\||\nabla u_p|^2\|_{L^{p'}(\Omega)} \le \tilde{c},$$

and this concludes the proof. □

2.2.4 Almost-uniform estimate of $\|\nabla u_p\|_\infty$ on $\partial\Omega$

In this section, we follow the strategy developed in [40] for the energy to obtain an almost-uniform estimate on the L^∞-norm of ∇u_p on $\partial\Omega$. Notice that it is not uniform because we have

$$-\operatorname{div}\left(G'(|\nabla u_p|^2)\nabla u_p\right) = \lambda_1(\vartheta_p)u_p, \qquad \text{where } G(t) := t + \frac{C_p}{p'}t^{p'}$$

and C_p is given by (2.12), so the quantity $\|u_p\|_{L^\infty(\Omega)}$ plays a crucial role in the equation.

Lemma 2.3 *If Ω is a bounded Lipschitz subset of $\mathbb{R}^2$ and $p \ge 1$, then*

$$u_p \in W^{2,r}(\Omega) \qquad \text{for all } r > 2.$$

Moreover, if Ω is $C^{1,1}$-regular, then $u_p \in C^1\left(\bar{\Omega}\right)$.

This is a well-known fact, and it follows, for example, from [91, Theorem 9.15] and a standard bootstrap argument to achieve higher regularity.

Remark 2.4 The Sobolev embedding theorem (see, for example, [80]) gives us another proof of the fact that for all $p > 1$ we have $u_p \in L^\infty(\Omega)$ and

$$\partial\Omega \in C^{2,\alpha} \implies u_p \in C^{1,\beta}(\bar{\Omega}) \quad \text{for some } \beta(\alpha) := \beta \in (0,1).$$

Notice that $u_p \in L^\infty(\Omega)$ does not means that there is a uniform estimate with respect to p, which is why in Lemma 2.5 and Lemma 2.6 we only obtain an almost-uniform estimate.

The main ingredient of this almost-uniform estimate is a weak maximum (or comparison) principle which we state below in the general form (see, for example, [80, Chapter 6]).

Lemma 2.4 (Weak maximum principle) *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set and let $G : [0,\infty) \to [0,\infty)$ be a convex function such that $G'(0) > 0$.*

(i) If u_Ω is the unique positive solution with $\|u_\Omega\|_{L^2(\Omega)} = 1$ of

$$\begin{cases} -\operatorname{div}(G'(|\nabla u|^2)\nabla u) = \lambda_1(\Omega, G)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then for any $\omega \subset \Omega$ bounded open subset, we have $u_\Omega \ge u_\omega$.

(ii) Let u_Ω be as above and denote by $\bar{u}$ the unique positive solution with $\|\bar{u}\|_{L^2(\Omega)} = 1$ of

$$\begin{cases} -\operatorname{div}(G'(|\nabla u|^2)\nabla u) = \lambda_1(\Omega, G)\|u_\Omega\|_{L^\infty(\Omega)} & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\bar{u} \geq u_\Omega \geq u_\omega$ for every $\omega \subseteq \Omega$.

Remark 2.5 The symbol $\lambda_1(\Omega, G)$ denotes the principal eigenvalue in Ω of the elliptic operator on the left-hand side, which depends on the function G.

We are now ready to construct a barrier for ∇u_p, starting with the following pointwise estimate:

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with $\partial\Omega \in C^{1,\alpha}_{\mathrm{loc}}$ and suppose that it satisfies the external ball condition at some $x_0 \in \partial\Omega$ with radius $\rho > 0$. Then there is a positive constant $c(|\Omega|, L) := c$ such that*

$$\frac{1 + C_p |\nabla u_p|^{2(p'-1)}(x_0)}{\|u_p\|_{L^\infty(\Omega)}} \leq c \left(1 + \frac{\operatorname{diam}\Omega}{\rho}\right)^{d-1} \operatorname{diam}\Omega, \tag{2.11}$$

where C_p is the constant given by

$$C_p := mL \left(\int_\Omega |\nabla u_p|^{2p'}\, dx\right)^{-1/p}. \tag{2.12}$$

Proof. The auxiliary function

$$G(t) := t + \frac{C_p}{p'} t^{p'}$$

is convex and, taking into account that $p' > 1$, we have

$$G'(t) = 1 + C_p t^{p'-1} \implies G'(0) = 1 > 0,$$

so it can be used to apply Lemma 2.4. Moreover, we can assume without loss of generality that the center of the external ball is the origin so that we have

$$\Omega \subset B_{R+\rho} \setminus \overline{B_\rho} := C_{R,\rho}, \qquad \text{where } R = \operatorname{diam}\Omega.$$

If we denote by u_G the unique positive solution of

$$-\operatorname{div}(G'(|\nabla u|^2)\nabla u) = \lambda_1(\vartheta_p)\|u_p\|_{L^\infty(\Omega)} \qquad \text{in } \Omega$$

with $\|u_G\|_{L^2(\Omega)} = 1$, then by Lemma 2.4 we have $u_G \geq u_p$. In a similar fashion, let U denote the unique positive solution of

$$-\operatorname{div}(G'(|\nabla u|^2)\nabla u) = \lambda_1(C_{R,\rho}, \vartheta_p)\|u_p\|_{L^\infty(\Omega)} \qquad \text{in } \Omega$$

with $\|U\|_{L^2(\Omega)} = 1$, so by Lemma 2.4 we have $U \geq u_G \geq u_p$. Moreover, the function U is radially symmetric and, therefore, in polar coordinates we have

$$\begin{cases} -r^{1-d}\partial_r(r^{d-1}G(|U'|^2)U') = \lambda_1(C_{R,\rho}, \vartheta_p)\|u_p\|_{L^\infty(\Omega)} & \text{for } r \in (\rho, \rho + R), \\ U(\rho) = U(\rho + R) = 0. \end{cases}$$

If ρ_1 denotes the radius for which U achieves its maximum value, then we can integrate the equation and deduce the following estimate:

$$\rho^{d-1}\left(1 + C_p|U'|^{2(p'-1)}\right)U'(\rho) \leq \lambda_1(C_{R,\rho}, \vartheta_p)\|u_p\|_{L^\infty(\Omega)} \frac{R(R+\rho)^d}{d}.$$

The left-hand side can be estimated from below using the fact that $\partial_r U$ is positive, while for the right-hand side, we use the inclusion property of eigenvalues to infer that

$$\Omega \subset C_{R,\rho} \implies \lambda_1(C_{R,\rho}, \vartheta_p) \leq \lambda_1(\Omega, \vartheta_p) = \lambda_1(\vartheta_p).$$

Finally, we use Lemma 2.2 to conclude that the estimate (2.11) holds. □

If Ω satisfies the external ball condition uniformly, then we can apply this result at every boundary point and deduce the following almost uniform estimate.

Lemma 2.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with a $C^{1,\alpha}_{\text{loc}}$ boundary. Then the following holds:*

(a) If Ω is convex, then

$$\frac{1 + C_p \|\nabla u_p\|^{2(p'-1)}_{L^\infty(\partial\Omega)}}{\|u_p\|_{L^\infty(\Omega)}} \leq \tilde{c} \operatorname{diam} \Omega,$$

where $\tilde{c}$ is the constant given in Lemma 2.2.

(b) If Ω satisfies the uniform external ball condition with radius $\rho > 0$, then

$$\frac{1 + C_p \|\nabla u_p\|^{2(p'-1)}_{L^\infty(\partial\Omega)}}{\|u_p\|_{L^\infty(\Omega)}} \leq \tilde{c} \left(1 + \frac{\operatorname{diam} \Omega}{\rho}\right)^{d-1} \operatorname{diam} \Omega.$$

Notice that the estimate is more precise when Ω is convex but, as we discuss in Remark 2.7, it is not enough to prove that the optimal density belongs to $L^\infty(\Omega)$.

2.2.5 Uniform estimate of ϑ_p in $L^r(\Omega)$

In this section, we prove that the norm in $L^r(\Omega)$ of ϑ_p can be estimated uniformly with respect to p, at least for values close to $p = 1$.

Remark 2.6 Let $r \in [1, \infty)$. Then by Hölder's inequality we have

$$\|u_p\|_{L^r(\Omega)} \leq |\Omega|^{1/r} \|u_p\|_{L^\infty(\Omega)} \leq \max\{|\Omega|, 1\} \|u_p\|_{L^\infty(\Omega)},$$

which means that it is enough to find a uniform estimate on the norm of ϑ_p in $L^{r_0}(\Omega)$ with $r_0 < \infty$, to deduce that it also holds in $L^r(\Omega)$ for all $r \in [1, \infty)$.

The first step is to prove a modification of De Pascale-Evans-Pratelli's a priori estimate for smooth domains, stated in [65] for the energy problem.

Lemma 2.7 (A priori estimate) *Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded open set with finite volume, fix $r \geq 2$ and let $G : [0, +\infty) \longrightarrow [0, +\infty)$ be a convex function satisfying $G'(0) > 0$. If*

$$u \in C^1(\bar{\Omega}) \cap H^2_{\text{loc}}(\Omega)$$

is the unique positive solution with fixed L^2-norm of the boundary value problem

$$\begin{cases} \operatorname{div}(G'(|\nabla u|^2)\nabla u) = \lambda_1 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

then for every $\epsilon \in (0, 1)$ the following estimate holds:

$$\int_\Omega \left|G'(|\nabla u|^2)\right|^r |\nabla u|^2 \, dx \leq 3\epsilon \left\|G'(|\nabla u|^2)\right\|^r_{L^r(\Omega)} + |\Omega| \left(\frac{(r-1)^r}{\epsilon^{r-1}} + \epsilon^{1-2r}\right) \lambda_1^r \|u\|^{2r}_{L^\infty(\Omega)}$$
$$\cdots - \frac{(r-1)^2 \|u\|^2_{L^\infty(\Omega)}}{\epsilon} \int_{\partial\Omega} HG'(|\nabla u|^2)^r |\nabla u|^2 \, d\mathcal{H}^{d-1},$$

where H denotes the mean curvature of $\partial\Omega$ with respect to the outer normal.

Proof. First, notice that $u \in C^\infty(\bar{\Omega})$ because Ω is smooth. If we introduce the auxiliary function

$$\sigma := G'\left(|\nabla u|^2\right),$$

then we can use $\varphi := \sigma^{r-1}u \in H_0^1(\Omega)$ as a test function for the equation (2.13). Indeed, a simple application of integration by parts leads to the identity

$$\int_\Omega \sigma^r |\nabla u|^2 \, dx + (r-1)\int_\Omega u\sigma^{r-1}(\nabla u \cdot \nabla\sigma)\, dx = \lambda_1 \int_\Omega \sigma^{r-1}|u|^2\, dx, \tag{2.14}$$

and, using the Hölder inequality together with Remark 2.6, we can estimate the right-hand side as

$$\lambda_1 \int_\Omega \sigma^{r-1}|u|^2\, dx \le \lambda_1 |\Omega|^{1/r} \|u\|^2_{L^\infty(\Omega)} \|\sigma\|^{r-1}_{L^r(\Omega)}.$$

However, it is not possible to do the same for the left-hand side since the integral

$$\int_\Omega u\sigma^{r-1}(\nabla u \cdot \nabla \sigma)\, dx$$

cannot be estimated directly. That said, we can test the equation (2.13) against a different function, for example $\psi := \operatorname{div}(\sigma^{r-1}\nabla u)$; it turns out that

$$\begin{aligned}
\int_\Omega \operatorname{div}(\sigma\nabla u)\psi\, dx &= -\lambda_1 \int_\Omega \operatorname{div}(\sigma^{r-2}\sigma\nabla u) u\, dx \\
&= -\lambda_1 \int_\Omega \left(\sigma^{r-2}\operatorname{div}(\sigma\nabla u)u + (r-2)\sigma^{r-2}(\nabla u\cdot\nabla\sigma)u\right)\, dx \\
&\le \lambda_1 \int_\Omega \left(\lambda_1\sigma^{r-2}|u|^2 + (r-2)\sigma^{r-2}|\nabla u\cdot\nabla\sigma||u|\right)\, dx.
\end{aligned}$$

The left-hand side integral can be rewritten more explicitly if we integrate by parts twice (to get rid of the divergence operator), obtaining

$$\begin{aligned}
\int_\Omega \operatorname{div}(\sigma\nabla u)\psi\, dx &= -\int_\Omega \sigma\nabla u\cdot\nabla\psi\, dx + \int_{\partial\Omega} \sigma\psi u_\nu\, d\mathcal{H}^{d-1} \\
&= \int_\Omega (\sigma u_i)_j(\sigma^{r-1}u_j)_i\, dx + \int_{\partial\Omega} \sigma^r(u_\nu\Delta u - u_i u_{ij}\nu_j)\, d\mathcal{H}^{d-1} \\
&= \int_\Omega (\sigma u_i)_j(\sigma^{r-1}u_j)_i\, dx + \int_{\partial\Omega} \sigma^r(u_\nu\Delta u - u_{\nu\nu})u_\nu\, d\mathcal{H}^{d-1} \\
&= \int_\Omega (\sigma u_i)_j(\sigma^{r-1}u_j)_i\, dx + \int_{\partial\Omega} \sigma^r u_\nu(u_\nu\Delta u - u_{\nu\nu})\, d\mathcal{H}^{d-1},
\end{aligned}$$

where we use the notation

$$u_\nu := \nabla u\cdot\nu \quad \text{and} \quad u_{\nu\nu} := \operatorname{Hess}(u)\nu\cdot\nu$$

for the first-order and second-order derivatives in the direction of the exterior normal ν to $\partial\Omega$ respectively. It follows that

$$\begin{aligned}
\int_\Omega \operatorname{div}(\sigma\nabla u)\psi\, dx = \int_\Omega \sigma^r \|\operatorname{Hess}(u)\|_2^2\, dx &+ (r-1)\int_\Omega \sigma^{r-2}|\nabla u\cdot\nabla\sigma|\, dx \\
&\cdots + r\int_\Omega \sigma^{r-1}\sigma_j u_i u_{ij}\, dx + \int_{\partial\Omega} \sigma^r u_\nu(\Delta u - u_{\nu\nu})\, d\mathcal{H}^{d-1},
\end{aligned}$$

where the 2-norm associated to the Hessian matrix is given by

$$\|\operatorname{Hess}(u)\|_2^2 := \sum_{i,j=1}^{d} u_{ij}^2.$$

Since u is smooth up to the boundary of Ω, we can decompose the Laplace operator as the sum of the normal and tangential part, namely

$$\Delta u = u_{\nu\nu} + H u_\nu(x) \qquad \text{for all } x \in \partial\Omega,$$

where H denotes the mean curvature of $\partial\Omega$. This implies the estimate

$$(r-1)\int_\Omega \sigma^{r-2}|\nabla u\cdot\nabla\sigma|^2\,dx + \int_{\partial\Omega} H\sigma^r|\nabla u|^2\,d\mathcal{H}^{d-1} \le \int_\Omega \operatorname{div}(\sigma\nabla u)\psi\,dx \tag{2.15}$$

since, as one can easily check, we have

$$r\int_\Omega \sigma^{r-1}\sigma_j u_i u_{ij}\,dx + \int_\Omega \sigma^r\|\operatorname{Hess}(u)\|_2^2\,dx \ge 0$$

being the sum of two non-negative numbers (recall that $\sigma_j u_i u_{ij} \ge 0$ follows from the assumptions on the convex function G). If we plug (2.15) into (2.14) and use Hölder inequality with

$$p = \frac{r}{2} \qquad \text{and} \qquad p' = \frac{r}{r-2}$$

we get the following estimate:

$$\begin{aligned}\int_\Omega \sigma^{r-2}|\nabla u\cdot\nabla\sigma|^2\,dx &\le \lambda_1^2\int_\Omega |u|^2\sigma^{r-2}\,dx - \int_{\partial\Omega} H\sigma^r|\nabla u|^2\,d\mathcal{H}^{d-1}\\ &\le \lambda_1^2\|u\|_{L^r(\Omega)}^2\|\sigma\|_{L^r(\Omega)}^{r-2} - \int_{\partial\Omega} H\sigma^r|\nabla u|^2\,d\mathcal{H}^{d-1}.\end{aligned}$$

The conclusion now follows by combining the inequalities obtained so far with the identity (2.14) and repeatedly applying Young's inequality

$$A^\alpha B^\beta \le \epsilon\alpha A + \epsilon^{\alpha/\beta}\beta B,$$

which is valid for all $\epsilon \in (0,1)$ and $\alpha + \beta = 1$ with $\alpha \ge 0$ and $\beta > 0$. In particular, it turns out that

$$\begin{aligned}\int_\Omega \sigma^r|\nabla u|^2\,dx &\le \|u\|_{L^\infty(\Omega)}\int_\Omega \sigma^{r-1}|\nabla u\cdot\nabla\sigma|\,dx + \lambda_1\|u\|_{L^\infty(\Omega)}^2|\Omega|^{1/r}\|\sigma\|_{L^r(\Omega)}^{r-1}\\ &\le \epsilon\|\sigma\|_{L^r(\Omega)}^r + \frac{(r-1)^2\|u\|_{L^\infty(\Omega)}^2}{\epsilon}\int_\Omega \sigma^{r-2}|\nabla u\cdot\nabla\sigma|\,dx\\ &\qquad \cdots + \lambda_1\|u\|_{L^\infty(\Omega)}^2|\Omega|^{1/r}\|\sigma\|_{L^r(\Omega)}^{r-1}\\ &\le \epsilon\|\sigma\|_{L^r(\Omega)}^r + \lambda_1^2\frac{(r-1)^2\|u\|_{L^\infty(\Omega)}^2}{\epsilon}\|u\|_{L^r(\Omega)}^2\|\sigma\|_{L^r(\Omega)}^{r-2}\\ &\qquad \cdots + \lambda_1\|u\|_{L^\infty(\Omega)}^2|\Omega|^{1/r}\|\sigma\|_{L^r(\Omega)}^{r-1} - \frac{(r-1)^2\|u\|_{L^\infty(\Omega)}^2}{\epsilon}\int_{\partial\Omega} H\sigma^r|\nabla u|^2\,d\mathcal{H}^{d-1}\\ &\le 3\epsilon\|\sigma\|_{L^r(\Omega)}^r + |\Omega|\left(\frac{(r-1)^r}{\epsilon^{r-1}} + \epsilon^{1-2r}\right)\lambda_1^r\|u\|_{L^\infty(\Omega)}^{2r}\\ &\qquad \cdots - \frac{(r-1)^2\|u\|_{L^\infty(\Omega)}^2}{\epsilon}\int_{\partial\Omega} H\sigma^r|\nabla u|^2\,d\mathcal{H}^{d-1},\end{aligned}$$

and this concludes the proof. □

This a priori estimate applies to smooth domains only. Therefore, our next step is to prove that smooth domains can approximate any Ω satisfying Theorem 2.1's assumptions so that the inequality of Lemma 2.7 also passes to the limit.

Lemma 2.8 *Let Ω be a bounded open set satisfying the uniform external ball condition and let Ω_n be a sequence of smooth open sets with finite volume such that*

$$\Omega_n \supseteq \Omega \qquad \text{and} \qquad |\Omega_n \setminus \Omega| \xrightarrow{n\to+\infty} 0.$$

Fix $p \in [1,\infty)$ and let u_p be the minimizer of F_p on Ω and u_p^n the minimizers of F_p on Ω_n, all positive and with norm in $L^2(\Omega)$ equal to one. Then

$$u_p^n \xrightarrow{n\to+\infty} u_p \qquad \text{strongly in } H^1(\mathbb{R}^d) \text{ and } W^{1,2p'}(\mathbb{R}^d).$$

The proof of this result is standard and requires minimal changes with respect to the energy case, which is dealt with in [40, Lemma 3.9].

Proposition 2.4 *Let $\Omega \subset \mathbb{R}^2$ be a set with finite perimeter and satisfying the uniform external ball condition with radius $R > 0$. For every $r \geq 2$, there are constants*

$$\delta(\Omega) := \delta \qquad \text{and} \qquad C(r, \operatorname{Per}\Omega, \lambda_1, \operatorname{diam}\Omega, R, \|u_1\|_{L^\infty(\Omega)}) := C,$$

such that the following holds:

$$\|\vartheta_p\|_{L^r(\Omega)} \leq C \qquad \text{for all } p \in (1, 1+\delta). \tag{2.16}$$

Proof. Suppose that Ω is smooth, let G_p be the family of functions defined as

$$G_p(t) := t + \frac{C_p}{p'} t^{p'},$$

and notice that u_p can be characterized as the unique positive minimizer with $\|u_p\|_{L^2(\Omega)} = 1$ of the corresponding integral functional

$$H_0^1(\Omega) \ni u \longmapsto \int_\Omega G_p(|\nabla u|^2)\, dx \in \mathbb{R}.$$

A simple computation shows that

$$G_p'(|\nabla u_p|^2) - 1 = \vartheta_p,$$

and, since the mean curvature H of a smooth domain satisfies $H \geq -R$, we can use De Pascale-Evans-Pratelli's a priori estimate (Lemma 2.7) to infer that

$$\int_\Omega \sigma_p^r |\nabla u_p|^2\, dx \leq 3\epsilon \|\sigma_p\|_{L^r(\Omega)}^r + |\Omega| \left(\frac{(r-1)^r}{\epsilon^{r-1}} + \epsilon^{1-2r} \right) \lambda_1(G_p)^r \|u_p\|_{L^\infty(\Omega)}^{2r}$$
$$\cdots - \frac{(r-1)^2 \|u_p\|_{L^\infty(\Omega)}^2}{\epsilon R} \int_{\partial\Omega} \sigma_p^r |\nabla u_p|^2\, d\mathcal{H}^{d-1}.$$

Moreover, from the definition of G_p it follows that $\lambda_1(G_p) = \lambda_1(\vartheta_p)$, which is uniformly bounded by a constant $\tilde{c}$ (see Lemma 2.2). Now consider the decomposition of Ω given by

$$S := \left\{ x \in \bar{\Omega} \,:\, |\nabla u_p(x)| \le \|\nabla u_p\|_{L^{2p'}(\Omega)} \right\},$$
$$B := \left\{ x \in \bar{\Omega} \,:\, |\nabla u_p(x)| > \|\nabla u_p\|_{L^{2p'}(\Omega)} \right\},$$

and notice that

$$\sigma_p := G_p(|\nabla u_p|^2) \le 1 + mL \qquad \text{on } S.$$

The idea is to use this decomposition to estimate the three terms above. For the first one we have

$$\begin{aligned}\|\sigma_p\|^r_{L^r(\Omega)} &= \|\sigma_p\|^r_{L^r(S)} + \|\sigma_p\|^r_{L^r(B)} \\ &\le (1+mL)^r|\Omega| + \|\nabla u_p\|^{-2}_{L^{2p'}(\Omega)} \int_B \sigma_p^r |\nabla u_p|^2\,dx,\end{aligned}$$

while for the third one we can use the almost-uniform estimate (b) of Lemma 2.6 and obtain

$$\begin{aligned}\int_{\partial\Omega} \sigma_p^r |\nabla u_p|^2\,d\mathcal{H}^{d-1} &= \int_{S\cap\partial\Omega} \sigma_p^r |\nabla u_p|^2\,d\mathcal{H}^{d-1} + \int_{B\cap\partial\Omega} \sigma_p^r |\nabla u_p|^2\,d\mathcal{H}^{d-1} \\ &\le (1+mL)^{r-2} \int_{S\cap\partial\Omega} \sigma_p^2 |\nabla u_p|^2\,d\mathcal{H}^{d-1} + \|u_p\|^{2-r}_{L^{2p'}(\Omega)} \int_{B\cap\partial\Omega} \sigma_p^r |\nabla u_p|^r\,d\mathcal{H}^{d-1} \\ &\le (1+mL)^{r-2}\operatorname{Per}\Omega \left[\tilde{c}\left(1 + \frac{\operatorname{diam}\Omega}{\rho}\right)^{d-1} \|u_p\|_{L^\infty(\Omega)} \operatorname{diam}\Omega \right]^2 \\ &\qquad \cdots + \|\nabla u_p\|^{2-r}_{L^{2p'}(\Omega)} \operatorname{Per}\Omega \left[\tilde{c}\left(1 + \frac{\operatorname{diam}\Omega}{\rho}\right)^{d-1} \|u_p\|_{L^\infty(\Omega)} \operatorname{diam}\Omega \right]^r.\end{aligned}$$

Now let $\epsilon := {}^1\!/\!_6\|\nabla u_p\|^2_{L^{2p'}(\Omega)}$ in the initial inequality and rearrange the terms in such a way that the following holds:

$$\begin{aligned}\frac{1}{2}\int_\Omega \sigma_p^r |\nabla u_p|^2\,dx &\le \frac{1}{2}\|\nabla u_p\|^2_{L^{2p'}(\Omega)}(1+mL)^r|\Omega| \\ &\cdots + 6^{2r-1}|\Omega| \left(\frac{(r-1)^r}{\|\nabla u_p\|^{2r-2}_{L^{2p'}(\Omega)}} + \frac{1}{\|\nabla u_p\|^{4r-2}_{L^{2p'}(\Omega)}} \right) \tilde{c}^r \|u_p\|^{2r}_{L^\infty(\Omega)} \\ &\cdots + \frac{6(r-1)^2\|u_p\|^2_{L^\infty(\Omega)}}{\|\nabla u_p\|^2_{L^{2p'}(\Omega)} R}(1+mL)^{r-2} C_2(\Omega)\|u\|^2_{L^\infty(\Omega)} \\ &\cdots + \frac{6(r-1)^2\|u_p\|^2_{L^\infty(\Omega)}}{\|\nabla u_p\|^2_{L^{2p'}(\Omega)} R} C_r(\Omega)\|\nabla u_p\|^{2-r}_{L^{2p'}(\Omega)}\|u_p\|^r_{L^\infty(\Omega)},\end{aligned}$$

where, for $s \ge 1$, we defined

$$C_s(\Omega) := \operatorname{Per}\Omega \left[\tilde{c}\left(1 + \frac{\operatorname{diam}\Omega}{R}\right)^{d-1} \operatorname{diam}\Omega \right]^s.$$

This inequality holds for smooth domains, so we can approximate any Ω satisfying the assumptions using Lemma 2.8 and pass to the limit. We obtain the same estimate up to a constant since

$$\operatorname{Per}\Omega_n \le 2\operatorname{Per}\Omega \qquad \text{and} \qquad \operatorname{diam}\Omega_n \le 2\operatorname{diam}\Omega.$$

Moreover, since u_p converges to u_1, we can always find a small $\delta > 0$ such that

$$\frac{1}{2}\|u_1\|_{L^\infty(\Omega)} \le \|u_p\|_{L^\infty(\Omega)} \le \frac{3}{2}\|u_1\|_{L^\infty(\Omega)} \qquad \text{for all } p \in (1, 1+\delta),$$

from which it follows that

$$\|\sigma_p\|_{L^r(\Omega)} \le C \qquad \text{for all } p \in (1, 1+\delta),$$

and this is enough to prove (2.16). Indeed, starting from the inequality

$$\int_\Omega \vartheta_p^r \, dx \le \int_\Omega \sigma_p^r \, dx,$$

we can exploit once again the decomposition $\Omega = S \cup B$ obtaining

$$\int_\Omega \sigma_p^r \, dx \le (1 + mL)^r |\Omega| + \|\nabla u_p\|_{L^{2p'}(\Omega)}^{-2} \int_B \sigma_p^r |\nabla u_p|^2 \, dx,$$

and the conclusion follows immediately. □

Remark 2.7 The estimate (2.16) is not uniform with respect to r since the constant C depends on it and, more precisely, we have

$$\lim_{r \to +\infty} C(r) = +\infty.$$

This means that (2.16) does not pass to the limit, and the reason is that there is a term that is asymptotically linear with respect to r, namely

$$C(r) \simeq \left[6^{2r-1} |\Omega| \frac{\tilde{c}^r (r-1)^r}{\|\nabla u_p\|_{L^{2p'}(\Omega)}^{2r-2}} \|u_p\|_{L^\infty(\Omega)}^{2r} \right]^{1/r} \simeq \tilde{c} r \qquad \text{as } r \longrightarrow +\infty.$$

Therefore, even if we assume that Ω is convex and use Lemma 2.6 (a), we still cannot get rid of this term as it has nothing to do with the boundary part of the integral.

2.2.6 Proof of Theorem 2.1 for non-convex domains

In this section, we combine all the ingredients to prove our main result for non-convex domains.

Proof of Theorem 2.1. Let $u_p \in W_0^{1,2p'}(\Omega)$ be the unique positive minimizer of F_p with $\|u_p\|_{L^2(\Omega)} = 1$ and let ϑ_p be the corresponding optimal density (Lemma 2.1) given by

$$\vartheta_p(x) = L |\nabla u_p|^{2(p'-1)}(x) \left(\int_\Omega |\nabla u_p|^{2p'} \, dx \right)^{-1/p}.$$

Now recall that in Proposition 2.4 we proved that for $\delta > 0$ small enough and any $r \in [2, \infty)$ there exists a constant $C := C(r)$ such that

$$m\|\vartheta_p\|_{L^p(\Omega)} = mL \qquad \text{and} \qquad \|\vartheta_p\|_{L^r(\Omega)} \le C \quad \text{for all } p \in (1, 1+\delta).$$

Consequently, the sequence $(\vartheta_p)_{p \in (1,1+\delta)}$ is uniformly bounded in $L^2(\Omega)$, and hence it converges weakly to a non-negative function $\bar{\vartheta} \in L^2(\Omega)$ that satisfies

$$\int_\Omega \bar{\vartheta} \, dx = \lim_{p \to 1^+} \int_\Omega \vartheta_p \, dx \le \liminf_{p \to 1^+} \|\vartheta_p\|_{L^p(\Omega)} |\Omega|^{1/p'} = L,$$

so $\bar{\vartheta}$ is admissible, in the sense that $\bar{\mu} := \bar{\vartheta}\, dx$ belongs to $\mathfrak{A}_L$. Now Proposition 2.3 asserts that u_p converges strongly in $H^1_0(\Omega)$ to u_1, so

$$\int_\Omega (1 + m\vartheta_p)\nabla u_p \cdot \nabla \varphi \, dx \xrightarrow{p\to 1^+} \int_\Omega (1 + m\bar{\vartheta})\nabla u_1 \cdot \nabla \varphi \, dx \qquad \text{for every } \varphi \in C^\infty_c(\Omega).$$

Moreover, it is easy to check that

$$\lambda_1(\vartheta_p) \int_\Omega u_p \varphi \, dx \xrightarrow{p\to 1^+} \lambda_1(\bar{\vartheta}) \int_\Omega u_1 \varphi \, dx,$$

so u_1 can equivalently be characterized as the unique positive solution with norm in $L^2(\Omega)$ equal to one of the boundary-value problem

$$\begin{cases} -\operatorname{div}((1 + \bar{\vartheta})\nabla u_1) = \lambda_1(\bar{\vartheta}) u_1 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

so integrating the equation by parts yields

$$\int_\Omega (1 + \bar{\vartheta})|\nabla u_1|^2 \, dx = \lambda_1(\bar{\vartheta}) \underbrace{\int_\Omega |u_1|^2 \, dx}_{=1} = \lambda_1(\bar{\vartheta}).$$

Denote for simplicity the left-hand side by $E(\bar{\vartheta}, u_1)$ and notice that, since u_p converges to u_1 strongly in L^2 and F_p Γ-converges to F_1, we have the identity

$$E(\bar{\vartheta}, u_1) = \lim_{p\to 1^+} \lambda_1(\vartheta_p) = \lim_{p\to 1^+} E(\vartheta_p, u_p) = \lim_{p\to 1^+} F_p(u_p) = F_1(u_1)$$

which, in turn, implies that

$$E(\bar{\vartheta}, u_1) = \min_{u\in H^1_0(\Omega)\setminus\{0\}} F_1(u).$$

Finally, we show (as in Lemma 2.1) that max and min can swap positions and we deduce that

$$\begin{aligned} \sup_{\vartheta\in\mathfrak{A}_L} \lambda_1(\vartheta) &= \sup_{\vartheta\in\mathfrak{A}_L} \min_{u\in H^1_0(\Omega)\setminus\{0\}} E(\vartheta, u) \\ &\le \min_{u\in H^1_0(\Omega)\setminus\{0\}} \sup_{\vartheta\in\mathfrak{A}_L} E(\vartheta, u) = \min_{u\in H^1_0(\Omega)\setminus\{0\}} F_1(u) = E(\bar{\vartheta}, u_1), \end{aligned}$$

which means that $\bar{\vartheta}$ is a solution of the maximization problem (2.3) since we proved above that it is an admissible competitor.

The higher integrability of $\bar{\vartheta}$ follows from the fact that ϑ_p belongs to $L^r(\Omega)$ for all $r \in [2, \infty)$ and

$$\|\vartheta_p\|_{L^r(\Omega)} \le C$$

is uniform with respect to $p \in (1, 1 + \delta)$. Moreover, taking $\bar{u} := u_1$, it is trivial to see that $\bar{\vartheta}$ is zero almost everywhere on the set

$$\left\{ x \in \Omega \,:\, |\nabla \bar{u}(x)| < \|\nabla \bar{u}\|_{L^\infty(\Omega)} \right\}.$$

Finally, the fact that $\bar{\vartheta} \in C^{1,\beta}(\bar{\Omega})$ when $\partial\Omega \in C^{2,\alpha}$ follows immediately from Lemma 2.1. □

2.3 Proof of Theorem 2.1 for convex domains

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex set with finite volume and consider the following max-min problem

$$\max \left\{ \inf_{u \in C_c^1(\Omega)\setminus\{0\}} J(u,\mu) \ : \ \mu \in \mathfrak{A}_L \right\}, \tag{2.17}$$

where

$$J(u,\mu) := \frac{\int_\Omega |\nabla u|^2\,dx + m\int_\Omega |\nabla u|^2\,d\mu}{\int_\Omega |u|^2\,dx}.$$

To simplify the notations, we also introduce the functional

$$\mathcal{J}(\mu) := \inf_{u \in C_c^1(\Omega)\setminus\{0\}} J(u,\mu). \tag{2.18}$$

Now let $\mu \in \mathfrak{A}_L$. The integral $\int_\Omega |\nabla u|^2\,d\mu$ is non-negative so we have

$$J(u,\mu) \geq \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx} \geq \lambda_1(\Omega) > 0$$

for all $u \in C_c^1(\Omega) \setminus \{0\}$; hence, if we take the inf over $C_c^1(\Omega)$, we get

$$\mathcal{J}(\mu) > 0.$$

This is an important difference with the energy problem because it implies that there are no admissible measures $\mu \in \mathfrak{A}_L$ such that $\mathcal{J}(\mu) = -\infty$.

Proposition 2.5 *Let Ω be as above. Then the problem (2.17) admits a solution $\bar{\mu} \in \mathfrak{A}_L$ satisfying*

$$\operatorname{spt}\bar{\mu} \subset \left\{ x \in \Omega \ : \ |\nabla \bar{u}(x)| = \|\nabla \bar{u}\|_{L^\infty(\Omega)} \right\},$$

where $\bar{u}$ is the unique positive solution with $\|\bar{u}\|_{L^2(\Omega)} = 1$ achieving the inf *in (2.18). Furthermore, we have*

$$\mathcal{J}(\bar{\mu}) = \min_{u \in W_0^{1,\infty}(\bar{\Omega})\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + mL\|\nabla u\|_{L^\infty(\Omega)}^2}{\int_\Omega |u|^2\,dx}. \tag{2.19}$$

Proof. The map $\mu \mapsto J(u,\mu)$ is weakly-$\star$ continuous so the functional

$$\mathfrak{A}_L \ni \mu \longmapsto \mathcal{J}(\mu) \in \mathbb{R}$$

is weakly-$\star$ upper semicontinuous. Moreover, the class $\mathfrak{A}_L$ is weakly-$\star$ compact so (2.17) admits a solution $\bar{\mu} \in \mathfrak{A}_L$. By definition of max and min we have the inequality

$$\max_{\mu \in \mathfrak{A}_L} \left\{ \inf_{u \in C_c^1(\Omega)\setminus\{0\}} J(u,\mu) \right\} \leq \inf_{u \in C_c^1(\Omega)\setminus\{0\}} \left\{ \sup_{\mu \in \mathfrak{A}_L} J(u,\mu) \right\}, \tag{2.20}$$

but the opposite inequality does not need to hold since the functional J is not concave. That said, we can solve the maximization problem on the right-hand side easily since

$$\max_{\mu \in \mathfrak{A}_L} J(u,\mu) = \frac{\int_\Omega |\nabla u|^2\,dx + mL\|\nabla u\|_{L^\infty(\Omega)}^2}{\int_\Omega |u|^2\,dx},$$

and this is achieved by choosing any admissible measure $\bar{\mu}$ with total variation L and satisfying

$$\operatorname{spt}\bar{\mu} \subset \left\{x \in \Omega \,:\, |\nabla \bar{u}(x)| = \|\nabla \bar{u}\|_{L^\infty(\Omega)}\right\}.$$

A simple computation now shows that

$$\begin{aligned}
\max_{\mu \in \mathfrak{A}_L} \left\{ \inf_{u \in C^1_c(\Omega)\setminus\{0\}} J(u,\mu) \right\} &\geq \inf_{u \in C^1_c(\Omega)} J(u,\bar{\mu}) \\
&= \inf_{u \in C^1_c(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + mL\|\nabla u\|^2_{L^\infty(\Omega)}}{\int_\Omega |u|^2\,dx} \\
&= \inf_{u \in C^1_c(\Omega)\setminus\{0\}} \left\{ \sup_{\mu \in \mathfrak{A}_L} J(u,\mu) \right\},
\end{aligned}$$

and this proves that (2.20) is actually an identity which, in turn, implies (2.19). □

Now that we know the existence of an optimal measure $\bar{\mu}$, we can investigate the minimization problem associated with the functional

$$J_1(u) = \int_\Omega |\nabla u|^2\,dx + mL\|\nabla u\|^2_{L^\infty(\Omega)}$$

where $u \in H^1_0(\Omega)$ satisfies the additional constraint $\|u\|_{L^2(\Omega)} = 1$ since, thanks to (2.19), this is completely equivalent to solving our initial problem.

Theorem 2.4 *The minimization problem*

$$\min\left\{J_1(u) \,:\, u \in H^1_0(\Omega),\ \|u\|_{L^2(\Omega)} = 1\right\}$$

admits a unique solution $\bar{u} \in W^{2,p}(\Omega)$ for all $p \in (2,\infty)$. Moreover, if Ω is convex, then $\bar{u} \in W^{2,\infty}(\Omega)$.

This is a well-known result and a proof can be found, for example, in the paper [79]. We are now ready to finish the proof of our main result for convex domains:

Proof of Theorem 2.1. Let $\bar{\mu}$ be as above. A standard result in elliptic regularity theory (see [18]) tells us that if Ω is either convex or $C^{2,\alpha}$-regular, then there is $\beta(\alpha) := \beta \in (0,1)$ such that

$$\bar{u} \in \arg\min J_1 \implies \bar{u} \in C^{2,\beta}(\bar{\Omega}).$$

By Theorem 2.4, we have that $\Delta\bar{u} \in L^p(\Omega)$ for all $p > 2$ and, in particular, $\Delta\bar{u} \in L^\infty(\Omega)$. Moreover, we can characterize $\bar{u}$ and $\bar{\mu}$ as solutions of the system

$$\begin{cases}
-\operatorname{div}(\bar{\mu}\nabla\bar{u}) = \Delta\bar{u} + \lambda_1\bar{u} & \text{if } x \in \Omega, \\
\bar{u} = 0 & \text{on } \partial\Omega, \\
|\nabla\bar{u}(x)| = \|\nabla\bar{u}\|_{L^\infty(\Omega)} & \text{if } x \in \operatorname{spt}\bar{\mu}.
\end{cases}$$

The right-hand side of the equation belongs to $L^\infty(\Omega)$, so by standard regularity results for the Monge-Kantorovich problem [65, 66, 140] we deduce that

$$\bar{\mu} \in L^p(\Omega) \quad \text{for all } p \in (2,+\infty],$$

and this concludes the proof. □

2.4 Existence and regularity with radial symmetry

This section shows that the optimization problem (2.3) has radially symmetric solutions when the domain is radial. Moreover, we give an explicit expression for $\bar{\vartheta}$ when $\Omega = B_1$.

Lemma 2.9 *If Ω is radially symmetric, then $\bar{\vartheta}$ and $\bar{u}$ are radially symmetric functions.*

Proof. Let u_p be the function given in Proposition 2.3 and recall that it is the unique positive solution with unitary L^2-norm of the minimization problem

$$\min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx + mL\|\nabla u\|^2_{L^{2p'}(\Omega)}}{\int_\Omega |u|^2\, dx}. \tag{2.21}$$

Denote by $u^\star$ the Steiner symmetrization of a function $u \in H_0^1(\Omega)$, and recall (Theorem 1.11) that, for every $1 \le p < \infty$, the Polya-Szegö's inequality yields

$$\int_{\Omega^\star} |\nabla u^\star|^p\, dx \le \int_\Omega |\nabla u|^p\, dx.$$

The set $\Omega^\star$ coincides with Ω because it is already radially symmetric, so by Corollary 1.1 we have

$$\int_\Omega |u|^2\, dx = \int_\Omega |u^\star|^2\, dx \qquad \text{for every } u \in H_0^1(\Omega).$$

In particular, if $u \in H_0^1(\Omega)$ is admissible for the minimization problem (2.21), then $u^\star$ is an even *better* competitor since we have proved that

$$\frac{\int_\Omega |\nabla u^\star|^2\, dx + mL\|\nabla u^\star\|^2_{L^{2p'}(\Omega)}}{\int_\Omega |u^\star|^2\, dx} \le \frac{\int_\Omega |\nabla u|^2\, dx + mL\|\nabla u\|^2_{L^{2p'}(\Omega)}}{\int_\Omega |u|^2\, dx}.$$

By minimality and uniqueness, we deduce that each u_p is radially symmetric. Moreover, we know that

$$u_p \xrightarrow{p \to 1^+} \bar{u} \qquad \text{strongly in } L^2(\Omega),$$

so, up to subsequences, we can assume that u_p converges almost everywhere to $\bar{u}$; therefore,

$$\bar{u}(x) = \lim_{p \to 1^+} u_p(x) \implies \bar{u} \text{ radially symmetric at a.e. } x \in \Omega.$$

Now, since $\|\bar{u}\|_{L^2(\Omega)} = 1$, we can characterize the corresponding optimal density $\bar{\vartheta}$ as the unique maximizer of the functional

$$\vartheta \longmapsto \int_\Omega (1 + \vartheta)\nabla \bar{u}\, dx.$$

Since $\bar{u}$ is radially symmetric, we can apply Theorem 1.11 and get

$$\int_\Omega (1 + \vartheta)\nabla \bar{u}\, dx \le \int_\Omega (1 + \vartheta)^* \nabla \bar{u}\, dx,$$

which allows us to conclude that $\bar{\vartheta}$ is also radially symmetric. □

At this point, we want to show that it is possible to find an explicit expression for the optimal density when Ω is the unit disk in $\mathbb{R}^2$. Indeed, we know that $\bar{u}$ and $\bar{\vartheta}$ satisfy the problem

$$\begin{cases} -\operatorname{div}\left((1+\bar{\vartheta})\nabla\bar{u}\right) = \lambda_1\bar{u} & \text{in } B_1 \\ \bar{u} = 0 & \text{on } \partial B_1, \end{cases} \tag{2.22}$$

and, since all functions are radially symmetric by Lemma 2.9, we can use polar coordinates to rewrite the equation as follows:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r(1+\bar{\vartheta})\bar{u}'\right) = \lambda_1\bar{u} \qquad \text{in } B_1. \tag{2.23}$$

Moreover, in Theorem 2.1 we proved that $\bar{\vartheta}$ is supported on the set on which $\nabla\bar{u}$ achieves its maximum value, so there must be $\bar{a} \in (0,1)$ such that

$$\bar{u}'(r) = -\|\nabla\bar{u}\|_{L^\infty(B_1)} \qquad \text{for all } r \in [\bar{a}, 1].$$

This means that, up to a multiplicative constant§, we have

$$\bar{u}(r) = 1 - r \qquad \text{for all } r \in [\bar{a}, 1].$$

If we now plug this expression back into (2.23), we obtain an equation for $\bar{\vartheta}$ (on the set on which it does not vanish) that can be solved easily; more precisely, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r(1+\bar{\vartheta})\right) = \lambda_1(1-r) \qquad \text{for all } r \in [\bar{a}, 1].$$

It follows that the optimal density is given by

$$\bar{\vartheta}(r) = \begin{cases} 0 & \text{if } r \in [0, \bar{a}), \\ -\frac{\lambda_1}{3}r^2 + \frac{\lambda_1}{2}r - 1 + \frac{\bar{a}}{r}\left(1 + \frac{\lambda_1}{3}\bar{a}^2 - \frac{\lambda_1}{2}\bar{a}\right) & \text{if } r \in [\bar{a}, 1]. \end{cases}$$

Now suppose that the value of λ_1 is known. We can find $\bar{a}$ by exploiting the integral condition on the density; more precisely, we have

$$\int_0^1 r\bar{\vartheta}(r)\,dr = \frac{L}{2\pi} \implies \lambda_1(\bar{a}) = 12\left(\frac{L/(2\pi) + (1/2)(\bar{a}-1)^2}{1 - 6\bar{a}^2 + 8\bar{a}^3 - 3\bar{a}^4}\right),$$

which leads to

$$\bar{a} = f^{-1}(\lambda_1(\bar{a})).$$

This equation admits a unique solution in the interval $(0,1)$, provided that λ_1 is bigger than or equal to the minimum of f, which is true for the eigenvalue given by (2.22).

Remark 2.8 In the energy problem, one can prove (see [40, Example 5.1]) that the optimal value $\bar{a}$ is the unique solution of the polynomial equation

$$a^{d+1} - (d+1)\left(1 + \frac{mL}{\omega_d}\right)a + d = 0.$$

Notice that for $L = 0$ the unique solution is $\bar{a} = 1$, and this is compatible with the fact that there is no reinforcement at all. The same is true in our framework since

$$\int_0^1 r\bar{\vartheta}(r)\,dr = \frac{L}{2\pi} = 0 \iff \bar{\vartheta}(r) = 0 \text{ for every } r \in [0,1],$$

which is equivalent to saying that $\bar{a} = 1$.

§ This is possible because our optimization problem admits a unique solution only if we fix the norm; otherwise, any scalar multiple of the same function works.

We can now recover the expression for $\bar{u}$ using the boundary condition naturally arising from the decomposition $[0, \bar{a}) \cup [\bar{a}, 1]$ and the Neumann condition at the origin. Indeed, we have

$$\bar{u}(r) = c_1 J_0\left(\sqrt{\lambda_1} r\right) + c_2 Y_0\left(\sqrt{\lambda_1} r\right) \qquad \text{for all } r \in [0, \bar{a}),$$

where J_0 and Y_0 are the first Bessel functions of first and second kind respectively. To find the two constants, we use the continuity of $\bar{u}$ at $r = \bar{a}$ obtaining

$$c_1 J_0\left(\bar{a}\sqrt{\lambda_1}\right) + c_2 Y_0\left(\bar{a}\sqrt{\lambda_1}\right) = 1 - \bar{a}.$$

Similarly, the Neumann boundary condition gives the equation

$$\lim_{r \to 0^+} \left[c_1 J_1(\sqrt{\lambda_1} r) + c_2 Y_1(\sqrt{\lambda_1} r)\right] = 0,$$

but, since $\lim_{r\to 0^+} Y_1(r) = -\infty$ and $\lim_{r\to 0^+} J_1(r) = 0$, this is satisfied if and only if $c_2 = 0$. If we now go back to the first condition, we find that

$$c_1 = \frac{1 - \bar{a}}{J_0(\bar{a}\sqrt{\lambda_1})},$$

which means that the optimal profile $\bar{u}$ is given by

$$\bar{u}(r) = \begin{cases} \frac{1-\bar{a}}{J_0(\bar{a}\sqrt{\lambda_1})} J_0(\sqrt{\lambda_1} r) & \text{if } r \in [0, \bar{a}), \\ 1 - r & \text{if } r \in [\bar{a}, 1]. \end{cases}$$

To conclude, in Figure 2.1 we show some examples of $\bar{\vartheta}$ for different values of the parameters obtained via numerical simulations. More precisely, we fix any λ_1 such that

$$\lambda_1 \geq j_{0,0}^2,$$

and we find the value of $\bar{a}$ as the unique solution in $(0, 1)$ of the following minimization problem:

$$\min_{a\in(0,1)} \frac{\left[\sqrt{\lambda_1}\frac{1-\bar{a}}{J_1(\bar{a}\sqrt{\lambda_1})}\right]^2 \int_0^a r\left[J_0(\sqrt{\lambda_1} r)\right]^2 dr + \frac{1}{2}(1-a^2) + \frac{mL}{2\pi}}{\left[\frac{1-\bar{a}}{J_0(\bar{a}\sqrt{\lambda_1})}\right]^2 \int_0^a r\left[J_0(\sqrt{\lambda_1} r)\right]^2 dr + \int_a^1 r(1-r)^2\, dr}.$$

Finally, since λ_1 and $\bar{a}$ are known, the length L can be recovered from the identity

$$\lambda_1 = 12\left(\frac{L/(2\pi) + (1/2)(\bar{a}-1)^2}{1 - 6\bar{a}^2 + 8\bar{a}^3 - 3\bar{a}^4}\right).$$

Notice that numerical simulations validate the regularity of $\bar{\vartheta}$ obtained Theorem 2.1 because $\bar{a} \in (0, 1)$ turns out to be the unique one for which w ehave

$$-\frac{(1-\bar{a})\sqrt{\lambda}}{J_0(\bar{a}\sqrt{\lambda_1})} J_1(\sqrt{\lambda_1} r) = \lim_{r\to\bar{a}^-} \bar{u}'(r) = \lim_{r\to\bar{a}^+} \bar{u}'(r) = -1.$$

Remark 2.9 In the energy problem (see [40, Example 5.1]), the optimal density in the radially symmetric case is linear with respect to r, namely it is given by

$$\bar{\vartheta}_f(r) = \left(\frac{r}{\bar{a}} - 1\right)^+ \qquad \text{for all } r \in [0, 1].$$

However, in our framework $\bar{\vartheta}$ is not linear (it depends also on r^2 and r^{-1}) and it is not even monotone increasing since we can prove that there exists $\bar{r} \in (\bar{a}, 1)$ such that

$$\vartheta\,\big|_{(\bar{a},\bar{r})} \text{ is increasing} \qquad \text{and} \qquad \vartheta\,\big|_{(\bar{r},1)} \text{ is decreasing.}$$

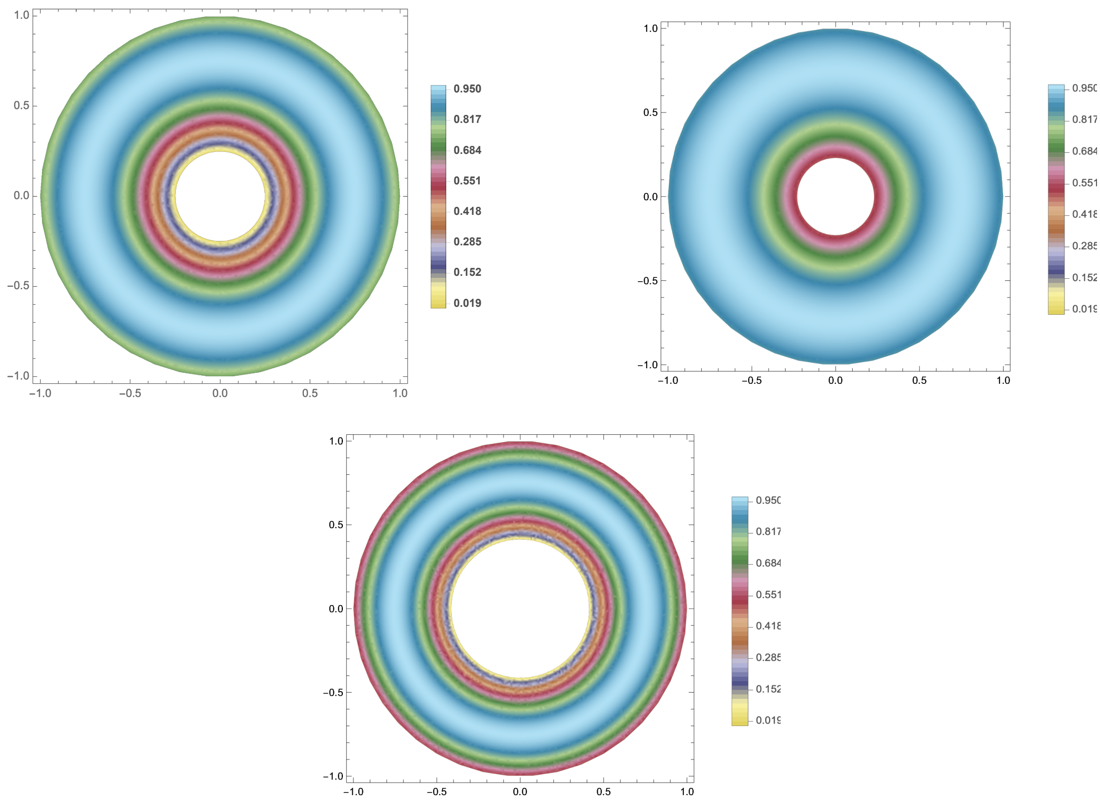

Figure 2.1: Level sets of the optimal density $\bar{\vartheta}$ on the unit disk $\Omega = B(0, 1)$. Parameters from left to right:

$$(\lambda_1, m) = (10, 5), \quad (\lambda_1, m) = (10000, 10) \quad \text{and} \quad (\lambda_1, m) = (7, 1).$$

It is interesting to notice that, depending on the value of λ_1, both the region on which $\bar{\vartheta}$ is equal to zero and the rate of decrease close to the boundary change significantly.

2.5 Proof of Theorem 2.2: the connected case

In this section, we consider the maximization problem (2.5) in which μ ranges among all measures with support $S := \operatorname{spt}\mu$ closed, connected, and satisfying

$$\mu \geq \mathcal{H}^1 \llcorner S.$$

We closely follow the strategy proposed in [2], which deals with optimizing the energy when an external force acts on Ω, and adapt it to the eigenvalue problem.

We already proved in Proposition 2.2 that there exists a solution $\bar{\mu}$ in the class $\mathfrak{A}_L^c$ to (2.5), so we only need to show that we can find a function $\bar{\vartheta} \in L^1(\Omega)$ such that

$$\bar{\mu} = \bar{\vartheta}\,\mathcal{H}^1 \llcorner S, \qquad \text{with } S := \operatorname{spt}\bar{\mu}.$$

To achieve this, we start with some technical results and use them in Proposition 2.6 to deduce that, if we denote by μ^a the absolutely continuous part of μ, then we have

$$\lambda_1(\mu) = \lambda_1(\mu^a).$$

Lemma 2.10 *Let K be a compact set in $\mathbb{R}^2$ with $\mathcal{H}^1(K) = 0$. For all $\epsilon > 0$ there exists a function $\phi_\epsilon \in C^\infty$ satisfying the following properties:*

(1) $|\phi_\epsilon(x) - x| \leq \epsilon$ *for all* $x \in \mathbb{R}^2$;

(2) $\nabla\phi_\epsilon = 0$ *on a neighborhood of* K;

(3) $|\nabla\phi_\epsilon(x)| \leq 1$ *for all* $x \in \mathbb{R}^2$.

If, in addition, we fix $r > 0$, then we can require that $\nabla\phi_\epsilon(x)$ is the 2×2 identity matrix for every $x \notin A_\epsilon$, where A_ϵ is an open set satisfying

$$|A_\epsilon \cap (-r, r)^2| \leq \epsilon.$$

This is a standard result and a proof can be found, for example, in [2, Lemma 3.5].

Lemma 2.11 *Let $\mu, \mu' \in \mathcal{A}_L^c$ be two admissible competitors and assume that*

$$\mu = \mu' + \lambda,$$

where λ is a positive measure supported on a Borel set E with $\mathcal{H}^1(E) = 0$. Then, for every $u \in C_c^\infty(\Omega)$ and every $\delta > 0$, there exists $v \in C_c^\infty(\Omega)$ such that

$$\|v - u\|_\infty \leq \delta \qquad \textit{and} \qquad \int_\Omega |\nabla v|^2 \, d\mu \leq \int_\Omega |\nabla u|^2 \, d\mu' + \delta.$$

Proof. The same argument used in the energy problem [2, Lemma 3.6] works; we will go over the main points here for completeness. Let $\epsilon > 0$ be a fixed parameter that we will choose later and let $K \subset E$ be a compact set such that

$$\lambda(\bar{\Omega} \setminus K) \leq \epsilon.$$

Let $\phi \in C^\infty(\mathbb{R}^2)$ be the function given by Lemma 2.10 and define

$$v(x) := u \circ \phi(x).$$

This is a smooth function with compact support contained in Ω if ϵ is small enough, and, as a consequence of Lemma 2.10 (1), we have the inequality

$$|v(x) - u(x)| = |u(\phi(x)) - u(x)| \leq L_u |\phi(x) - x| \leq L_u \epsilon,$$

where L_u is the Lipschitz constant of u. Taking the sup over x yields

$$\|v - u\|_\infty \leq \delta,$$

provided that we choose ϵ small enough, for example $0 < \epsilon \leq {}^{\delta}/_{L_u}$. We can also estimate the absolute value of the gradient of v with the one of u using the chain rule

$$|\nabla v(x)| = |\nabla u(\phi(x))| \underbrace{|\nabla\phi(x)|}_{\leq 1} \leq |\nabla u(x)| + |\nabla u(\phi(x)) - \nabla u(x)| \leq |\nabla u(x)| + L_{\nabla u}\epsilon,$$

where $L_{\nabla u}$ is the Lipschitz constant of ∇u and $|\nabla\phi(x)| \leq 1$ by Lemma 2.10 (3). Finally, since $\nabla v = 0$ on the set K, we can estimate the integral as follows:

$$\begin{aligned}\int_\Omega |\nabla v|^2\, d\mu &\le \int_{\Omega\setminus K} |\nabla u|^2\, d\mu + L_u \epsilon \\ &\le \int_\Omega |\nabla u|^2\, d\mu' + \int_{\Omega\setminus K} |\nabla u|^2\, d\lambda + C\epsilon \\ &\le \int_\Omega |\nabla u|^2\, d\mu' + C'\epsilon.\end{aligned}$$

This concludes the proof since we can choose $0 < \epsilon \le \delta/C$, if it is not small enough already. □

Lemma 2.12 *Let μ and μ' be as in Lemma 2.11. Then, for every $u \in C_c^\infty(\Omega)$ and every $\delta > 0$, there exists a function $v \in C_c^\infty(\Omega)$ such that*

$$\|v - u\|_\infty \le \delta \qquad \textit{and} \qquad J(\mu, v) \le J(\mu', u) + \delta,$$

where

$$J(\nu, w) = \frac{\int_\Omega |\nabla w|^2\, dx + m \int_\Omega |\nabla w|^2\, d\nu}{\int_\Omega |w|^2\, dx}.$$

Proof. To ease the notations, we introduce the auxiliary measures

$$\tilde\mu := dx + m\mu \qquad \text{and} \qquad \tilde\mu' := dx + m\mu'$$

so that we can write the numerators of J as follows:

$$\begin{aligned}\int_\Omega |\nabla v|^2\, dx + m \int_\Omega |\nabla v|^2\, d\mu &= \int_\Omega |\nabla v|^2\, d\tilde\mu, \\ \int_\Omega |\nabla u|^2\, dx + m \int_\Omega |\nabla u|^2\, d\mu' &= \int_\Omega |\nabla u|^2\, d\tilde\mu'.\end{aligned}$$

Fix $\bar\delta \le \delta$ to be chosen later. Let v be the function given by Lemma 2.11 with $\tilde\mu$, $\tilde\mu'$ and $\bar\delta$; to prove that $J(\mu, v) \le J(\mu', u) + \delta$, we notice that the estimate $\|u - v\|_\infty \le \bar\delta$ implies

$$\int_\Omega |u|^2\, dx \le \int_\Omega |v|^2\, dx + \int_\Omega |u - v|^2\, dx \le \int_\Omega |v|^2 + |\Omega|\bar\delta^2,$$

which takes care of the denominators. Similarly, we can use the integral estimate of Lemma 2.11 to deduce an inequality for the numerators, i.e.,

$$\int_\Omega |\nabla v|^2\, d\tilde\mu \le \int_\Omega |\nabla u|^2\, d\tilde\mu' + \bar\delta.$$

If we put everything together, we get

$$J(\mu, v) = \frac{\int_\Omega |\nabla v|^2\, d\tilde\mu}{\int_\Omega |v|^2\, dx} \le \frac{\int_\Omega |\nabla u|^2\, d\tilde\mu'}{\int_\Omega |u|^2\, dx - |\Omega|\bar\delta^2}, \tag{2.24}$$

so, if we consider the Taylor expansion of the right-hand side at $\bar\delta = 0$, we deduce that

$$\frac{\int_\Omega |\nabla u|^2\, d\tilde\mu'}{\int_\Omega |u|^2\, dx - |\Omega|\bar\delta^2} \le \frac{\int_\Omega |\nabla u|^2\, d\tilde\mu'}{\int_\Omega |u|^2\, dx} + C|\Omega|\bar\delta^2 + \mathcal{O}(\bar\delta^4) = J(\mu', u) + C|\Omega|\bar\delta^2 + \mathcal{O}(\bar\delta^4),$$

and this, together with (2.24), concludes the proof if we take $\bar\delta$ small enough. □

We now have all the ingredients necessary for the proof of Theorem 2.2, which is an immediate consequence of the following result:

Proposition 2.6 *Let $\mu, \mu' \in \mathscr{A}_L^c$ be as in Lemma 2.11. Then*

$$\lambda_1(\mu) = \lambda_1(\mu').$$

In particular, the optimal measure solving (2.5) is absolutely continuous with respect to $\mathscr{H}^1 \llcorner S$.

Proof. The inequality $\lambda_1(\mu') \geq \lambda_1(\mu)$ is a consequence of the fact that $\lambda_1(\cdot)$ is increasing, while the opposite one is given by Lemma 2.12. □

2.5.1 Analysis of boundary points

Let $\bar{\mu} = \bar{\vartheta}\,\mathscr{H}^1 \llcorner S$ be the optimal measure obtained in Theorem 2.2 and let $\bar{u}$ be a solution of the associated minimization problem

$$\lambda_1(S) = \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + m\int_S \bar{\vartheta}|\nabla_\tau u|^2\,d\mathscr{H}^1}{\int_\Omega |u|^2\,dx}. \tag{2.25}$$

The first variation of the functional is equal to zero at $\bar{u}$, which means that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \frac{\int_\Omega |\nabla(u+\epsilon v)|^2\,dx + m\int_S \bar{\vartheta}|\nabla_\tau(u+\epsilon v)|^2\,d\mathscr{H}^1}{\int_\Omega |u+\epsilon v|^2\,dx} = 0 \qquad \text{for all } v \in H_0^1(\Omega),$$

and a straightforward computation leads to the following equality:

$$\int_\Omega \bar{u}^2\,dx \left[\int_\Omega \nabla\bar{u}\cdot\nabla v\,dx + m\int_S \bar{\vartheta}(\nabla_\tau\bar{u}\cdot\nabla_\tau v)\,d\mathscr{H}^1\right]$$
$$\cdots - \int_\Omega \bar{u}v\,dx \left[\int_\Omega |\nabla\bar{u}|^2\,dx + m\int_S \bar{\vartheta}|\nabla_\tau\bar{u}|^2\,d\mathscr{H}^1\right] = 0.$$

If we let $\bar{u}$ be the minimizer of (2.25) with L^2-norm equal to one, we can plug the identity

$$\lambda_1(S) = \int_\Omega |\nabla\bar{u}|^2\,dx + m\int_S \bar{\vartheta}|\nabla_\tau\bar{u}|^2\,d\mathscr{H}^1$$

into the first variation above to obtain

$$\int_\Omega \nabla\bar{u}\cdot\nabla v\,dx + m\int_S \bar{\vartheta}(\nabla_\tau\bar{u}\cdot\nabla_\tau v)\,d\mathscr{H}^1 - \lambda_1(S)\int_\Omega \bar{u}v\,dx = 0 \qquad \text{for all } v \in H_0^1(\Omega). \tag{2.26}$$

To find the Euler-Lagrange equations, we need v to replace ∇v in all the integrals above; therefore, integrating by parts the first term, we get

$$\int_\Omega \nabla\bar{u}\cdot\nabla v\,dx = -\int_\Omega \Delta\bar{u}v\,dx + \int_S \left[\frac{\partial\bar{u}}{\partial\nu}\right] v\,d\mathscr{H}^1,$$

where we used the notation

$$\left[\frac{\partial\bar{u}}{\partial\nu}\right] := \partial_+\bar{u} + \partial_-\bar{u}.$$

This term does not depend on the choice of an orientation for S since $\partial_\pm u$ are, respectively, the positive and negative derivative of $\bar{u}$ on S. Now notice that

$$\nabla_\tau\bar{u}\cdot\nabla_\tau v = (\nabla\bar{u}\cdot\tau)\tau\cdot(\nabla v\cdot\tau)\tau = (\nabla\bar{u}\cdot\tau)\cdot(\nabla v\cdot\tau),$$

so we can integrate by parts the boundary term of (2.26) and obtain

$$m \int_S \bar{\vartheta}(\nabla_\tau \bar{u} \cdot \nabla_\tau v)\, d\mathcal{H}^1 = -m \int_S \mathrm{div}_\tau(\bar{\vartheta}\nabla_\tau \bar{u}) v \, d\mathcal{H}^1 + m \left[v \bar{\vartheta} \nabla_\tau \bar{u} \right]_{S^\#},$$

where $-\mathrm{div}_\tau(-\nabla_\tau)$ is the *Laplace-Beltrami operator* on S and $S^\#$ is the set of terminal-type and branching-type points of S.

Proposition 2.7 *If $\bar{u} \in H^1_0(\Omega) \cap C^2(\bar{\Omega})$ is a minimum point of (2.25), then (up to a multiplicative constant) it is the unique solution of the following boundary-value problem:*

$$\begin{cases} -\Delta \bar{u} = \lambda_1(S) u & \text{in } \Omega \setminus S, \\ [\partial \bar{u}/\partial \nu] - m \ \mathrm{div}_\tau(\bar{\vartheta}\nabla_\tau \bar{u}) = 0 & \text{in } S, \\ \bar{u} = 0 & \text{on } \partial\Omega, \\ \bar{\vartheta}(x)\nabla_\tau \bar{u}(x) = 0 & \text{if } x \in S^\#. \end{cases}$$

It is worth remarking that the points in $S^\#$ can be of three different kinds and, in general, we should expect all of them to occur:

- **Dirichlet.** If $x \in S^\# \cap \partial\Omega$, then $u(x) = 0$.
- **Neumann.** If $x \in S^\#$ is a terminal point of S, then $\nabla_\tau u(x) = 0$.
- **Kirchhoff.** If $x \in S^\#$ is a branching point of S, then

$$\sum_i \nabla_{\tau_i} u^i(x) = 0,$$

  where u^i is the trace of u over the i-th branch of S ending at x and τ_i the tangent vector.

As a consequence, using Proposition 2.7 and Theorem 2.2, it is easy to verify that [2, Proposition 4.1], obtained in the energy problem, also holds in the eigenvalue framework.

Proposition 2.8 *Let $\bar{\mu}$ be a solution of the maximization problem (2.6) and let $\bar{u}$ be the unique positive solution of (2.25) with $\|\bar{u}\|_{L^2(\Omega)} = 1$. Then there exists a constant $c > 0$ such that*

$$\begin{cases} |\nabla_\tau \bar{u}| = c & \text{for } \mathcal{H}^1\text{-a.e. } x \in \{\bar{\vartheta}(x) > 1\}, \\ |\nabla_\tau \bar{u}| \le c & \text{for } \mathcal{H}^1\text{-a.e. } x \in \{\bar{\vartheta}(x) = 1\}. \end{cases}$$

2.6 Open problems

In conclusion, we discuss some open questions related to the problems considered in [35] and suggest possible ways to approach them, referring to the energy case.

Problem 2.1 In Theorem 2.1 we proved that, when Ω is regular enough (for example, with boundary of class $C^{1,\alpha}$), the optimization problem

$$\max_{\mu \in \mathfrak{A}_L} \lambda_1(\mu)$$

admits a solution of the form $\bar{\mu} = \bar{\vartheta}\, dx$, with $\bar{\vartheta} \in C^{1,\beta}(\bar{\Omega})$ for some $\beta \in (0,1)$ depending on α.

It would be interesting to know if additional regularity properties hold in general. Indeed, in the radially symmetric case we showed that

$$\bar{\vartheta}(r) = \begin{cases} 0 & \text{if } r \in [0, \bar{a}), \\ -\frac{\lambda_1}{3} r^2 + \frac{\lambda_1}{2} r - 1 + \frac{\bar{a}}{r}\left(1 + \frac{\lambda_1}{3}\bar{a}^2 - \frac{\lambda_1}{2}\bar{a}\right) & \text{if } r \in [\bar{a}, 1], \end{cases}$$

which is smooth everywhere except at $r = \bar{a}$ where, taking into account that the value of $\bar{a}$ is difficult to find explicitly, we can only verify up to the $C^{1,\beta}$-regularity.

Problem 2.2 In Theorem 2.2 we proved that (2.3) admits a solution of the form

$$\bar{\mu} = \bar{\vartheta}\,\mathcal{H}^1 \llcorner S \qquad \text{for some } S \subset \Omega \text{ closed and connected,}$$

where $\bar{\vartheta} \in L^1(S)$, with $\mathcal{L}(S) \le L$ and $\bar{\vartheta} \ge 1$ on S.

We expect both S and $\bar{\vartheta}$ to be more regular (even without additional assumptions), but this seems quite challenging to prove following the strategy proposed here. In particular, it would be interesting to prove or disprove the regularity of S up to a finite number of branching points (see Proposition 2.7 for more details) for which the *Kirchhoff rule* holds:

$$\sum_{i=1}^{\ell} \nabla_{\tau_i} u = 0.$$

Problem 2.3 For the optimal set given by Theorem 2.2, several optimality conditions are worth investigating, for example, the following ones, as they appear in other problems (e.g., the energy problem) studied in optimal transport and structural mechanics [39, 42, 53].

(a) Does S contain closed loops (subsets homeomorphic to the circle $\mathbb{S}^1$)?

(b) Do the branching points of S only have three branches as in the energy case, or is a higher number possible?

(c) Does S intersect the boundary $\partial\Omega$?

(d) Is it possible, in some cases, that there is a nontrivial subset $K \subset S$ on which we have $\bar{\vartheta} > 1$ strictly?

In the energy problem, any optimal set S does not contain any closed loop, has at most three branches at branching points, and intersects the boundary (possibly multiple times). A proof of these facts is not trivial and requires an entirely different approach, for example, by adapting the techniques of [53] to the eigenvalue problem (although there are some technical issues to overcome).

That said, it should be possible to prove or disprove (d) by direct computation. Indeed, if S is the optimal set with $\mathcal{L}(S) = L$, then we can remove a portion of the set to obtain a competitor

$$S_\epsilon := S \setminus B_{r_\epsilon}(x_0),$$

where $r_\epsilon > 0$ is chosen in such a way that $\mathcal{L}(S_\epsilon) = L - \epsilon < L$. Accordingly, we consider a perturbation of the optimal density given by

$$\bar{\vartheta}_\epsilon := \begin{cases} \bar{\vartheta}(x) & \text{if } x \in S_\epsilon \setminus K_\epsilon, \\ \bar{\vartheta}(x) + \epsilon & \text{if } x \in K_\epsilon, \end{cases}$$

where K_ϵ is a portion of S_ϵ chosen in such a way that the constraint is saturated, i.e.,

$$\int_{S_\epsilon} \bar{\vartheta}_\epsilon(x)\, d\mathcal{H}^1 = L.$$

The idea is to choose S_ϵ and K_ϵ carefully and prove that the measure

$$\tilde{\mu} := \bar{\vartheta}_\epsilon\, \mathcal{H}^1 \llcorner S_\epsilon$$

is a better competitor than $\bar{\mu} = \bar{\vartheta}\, \mathcal{H}^1 \llcorner S$, which, in turn, would imply that the optimal density is > 1 on a nontrivial set. The downside of this strategy is that

$$\lambda_1(\tilde{\mu}) \geq \lambda_1(\bar{\mu})$$

is a very difficult inequality to prove: the choice of S_ϵ is delicate since it is not clear, a priori, which portion of S is convenient (in terms of the maximization of the eigenvalue) to remove.

Problem 2.4 We proved that the maximization problem in which we allow sets of at most N connected components has a solution of the form

$$\bar{\mu} = \sum_{i=1}^{\ell} \bar{\vartheta}_i \mathcal{H}^1 \llcorner S_i,$$

with $\ell \leq N$, but we did not establish whether or not $\ell = N$.

A possible way to approach this problem is the following. Let $N = 2$, suppose that S is connected (i.e., $\ell = 1$), and consider the perturbation

$$S_\epsilon := S \setminus B_\epsilon(x_0),$$

where $x_0 \in S$ and $\epsilon > 0$ is small enough. By construction S_ϵ has two connected components and, as above, we can modify $\bar{\vartheta}$ in such a way that

$$\int_{S_\epsilon} \bar{\vartheta}_\epsilon \, d\mathcal{H}^1 = L.$$

This is achieved by redistributing the mass lost when a portion of S is removed, but this leads to difficult estimates since it is unclear how to do it in general.

Problem 2.5 The numerical treatment of both optimization problems presents several difficulties since many local maxima are possible, and global optimization algorithms are too slow to compute the "reinforced" first eigenvalue. That said, in the case of energy optimization (see [2] and [40]), it is possible to implement efficient algorithms that allow a certain degree of numerical analysis.

Shape optimization problems for functionals with a boundary integral 3

This chapter discusses the results obtained in [36], which focuses on a shape optimization problem for functionals with a boundary integral term. The prototype for our class of integral functionals is obtained by solving the energy PDE with Robin boundary conditions, i.e.,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \beta u + \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

and minimizing the corresponding energy

$$E(\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega f u\,dx + \frac{\beta}{2}\int_{\partial\Omega} u^2\,d\mathcal{H}^{d-1}.$$

The domain Ω ranges in a class of admissible sets with "good" compactness properties to guarantee the existence of a solution; more precisely, our goal is to solve the following:

Problem. Let $D \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Find a domain Ω that solves the minimization problem

$$\inf\left\{\mathcal{J}(\Omega)\ :\ \Omega \subset D \text{ and } \Omega \text{ Lipschitz}\right\}, \tag{3.2}$$

where the *shape functional* $\mathcal{J}$ is defined by setting

$$\mathcal{J}(\Omega) := \min_{u\in W^{1,p}(\Omega)}\left\{\int_\Omega j(x,u,\nabla u)\,dx + \int_{\partial\Omega} g(x,u)\,d\mathcal{H}^{d-1}\right\}.$$

Here $p > 1$, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, and the integrands j and g satisfy certain properties - see Subsection 3.1.2 for more details -.

The prototype problem discussed above can also be written in the form (3.2), which corresponds to the following integrands:

$$j(x,s,z) = \frac{1}{2}|z|^2 - f(x)s + c \qquad \text{and} \qquad g(x,s) = \frac{\beta}{2}s^2.$$

Note that the scalar c is the Lagrange multiplier associated with the natural measure constraint on Ω, which in our case is $|\Omega| \le |D|$ since $\Omega \subset D$.

Remark 3.1 The shape optimization problem in which the Dirichlet boundary condition

$$u = 0 \qquad \text{on } \partial\Omega$$

replaces the Robin boundary condition and $g(x,0) = 0$ has been considered in [41]. However, the strategy proposed there is different since the boundary integral disappears.

3.1 Formulation of the problem and main results

A shape optimization problem similar to (3.2) was studied by Bucur-Giacomini in [24]. They considered a prototype problem (3.1) with $f = 0$ and Dirichlet boundary condition of the form

$$u = u_0 \qquad \text{on some } D_0 \subset D,$$

and proved the existence of a minimizer as well as some regularity properties. The main difference is that in our framework, there is a linear term (with respect to u) in the first integrand, namely

$$j(x, u, \nabla u) = \frac{1}{2}|\nabla u|^2 - \underline{f(x)u} + c.$$

This raises several technical problems (mainly with some estimates), which will require additional assumptions or different strategies.

The key idea, introduced in [24], is to consider a relaxation (see Section 1.4) of the initial problem which is obtained by extending all functions $u \in W^{1,p}(\Omega)$ to zero outside Ω, namely

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

These functions are not obtained through the extension operator, so we cannot expect them to be elements of $W^{1,p}(\mathbb{R}^d)$ as the following example shows:

Example 3.1 If Ω is a regular domain, then the functions in $H^1(\Omega)$ that can be extended to zero and still belong to $H^1(\mathbb{R}^d)$ are precisely the functions in $H^1_0(\Omega)$. Consequently, any

$$u \in H^1(\Omega) \setminus H^1_0(\Omega)$$

is a counterexample to the assertion above. Indeed, if we consider $\Omega = (0, 1) \subset \mathbb{R}$ and

$$u(x) = 1 \qquad \text{for all } x \in \Omega,$$

then $u \in H^1(\Omega)$, but the extension $\tilde{u}$ is not continuous and therefore it does not belongs to $H^1(\mathbb{R})$ since by Theorem 4.7 the embedding $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ is continuous.

That said, the extended function $\tilde{u}$ belongs to $\mathrm{SBV}(\mathbb{R}^d)$ (see Section 1.5 for more details); therefore, the boundary integral is not well-defined and must be replaced by

$$\int_{\partial\Omega} g(x, u)\, d\mathcal{H}^{d-1} \to \int_{J_u} \left[g(x, u^+) + g(x, u^-)\right] \, d\mathcal{H}^{d-1},$$

where $u^\pm$ and J_u are, respectively, the traces of u and the jump set (see Definition 1.13).

The main result of [36] is the existence of an optimal shape $\bar{\Omega} \subset D$ which is open, has finite perimeter and Lipschitz boundary. We first consider the relaxation of (3.2) to the class

$$\mathcal{A}(D) := \left\{\Omega \subset D \;:\; \Omega \text{ open, } \partial\Omega \text{ is } \mathcal{H}^{d-1}\text{-rectifiable and } \mathcal{H}^{d-1}(\partial\Omega) < \infty\right\},$$

on which the integral functional takes the form

$$\mathcal{J}(\Omega, u) := \int_\Omega j(x, u, \nabla u)\, dx + \int_{\partial\Omega} \left[g(x, u^+) + g(x, u^-)\right] \, d\mathcal{H}^{d-1}$$

and, for any admissible competitor $\Omega \in \mathcal{A}(D)$, we set

$$\mathcal{J}(\Omega) := \inf_{u \in W^{1,p}(\Omega)} \mathcal{J}(\Omega, u). \tag{3.3}$$

3.1.1 Outline of the chapter

The first step is to show that the shape optimization problem $\min_{\Omega\in\mathcal{A}(D)} \mathcal{J}(\Omega)$ can be reformulated in terms of the *free discontinuity functional*

$$\mathcal{F}(u) := \int_{\mathbb{R}^d\cap\{u\neq 0\}} j(x,u,\nabla u)\,dx + \int_{J_u} \left[g(x,u^+)+g(x,u^-)\right]\,d\mathcal{H}^{d-1}, \tag{3.4}$$

which is defined on the functional space

$$\mathcal{F}_D := \left\{u \in \mathrm{SBV}(\mathbb{R}^d) \;:\; u = 0 \text{ on } \mathbb{R}^d \setminus D\right\}.$$

More precisely, we prove in Lemma 3.1 that any solution $\bar{u}$ of the optimization problem

$$\min\left\{\mathcal{F}(u) \;:\; u \in \mathcal{F}_D\right\} \tag{3.5}$$

leads to a minimizer of (3.3) simply by setting

$$\bar{\Omega} := \{\bar{u} \neq 0\}.$$

We cannot solve the minimization problem (3.5) directly because the free discontinuity functional $\mathcal{F}$ is not coercive on $\mathcal{F}_D$. Therefore, we introduce the larger functional space

$$\mathcal{X}_D := \left\{u \;:\; u \vee \epsilon,\; u \wedge (-\epsilon) \in \mathrm{GSBV}(\mathbb{R}^d) \text{ for all } \epsilon > 0 \text{ and } u = 0 \text{ on } \mathbb{R}^d \setminus D\right\},$$

where $\mathrm{GSBV}(\mathbb{R}^d)$ is given by Definition 1.15, and in Subsection 3.2.2 we show that the corresponding minimization problem

$$\min\left\{\mathcal{F}(u) \;:\; u \in \mathcal{X}_D\right\}$$

admits a solution $\bar{u} \in \mathcal{X}_D$ which, a priori, does not belong to $\mathcal{F}_D$. Moreover, if we also assume (3.9) - which corresponds to f decreasing - we can prove that $\bar{u}$ is non-negative.

The next step is to show that, under additional assumptions, the function $\bar{u}$ belongs to $L^\infty(\mathbb{R}^d)$ (see Lemma 3.4) and there exists $\alpha > 0$ such that

$$\bar{u} \geq \alpha \qquad \text{on } \{\bar{u} > 0\},$$

which follows from Theorem 3.4. This means that $\bar{u} \in \mathcal{F}_D$ and, consequently, we have

$$\bar{\Omega} = \{\bar{u} > 0\} \implies \operatorname{Per}\bar{\Omega} < \infty,$$

so, to prove that $\bar{\Omega}$ is a minimizer of (3.3) in $\mathcal{A}(D)$, we only need to show that $\bar{\Omega}$ is open.

Finally, in Section 3.4, we follow the strategy of [24] strictly and use a standard argument concerning almost quasi-solutions of a *Mumford-Shah-type functional*

$$MS(u) := \int_{\mathbb{R}^d} f(x,\nabla u)\,dx + \mathcal{H}^{d-1}(J_u),$$

to prove that $\bar{\Omega}$ is open, and thus belongs to $\mathcal{A}(D)$. In addition, we find a careful approximation of $\bar{\Omega}$, which consists of Lipschitz domains with polyhedral jump sets, and we show that

$$\mathcal{J}(\bar{\Omega}) = \inf_{\substack{\Omega\in\mathcal{A}(D)\\ \Omega \text{ Lipschitz}}} \mathcal{J}(\Omega),$$

which means that $\bar{\Omega}$ is not only a minimizer of (3.3) but also a solution to the initial problem (3.2).

3.1.2 Assumptions on j and g

Throughout this chapter, we indicate by $|\cdot|$ the d-dimensional Lebesgue measure and by $\mathbb{1}_E$ the characteristic function of a set $E \subset \mathbb{R}^d$, which is given by

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

We now describe the assumptions on j and g that are sufficient (but some may not be necessary) to obtain our main results. These are modeled on the integrands

$$j(x, u, \nabla u) = |\nabla u|^p - f(x)u + 1,$$

where f is a function that belongs to a Lebesgue space, and

$$g(x, u) = \beta |u|^q$$

for some $\beta > 0$ and $p \geq q > 1$. Notice that for $p = q = 2$ we obtain the prototype problem (3.1).

Remark 3.2 We divide the assumptions so that we can state each result with as few as possible, but the main idea is that for the existence of a solution to (3.2), all of them are used.

Therefore, we assume that the function j satisfies some (or all) of the following properties:

(j1) We have that

- $x \mapsto j(x, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^d$;
- $(s, \xi) \mapsto j(x, s, \xi)$ is lower semicontinuous for almost every $x \in \mathbb{R}^d$;
- $\xi \mapsto j(x, s, \xi)$ is convex for all $s \in \mathbb{R}$ and almost every $x \in \mathbb{R}^d$.

(j2) The function $x \mapsto j(x, 0, 0)$ is non-negative a.e. and belongs to $L^1(\mathbb{R}^d)$.

(j3) There are $p > 1$ and $L > 0$ such that

$$j(x, s, \xi) - j(x, s, 0) \geq L|\xi|^p, \tag{3.6}$$

and functions $f \in L^\infty(\mathbb{R}^d)$ and $a \in L^1(\mathbb{R}^d)$ for which

$$j(x, s, 0) \geq -f(x)|s|^q - a(x) \qquad \text{for almost every } x \in \mathbb{R}^d. \tag{3.7}$$

Moreover, the uniform norm of f is bounded above by

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq \frac{L}{2q} \lambda_{\frac{\beta_1}{L}, q}(B_D), \tag{3.8}$$

where q and β_1 are given in (g3), B_D is any ball of volume $|D|$, and the *principal Robin eigenvalue* is defined as

$$\lambda_{b,\alpha}(B_D) := \min_{u \in W^{1,\alpha}(B) \setminus \{0\}} \frac{\int_{B_D} |\nabla u|^\alpha \, dx + b \int_{\partial B_D} |u|^\alpha \, d\mathcal{H}^{d-1}}{\int_{B_D} |u|^\alpha \, dx},$$

where $b > 0$ and $\alpha > 1$ are fixed parameters.

(j4) There exists $\epsilon_0 > 0$ such that for almost every $x \in \mathbb{R}^d$ we have

$$j(x, s, 0) - j(x, t, 0) \geq 0 \qquad \text{for all } s < t < \epsilon_0. \tag{3.9}$$

Furthermore, if q is the exponent given in (g3), then it satisfies $q \leq p$ and

$$q > \max\left\{1, \frac{p}{2p-1}\left[p + \frac{p-1}{(d-1)p'}\frac{2}{1+\sqrt{1+\frac{4}{p'(d-1)}}}\right]\right\}, \tag{3.10}$$

where p' is the conjugate exponent of p.

(j5) The lower bound (3.6) is an equality, namely we have

$$j(x,s,\xi) - j(x,s,0) = L|\xi|^p.$$

Moreover, there are costants $M_0, C_j > 0$ such that

$$j(x,s,0) - j(x,t,0) \geq -C_j|s|^q \qquad \text{for all } s \geq t > M_0. \tag{3.11}$$

Similarly, we require that g satisfies some (or all) of the following properties:

(g1) We have
- $x \mapsto g(x,s)$ is measurable for all $s \in \mathbb{R}$;
- $s \mapsto g(x,s)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^d$.

(g2) For every $x \in \mathbb{R}^d$, it turns out that $g(x,0) \leq 0$ and $g(x,0) \in L^1(\mathbb{R}^d)$.

(g3) There exists a continuous positive function

$$\beta_1 : \mathbb{R}^d \longrightarrow (0,\infty) \qquad \text{with } \beta_1 := \min_{x\in\mathbb{R}^d} \beta_1(x) > 0$$

such that for almost every $x \in \mathbb{R}^d$ and every $s \in \mathbb{R}$ we have

$$g(x,s) \geq \beta_1(x)|s|^q. \tag{3.12}$$

(g4) There exists a continuous positive function

$$\beta_2 : \mathbb{R}^d \longrightarrow (0,\infty) \qquad \text{with } \beta_2 := \max_{x\in\mathbb{R}^d} \beta_2(x) > \beta_1$$

such that for almost every $x \in \mathbb{R}^d$ and every $s \in \mathbb{R}$ we have

$$\beta_2(x)|s|^q \geq g(x,s) \geq \beta_1(x)|s|^q.$$

3.1.3 Main results

As mentioned above, the first step is to find a solution to

$$\min\{\mathcal{F}(u) \,:\, u \in \mathcal{X}_D\}.$$

Notice that, since $\mathcal{F}$ is coercive in $\mathcal{X}_D$ by construction, the existence of a minimizer is achieved under very mild assumptions on both j and g

Theorem 3.1 *Under the assumptions (j1)–(j3) and (g1)–(g3), the minimization problem*

$$\min\{\mathcal{F}(u) \,:\, u \in \mathcal{X}_D\} \tag{3.13}$$

admits a solution $\bar{u} \in \mathcal{X}_D$. Moreover, if we set

$$\bar{\Omega} := \{\bar{u} \neq 0\},$$

then the relaxed shape optimization problem associated to (3.3) is solved by $\bar{\Omega}$ on the class of admissible sets

$$\{\Omega \subseteq D \;:\; \Omega \text{ measurable}\} \supset \mathcal{A}(D).$$

The proof of the second assertion follows immediately from Lemma 3.1, once we show that a solution to (3.13) actually exists. Notice that

$$\mathcal{J}(\bar{\Omega}) = \min_{\substack{\Omega \subset D \\ \Omega \text{ measurable}}} \mathcal{J}(\Omega)$$

does not mean that $\bar{\Omega}$ is also a solution in $\mathcal{A}(D)$ since, a priori, it may not belong to $\mathcal{A}(D)$. However, if we can prove that $\bar{\Omega} \in \mathcal{A}(D)$, it follows immediately that

$$\{\Omega \subseteq D \;:\; \Omega \text{ measurable}\} \supset \mathcal{A}(D) \implies \mathcal{J}(\bar{\Omega}) = \min_{\Omega \in \mathcal{A}(D)} \mathcal{J}(\Omega).$$

Therefore, the next step is to prove that $\bar{u}$ belongs to $\mathfrak{F}_D$ (and $\bar{\Omega}$ has a finite perimeter), but this is not obvious and requires additional assumptions on j and g.

Theorem 3.2 *Let $\bar{u}$ be given by Theorem 3.1. Under the assumptions (j1)–(j4) and (g1)–(g4), the function $\bar{u}$ is non-negative and solves the minimization problem (3.5); consequently,*

$$\bar{\Omega} = \{\bar{u} > 0\} \implies \operatorname{Per} \bar{\Omega} < \infty.$$

In particular, the set $\bar{\Omega}$ minimizes the shape functional $\mathcal{J}$ on the class of admissible sets

$$\left\{\Omega \subset D \;:\; \Omega \text{ measurable},\; \partial\Omega \text{ is } \mathcal{H}^{d-1}\text{-rectifiable with } \mathcal{H}^{d-1}(\partial\Omega) < \infty\right\} \supset \mathcal{A}(D).$$

Finally, following the strategy proposed in [24], we show that $\bar{\Omega}$ is open (hence, solving the relaxation on $\mathcal{A}(D)$) and, with some additional effort, that it is also a solution to the initial problem (3.2).

Theorem 3.3 *Let $\bar{u}$ be given by Theorem 3.1. Under the assumptions (j1)–(j5) and (g1)–(g4), the set $\bar{\Omega}$ is open and, in addition, we have*

$$\inf_{\substack{\Omega \in \mathcal{A}(D) \\ \Omega \text{ Lipschitz}}} \mathcal{J}(\Omega) = \mathcal{J}(\bar{\Omega}).$$

3.2 Proof of Theorem 3.1: existence of an optimal profile in $\mathcal{X}_D$

This section aims to prove Theorem 3.1 following the strategy proposed in [41] for the Dirichlet boundary conditions, which consists in reducing the shape optimization problem

$$\min\{\mathcal{J}(\Omega) \;:\; \Omega \in \mathcal{A}(D)\}$$

to an auxiliary problem that does not depend on Ω, which is given by

$$\min\{\mathcal{F}(u) \;:\; u \in \mathfrak{F}_D\},$$

where $\mathcal{F}(u)$ is the free discontinuity functional (3.4). Recall that the initial problem (3.2) requires us to minimize the shape functional

$$\mathcal{J}(\Omega) = \min\left\{\int_\Omega j(x,u,\nabla u)\,dx + \int_{\partial\Omega} g(x,u)\,d\mathcal{H}^{d-1} \;:\; u \in W^{1,p}(\Omega)\right\}$$

among all Lipschitz domains Ω contained in D; however, the lack of compactness for minimizing sequences makes it impossible to proceed directly. Thus we consider the functional

$$\mathcal{J}(\Omega) = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega j(x, u, \nabla u)\, dx + \int_{J_u} \left[g(x, u^+) + g(x, u^-) \right] d\mathcal{H}^{d-1} \right\},$$

and the corresponding minimization problem with Ω contained in D and measurable. Notice that, since we are allowing irregular sets to be competitors, the boundary term

$$\int_{\partial\Omega} g(x,u)\, d\mathcal{H}^{d-1}$$

is not well-defined, and hence we generalize it using the traces $u^\pm$ and the jump set (which is contained in $\partial\Omega$ by construction). Moreover, we use the notation

$$u(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

since the relaxation of our initial problem is obtained, as explained in Section 3.1, by extending to zero outside of Ω the functions in $W^{1,p}(\Omega)$.

3.2.1 Reduction to the auxiliary problem

The goal of this section is to prove that any solution to the relaxed minimization problem introduced above can be characterized as

$$\bar{\Omega} = \{\bar{u} \neq 0\},$$

where $\bar{u}$ minimizes the free discontinuity functional $\mathcal{F}$. The main advantage of getting rid of the shape as a variable is that the "new" class of admissible competitors is

$$\mathcal{F}_D = \left\{ u \in \mathrm{SBV}(\mathbb{R}^d) \ : \ u = 0 \text{ on } \mathbb{R}^d \setminus D \right\},$$

which means that we can apply the compactness theory for $\mathrm{SBV}(\mathbb{R}^d)$ developed in Section 1.5 to prove the existence of a solution via the direct method in the calculus of variations.

Lemma 3.1 *Suppose that (j2) and (g2) hold and let* $\bar{u}$ *be a solution of*

$$\min \left\{ \mathcal{F}(u) \ : \ u \in \mathcal{F}_D \right\}.$$

Then the set $\bar{\Omega} := \{\bar{u} \neq 0\}$ *is a solution to the minimization problem* $\min_{\Omega \in \mathcal{A}(D)} \mathcal{J}(\Omega)$.

Remark 3.3 Notice that the same conclusion holds if $\bar{u}$ minimizes $\mathcal{F}(u)$ in $\mathcal{X}_D$, but the corresponding optimal shape $\bar{\Omega}$ is a solution in a larger class of admissible sets, i.e.,

$$\mathcal{J}(\bar{\Omega}) = \min_{\substack{\Omega \subset D \\ \Omega \text{ measurable}}} \mathcal{J}(\Omega).$$

We will use both characterizations since we first show that there exists a solution $\bar{u} \in \mathcal{X}_D$, which corresponds to $\bar{\Omega}$ measurable, and then prove the properties

$$\bar{u} \in L^\infty \quad + \quad \bar{u} \geq \alpha > 0$$

to deduce that $\bar{u} \in \mathfrak{F}_D$ and, consequently, that $\bar{\Omega} \in \mathcal{A}(D)$.

Proof. Let $u \in W^{1,p}(\Omega)$ and extend it to zero outside of Ω. It is easy to verify that

$$\begin{aligned}\int_\Omega j(x,u,\nabla u)\,dx &= \int_{\mathbb{R}^d} j(x,u,\nabla u)\,dx - \int_{\mathbb{R}^d\setminus\Omega} j(x,0,0)\,dx\\ &= \int_{\mathbb{R}^d} j(x,u,\nabla u)\,dx + \int_{\Omega} j(x,0,0)\,dx - \int_{\mathbb{R}^d} j(x,0,0)\,dx\\ &= \int_{\{u\neq 0\}} j(x,u,\nabla u)\,dx + \int_{\Omega} j(x,0,0)\,dx\\ &\geq \int_{\{u\neq 0\}} j(x,u,\nabla u)\,dx\end{aligned}$$

since $j(x,0,0) \geq 0$ by assumption (j2). Similarly, by (g2) we get

$$\begin{aligned}\int_{\partial\Omega} g(x,u)\,d\mathcal{H}^{d-1} &= \int_{J_u} \left[g(x,u^+) + g(x,0)\right]\,d\mathcal{H}^{d-1} - \int_{\partial\Omega} g(x,0)\,d\mathcal{H}^{d-1}\\ &\geq \int_{J_u} \left[g(x,u^+) + g(x,0)\right]\,d\mathcal{H}^{d-1}.\end{aligned}$$

Notice that, since u does not belong to $W^{1,p}(\mathbb{R}^d)$, the left trace u^- might not be equal to zero on J_u and therefore, it is necessary to replace the last integral with

$$\int_{J_u} \left[g(x,u^+) + g(x,u^-)\right]\,d\mathcal{H}^{d-1}.$$

Now let $\bar{u} \in \mathfrak{F}_D$ be a minimizer of $\mathcal{F}$. For any $\Omega \in \mathcal{A}(D)$, there is u_Ω that achieves the inf in $\mathcal{J}(\Omega)$, allowing us to write the functional as follows:

$$\mathcal{J}(\Omega) = \int_\Omega j(x,u_\Omega,\nabla u_\Omega)\,dx + \int_{\partial\Omega} \left[g(x,u_\Omega^+) + g(x,u_\Omega^-)\right]\,d\mathcal{H}^{d-1}.$$

Using the minimality of $\bar{u}$ and the estimates obtained above we get

$$\mathcal{J}(\Omega) \geq \mathcal{F}(u_\Omega) \geq \mathcal{F}(\bar{u}) \geq \mathcal{J}(\bar{\Omega}) \qquad \text{for all } \Omega \in \mathcal{A}(D),$$

which is enough to conclude the proof since it implies that

$$\min_{\Omega\in\mathcal{A}(D)} \mathcal{J}(\Omega) = \mathcal{J}(\bar{\Omega}).$$

□

The last ingredient that plays a crucial role in proving the existence of a minimizer in $\mathcal{X}_D$ is the following *Poincaré-type inequality*, which was established in [23] for $p = 2$ and $\alpha \in [1,2]$, though it can be extended with minimal changes to any $p > 1$ by working in the space

$$\mathrm{SBV}^{1/p}(\mathbb{R}^d) := \left\{u \in L^p(\mathbb{R}^d) \,:\, u \geq 0 \text{ a.e. in } \mathbb{R}^d \text{ and } u^p \in \mathrm{SBV}(\mathbb{R}^d)\right\}.$$

Lemma 3.2 *Let $p > 1$, $\alpha \in [1,p]$ and $b, m > 0$. For any $u \in \mathrm{SBV}(\mathbb{R}^d)$ that satisfies $|\{u \neq 0\}| \leq m$, we have*

$$\frac{\int_{\mathbb{R}^d} |\nabla u|^p\,dx + b\int_{J_u} \left[|u^+|^p + |u^-|^p\right]\,d\mathcal{H}^{d-1}}{\|u\|^p_{L^\alpha(\mathbb{R}^d)}} \geq \lambda_{b,\alpha}(B_m), \tag{3.14}$$

where B_m is a ball of measure m and $\lambda_{b,\alpha}(B)$ the principal Robin eigenvalue. Moreover, the equality holds if and *only if u is the first eigenfunction, i.e. the solution (up to a multiplicative constant) of*

$$\lambda_{b,\alpha}(B_m) = \min_{v\in W^{1,p}(B_m)\setminus\{0\}} \frac{\int_{B_m}|\nabla v|^p\,dx + b\int_{\partial B_m}|v|^p\,d\mathcal{H}^{d-1}}{\left(\int_{B_m}|v|^\alpha\,dx\right)^{p/\alpha}}.$$

3.2.2 Proof of Theorem 3.1

The goal of this section is to put together all the technical results presented so far to show that the free discontinuity functional $\mathcal{F}$ has a minimizer $\bar{u}$ in the class $\mathcal{X}_D$; we achieve this by proving that if we consider the notion of convergence

$$u_n \xrightarrow{\mathcal{X}_D} u \iff \begin{cases} u_n \vee \epsilon \xrightarrow{\mathrm{GSBV}(\mathbb{R}^d)} u \vee \epsilon \\ u_n \wedge (-\epsilon) \xrightarrow{\mathrm{GSBV}(\mathbb{R}^d)} u \wedge (-\epsilon) \end{cases} \qquad \text{for all } \epsilon > 0,$$

then the functional $\mathcal{F}$ is lower semicontinuous and coercive in $\mathcal{X}_D$.

Proof of Theorem 3.1. We divide the proof into two steps: coercivity and lower semicontinuity.

Step 1. *Coercivity of the functional $\mathcal{F}$*

Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{X}_D$ be a sequence on which $\mathcal{F}$ is uniformly bounded, i.e., there exists a positive constant C that does not depend on n such that we have

$$\mathcal{F}(u_n) \le C.$$

To prove that $\mathcal{F}$ is coercive, we need to show that the sequence u_n is uniformly bounded with respect to the $\mathcal{X}_D$ norm or, equivalently, that

$$\|u_n \vee \epsilon\|_{\mathrm{GSBV}(\mathbb{R}^d)} + \|u_n \wedge (-\epsilon)\|_{\mathrm{GSBV}(\mathbb{R}^d)} \le \tilde{C} \qquad \text{for all } \epsilon > 0,$$

where $\tilde{C}$ may depend on ϵ, but not on n. Fix $\epsilon > 0$ and consider the sequence of truncated functions

$$v_{n,\epsilon} := (u_n - \epsilon) \vee 0 + (u_n + \epsilon) \wedge 0.$$

It is easy to verify that $v_{n,\epsilon} \in \mathrm{GSBV}(\mathbb{R}^d)$, but before we can go any further we need to show that there exists a positive constant C_1, that may depend on ϵ, such that

$$\mathcal{F}(u_n) \le C \implies \mathcal{F}(v_{n,\epsilon}) \le C_1.$$

This is easy to prove, but it is more convenient to obtain a more accurate estimate for $v_{n,\epsilon}$. Indeed, if we combine the assumptions (3.6) and (3.12) with the fact that $\mathcal{F}(u_n) \le C$, we get

$$L\int_{\mathbb{R}^d}|\nabla u_n|^p\,dx + \beta_1\int_{J_{u_n}}\left[|u_n^+|^q + |u_n^-|^q\right]d\mathcal{H}^{d-1} \le C' - \int_{\{u_n\neq 0\}} j(x,u_n,0)\,dx, \tag{3.15}$$

where $\beta_1 := \min_{x\in\mathbb{R}^d}\beta_1(x) > 0$, so our goal is to prove that a similar estimate holds for the corresponding truncated function $v_{n,\epsilon}$. For this, we use (3.7) to obtain

$$C' - \int_{\{u_n\neq 0\}} j(x,u_n,0)\,dx \le C_2 + \int_{\mathbb{R}^d} f(x)|u_n|^q\,dx, \qquad \text{with } C_2 := C' + \int_{\mathbb{R}^d} a(x)\,dx,$$

so the estimate (3.15) can be rewritten as follows:

$$L \int_{\mathbb{R}^d} |\nabla u_n|^p \, dx + \beta_1 \int_{J_{u_n}} [|u_n^+|^q + |u_n^-|^q] \, d\mathcal{H}^{d-1} \leq C_2 + \int_{\mathbb{R}^d} f(x)|u_n|^q \, dx. \tag{3.16}$$

By truncation,- $\nabla v_{n,\epsilon}$ coincides with ∇u_n on the set $\{|u_n| \geq \epsilon\}$ and is equal to zero outside; thus, the following inequality is trivially satisfied:

$$\int_{\mathbb{R}^d} |\nabla u_n|^p \, dx \geq \int_{\mathbb{R}^d} |\nabla v_{n,\epsilon}|^p \, dx.$$

The jump set of $v_{n,\epsilon}$ is a subset of J_{u_n} since truncating does not create more jump points (but, instead, removes the ones *close* to the origin); moreover, we have that

$$|v_{n,\epsilon}| > |u_n| \iff |u_n| \in [0, \epsilon),$$

but, as we mentioned above, there are no jump points there since $J_{v_{n,\epsilon}} \cap [0, \epsilon) = \emptyset$. Finally, putting everything together leads to the following estimate for the jump integral:

$$\int_{J_{u_n}} [|u_n^+|^q + |u_n^-|^q] \, d\mathcal{H}^{d-1} \geq \int_{J_{v_{n,\epsilon}}} \left[|v_{n,\epsilon}^+|^q + |v_{n,\epsilon}^-|^q\right] \, d\mathcal{H}^{d-1}.$$

The right-hand side of (3.16), on the other hand, is harder to estimate because an inequality such as

$$\int_{\mathbb{R}^d} f(x)|u_n|^q \, dx \leq \int_{\mathbb{R}^d} f(x)|v_{n,\epsilon} \pm \epsilon|^q \, dx$$

is not valid for any datum f that takes negative values in a non-negligible set contained in the support of u_n (which can occur in our framework). Nonetheless, by assumption (3.8) we have

$$\int_{\mathbb{R}^d} f(x)|u_n|^q \, dx \leq \|f\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |u_n|^q \, dx,$$

and we can estimate $|u_n|^q$ using the definition of $v_{n,\epsilon}$, i.e.,

$$\begin{aligned} |u_n| &\leq |v_{n,\epsilon} + \epsilon| \quad \text{on } \{u_n > 0\}, \\ |u_n| &\leq |v_{n,\epsilon} - \epsilon| \quad \text{on } \{u_n < 0\}. \end{aligned}$$

It turns out that

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)|u_n|^q \, dx &\leq \|f\|_{L^\infty(\mathbb{R}^d)} \left[\int_{\{u_n \geq 0\}} |v_{n,\epsilon} + \epsilon|^q \, dx + \int_{\{u_n \leq 0\}} |v_{n,\epsilon} - \epsilon|^q \, dx\right] \\ &\leq q\|f\|_{L^\infty(\mathbb{R}^d)} \left[\int_{\mathbb{R}^d} |v_{n,\epsilon}|^q \, dx + \epsilon^q |D|\right] \end{aligned}$$

since the support of u_n is a subset of D by definition of $\mathcal{X}_D$. If we now put all the estimates obtained so far together, we finally obtain an estimate similar to (3.16) for $v_{n,\epsilon}$, namely

$$\begin{aligned} L \int_{\mathbb{R}^d} |\nabla v_{n,\epsilon}|^p \, dx + \beta_1 \int_{J_{v_{n,\epsilon}}} &\left[|v_{n,\epsilon}^+|^q + |v_{n,\epsilon}^-|^q\right] \, d\mathcal{H}^{d-1} \\ &\leq C_2 + q\|f\|_{L^\infty(\mathbb{R}^d)} \left[\int_{\mathbb{R}^d} |v_{n,\epsilon}|^q \, dx + \epsilon^q |D|\right]. \end{aligned} \tag{3.17}$$

The next step is to estimate the left-hand side from below using the Poincaré-type inequality (3.14), but it requires gradient and traces to have the same power, so we exploit the fact that

$$a^p \geq a^q - 1 \qquad \text{for every non-negative } a \in \mathbb{R} \text{ and every } p \geq q,$$

to estimate the first integral of (3.17) as follows:

$$\int_{\mathbb{R}^d} |\nabla v_{n,\epsilon}|^p \geq \int_{R^d} |\nabla v_{n,\epsilon}|^q - |D|.$$

Notice that the assumption spt $u_n \subset D$ is crucial here since, otherwise, the integral of -1 would be infinite, making the estimate useless. If we let $C_3 := C_2 + L|D|$, we can rewrite (3.17) as

$$\int_{\mathbb{R}^d} |\nabla v_{n,\epsilon}|^q \, dx + \frac{\beta_1}{L} \int_{J_{v_{n,\epsilon}}} \left[|v_{n,\epsilon}^+|^q + |v_{n,\epsilon}^-|^q \right] \, d\mathcal{H}^{d-1} \\ \leq \frac{C_3}{L} + \frac{q\|f\|_{L^\infty(\mathbb{R}^d)}}{L} \left[\int_{\mathbb{R}^d} |v_{n,\epsilon}|^q \, dx + \epsilon^q |D| \right],$$

which is what we wanted because now gradient and traces have the same power q. Since $v_{n,\epsilon}$ belongs to $\mathrm{SBV}(\mathbb{R}^d)$, we can apply (3.14) and obtain the inequality

$$\|v_{n,\epsilon}\|^q_{L^q(\mathbb{R}^d)} \left[\lambda_{\frac{\beta_1}{L},q}(B) - \frac{q\|f\|_{L^\infty(\mathbb{R}^d)}}{L} \right] \leq C''_L + \frac{q\|f\|_{L^\infty(\mathbb{R}^d)}}{L} \epsilon^q |D|,$$

which, taking into account that $2q\|f\|_{L^\infty(\mathbb{R}^d)} \leq L\,\lambda_{\frac{\beta_1}{L},q}(B)$ by assumption (3.8), leads to the following (linear with respect to ϵ) bound from above:

$$\|v_{n,\epsilon}\|_{L^q(\mathbb{R}^d)} \leq \tilde{C}_\epsilon := \tilde{C}_1 + \tilde{C}_2 \epsilon.$$

The constants do not depend on n, so $v_{n,\epsilon}$ is uniformly bounded in $L^q(\mathbb{R}^d)$. To apply Theorem 1.14, we also need an estimate on the measure of the jump set; for this, we start from

$$\int_{J_{u_n}} [|u_n^+|^q + |u_n^-|^q] \, d\mathcal{H}^{d-1} \leq \frac{\tilde{C}_1 + \tilde{C}_2 \epsilon}{\beta_1},$$

which is obtained by plugging the uniform L^q-bound in (3.16). Indeed, if we denote by $J_{u_n}^{\geq \epsilon}$ the set of all jumps of u_n which are bigger than or equal to ϵ, we immediately deduce that

$$\epsilon^q \mathcal{H}^{d-1} \left(J_{u_n}^{\geq \epsilon} \right) \leq \frac{\tilde{C}_1 + \tilde{C}_2 \epsilon}{2\beta_1}$$

since $|u_n^+|^q + |u_n^-|^q \geq 2\epsilon^q$ by definition of $J_{u_n}^{\geq \epsilon}$. Therefore, for ϵ small enough we have

$$\mathcal{H}^{d-1} \left(J_{u_n}^{\geq \epsilon} \right) \leq \frac{\tilde{C}_1}{2\beta_1} \epsilon^{-q} + \mathcal{O}(\epsilon^{-q+1}),$$

and, using the set identity $J_{v_{n,\epsilon}} = J_{u_n}^{\geq \epsilon}$, we get a uniform bound for the measure of the jump set:

$$\mathcal{H}^{d-1} \left(J_{v_n,\epsilon} \right) \leq \frac{\tilde{C}_1}{2\beta_1} \epsilon^{-q} + \mathcal{O}(\epsilon^{-q-1}).$$

We have now verified all the assumptions necessary to apply Theorem 1.14, which gives (for $\epsilon > 0$ fixed) the existence of a subsequence $v_{n_k,\epsilon}$ such that

$$v_{n_k,\epsilon} \xrightarrow{k \to \infty} u_\epsilon \in \mathrm{GSBV}(\mathbb{R}^d)$$

with respect to the topology of $L^1(\mathbb{R}^d)$. A standard diagonal argument shows that there exists $\bar{u} \in \mathcal{X}_D$ such that for every $\epsilon > 0$ we have

$$u_\epsilon = \bar{u} \vee \epsilon,$$

which means that the limit of each subsequence is the ϵ-truncation of the same function $\bar{u}$ that belongs to the class $\mathcal{X}_D$ as required. Finally, the coercivity of $\mathcal{F}$ follows immediately since

$$\|u_n\|_{\mathcal{X}_D} = \lim_{\epsilon\to 0^+} \|v_{n,\epsilon}\|_{\mathrm{GSBV}(\mathbb{R}^d)} \leq \lim_{\epsilon\to 0^+} \|u_\epsilon\|_{\mathrm{GSBV}(\mathbb{R}^d)} = \|\bar{u}\|_{\mathcal{X}_D} < \infty.$$

Step 2. *Lower semicontinuity of the functional $\mathcal{F}$*

Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{X}_D$ be a sequence converging to some u and let $v_{n,\epsilon}$ be defined as above. For simplicity, consider the decomposition of the free discontinuity functional

$$\mathcal{F}(u) = \mathcal{F}_1(u) + \mathcal{F}_2(u),$$

where

$$\mathcal{F}_2(u) := \int_{\mathbb{R}^d} j(x,u,0)\,dx \qquad \text{and} \qquad \mathcal{F}_1(u) := \mathcal{F}(u) - \mathcal{F}_2(u).$$

The functional $\mathcal{F}_1$ satisfies the assumptions of Theorem 1.13, so $\mathcal{F}_1$ is lower semicontinuous and

$$\liminf_{n\to\infty} \mathcal{F}_1(v_{n,\epsilon}) \geq \mathcal{F}_1(v_\epsilon),$$

where v_ϵ is the limit of $v_{n,\epsilon}$ in $\mathrm{GSBV}(\mathbb{R}^d)$ for $\epsilon > 0$ fixed. It remains to prove that $\mathcal{F}_2$ is also lower semicontinuous or, equivalently, that

$$\int_{\mathbb{R}^d} j(x,v_\epsilon,0)\,dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} j(x,v_{n,\epsilon},0)\,dx.$$

By Fatou's Lemma and the assumption (3.7), we have

$$\int_{\mathbb{R}^d} j(x,v_\epsilon,0)\,dx + \int_{\mathbb{R}^d} -f(x)|v_\epsilon|^q\,dx \leq \liminf_{n\to\infty} \left[\int_{\mathbb{R}^d} j(x,v_{n,\epsilon},0)\,dx + \int_{\mathbb{R}^d} -f(x)|v_{n,\epsilon}|^q\right],$$

so it is sufficient to show that $v_{n,\epsilon}$ converges to v_ϵ strongly in $L^q(\mathbb{R}^d)$. For this, notice that we can assume without loss of generality that there exists $N \in \mathbb{N}$ such that

$$\mathcal{F}(u_n) \leq 0 \qquad \text{for all } n \geq N$$

since u_n is a minimizing sequence and $\mathcal{F}(0) = 0$. This means that we can repeat the argument used in the previous step to deduce that

$$\|v_{n,\epsilon} - v_\epsilon\|_{L^q(\mathbb{R}^d)} \longrightarrow 0$$

for all $\epsilon > 0$ small enough, as a consequence of Lemma 1.1. □

Remark 3.4 The function $\bar{u}$ obtained in the first step may not belong to $\mathrm{GSBV}(\mathbb{R}^d)$ since the estimate on the measure of the jump set tells us that

$$\mathcal{H}^{d-1}(J_{u_\epsilon}) \lesssim \epsilon^{-q},$$

and this does not give any helpful information if we take the limit as $\epsilon \to 0^+$ because the right-hand side blows up to $+\infty$.

Remark 3.5 The assumption $p \geq q$ is merely technical, and therefore we expect that (with some effort) it can be removed. The reason is that if $p < q$, then we have the estimate

$$1 + |v_{n,\epsilon}|^q \geq c_{p,q}|v_{n,\epsilon}|^p,$$

but the integral of -1 now yields $-\mathscr{H}^{d-1}(J_{v_{n,\epsilon}})$. This is a big issue because, even though we can find a constant $c_{n,\epsilon} > 0$ such that

$$\mathscr{H}^{d-1}(J_{v_{n,\epsilon}}) \le c_{n,\epsilon},$$

we have no information on the behavior of $c_{n,\epsilon}$ as $n \longrightarrow +\infty$. Consequently, arguing as we did in the proof above, we would end up with the estimate

$$\|v_{n,\epsilon}\|_{L^p(\mathbb{R}^d)} \le c'_{n,\epsilon},$$

which may not be uniform, and thus cannot be used for the compactness result. Nonetheless, if one can prove that the sequence $c_{n,\epsilon}$ does not blow up as $n \longrightarrow +\infty$, for example

$$\lim_{n \to +\infty} c_{n,\epsilon} = C_\epsilon < \infty \qquad \text{for every } \epsilon > 0 \text{ small enough},$$

then the same argument above would automatically apply to the case $1 < p < q$.

To conclude this section, we show that it is possible to ensure that $\bar{u}$ is a non-negative function at the cost of an additional assumption on j.

Lemma 3.3 *Suppose that the assumptions of Theorem 3.1 hold. If j also satisfies (j4), then the optimal profile $\bar{u}$ is a non-negative function. In particular, we have*

$$\bar{\Omega} = \{\bar{u} \neq 0\} = \{\bar{u} > 0\}.$$

Proof. Let $E := \{\bar{u} < 0\} \subset \bar{\Omega}$, consider the competitor defined by

$$\bar{u}_E(x) := \begin{cases} \bar{u}(x) & \text{if } x \in \mathbb{R}^d \setminus E, \\ 0 & \text{if } x \in E, \end{cases}$$

and notice that, taking $L = 1$ for simplicity, we have

$$\begin{aligned} \mathscr{F}(\bar{u}) - \mathscr{F}(\bar{u}_E) &\ge \int_E \left[j(x, \bar{u}, \nabla \bar{u}) - j(x, 0, 0) \right] \, dx \\ &\ge \int_E \left[|\nabla \bar{u}|^p + j(x, \bar{u}, 0) - j(x, 0, 0) \right] \, dx. \end{aligned}$$

The term $j(x, \bar{u}, 0) - j(x, 0, 0)$ is non-negative because $s \mapsto j(x, s, 0)$ is non-increasing by (3.9), so we have the inequality

$$\mathscr{F}(\bar{u}) - \mathscr{F}(\bar{u}_E) \ge \int_E |\nabla \bar{u}|^p \, dx \ge 0,$$

and this is enough to conclude the proof by the minimality of $\bar{u}$ for $\mathscr{F}$. □

3.3 Proof of Theorem 3.2: the optimal shape $\bar{\Omega}$ has finite perimeter

The goal of this section is to prove that under additional assumptions on j and g, the optimal profile $\bar{u}$ obtained above belongs to $\mathfrak{F}_D$, which, in turn, implies that

$$\bar{\Omega} = \{\bar{u} \neq 0\},$$

has finite perimeter. To achieve this, it is sufficient to show that

$$\bar{u} \in L^\infty(\mathbb{R}^d) \qquad \text{and} \qquad \bar{u} \ge \alpha > 0.$$

3.3.1 Bound on the uniform norm of $\bar{u}$

To show that $\bar{u} \in L^\infty(\mathbb{R}^d)$, we adapt the strategy proposed in the proof of [28, Theorem 12] using the Poincaré-type inequality (3.14). Indeed, by minimality we have

$$\mathcal{F}(\bar{u}) \leq \mathcal{F}(\bar{u} \wedge M),$$

which, with some effort, leads to the inequality

$$\int_{\mathbb{R}^d} |\bar{u} - M|^q \, dx \geq |\Omega|^{-q/q'} \left[\int_{\Omega_M} (\bar{u} - M) \, dx \right]^q,$$

where $\Omega_M := \bar{\Omega} \cap \{\bar{u} > M\}$. At this point, the idea of [28] is to consider the rescaling of Ω_M given by

$$r(M) := \frac{|D|^{1/d}}{|\Omega_M|^{1/d}} \qquad \text{and} \qquad \Omega_M^\# := r(M) \cdot \Omega_M, \tag{3.18}$$

plug it into the inequality above to make it more accurate and conclude that $\bar{u} \in L^\infty(\mathbb{R}^d)$.

Lemma 3.4 *Under the assumptions (j1)–(j4) and (g1)–(g3), there exists a positive constant M such that*

$$\|\bar{u}\|_\infty \leq M.$$

Proof. We proved that $\bar{u}$ minimizes $\mathcal{F}(u)$ on $\mathcal{X}_D$, so we have

$$\mathcal{F}(\bar{u}) \leq \mathcal{F}(\bar{u} \wedge M) \qquad \text{for all } M > 0,$$

which, using the notation for Ω_M introduced above, can be rewritten as follows:

$$\int_{\Omega_M} \big(j(x, \bar{u}, \nabla \bar{u}) - j(x, M, 0) \big) \, dx + \int_{J_{\bar{u}} \cap \{\bar{u}^- > M\}} \left[g(x, \bar{u}^+) + g(x, \bar{u}^-) \right] \, d\mathcal{H}^{d-1}$$
$$\cdots + \int_{J_{\bar{u}} \cap \{\bar{u}^- < M < \bar{u}^+\}} \left[g(x, \bar{u}^+) \right] \, d\mathcal{H}^{d-1} \leq 0.$$

By Lemma 3.3 the function $\bar{u}$ is non-negative, so

$$\int_{J_{\bar{u}} \cap \{\bar{u}^- < M < \bar{u}^+\}} \left[g(x, \bar{u}^+) \right] \, d\mathcal{H}^{d-1} \geq 0$$

and the inequality above can simply be rewritten as

$$\int_{\Omega_M} \big(j(x, \bar{u}, \nabla \bar{u}) - j(x, M, 0) \big) \, dx + \int_{J_{\bar{u}} \cap \{\bar{u}^- > M\}} \left[g(x, \bar{u}^+) + g(x, \bar{u}^-) \right] \, d\mathcal{H}^{d-1} \leq 0.$$

Notice that this is possible because the loss of information due to this simplification does not hinder our strategy. In any case, we can now apply (3.12) to estimate the jump term as

$$\int_{J_{\bar{u}} \cap \Omega_M} \left[g(x, \bar{u}^+) + g(x, \bar{u}^-) \right] \, d\mathcal{H}^{d-1} \geq \beta_1 \int_{J_{\bar{u}} \cap \Omega_M} \left[|\bar{u}^+|^q + |\bar{u}^-|^q \right] \, d\mathcal{H}^{d-1},$$

while for the integral on Ω_M we can add and subtract $j(x, \bar{u}, 0)$ obtaining

$$\int_{\Omega_M} \big(j(x, \bar{u}, \nabla \bar{u}) \pm j(x, \bar{u}, 0) - j(x, M, 0) \big) \, dx,$$

and then apply the assumption (3.6) to infer

$$\int_{\Omega_M} \big(j(x,\bar{u},\nabla\bar{u}) - j(x,\bar{u},0)\big)\, dx \geq L\int_{\Omega_M} |\nabla\bar{u}|^p\, dx \geq L\int_{\Omega_M} |\nabla\bar{u}|^q\, dx - L|\Omega_M|.$$

Putting everything together yields

$$L\int_{\Omega_M} |\nabla\bar{u}|^q\, dx + \beta_1\int_{J_{\bar{u}}\cap\Omega_M} [|\bar{u}^+|^q + |\bar{u}^-|^q]\, d\mathcal{H}^{d-1} \leq L|\Omega_M| + \int_{\Omega_M} \big(j(x,M,0) - j(x,\bar{u},0)\big)\, dx,$$

which, as a consequence of (3.11), can be rewritten as follows:

$$\int_{\Omega_M} |\nabla\bar{u}|^q\, dx + \frac{\beta_1}{L}\int_{J_{\bar{u}}\cap\Omega_M} [|\bar{u}^+|^q + |\bar{u}^-|^q]\, d\mathcal{H}^{d-1} \leq \left(1 + \frac{C_j}{L}M^q\right)|\Omega_M|.$$

Let $v := \max\{\bar{u} - M, 0\}$. The gradients of v and $\bar{u}$ coincide at every $x \in \mathbb{R}^d$, so we have

$$\int_{\Omega_M} |\nabla\bar{u}|^q\, dx = \int_{\mathbb{R}^d} |\nabla v|^q\, dx.$$

Similarly, notice that the jump sets coincide ($J_v = J_{\bar{u}} \cap \Omega_M$), while the size of the jumps is necessarily equal or smaller (due to the definition of v), and thus

$$\int_{J_{\bar{u}}\cap\Omega_M} [|\bar{u}^+|^q + |\bar{u}^-|^q]\, d\mathcal{H}^{d-1} \geq \int_{J_v} [|v^+|^q + |v^-|^q]\, d\mathcal{H}^{d-1}.$$

The idea is to replace $\bar{u}$ with v because the latter belongs to $L^\infty(\mathbb{R}^d)$, and thus we can apply the Poincaré-type inequality (3.14) to v, obtaining

$$\int_{\Omega_M} |\nabla\bar{u}|^q\, dx + \frac{\beta_1}{L}\int_{J_{\bar{u}}\cap\Omega_M} [|\bar{u}^+|^q + |\bar{u}^-|^q]\, d\mathcal{H}^{d-1} \geq \lambda_{\frac{\beta_1}{L},q}(\Omega_M)\int_{\mathbb{R}^d} |v|^q\, dx.$$

Since v coincides with $\bar{u} - M$ on Ω_M and is identically zero on $\mathbb{R}^d \setminus \Omega_M$, we can rewrite the right-hand side in terms of $\bar{u}$ as follows:

$$\int_{\mathbb{R}^d} |v|^q\, dx = \int_{\Omega_M} |\bar{u} - M|^q\, dx.$$

We can estimate this integral from below using Hölder's inequality

$$\int_{\Omega_M} (\bar{u} - M)\, dx \leq |\Omega|^{1/q'}\left[\int_{\Omega_M} |\bar{u} - M|^q\, dx\right]^{1/q},$$

which, in turn, implies that

$$\int_{\mathbb{R}^d} |\bar{u} - M|^q\, dx \geq |\Omega|^{-q/q'}\left[\int_{\Omega_M} (\bar{u} - M)\, dx\right]^q.$$

Now let $f(M) := \int_{\Omega_M} (\bar{u} - M)\, dx$. Then the inequality obtained above can be rewritten as

$$\lambda_{\frac{\beta_1}{L},q}(\Omega_M) f(M)^q \leq \left(1 + \frac{C_j}{L}M^q\right)|\Omega_M|^q,$$

but this is not accurate enough because $f(M)$ and $|\Omega_M|$ have the same exponent; therefore, as we mentioned before, we introduce the rescaling $\Omega_M^\#$ and obtain

$$\lambda_{\frac{\beta_1}{Lr(M)^{q-1}},\, q}\left(\Omega_M^\#\right) f(M)^q \leq \left(1 + \frac{C_j}{L}M^q\right)|\Omega_M^\#|^q,$$

which, using the definition of $r(M)$ given in (3.18), is equivalent to

$$\lambda_{\frac{\beta_1}{Lr(M)^{q-1}},q}(\Omega_M^\#) f(M)^q r(M)^{q-1} |D|^{\frac{1}{d}} \le \left(1 + \frac{C_j}{L} M^q\right) |\Omega_M|^{q+\frac{1}{d}}.$$

Now notice that the derivative of $f(M)$ is given by

$$f'(M) = -\int_{\Omega_M} dx = -|\Omega_M|,$$

so we can rewrite the inequality above as

$$\begin{aligned} -\frac{f'(M)}{f(M)^{\frac{qd}{qd+1}}} &\ge \left[\left(1 + \frac{C_j}{L} M^q\right)^{-1} \lambda_{\frac{\beta_1}{Lr(M)^{q-1}},q}(\Omega_M^\#) r(M)^{q-1}\right]^{\frac{d}{qd+1}} |D|^{\frac{1}{qd+1}} \\ &\ge \left[\left(1 + \frac{C_j}{L} M^q\right)^{-1} \lambda_{\frac{\beta_1}{Lr(M)^{q-1}},q}(B_D) r(M)^{q-1}\right]^{\frac{d}{qd+1}} |D|^{\frac{1}{qd+1}}, \end{aligned}$$

where B_D is any ball of volume equal to $|D|$. Indeed, the principal eigenvalue $\lambda_{b,q}(\cdot)$ is minimized (at fixed volume) by the ball, so

$$\lambda_{b,q}(\Omega_M) \ge \lambda_{b,q}(B) \qquad \text{for every ball } B \text{ such that } |B| = |\Omega_M|.$$

Moreover, for every $t > 0$ we have

$$\lambda_{b,q}(tC) = t^{-2} \lambda_{b,q}(C),$$

which means that $r \mapsto \lambda_{b,q}(B_r)$ is a monotone decreasing function. In particular, since Ω_M is a subset of D, we can use the estimate

$$\lambda_{\frac{\beta_1}{Lr(M)^{q-1}},q}(\Omega_M^\#) \ge \lambda_{\frac{\beta_1}{Lr(M)^{q-1}},q}(B_D),$$

where B_D is any ball of volume $|D|$, to prove the claim. If we now integrate between 0 and $T < \|\bar{u}\|_\infty$, taking into account that f is a non-negative function, we obtain

$$f(0)^{1/(qd+1)} \ge c_{|D|,q} \int_0^T \left[\left(1 + \frac{C_j}{L} M^q\right)^{-1} \lambda_{\frac{\beta_1}{Lr(M)^{q-1}},q}(B) r(M)^{q-1}\right]^{d/(qd+1)} dM.$$

The left-hand side is bounded from above because $f(0) \le \|\bar{u}\|_{L^1(\mathbb{R}^d)}$ by definition; on the other hand, for the right-hand side we can apply [28, Lemma 13] to infer that

$$\lim_{M \to \|\bar{u}\|_\infty} \left(1 + \frac{C_j}{L} M^q\right)^{-1} \lambda_{\frac{\beta_1}{Lr(M)^{q-1}},q}(B) r(M)^{q-1} = \left(1 + \frac{C_j}{L} \|\bar{u}\|_\infty^q\right)^{-1} \frac{\beta_1}{L} d.$$

This concludes the proof of $\|\bar{u}\|_\infty < \infty$ since

$$(M^{-q})^{d/(qd+1)} = M^{-qd/(qd+1)} \qquad \text{and} \qquad -\frac{qd}{qd+1} \in (-1, 0).$$

□

3.3.2 Bound from below on the optimal function $\bar{u}$

We proved that $\bar{u} \in L^\infty(\mathbb{R}^d)$, so to deduce that it belongs to $\mathrm{SBV}(\mathbb{R}^d)$ it is sufficient to show that there exists $\alpha > 0$ such that

$$\bar{u} \geq \alpha \qquad \text{for almost every } x \in \{\bar{u} > 0\}.$$

We follow the strategy proposed in [24, Theorem 3.5], but our framework has two different exponents (p and q), so additional terms will appear in the estimates.

Definition 3.1 (Supersolution) *We say that $w \in \mathcal{F}_D \cap \{u \geq 0\}$ is a **supersolution** for the functional*

$$\mathcal{L}(u) = \int_{\mathbb{R}^d} \left[j(x,u,\nabla u) - j(x,0,0) \right] dx + \int_{J_u} \left[g(x,u^+) + g(x,u^-) \right] d\mathcal{H}^{d-1}$$

if, for every $v \in \mathcal{F}_D \cap \{u \geq 0\}$ with $0 \leq w \leq v$, we have

$$\mathcal{L}(w) \leq \mathcal{L}(v).$$

The functional $\mathcal{L}$ is similar to $\mathcal{F}$, but the Lagrange multiplier does not appear because there is no constraint on the volume of admissible sets. The first step is to prove that

$$\bar{u}_{\geq 0} := \bar{u} \vee 0 \qquad \text{and} \qquad -\bar{u}_{\leq 0} := -(\bar{u} \wedge 0)$$

are both supersolutions for the functional $\mathcal{L}$. This is crucial because, under the assumptions of Theorem 3.2, we have that Lemma 3.3 applies, and hence

$$\bar{u} = \bar{u}_{\geq 0}.$$

This means that $\bar{u}$ is also a supersolution for $\mathcal{L}$, and therefore the bound from below, which we obtain in Theorem 3.4, holds for $\bar{u}$.

Lemma 3.5 *Under the assumptions (j1)–(j3) and (g1)–(g3), the functions $\bar{u}_{\geq 0}$ and $-\bar{u}_{\leq 0}$ are supersolutions for the functional $\mathcal{L}$ in the sense of Definition 3.1.*

Proof. We argue by contradiction. Suppose that $\bar{u}_{\geq 0}$ is not a supersolution for $\mathcal{L}$ and let $v \in \mathcal{F}_D$ be an admissible function (i.e., $0 \leq \bar{u}_{\geq 0} \leq v$) such that

$$\mathcal{L}(v) < \mathcal{L}(\bar{u}_{\geq 0}). \tag{3.19}$$

To find a contradiction it is sufficient to show that we can define a function $\bar{v}$ satisfying $\mathcal{F}(\bar{v}) < \mathcal{F}(\bar{u})$, which is impossible because $\bar{u}$ minimizes $\mathcal{F}$. For this, we let

$$\bar{v}(x) := \begin{cases} \bar{u}(x) & \text{if } \bar{u}(x) \leq 0 \\ v(x) & \text{if } \bar{u}(x) > 0 \end{cases}$$

and we estimate the difference $\mathcal{F}(\bar{u}) - \mathcal{F}(\bar{v})$ as follows:

$$\mathcal{F}(\bar{u}) - \mathcal{F}(\bar{v}) \geq \int_{\{\bar{u}>0\}} \left[j(x,\bar{u},\nabla\bar{u}) - j(x,v,\nabla v) \right] dx$$
$$\cdots + \int_{\{\bar{u}>0\}\cap J_{\bar{u}}} \left[g(x,\bar{u}^+) + g(x,\bar{u}^-) \right] d\mathcal{H}^{d-1} - \int_{\{\bar{u}>0\}\cap J_v} \left[g(x,v^+) + g(x,v^-) \right] d\mathcal{H}^{d-1}.$$

The right-hand side is equal to $\mathcal{L}(\bar{u}_{\geq 0}) - \mathcal{L}(v)$, so we can use (3.19) to deduce that

$$\mathcal{F}(\bar{u}) - \mathcal{F}(\bar{v}) \geq \mathcal{L}(\bar{u}_{\geq 0}) - \mathcal{L}(v) > 0,$$

which is the desired contradiction. □

Theorem 3.4 *Suppose that (j1)–(j4) and (g1)–(g3) hold and let u be a supersolution for $\mathscr{L}$ in the sense of Definition 3.1. Then there exists a positive constant α such that*

$$u \geq \alpha \qquad \text{almost everywhere on } \{u > 0\}. \tag{3.20}$$

Proof. Fix $\epsilon > 0$ and let $u_\epsilon := u \vee \epsilon$. Then $u_\epsilon \in \mathscr{F}_D$ is a competitor as a supersolution for $\mathscr{L}$, which means that, by the minimality of u, we have

$$\mathscr{L}(u) \leq \mathscr{L}(u_\epsilon).$$

This inequality can be rewritten as

$$\int_{\mathbb{R}^d} \big[j(x,u,\nabla u) - j(x,0,0)\big]\, dx + \int_{J_u} \big[g(x,u^+) + g(x,u^-)\big]\, d\mathscr{H}^{d-1}$$
$$\cdots \leq \int_{\mathbb{R}^d} \big[j(x,u_\epsilon,\nabla u_\epsilon) - j(x,0,0)\big]\, dx + \int_{J_{u_\epsilon}} \big[g(x,u_\epsilon^+) + g(x,u_\epsilon^-)\big]\, d\mathscr{H}^{d-1},$$

so, using the assumptions (j2), (g2) and (g4), we get

$$\beta_2 \epsilon^p \mathscr{H}^{d-1}\left(\partial^e\{u > \epsilon\} \setminus J_u\right) \geq \int_{\{u \leq \epsilon\}} \big[|\nabla u|^p + j(x,u,0) - j(x,\epsilon,0)\big]\, dx$$
$$\cdots + \beta_1 \int_{J_u \cap \{u^- < u^+ \leq \epsilon\}} [|u^+|^q + |u^-|^q]\, d\mathscr{H}^{d-1},$$

where ∂^e denotes the *external boundary* (see [107]). The reason is that the jump term on $J_u \cap \{u^- \geq \epsilon\}$ is the same on both sides, while the integral

$$\int_{J_u \cap \{u^- \leq \epsilon < u^+\}} \big[g(x,u^+) + g(x,u^-)\big]\, d\mathscr{H}^{d-1}$$

is non-negative and does not give any helpful information; thus, it can be replaced by zero. Moreover, the assumption (j4) tells us that $j(x,\cdot,0)$ is decreasing, so

$$\int_{\{u \leq \epsilon\}} \big[j(x,u,0) - j(x,\epsilon,0)\big]\, dx \geq 0,$$

which means that this term can also be replaced by zero in the estimate above. It follows that

$$\beta_2 \epsilon^q \mathscr{H}^{d-1}\left(\partial^e\{u > \epsilon\} \setminus J_u\right) \geq L \int_{\{u \leq \epsilon\}} |\nabla u|^p\, dx + \beta_1 \int_{\{u^- < u^+ \leq \epsilon\} \cap J_u} [|u^+|^q + |u^-|^q]\, d\mathscr{H}^{d-1},$$

and therefore, for almost every δ with $0 < \delta < \epsilon$, we have

$$L \int_{\{u < \epsilon\}} |\nabla u|^p\, dx + \beta_1 \delta^q \mathscr{H}^{d-1}\left(\partial^e\{\delta < u < \epsilon\} \cap J_u\right) \leq \beta_2 \epsilon^q \mathscr{H}^{d-1}\left(\partial^e\{u > \epsilon\} \setminus J_u\right).$$

We will refer to this inequality multiple times throughout this proof, so to ease the notations, we introduce the auxiliary functions

$$E(\epsilon) := \int_{\{u \leq \epsilon\}} |\nabla u|^p\, dx, \qquad \gamma(\delta,\epsilon) := \mathscr{H}^{d-1}\left(\partial^e\{\delta < u < \epsilon\} \cap J_u\right),$$

$$h(\epsilon) := \mathscr{H}^{d-1}\left(\partial^e\{u > \epsilon\} \setminus J_u\right).$$

The inequality above can be rewritten as follows:

$$E(\epsilon) + \frac{\beta_1}{L}\delta^q \gamma(\delta, \epsilon) \leq \frac{\beta_2}{L}\epsilon^q h(\epsilon). \tag{3.21}$$

Our goal is to adapt the proof of [24, Theorem 3.5], but we need to be careful because there are two different exponents (p and q); indeed, notice that

$$E(\epsilon) \leq \frac{\beta_2}{L}\epsilon^q h(\epsilon) \implies \left[\int_{\{u<\epsilon\}} |\nabla u|^p \, dx\right]^{1/p} \leq \left(\frac{\beta_2}{L}\right)^{1/p} \epsilon^{q/p} h(\epsilon)^{1/p},$$

so the ratio q/p plays a fundamental role in the following, explaining why we need the technical assumption (3.10), at least with the strategy proposed here.

Step 0. *Setting of the problem*

For $\eta > 0$, we define the sequences

$$\epsilon_i := \frac{5}{6}\eta + \frac{2^{-i}}{6}\eta \qquad \text{and} \qquad \delta_i = \frac{2}{3}\eta - \frac{2^{-i}}{6}\eta,$$

and denote the respective limits as follows:

$$\epsilon_\infty := \lim_{i\to+\infty} \epsilon_i = \frac{5}{6}\eta \qquad \text{and} \qquad \delta_\infty := \lim_{i\to+\infty} \delta_i = \frac{2}{3}\eta.$$

Moreover, for any $0 < \delta < \epsilon$, we introduce the set $\Omega(\delta, \epsilon) := \{\delta < u < \epsilon\}$. We claim that it is enough to show that there exists $\eta_0 > 0$ such that

$$|\Omega(\delta_\infty, \epsilon_\infty)| \int_{\delta_\infty}^{\epsilon_\infty} h(s)\, ds = 0 \qquad \text{for all } \eta < \eta_0. \tag{3.22}$$

Indeed, the isoperimetric inequality (1.10) applied to $\Omega(\delta, \epsilon)$ gives

$$|\Omega(\delta, \epsilon)|^{(d-1)/d} \leq C_d \left(h(\epsilon) + h(\delta) + \gamma(\delta, \epsilon)\right), \tag{3.23}$$

which means that, if we use (3.21) with $\eta/2 < \delta < \epsilon < \eta$, we get

$$|\Omega(\delta, \epsilon)|^{(d-1)/d} \leq \frac{\beta_2}{\beta_1} C_d \left(1 + 2^q\right)\left[h(\epsilon) + h(\delta)\right].$$

This, together with (3.22), allows us to deduce that

$$|\Omega(\delta_\infty, \epsilon_\infty)| = 0,$$

which means that u must be larger than or equal to $(5/6)\eta_0$ almost everywhere on its support, concluding the proof of (3.20) by taking $\alpha := (5/6)\eta_0$.

Step 1. *The main inequalities*

For every $i \in \mathbb{N}$, let

$$a_i := \int_{\delta_i}^{\epsilon_i} h(s)\, ds \qquad \text{and} \qquad b_i := |\Omega(\delta_i, \epsilon_i)|.$$

We claim that there are positive constants c_1 and c_2 such that

$$a_i \leq c_1 \left[\frac{2^i}{\eta^{1-q/p}}\right] a_{i-1}(b_{i-1})^{1/(dp')} \qquad \text{and} \qquad b_i \leq c_2 \left(\frac{2^i}{\eta}\right)^{d/(d-1)} (a_{i-1})^{d/(d-1)}. \tag{3.24}$$

The estimate on b_i is obtained as in [24, Theorem 3.5], so we only focus on the first one and point out the main differences. Indeed, using the coarea formula (see, e.g., [8]), Hölder's inequality, and the isoperimetric inequality (3.23), we get the following chain of inequalities:

$$\begin{aligned}\int_\delta^\epsilon h(s)\,ds &= \int_{\Omega(\delta,\epsilon)} |\nabla u|\,dx \le |\Omega(\delta,\epsilon)|^{1/p'} \|\nabla u\|_{L^p(\Omega(\delta,\epsilon))}\\ &\le |\Omega(\delta,\epsilon)|^{1/p'} \left(\frac{\beta_2}{L}\right)^{1/p} \epsilon^{q/p} h(\epsilon)^{1/p}\\ &= |\Omega(\delta,\epsilon)|^{1/(dp')+(d-1)/(dp')} \left(\frac{\beta_2}{L}\right)^{1/p} \epsilon^{q/p} h(\epsilon)^{1/p}\\ &\le |\Omega(\delta,\epsilon)|^{1/(dp')} \left[C_{d,\beta}(1+2^q)\right]^{1/p'} \left(\frac{\beta_2}{L}\right)^{1/p} [h(\epsilon)+h(\delta)]^{1/p'} \epsilon^{q/p} h(\epsilon)^{1/p}\\ &\le |\Omega(\delta,\epsilon)|^{1/(dp')} \left[C_{d,\beta}(1+2^q)\right]^{1/p'} \left(\frac{\beta_2}{L}\right)^{1/p} \epsilon^{q/p} [h(\epsilon)+h(\delta)]\,.\end{aligned}$$

Remark 3.6 For the last step, we used the obvious inequality

$$h(\epsilon)^{1/p} \le [h(\epsilon)+h(\delta)]^{1/p}$$

with the identity $1/p + 1/p' = 1$; however, it is likely possible to find a more refined inequality and, in turn, to significantly improve the first estimate in (3.24).

If we integrate with respect to ϵ on $[\epsilon_i, \epsilon_{i-1}]$ and δ on $[\delta_{i-1}, \delta]$, we get η^2 on the left-hand side and

$$\epsilon^{\frac{q}{p}}\eta \le \eta^{\frac{q}{p}+1}$$

on the right-hand side; this is enough to conclude the proof of the claim.

Step 2. *Combining the main inequalities.*

We claim that there exists $\zeta > 0$ such that, for every $n \in \mathbb{N}$, $U_n := a_n^\zeta b_n$ satisfies the inequality

$$U_n \le \frac{\tilde{c}}{\eta^{d/(d-1)+\zeta(1-q/p)}} A^n U_{n-1}^{\vartheta},$$

where c, A are positive constants and $\vartheta > 1$. We can prove this as in [24, Theorem 3.5], i.e., by taking ζ and ϑ solutions of the system

$$\begin{cases}\zeta + d/(d-1) = \vartheta\zeta,\\ \zeta/(dp') = \vartheta.\end{cases}$$

If we substitute the second equation into the first one and solve for ζ, we get a unique admissible ($\zeta > 0$) solution, which is given by

$$\zeta = dp' \left[\frac{1+\sqrt{1+\frac{4}{(d-1)p'}}}{2}\right]. \tag{3.25}$$

At this point, we can plug this value into the second equation and find an explicit expression for ϑ, i.e.,

$$\vartheta = \frac{\zeta}{dp'} = \frac{1+\sqrt{1+\frac{4}{(d-1)p'}}}{2} > 1.$$

Step 3. *Decay estimate for the energy $E(\epsilon)$*

We claim that there are positive constants ϵ_0 and c_0 such that

$$E(\epsilon) \le c_0 \epsilon^{\frac{p(q-1)}{p-1}} \qquad \text{for all } \epsilon \le \epsilon_0. \tag{3.26}$$

There is a crucial difference with [24] because the exponent has an additional factor $(q-1)/(p-1)$, which is controlled* by (3.10). To prove the claim, we use (3.21) to deduce that

$$E(\epsilon) \le \frac{\beta_2}{L} \epsilon^q h(\epsilon),$$

applying the coarea formula and Hölder's inequality as above to obtain

$$\begin{aligned} \epsilon E(\epsilon) &\le \int_\epsilon^{2\epsilon} E(s)\, ds \le \frac{\beta_2}{L}(2\epsilon)^q \int_\epsilon^{2\epsilon} h(s)\, ds \\ &= \frac{\beta_2}{L}(2\epsilon)^q \int_{\Omega(\epsilon,2\epsilon)} |\nabla u|\, dx \\ &\le \frac{\beta_2}{L}(2\epsilon)^q |\Omega(\epsilon,2\epsilon)|^{1/p'} E(2\epsilon)^{1/p}. \end{aligned}$$

If we now take ϵ_0 so small that $(\beta_2/L)2^q|\Omega(0,2\epsilon_0)|^{1/p'} \le 1$, then we get

$$E(\epsilon) \le \epsilon^{q-1} E(2\epsilon)^{\frac{1}{p}} \qquad \text{for all } \epsilon \le \epsilon_0.$$

Finally, a standard iterative argument (see, e.g., [24, Lemma 3.6]), proves (3.26).

Step 4. *Conclusion*

We claim that we can find $\eta > 0$ such that

$$U_0 \le \kappa^{-1/(\vartheta-1)} A^{-1/(\vartheta-1)^2} \eta^{(q-1)/(p-1)} \zeta, \tag{3.27}$$

where ζ is given by (3.25). Indeed, by Hölder's inequality and the coarea formula we have

$$\begin{aligned} U_0 &= \left|\Omega\left(\frac{\eta}{2},\eta\right)\right| \left[\int_{\eta/2}^{\eta} h(s)\, ds\right]^\zeta = \left|\Omega\left(\frac{\eta}{2},\eta\right)\right| \left[\int_{\Omega(\frac{\eta}{2},\eta)} |\nabla u|\, dx\right]^\zeta \\ &\le \left|\Omega\left(\frac{\eta}{2},\eta\right)\right| \left[E(\eta)^{\frac{1}{p}} \left|\Omega\left(\frac{\eta}{2},\eta\right)\right|^{\frac{1}{p'}}\right]^\zeta \\ &= \left|\Omega\left(\frac{\eta}{2},\eta\right)\right|^{1+\zeta/p'} E(\eta)^{\zeta/p}, \end{aligned}$$

so, if we apply the decay estimate (3.26), we conclude that

$$U_0 \le c_0' \left|\Omega\left(\frac{\eta}{2},\eta\right)\right|^{1+\zeta/p'} \eta^{\frac{q-1}{p-1}\zeta}.$$

Consequently, the claim (3.27) follows if we choose $\eta > 0$ small enough to have

$$c_0' \left|\Omega\left(\frac{\eta}{2},\eta\right)\right|^{1+\zeta/p'} \le \kappa^{-1/(\vartheta-1)} A^{-1/(\vartheta-1)^2}.$$

* This step of the proof requires the technical assumption (3.10) to work, but it is unlikely to be necessary; indeed, we expect that it can be removed entirely (or, at least, significantly weakened) with a different strategy.

Now, to conclude the proof of (3.22), it is sufficient to show that

$$\lim_{n\to\infty} U_n = 0.$$

Putting together the iterative formula given in Step 2 and (3.27), we get the following inequality:

$$\begin{aligned} U_n &\le \kappa^{\frac{\vartheta^n-1}{\vartheta-1}} \eta^{-\frac{\vartheta^n-1}{\vartheta-1}\left[\frac{d}{d-1}+\zeta\left(1-\frac{q}{p}\right)\right]} A^{\frac{\vartheta(\vartheta^n-1)-n\vartheta+n}{(\vartheta-1)^2}} (U_0)^{\vartheta^n} \\ &\le \kappa^{-\frac{1}{\vartheta-1}} A^{\frac{\vartheta^n}{\vartheta-1}} \eta^{\left[\zeta\frac{q-1}{p-1}-\frac{d}{d-1}-\zeta\left(1-\frac{q}{p}\right)\right]\vartheta^n} \\ &\le \kappa_1 \left[A^{\frac{1}{\vartheta-1}} \eta^{\zeta\left(\frac{q-1}{p-1}+\frac{q}{p}-1\right)-\frac{d}{d-1}}\right]^{\vartheta^n}. \end{aligned}$$

Therefore, we have $U_n \to 0$ as $n \to +\infty$ if we prove that the quantity inside is strictly less than one up to replacing η with a smaller value or, equivalently, that

$$\zeta\left(\frac{q-1}{p-1}+\frac{q}{p}-1\right)-\frac{d}{d-1} > 0.$$

This can easily be verified to be equivalent to the technical assumption (3.10) using the explicit expression for ζ given by (3.25), concluding the proof. □

Remark 3.7 The assumption (3.10) seems somewhat restrictive, but numerical simulations show that

$$\zeta\left(\frac{q-1}{p-1}+\frac{q}{p}-1\right)-\frac{d}{d-1} > 0$$

is verified in a significant portion of the plane (see Figure 3.1).

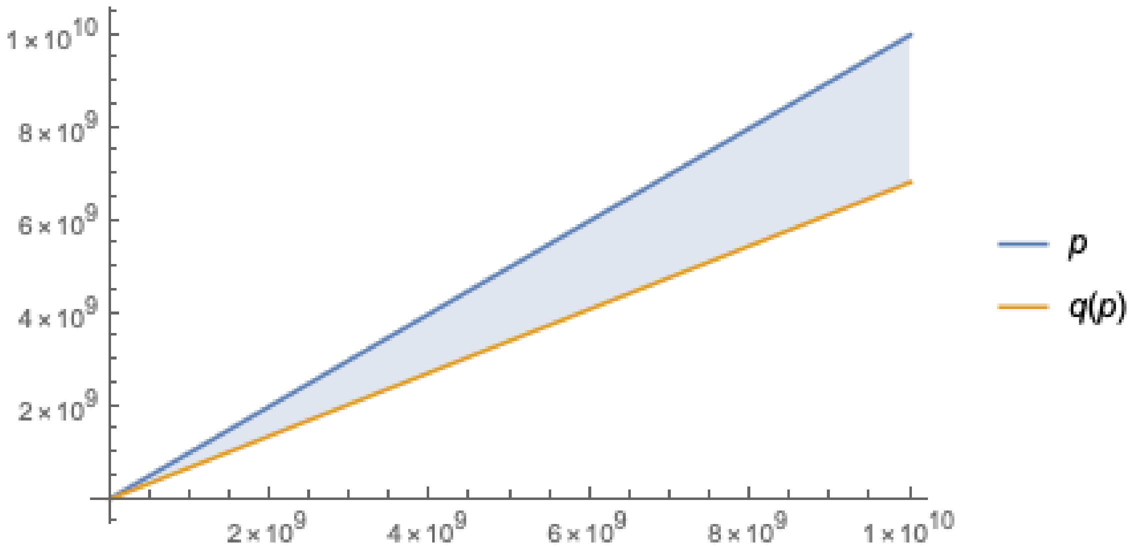

Figure 3.1: The range of admissible values $q > 1$ satisfying the technical assumption (3.10) for $d = 3$ is represented by the area between the two lines.

We are now ready to prove that the optimal function $\bar{u}$ belongs to $\mathrm{SBV}(\mathbb{R}^d)$ and, as a consequence, that the corresponding optimal set $\bar{\Omega} = \{\bar{u} > 0\}$ has finite perimeter.

Proof of Theorem 3.2. The function $\bar{u}$ is a supersolution for $\mathcal{L}$, so by Theorem 3.4 there exists a positive constant α such that we can write

$$\bar{u} = \bar{u} \vee \alpha.$$

Moreover, we know that $\bar{u} \in \mathcal{X}_D$ means that $\bar{u} \vee \epsilon \in \mathrm{SBV}(\mathbb{R}^d)$ for every $\epsilon > 0$, so this immediately implies that $\bar{u} \in \mathrm{SBV}(\mathbb{R}^d)$. Now notice that the perimeter of $\bar{\Omega}$ can be estimated by

$$\operatorname{Per}\bar{\Omega} = \sup\left\{\int_{\mathbb{R}^d} \operatorname{div}(\phi)\mathbb{1}_{\bar{\Omega}}(x)\,dx \;:\; \phi \in C_c^1(\mathbb{R}^d) \text{ and } \|\phi\|_\infty \le 1\right\} \le \alpha^{-1}\|\bar{u}\|_{\mathrm{BV}(\mathbb{R}^d)},$$

where $\mathbb{1}_{\bar{\Omega}}$ is the characteristic function of $\bar{\Omega}$. The right-hand side is finite because $\bar{u} \in \mathrm{SBV}(\mathbb{R}^d)$ and $\alpha > 0$, so we finally proved that

$$\bar{u} \in \mathrm{SBV}(\mathbb{R}^d) \implies \operatorname{Per}\bar{\Omega} < \infty.$$

□

3.4 Proof of Theorem 3.3: the optimal shape $\bar{\Omega}$ is open

To prove that the optimal set $\bar{\Omega}$ is open, we first need to investigate the topological properties of the corresponding jump set $J_{\bar{u}}$. Recall that the prototype case is the integrand

$$j(x, u, \nabla u) = |\nabla u|^p - f(x)u + 1,$$

so we cannot apply the results obtained in [24, Section 4] directly since the linear term leads to several issues that need to be addressed. From now on, we will assume (j5), which asserts that

$$j(x, s, \xi) - j(x, s, 0) = L|\xi|^p \qquad \text{for all } s, \xi \in \mathbb{R}$$

but, as we will see later, this assumption could be replaced by a weaker one. Moreover, we consider the Mumford-Shah functional obtained by removing the linear term and taking $g = 1$, i.e.,

$$MS(u) := L\int_{\mathbb{R}^d} |\nabla u|^p\,dx + \mathcal{H}^{d-1}(J_u).$$

The function $\bar{u}$ is not a minimizer of this functional, so we now introduce a weaker notion of the minimum that is satisfied by a suitable rescaling of $\bar{u}$.

Definition 3.2 *Let $u \in \mathrm{SBV}^p_{\mathrm{loc}}(\mathbb{R}^d)$ be a function such that*

$$u = 0 \qquad \text{on } \mathbb{R}^d \setminus D.$$

We say that u is an almost-quasi minimizer for the functional $MS(\cdot)$ with Dirichlet boundary conditions if there are $\Lambda \ge 1$, $\vartheta > 0$ and $c_\vartheta > 0$ such that

$$\int_{B_\rho(x_0)} L|\nabla u|^p\,dx + \mathcal{H}^{d-1}\left(J_u \cap \bar{B}_\rho(x_0)\right) \le \int_{B_\rho(x_0)} L|\nabla v|^p\,dx + \Lambda\mathcal{H}^{d-1}\left(J_v \cap \bar{B}_\rho(x_0)\right) + c_\vartheta \rho^{d-1+\vartheta}$$

for all $B_\rho(x_0) \Subset D$ and for every $v \in \mathrm{SBV}^p_{\mathrm{loc}}(\mathbb{R}^d)$, $v = 0$ in $\mathbb{R}^d \setminus D$ satisfying the condition

$$\{v \ne u\} \subseteq B_\rho(x_0).$$

Theorem 3.5 *Let $u \in \mathrm{SBV}^p_{\mathrm{loc}}(D)$ be an almost-quasi minimizer of $MS(\cdot)$ with Dirichlet boundary conditions in the sense of the definition above. Then*

$$\mathcal{H}^{d-1}\left((\bar{J}_u \setminus J_u) \cap D\right) = 0,$$

which means that the jump set of u is essentially closed in D.

This result was proved in [24, Theorem 2.3] under more general assumptions (which is the reason why (j5) is not optimal) that are satisfied by the function

$$h(x, \nabla u) =: j(x, u, \nabla u) - j(x, u, 0) = L|\nabla u|^p.$$

Proposition 3.1 *Under the assumptions (j1)–(j5) and (g1)–(g4), if $\bar{u} \in \mathrm{SBV} \cap L^\infty(\mathbb{R}^d)$ is the minimizer of $\mathcal{F}$ given by Theorem 3.1 and $\alpha > 0$ the constant in Theorem 3.2 such that*

$$\bar{u} \geq \alpha \qquad \text{almost everywhere on } \{\bar{u} > 0\},$$

then the rescaling $\tilde{u} := (2\beta_1)^{1/q} \alpha \bar{u}$ is an almost-quasi minimizer of the Mumford-Shah functional with Dirichlet boundary conditions on D in the sense of Definition 3.2.

Proof. Let $B_\rho(x_0) \subset D$ and take any $v \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^d)$ with $v = 0$ on $\mathbb{R}^d \setminus D$ satisfying

$$\{v \neq \bar{u}\} \subseteq B_\rho(x_0).$$

Without loss of generality we can replace v with $w := (v \wedge M) \vee 0$, where $M \geq \|\bar{u}\|_\infty$ is given by Lemma 3.4. As a consequence, the function w is a competitor for $\mathcal{F}$ and, by minimality,

$$\mathcal{F}(\bar{u}) \leq \mathcal{F}(w),$$

which immediately translates to

$$\int_{\mathbb{R}^d} L|\nabla \bar{u}|^p \, dx + \int_{\mathbb{R}^d} j(x, \bar{u}, 0) \, dx + \int_{J_{\bar{u}}} \left[g(x, \bar{u}^+) + g(x, \bar{u}^-) \right] \, d\mathcal{H}^{d-1}$$
$$\leq \int_{\mathbb{R}^d} L|\nabla w|^p \, dx + \int_{\mathbb{R}^d} j(x, w, 0) \, dx + \int_{J_w} \left[g(x, w^+) + g(x, w^-) \right] \, d\mathcal{H}^{d-1}.$$

Moreover, the functions $\bar{u}$ and v coincide everywhere outside of $B_\rho(x_0)$ and w is obtained by truncation outside of the image of u, which means that

$$\{w \neq \bar{u}\} \subset \{v \neq \bar{u}\} \subseteq B_\rho(x_0).$$

Using this property and (g4) leads to the following estimate:

$$\int_{B_\rho(x_0)} L|\nabla \bar{u}|^p \, dx + \int_{B_\rho(x_0)} \left[j(x, \bar{u}, 0) - j(x, w, 0) \right] \, dx + 2\beta_1 \alpha^q \mathcal{H}^{d-1} \left(J_{\bar{u}} \cap \bar{B}_\rho(x_0) \right)$$
$$\leq \int_{B_\rho(x_0)} L|\nabla w|^p \, dx + 2\beta_2 M^q \mathcal{H}^{d-1} \left(J_w \cap \bar{B}_\rho(x_0) \right) + \gamma \omega_d \rho^d.$$

We apply (3.11) to estimate the second integral as

$$\int_{B_\rho(x_0)} \left[j(x, \bar{u}, 0) - j(x, w, 0) \right] \, dx \geq -C_j |B_\rho(x_0)| \|w\|^q_{L^q(\mathbb{R}^d)} \geq -C'_j \rho^d,$$

and this is enough to conclude since

$$\int_{B_\rho(x_0)} L|\nabla \bar{u}|^p \, dx + 2\beta_1 \alpha^q \mathcal{H}^{d-1} \left(J_{\bar{u}} \cap \bar{B}_\rho(x_0) \right)$$
$$\leq \int_{B_\rho(x_0)} L|\nabla w|^p \, dx + 2\beta_2 M^q \mathcal{H}^{d-1} \left(J_w \cap \bar{B}_\rho(x_0) \right) + C''_{j,d} \rho^d,$$

where we define $C''_{j,d} := \gamma \omega_d + C'_j$. □

We now have all the ingredients to prove Theorem 3.3, which asserts that $\bar\Omega = \{\bar u \neq 0\}$ is open (and thus belongs to $\mathcal{A}(D)$) and, moreover, that it solves the initial shape optimization problem

$$\inf_{\substack{\Omega\in\mathcal{A}(D)\\ \Omega \text{ Lipschitz}}} \mathcal{J}(\Omega) = \mathcal{J}(\bar\Omega). \tag{3.28}$$

Proof of Theorem 3.3. First, we notice that the jump set is invariant under rescaling, i.e.,

$$J_{\bar u} = J_{\gamma \bar u} \qquad \text{for all } \gamma > 0,$$

so combining Theorem 3.5 and Proposition 3.1 yields

$$\mathcal{H}^{d-1}\left(\left(\overline{J_{\bar u}} \setminus J_{\bar u}\right) \cap D\right) = 0. \tag{3.29}$$

This means that $J_{\bar u}$ is essentially closed in D and, as a consequence of (3.20), we also have

$$\mathcal{H}^{d-1}(J_{\bar u}) < \infty.$$

Now notice that the optimal shape $\bar\Omega$ corresponds to the unique (by minimality) connected component of $D \setminus J_{\bar u}$ on which $\bar u$ does not vanish. It follows that

$$\partial\bar\Omega \subseteq \left(\overline{J_{\bar u}} \cap D\right) \cup \partial D \implies \partial\bar\Omega \cap \bar\Omega = \emptyset,$$

which proves that $\bar\Omega$ is an open set. In addition, by (3.29) we have

$$\mathcal{H}^{d-1}\left((\partial\bar\Omega \setminus J_{\bar u}) \cap D\right) = 0,$$

so, using the assumption (g4), we obtain the following identity:

$$\int_{\partial\bar\Omega\setminus J_{\bar u}} \left[g(x,\bar u^+) + g(x,\bar u^-)\right] \, d\mathcal{H}^{d-1} = 0.$$

Moreover, the traces $\bar u^+$ and $\bar u^-$ are equal to zero almost everywhere on $\partial\bar\Omega \cap \partial D$ since D is Lipschitz and $\bar u = 0$ outside of $\bar\Omega$ by construction.

To conclude the proof of this theorem, we only need to verify that $\bar\Omega$ is also a solution to the initial shape optimization problem (3.2), which is achieved by showing that

$$\inf_{\substack{\Omega\in\mathcal{A}(D)\\ \Omega \text{ Lipschitz}}} \mathcal{J}(\Omega) = \mathcal{J}(\bar\Omega).$$

For $\epsilon > 0$ fixed, we apply Proposition 3.2 to find an admissible competitor $w \in \mathcal{F}_D$ for the free discontinuity functional $\mathcal{F}$, with jump set satisfying $J_w \subset D$, such that

$$\mathcal{F}(w) < \mathcal{F}(\bar u) + \epsilon.$$

By [56, Theorem 3.1], we can find a sequence $w_k \in \mathcal{F}_D \cap W^{1,p}(D \setminus J_{w_k})$, with jump sets J_{w_k} essentially closed and polyhedral (i.e., given by the union of the intersection with D of a finite number of $(d-1)$-simplexes), such that the following holds:

$$w_k \xrightarrow{k\to\infty} w \quad \text{and} \quad \nabla w_k \xrightarrow{k\to\infty} \nabla w \qquad \text{strongly in } L^p(\mathbb{R}^d),$$
$$\mathcal{F}(w_k) \xrightarrow{k\to\infty} \mathcal{F}(w).$$

We can also assume without loss of generality that the measures of the supports converge; if not, it is sufficient to replace w_k with $(w_k - t_k)_+$ for a suitable sequence $t_k \in \mathbb{R}$ with $t_k \to 0$. The idea is to widen the jump set by making small holes H_k with polyhedral boundary in such a way that the resulting set belongs to the class of admissible sets, i.e.,

$$\Omega_k = D \setminus \bar{H}_k \in \mathcal{A}(D).$$

Therefore, the set Ω_k is a competitor for the minimization problem (3.28) since it has Lipschitz boundary. In addition, the restriction of w_k to Ω_k is admissible for minimizing $\mathcal{F}$, so we have

$$\mathcal{F}\left(w_k\big|_{\Omega_k}\right) \geq \mathcal{F}(\bar{u}).$$

If we now consider the holes small enough and k so large that we have

$$\begin{aligned}
\mathcal{J}(\Omega_k) &\leq \int_{\Omega_k} j(x, w_k, \nabla w_k)\, dx + \int_{\partial\Omega_k} g(x, w_k)\, d\mathcal{H}^{d-1} \\
&\leq \mathcal{F}\left(w_k\big|_{\Omega_k}\right) + \epsilon \leq \mathcal{F}(w) + 2\epsilon \\
&< \mathcal{F}(\bar{u}) + 3\epsilon = \mathcal{J}(\bar{\Omega}) + 3\epsilon,
\end{aligned}$$

then the result follows by taking a sequence $\epsilon_n \to 0$ and selecting the corresponding $k_n = k$. □

Proposition 3.2 *Let $v \in \mathcal{F}_D \cap L^\infty(\mathbb{R}^d)$ with $\mathcal{H}^{d-1}(J_v) < \infty$. For every $\epsilon > 0$ there is $w \in \mathcal{F}_D \cap L^\infty(\mathbb{R}^d)$ such that $J_w \subset D$, $\mathcal{H}^{d-1}(J_w) < \infty$ and*

$$\mathcal{F}(w) < \mathcal{F}(v) + \epsilon.$$

The proof follows the same argument used in [24, Proposition 3.12] with $B = \emptyset$, but there is a significant difference which is the way auxiliary functions are defined; more precisely, we let

$$v_i^\xi(y) := \begin{cases} v_i(y', y_d + \xi) & \text{if } y_d < f_i(y') - \xi, \\ \|v\|_\infty \psi_i(y', f_i(y')) & \text{if } y_d \geq f_i(y') - \xi, \end{cases}$$

where ψ_i is the partition of unity introduced in the referenced paper. This does not change the next step of the proof because the integral term

$$\int_{\mathbb{R}^d} \left[j(x, v_i^\xi, 0) - j(x, v_i, 0) \right] dx$$

can be easily estimated with a constant when $\xi > 0$ is sufficiently small, since the support of v_i^ξ can be chosen as close as we need to $\operatorname{spt} v_i$.

3.5 Open problems and additional remarks

In this final section, we want to discuss some open questions and remarks that naturally arise from the shape optimization problem (3.2).

Problem 3.1 Under relatively mild assumptions, we obtained the existence of an optimal domain $\bar{\Omega}$ in the class of measurable subsets of D. In addition, we proved properties such as

$$\operatorname{Per} \bar{\Omega} < \infty \qquad \text{and} \qquad \bar{\Omega} \text{ open},$$

under stronger assumptions. Thus, it would be interesting to examine the properties of optimal domains further and, more precisely, answer the following questions:

(a) Is it possible to have higher regularity, for example $\partial\bar{\Omega} \in C^{1,\alpha}$?

(b) Does $\bar{\Omega}$ have internal fractures (modeled as one-dimensional sets) or cuspids at the boundary?

It is worth remarking that question (b) can be answered partially. Indeed, according to the results obtained in [25], we have the following properties:

- if $d = 2$, cuspidal domains are (in principle) possible only if they can be parameterized around the cuspid as $x_2 = |x_1|^\alpha$ with $\alpha < 2$;
- the optimal set $\bar{\Omega}$ does not contain uncountably many fractures (otherwise, one can remove some of them and obtain a better solution).

As for the first question, if we consider the prototype problem given by

$$j(x,s,z) = |z|^p - f(x)s + 1 \qquad \text{and} \qquad g(x,s) = |s|^p, \tag{3.30}$$

for some $p > 1$, it is worth investigating if one can achieve $C^{1,\alpha}$-regularity of the free boundary assuming, for example, f non-negative and bounded, or if more assumptions are needed.

Problem 3.2 In the prototype problem (3.30), a fundamental assumption to prove that $\bar{\Omega}$ is open and has a finite perimeter is the non-negativity and boundedness of the datum f. It would be interesting to see if our results still hold with weaker assumptions on f, for example

$$D = E \sqcup F \text{ with } f \geq 0 \text{ on } D \text{ and } f \leq 0 \text{ on } E \qquad \text{and} \qquad f \in L^1(D).$$

As a suggestion, in the following example, modeled on a similar question for Dirichlet boundary conditions (see [41]), we show that this is the case.

Example 3.2 Fix $\Omega_0 \subset D$ and recall that the functional with $p = q = 2$ is given by

$$\mathcal{J}(\Omega) = \inf\left\{ \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - f(x)u + 1 \right) dx + \frac{\beta}{2}\int_{\partial\Omega} |u|^2 \, dx \ : \ u \in W^{1,2}(\Omega) \right\}.$$

Now let w be the **torsion function** associated to the selected domain Ω_0, namely the solution to the boundary-value problem

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega_0, \\ w = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Since we have not assumed Ω_0 to be regular (indeed, it can be as irregular as we want), the solution w is intended in the weak sense. In other words, it is the unique minimizer of

$$\int_{\Omega_0} \left(\frac{1}{2}|\nabla u|^2 - u \right) dx + \frac{\beta}{2}\int_{J_u} \left[|u^+|^2 + |u^-|^2 \right] dx$$

in the space $\mathrm{GSBV}(\Omega_0)$. Let $f := -\Delta(w^2)$. A formal computation shows that

$$f = -\left[w\Delta w + |\nabla w|^2 \right] = \left[w - |\nabla w|^2 \right],$$

where the second equality follows using the fact that $\Delta w = -1$. In any case, it is easy to see that the following properties hold:

- The function f does not belong to L^∞, but it is always (at least) a distribution depending on the regularity of the torsion function w (and thus of Ω_0).
- The function f belongs to $L^1(D)$ if the chosen domain Ω_0 satisfies low-regularity assumptions.
- The sign of f depends on the quantity $w - |\nabla w|^2$ which, in general, is not positive almost everywhere.

Now let $\bar{u} := w^2$. A priori $\bar{u}$ may not belong to GSBV(Ω_0), but using the chain rule for the weak derivative we obtain the identity

$$D\bar{u} = w\nabla w \cdot dx + w^2 \cdot d\mathcal{H}^{d-1} \llcorner J_w,$$

so it is easy to verify that $\bar{u}$ minimizes on $\mathcal{X}_D$ the functional

$$\int_{\Omega_0} \left(\frac{1}{2}|\nabla u|^2 - f(x)u + 1\right) dx + \beta \int_{J_u} \left[|u^+|^2 + |u^-|^2\right] dx$$

and, as a consequence of the reduction result (see Lemma 3.1), we have

$$\Omega_0 = \{w \neq 0\} = \{\bar{u} > 0\}.$$

Shape optimization problem in control form and regularity of the free boundary

Shape optimization problems in control form | 4

Let $D \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and, for every $\Omega \subset D$, which will be our control variable, denote by u_Ω the corresponding state variable, solution of the following PDE:

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \overline{D} \setminus \Omega, \end{cases} \tag{4.1}$$

where $p > 1$ is given, $f \in W^{-1,p'}(D)$ and $-\Delta_p$ is the p-Laplacian defined by

$$-\Delta_p u := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

The function u_Ω can also be characterized as the solution to a variation problem. More precisely, multiplying the equation by u and integrating by parts yields

$$\int_D f u \, dx = -\int_D u \Delta_p u \, dx = \int_D \frac{1}{p}|\nabla u|^p \, dx,$$

which means that u_Ω is the unique (up to a multiplicative constant) solution of

$$\min\left\{\int_D \left(\frac{1}{p}|\nabla u|^p - f(x)u\right) dx \ : \ u \in W_0^{1,p}(D), \ u = 0 \text{ on } \overline{D} \setminus \Omega\right\}.$$

This chapter aims to discuss the results obtained in [38], in which we study the existence and properties of solutions to shape optimization problems of the form

$$\min\left\{J(u_\Omega) \ : \ \Omega \text{ open}, \ \Omega \subset D\right\}, \tag{4.2}$$

for cost functionals independent of ∇u, i.e.,

$$J(u) = \int_D j(x,u) \, dx, \tag{4.3}$$

where $j : D \times \mathbb{R} \to \mathbb{R}$ is a given function with $s \mapsto j(x,s)$ lower semicontinuous. The prototype problem is the one associated with the cost

$$j(x,s) = -g(x)s + \lambda \, \mathbb{1}_{(0,+\infty)}(s),$$

where λ is a non-negative constant (the Lagrange multiplier) and $g : D \to \mathbb{R}$ a given function that satisfy some regularity properties. If we plug this into (4.3), we get

$$J(u_\Omega) = -\int_D g(x)u_\Omega(x) \, dx + \lambda \, |\{u_\Omega > 0\}|, \tag{4.4}$$

which is our model cost functional. We now state the main theorem of [38] for the prototype problem, but the same result is valid in a much more general setting (see Section 4.1).

Theorem 4.1 *Let $\lambda > 0$, $g \in L^r(D)$ for some $r > 1$ be a non-negative measurable function and J the model functional (4.4). Suppose that f is non-negative and*

$$f \in L^q(D) \qquad \text{with } q > \frac{d}{p} \text{ and } q \geq 1.$$

Then the following assertions hold true:

(i) If there is a constant $C > 0$ *such that*

$$f(x) \leq Cg(x) \qquad \text{for every } x \in D,$$

then there is an open set $\Omega_{\text{opt}} \subset D$ *solution to the problem* (4.2).

(ii) If Ω_{opt} *is a solution to* (4.2)*, then it has a finite perimeter:*

$$\text{Per}\left(\Omega_{\text{opt}}\right) < \infty.$$

Remark 4.1 When $f = g$, our problem reduces to a free boundary problem (see Section 4.6), which means that the optimal state function can also be characterized as the solution of

$$\min\left\{\frac{1}{p}\int_D |\nabla u|^p \, dx - \int_D f(x)u \, dx + \frac{p-1}{p}\lambda|\{u \neq 0\}| \; : \; u \in W_0^{1,p}(D)\right\}.$$

This allows one to study the properties of the optimal set Ω_{opt} using well-known free boundary regularity techniques. The main novelty of [38] is that the problem (4.2) cannot be written as a variational problem in $W_0^{1,p}(D)$ since we have $f \neq g$.

In particular, this means that the state function corresponding to an optimal shape is, a priori, only optimal among all functions that satisfy (4.1) on other admissible domains. This is the main difficulty in studying this functional, which was first introduced in [45], where the minimizer's existence and some regularity properties were obtained in the case $p = 2$.

Regularity of the free boundary for $p = 2$

We expect that the regularity of the free boundary obtained in Theorem 4.1 can be improved, but this is very challenging for a generic value of $p > 1$. However, for $p = 2$, the Laplace operator

$$\Delta u = \text{div}(\nabla u)$$

is linear, and we can take advantage of plenty of results for subharmonic functions. This will be discussed in more details in Chapter 7 since we first need (in Chapter 5 and Chapter 6) to develop a general regularity theory for viscosity solutions.

In the remainder of this section, we introduce the notions that are necessary to at least state the main result for $p = 2$ so that we can compare it with Theorem 4.1. For this, consider the functional

$$J(u) = \int_{\mathbb{R}^d} \left(-g(x)u + \mathbb{1}_{\{u>0\}}(x)\right) \, dx,$$

which is obtained from (4.4) by setting $\lambda = 1$ and replacing D with $\mathbb{R}^d$. The reason is that we use a local definition of the minimizer, so there is no need to work inside D.

Definition 4.1 (Local minimizer) *A set* $\Omega \subset \mathbb{R}^d$ *of finite measure is optimal in a ball* $B \subset \mathbb{R}^d$*, if*

$$J(u_\Omega) \leq J(u_\omega) \qquad \text{for every } \omega \text{ such that } \Omega \Delta \omega \Subset B.$$

The main result of [37] is the following. However, weaker assumptions on the regularity of f and g are necessary if we only want to prove the first assertion (for example, the same of Theorem 4.1).

Theorem 4.2 *Let* $d \geq 2$ *and let* $\Omega \subset \mathbb{R}^d$ *be optimal in a ball* $B \subset \mathbb{R}^d$ *in the sense of Definition 4.1. Suppose that the following conditions hold:*

(a) $f, g \in C_b^2(B)$*;*

(b) $f \geq 0$ *in* $\Omega \cup B$*, and* $\bar{u} > 0$ *in* Ω*;*

(c) there are constants $C_1, C_2 > 0$ *such that*

$$C_1 g(x) \leq f(x) \leq C_2 g(x) \qquad \text{for every } x \in \Omega \cup B.$$

Then there is a closed set $S \subset \partial\Omega \cap B$ *such that*

(i) $(\partial\Omega \cap B) \setminus S \in C^{1,\alpha}$ *for some* $\alpha \in (0, 1]$*;*

(ii) S *is empty if* $d \leq 4$*, and* $\dim_{\mathcal{H}}(S) \leq d - 5$ *if* $d \geq 5$*.*

We will prove (i) in Section 7.5 and (ii) in Subsection 7.6.3, while the remainder of this chapter is dedicated to the problem in the case $p > 1$.

4.1 Formulation of the problem and main results

This section aims to present the main results obtained in [38] in the most general framework possible and show that the model functional (4.4) is well within the assumptions of each one.

4.1.1 Existence of optimal open shapes for $p > d$

In Section 4.2, we prove the existence of an optimal set Ω_{opt} for $p > d$, which requires very mild assumptions on the integrand j. Indeed, in this case, any state function u_Ω satisfying

$$\begin{cases} -\Delta_p u_\Omega = f & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \overline{D} \setminus \Omega, \end{cases}$$

is continuous by Sobolev embedding theorem (Theorem 4.7) since we know that

$$W^{1,p}(D) \hookrightarrow C^0(\overline{D})$$

is a (compact) continuous inclusion for these values of p and d. Moreover, we will see that the optimal shape solving (4.2) can be written as

$$\Omega_{\text{opt}} = \{\bar{u} \neq 0\},$$

where $\bar{u}$ satisfies (4.1) with $\Omega = \Omega_{\text{opt}}$. It is interesting to notice that in the case $p > d$ we can consider a more general shape optimization problem, namely

$$\min \left\{ J(u_\Omega) \ : \ \Omega \text{ open}, \ \Omega \subset D, \ |\Omega| \leq m \right\}, \tag{4.5}$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$, $m \in (0, |D|]$ is given, u_Ω is the state variable and J is a functional of the form (4.3). Our main result is the following:

Theorem 4.3 *Let* $D \subset \mathbb{R}^d$ *be a bounded open set and* $0 < m \leq |D|$*. Let* $p > d$ *and suppose that for every* $M > 0$ *there exists a function* $a_M \in L^1(D)$ *such that*

$$-a_M(x) \leq j(x, s) \qquad \text{for almost every } x \in \mathbb{R}^d \text{ and every } |s| \leq M. \tag{4.6}$$

Then the minimization problem (4.5) *admits a solution* $\Omega_{\text{opt}} = \{\bar{u} \neq 0\}$*.*

Remark 4.2 This result applies to the model functional (4.4) if we assume, for example, that $g \in L^1(D)$ and $\lambda \in \mathbb{R}$. Indeed, in this case, we can take

$$a_M(x) = M|g(x)| + |\lambda|,$$

and this is well within the assumptions of Theorem 4.1.

4.1.2 Existence of optimal quasi-open sets for $p \leq d$

In Subsection 4.3.3, we study the shape optimization problem (4.2) for $p \leq d$. Notice that the Sobolev embedding theorem no longer applies in this case; thus, the state variable

$$-\Delta_p u_\Omega = f$$

may not be continuous. Indeed, the existence of optimal shapes in the class

$$\mathcal{A}(D) := \{\Omega \subset D \ : \ \Omega \text{ open}\} \tag{4.7}$$

is not guaranteed with the assumptions of Theorem 4.3, so we consider the relaxation to p-capacitary measures (Subsection 4.3.2) and prove the existence of an optimal measure

$$\bar{\mu} = \infty_{D\setminus\Omega_{\text{opt}}}$$

This suggests that it is necessary to first consider a larger set of admissible sets, namely

$$\mathcal{A}_{p,m}(D) := \{\Omega \subset D \ : \ \Omega \text{ is } p\text{-quasi-open}, \ |\Omega| \leq m\},$$

which is defined in more details in Section 4.3. Therefore, we consider the minimization problem

$$\min_{\Omega\in\mathcal{A}_{p,m}(D)} J(\Omega), \tag{4.8}$$

where $m \in (0, |D|]$ is given, and we obtain the following result:

Theorem 4.4 *Let D be a bounded open set in $\mathbb{R}^d$, $p \leq d$ and $f \geq 0$. Suppose, in addition, that the cost integrand j satisfies the following properties:*

(i) There are $a \in L^1(D)$ and $c \in \mathbb{R}$ such that

$$a(x) - c|s|^r \leq j(x,s), \qquad \text{where} \quad \begin{cases} 0 < r < \frac{dp}{d-p} & \text{for } p < d, \\ 0 < r < +\infty & \text{for } p = d. \end{cases}$$

(ii) The function j can be written as

$$j(x,s) = j_0(x,s) + \lambda\, \mathbb{1}_{(0,+\infty)}(s),$$

where λ is non-negative and $j_0(x,\cdot)$ is non-increasing for a.e. $x \in \mathbb{R}^d$.

Then, for any $m \in (0, |D|]$, there exists a solution Ω_{opt} to (4.8). Moreover, if $\lambda = 0$, then

$$|\Omega_{\text{opt}}| = m.$$

Remark 4.3 The model functional (4.3) satisfies the conditions of Theorem 4.4 if we assume, for example, that for some $\ell > 1$ we have

$$\lambda \geq 0 \quad \text{and} \quad g \geq 0 \text{ and } g \in L^\ell(D).$$

Indeed, in this case, the function

$$j_0(x,s) = -g(x)s$$

is non-increasing and the property (i) is satisfied with $a(x) = -|g(x)|^\ell$, $c = 1$ and

$$r = \frac{\ell}{\ell - 1}.$$

Indeed, by Young's inequality we get

$$j(x,s) = -g(x)s + \lambda\, \mathbb{1}_{(0,+\infty)}(s) \geq -g(x)s \geq -|g(x)|^\ell - |s|^r = a(x) - s^r,$$

and this is enough to conclude that Theorem 4.4 applies to (4.4).

4.1.3 Existence of optimal open sets for $p \leq d$

In Subsection 4.3.4 we show that, under additional assumptions on j, if we remove* the measure constraint, then the optimal p-quasi-open set Ω_{opt} given in Theorem 4.4 is open, and hence

$$J(\Omega_{opt}) = \min_{\Omega \in \mathcal{A}(D)} J(\Omega),$$

where $\mathcal{A}(D)$ is the class of admissible sets defined in (4.7). More precisely, we require that j has a growth condition which, for the model functional, is given by

$$g(x) \geq c f(x) \qquad \text{for every } x \in D.$$

Moreover, we require $f \in L^q(D)$ for some q that depends on d and p since we will deduce that the optimal shape $\Omega_{opt} = \{\bar{u} > 0\}$ is open by proving that $\bar{u}$ is α-Hölder continuous.

Remark 4.4 The class of open sets is dense in the space of p-quasi-open sets, so putting Theorem 4.4 and Theorem 4.5 together proves the existence for the shape optimization problem (4.2) in $\mathcal{A}(D)$.

Theorem 4.5 *Let D be a bounded open set in $\mathbb{R}^d$, $p \leq d$, $m = |D|$ and $f \in L^q(D)$ for some $q > d/p$. Suppose that the cost integrand j can be written as*

$$j(x,s) = j_0(x,s) + \lambda\, \mathbb{1}_{(0,+\infty)}(s),$$

where $\lambda > 0$ and j_0 satisfies the following properties:

(a) $j_0(x,0) = 0$ *for almost every $x \in D$;*

(b) there are constants $c > 0$ and $\epsilon > 0$ such that

$$\frac{j_0(x,t) - j_0(x,s)}{t - s} \leq -c f(x) \tag{4.9}$$

for almost every $x \in D$ and all $s,t \in \mathbb{R}$ satisfying $0 < t - s < \epsilon$.

Then every solution of (4.8) is an open set, and hence (4.2) has a solution in the class $\mathcal{A}(D)$.

Example 4.1 If we do not require f to be regular enough, then Ω_{opt} given in Theorem 4.4 is not open in general. See [41, Example 4.3] for a counterexample.

Remark 4.5 The model functional (4.3) satisfies the conditions of Theorem 4.4 if we assume, as mentioned above, that $\lambda > 0$ and

$$g(x) \geq c f(x) \qquad \text{for almost every } x \in D.$$

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* The measure constraint has to be removed (by setting, for example, $m = |D|$) because it leads to some issues when we prove that $\bar{u}$ belongs to $C^{0,\alpha}$. However, it is possible that with a different strategy, getting rid of it is unnecessary.

4.1.4 Finite perimeter of optimal sets for $p > 1$

Since the existence of an optimal set Ω_{opt} has been established for all $p > 1$, in Section 4.5 we show that under very mild assumptions we have

$$\text{Per}\left(\Omega_{\text{opt}}\right) < \infty.$$

The strategy used to prove this result was first introduced by Bucur in [20] and [29] for the optimization of the k-th eigenvalue of the Laplace operator.

Theorem 4.6 *Let D be a bounded open set in $\mathbb{R}^d$. Assume that j can be written as*

$$j(x,s) = j_0(x,s) + \lambda\, \mathbb{1}_{(0,+\infty)}(s),$$

with $\lambda > 0$ and $j(x,0) = 0$ for almost every $x \in D$. For any $p > 1$, let Ω_{opt} be a solution to the minimization problem without measure constraint, namely

$$J(\Omega_{\text{opt}}) = \min_{\Omega \in \mathcal{A}(D)} J(\Omega),$$

and suppose that either one of the following two sets of assumptions is satisfied:

(1) We have $f \in W^{-1,p'}(D)$, $f \geq 0$ and there are $a \in L^1(D)$ and $\epsilon_0, c > 0$ such that

$$\frac{|j_0(x,s+\epsilon) - j_0(x,s)|}{\epsilon} \leq a(x) + c|s|^{p^*} \tag{4.10}$$

for all $s \in \mathbb{R}$, for almost every $x \in D$ and for all $\epsilon \leq \epsilon_0$.

(2) We have $f \in L^q(D)$ for some $q > d/p$, $f \geq 0$ and there are $a(\cdot,s) \in L^1(D)$, non-decreasing and continuous in s, and $\epsilon_0 > 0$ such that

$$\frac{|j_0(x,s+\epsilon) - j_0(x,s)|}{\epsilon} \leq a(x,s) \tag{4.11}$$

for all $s \in \mathbb{R}$, almost every $x \in D$ and all $\epsilon \leq \epsilon_0$.

Then the set Ω_{opt} has finite perimeter.

The assumption $j(x,0) = 0$ is not necessary, but it helps simplify the expressions occurring in the proof. To see that this is not a restrictive hypothesis, it is sufficient to notice that

$$\tilde{J}(u) := J(u) - \int_D j(x,0)\,dx$$

satisfies $\tilde{j}(x,0) = 0$ and has the same minimizers of J since they only differ for a constant term.

Remark 4.6 The model functional (4.3) satisfies the conditions of Theorem 4.6. More precisely, we can easily verify that both sets of assumptions are fulfilled:

(1) For the condition (4.10) we simply notice that

$$\frac{|-g(x)(s+\epsilon) + g(x)s|}{\epsilon} = |g(x)|,$$

so it is sufficient to take $a(x) := g(x)$ and assume $g \in L^1(D)$.

(2) For the condition (4.11) we can take

$$a(x,s) := |g(x)|,$$

which does not depend on s, so it is trivially non-decreasing and continuous in s.

4.1.5 Minimum problem on γ-compact classes

In Section 4.4, we consider the same minimization problem

$$\min J(\Omega),$$

but Ω ranges in some classes (compact with respect to a particular notion convergence) that encode different kinds of geometrical restrictions, for example

$$\mathfrak{A}_{convex} := \{\Omega \subset D \ : \ \Omega \text{ convex}\}.$$

More precisely, we show that the γ_p-convergence is equivalent to the Hausdorff complementary convergence and exploit the compactness properties of the latter to infer that

$$\min_{\Omega \in \mathfrak{A}_{convex}} J(\Omega),$$

admits a solution $\Omega_{\text{opt}} \in \mathfrak{A}_{convex}$, and that the same is valid in other geometrical classes. Moreover, we consider a class which is only of topological type, i.e.,

$$\mathcal{O}_k := \left\{\Omega \subset D \ : \ \Omega \text{ open and has } \textbf{at most } k \text{ connected components}\right\},$$

and prove that, for $p > d-1$, the minimization problem $\min_{\Omega\in\mathcal{O}_k} J(\Omega)$ has a solution for every $k \in \mathbb{N}$.

4.2 Proof of Theorem 4.3: existence of an optimal set for $p > d$

Consider the shape minimization problem (4.2) with the class of admissible sets given by

$$\mathcal{A}_m(D) = \left\{\Omega \subset D \ : \ \Omega \text{ open and } |\Omega| \leq m\right\},$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$ and $0 < m \leq |D|$ is given.

As we mentioned above, the assumption $p > d$ plays a crucial role since it allows us to exploit the Sobolev embedding theorem, which we now recall for completeness (see [80] for a proof).

Theorem 4.7 *Let $A \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $k \geq 1$ and $1 \leq p \leq \infty$. Then the following inclusions are continuous:*

- *If $kp < d$, then $H^{k,p}(A) \hookrightarrow L^q(A)$ for all $1 \leq q \leq dp/(d-kp)$.*
- *If $kp = d$, then $H^{k,p}(A) \hookrightarrow L^q(A)$ for all $1 \leq q < \infty$.*
- *If $kp > d$, then $H^{k,p}(A) \hookrightarrow C^{0,\alpha}(\bar{A})$, where*

$$\alpha = \begin{cases} k - \frac{d}{p} & \text{if } k - d/p < 1, \\ \alpha_0 \in [0,1) & \text{if } k - d/p = 1 \text{ and } p > 1, \\ 1 & \text{if } k - d/p > 1. \end{cases}$$

Furthermore, the following inclusions are also compact:

- *If $kp < d$, then $H^{k,p}(A) \hookrightarrow L^q(A)$ for all $1 \leq q < dp/(d-kp)$.*
- *If $kp = d$, then $H^{k,p}(A) \hookrightarrow L^q(A)$ for all $q \in [1,\infty)$.*
- *If $kp > d$, then $H^{k,p}(A) \hookrightarrow C^0(\bar{A})$.*

The last technical result we need to prove the existence is the classical Poincaré inequality, which we also recall below for completeness.

Theorem 4.8 (Poincaré) *Let $1 \leq p < \infty$ and let A be a bounded*[†] *subset of $\mathbb{R}^d$. Then there exists a constant $C > 0$, depending only on A and p, such that for every $u \in W_0^{1,p}(A)$ we have*

$$\|u\|_{L^p(A)} \leq C\|\nabla u\|_{L^p(A)}. \tag{4.12}$$

For a proof of this result we refer to [80], while a discussion on the optimal constant can be found in [118, Chapter 12.2]. We are now ready to prove the main theorem of this section:

Proof of Theorem 4.3. Let Ω_n be a minimizing sequence in $\mathscr{A}_m(D)$ and let u_n be the corresponding state variables given by (4.1). Multiplying the equation by u_n yields

$$(-\Delta_p u_n)u_n = f u_n,$$

so we can integrate by parts and obtain the following identity:

$$\int_D |\nabla u_n|^p \, dx = \langle f, u_n \rangle \leq \|f\|_{W^{-1,p'}(D)} \|u_n\|_{W^{1,p}(D)}.$$

Notice that f belongs to $W^{-1,p'}(D)$, so the inequality above follows from the fact that the right-hand side is a dual pairing. On the other hand, by Poincaré inequality (4.12), we get

$$\int_D |u_n|^p \, dx \leq C \int_D |\nabla u_n|^p \, dx.$$

If we now combine these two inequalities together, we obtain

$$\|u_n\|_{W^{1,p}(D)}^p \leq (1 + C) \int_D |\nabla u_n|^p \, dx \leq (1 + C)\|f\|_{W^{-1,p'}(D)} \|u_n\|_{W^{1,p}(D)},$$

which, since $p > 1$ strictly, immediately implies

$$\|u_n\|_{W^{1,p}(D)}^{p-1} \leq (1 + C)\|f\|_{W^{-1,p'}(D)}.$$

Therefore, the sequence u_n is uniformly bounded in $W^{1,p}(D)$ and, by Theorem 4.7, compact in $C^{0,\alpha}(D)$ for some $\alpha > 0$; consequently, up to subsequences, we have

$$\|u_n - \bar{u}\|_\infty \xrightarrow{n \to \infty} 0 \qquad \text{for some } \bar{u} \in C^{0,\alpha}(D).$$

It is now easy to verify that the set

$$\Omega_{\text{opt}} := \{\bar{u} \neq 0\}$$

is open and belongs to $\mathscr{A}_m(D)$; moreover, the function $\bar{u}$ verifies (4.1) with $\Omega = \Omega_{\text{opt}}$. To conclude the proof, notice that by (4.6), we have

$$J(\bar{u}) \leq \liminf_{k \to \infty} J(u_{n_k}),$$

which means that Ω_{opt} is a minimizer for $J(\Omega)$ in the class $\mathscr{A}_m(D)$, proving the claim. □

Problem 4.1 Can we expect symmetric solutions if all the data are symmetric?

The answer to this question is negative and, in general, we cannot expect any symmetry property, as the following example shows (see Section 4.6 for the definition of the supremal problem):

[†] It is enough to assume A bounded in at least one direction in $\mathbb{R}^d$.

Example 4.2 Let $d = 2$, $D = B_1$, $p = \infty$ and consider the supremal optimization problem with $j(x,u) = -u$ and mixed Dirichlet-Neumann boundary conditions. Then $u_\Omega = d(x, D \setminus \Omega)$, and the minimization problem can be written as follows:

$$\min\left\{\|d(x, D \setminus \Omega)\|_\infty \ : \ |\Omega| \leq m\right\}.$$

The optimal solution Ω_{opt} is unique up to rotations and is represented on the side by the shaded region. It is given by the intersection

$$\Omega_{opt} = B_1 \cap B_{r_m}(\bar{x}),$$

where $\bar{x} \in \partial D$ and r_m is the unique radius such that the area of the intersection is equal to m.

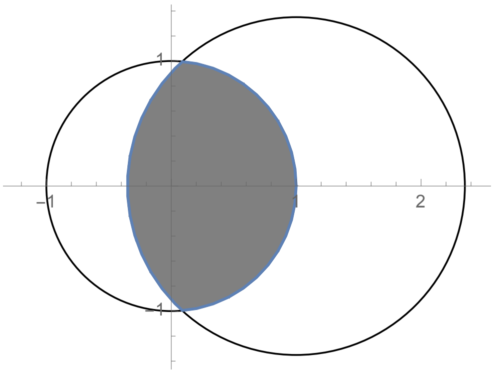

Figure 1. The colored region represents an optimal Ω_{opt} with $\bar{x} = (1,0)$ and $m \approx 2$ ($r_m \approx 1.351$).

4.3 Existence of a solution in the case $p \leq d$

The case $p \leq d$ is much more delicate because minimizing sequences Ω_n have a significantly different behavior due to the lack of compactness. More precisely, we have

$$\#(\Omega_n) \xrightarrow{n \to +\infty} \infty,$$

where $\#(\cdot)$ denotes the number of connected components. This means that Ω_n converges in a suitable sense to a relaxed solution which is, in general, not a domain but a **capacitary measure**.

In particular, a solution to the shape optimization problem in $\mathfrak{A}(D)$ may not exist for $p \leq d$ without additional assumptions; indeed, several counterexamples are known in the literature (see, for example, [22, Section 4.2]). The reader interested in having a complete view of relaxed shape optimization problems with Dirichlet conditions on the free boundary and capacitary measure may see, for example, the book [22] and the articles [31, 33].

4.3.1 Preliminaries

This section aims to present some notions and basic properties from the theory of p-capacitary measures and γ_p-convergence, which play a crucial role in the proof of Theorem 4.4.

Definition 4.2 *A set $\Omega \subset D$ is said **p-quasi-open** if there exists a function $u \in W^{1,p}(D)$ such that*

$$\Omega = \{u > 0\}.$$

*Similarly, a set $K \subset D$ is said **p-quasi-closed** if there exists a function $u \in W^{1,p}(D)$ such that*

$$K = \{u = 0\}.$$

We are now ready to introduce the notion of p-capacitary measure (see Definition 1.12 for the definition of p-capacity, which is necessary for the following).

Definition 4.3 *A non-negative regular Borel measure μ on $\mathbb{R}^d$ (possibly taking the value $+\infty$) is a **p-capacitary measure** if, for every Borel set $E \subset \mathbb{R}^d$, we have*

$$\mathrm{cap}_p(E) = 0 \implies \mu(E) = 0 \qquad \text{and} \qquad \mu(E) = \inf\left\{\mu(\Omega) \ : \ \Omega \supset E, \ \Omega \ p\text{-quasi open}\right\}.$$

We denote by $\mathcal{M}_p$ the class of all p-capacitary measures on $\mathbb{R}^d$. This includes a large variety of measures, for example:

- all measures of the form $a(x)\,dx$ with $a \in L^1_{\text{loc}}(\mathbb{R}^d)$;
- all measures of the form $b(x)\mathcal{H}^k \llcorner S$, where S is a smooth k-dimensional surface, $\mathcal{H}^k$ is the k-dimensional Hausdorff measure, $k > d-p$, and b is locally integrable on S;
- all measures of the form
$$\infty_K(E) = \begin{cases} 0 & \text{if } \operatorname{cap}_p(E \cap K) = 0 \\ +\infty & \text{otherwise} \end{cases}$$
where K is a p-quasi-closed set in $\mathbb{R}^d$.

The following results summarize essential properties of the class $\mathcal{M}_p$ that will be useful to prove the existence of a p-quasi-open optimal set. A proof can be found in [22].

Lemma 4.1 *Let $\mu \in \mathcal{M}_p$. Then the subspace of function $u \in W^{1,p}(\mathbb{R}^d)$ such that*

$$\|u\|_{W^{1,p}_\mu} := \left(\int |\nabla u|^p \, dx + \int |u|^p \, d\mu \right)^{1/p} < \infty$$

is a well-defined Sobolev space, which is usually denoted by $W^{1,p}_\mu(\mathbb{R}^d)$.

Lemma 4.2 *Let $\mu \in \mathcal{M}_p(D)$ and let f be in the dual space of $W^{1,p}_\mu(D)$. Then the PDE*

$$-\Delta_p u + \mu |u|^{p-2} u = f \qquad \text{in } \Omega \tag{4.13}$$

is well-defined in the weak sense, namely $u \in W^{1,p}_\mu(D)$, and

$$\int_D |\nabla u|^{p-2} \nabla u \nabla v \, dx + \int |u|^{p-2} uv \, d\mu = \langle f, v \rangle \qquad \text{for all } v \in W^{1,p}_\mu(D).$$

Moreover, the equation above admits a unique solution $u_{\mu,f}$, which can equivalently be characterized as the unique minimum point of the functional

$$F(u) := \int_D \frac{1}{p} |\nabla u|^p \, dx + \int_D \frac{1}{p} |u|^p \, d\mu - \langle f, u \rangle.$$

Remark 4.7 Let Ω be a p-quasi-open set. If we consider the p-capacitary measure

$$\mu := \infty_{D \setminus \Omega},$$

then (4.13) becomes the state equation (4.1) with Dirichlet boundary conditions on $D \setminus \Omega$, i.e.,

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } D \setminus \Omega, \\ \partial_\nu u = 0 & \text{on } \partial D \setminus (D \setminus \Omega). \end{cases}$$

If we denote by $u_{\mu,f}$ the unique solution of (4.13) given by Lemma 4.2 for $\mu \in \mathcal{M}_p$ and f fixed, then the following monotonicity properties hold:

Lemma 4.3 *The map $\mu \mapsto u_{\mu,f}$ is decreasing for any $f \geq 0$, while the map $f \mapsto u_{\mu,f}$ is increasing for every*

$\mu \in \mathcal{M}_p(D)$. More precisely, we have

$$\begin{cases} \mu_1 \ge \mu_2 \implies u_{\mu_1,f} \le u_{\mu_2,f} & \text{if } f \ge 0, \\ f_1 \le f_2 \implies u_{\mu,f_1} \le u_{\mu,f_2} & \text{for every } \mu \in \mathcal{M}_p. \end{cases}$$

Proof. Let $f \ge 0$ and set $u_i := u_{\mu_i,f}$. The monotonicity with respect to μ can be rewritten as

$$\mu_1 \ge \mu_2 \implies u_1 \le u_2,$$

so to show this implication, we exploit the variational characterization of the solutions u_i as the unique minima of the corresponding functionals

$$F_i(u) = \int_D \frac{1}{p}|\nabla u|^p\,dx + \int_D \frac{1}{p}|u|^p\,d\mu_i - \langle f, u\rangle.$$

More precisely, it is sufficient to prove that

$$F_1(u_1 \wedge u_2) \le F_1(u_1) \tag{4.14}$$

since, taking into account that u_1 is the unique minimum point of F_1, we have

$$u_1 \wedge u_2 = u_1 \implies u_1 \le u_2.$$

To prove the claim (4.14), we first notice that

$$F_1(u \wedge v) + F_1(u \vee v) = F_1(u) + F_1(v),$$

so it is completely equivalent to show that

$$F_1(u_2) \le F_1(u_1 \vee u_2).$$

The left-hand side can be rewritten as

$$F_1(u_2) = F_2(u_2) + \int_D \frac{1}{p}|u_2|^p\,d(\mu_1 - \mu_2),$$

while the right-hand side as

$$F_1(u_1 \vee u_2) = F_2(u_1 \vee u_2) + \int_D \frac{1}{p}|u_1 \vee u_2|^p\,d(\mu_1 - \mu_2),$$

which means that

$$F_1(u_2) \le F_1(u_1 \vee u_2) \iff F_2(u_2) \le F_2(u_1 \vee u_2) + \int_D \frac{1}{p}\left(|u_1 \vee u_2|^p - |u_2|^p\right)\,d(\mu_1 - \mu_2).$$

Since $f \ge 0$, the maximum principle implies that $u_i \ge 0$ for $i = 1, 2$, and this is enough to conclude the proof of the claim (4.14) because

$$|u_1 \vee u_2| \ge |u_2| \quad + \quad u_2 \text{ minimum of } F_2 \implies F_2(u_2) \le F_2(u_1 \vee u_2).$$

In a similar way we can prove the monotonicity with respect to f. Let $\mu \in \mathcal{M}_p$, set $u_i := u_{\mu,f_i}$ and consider the corresponding functionals

$$F_i(u) = \int_D \frac{1}{p}|\nabla u|^p\,dx + \int_D \frac{1}{p}|u|^p\,d\mu - \langle f_i, u\rangle.$$

We need to prove that

$$F_1(u_1 \wedge u_2) \leq F_1(u_1),$$

but, arguing as above, it is completely equivalent to

$$F_1(u_2) \leq F_1(u_1 \vee u_2). \tag{4.15}$$

The left-hand side can be rewritten as

$$F_1(u_2) = F_2(u_2) + \langle f_2 - f_1, u_2 \rangle,$$

while the right-hand side is given by

$$F_1(u_1 \vee u_2) = F_2(u_1 \vee u_2) + \langle f_2 - f_1, u_1 \vee u_2 \rangle,$$

which means that (4.15) is also equivalent to

$$F_2(u_2) \leq F_2(u_1 \vee u_2) + \langle f_2 - f_1, u_1 \vee u_2 - u_2 \rangle.$$

Finally, this inequality is an obvious consequence of $(u_1 \vee u_2) - u_2 \geq 0$ and the fact that u_2 is the unique minimizer of F_2. □

We consider the subclass $\mathcal{M}_p(D)$ of p-capacitary measures supported on $\bar{D}$ and notice that it can be endowed with a very natural notion convergence, the *γ_p-convergence*.

Definition 4.4 *A sequence $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{M}_p(D)$ γ_p-converges to some $\mu \in \mathcal{M}_p(D)$ if and only if*

$$\|u_{\mu_n,1} - u_{\mu,1}\|_{L^p(D)} \xrightarrow{n\to+\infty} 0.$$

It is easy to verify that the γ_p-convergence implies a slightly weaker notion of convergence for a general energy; namely we have

$$\|u_{\mu_n,1} - u_{\mu,1}\|_{L^p(D)} \to 0 \quad \Longrightarrow \quad \|u_{\mu_n,f} - u_{\mu,f}\|_{L^p(D)} \to 0$$

for every $f \in W^{-1,p'}(D)$.

Remark 4.8 If we endow $\mathcal{M}_p(D)$ with the distance

$$d_{\gamma_p}(\mu,\nu) := \|u_{\mu,1} - u_{\nu,1}\|_{L^p(D)},$$

then we get a **compact** metric space which notion of convergence is given by Definition 4.4.

To conclude, it is essential to remark that the following subclasses are dense (with respect to the γ_p-convergence) in $\mathcal{M}_p(D)$:

- the class of measures $a(x)\,dx$ with a smooth function;
- the class of measures ∞_K with K smooth closed set.

In addition, if μ_n γ_p-converges to μ, the Dirichlet regions $\{\mu_n = \infty\}$ fulfill the following semicontinuity property:

$$\limsup_{n\to+\infty} |\{\mu_n = \infty\}| \leq |\{\mu = \infty\}|. \tag{4.16}$$

See [22, Proposition 5.3.6] for a more detailed discussion and a proof of this fact.

4.3.2 Existence of an optimal measure in $\mathcal{M}_p(D)$

In this section, we consider the relaxation of the minimization problem (4.2) to p-capacitary measures and prove the existence of a solution. More precisely, we focus on

$$\min \left\{ J(u_{\mu,f}) \;:\; u_{\mu,f} \text{ solves (4.13) and } \mu \in \mathfrak{A}_{p,m} \right\}, \tag{4.17}$$

where the admissible class of measures is defined as follows:

$$\mathfrak{A}_{p,m} := \left\{ \mu \in \mathcal{M}_p(D) \;:\; |\{\mu < \infty\}| \le m \right\}.$$

The following result shows that, under very mild assumptions, a minimizer μ_{opt} exists for the relaxed problem (4.17). However, a priori, we do not know if it takes the form

$$\mu_{\text{opt}} = \infty_K \tag{4.18}$$

for some p-quasi-closed set K; indeed, further assumptions will be necessary to ensure that the optimal measure takes the form (4.18).

It is worth remarking that this strategy is not new, and a similar argument was used in [34] in the context of optimal potentials for Schrödinger operators.

Theorem 4.9 *Let $p \le d$ and let $f \in W^{-1,p'}(D)$. Assume that there exists $a \in L^1(D)$ such that the cost integrand j satisfies the assumption*

$$a(x) - c|s|^r \le j(x,s) \qquad \text{with } r < \begin{cases} dp/(d-p) & \text{if } p < d, \\ +\infty & \text{if } p = d. \end{cases} \tag{4.19}$$

Then the relaxed minimization problem (4.17) admits a solution $\mu_{\text{opt}} \in \mathcal{M}_p(D)$.

Proof. Let $(\mu_n)_{n\in\mathbb{N}} \in \mathfrak{A}_{p,m}$ be a minimizing sequence. Since $\mathcal{M}_p(D)$ is compact with respect to the γ_p-convergence, we may assume up to subsequences that there is $\mu_{\text{opt}} \in \mathcal{M}_p(D)$ such that

$$\mu_n \xrightarrow{\gamma_p} \mu_{\text{opt}}.$$

The limit measure μ_{opt}, a priori, may not belong to the class $\mathfrak{A}_{p,m}$; however, the upper semicontinuity property (4.16) gives

$$\limsup_{n\to+\infty} |\{\mu_n = \infty\}| \le |\{\mu_{\text{opt}} = \infty\}| \implies |\{\mu_{\text{opt}} < \infty\}| \le m,$$

which means that $\mu_{\text{opt}} \in \mathfrak{A}_{p,m}$. Moreover, the γ_p-convergence implies

$$u_{\mu_n,f} \xrightarrow{n\to+\infty} u_{\mu_{\text{opt}},f} \qquad \text{weakly in } W^{1,r}(D),$$

so, using the assumption (4.19), we deduce that

$$\|u_{\mu_n,f} - u_{\mu_{\text{opt}},f}\|_{L^r(D)} \xrightarrow{n\to+\infty} 0.$$

Finally, by Fatou's lemma we have

$$J(u_{\mu_{\text{opt}},f}) \le \liminf_{n\to+\infty} J(u_{\mu_n,f}),$$

and this is enough to infer that μ_{opt} is a solution to the minimization problem (4.17). □

4.3.3 Existence of a p-quasi-open optimal set

We are now ready to prove Theorem 4.4, which asserts that, under the additional assumption

$$j(x,s) = j_0(x,s) + \lambda\, \mathbb{1}_{(0,+\infty)}(s),$$

where $\lambda \geq 0$ and $s \mapsto j_0(x,s)$ non-increasing for a.e. x, the optimal measure μ_{opt} obtained above takes the desired form, namely

$$\mu_{\text{opt}} = \infty_{D\setminus\Omega_{\text{opt}}},$$

where Ω_{opt} is a p-quasi-open set which solves the minimization problem (4.5) for any $m \in (0, |D|]$.

Proof of Theorem 4.4. Let $\mu_{\text{opt}} \in \mathcal{M}_p(D)$ be the optimal measure given by Theorem 4.9 and $\bar{u} := u_{\mu_{\text{opt}},f}$ the corresponding state function, i.e., the solution of (4.13). Then, the set

$$\Omega_{\text{opt}} := \{\bar{u} > 0\}$$

is p-quasi-open since $\bar{u} \in W^{1,p}(D)$ and, by the maximum principle, we have

$$f \geq 0 \implies \bar{u} \geq 0.$$

Moreover, the upper semicontinuity property (4.16) yields

$$|\Omega_{\text{opt}}| \leq m \implies \tilde{\mu} := \infty_{D\setminus\Omega_{\text{opt}}} \in \mathfrak{A}_{p,m}.$$

By Lemma 4.3, we have $u_{\tilde{\mu},f} \geq \bar{u}$ since, by construction, $\tilde{\mu} \leq \mu_{\text{opt}}$. Therefore, we can exploit the monotonicity assumption (ii) to deduce the inequality

$$j_0(x, u_{\tilde{\mu},f}) \leq j_0(x, \bar{u}),$$

which, in turn, implies that

$$J(u_{\tilde{\mu},f}) \leq J(\bar{u}).$$

This proves that $\tilde{\mu} = \infty_{D\setminus\Omega_{\text{opt}}}$ is a minimizer, so we only need to verify that it also saturates the volume constraint. We argue by contradiction. Assume $|\Omega_{\text{opt}}| < m$ and let $\hat{\Omega}$ be any set satisfying

$$\hat{\Omega} \supset \Omega_{\text{opt}} \qquad \text{and} \qquad |\hat{\Omega}| = m.$$

By construction $\infty_{D\setminus\hat{\Omega}} \leq \infty_{D\setminus\Omega_{\text{opt}}}$, so Lemma 4.3 gives $u_{\infty_{D\setminus\hat{\Omega}},f} \geq u_{\infty_{D\setminus\tilde{\Omega}},f}$, hence the optimality of $\hat{\Omega}$ using again the monotonicity property (ii). ☐

Remark 4.9 The assumption $s \mapsto j(x,s)$ non-increasing is necessary. Indeed, the paper [22, Section 4.2] considers the case

$$f = 1 \qquad \text{and} \qquad j(x,s) = |s-c|^2,$$

and shows that the optimal measure in $\mathcal{M}_p(D)$ does not take the form $\mu = \infty_{D\setminus\Omega}$ for $c > 0$ small.

Remark 4.10 In some cases, the proof of the fact that the optimal measure takes the form $\infty_{D\setminus Om}$ becomes trivial; for example, if we assume that

$$j(x,s) \geq j(x,0) \qquad \text{for all } (x,s) \in \mathbb{R}^d \times \mathbb{R},$$

then the measure $\mu = \infty_{\bar{D}}$, which corresponds to $\Omega = \emptyset$, gives $u_{\mu,f} \equiv 0$ and solves the problem.

Lemma 4.4 *Let $\mu \in \mathcal{M}_p(D)$ be a non-negative measure. If $f \in L^q(D)$ for some $q > d/p$, then $u_{\mu,f} \in L^\infty(D)$.*

Proof. We can assume that $f \geq 0$. By Lemma 4.3, we have that

$$\mu \geq 0 \implies u_{\mu,f} \leq u_{0,f},$$

where $u_{0,f}$ is the unique solution (in the weak sense) of the equation

$$-\Delta_p u_{0,f} = f \quad \text{if } x \in D.$$

By modifying the proof of [89, Theorem 8.17] to hold for any $p > 1$, we find that

$$f \in L^q(D) \text{ with } q > \frac{d}{p} \implies u_{0,f} \in L^\infty(D),$$

and this is enough to infer that $u_{\mu,f} \in L^\infty(D)$ since $\|u_{\mu,f}\|_{L^\infty(D)} \leq \|u_{0,f}\|_{L^\infty(D)} < \infty$. □

4.3.4 Existence of an open optimal set

Let Ω_{opt} be a solution to (4.8) given by Theorem 4.5 with $m = |D|$ to remove the measure constraint, and recall that we can write

$$\Omega_{\text{opt}} = \{\bar{u} > 0\},$$

where $\bar{u}$ is obtained as the solution of the minimization problem

$$\min\left\{\int_D \left(j(x,u) + \lambda\, \mathbb{1}_{\{u>0\}}(x)\right)\,dx \;:\; \Delta_p u + f \geq 0,\; u \in W_0^{1,p}(D)\right\}. \tag{4.20}$$

This section aims to prove the theorem following a strategy developed in [58, Section 3]. First, we recall a well-known result in Morrey-Campanato spaces (see [125] for more details):

Theorem 4.10 (Morrey) *Let $u \in W^{1,p}(D)$ and $\alpha \in (0,1]$. Suppose that there exists $M > 0$ such that*

$$\int_{B_r} |\nabla u|\,dx \leq M r^{\frac{d}{p}-1+\alpha} \qquad \text{for every } B_r \subset D.$$

Then $u \in C^{0,\alpha}(D)$ and there exists a constant $C := C(d,\alpha) > 0$ such that

$$\operatorname{osc}_{B_r}[u] \leq CMr^\alpha \qquad \text{for every } B_r \subset D,$$

*where the **oscillation** of a function u on an open set U is given by*

$$\operatorname{osc}_U[u] := \sup_{x\in U} u(x) - \inf_{x\in U} u(x). \tag{4.21}$$

We are now ready to prove that $\Omega_{\text{opt}} = \{\bar{u} > 0\}$ is open. We achieve this by showing that there is $\alpha \in (0,1]$ such that the optimal profile $\bar{u}$ satisfies

$$\|\nabla \bar{u}\|_{L^p(B_\rho)} \leq C\rho^{d/p-(1-\alpha)} \qquad \text{for all } \rho > 0,$$

so we can apply Theorem 4.10 and deduce that $\bar{u}$ is α-Hölder continuous.

Proof of Theorem 4.5. First, notice that the minimization problem we are considering (4.8) does not have any constraint on the volume, so the cost functional takes the form

$$j(x,u) = j_0(x,u) + \lambda\, \mathbb{1}_{(0,+\infty)}(u)$$

with λ strictly positive, and thus we can assume $\lambda = 1$ without loss of generality.

Let $w \in W_0^{1,p}(D)$ be a function satisfying $w \geq \bar{u}$, where $\bar{u}$ is the optimal profile obtained as a weak solution (in the sense of (4.20)) of the following problem:

$$\begin{cases} -\Delta_p \bar{u} = f & \text{in } \Omega_{\text{opt}}, \\ \bar{u} \in W_0^{1,p}(\Omega_{\text{opt}}). \end{cases}$$

Notice that the function w is not a competitor for the auxiliary problem (4.20), but it is possible to construct one starting from the set

$$\Omega_w := \{w > 0\} \supset \Omega_{\text{opt}},$$

which is p-quasi-open by construction. Therefore, let u_w be the solution of

$$\begin{cases} -\Delta_p u_w = f & \text{in } \Omega_w, \\ u_w \in W_0^{1,p}(\Omega_w), \end{cases}$$

and notice that Ω_w is a competitor for the functional $J(\Omega)$, which means that we can exploit the minimality of Ω_{opt} to deduce the following inequality:

$$\int_{\Omega_{\text{opt}}} j(x, \bar{u})\, dx + |\{\bar{u} > 0\}| \leq \int_{\Omega_w} j(x, u_w)\, dx + |\{u_w > 0\}|.$$

Since $\bar{u}$ is zero outside of Ω_{opt} and $\Omega_{\text{opt}} \subset \Omega_w$, we can write both integrals on Ω_w,

$$\int_{\Omega_w} \big(j(x, \bar{u}) - j(x, u_w)\big)\, dx \leq |\{u_w > 0\}| - |\{\bar{u} > 0\}|,$$

and use the assumption (4.9) to estimate the left-hand side, obtaining

$$\int_{\Omega_w} f\,(u_w - \bar{u})\; dx \leq \frac{1}{c}\left[|\{u_w > 0\}| - |\{\bar{u} > 0\}|\right]. \tag{4.22}$$

Now, we multiply the state equations for $\bar{u}$ and u_w respectively to find the identities

$$\int_{\Omega_w} |\nabla u_w|^p\, dx = \int_{\Omega_w} f u_w\, dx \qquad \text{and} \qquad \int_{\Omega_w} |\nabla \bar{u}|^p\, dx = \int_{\Omega_w} f \bar{u}\, dx,$$

and we can apply (4.22) to estimate the difference as follows:

$$\int_{\Omega_w} \frac{1}{p}|\nabla \bar{u}|^p\, dx - \int_{\Omega_w} \frac{1}{p}|\nabla u_w|^p\, dx \leq \frac{1}{cp}\left[|\{u_w > 0\}| - |\{\bar{u} > 0\}|\right].$$

If we put both inequalities together, we get

$$\int_{\Omega_w} \left(\frac{1}{p}|\nabla \bar{u}|^p - f\bar{u}\right) dx - \int_{\Omega_w} \left(\frac{1}{p}|\nabla u_w|^p - f u_w\right) dx \leq \frac{p-1}{pc}\left[|\{u_w > 0\}| - |\{\bar{u} > 0\}|\right]. \tag{4.23}$$

Now, recall that u_w can also be characterized as the unique minimizer of the functional

$$W_0^{1,p}(\Omega_w) \ni u \mapsto \int_{\Omega_w} \left(\frac{1}{p}|\nabla u|^p - f u\right) dx,$$

so we can use w as a competitor and deduce that

$$\int_{\Omega_w} \left(\frac{1}{p}|\nabla u_w|^p - f u_w\right) dx \leq \int_{\Omega_w} \left(\frac{1}{p}|\nabla w|^p - f w\right) dx.$$

If we plug this inequality into (4.23), we get

$$\int_{\Omega_w} \left(\frac{1}{p}|\nabla \bar{u}|^p - f\bar{u}\right) dx - \int_{\Omega_w} \left(\frac{1}{p}|\nabla w|^p - fw\right) dx \leq \frac{1}{c} \left[|\{w > 0\}| - |\{\bar{u} > 0\}|\right],$$

but the function w only satisfies the property $w \geq \bar{u}$, so we need to make a more refined choice. More precisely, for any ball $B_r \subset D$, let w be the unique solution of

$$\begin{cases} -\Delta_p w = f & \text{in } B_r, \\ w = \bar{u} & \text{on } D \setminus \partial B_r. \end{cases} \tag{4.24}$$

In this case, the functions $\bar{u}$ and w coincide outside of B_r, so the inequality above takes the form

$$\frac{1}{p} \int_{B_r} \left(|\nabla \bar{u}|^p - |\nabla w|^p\right) dx \leq \int_{B_r} f(\bar{u} - w)\, dx + \frac{1}{c}|B_r \cap \{\bar{u} = 0\}|.$$

To estimate the first integral on the right-hand side, we apply Hölder inequality twice (since we want to use the assumption $f \in L^q(D)$ and, simultaneously, find the *best* norm of $\bar{u} - w$) to get

$$\begin{aligned} \int_{B_r} f(\bar{u} - w)\, dx &\leq \|\bar{u} - w\|_{L^{p^*}(B_r)} \left(\int_{B_r} f^{dp/(dp+p-d)}\right)^{1+1/d-1/p} \\ &\leq C\|\bar{u} - w\|_{L^{p^*}(D)} \|f\|_{L^q(D)} r^{\alpha_{p,q}}, \end{aligned}$$

where C is a constant that depends on p, q, and d (more precisely, the volume of the unit ball B_1 to a specific power), and we define

$$\alpha_{p,q} := \frac{dp + p - d}{dp} - \frac{1}{q}.$$

We now have to distinguish between the case $1 < p \leq 2$ and $p > 2$ because we obtain two slightly different estimates (although the strategy is the same). More precisely, we have

$$\int_{B_r} |\nabla(\bar{u} - w)|^p\, dx \leq C\left[\|\bar{u} - w\|_{L^{p^*}(D)} \|f\|_{L^q(D)} r^{\alpha_{p,q}} + |B_r \cap \{\bar{u} = 0\}|\right] \tag{4.25}$$

for $p > 2$, while for $1 < p \leq 2$ we get

$$\int_{B_r} |\nabla(\bar{u} - w)|^p\, dx \leq C\left[\|\bar{u} - w\|_{L^{2^*}(D)} \|f\|_{L^q(D)} r^{\alpha_{2,q}} + |B_r \cap \{\bar{u} = 0\}|\right]^{\frac{p}{2}} \left[\int_{B_r} |\nabla \bar{u}|^p\, dx\right]^{1-\frac{p}{2}} \tag{4.26}$$

Now, following [58, Section 3], we let $M := \|\bar{u}\|_{L^\infty(B_1)}$, $B := B_r(x_0)$ with $x_0 \in B_{7/8}$, and $0 < r \leq 1/16$. If we denote by w the solution of (4.24) in B_r, then we can write

$$w = w_h + w_n,$$

where w_h is the p-harmonic part of w, namely the solution of

$$\begin{cases} -\Delta_p w_h = 0 & \text{in } B, \\ w_h = \bar{u} & \text{on } D \setminus \partial B, \end{cases}$$

and w_n is simply given by $w - w_h$. Since w_h is p-harmonic and $w_h - \bar{u} \in W_0^{1,p}(B)$, a standard result in the theory of harmonic functions yields

$$\sup_{1/2\, B} |\nabla w_h| \leq \left[\frac{C}{r^d} \int_B |\nabla w_h|^p\right]^{1/p} \leq \frac{CM}{r}.$$

To estimate the gradient of w_n, we multiply $-\Delta_p w_n = f$ by w_n and integrate by parts, obtaining

$$\int_B |\nabla w_n|^p \, dx = \int_B f w_n \, dx,$$

which, applying Hölder's inequality to the right-hand side, yields

$$\int_B |\nabla w_n|^p \, dx \le \left[\int_B f^q \, dx\right]^{1/q} \left[\int_B w_n^{q'} \, dx\right]^{1/q'} \le \|f\|_{L^q(D)} M r^{d/q'}.$$

Let $\epsilon > 0$ be a small parameter such that $r^\epsilon \le 1/2$. We can estimate the norm of $\nabla \bar{u}$ on $B_{r^{1+\epsilon}}$ using the triangular inequality as follows:

$$\|\nabla \bar{u}\|_{L^p(B_{r^{1+\epsilon}})} \le \|\nabla(\bar{u} - w)\|_{L^p(B_{r^{1+\epsilon}})} + \|\nabla w_h\|_{L^p(B_{r^{1+\epsilon}})} + \|\nabla w_n\|_{L^p(B_{r^{1+\epsilon}})}.$$

Now, if $p > 2$, we can use (4.25) together with the estimates for ∇w_h and ∇w_n obtained above and the triangular inequality, to deduce that

$$\begin{aligned}\|\nabla \bar{u}\|_{L^p(B_{r^{1+\epsilon}})} &\le \|\nabla(\bar{u} - w)\|_{L^p(B_r)} + C r^{(1+\epsilon)d/p} C M r^{-(1+\epsilon)} + \|f\|_{L^q(D)} M r^{(1+\epsilon)d/q'} \\ &\le C' \left[r^{d/(pp^*)+\alpha_{p,q}/p} + r^{d/p} + r^{(1+\epsilon)(d/p-1)} + r^{(1+\epsilon)d/q'} \right],\end{aligned}$$

where the term $\|\nabla(\bar{u} - w)\|_{L^p(B_{r^{1+\epsilon}})}$ was replaced by$\|\nabla(\bar{u} - w)\|_{L^p(B_r)}$ in the first inequality because $\bar{u}$ and w coincide outside of B_r. Similarly, if $1 < p \le 2$, we can use (4.26) to deduce that

$$\|\nabla \bar{u}\|_{L^p(B_{r^{1+\epsilon}})} \le C' \left[\left(r^{d/2} + r^{\alpha_{2,q}/2 + d/2^*} \right) r^{d(1-p/2)/(pq')} + r^{(1+\epsilon)(d/p-1)} + r^{(1+\epsilon)d/q'} \right].$$

It is now easy to check that for every $1 < p \le d$ and every $q > d/p$ we have

$$\frac{d}{pp^*} + \frac{\alpha_{p,q}}{p} < \frac{d}{p},$$

so, if we let $\rho := r^{1+\epsilon}$, the inequalities above can be rewritten (with a different constant C) as

$$\|\nabla \bar{u}\|_{L^p(B_\rho)} \le C \rho^{d/p-(1-\alpha)},$$

where $\alpha > 0$ if $\epsilon > 0$ is chosen small enough. The conclusion now follows from a straightforward application of Theorem 4.10. □

4.4 Minimum problem on γ-compact classes

In general, without the assumptions we have seen in Section 4.2 and Section 4.3, the existence of an optimal set may fail. However, if we put geometrical restrictions on the class of admissible competitors, we can prove existence under very general assumptions with a different strategy.

There are several classes of admissible domains in the literature in which the existence of optimal shapes is well-known. We now briefly recall some of the most commonly used (see [22]):

(i) The class $\mathfrak{A}_{convex}$ of convex sets $\Omega \subset D$.

(ii) The class $\mathfrak{A}_{unif\ cone}$ of domains $\Omega \subset D$ satisfying a *uniform exterior cone property*, which means that for every $x_0 \in \partial\Omega$ there is a closed cone, with uniform height and opening, and with vertex in x_0, lying in the complement of Ω.

(iii) The class $\mathfrak{A}_{unif\ flat\ cone}$ of domains $\Omega \subset D$ satisfying a uniform *flat* cone condition, which is the same as above but we also allow cones to be k-flat, i.e., of dimension k with $k > d - p$.

(iv) The class $\mathfrak{A}_{cap\ density}$ of domains $\Omega \subset D$ satisfying a *uniform capacitary density condition*. More precisely, there are $c, r > 0$ (that do not depend on Ω) such that for every $x \in \partial\Omega$ we have

$$\frac{\operatorname{cap}_p\left(B_\rho(x) \setminus \Omega, B_{2\rho}(x)\right)}{\operatorname{cap}_p\left(B_\rho(x), B_{2\rho}(x)\right)} \geq c \quad \text{for every } \rho \in (0, r).$$

(v) The class $\mathfrak{A}_{unif\ Wiener}$ of domains $\Omega \subset D$ satisfying a *uniform Wiener condition*, which means that for every $x \in \partial\Omega$ and every $0 < r < R < 1$ we have

$$\int_r^R \left[\frac{\operatorname{cap}_p\left(B_\rho(x) \setminus \Omega, B_{2\rho}(x)\right)}{\operatorname{cap}_p\left(B_\rho(x), B_{2\rho}(x)\right)}\right]^{1/(p-1)} \frac{d\rho}{\rho} \geq g(r, R, x),$$

where $g : (0,1) \times (0,1) \times D \to \mathbb{R}^+$ is a fixed function (that does not depend on Ω) with the property that for every $R \in (0,1)$ it satisfies

$$\lim_{r \to 0^+} g(r, R, x) = +\infty \quad \text{locally uniformly w.r.t. } x.$$

It is easy to verify that these classes satisfy the following inclusions chain:

$$\mathfrak{A}_{convex} \subset \mathfrak{A}_{unif\ cone} \subset \mathfrak{A}_{unif\ flat\ cone} \subset \mathfrak{A}_{cap\ density} \subset \mathfrak{A}_{unif\ Wiener}\ ,$$

In addition, the γ_p-convergence for a sequence of domains Ω_n (in each class) is equivalent to the Hausdorff convergence of the complements

$$K_n := \overline{D} \setminus \Omega_n,$$

which is often referred to as *Hausdorff complementary (H^c) convergence*.

Remark 4.11 This notion of convergence is induced by the distance

$$d_{H^c}(\Omega_1, \Omega_2) := d_H(\Omega_1^c, \Omega_2^c),$$

where d_H is the standard Hausdorff distance defined by setting

$$d_H(K_1, K_2) := \sup_{x \in K_1}\left[\inf_{y \in K_2} |x - y|\right] \vee \sup_{x \in K_2}\left[\inf_{y \in K_1} |x - y|\right].$$

In particular, the properties below, which are well-known (see for instance [99]) for the Hausdorff convergence on the class

$$\mathcal{A}(D) = \left\{\Omega \subset D \ :\ \Omega \text{ open}\right\},$$

also hold for the classes above endowed with the γ_p-convergence:

- $(\mathfrak{A}, d_{H^c})$ is a compact metric space;
- if $\Omega_n \to \Omega$ in the H^c-convergence, then for every compact set $K \subset \Omega$, there exists $n_K \in \mathbb{N}$ such that $K \subset \Omega_n$ for every $n \geq n_K$;
- the Lebesgue measure is lower semicontinuous for the H^c-convergence;
- the map that associates to a set Ω the number of connected components of the set $\overline{D} \setminus \Omega$ is lower semicontinuous with respect to the H^c-convergence.

4.4.1 A class with topological constraints only

Geometrical restrictions are not the only conditions that can help achieve existence under general assumptions. Indeed, an interesting class was considered by Šverák in [143] and is given by

$$\mathcal{O}_k := \left\{ \Omega \subset D \ : \ \Omega \text{ open}, \# \left(\overline{D} \setminus \Omega \right) \le k \right\},$$

where $\#(E)$ denotes the number of connected components of a set E. Notice that both conditions (being open and having a bound on the number of connected components) are of topological type; thus, no geometrical restriction is imposed on competitors.

Remark 4.12 In the case $d = p = 2$ it is well-known that the Hausdorff complementary convergence implies the γ_2-convergence in $\mathcal{O}_k$.

This result is generally false in higher dimension, and capacity properties are crucial. Indeed, if we can prove that for $p > d$ we have

$$\mathrm{cap}_p\left(\{x\}\right) > 0 \qquad \text{for every } x \in D,$$

then Theorem 4.3 gives the existence of optimal shapes under very mild assumptions.

Proof. The idea is to argue as for Theorem 4.3 and exploit the fact that the embedding

$$W_0^{1,p}(\mathbb{R}^d) \hookrightarrow C^{0,\gamma}(\mathbb{R}^d) \quad \text{with } \gamma = 1 - d/p$$

is continuous for $p > d$ by Theorem 4.7, which means that, by composition, the embedding

$$W_0^{1,p}(\mathbb{R}^d) \hookrightarrow C^0(\mathbb{R}^d)$$

is also continuous. As a consequence, given $u \in C_c^\infty(\mathbb{R}^d)$, we have

$$\|u\|_{C^0(\mathbb{R}^d)} = \|u\|_\infty \le M\|u\|_{W_0^{1,p}(\mathbb{R}^d)}.$$

By the classical Poincaré inequality (4.12), the norm in $W_0^{1,p}(\mathbb{R}^d)$ is equivalent to the norm in $L^p(\mathbb{R}^d)$ of the gradient, which leads to

$$\|u\|_{C^0(\mathbb{R}^d)} = \|u\|_\infty \le M'\|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

Now recall that the p-capacity is defined as

$$\mathrm{cap}_p(K) = \inf\left\{ \int_{\mathbb{R}^d} |\nabla u|^p \, dx \ : \ u \in C_c^\infty(\mathbb{R}^d), \ u(x) \ge 1 \text{ for all } x \in K \right\},$$

so it is easy to verify that, for any nonempty set K (thus even a singlet) and any $u \in C_c^\infty(\mathbb{R}^d)$ satisfying the property $u(x) \ge 1$ on K, we have

$$\|\nabla u\|_{L^p(\mathbb{R}^d)} \ge \frac{\|u\|_\infty}{M'} \ge \frac{1}{M'} > 0.$$

Finally, if we take the infimum over u, we get a strictly positive number on the right-hand side, which ultimately means that $\mathrm{cap}_p(K) > 0$. □

When $p \le d$, the generalization of the Šverák result to higher dimensions was obtained in [27], where the theorem below is proved.

Theorem 4.11 *Let $d-1<p\le d$. If a sequence $(\Omega_n)_{n\in\mathbb{N}}\in\mathcal{O}_k$ converges in the Hausdorff complementary topology to some Ω, then*

$$\Omega_n \xrightarrow{\gamma_p} \Omega \qquad \text{and} \qquad \Omega\in\mathcal{O}_k.$$

As a consequence of this theorem, a large class of shape optimization problems admits solutions in the classes $\mathcal{O}_k$ above under very mild assumptions:

Corollary 4.1 *Let $d-1<p\le d$ and assume that there exist $a\in L^1(D)$ and $c>0$ such that*

$$-a(x)-c|s|^q\le j(x,s) \qquad \text{with } q<\frac{dp}{d-p} \quad (\text{any } q<\infty \text{ if } p=d).$$

Then, for every integer k, the shape optimization problem

$$\min\left\{\int_D j(x,u_\Omega)\,dx\ :\ \Omega\in\mathcal{O}_k\right\}$$

admits a solution Ω_{opt}, where u_Ω denotes the solution of (4.1).

4.5 Finite perimeter of optimal sets

In this section, we prove that any optimal shape Ω_{opt} has finite perimeter. Notice that the prototype

$$j(x,s)=-g(x)s+\mathbb{1}_{(0,+\infty)}(s) \qquad \text{with } g(x)\ge 0$$

is well within both sets of assumptions (i) and (ii) of Theorem 4.6.

Proof of Theorem 4.6. Let $\Omega_{\text{opt}}=\{\bar u>0\}$ be an optimal shape and recall that $\bar u$ can also be characterized as the unique solution of the minimization problem

$$\min\left\{\int_D \big(j_0(x,u)+\mathbb{1}_{\{u>0\}}(x)\big)\ dx\ :\ \Delta_p u+f\ge 0,\ u\in W_0^{1,p}(D)\right\}.$$

Let $\varphi\in W_0^{1,p}(D)$ be a non-negative test function. Then

$$\langle\Delta_p\bar u+f,\varphi\rangle\ge 0,$$

and a simple integration by parts gives the inequality

$$\big\langle f,\varphi\big\rangle\ge\big\langle|\nabla\bar u|^{p-2}\nabla\bar u,\nabla\varphi\big\rangle\,. \tag{4.27}$$

Let $\epsilon>0$ be small enough for (4.10) and (4.11) to hold and let $\Omega_\epsilon:=\{0<\bar u<\epsilon\}$. If we take as a test function $\varphi:=\bar u\wedge\epsilon$, then

$$\big\langle|\nabla\bar u|^{p-2}\nabla\bar u,\nabla\varphi\big\rangle=\int_D|\nabla\bar u|^{p-2}\nabla\bar u\cdot\nabla(\bar u\wedge\epsilon)\,dx=\int_{\Omega_\epsilon}|\nabla\bar u|^p\,dx,$$

and

$$\big\langle f,\varphi\big\rangle=\int_D f(\bar u\wedge\epsilon)\,dx\le\int_{\Omega_\epsilon}f\bar u\,dx+\int_{D\setminus\Omega_\epsilon}f\epsilon\,dx\le C\epsilon.$$

Therefore, if we plug these two back into (4.27), we get

$$\int_{\Omega_\epsilon}|\nabla\bar u|^p\,dx\le C\epsilon, \tag{4.28}$$

where we use C to denote a positive constant which may vary from line to line throughout the proof.

Let $\bar{v} := (\bar{u} - \epsilon)_+$ and notice that it satisfies $\Delta_p \bar{v} + f \geq 0$ in the weak sense, so $\bar{v}$ is a competitor for the minimization problem above. Using the minimality of $\bar{u}$, we get

$$\int_D \left(j_0(x, \bar{u}) + \chi_{\{\bar{u}>0\}}(x)\right) \, dx \leq \int_D \left(j_0(x, \bar{v}) + \chi_{\{\bar{v}>0\}}(x)\right) \, dx,$$

which, using the decomposition $\Omega_{\text{opt}} = \Omega_\epsilon \cup (\Omega_{\text{opt}} \setminus \Omega_\epsilon)$, can be rewritten as follows:

$$\int_{\Omega_\epsilon} \left(j_0(x, \bar{u}) + 1\right) \, dx + \int_{\Omega_{\text{opt}} \setminus \Omega_\epsilon} j_0(x, \bar{u}) \, dx \leq \int_{\Omega_{\text{opt}} \setminus \Omega_\epsilon} j_0(x, \bar{u} - \epsilon) \, dx.$$

To use the assumptions (4.10) or (4.11) we first move everything on the right-hand side and then add the integral of $j_0(x, 0)$ on Ω_ϵ on both sides to obtain

$$\begin{aligned}
\int_{\Omega_\epsilon} 1 \, dx &\leq \int_{\Omega_\epsilon} \left(1 + j_0(x, 0)\right) \, dx \\
&\leq \int_{\Omega_{\text{opt}} \setminus \Omega_\epsilon} \left[j_0(x, \bar{u} - \epsilon) - j_0(x, \bar{u})\right] \, dx + \int_{\Omega_\epsilon} \left[j_0(x, 0) - j_0(x, \bar{u})\right] \, dx,
\end{aligned}$$

where the first inequality follows from the fact that $j_0(x, 0) = 0$. We now divide the proof since there are minor differences in using the two sets of assumptions separately:

(i) In this case, we have

$$\int_{\Omega_\epsilon} |j_0(x, 0) - j_0(x, \bar{u})| \, dx \leq \int_{\Omega_\epsilon} a(x) \bar{u} \, dx \leq C\epsilon$$

since $\bar{u} < \epsilon$ on Ω_ϵ by definition, and $a \in L^1(D)$. Similarly, we have

$$\int_{\Omega_{\text{opt}} \setminus \Omega_\epsilon} \left|j_0(x, \bar{u} - \epsilon) - j_0(x, \bar{u})\right| \, dx \leq \int_{\Omega_{\text{opt}} \setminus \Omega_\epsilon} \epsilon \left[a(x) + c|\bar{u}|^{p^*}\right] \, dx \leq C\epsilon$$

as a consequence of the Sobolev embedding theorem (Theorem 4.7) for $u \in W_0^{1,p}$. On the other hand, the left-hand side is given by

$$\int_{\Omega_\epsilon} 1 \, dx = |\Omega_\epsilon|,$$

so putting everything together leads to the following estimate:

$$|\Omega_\epsilon| \leq C\epsilon. \tag{4.29}$$

(ii) In this case, the function $\bar{u}$ is bounded (it follows from Theorem 4.3 for $p > d$ and from Lemma 4.4 for $p \leq d$) and we have

$$\int_{\Omega_\epsilon} |j_0(x, 0) - j_0(x, \bar{u})| \, dx \leq \int_{\Omega_\epsilon} a(x, \bar{u}) \bar{u} \, dx \leq \epsilon C \|\bar{u}\|_\infty,$$

and

$$\int_{\Omega_{\text{opt}} \setminus \Omega_\epsilon} \left|j_0(x, \bar{u} - \epsilon) - j_0(x, \bar{u})\right| \, dx \leq \int_{\Omega_{\text{opt}} \setminus \Omega_\epsilon} \epsilon \left[a(x, \bar{u})\right] \, dx \leq \epsilon C \|\bar{u}\|_\infty.$$

In particular, the estimate (4.29) holds also in this case, but with a different constant.

Since both sets of assumptions lead to the same estimate (up to a constant), separating them is no longer necessary. If we now put (4.28) and (4.29) together and apply Hölder's inequality, we get

$$\int_{\Omega_\epsilon} |\nabla \bar{u}|\, dx \leq \left[\int_{\Omega_\epsilon} |\nabla \bar{u}|^p\, dx\right]^{1/p} |\Omega_\epsilon|^{1/p'} \leq (C\epsilon)^{1/p}(C\epsilon)^{1/p'} = C\epsilon,$$

so we finally deduce that Ω_{opt} has a finite perimeter using the strategy proposed in [20]. More precisely, the coarea formula (see, e.g., [8]) gives

$$\int_0^\epsilon \mathcal{H}^{d-1}(\partial^*\{\bar{u} > \epsilon\})\, dt = \int_{\Omega_\epsilon} |\nabla \bar{u}|\, dx \leq C\epsilon,$$

which means that there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ that goes to zero such that

$$\mathcal{H}^{d-1}(\partial^*\{\bar{u} > \delta_n\}) \leq C \quad \text{for all } n \in \mathbb{N}.$$

This is enough to conclude, since

$$\text{Per}\,\Omega_{\text{opt}} = \mathcal{H}^{d-1}(\partial^*\{\bar{u} > 0\}) = \lim_{n\to+\infty} \mathcal{H}^{d-1}(\partial^*\{\bar{u} > \delta_n\}) \leq C,$$

as the constant C does not depend on n. □

4.6 Remarks, open questions and related problems

The existence and regularity of optimal shapes in the case $f = g$ was already studied for all $p > 1$ since there is an underlying variational structure. Indeed, if we test (4.1) with u_Ω, we get

$$\int_D |\nabla u_\Omega|^p\, dx = \int_D f(x) u_\Omega\, dx,$$

and thus we can characterize u_Ω as the unique solution of

$$\min\left\{\frac{1}{p}\int_D |\nabla u|^p\, dx - \int_D f(x)u\, dx \;:\; u \in W_0^{1,p}(D)\right\}.$$

Therefore, the shape optimization problem (4.2) becomes "equivalent" to the free boundary problem

$$\min\left\{\frac{1}{p}\int_D |\nabla u|^p\, dx - \int_D f(x)u\, dx + \frac{p-1}{p}\lambda|\{u \neq 0\}| \;:\; u \in W_0^{1,p}(D)\right\} \tag{4.30}$$

in the following sense:

- if Ω_{opt} solves (4.2), then $\bar{u} := u_{\Omega_{\text{opt}}}$ is a solution to (4.30);
- if $\bar{u}$ solves (4.30), then the set $\{\bar{u} \neq 0\}$ is optimal for (4.2).

This equivalence is crucial because problems of the form (4.30) have been widely studied (especially in the case $p = 2$). For example, the existence of solutions $\bar{u} \in W^{1,p}$ is easy and implies that

$$\Omega := \{\bar{u} \neq 0\}$$

is a solution to (4.2) in the class of the p-quasi-open sets; existence in the class of open sets, however, requires more efforts because one has to study the regularity of the solutions of (4.30) - see [144] and the references therein for more details -.

In the general case $p \neq 2$, the regularity of $\bar{u}$ and of the corresponding free boundary $\partial\{\bar{u} > 0\}$ were first studied by Danielli and Petrosyan in [58] in the case $f = 0$, while for $f \geq 0$ the problem was discussed in [41].

Supremal functionals

A class of shape optimization problems related to the one from Theorem 4.1 are the ones in which the cost functional is given by

$$J(u_\Omega) := -\operatorname*{ess\,sup}_{x\in D} j(x, u_\Omega).$$

The existence of a solution can be obtained by the same argument as in the proofs of Theorem 4.3, in the case $p > d$, and of Theorem 4.4, in the case $p \le d$.

Mixed boundary conditions

The full Dirichlet boundary condition in (4.1) can be replaced by the mixed Dirichlet-Neumann condition

$$u = 0 \;\text{ on }\; D \setminus \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \;\text{ on }\; \partial D \cap \partial\Omega.$$

Since we would like to test very irregular domains Ω, this expression is only formal and has to be intended in a weak sense. More precisely, the state function is the unique solution of

$$\min\left\{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega uf\,dx \;:\; u \in W^{1,p}(D),\; u = 0 \text{ cap}_p\text{-a.e. on } D \setminus \Omega\right\}.$$

In this case, the existence results Theorem 4.3 and Theorem 4.4 still hold for the function $J(u_\Omega)$ if the measure constraint is **not** saturated, i.e., if we assume $m < |D|$.

This problem, along to similar ones with mixed boundary conditions, have been considered in [44].

Remark 4.13 The case $m = |D|$, on the other hand, is not particularly interesting because if we assume, for example, that f is a non-negative function, then

$$\inf\left\{J(u_\Omega) \;:\; \Omega \text{ open},\; \Omega \subset D\right\} = -\infty.$$

This can be verified for our model function (4.4) if we take g non-negative and $\lambda \ge 0$.

Problem 4.2 Shape optimization problems with the Neumann boundary condition on the free part (i.e., the part that does not lie on ∂D) require a completely different approach. At the moment, very little is known about them.

The limit problem as $p \to \infty$

If we take the limit as $p \to \infty$, the state function no longer satisfies a PDE but it can still be characterized as the unique solution of the following variational problem:

$$\min\left\{-\int_D f(x)u\,dx \;:\; |\nabla u| \le 1,\; u = 0 \text{ on } \overline{D} \setminus \Omega\right\}.$$

It is easy to verify that the solution is the distance function

$$u_\Omega(x) := d(x, \overline{D} \setminus \Omega).$$

In this way, the problems above, with $p = \infty$, are related to some optimization problems in mass transport theory; see, for instance, [39] and [43].

Further open problems

Problem 4.3 In Theorem 4.11, we proved the existence of an optimal shape in the class $\mathbb{O}_k$ under the assumption $d - 1 < p \leq d$. It would be interesting to see if this interval is optimal.

The boundary Harnack principle on optimal domains 5

In this chapter, we give an overview of the results obtained in [124], which will play a fundamental role in proving the assertion (ii) of Theorem 4.2 (see Lemma 7.8 and Proposition 7.7 for more details).

5.1 Introduction and main results

Our goal is to prove a *boundary Harnack principle* for domains that satisfy certain geometrical conditions naturally occurring in some variational problems. It is a crucial tool in proving the $C^{1,\alpha}$-regularity of free boundaries arising in vectorial free boundary and shape optimization problems (for some applications, see Section 5.5 and Subsection 7.4.2). That said, before we state our main result, we recall what it means for a domain $\Omega \subset \mathbb{R}^d$ to satisfy the boundary Harnack principle.

Definition 5.1 (Boundary Harnack principle) *Let $\Omega \subset B_1$. We say that the **boundary Harnack principle** holds in Ω, if there is $\alpha > 0$ such that, for every couple of functions*

$$u : B_1 \to \mathbb{R} \qquad \text{and} \qquad v : B_1 \to \mathbb{R}$$

which are continuous on B_1, positive and harmonic in $\Omega \cap B_1$ and vanishing identically on $B_1 \setminus \Omega$, the ratio

$$\frac{u}{v} : \Omega \to \mathbb{R}$$

can be extended to a $C^{0,\alpha}$-regular function on $B_{1/2} \cap \bar{\Omega}$.

We are now ready to state the main result of [124]. In the following, we will denote by $|E|$ the Lebesgue measure of a set E and by $d(\cdot, E)$ the distance from E, which is given by

$$d(x, E) := \inf_{y \in E} |x - y|.$$

Theorem 5.1 *Let $\Omega \subset B_1$ be an open set with $0 \in \partial\Omega$ and $\phi : B_1 \to \mathbb{R}$ a continuous function such that:*

(a) $\phi > 0$ on Ω, $\phi \equiv 0$ on $B_1 \setminus \Omega$ and ϕ is Lipschitz-continuous on B_1 with constant L;

(b) there exists a constant $\kappa > 0$ such that

$$\phi(x) \geq \kappa\, d\,(x, B_1 \setminus \Omega) \qquad \text{for every } x \in B_{1/2};$$

(c) we have the inequality $\Delta\phi \geq 0$ in sense of distributions in B_1;

(d) there is a constant $\mu > 0$ such that for every $x_0 \in \partial\Omega \cap B_1$, we have

$$|B_r(x_0) \setminus \Omega| \geq \mu |B_r(x_0)| \qquad \text{for every } r \in (0, 1 - |x_0|);$$

(e) there is a constant $\Lambda > 0$ such that for every $x_0 \in \partial\Omega \cap B_1$ and every $r \in (0, 1 - |x_0|)$, we have

$$\Big|\{0 < \phi < rt\} \cap B_r(x_0)\Big| \leq \Lambda t |B_r| \qquad \text{for every } t > 0;$$

(f) there is a constant $\eta > 0$ such that for every $x_0 \in \partial\Omega \cap B_1$ and every $r \in (0, 1 - |x_0|)$, we have

$$\sup_{x \in B_r(x_0)} \phi(x) \geq \eta r.$$

Then the boundary Harnack principle holds in Ω in the sense of Definition 5.1.

In Section 5.4, we will show that it is possible to deduce Theorem 5.1 immediately once we prove the following *boundary Harnack inequality*:

Theorem 5.2 *Under the same assumptions of Theorem 5.1, there are $M > 0$, $\delta \in (0, \eta]$ and $0 < \rho < R \leq 1$, depending on the dimension d and the constants from (a)–(f), such that the following holds:*

Boundary Harnack inequality. *Suppose that $u, v : B_1 \to \mathbb{R}$ are non-negative continuous functions satisfying*

$$\begin{cases} \Delta u = \Delta v = 0 & \text{in } \Omega \cap B_1, \\ u = v = 0 & \text{on } B_1 \setminus \Omega, \\ u(x_0) = v(x_0) & \text{for some } x_0 \in B_R \cap \{\phi > \delta R\}. \end{cases} \tag{5.1}$$

Then

$$M^{-1} v(x) \leq u(x) \leq M v(x) \qquad \text{for every } x \in B_\rho.$$

Consequently, if a domain $\Omega \subset B_1$ admits a function satisfying (a)–(f), then the *boundary Harnack inequality* holds on Ω. This general principle is well-known in the literature and was used, for instance, in [1], [52] and [130, 131]. However, in all these papers, the result is obtained by showing that the optimal domain Ω is NTA (see Definition 5.4) and deducing the boundary Harnack inequality by applying a well-known result by Jerison and Kenig that can be found in [103].

The main novelty of [124] is not the result but the strategy used to achieve it. Indeed, we give a direct proof that essentially uses only the mean value formula for harmonic functions and the classical Alt-Caffarelli-Friedman monotonicity formula for subharmonic functions [5].

To conclude this section, we define *non-tangentially accessible* (NTA) domains. First, we need to introduce the so-called *corkscrew condition*:

Definition 5.2 *An open set $\Omega \subset \mathbb{R}^d$ satisfies the interior (exterior) corkscrew condition if there are $r_0 > 0$ and $\varepsilon > 0$ such that for all $r \in (0, r_0)$ and $x \in \partial\Omega$, $B(x, r) \cap \Omega$ ($B(x, r) \cap (\mathbb{R}^d \setminus \Omega)$) contains a ball of radius εr.*

Moreover, we say that an open set Ω satisfies the corkscrew condition if it satisfies both the interior and the exterior corkscrew condition.

Definition 5.3 (Harnack-chain) *Let $\Omega \subset \mathbb{R}^d$ be an open set and $\alpha \geq 1$. A sequence of balls $B_0, \ldots, B_k \subset \Omega$ is an α-Harnack chain if, for all $i = 1, \ldots, k$, we have*

$$B_i \cap B_{i-1} \neq \emptyset \qquad \text{and} \qquad \alpha^{-1} d(B_i, \partial\Omega) \leq r(B_i) \leq \alpha d(B_i, \partial\Omega).$$

Definition 5.4 *A bounded open set Ω is a NTA domain in the metric space $\mathbb{R}^d$ if the following holds:*

- *Ω satisfies the corkscrew condition;*
- *there exists $\alpha \geq 1$ such that, for all $\eta > 0$ and all $x, y \in \Omega$ for which we have*

$$d(x, \partial\Omega),\ d(y, \partial\Omega) \geq \eta \qquad \text{and} \qquad d(x, y) \leq C\eta$$

  for some $C > 0$, there exists an α-Harnack chain $B_0, \ldots, B_k \subset \Omega$ with $x \in B_0$, $y \in B_k$; in addition, the length of the chain k may depend on C, but not on η.

5.1.1 Outline of the chapter and further remarks

In Section 5.2 we prove interior Harnack inequalities close and away from the boundary since they are fundamental for our main result and encode the geometric properties of the domain we are considering (in particular, see Lemma 5.2 and Lemma 5.4).

In Section 5.3 we prove the boundary Harnack inequality (Theorem 5.1) following step-by-step the strategy developed in the recent paper [74] by De Silva and Savin. Finally, in Section 5.4 we show how to deduce the boundary Harnack principle from Theorem 5.2.

Notice that, in general, the boundary Harnack principle on a domain Ω is obtained as a consequence of the validity of the boundary Harnack inequality at any scale and for any couple of non-negative functions u and v satisfying (5.1) on a rescaling of Ω. This implication is well-known in the literature (see, for example, [103]) and in Section 5.4 we give a short proof of this fact in our setting.

Remark 5.1 The assumptions of Theorem 5.1 are scale-invariant. Indeed, let Ω and ϕ be as in Theorem 5.1 and let $x_0 \in \partial\Omega \cap B_1$. If we define the corresponding rescalings centered at x_0 as

$$\Omega_{r,x_0} := \frac{1}{r}(\Omega - x_0) \quad \text{and} \quad \phi_{r,x_0} := \frac{\phi(x_0 + rx)}{r},$$

for $r \in (0, 1 - |x_0|)$, then it is easy to verify that $\Omega_{r,x_0} \subset B_1$ and that the properties (a)–(e) are satisfied with the same constants.

Remark 5.2 The assumption (f) is only needed to ensure that the set

$$B_R \cap \{\phi > \delta R\},$$

appearing in (5.1), is non-empty, at least for $\delta > 0$ small enough. Indeed, if we assume that the couple (Ω, ϕ) satisfies (a) and (f) of Theorem 5.1, then

$$B_r(x_0) \cap \{\phi > r\delta\} \neq \emptyset$$

for every $x_0 \in \partial\Omega \cap B_1$, $r \in (0, 1 - |x_0|)$ and $\delta \in (0, \eta)$.

Several versions of the boundary Harnack inequality (B.H.I.) appeared recently in the literature, for instance

- in [121], the authors established a B.H.I. on the class of nodal domains of solutions to uniformly elliptic equations in divergence form;
- in [3], it was proved for solutions with right-hand side on sufficiently flat Lipschitz domains.

To conclude this section, we also refer to [73] for a higher-order boundary Harnack principle.

5.2 Harnack chains and interior Harnack inequalities

This section aims to prove the existence of Harnack chains (see Definition 5.3) and, consequently, deduce Harnack-type inequalities close and away from the boundary.

5.2.1 Harnack inequality close to the boundary

We now show how to construct short Harnack chains starting from a point close to the boundary of Ω and deduce (by iteration) the Harnack inequality close to the boundary in Lemma 5.2.

Lemma 5.1 *Let $\Omega \subset B_1$ with $0 \in \partial\Omega$ and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions (a)–(c) of Theorem 5.1, let $x_0 \in \{\phi > 0\} \cap B_1$ be a point such that*

$$r := d(x_0, \partial\Omega) < \frac{1}{3}(1 - |x_0|),$$

and let z_0 be a projection of x_0 onto $\partial\Omega$. Then there is $y_0 \in \partial B_r(x_0)$ such that:

(i) *$\phi(y_0) \geq (1+\sigma)\phi(x_0)$, where $\sigma > 0$, depends only on the dimension d, and the constants L from assumption (a) and κ from assumption (b);*

(ii) *there is a constant $c_{\mathcal{H}} > 1$, depending only on the dimension d and the constants L and κ, such that for every positive harmonic function $w : \Omega \to \mathbb{R}$, we have*

$$c_{\mathcal{H}}^{-1} w(y_0) \leq w(x_0) \leq c_{\mathcal{H}} w(y_0).$$

Proof. Let $\epsilon > 0$ and notice that, since ϕ is a subharmonic function by (c), we can estimate the difference between the average on $\partial B_r(x_0)$ and $\phi(x_0)$ as follows:

$$\fint_{\partial B_r(x_0)} \phi(x)\,dx - \phi(x_0) = \frac{1}{d\omega_d}\int_0^r s^{1-d}\Delta\phi(B_s(x_0))\,ds \geq 0.$$

Now, let the radius $\rho > 0$ be such that

$$\mathcal{H}^{d-1}\left(\partial B_r(x_0)\cap B_\rho(z_0)\right) = \epsilon^{d-1}\mathcal{H}^{d-1}\left(\partial B_r(x_0)\right),$$

and choose $\epsilon > 0$ satisfying $\rho \leq 2\epsilon r$. Then, by the Lipschitz continuity of ϕ, we get

$$\phi(x) \leq 2L\epsilon r \qquad \text{for every } x \in \partial B_r(x_0)\cap B_\rho(z_0),$$

where L is the Lipschitz constant from (a). If we now let

$$M := \max\left\{\phi(x) \,:\, x \in \partial B_r(x_0)\right\},$$

we can put everything together (included the assumption (b)) with the estimate above to obtain

$$\begin{aligned}\phi(x_0) \leq \fint_{\partial B_r(x_0)} \phi(x)\,dx &\leq \frac{1}{r^{d-1}}\left((2\epsilon r)^{d-1}2L\epsilon r + M\left(r^{d-1} - (2\epsilon r)^{d-1}\right)\right)\\ &\leq \frac{1}{r^{d-1}}\left((2\epsilon r)^{d-1}\frac{2L\epsilon}{\kappa}\phi(x_0) + M\left(r^{d-1} - (2\epsilon r)^{d-1}\right)\right)\\ &\leq (2\epsilon)^{d-1}\frac{2L\epsilon}{\kappa}\phi(x_0) + M\left(1-(2\epsilon)^{d-1}\right),\end{aligned}$$

which, in turn, implies that

$$\left(1 - \epsilon^{d-1}\frac{2^d L\epsilon}{\kappa}\right)\phi(x_0) \leq \left(1-\epsilon^{d-1}\right)M.$$

We need the factor on the left-hand side to be positive, so we modify (if necessary) the value of ϵ in such a way that the following holds:

$$\frac{2^d L\epsilon}{\kappa} \leq \frac{1}{2^{d-1}} \implies 1 - \epsilon^{d-1}\frac{2^d L\epsilon}{\kappa} \geq 1 - \left(\frac{\epsilon}{2}\right)^{d-1}.$$

Finally, notice that any point $y_0 \in \partial B_r(x_0)$ that achieves the maximum (i.e., such that $M = \phi(y_0)$) satisfies the condition (i); more precisely, we have

$$(1+\sigma)\phi(x_0) \leq \phi(y_0) \qquad \text{with} \quad 1+\sigma := \frac{1}{1-(2\epsilon)^{d-1}}\left(1-\epsilon^{d-1}\right).$$

In order to prove (ii), we notice that by the Lipschitz continuity of ϕ, we have

$$d(y_0, B_1\setminus\Omega) \geq \frac{1}{L}\phi(y_0) \geq \frac{1}{L}\phi(x_0) \geq \frac{\kappa}{L}r,$$

which means that

$$B_r(x_0) \cap B_{\frac{r\kappa}{L}}(y_0) \subset \Omega,$$

and the thesis follows by the classical Harnack inequality [80, Chapter 6]. □

Consequently, by iterating this result, we obtain the following Harnack-type inequality that holds for points close enough to the boundary.

Lemma 5.2 *Under the same assumptions of Lemma 5.1, there are constants $A \geq 1$ and $\delta_0 > 0$, depending only on d, L and κ such that, for every positive harmonic function $w : \Omega \longrightarrow \mathbb{R}$, we have*

$$\sup_{B_{1/2} \cap \{\phi > \frac{\delta}{2}\}} w \leq A \sup_{B_1 \cap \{\phi > \delta\}} w \qquad \textit{and} \qquad \inf_{B_{1/2} \cap \{\phi > \frac{\delta}{2}\}} w \geq \frac{1}{A} \inf_{B_1 \cap \{\phi > \delta\}} w$$

for every $\delta \in (0, \delta_0]$.

Proof. Let $x_0 \in B_{1/2} \cap \{\phi > \frac{\delta}{2}\}$. If $\phi(x_0) > \delta$, then for any $A \geq 1$ we have

$$\frac{1}{A} \inf_{B_1 \cap \{\phi > \delta\}} w \leq w(x_0) \qquad \text{and} \qquad w(x_0) \leq A \sup_{B_1 \cap \{\phi > \delta\}} w,$$

so we only need to consider the case $x_0 \in B_{1/2} \cap \{\frac{\delta}{2} < \phi \leq \delta\}$. Let x_1 be the point y_0 obtained with Lemma 5.1 starting from x_0. Then

$$\phi(x_1) \geq (1 + \sigma)\phi(x_0) \geq (1 + \sigma)\frac{\delta}{2},$$

and, by construction, we have $x_1 \in \partial B_r(x_0)$ with $r := d(x_0, B_1 \setminus \Omega)$. It follows from (b) that

$$|x_0 - x_1| = r \leq \frac{1}{\kappa}\phi(x_0) \leq \frac{\delta}{\kappa}$$

and, if $x_1 \in \{\phi \leq \delta\}$, we can repeat the same procedure to find a point x_2. Iterating this argument, we obtain a sequence of points x_n such that

$$x_n \in B_{r_n} \cap \left\{\frac{\delta}{2}(1 + \sigma)^n < \phi \leq \delta\right\} \qquad \text{with } r_n := \frac{1}{2} + n\frac{\delta}{\kappa},$$

satisfying the estimates

$$c_{\mathcal{H}}^{-n}\, w(x_n) \leq w(x_0) \leq c_{\mathcal{H}}^{n}\, w(x_n),$$

where $c_{\mathcal{H}} > 1$ is the Harnack constant from (ii) of Lemma 5.1. Now, let N to be the largest integer for which $x_N \in B_1 \cap \{\phi \leq \delta\}$ and to which we can still apply Lemma 5.1 to obtain a point x_{N+1}. Then

$$\frac{1}{2}(1 + \sigma)^N \leq 1,$$

which implies that N is bounded above by

$$N \leq \frac{1}{\log_2(1 + \sigma)}.$$

As a consequence, the radius associated to x_{N+1} satisfies

$$r_{N+1} \leq \frac{1}{2} + (N + 1)\frac{\delta}{\kappa} \leq \frac{1}{2} + \left(\frac{1}{\log_2(1 + \sigma)} + 1\right)\frac{\delta_0}{\kappa},$$

so, by choosing $\delta_0 > 0$ small enough, we can assume $r_{N+1} \leq 3/4$ and apply Lemma 5.1 to x_{N+1} to obtain another point x_{N+2}, which leads to the end of the iterative procedure because

$$x_{N+2} \in \{\phi > \delta\}.$$

Therefore, we have

$$c_{\mathcal{H}}^{-(N+1)} \min_{B_1 \cap \{\phi > \delta\}} w \leq c_{\mathcal{H}}^{-(N+1)} w(x_{N+1}) \leq w(x_0)$$

and

$$w(x_0) \leq c_{\mathcal{H}}^{N+1} w(x_{N+1}) \leq c_{\mathcal{H}}^{N+1} \max_{B_1 \cap \{\phi > \delta\}} w.$$

Finally, the results follows by taking $A := c_{\mathcal{H}}^{N+1} \geq 1$ and x_0 as the point at which the maximum (resp. the minimum) of the function w is achieved in the set $B_1 \cap \{\phi \geq \delta/2\}$. □

5.2.2 Harnack inequality away from the boundary

The goal of this section is to prove an interior Harnack-type inequality. As above, we will deduce it starting from the construction of interior Harnack chains, which is achieved by combining (c) with the monotonicity formula of Alt-Caffarelli-Friedman [5].

Lemma 5.3 *Suppose that $\Omega \subset B_1$, with $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions (a), (c) and (d) of Theorem 5.1. For every $\delta \in (0, 2L)$ there is $\tau \in (0, 1)$ such that the following holds:*

> ***Claim.*** *For every $R \in (0, 1)$ and every couple of points $x_1, x_2 \in B_{\tau R} \cap \{\phi > \delta R\}$, there is a curve connecting x_1 to x_2 in $B_R \cap \{\phi > (\delta/2)R\}$.*

Proof. We argue by contradiction. Fix $\tau \in (0, 1)$ and suppose that $x_1, x_2 \in B_{\tau R} \cap \{\phi > \delta R\}$ lie in two different connected components, which we denote by Ω_1 and Ω_2, of

$$B_R \cap \left\{\phi > \frac{\delta}{2} R\right\}.$$

Let ϕ_1 and ϕ_2 be the restrictions of the function $(\phi - {}^{\delta}\!/_{2}\, R)_+$ to Ω_1 and Ω_2 respectively. It is easy to verify that both are L-Lipschitz, where L is given in (a), and satisfy

$$\phi_j(x_j) \geq \frac{\delta}{2} R \qquad \text{for } j = 1, 2.$$

Moreover, for every $r \in [\tau R, R]$, there is $x_r \in \partial B_r$ such that

$$\phi_1(x_r) = \phi_2(x_r) = 0$$

since it is sufficient to take $x_r \in \{\phi < (\delta/2)R\}$. Now let

$$\psi_j := (\phi_j - \frac{3\delta}{4} R)_+ \qquad \text{for } j = 1, 2,$$

and notice that each ψ_j is L-Lipschitz, harmonic where positive, and satisfies

$$\psi_j(x_j) \geq \frac{\delta}{4} R \qquad \text{and} \qquad \psi_j \equiv 0 \text{ on } B_{\delta R/4L}(x_r)$$

for every $r \in [\tau R, R]$. If δ is small enough, this implies the density estimate

$$\alpha(r) := \frac{\mathcal{H}^{d-1}\Big(\{\psi_1 = \psi_2 = 0\} \cap \partial B_r\Big)}{\mathcal{H}^{d-1}(\partial B_r)} \geq F\left(\frac{\delta^{d-1}}{(4L)^{d-1}}\right) \qquad \text{for every } r \in [\tau R, R],$$

where $F : [0, +\infty) \longrightarrow \mathbb{R}$ is a continuously differentiable increasing function depending only on d and such that $F(0) = 0$ and $F'(0) > 0$. Consider the function

$$\Phi(r) := \frac{1}{r^4}\left(\int_{B_r} \frac{|\nabla\psi_1|^2}{|x|^{d-2}}\,dx\right)\left(\int_{B_r} \frac{|\nabla\psi_2|^2}{|x|^{d-2}}\,dx\right),$$

and notice that, by [5] (see also [1, Lemma 4.3]), we have

$$\frac{d}{dr}\left[\ln\Phi(r)\right] \geq \frac{1}{r}G\left(\alpha(r)\right),$$

where $G : [0, +\infty) \longrightarrow \mathbb{R}$ is a positive increasing convex function with $G(0) = 0$ and $G'(0) > 0$. As a consequence, for $\delta > 0$ small enough, we have

$$\frac{d}{dr}\left[\ln\Phi(r)\right] \geq C_d\frac{1}{r}\frac{\delta^{d-1}}{(4L)^{d-1}},$$

so integrating both sides and taking into account that $\phi_1(0) = 0 = \phi_2(0)$ leads to the estimate

$$\left[\fint_{B_r} |\nabla\psi_1|^2\,dx\right]\left[\fint_{B_r} |\nabla\psi_2|^2\,dx\right] \leq \Phi(r) \leq \left(\frac{r}{R}\right)^{\beta}\Phi(R), \qquad \text{for every } r \in [\tau R, R],$$

where $\beta := C_d(\delta/(4L))^{d-1}$. Moreover, by the density estimate (d) and the classical Poincaré inequality (4.12), we deduce the following estimate on the average of ψ_j^2:

$$\frac{1}{r^4}\left[\fint_{B_r} \psi_1^2\,dx\right]\left[\fint_{B_r} \psi_2^2\,dx\right] \leq \left(\frac{r}{R}\right)^{\beta}\Phi(R), \qquad \text{for every } r \in [\tau R, R].$$

To conclude we also need an estimate from below. By the Lipschitz continuity (a), we have

$$\psi_i(x) \geq \frac{\delta}{4}R - L|x - x_i| \qquad \text{in } B_\rho(x_i) \text{ for } i = 1, 2.$$

If we now choose $\rho := \delta R/(4L)$, then it is easy to verify that $\rho \leq R/2$; therefore, we have

$$\begin{aligned}\int_{B_r} \psi_i^2\,dx \geq \int_{B_r\cap B_\rho(x_i)} \psi_i^2\,dx &\geq \int_{B_r\cap B_\rho(x_i)} \left(\frac{\delta}{4}R - L|x - x_i|\right)^2 dx \\ &\geq c_d\int_{B_\rho(x_i)} \left(\frac{\delta}{4}R - L|x - x_i|\right)^2 dx,\end{aligned}$$

where c_d is a dimensional constant. Now, a straightforward computation gives

$$\int_{B_\rho(x_i)}\left(\frac{\delta}{4}R - L|x-x_i|\right)^2 dx \geq \frac{1}{|B_\rho|}\left(\int_{B_\rho}\left(\frac{\delta}{4}R - L|x|\right)dx\right)^2$$
$$= \frac{(d\omega_d)^2}{\omega_d\rho^d}\left(\int_0^\rho s^{d-1}\left(\frac{\delta}{4}R - sL\right)ds\right)^2$$
$$= \frac{(d\omega_d)^2}{\omega_d\rho^d}\left(L\int_0^\rho s^{d-1}\left(\rho - s\right)ds\right)^2$$
$$= \frac{\omega_d d^2}{(d+1)^2}L^2\rho^{d+2}.$$

It follows that

$$\frac{1}{r^2}\fint_{B_r}\psi_i^2\,dx \geq C_d\frac{\delta^{d+2}}{L^d}\left(\frac{R}{r}\right)^{d+2}$$

and, by the Lipschitz continuity of ψ_1 and ψ_2, we obtain the inequality

$$C_d\frac{\delta^{2d+4}}{L^{2d}} \leq \left(\frac{r}{R}\right)^{d+2+\beta}\Phi(R) \leq \left(\frac{r}{R}\right)^{d+2+\beta}\frac{\omega_d^2L^4}{4} \qquad \text{for every } r\in[\tau R, R].$$

In particular, if we take $r = \tau R$, we deduce that

$$\tau^{d+2+\beta} \geq C_d\left(\frac{\delta}{L}\right)^{2d+4},$$

which is a contradiction if τ is small enough. □

We are now ready to state and prove the interior Harnack-type inequality:

Lemma 5.4 *Suppose that $\Omega\subset B_1$, with $0\in\partial\Omega$, and $\phi : B_1 \longrightarrow \mathbb{R}$ satisfy the conditions (a), (c) and (d) of Theorem 5.1. Then for every $\delta > 0$ there is R_0 for which the following holds:*

> ***Claim.*** *For every $R\in(0,R_0]$, there is a constant $C_{\mathcal{H}} := C_{\mathcal{H}}(\delta, R) > 0$ such that for every positive harmonic function $w : \Omega\cap B_1 \longrightarrow \mathbb{R}$ satisfying*
>
> $$w \geq 0 \quad in \quad \Omega\cap B_1 \qquad and \qquad \Delta w = 0 \quad in \quad \Omega\cap B_1,$$
>
> *we have*
>
> $$\inf_{\{\phi>\delta R\}\cap B_R} w \geq C_{\mathcal{H}} \sup_{\{\phi>\delta R\}\cap B_R} w.$$

To prove Lemma 5.4 it is sufficient to show that there are $N\in\mathbb{N}$ and $R>0$ such that, for every pair of points $x_0, y_0 \in \{\phi > \delta R\}\cap B_R$, there are a curve $\gamma : [0,1] \longrightarrow B_1$ satisfying

$$\gamma(0) = x_0 \qquad \text{and} \qquad \gamma(1) = y_0,$$

and a family of balls $\{B_r(x_j) : j = 1,\dots,N\}$ with the following properties:

- $x_j \in \gamma([0,1])$ for every $j = 1,\dots,N$;
- $B_{2r}(x_j) \subset \Omega$ for every $j = 1,\dots,N$;
- the family $\{B_r(x_j) : j = 1,\dots,N\}$ is an open covering of $\gamma([0,1])$.

The existence of such a family is an immediate consequence of Lemma 5.3 together with a standard covering theorem (see, for example, [82, Section 1.3]).

5.3 Boundary Harnack inequality

In this section, we prove the boundary Harnack inequality (Theorem 5.2) following the recent proof given by De Silva-Savin in [74], which is divided into the following three main steps:

(s1) we prove Lemma 5.6, from which the boundary Harnack inequality follows by an iteration procedure;
(s2) we prove Proposition 5.1, which allows us to start the iteration procedure. It is obtained as a consequence of Lemma 5.7 and the Harnack-type estimate Lemma 5.8;
(s3) we combine the results from the previous points to obtain the proof of Theorem 5.2.

Remark 5.3 For general operators Lemma 5.8 is contained in the proof of the Krylov-Safonov's Theorem [117] (see also [48, Theorem 4.8] and [74, Theorem 1.3]), while in our case it is a consequence of the mean-value formula.

First step: oscillation estimate and iteration lemma

The main result of this step is obtained as a consequence of Lemma 5.2 and the oscillation lemma from the De Giorgi's theorem [90, Chapter 8], which we now recall:

Lemma 5.5 (Oscillation) *Let $w : B_1 \longrightarrow \mathbb{R}$ be a subharmonic function, bounded in $[0,1]$ and such that*

$$\left|\{w = 0\} \cap B_{1/4}\right| \geq \mu \left|B_{1/4}\right|$$

for some $\mu > 0$. Then there exists $c := c(\mu, d) > 0$ such that

$$w \leq 1 - c \quad \text{on } B_{1/2}. \tag{5.2}$$

Proof. By the mean value formula, for every $x_0 \in B_{1/4}$ we have

$$w(x_0) \leq \frac{1}{|B_{1/2}|}\int_{B_{1/2}(x_0)} w(x)\,dx \leq \frac{1}{|B_{1/2}|}\Big(|B_{1/2}| - \mu|B_{1/4}|\Big) = 1 - \frac{\mu}{2^d}.$$

Now let $y_0 \in B_{1/2}$. Since $B_{1/2}(y_0) \cap B_{1/4}$ contains at least a ball of radius $1/8$, we can exploit the previous estimate in $B_{1/4}$ to obtain the estimate

$$w(y_0) \leq \frac{1}{|B_{1/2}|}\int_{B_{1/2}(y_0)} w(x)\,dx \leq \frac{1}{|B_{1/2}|}\Big(|B_{1/2}| - \frac{\mu}{2^d}|B_{1/8}|\Big) = 1 - \frac{\mu}{8^d},$$

which is precisely (5.2) with $c = 8^{-d}\mu$. □

Lemma 5.6 *Let $\Omega \subset B_1$, with $0 \in \partial\Omega$, and $\phi : B_1 \longrightarrow \mathbb{R}$ satisfy the conditions (a)–(d) of Theorem 5.1. Then there are constants $\delta_1, a > 0$ depending on d and (a), (b) and (d), for which the following holds:*

Claim. *Suppose that $w : B_1 \to \mathbb{R}$ is a continuous function satisfying*

$$\begin{cases} \Delta w = 0 & \text{in } B_1 \cap \{\phi > 0\} \\ w = 0 & \text{on } B_1 \cap \{\phi = 0\} \\ w \geq M & \text{on } B_1 \cap \{\phi > \delta\} \\ w \geq -1 & \text{on } B_1 \cap \{0 < \phi \leq \delta\}, \end{cases} \tag{5.3}$$

for some $\delta \in (0, \delta_1]$ and some $M > 0$. Then we have

$$\begin{cases} \Delta w = 0 & \text{in } B_{1/2} \cap \{\phi > 0\} \\ w = 0 & \text{on } B_{1/2} \cap \{\phi = 0\} \\ w \geq aM & \text{on } B_{1/2} \cap \{\phi > \frac{\delta}{2}\} \\ w \geq -a & \text{on } B_{1/2} \cap \{0 < \phi \leq \frac{\delta}{2}\}. \end{cases}$$

In particular, if we let w^+ and w^- be respectively the positive and negative part of w, we get the following estimates:

$$\sup_{B_{1/2}\cap\Omega} w^- \leq a \qquad \text{and} \qquad \inf_{B_{1/2}\cap\{\phi>\delta/2\}} w^+ \geq aM \geq M \sup_{B_{1/2}\cap\{\phi\leq\delta/2\}} w^-.$$

Proof. By (5.3), the function $w + 1$ is harmonic on $B_1 \cap \Omega$ and non-negative. If we take δ_1 to be the constant δ_0 from Lemma 5.2, then it turns out that

$$\min_{B_{1/2}\cap\{\phi>\frac{\delta}{2}\}} (w+1) \geq \frac{1}{A} \min_{B_1\cap\{\phi>\delta\}} (w+1) \geq \frac{1}{A}(M+1).$$

Therefore, if we choose a to be such that

$$a \leq \frac{1}{2A} \qquad \text{and} \qquad M = 2A,$$

then we get

$$\min_{B_{1/2}\cap\{\phi>\frac{\delta}{2}\}} w \geq \frac{1}{A}(M+1) - 1 \geq 1 \geq aM.$$

On the other hand, by the density bound (d) and the classical De Giorgi's oscillation lemma (see [74, Theorem 1.2]) applied to w^-, we get

$$\sup_{B_{1/2}} w^- \leq (1-c) \sup_{B_1} w^- \leq 1 - c,$$

where $c \in (0, 1)$ is the dimensional constant from Lemma 5.5. Thus, the conclusion follows by taking a to be the minimum between $1/(2A)$ and $1 - c$. □

Second step: growth lemma and Krylov-Safonov-type estimate

We now prove a bound which allows us to start the iterative procedure. Notice that this is the only point of the proof in which the assumption (e) is used.

Proposition 5.1 *Suppose that $\Omega \subset B_1$, with $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions (a)–(c) and (e) of Theorem 5.1. Then, there are $C, \delta_2 > 0$ depending on d and (a)–(e), for which the following holds:*

> ***Claim.*** *If $w : B_1 \longrightarrow \mathbb{R}$ is a non-negative continuous function satisfying*
> $$\begin{cases} \Delta w = 0 & \text{in } B_1 \cap \{\phi > 0\} \\ w = 0 & \text{on } B_1 \cap \{\phi = 0\} \\ w \le 1 & \text{on } B_1 \cap \{\phi \ge \delta_2\}, \end{cases}$$
> *then*
> $$w \le C \qquad \text{in } B_{1/4}.$$

We obtain the proof of this proposition combining two technical results. The first lemma is a consequence of the Harnack-type inequality close to the boundary (see Lemma 5.2).

Lemma 5.7 *Let $\Omega \subset B_1$, with $0 \in \partial\Omega$, and $\phi : B_1 \longrightarrow \mathbb{R}$ satisfy the conditions (a)–(c) of Theorem 5.1. Then there are constants $\delta_2, C, p > 0$ depending on d, L and κ from (a) and (b), for which the following holds:*

> ***Claim.*** *For every $\delta \in (0, \delta_2]$ and every positive harmonic $w : \Omega \longrightarrow \mathbb{R}$ such that*
> $$w \le 1 \qquad \text{on } B_1 \cap \{\phi > \delta\},$$
> *we have*
> $$w \le C\phi^{-p} \qquad \text{on } B_{1/2} \cap \Omega.$$

Remark 5.4 Notice that the constants δ_2 from Lemma 5.7 and Proposition 5.1 are the same.

Proof of Lemma 5.7. First, notice that for every $\delta < \tilde{\delta} \in (0, \delta_2]$ we have

$$w \le 1 \text{ on } B_1 \cap \{\phi > \delta\} \implies w \le 1 \text{ on } B_1 \cap \{\phi > \tilde{\delta}\},$$

so it is sufficient to prove the claim for $\delta = \delta_2$. Let $x_0 \in B_{1/2} \cap \Omega$ and $\ell \ge 1$ be a fixed constant that we will choose later. If x_0 is such that

$$\phi(x_0) \ge \ell\delta_2,$$

then it is enough to choose any $C \ge L^p$ since by the Lipschitz bound (a) and the fact that $x_0 \in \{\phi > \delta_2\}$, we have the inequality

$$w(x_0) \le 1 \le L^p \left(\max_{B_{1/2}} \phi\right)^{-p} \le L^p \phi(x_0)^{-p}.$$

Therefore, assume $\phi(x_0) \le \ell\delta_2$. Let z_0 be the projection of x_0 on $\partial\Omega \cap B_1$. By (b), we have

$$r := |x_0 - z_0| \le \frac{1}{\kappa}\phi(x_0) \le \frac{\ell\delta_2}{\kappa}.$$

Thus, if $\ell\delta_2 > 0$ is small enough (for example, $\ell\delta_2 \le \kappa/8$), then we have $r \le 1/8$ and, in particular, $B_{2r}(z_0) \subset B_1$. Moreover, by (b) we get

$$\frac{\kappa}{2} 2r = \kappa|x_0 - z_0| \le \phi(x_0)$$

and so, since $\ell\delta_2 \le \frac{\kappa}{2}$, the following holds:

$$x_0 \in B_{2r}(z_0) \cap \left\{\phi > \kappa r\right\} \subset B_{2r}(z_0) \cap \left\{\phi > 2\ell\delta_2 r\right\}.$$

Now assume that $\ell\delta_2 \le \delta_0$, where δ_0 is the threshold from Lemma 5.2, and let $n \ge 1$ be the only integer such that

$$2^n r \le \frac{1}{4} < 2^{n+1} r.$$

Then $B_{2^n r}(z_0) \subset B_1$, and we can iterate the estimate from Lemma 5.2 obtaining

$$w(x_0) \le \max_{B_{2r}(z_0)\cap\{\phi>2r\ell\delta_2\}} w \le A^{n-1} \max_{B_{2^n r}(z_0)\cap\{\phi>2^n\ell\delta_2 r\}} w \le A^{n-1} \max_{B_1\cap\{\phi>\frac{\ell\delta_2}{8}\}} w,$$

which means that we can take $\ell = 8$. Moreover, since

$$2 \le \kappa\phi(x_0)^{-1},$$

by choosing $p > 0$ such that $A^{n-1} = 2^p > 1$, we get

$$w(x_0) \le A^{n-1} \max_{B_1\cap\{\phi>\delta_2\}} w \le 2^p \max_{B_1\cap\{\phi>\delta_2\}} w \le \kappa^p \phi(x_0)^{-p},$$

proving the claim. Notice that it is enough to choose δ_2 and C as

$$\delta_2 \le \min\left\{\frac{\kappa}{64}, \frac{\delta_0}{8}\right\} \qquad \text{and} \qquad C = \max\{\kappa^p, L^p\}.$$

□

Corollary 5.1 *There exists $\delta_1 > 0$ such that for every $\delta \in (0, \delta_1]$ there is a constant T, depending on d, δ, L, κ and Λ for which the following holds. For every positive harmonic function $w : \Omega \to \mathbb{R}$, satisfying*

$$w \le 1 \qquad \text{on } B_1 \cap \{\phi > \delta\},$$

we have $w \le T$ on $B_{1/4}$.

In the proof of Proposition 5.1 we need the following *Krylov-Safonov-type estimate*, which was also used in [74, Theorem 1.3]. In our framework, there is a simple proof based on the mean-value formula.

Lemma 5.8 (A Krylov-Safonov-type estimate) *Suppose that Ω is an open set in B_1 and that the continuous* function $w : B_1 \to \mathbb{R}$ is such that:*

- *w is non-negative on B_1 and vanishes identically on $B_1 \setminus \Omega$;*
- *w is harmonic in Ω and subharmonic in B_1;*
- *Ω satisfies the exterior density bound (d) in B_1;*
- *there is $\epsilon > 0$ such that $\int_{B_1} w^\epsilon \, dx \le 1$.*

Then there is a constant $M > 0$, depending on d, the density bound μ from (d) and ϵ, such that

$$w \le M \qquad \text{in } B_{1/2}.$$

Proof. Let $x_0 \in B_{1/2} \cap \Omega$, $R := \text{dist}(x_0, \partial\Omega)$ and $M := w(x_0) > 0$. We fix $\delta := \epsilon/(2d)$ and, for simplicity, divide the proof into two cases:

(1) Assume that $2R \ge M^{-\delta}$. The function w is harmonic and positive on $B_R(x_0)$, so by the classical Harnack inequality [80, Chapter 6] there exists $c := c(d) > 0$ such that

$$w \ge cM \qquad \text{in } B_{R/2}(x_0).$$

* We notice that this assumption is not restrictive as below we will also assume w is harmonic in Ω and that Ω satisfies an exterior density bound.

It follows that

$$\begin{aligned} 1 \geq \int_{B_1} w^\epsilon \, dx &\geq \int_{B_{R/2}(x_0)} w^\epsilon \, dx \\ &\geq \left|B_{R/2}\right| (cM)^\epsilon \\ &\geq \frac{\omega_d \, c^\epsilon}{4^d} M^{-d\delta+\epsilon} = \frac{\omega_d \, c^\epsilon}{4^d} M^{\frac{\epsilon}{2}}, \end{aligned}$$

which means that in this case there is a constant $C_{d,\epsilon}$ such that $M \leq C_{d,\epsilon}$.

(2) Suppose, by contradiction, that $2R \leq M^{-\delta}$ and $M > C_{d,\epsilon}$, where $C_{d,\epsilon}$ is the constant defined above. If z_0 denotes the projection of x_0 on $\partial\Omega \cap B_1$, then

$$B_{M^{-\delta}/2}(z_0) \subseteq B_{M^{-\delta}}(x_0) \subset B_1.$$

In particular, since $\Delta w \geq 0$ in B_1, the mean value formula gives

$$\begin{aligned} M = w(x_0) &\leq \frac{1}{|B_{M^{-\delta}}|} \int_{B_{M^{-\delta}}(x_0)} w(x) \, dx \\ &\leq \frac{|B_{M^{-\delta}}(x_0) \cap \Omega|}{|B_{M^{-\delta}}|} \|w\|_{L^\infty(B_{M^{-\delta}}(x_0))}, \end{aligned}$$

which, by the density estimate (d) in the ball $B_{M^{-\delta}/2}(z_0)$, leads to

$$M \leq (1 - 2^{-d}\mu) \|w\|_{L^\infty(B_{M^{-\delta}}(x_0))} \leq \frac{1}{1 + 2^{-d}\mu} \|w\|_{L^\infty(B_{M^{-\delta}}(x_0))}.$$

Consequently, there exists a point $x_1 \in B_{M^{-\delta}}(x_0)$ such that

$$w(x_1) \geq (1 + 2^{-d}\mu) M,$$

and, iterating, we obtain a sequence of points $x_n \in \Omega \cap B_1$ satisfying the following properties:

$$w(x_{n+1}) \geq M(1 + 2^{-d}\mu)^n \qquad \text{and} \qquad |x_{n+1} - x_n| \leq \frac{1}{M^\delta (1 + 2^{-d}\mu)^{n\delta}}.$$

Now, if we choose $M > 0$ large enough, then

$$\sum_{n=0}^{+\infty} \frac{1}{M^\delta (1 + 2^{-d}\mu)^{n\delta}} \leq \frac{1}{4},$$

so x_n is defined for every $n \geq 1$ (it never leaves $\Omega \cap B_{3/4}$), but this is impossible since $w(x_n) \to \infty$.

□

We are now ready to prove the main result of this step:

Proof of Proposition 5.1. It is sufficient to show that there are $\alpha > 0$ and $C > 0$ such that

$$\int_{B_{1/2}} w^\alpha \, dx \leq C.$$

Indeed, by Lemma 5.7 and assumption (e) (in which Λ appears), we have that

$$\begin{aligned}
\int_{B_{1/2}} (C\phi^{-p})^\alpha\,dx &= C^\alpha p\alpha \int_0^{+\infty} t^{\alpha p-1}|\{\phi^{-1} > t\}\cap B_{1/2}|\,dt \\
&\le C^\alpha p\alpha \left(|B_{1/2}| \int_0^1 t^{\alpha p-1}\,dt + \int_1^{+\infty} t^{\alpha p-1}|\{\phi^{-1} > t\}\cap B_{1/2}|\,dt\right) \\
&\le C^\alpha p\alpha \left(\frac{1}{\alpha p}|B_{1/2}| + \int_1^{+\infty} t^{\alpha p-1}|\{\phi < 1/t\}\cap B_{1/2}|\,dt\right) \\
&\le C^\alpha p\alpha \left(\frac{1}{\alpha p}|B_{1/2}| + \Lambda|B_{1/2}| \int_1^{+\infty} t^{\alpha p-2}\,dt\right) \\
&= C^\alpha |B_{1/2}| \left(1 + \Lambda\frac{\alpha p}{1-\alpha p}\right),
\end{aligned}$$

so we can choose $\alpha = 1/(2p)$, and the conclusion follows from Lemma 5.8. □

Third step: conclusion of the proof

We are now ready to prove Theorem 5.2. For this, we show that it is possible to choose M and $\delta > 0$ in such a way that the iterative procedure in Lemma 5.6 can start.

Lemma 5.9 *Suppose that $\Omega \subset B_1$, with $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions (a)–(e) of Theorem 5.1. Let $R \in (0, R_0]$, where R_0 is the radius from Lemma 5.4. Then there are constants*

$$C_* > 0 \qquad \textit{and} \qquad \delta \le \min\{\eta, \delta_1, \delta_2\}^\dagger,$$

depending on the dimension d, the radius R, and the constants from the assumptions (a)–(e), such that for every couple of non-negative continuous functions

$$u, v : B_1 \to \mathbb{R}$$

satisfying

$$\begin{cases} \Delta u = \Delta v = 0 & \textit{in } \Omega\cap B_1 \\ u = v = 0 & \textit{on } B_1\setminus\Omega \\ u(x_0) = v(x_0) & \textit{for some } x_0 \in B_R \cap \{\phi > \delta R\}, \end{cases}$$

we have that the two functions

$$w_1 := C_* u - v \qquad \textit{and} \qquad w_2 := C_* v - u$$

fulfill the assumptions of Lemma 5.6.

Proof. First, notice that by Lemma 5.4 there is a constant C (depending also on R) such that

$$\frac{1}{C} \le u, v \le C \qquad \text{on } B_R \cap \{\phi > \delta R\}.$$

Thus, by Proposition 5.1, there is a constant $\Lambda > 0$ such that

$$v \le \Lambda \qquad \text{in } B_{R/4},$$

† δ_1 and δ_2 are the constants from Lemma 5.6 and Proposition 5.1, while η is the constant from (f) of Theorem 5.1.

and a constant $\lambda > 0$ such that

$$u \leq \lambda \qquad \text{in } B_{R/4} \cap \left\{\phi > \frac{\delta}{4}R\right\}.$$

Thus, for some $C_1, C_2 > 0$ large enough, the functions $C_1 u - v$ and $C_2 v - u$ satisfy the assumption of Lemma 5.6. Finally, the result follows by taking $C_* := \max\{C_1, C_2\}$. □

We are finally ready to put together everything we have proved so far to establish the boundary Harnack inequality:

Proof of Theorem 5.2. If we choose R and δ small enough, we can apply Lemma 5.9 in a neighborhood of the origin. In particular, there are $\rho \in (0, R)$ and $\delta > 0$ such that

$$\sup_{B_{R/4}(x_0)\cap\Omega} (C_* u - v)^- \leq a \qquad \text{and} \qquad \inf_{B_{R/4}(x_0)\cap\{\phi > \frac{R}{4}\delta\}} (C_* u - v)^+ > 0,$$

for every $x_0 \in B_\rho$. Iterating Lemma 5.6 (up to a dilatation and rescaling), we obtain

$$C_* u - v \geq 0 \qquad \text{in } B_{r_n}(x_0) \cap \{\phi > r_n \delta\}$$

for every $n \geq 0$, where we set $r_n := R2^{-2-n}$. Now, it is sufficient to notice that for $\rho > 0$ small enough the following collection of sets

$$\left\{B_{r_n}(x_0) \cap \{\phi > r_n\delta\} \ : \ x_0 \in \partial\Omega \cap B_\rho,\ n \geq 0\right\}$$

is a covering of B_ρ. By repeating the same argument with $C_* v - u$, we conclude the proof. □

5.4 Proof of Theorem 5.1: the boundary Harnack principle

We are now ready to prove that the boundary Harnack principle (Theorem 5.1) can be obtained as a consequence of the boundary Harnack inequality (Theorem 5.2). More precisely, we show that given any two harmonic functions $u, v : \Omega \to \mathbb{R}$ vanishing identically on the boundary $\partial\Omega$, the ratio

$$\frac{u}{v} : \Omega \to \mathbb{R}$$

is α-Hölder continuous up to the boundary. We achieve this as a consequence of Proposition 5.2, which is well-known in the literature (see, e.g., [108, Corollary 1.3.8]); here, we give a detailed proof for the sake of completeness.

Definition 5.5 *Let $\Omega \subset B_1$ be an open set. We say that Ω has the* ***BH half-space property*** *if there is a constant $M > 0$ such that for every $x_0 \in \partial\Omega \cap B_{1/2}$ and $r \in (0, 1 - |x_0|)$, there exists $P_r(x_0) \in B_r(x_0) \cap \Omega$ with the following property:*

> ***BH property.*** *For every pair of continuous non-negative functions $u, v : B_r(x_0) \to \mathbb{R}$ satisfying*
>
> $$\begin{cases} \Delta u = \Delta v = 0 & \text{in } B_r(x_0) \cap \Omega, \\ u = v = 0 & \text{on } B_r(x_0) \setminus \Omega, \\ u(P_r(x_0)) = v(P_r(x_0)), \end{cases}$$
>
> *we have that*
>
> $$M^{-1} \leq \frac{u(x)}{v(x)} \leq M \qquad \text{for every } x \in B_{r/2}(x_0) \cap \Omega.$$

Proposition 5.2 *Let $\Omega \subset B_1$ be an open set satisfying the BH half-space property. Then there are $\alpha, C > 0$, depending on c, M and d, such that for every pair of continuous non-negative functions*

$$u, v : B_1 \longrightarrow \mathbb{R}$$

satisfying

$$\begin{cases} \Delta u = \Delta v = 0 & \text{in } B_1 \cap \Omega, \\ u = v = 0 & \text{on } B_1 \setminus \Omega, \\ u(P_1(0)) = v(P_1(0)) > 0, \end{cases}$$

the following α-Hölder estimate holds:

$$\left| \frac{u(x)}{v(x)} - \frac{u(y)}{v(y)} \right| \le C|x-y|^\alpha \qquad \text{for every } x, y \in B_{1/4} \cap \Omega. \tag{5.4}$$

In order to prove this proposition, it is sufficient to estimate the oscillation of $\frac{u}{v}$ from one scale to another with a constant strictly smaller than one.

Lemma 5.10 *Let $\Omega \subset B_1$ be an open set satisfying the BH half-space property. Then for every $x_0 \in \partial\Omega \cap B_{1/2}$, every $r \le \frac{1}{2}$, and every pair of continuous non-negative functions $u, v : B_r(x_0) \longrightarrow \mathbb{R}$ satisfying*

$$\begin{cases} \Delta u = \Delta v = 0 & \text{in } B_r(x_0) \cap \Omega \\ u = v = 0 & \text{on } B_r(x_0) \setminus \Omega, \end{cases}$$

we have

$$\operatorname{osc}_{\Omega \cap B_{r/2}(x_0)} \left[\frac{u}{v} \right] \le \left(1 - \frac{1}{2M} \right) \operatorname{osc}_{\Omega \cap B_r(x_0)} \left[\frac{u}{v} \right],$$

where M is the constant given by Definition 5.5 and the oscillation is defined in (4.21).

Proof. Set $P_r := P_r(x_0)$ and consider the quantities

$$m_r := \inf_{\Omega \cap B_r(x_0)} \frac{u}{v} \qquad \text{and} \qquad M_r := \sup_{\Omega \cap B_r(x_0)} \frac{u}{v}.$$

Suppose that $u(P_r)/v(P_r) \ge (M_r + m_r)/2$. Then the functions $u - m_r v$ and v are harmonic, non-negative in $B_r(x_0)$ and satisfy the estimate

$$u(P_r) - m_r v(P_r) \ge \frac{M_r - m_r}{2} v(P_r).$$

Therefore, using the BH half-space property (Definition 5.5), we get

$$u - m_r v \ge \frac{1}{M} \frac{M_r - m_r}{2} v \qquad \text{in } B_{r/2}(x_0),$$

so we can divide by v and take the inf over Ω to obtain

$$\inf_{\Omega \cap B_{r/2}(x_0)} \frac{u}{v} \ge m_r + \frac{1}{M} \frac{M_r - m_r}{2}.$$

On the other hand, the supremum on $B_{r/2}(x_0)$ is smaller than or equal to (by definition) the supremum on the ball of radius r, i.e.,

$$\sup_{\Omega \cap B_{r/2}(x_0)} \frac{u}{v} \le \sup_{\Omega \cap B_r(x_0)} \frac{u}{v} = M_r,$$

which means that we can estimate the oscillation on $B_{r/2}(x_0)$ as follows:

$$\operatorname{osc}_{\Omega\cap B_{r/2}(x_0)}\left[\frac{u}{v}\right] \le M_r - \left(m_r + \frac{1}{M}\frac{M_r - m_r}{2}\right) = (M_r - m_r)\left(1 - \frac{1}{2M}\right).$$

The same strategy (estimating the supremum instead of the infimum) can be employed if we assume that $u(P_r)/v(P_r) \le (M_r + m_r)/2$. Indeed, we have

$$M_r v - u \ge \frac{1}{M}\frac{M_r - m_r}{2} v \qquad \text{in } B_{r/2}(x_0),$$

which implies that

$$\sup_{\Omega\cap B_{r/2}(x_0)} \frac{u}{v} \le M_r - \frac{1}{M}\frac{M_r - m_r}{2}.$$

On the other hand, the inf over $B_{r/2}(x_0)$ is always larger than or equal to the inf over the ball of radius r; more precisely, we have

$$\inf_{\Omega\cap B_{r/2}(x_0)} \frac{u}{v} \ge \inf_{\Omega\cap B_r(x_0)} \frac{u}{v} = m_r,$$

which means that the oscillation on $B_{r/2}(x_0)$ can be estimate as

$$\operatorname{osc}_{\Omega\cap B_{r/2}(x_0)}\left[\frac{u}{v}\right] \le \left(M_r - \frac{1}{M}\frac{M_r - m_r}{2}\right) - m_r = (M_r - m_r)\left(1 - \frac{1}{2M}\right),$$

concluding the proof. □

Proof of Proposition 5.2. We will prove the following claim:

> **Claim.** There is a constant $c \in (0,1)$ such that for every $x_0 \in \overline{\Omega} \cap B_{1/2}$, every $r \le 1/2$, and every pair of continuous and non-negative functions
>
> $$u, v : B_r(x_0) \longrightarrow \mathbb{R}$$
>
> satisfying
>
> $$\Delta u = \Delta v = 0 \text{ in } B_r \cap \Omega \qquad \text{and} \qquad u = v = 0 \text{ on } B_r \setminus \Omega,$$
>
> we have that
>
> $$\operatorname{osc}_{\Omega\cap B_{\frac{r}{16}}(x_0)}\left[\frac{u}{v}\right] \le (1-c)\operatorname{osc}_{\Omega\cap B_r(x_0)}\left[\frac{u}{v}\right].$$

In order to prove this claim we consider two different scenarios:

(1) Suppose that there is a point $y_0 \in \partial\Omega \cap B_{r/8}(x_0)$. Then we have that $B_{r/2}(y_0) \subset B_r(x_0)$ and, by Lemma 5.10, we can estimate the oscillation on a smaller ball as follows:

$$\begin{aligned}\operatorname{osc}_{B_{r/4}(y_0)\cap\Omega}\left[\frac{u}{v}\right] &\le \left(1 - \frac{1}{2M}\right)\operatorname{osc}_{B_{r/2}(y_0)\cap\Omega}\left[\frac{u}{v}\right] \\ &\le \left(1 - \frac{1}{2M}\right)\operatorname{osc}_{B_r(x_0)\cap\Omega}\left[\frac{u}{v}\right].\end{aligned}$$

Now, since $B_{r/8}(x_0) \subset B_{r/4}(y_0)$, we conclude that

$$\operatorname{osc}_{B_{r/8}(x_0)\cap\Omega}\left[\frac{u}{v}\right] \le (1-c)\operatorname{osc}_{B_r(x_0)\cap\Omega}\left[\frac{u}{v}\right], \qquad \text{with } c = \frac{1}{2M}.$$

(2) Suppose that $B_{r/8}(x_0) \subset \Omega$. Then the classical (interior) Harnack inequality tells us that

$$\operatorname{osc}_{B_{r/16}(x_0)\cap\Omega}\left[\frac{u}{v}\right] \leq (1-C_d)\operatorname{osc}_{B_{r/8}(x_0)\cap\Omega}\left[\frac{u}{v}\right] \leq (1-C_d)\operatorname{osc}_{B_r(x_0)\cap\Omega}\left[\frac{u}{v}\right],$$

where $C_d \in (0,1)$ is a dimensional constant given by the Harnack inequality.

The α-Hölder estimate (5.4) now follows by a standard argument since we have a bound on the oscillation which can be iterated. □

5.5 Some applications

To conclude this chapter, we discuss two examples (arising from shape optimization problems) of domains that satisfy the conditions of Theorem 5.1. For further applications, see [109] and [110].

5.5.1 The vectorial free boundary problem

Consider the functional

$$\mathcal{F}(U) := \int_{B_1} |\nabla U|^2\,dx + |\{|U|>0\}|,$$

defined for any vector-valued function $U : B_1 \longrightarrow \mathbb{R}^k$.

Definition 5.6 *A function $U : B_1 \longrightarrow \mathbb{R}^k$ is a solution of the vectorial problem if it minimizes $\mathcal{F}$ among all vector-valued functions with prescribed values on ∂B_1.*

Definition 5.7 *A vector-valued function $U : B_1 \longrightarrow \mathbb{R}^k$ is **non-degenerate** if there is a component u_i which is strictly positive in $\{|U>0|\} \cap B_1$.*

The non-degenerate case was first studied in [52], [115] and [130], while the regularity of the flat free boundaries in the degenerate case was first obtained in [116]; see also [75] for a different approach, and [142] for an analysis of the singular part of the free boundaries in dimension two.

In any case, the proofs in [52], [130] and [131], of the $C^{1,\alpha}$-regularity of the flat free boundaries are all based on the Boundary Harnack principle, which allows transforming the free boundary condition

$$\sum_{j=1}^{k} |\nabla u_j|^2 = 1 \qquad \text{on } \partial\{|U>0|\} \cap B_1,$$

into a condition of the form

$$|\nabla u_j| = g(x) \qquad \text{on } \partial\{|U>0|\} \cap B_1,$$

involving just one component of U and an auxiliary Hölder-continuous function $g : \partial\Omega \longrightarrow \mathbb{R}$. To prove that the Boundary Harnack principle holds on $\Omega_U := \{|U|>0\}$, in [52] it was shown that Ω_U is an NTA domain (Definition 5.4), while in [130] it was proved that Ω_U is Reifenberg-flat; in both cases, the conclusion followed from [103].

Consequently, our main result (Theorem 5.1) offers an alternative approach. In fact, the modulus $|U|$ of a variational solution U satisfies the conditions of Theorem 5.1. In fact, (a) and (c) are clearly satisfied. For the Lipschitz continuity and the non-degeneracy (f) of $|U|$ we refer to [130], while (e) was proved in [131, Section 2.2]. Moreover, in the non-degenerate case, in [115] it was shown that up to a constant one can bound $|U|$ from above with u_1. Thus, (b) is an immediate consequence from the classical interior Harnack inequality and the non-degeneracy of $|U|$. Finally, the exterior density estimate (d) was proved in [130].

5.5.2 Subsolutions and supersolutions

Fix $\Lambda > 0$. For every non-negative function $u : B_1 \to \mathbb{R}$, we define the functional

$$\mathcal{F}_\Lambda(u) := \int_{B_1} |\nabla u|^2 \, dx + \Lambda \, |\{u > 0\}| \, .$$

Definition 5.8 *A non-negative function $u : B_1 \to \mathbb{R}$ is a supersolution of $\mathcal{F}_\Lambda$ if*

$$\mathcal{F}_\Lambda(u) \le \mathcal{F}_\Lambda(v)$$

for every non-negative $v : B_1 \to \mathbb{R}$ such that

$$v = u \quad \text{on } \partial B_1 \qquad \text{and} \qquad u \le v .$$

Similarly, we say that u is a subsolution if the condition $u \le v$ is replaced by $v \le u$.

Notice that if u is at the same time a sub- and a supersolution of $\mathcal{F}_\Lambda$, then u is a minimizer of $\mathcal{F}_\Lambda$ and thus, by the classical result of Alt and Caffarelli [4], the free boundary $\partial\{u > 0\}$ is smooth in B_1 up to a set (of singular points) of "small" Hausdorff dimension.

Problem 5.1 If u is a subsolution for $\mathcal{F}_\lambda$ and a supersolution for $\mathcal{F}_\Lambda$ with $\lambda \neq \Lambda$, then nothing is known about the local structure of the free boundary.

Although this is an open problem, from the analysis in [4, Sections 3 and 4] (and [145]), one can easily check that we have the following result.

Proposition 5.3 *Fix $0 < \lambda < \Lambda$ are let $u \in H^1(B_1)$ be a non-negative function, which is a subsolution for $\mathcal{F}_\lambda$ and a supersolution for $\mathcal{F}_\Lambda$. Then u satisfies the conditions (a)–(f) of Theorem 5.1 and the Boundary Harnack principle holds on the set*

$$\Omega_u = \{u > 0\} \cap B_1 .$$

Epsilon-regularity for a free boundary system 6

The goal of this chapter is to describe the results obtained in [123], which will play a crucial role in the proof of the $C^{1,\alpha}$-regularity of the free boundary in Section 7.5.

6.1 Main results and some remarks

Let $u, v \in C^0(B_1)$ be two continuous non-negative functions on the unit ball in $\mathbb{R}^d$ such that

$$\Omega := \{u > 0\} = \{v > 0\}.$$

Suppose that u and v are also solutions of the free boundary problem

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{6.1}$$
$$-\Delta v = 0 \quad \text{in } \Omega, \tag{6.2}$$
$$\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = 1 \quad \text{on } \partial\Omega \cap B_1, \tag{6.3}$$

where the two equations (6.1)-(6.2) hold in the classical sense in Ω. On the other hand, since we will not assume Ω regular, the boundary condition (6.3) is to be intended in a generalized sense. Following the classical approach of Caffarelli [46, 47], for simplicity, in the introduction and in Theorem 6.1, we will assume that (6.3) holds in the sense of Definition 6.1 below.

The main result of [123] applies to a more general notion of solutions. However, we postpone the discussion to Section 6.2 to avoid technicalities.

Definition 6.1 (Definition of solutions) *We say that (6.3) holds, if at any point $x_0 \in \partial\Omega \cap B_1$ at which $\partial\Omega$ admits a one-sided tangent ball*, we have that the functions u and v can be expanded as*

$$u(x) = \alpha\left((x - x_0)\cdot\nu\right)_+ + o\left(|x - x_0|\right),$$
$$v(x) = \beta\left((x - x_0)\cdot\nu\right)_+ + o\left(|x - x_0|\right),$$

where ν is a unit vector and $\alpha, \beta \in \mathbb{R}$ positive numbers such that $\alpha\beta = 1$.

Our main result is a regularity theorem that applies to solutions that are sufficiently ϵ-flat in the sense of the following definition:

Definition 6.2 (Flatness) *We say that u and v are ϵ-flat in B_1, if there are a unit vector $\nu \in \partial B_1$ and positive constants α and β with $\alpha\beta = 1$, such that*

$$\alpha\left(x\cdot\nu - \epsilon\right)_+ \le u(x) \le \alpha\left(x\cdot\nu + \epsilon\right)_+ \qquad \text{for every } x \in B_1,$$
$$\beta\left(x\cdot\nu - \epsilon\right)_+ \le v(x) \le \beta\left(x\cdot\nu + \epsilon\right)_+ \qquad \text{for every } x \in B_1.$$

We will also say that u and v are ϵ-flat in the direction ν.

Theorem 6.1 *There is $\epsilon_0 > 0$ such that, if u and v are non-negative continuous functions on B_1 which are solutions of (6.1)-(6.2)-(6.3) and ϵ-flat in B_1, for some $\epsilon \in (0, \epsilon_0]$, then $\partial\Omega$ is $C^{1,\alpha}$ in $B_{1/2}$.*

* For a precise definition of one-sided tangent ball, see Definition 6.3.

Notice that Theorem 6.1 follows directly from Theorem 6.3 (which is proved in Subsection 6.2.3) as it is the same result for a more general notion of solutions, which we define in Section 6.2 in terms of the blow-ups of u and v. The proof of Theorem 6.3 will be given in Section 6.3 and Section 6.4.

The rest of the introduction is organized as follows. In Subsection 6.1.1, we discuss the relation of the system (6.1)-(6.2)-(6.3) to the well-known one-phase, two-phase and vectorial Bernoulli problems, with which it shares several key features. In Subsection 6.1.2, we explain the overall strategy and the novelties of the proof and, finally, in Subsection 6.1.3 we discuss the applications of our result to the theory of shape optimization.

6.1.1 The classical one-phase, two-phase and vectorial problems

The free boundary problem (6.1)-(6.2)-(6.3) is a vectorial analogue of the classical one-phase problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega := \{u > 0\}, \\ |\nabla u| = 1 & \text{on } \partial\Omega \cap B_1, \end{cases} \tag{6.4}$$

which was introduced by Alt and Caffarelli in [4]. Later, in a series of papers (see [46, 47] and the book [51]), Caffarelli studied the following two-phase problem, in which the solution is given by a single function $u : B_1 \to \mathbb{R}$ that changes sign:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_+ := \{u > 0\} \quad \text{and} \quad \Omega_- := \{u < 0\}\,, \\ (\partial_n^+ u)^2 - (\partial_n^- u)^2 = 1 & \text{on } \partial\Omega_+ \cap \partial\Omega_- \cap B_1. \end{cases} \tag{6.5}$$

Here, the transmission condition on the boundary $\partial\Omega_+ \cap \partial\Omega_-$ is defined in terms of Taylor expansions at points with a one-sided tangent ball (contained in Ω_+ or Ω_-), precisely as in Definition 6.1, while n denotes the normal to $\partial\Omega_+ \cap \partial\Omega_-$ at such points.

Recently, De Silva [68] gave a different proof to the one-phase ϵ-regularity theorem from [4]. The method found application to several generalizations of (6.5) (see [69], [70] and [71]), and opened the way to the original two-phase problem of Alt-Caffarelli-Friedman [5], for which the $C^{1,\alpha}$ regularity of the free boundary in every dimension was proved only recently in [67] by a similar argument.

Inspired by a problem arising in the theory of shape optimization, a cooperative vectorial version of the one-phase problem was introduced in [52], [115], and [130]. In this case, the solutions are functions

$$U = (u_1, \dots, u_k) : B_1 \to \mathbb{R}^k$$

satisfying

$$\begin{cases} -\Delta U = 0 & \text{in } \Omega := \{|U| > 0\}, \\ \sum_{j=1}^k |\nabla u_j|^2 = 1 & \text{on } \partial\Omega \cap B_1. \end{cases} \tag{6.6}$$

The regularity of the vectorial free boundaries turned out to be quite challenging, especially when it comes to viscosity solutions. This is mainly because the regularity techniques, based on the maximum principle and the comparison of the solutions with suitable test functions (see, for instance, [68] and [67]), are, in general, hard to implement in the case of systems. Epsilon-regularity theorems for the vectorial problem (6.6) were proved in [52, 115, 116, 130, 131, 142] and more recently, in [75], where the regularity of the flat free boundaries was obtained directly for viscosity solutions.

Several properties are specific to this problem. First of all, as (6.4) and (6.5), it has an underlying variational structure, which allows the use of purely variational techniques as the epiperimetric inequality [142] and which allows proving the regularity of minimizers (and almost-minimizers). Second, the problem is invariant with respect to vertical rotations (that is, if U is a solution and if A is an orthogonal matrix, then AU is also a solution), which in particular allows to write the free boundary condition in (6.6) in terms of the modulus $|U|$ as $|\nabla|U|| = 1$ on $\partial\Omega$ (see [130] and [75]).

6.1.2 Outline of the chapter

The free boundary problem (6.1)-(6.2)-(6.3) arises in the study of a whole class of shape optimization problems, which we discuss in more details in Subsection 6.1.3. However, unlike the one-phase [4], the two-phase [5] and the vectorial problem [52, 130], it does not have an underlying variational structure in terms of u and v, which means that strategies based on the epiperimetric inequality [142] are not useful here. Therefore, we prove an improvement-of-flatness theorem, from which Theorem 6.1 follows by a standard argument (see [145] for more details).

Theorem 6.2 (Improvement of flatness) *There are dimensional constants $\epsilon_0 > 0$ and $C > 0$ such that the following holds:*

> ***Improvement of flatness.*** *Let $u, v \in C^0(B_1)$ be two non-negative functions, which are also solutions to (6.1)-(6.2)-(6.3), and assume $0 \in \partial\Omega$. If u and v satisfy, for some $\epsilon < \epsilon_0$,*
>
> $$(x_d - \epsilon)_+ \le u(x) \le (x_d + \epsilon)_+ \qquad \textit{and} \qquad (x_d - \epsilon)_+ \le v(x) \le (x_d + \epsilon)_+$$
>
> *in B_1, then there are a unit vector $\nu \in \mathbb{R}^d$ with $|\nu - e_d| \le C\epsilon$ and a radius $\rho \in (0,1)$ such that*
>
> $$\tilde{\alpha}\left(x \cdot \nu - \frac{\epsilon}{2}\right)_+ \le \frac{u(\rho x)}{\rho} \le \tilde{\alpha}\left(x \cdot \nu + \frac{\epsilon}{2}\right)_+ \quad \textit{and} \quad \tilde{\beta}\left(x \cdot \nu - \frac{\epsilon}{2}\right)_+ \le \frac{v(\rho x)}{\rho} \le \tilde{\beta}\left(x \cdot \nu + \frac{\epsilon}{2}\right)_+$$
>
> *for all $x \in B_1$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are such that $\tilde{\alpha}\tilde{\beta} = 1$, $|1 - \tilde{\alpha}| \le C\epsilon$ and $|1 - \tilde{\beta}| \le C\epsilon$.*

Remark 6.1 We actually prove a more general improvement-of-flatness result in Theorem 6.5.

To prove this result, we use the general strategy developed by De Silva in [68] for viscosity solutions of the one-phase problem, which consists of two main ingredients: a *partial Harnack inequality* and the *analysis of the linearized problem*.

We argue by contradiction, taking a sequence of solutions (u_n, v_n), which are ϵ_n-flat with $\epsilon_n \to 0$, and considering the corresponding linearizing sequence

$$\widetilde{u}_n(x) = \frac{u_n(x) - x_d}{\epsilon_n} \qquad \text{and} \qquad \widetilde{v}_n(x) = \frac{v_n(x) - x_d}{\epsilon_n}.$$

The first step is to show that they converge to some u_∞ and v_∞, and this is done by proving a *partial Harnack inequality* (see Section 6.3). Roughly speaking, we show that for any couple (u, v) of ϵ-flat solutions, with $\epsilon \le \epsilon_0$, there is $c \in (0,1)$ such that either

$$(x_d - (1-c)\epsilon)_+ \le u(x) \le (x_d + \epsilon)_+ \quad \text{and} \quad (x_d - (1-c)\epsilon)_+ \le v(x) \le (x_d + \epsilon)_+ \quad \text{in } B_{1/2}, \tag{6.7}$$

or

$$(x_d - \epsilon)_+ \le u(x) \le (x_d + (1-c)\epsilon)_+ \quad \text{and} \quad (x_d - \epsilon)_+ \le v(x) \le (x_d + (1-c)\epsilon)_+ \quad \text{in } B_{1/2}. \tag{6.8}$$

In other words, the flatness is improved either above or below, in the same direction e_d, but without the scaling factor that would allow us to iterate without going above the threshold ϵ_0.

The second step is to show that the u_∞ and v_∞ are solutions to a PDE problem (the so-called *linearized problem* or *limit problem*), from which one can obtain an oscillation decay for u_∞ and v_∞ that can then be transferred back to $\widetilde{u}_n$ and $\widetilde{v}_n$, for some n large enough. In our case, the linearized problem is the following system of PDEs

$$\begin{cases} \Delta u_\infty = \Delta v_\infty = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ u_\infty = v_\infty \quad \text{and} \quad \partial_{x_d} u_\infty + \partial_{x_d} v_\infty = 0 & \text{on } B_1 \cap \{x_d = 0\}, \end{cases}$$

which we discuss in more details in Lemma 6.3 and Section 6.4. In our framework, the partial Harnack inequality is the most challenging part of the proof. Indeed, in the one-phase and two-phase (see [68], [69–71], and [67]), the validity of (6.7)-(6.8) is obtained by constructing explicit competitors, which are essentially variations of the constructions in [68]. However, in our case, the functions u and v are not solutions to any free boundary problem (if considered separately), meaning we cannot construct explicit competitors.

That said, inspired by [75], it is possible to use appropriate combinations of u and v, which are crucial in both steps of our proof. More precisely, we show that if u and v are viscosity solutions to (6.1)-(6.2)-(6.3), then $\sqrt{uv}$ and $\frac{1}{2}(u+v)$ are respectively a viscosity subsolution and viscosity supersolution of the one-phase problem (6.4) (see Lemma 6.1 and Remark 6.10)[†]. Moreover, it is easy to check that $\sqrt{uv}$ and $\frac{1}{2}(u+v)$ inherit the flatness of u and v. Thus, by using the competitors from [68] on these functions, we obtain the following dichotomy:

the flatness of $\frac{1}{2}(u+v)$ is improved from above or *the flatness of $\sqrt{uv}$ is improved from below.*

This information cannot be transferred back to u and v by algebraic manipulations since, for example, a bound from below on $\sqrt{uv}$ does not imply a bound from below for both u and v. However, we notice that the improved flatness of $\sqrt{uv}$ or $\frac{1}{2}(u+v)$ implies that in $B_{1/2}$ the boundary $\partial\Omega$ is trapped between two nearby translations of a half-space, which are distant at most $(2-c)\epsilon$. Using this geometric information and a comparison argument based on the boundary Harnack principle, in Lemma 6.2, we obtain that also the flatness of u and v improves in $B_{1/2}$.

6.1.3 On the boundary condition $(\partial_n u)(\partial_n v) = 1$ and its relation to a shape optimization problem

Our result applies to a whole class of shape optimization problems, that is, variational problems

$$\min\left\{J(\Omega)\ :\ \Omega \in \mathcal{A}\right\},$$

where $\mathcal{A}$ is an admissible class of subsets of $\mathbb{R}^d$. Typically, we consider sets of fixed measure contained in a given bounded open set $D \subset \mathbb{R}^d$, while the functional J satisfies

$$\omega \subseteq \Omega \implies J(\omega) \geq J(\Omega),$$

and depends on the resolvent of an elliptic operator with Dirichlet boundary conditions. The shape functionals are usually related to models in Engineering, Mechanics and Material Sciences and in most of the cases fall in one of the following main classes:

- *spectral functionals*;
- *integral functionals*.

For more details, we refer to the books [21, 97, 98, 100], the survey paper [30] and the articles [111, 139]. The spectral functionals are functionals of the form

$$J(\Omega) = F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)),$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is a monotone function and $\lambda_1(\Omega), \dots, \lambda_k(\Omega)$ are the eigenvalues of the Laplacian on Ω with Dirichlet boundary condition. The regularity and the local structure of these optimal sets were studied in [26], [115, 116], [130] (see also [19] and [132] for the special cases $J(\Omega) = \lambda_1(\Omega)$ and $J(\Omega) = \lambda_2(\Omega)$), and are related to the vectorial Bernoulli problem from [52], [130] and [131]. An ϵ-regularity theorem for general spectral functionals was obtained in [116].

[†] This situation is similar to the one of the vectorial problem (6.6) in which each of the components of U is a viscosity supersolution, while, as it was shown in [130], the modulus $|U|$ satisfies $|\nabla|U|| = 1$ on $\partial\{|U| > 0\}$ and is a viscosity subsolution of (6.4); this was used in [75] to prove a partial Harnack inequality and an ϵ-regularity theorem for solutions of (6.6).

The integral functionals, on the other hand, can be written in the form

$$J(\Omega) = \int_D j(u_\Omega, x)\, dx, \tag{6.9}$$

where $j : \mathbb{R} \times D \longrightarrow \mathbb{R}$ is a given function and the state function u_Ω is the unique solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \in H^1_0(\Omega), \end{cases}$$

the right-hand side $f : D \to \mathbb{R}$ being a fixed measurable function.

As it was observed already in [45], a free boundary system of the form (6.1)-(6.2)-(6.3) naturally arises in the computation of the first variation of J. This is easy to see if one formally computes the first variation of J for smooth sets. Indeed, if Ω is a smooth optimal set and $\xi \in C^\infty(D; \mathbb{R}^d)$ a compactly supported vector field, then we can define the family of sets

$$\Omega_t := (Id + t\xi)(\Omega),$$

and the corresponding family of state functions $u_t := u_{\Omega_t}$. Then the first variation of J is given by

$$\begin{aligned} \delta J(\Omega)[\xi] := \frac{d}{dt}\Big|_{t=0} J(\Omega_t) &= \frac{d}{dt}\Big|_{t=0} \int_{\Omega_t} j(u_t, x)\, dx \\ &= \frac{d}{dt}\Big|_{t=0} \left[\int_D \big(j(u_t, x) - j(0, x)\big)\, dx + \int_{\Omega_t} j(0, x)\, dx \right] \\ &= \int_\Omega u' \frac{\partial j}{\partial u}(u_\Omega, x)\, dx + \int_{\partial\Omega} j(0, x)\, \xi \cdot n_\Omega \, d\mathcal{H}^{d-1}, \end{aligned}$$

where $n_\Omega(x)$ to be the exterior normal at $x \in \partial\Omega$ and the formal derivative u' (of u_t at $t = 0$) is the solution of the boundary value problem

$$\begin{cases} -\Delta u' = 0 & \text{in } \Omega, \\ u' = -\xi \cdot \nabla u_\Omega & \text{on } \partial\Omega, \end{cases}$$

in which the condition on $\partial\Omega$ is a consequence of the fact that, given $x \in \partial\Omega$, we have

$$u_t(x + t\xi(x)) = 0 \qquad \text{for every } t \in \mathbb{R}.$$

We next define the function

$$g(x) := -\frac{\partial j}{\partial u}(u_\Omega(x), x),$$

and the solution v_Ω of the problem

$$\begin{cases} -\Delta v = g & \text{in } \Omega, \\ v \in H^1_0(\Omega). \end{cases}$$

Remark 6.2 Notice that, to have the monotonicity of J, it is natural to assume $f \geq 0$ and $\partial_u j(x, u) \leq 0$, which, in turn, implies $g \geq 0$ and that both u_Ω and v_Ω are non-negative. On the other hand, if f and $\partial_u j(x, u)$ change sign, then, in general, an optimal set might not exist (see, for instance, [45]).

In order to complete the computation of $\delta J(\Omega)[\xi]$, we integrate by parts in Ω, obtaining

$$-\int_\Omega u' g(x)\, dx = \int_\Omega u' \Delta v_\Omega \, dx = -\int_\Omega \nabla u' \cdot \nabla v_\Omega \, dx + \int_{\partial\Omega} u' \frac{\partial u_\Omega}{\partial n} = \int_{\partial\Omega} u' \frac{\partial u_\Omega}{\partial n}.$$

Now, since ∇u_Ω is parallel to n_Ω at the boundary, we have

$$u' = -\xi \cdot \nabla u_\Omega = -(\xi \cdot n_\Omega)(n_\Omega \cdot \nabla u_\Omega),$$

and thus the first variation of J is given by

$$\delta J(\Omega)[\xi] = \int_{\partial\Omega} \left(-\frac{\partial u_\Omega}{\partial n} \frac{\partial v_\Omega}{\partial n} + j(0,x) \right) n_\Omega \cdot \xi \, .$$

Since the vector field ξ is arbitrary and Ω is a minimizer among the sets of prescribed measure, in a neighborhood $B_r(x_0)$ of a point x_0 of the free boundary $\partial\Omega \cap D$, u_Ω and v_Ω are solutions of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \cap B_r(x_0), \\ -\Delta v = g & \text{in } \Omega \cap B_r(x_0), \\ (\partial_n u)(\partial_n v) = c + j(0,x) & \text{on } \partial\Omega \cap B_r(x_0), \end{cases}$$

where c is a positive constant.

Remark 6.3 Our definition Definition 6.1 is a generalization of the notion proposed by Caffarelli in the context of a two-phase free boundary problem [46, 47].

Remark 6.4 In Theorem 6.1, we do not assume that the functions u and v are minimizers of a functional or solutions of a shape optimization problem of any kind, so this result is of independent interest and can be seen as a one-phase vectorial version of the classical results of Caffarelli [46, 47].

6.2 On the viscosity formulations of solution

In this section, we briefly discuss the boundary condition (6.3) since, as mentioned above, we will prove a more general theorem for generalized solutions in the sense of Definition 6.5.

Definition 6.3 (One sided tangent balls) *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $x_0 \in \partial\Omega$. We say that Ω admits a one-sided tangent ball at x_0 if one of the following conditions hold:*

(i) ***interior:*** *there are $r > 0$ and $y_0 \in \Omega$ such that*

$$B_r(y_0) \subset \Omega \qquad \textit{and} \qquad \partial B_r(y_0) \cap \partial\Omega = \{x_0\};$$

(ii) ***exterior:*** *there are $r > 0$ and $y_0 \in \mathbb{R}^d \setminus \overline{\Omega}$ such that*

$$B_r(y_0) \subset \mathbb{R}^d \setminus \overline{\Omega} \qquad \textit{and} \qquad \partial B_r(y_0) \cap \partial\Omega = \{x_0\}.$$

Moreover, we use the notation ν_{x_0,y_0} to denote the exterior normal, which is given by

$$\frac{y_0 - x_0}{|y_0 - x_0|} \quad \textit{if the ball is interior}, \qquad -\frac{y_0 - x_0}{|y_0 - x_0|} \quad \textit{if the ball is exterior}. \tag{6.10}$$

When Ω is regular, we notice that the vector ν_{x_0,y_0} is the inner normal to $\partial\Omega$ at x_0, while for non-smooth domains, it may depend on the ball $B_r(y_0)$.

Let $Q : B_1 \to \mathbb{R}$ be a $C^{0,\alpha}$-regular function (for some $\alpha > 0$) and suppose that there is $C_Q \geq 1$ such that

$$C_Q^{-1} \leq Q(x) \leq C_Q \qquad \text{for all } x \in B_1.$$

Then we can generalize the notion of solutions given in Definition 6.1 as follows:

Definition 6.4 (Definition of solutions, I) *Given two continuous non-negative functions $u, v : B_1 \longrightarrow \mathbb{R}$ with the same support $\Omega = \{u > 0\} = \{v > 0\}$, we say that*

$$\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = Q \qquad \text{on } \partial\Omega \cap B_1,$$

if at any point $x_0 \in \partial\Omega \cap B_1$, for which $\partial\Omega$ admits a one-sided tangent ball at x_0, we have that the functions u and v can be expanded as

$$u(x) = \alpha\left((x - x_0)\cdot \nu\right)_+ + o\left(|x - x_0|\right),$$
$$v(x) = \beta\left((x - x_0)\cdot \nu\right)_+ + o\left(|x - x_0|\right),$$

where ν is given by (6.10) and α, β are positive real numbers such that

$$\alpha\beta = Q(x_0).$$

Remark 6.5 This definition implies that if $\partial\Omega$ admits a one-sided tangent ball at $x_0 \in \partial\Omega$, it is unique and thus excludes a priori domains with angles and cusps. Moreover, the blow-ups of u and v are unique at such points.

That said, a priori, we do not know whether the domain has cusps or not, so we need a more general definition that works in the most general framework possible.

6.2.1 A more general notion of solution

For every $x_0 \in \partial\Omega \cap B_1$ and every $r > 0$ small enough, we define

$$u_{r,x_0}(x) = \frac{1}{r}u(x_0 + rx) \qquad \text{and} \qquad v_{r,x_0}(x) = \frac{1}{r}v(x_0 + rx).$$

Throughout the paper we will also adopt the notation $u_r := u_{r,0}$ and $v_r := v_{r,0}$. Therefore, we can express Definition 6.4 in terms of the rescalings u_{r,x_0} and v_{r,x_0} in the following way:

Remark 6.6 Let $u, v : B_1 \longrightarrow \mathbb{R}$ be two non-negative continuous functions with the same support

$$\Omega := \{u > 0\} = \{v > 0\}.$$

Then the following are equivalent:

(i) $(\partial_n u)(\partial_n v) = Q$ on $\partial\Omega \cap B_1$ in the sense of Definition 6.4;

(ii) at any point $x_0 \in \partial\Omega \cap B_1$, for which one of the conditions (i) and (ii) of Definition 6.3 hold, we have that u_{r,x_0} and v_{r,x_0} converge uniformy in B_1 as $r \longrightarrow 0$ respectively to

$$x \mapsto \alpha\left((x - x_0)\cdot \nu\right)_+ \qquad \text{and} \qquad x \mapsto \beta\left((x - x_0)\cdot \nu\right)_+,$$

where $\alpha, \beta > 0$ are such that $\alpha\beta = Q(x_0)$ and $\nu \in \mathbb{R}^d$ is the unit vector given by (6.10).

In particular, Remark 6.6 implies that Definition 6.4 can be generalized as follows.

Definition 6.5 (Definition of solutions, II) *Given two continuous non-negative functions $u, v : B_1 \longrightarrow \mathbb{R}$ with the same support $\Omega = \{u > 0\} = \{v > 0\}$, we say that*

$$\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = Q \qquad \text{on } \partial\Omega \cap B_1,$$

if at any point $x_0 \in \partial\Omega \cap B_1$, for which one of the conditions (i) and (ii) of Definition 6.3 holds, there are

- *a decreasing sequence $r_n \longrightarrow 0$;*

- *two positive constants $\alpha > 0$ and $\beta > 0$ such that $\alpha\beta = Q(x_0)$;*
- *a unit vector $\nu \in \mathbb{R}^d$;*

such that the sequences u_{r_n,x_0} and v_{r_n,x_0} converge uniformly in B_1 respectively to

$$u_0(x) := \alpha\,(x\cdot\nu)_+ \qquad \text{and} \qquad v_0(x) := \beta\,(x\cdot\nu)_+\,.$$

Remark 6.7 We say that u_0 and v_0 are blow-up limits of u and v at x_0. We notice that the blow-up limits at a point may not be unique since, a priori, they may depend on the sequence $r_n \to 0$.

Remark 6.8 The sequence $r_n \to 0$ from Definition 6.5 may depend on the tangent ball $B_r(y_0)$ at x_0 which is not necessarily unique. Thus, we do not assume that the blow-ups of u and v at x_0 and the tangent ball $B_r(y_0)$ are unique.

6.2.2 Optimality conditions in viscosity sense

We are now ready to give the viscosity formulation (Lemma 6.1) of the free boundary condition in the generalized definition, which will play a crucial later in proving the main results.

Definition 6.6 *Let $u : B_1 \to \mathbb{R}$ be a continuous non-negative function, $\varphi \in C^\infty(\mathbb{R}^d)$ be given and*

$$\varphi_+(x) := \max\{\varphi(x), 0\}.$$

- *We say that φ_+ touches u from below at $x_0 \in \partial\{u > 0\} \cap B_1$ if $u(x_0) = \varphi(x_0) = 0$ and*

$$\varphi_+(x) \le u(x) \qquad \text{for every } x \text{ in a neighborhood of } x_0.$$

- *We say that φ_+ touches u from above at $x_0 \in \partial\{u > 0\} \cap B_1$ if $u(x_0) = \varphi(x_0) = 0$ and*

$$\varphi_+(x) \ge u(x) \qquad \text{for every } x \text{ in a neighborhood of } x_0.$$

As mentioned in the introduction, we cannot prove the partial Harnack inequality (Theorem 6.4) for u and v because there is no underlying variational structure. However, since

$$\sqrt{uv} \qquad \text{and} \qquad \frac{u+v}{2}$$

are, respectively, a subsolution and a supersolution of the one-phase problem, we can prove the following viscosity lemma:

Lemma 6.1 *Suppose that u and v satisfy*

$$\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = 1 \qquad \text{on } \partial\Omega \cap B_1$$

in the sense of Definition 6.5. Then the following holds:

(a) If φ_+ touches $\sqrt{uv}$ from below at a point $x_0 \in B_1 \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \le 1$.

(b) If φ_+ touches $\sqrt{uv}$ from above at a point $x_0 \in B_1 \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \ge 1$.

(c) If a and b are constants such that

$$a, b > 0 \qquad \text{and} \qquad ab = 1,$$

and if φ_+ touches the function $w_{ab} := \frac{1}{2}(au + bv)$ from above at $x_0 \in B_1 \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \ge 1$.

Proof. We start by proving (c). Suppose that φ_+ touches w_{ab} from above at $x_0 \in \partial\Omega$. Then there is a ball touching $\partial\Omega$ at x_0 from outside (in the sense of Definition 6.3 (ii)). But then, by Definition 6.5, there are blow-up limits of u and v given respectively by

$$u_0(x) = \alpha\,(x \cdot \nu)_+ \qquad \text{and} \qquad v_0(x) = \beta\,(x \cdot \nu)_+ \ . \tag{6.11}$$

Moreover, since φ is smooth, the blow-up of φ_+ is given by

$$\varphi_0(x) = \big(x \cdot \nabla\varphi(x_0)\big)_+ \,. \tag{6.12}$$

By hypothesis, we have that φ_0 touches from above (at the origin) the function

$$x \longmapsto \frac{1}{2}\,(a u_0(x) + b v_0(x)) = \frac{a\alpha + b\beta}{2}\,(x \cdot \nu)_+ \,,$$

so, since $a\alpha + b\beta \geq 2\sqrt{\alpha a \beta b} = 2$, we have that φ_0 touches from above also the function

$$x \longmapsto (x \cdot \nu)_+ \,.$$

Thus, $\nabla\varphi(x_0) = \nu$ and, in particular, $|\nabla\varphi(x_0)| \geq 1$. We next prove (a) and (b). If φ_+ touches $\sqrt{uv}$ from below (resp. above) at $x_0 \in \partial\Omega$, then $\partial\Omega$ has an interior (resp. exterior) tangent ball at x_0. In particular, by Definition 6.5, u and v have blow-ups u_0 and v_0 given by (6.11). But then the function

$$\sqrt{u_0 v_0} = (x \cdot \nu)_+$$

is a blow-up limit of $\sqrt{uv}$ and, using again that the blow-up of φ_+ is (6.12), we conclude. □

6.2.3 Generalization of the main result

We are now ready to state the generalization of Theorem 6.1. The presence of f, g, and Q leads to minor technical adjustments to the proof, so we will not go through it.

Theorem 6.3 *Let $f, g \in L^\infty(B_1)$ be non-negative functions and $Q \in C^{0,\alpha}(B_1)$. Suppose that $u, v \in C(B_1)$ are non-negative solutions of the system*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ -\Delta v = g & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}\dfrac{\partial v}{\partial n} = Q & \text{on } \partial\Omega \cap B_1, \end{cases} \tag{6.13}$$

where the free boundary condition holds in the sense of Definition 6.5. Then there is $\epsilon > 0$ such that, if u and v are ϵ-flat in B_1 and we have the estimates

$$\|f\|_{L^\infty(B_1)} + \|g\|_{L^\infty(B_1)} \leq \epsilon^2 \qquad \text{and} \qquad \|Q(x) - 1\|_{L^\infty(B_1)} \leq \epsilon,$$

then $\partial\Omega$ is $C^{1,\alpha}$ in $B_{1/2}$.

Remark 6.9 The positivity assumption on f and g is technical and is only required for the estimate on the Laplacian of $\sqrt{uv}$ in Remark 6.10. Without this assumption, one should know that the functions u and v are comparable on Ω, i.e., that u/v is bounded away from zero and infinity.

6.3 A partial Harnack inequality

In this section, we prove the partial Harnack inequality for solutions to (6.13) following the strategy proposed in [68]. Indeed, as already mentioned, the boundary condition does not allow us to work with u and v directly, so we study the improvement of flatness of the auxiliary functions

$$\frac{u+v}{2} \qquad \text{and} \qquad \sqrt{uv},$$

in order to trap the boundary $\partial\Omega$ between nearby translations of a half-space.

Remark 6.10 This strategy relies on the fact that the auxiliary functions are, respectively, subsolution and supersolution for the scalar one-phase problem (6.4). Indeed, if u and v are harmonic in Ω and satisfy Lemma 6.1 (c), then $w := {}^1\!/\!_2(u+v)$ is a subsolution since

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega, \\ |\nabla w| \geq 1 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, the function $z := \sqrt{uv}$ is a supersolution since

$$\begin{cases} -\Delta z \geq 0 & \text{in } \Omega, \\ |\nabla z| = 1 & \text{on} \partial\Omega. \end{cases}$$

The boundary condition follows again from Lemma 6.1, while the superharmonicity in Ω from the fact that, if $u, v : \Omega \to \mathbb{R}$ are positive and superharmonic on an open set Ω, then

$$\begin{aligned} \Delta\left(\sqrt{uv}\right) &= \operatorname{div}\left(\frac{u\nabla v + v\nabla u}{2\sqrt{uv}}\right) \\ &= \frac{u\Delta v + 2\nabla u \cdot \nabla v + v\Delta u}{2\sqrt{uv}} - (u\nabla v + v\nabla u) \cdot \frac{u\nabla v + v\nabla u}{4(uv)^{3/2}} \\ &= \frac{2uv\,(u\Delta v + v\Delta u) + 4uv\nabla u \cdot \nabla v - |u\nabla v + v\nabla u|^2}{4(uv)^{3/2}} \\ &= \frac{(u\Delta v + v\Delta u)}{2\sqrt{uv}} - \frac{|u\nabla v - v\nabla u|^2}{4(uv)^{3/2}} \leq \frac{(u\Delta v + v\Delta u)}{2\sqrt{uv}} \leq 0. \end{aligned}$$

We now show that an improvement-of-flatness result for u and v follows immediately once we can trap the set Ω between two nearby translations of a half-space.

Lemma 6.2 *Let $\epsilon > 0$ and $\phi \in C(B_1)$ be a non-negative solution of*

$$-\Delta\phi = f \qquad \text{in } B_1 \cap \{\phi > 0\},$$

with $f \in L^\infty(B_1)$. Assume that

$$\gamma(x_d + a)_+ \leq \phi(x) \leq \gamma(x_d + a + \epsilon)_+ \qquad \text{for all } x \in B_1, \tag{6.14}$$

with $|a| \leq 1$ and, for some $C \in (0,1)$ universal constant, either one of the following inclusions holds:

$$\Omega \supset \{x_d + a + C\epsilon > 0\} \cap B_{1/4} \qquad \text{or} \qquad \Omega \subset \{x_d + a + (1-C)\epsilon > 0\} \cap B_{1/4}. \tag{6.15}$$

Then there are $\delta, \rho \in (0,1)$ dimensional constants such that either

$$\phi(x) \geq \gamma(x_d + a + \delta\epsilon)_+ \qquad \text{or} \qquad \phi(x) \leq \gamma(x_d + a + (1-\delta)\epsilon)_+,$$

holds for every $x \in B_\rho$.

Proof. Suppose that the first inclusion in (6.15) holds, i.e.,

$$\Omega \supset \{x_d + a + C\epsilon > 0\} \cap B_{1/4}.$$

Set $D = B_{1/4} \setminus \{x_d \le -a - C\epsilon\}$, and consider the function

$$\begin{cases} \Delta\varphi = 0 & \text{in } D \\ \varphi = 0 & \text{in } B_{1/4} \setminus D \\ \varphi = w & \text{on } \partial B_{1/4} \end{cases} \qquad \text{with } w := \phi - \frac{1}{2}(x_d + a + C\epsilon)_+^2 \|f\|_{L^\infty(B_1)}.$$

Since $\phi > 0$ in D, we have $-\Delta\phi \ge -\|f\|_{L^\infty(B_1)}$ in D, and thus $-\Delta w \ge 0$ in D. Then, by applying the maximum principle in D, we get

$$\varphi \le w \le \phi \qquad \text{in } D.$$

On the other hand, since $\phi \ge 0 = \varphi$ in $B_{1/4} \setminus D$, it follows that $\varphi \le \phi$ in $B_{1/4}$. Thus, we claim that

$$\varphi \ge \gamma(x_d + a + \delta\epsilon)_+ \qquad \text{for all } x \in B_{1/32}, \tag{6.16}$$

for some universal $\delta \in (0,1)$, from which the desired inequality follows. By (6.14) we know that

$$\gamma(x_d + a + C\epsilon)_+ \le \gamma\left(x_d + a + \frac{1+C}{2}\epsilon\right)_+ \le \phi + \frac{1+C}{2}\gamma\epsilon \qquad \text{in } \overline{B}_{1/4},$$

and, since $\varphi = \phi$ on $\partial B_{1/4}$, we get

$$\gamma(x_d + a + C\epsilon)_+ \le \varphi + \frac{1+C}{2}\gamma\epsilon \qquad \text{on } \partial B_{1/4}.$$

Therefore, applying the maximum principle in D yields

$$\gamma(x_d + a + C\epsilon)_+ - \varphi \le \frac{1+C}{2}\gamma\epsilon \qquad \text{in } B_{1/4}.$$

Consider now the function

$$\begin{cases} -\Delta h = 0 & \text{in } D, \\ h = 0 & \text{in } B_{3/16} \setminus D, \\ h = \frac{1+C}{2}\gamma\epsilon & \text{on } \partial B_{3/16}. \end{cases}$$

Clearly, $0 \le h \le \frac{1+C}{2}\gamma\epsilon$, and by the maximum principle

$$\gamma(x_d + a + C\epsilon)_+ - \varphi \le h \qquad \text{in } B_{3/16}. \tag{6.17}$$

Now, by the boundary Harnack inequality, if we set $\bar{x} = (1/8)e_d$, we get

$$h(x) \le C_1 \frac{h(\bar{x})}{(1/8 + a + C\epsilon)_+}(x_d + a + C\epsilon)_+ \le C_2\gamma\epsilon(x_d + a + C\epsilon)_+ \qquad \text{in } B_{1/8},$$

for some universal constants $C_1, C_2 > 0$. This last inequality, together with (6.17), leads to

$$(1 - C_2\epsilon)\gamma(x_d + a + C\epsilon)_+ \le \varphi \qquad \text{in } B_{1/8}.$$

On the other hand, since there exists $\delta \in (0,1)$, $\delta < C$ and $\rho < 1/8$ such that

$$(x_d + a + \delta\epsilon)_+ \le (1 - C_2\epsilon)(x_d + a + C\epsilon)_+ \qquad \text{for all } x \in B_\rho,$$

we conclude the proof of the claim (6.16). Now suppose that the second inclusion of (6.24) holds, namely we have

$$\Omega \subset \{x_d + a + (1-C)\epsilon > 0\} \cap B_{1/4}.$$

Set $D = B_{1/4} \setminus \{x_d \le -a - (1-C)\epsilon\}$ and consider the function

$$\begin{cases} \Delta\varphi = 0 & \text{in } D \\ \varphi = 0 & \text{in } B_{1/4} \setminus D \\ \varphi = w & \text{on } \partial B_{1/4} \end{cases} \qquad \text{with } w = \phi + \frac{1}{2}(x_d + a + (1-C)\epsilon)_+^2 \|f\|_{L^\infty(B_1)}.$$

Notice that $w > 0$ and $-\Delta w \le 0$ in D. Therefore, by maximum principle

$$\varphi \ge w \qquad \text{in } B_{1/4},$$

and, since $w \ge \phi$ in $B_{1/4}$, we have $\varphi \ge \phi$ in $B_{1/4}$. We claim that

$$\varphi \le \gamma(x_d + a + (1-\delta)\epsilon)_+ \qquad \text{for all } x \in B_{1/32},$$

for some $\delta \in (0,1)$, from which the desired inequality follows. By (6.14) we know that

$$\phi - \frac{1+C}{2}\gamma\epsilon \le \gamma\left(x_d + a + \frac{1-C}{2}\epsilon\right)_+ \le \gamma\,(x_d + a + (1-C)\epsilon)_+ \qquad \text{in } \overline{B}_{1/4},$$

and, since $\varphi = \phi$ on $\partial B_{1/4}$, we deduce that

$$\phi - \gamma(x_d + a + (1-C)\epsilon)_+ \le \frac{1+C}{2}\gamma\epsilon \qquad \text{on } \partial B_{1/4}.$$

Therefore, by applying the maximum principle in D, we get

$$\phi - \gamma(x_d + a + (1-C)\epsilon)_+ \le \frac{1+C}{2}\gamma\epsilon \qquad \text{in } B_{1/4}.$$

Consider now the function

$$\begin{cases} -\Delta h = 0 & \text{in } D \\ h = 0 & \text{in } B_{3/16} \setminus D \\ h = \frac{1+C}{2}\gamma\epsilon & \text{on } \partial B_{3/16}. \end{cases}$$

Clearly, $0 \le h \le \frac{1+C}{2}\gamma\epsilon$, and by the maximum principle

$$\phi - \gamma(x_d + a + (1-C)\epsilon)_+ \le h \qquad \text{in } B_{3/16}. \tag{6.18}$$

Now, by the boundary Harnack inequality, if we set $\bar{x} = 1/8e_d$, we find that

$$h(x) \le C_1 \frac{h(\bar{x})}{(1/8 + a + (1-C)\epsilon)_+}(x_d + a + (1-C)\epsilon)_+ \le C_2\gamma\epsilon(x_d + a + (1-C)\epsilon)_+ \qquad \text{in } B_{1/8}$$

for some universal constants $C_1, C_2 > 0$. This last inequality, together with (6.18), leads to

$$\phi \le (1 + C_2)\gamma(x_d + a + (1-C)\epsilon)_+ \qquad \text{in } B_{1/8}.$$

On the other hand, since there is $\delta \in (0,1)$ and $\delta < C$ such that

$$(1 + C_2)(x_d + a + (1-C)\epsilon)_+ \le (x_d + a + (1-\delta)\epsilon)_+ \qquad \text{for all } x \in B_R,$$

we conclude the proof. □

We now prove the partial Harnack inequality for solutions to (6.13). This result will later be applied to the rescalings u_{r,x_0} and v_{r,x_0} at some point $x_0 \in \bar{\Omega}$.

Theorem 6.4 (Partial Harnack) *Given $K > 0$, there are $\epsilon_0, \rho > 0$ such that the following holds. If u and v are solutions of* (6.13) *in the sense of Definition 6.5 with $0 \in \bar{\Omega}$ and*

$$\begin{aligned} \alpha(x_d + a)_+ \le u(x) \le \alpha(x_d + b)_+ \qquad &\textit{for all } x \in B_1, \\ \beta(x_d + a)_+ \le v(x) \le \beta(x_d + b)_+ \qquad &\textit{for all } x \in B_1, \end{aligned}$$

for some α and β satisfying

$$0 < \alpha, \beta \le K \qquad \textit{and} \qquad \alpha\beta = 1,$$

and some a and b such that

$$|a|, |b| < \frac{1}{10} \qquad \textit{and} \qquad b - a \le \epsilon_0,$$

then there are $\tilde{a}, \tilde{b}$ satisfying $\tilde{b} - \tilde{a} \le (1-\delta)(b-a)$ for some $\delta > 0$ such that

$$\begin{aligned} \alpha(x_d + \tilde{a})_+ \le u(x) \le \alpha(x_d + \tilde{b})_+ \\ \beta(x_d + \tilde{a})_+ \le v(x) \le \beta(x_d + \tilde{b})_+ , \end{aligned}$$

for all $x \in B_\rho$, with $\rho < 1/8$ depending on d only.

Proof. As in [68], let $\bar{x} := e_d/5$ and consider the function $w : \mathbb{R}^d \to \mathbb{R}$ defined as

$$w(x) := \begin{cases} 1 & \text{if } x \in B_{1/20}(\bar{x}), \\ 0 & \text{if } x \notin B_{3/4}(\bar{x}), \\ \bar{c}\left(|x - \bar{x}|^{-d} - (3/4)^{-d}\right) & \text{if } x \in B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x}), \end{cases}$$

where $\bar{c} := 20^d - (4/3)^d$. Notice that the function w is nonzero exactly on $B_{3/4}(\bar{x})$ and it satisfies the following properties on $B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})$:

(w1) it is subharmonic since $\Delta w(x) = 2d\bar{c}|x - \bar{x}|^{-d-2} \ge 2d\bar{c}(3/4)^{-d-2} > 0$;

(w2) $\partial_{x_d} w$ is striclty positive on the half-space $\{x_d < 1/10\}$.

Step 1. *Invariant transformation and flatness estimates.*

By assumption, we have

$$\frac{1}{10K} \le u(\bar{x}) \le 10K \qquad \text{and} \qquad \frac{1}{10K} \le v(\bar{x}) \le 10K,$$

so there is a constant $c \in [(100K)^{-1}, 100K]$ such $\tilde{u} := cu$ and $\tilde{v} := c^{-1}v$ satisfy

$$\tilde{u}(\bar{x}) = \tilde{v}(\bar{x}),$$

and are also a solution to (6.1)-(6.2)-(6.3). Moreover, the flatness is preserved since

$$\begin{cases} \tilde{\alpha}(x_d + a)_+ \le \tilde{u}(x) \le \tilde{\alpha}(x_d + b)_+ & \text{for all } x \in B_1, \\ \tilde{\beta}(x_d + a)_+ \le \tilde{v}(x) \le \tilde{\beta}(x_d + b)_+ & \text{for all } x \in B_1, \end{cases} \tag{6.19}$$

where

$$\tilde{\alpha} := c\alpha \qquad \text{and} \qquad \tilde{\beta} := c^{-1}\beta.$$

Now, let $\epsilon := b - a < \epsilon_0$ and, without loss of generality, assume

$$\tilde{\alpha} := 1 + \delta \ge 1 \ge \frac{1}{1+\delta} = \tilde{\beta}.$$

Then, since $\tilde{u}(\bar{x}) = \tilde{v}(\bar{x})$, by (6.19) we have

$$\tilde{\alpha}\left(\frac{1}{5} + a\right) \leq \tilde{\beta}\left(\frac{1}{5} + b\right),$$

from which it follows that

$$1 + \delta = \tilde{\alpha} \leq \left[\frac{1/5 + b}{1/5 + a}\right]\tilde{\beta} \leq \frac{1/5 + b}{1/5 + a} = 1 + \frac{b - a}{1/5 + a} \leq 1 + 10\epsilon.$$

In particular, this implies that $\delta \leq 10\epsilon$ and

$$1 \geq \tilde{\beta} = \frac{1}{1+\delta} \geq 1 - \delta \geq 1 - 10\epsilon,$$

which, finally, yields the following estimate:

$$1 \leq \frac{\tilde{\alpha} + \tilde{\beta}}{2} = \frac{1}{2}\left(1 + \delta + \frac{1}{1+\delta}\right) \leq 1 + \delta^2 \leq 1 + 100\epsilon^2.$$

This, together with (6.19), implies that for all $x \in B_1$ we have

$$\begin{cases} (x_d + a)_+ \leq & (\tilde{u}(x)\tilde{v}(x))^{1/2} & \leq (x_d + b)_+ \\ (x_d + a)_+ \leq & \frac{1}{2}\left(\tilde{u}(x) + \tilde{v}(x)\right) & \leq (1 + 100\epsilon^2)(x_d + b)_+, \end{cases} \tag{6.20}$$

with

$$\tilde{u}(\bar{x}) = \tilde{v}(\bar{x}) = (\tilde{u}(\bar{x})\tilde{v}(\bar{x}))^{1/2} = \frac{\tilde{u}(\bar{x}) + \tilde{v}(\bar{x})}{2}.$$

Now, using again (6.19) and choosing ϵ_0 such that $\delta \leq 10\epsilon \leq 10\epsilon_0 \leq 1/2$, we have

$$|\tilde{u} - \tilde{v}| \leq 2\epsilon \qquad \text{in } B_1.$$

Moreover, since $a, b \leq 1/10$, we also have the estimate

$$1 \geq \tilde{u} \geq \frac{1}{40} \qquad \text{in } B_{1/20}(\bar{x}),$$

and this implies that we can choose ϵ_0 small enough such that on $B_{1/20}(\bar{x})$ there holds

$$0 \leq \frac{\tilde{u} + \tilde{v}}{2} - (\tilde{u}\,\tilde{v})^{1/2} = \tilde{u}\left(1 + \frac{1}{2}\frac{\tilde{v} - \tilde{u}}{\tilde{u}}\right) - \tilde{u}\sqrt{1 + \frac{\tilde{v} - \tilde{u}}{\tilde{u}}} \leq C\epsilon^2, \tag{6.21}$$

where C is a positive constant that depends on the dimension d only.

Step 2. *Gaining space for the domain* Ω.

We now argue as in [68] and [75] by considering separately the following two cases:

$$\frac{\tilde{u}(\bar{x}) + \tilde{v}(\bar{x})}{2} \geq \frac{\epsilon}{2} + (\bar{x}_d + a)_+ \qquad \text{and} \qquad \frac{\tilde{u}(\bar{x}) + \tilde{v}(\bar{x})}{2} \leq \frac{\epsilon}{2} + (\bar{x}_d + a)_+.$$

In the first case, since $|a| < 1/10$, we have $B_{1/10}(\bar{x}) \subset \{x_d + a > 0\}$, and so the function

$$h(x) := \frac{\tilde{u}(x) + \tilde{v}(x)}{2} - (\bar{x}_d + a)_+$$

is non-negative and solves a uniformly elliptic equation in $B_{1/10}(\bar{x})$ with right-hand side bounded from above and below by ϵ^2. Therefore, since $h(\bar{x}) \geq \frac{\epsilon}{2}$, the classical Harnack inequality yields

$$h \geq C_{\mathscr{H}}\epsilon \qquad \text{in } B_{1/20}(\bar{x}),$$

where $C_{\mathscr{H}}$ is a positive constant that depends on d only. Now, using (6.21) and choosing ϵ_0 small enough (depending on the dimension), we get that

$$(\tilde{u}\tilde{v})^{1/2} - (x_d + a) \geq \frac{1}{2}C_{\mathscr{H}}\epsilon \qquad \text{in } B_{1/20}(\bar{x}).$$

Now consider the family of functions

$$\psi_t(x) := x_d + a + \frac{1}{2}C_{\mathscr{H}}\epsilon(w(x) - 1) + \frac{1}{2}C_{\mathscr{H}}\epsilon t,$$

defined for $t \geq 0$ and $x \in B_1$. So far we proved that

$$(\tilde{u}(x)\tilde{v}(x))^{1/2} > \big(\psi_t(x)\big)_+ \qquad \text{for every } x \in B_{1/20}(\bar{x}) \text{ and every } t < 1.$$

We will show that the same inequality holds for every $x \in B_1$. Notice that the family of functions ψ_t satisfies, as a consequence of (w-i) and (w-ii), the following properties:

(ψ1) $\Delta\psi_t \geq C\epsilon > 0$ on $B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})$;

(ψ2) $|\nabla\psi_t|(x) > 1$ on $\Big(B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})\Big) \cap \{x_d < 1/10\}$.

We argue by contradiction. Suppose that for some $t < 1$ there exists $y \in B_1$ such that ψ_t touches $(\tilde{u}\tilde{v})^{1/2}$ from below at y. By Remark 6.10 and (ψ1), we have

$$y \notin \Omega \cap \Big(B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})\Big).$$

On the other hand, by (ψ2) and Lemma 6.1, we have

$$y \notin \partial\Omega \cap \Big(B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})\Big),$$

and this is a contradiction. Consequently, for every $x \in B_1$ there holds

$$(\tilde{u}(x)\tilde{v}(x))^{1/2} > \left(x_d + a + \frac{1}{2}C_{\mathscr{H}}\epsilon w(x)\right)_+,$$

and, in particular, it follows that

$$\Omega \supset \left\{x \in B_1 \ : \ x_d + a + \frac{1}{2}C_{\mathscr{H}}\epsilon w(x) > 0\right\}. \tag{6.22}$$

For the second case, we first notice that it is equivalent to

$$\frac{\epsilon}{2} \leq \bar{x}_d + b - \frac{\tilde{u}(\bar{x}) + \tilde{v}(\bar{x})}{2}.$$

Using this estimate, (6.20) and the Harnack inequality in $B_{1/10}(\bar{x})$, we get that

$$(1 + 100\epsilon^2)(x_d + b) - \frac{\tilde{u} + \tilde{v}}{2} \geq C_{\mathscr{H}}\epsilon \qquad \text{in } B_{1/20}(\bar{x}).$$

Now consider the family of functions

$$\eta_t(x) := (1 + 100\epsilon^2)(x_d + b) - C_{\mathscr{H}}\epsilon(w(x) - 1) - C_{\mathscr{H}}\epsilon t,$$

defined for $t \geq 0$ and $x \in B_1$. Then

$$\frac{\tilde{u}(x) + \tilde{v}(x)}{2} > \big(\eta_t(x)\big)_+ \qquad \text{for every } x \in B_{1/20}(\bar{x}) \text{ and every } t < 1.$$

Let us prove that the same inequality holds for every $x \in B_1$. Notice that, for every $t > 0$, we have

- (η1) $\Delta \eta_t < -C\epsilon$ on $B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})$;
- (η2) $|\nabla \eta_t|(x) < 1$ on $\Big(B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})\Big) \cap \{x_d < 1/10\}$.

We argue once again by contradiction. Suppose that for some $t < 1$ there exists $z \in B_1$ such that η_t touches from above $1/2(\tilde{u} + \tilde{v})$ at z. By (η1), we have

$$z \notin \Omega \cap \Big(B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})\Big).$$

On the other hand, by (η2) and Lemma 6.1, we have

$$z \notin \partial\Omega \cap \Big(B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})\Big),$$

and this is a contradiction. As a consequence, we have

$$\frac{\tilde{u}(x) + \tilde{v}(x)}{2} \leq \big((1 + 100\epsilon^2)(x_d + b) - C_{\mathcal{H}}\epsilon w(x)\big)_+ \qquad \text{for every } x \in B_1,$$

and, in particular,

$$\Omega \subset \big\{x \in B_1 \ : \ (1 + 100\epsilon^2)(x_d + b) - C_{\mathcal{H}}\epsilon w(x) > 0\big\}.$$

Finally, choosing ϵ_0 small enough and using that w is bounded away from zero in $B_{1/4}$, we get

$$\Omega \subset \left\{x \in B_{1/4} \ : \ x_d + b - \frac{1}{2}C_{\mathcal{H}}\epsilon w(x) > 0\right\}. \tag{6.23}$$

Step 3. *Conclusion of the proof*

So far we proved that we have one of the two inclusions (6.22) and (6.23); more precisely, there is a constant $C > 0$ such that either

$$\Omega \supset \{x_d + a + C\epsilon > 0\} \cap B_{1/4} \qquad \text{or} \qquad \Omega \subset \{x_d + b - C\epsilon > 0\} \cap B_{1/4}. \tag{6.24}$$

Then, by applying Lemma 6.2 to both u and v (replacing γ respectively with α and β), we find a universal constant $\delta \in (0, 1)$ such that either

$$\begin{cases} u(x) \leq \alpha(x_d + b - \delta(b - a))_+ \\ v(x) \leq \beta(x_d + b - \delta(b - a))_+ \end{cases} \quad \text{or} \quad \begin{cases} u(x) \geq \alpha(x_d + a + \delta(b - a))_+ \\ v(x) \geq \beta(x_d + a + \delta(b - a))_+ \end{cases}$$

for every $x \in B_\rho$, with $\rho < 1/8$ that depends on the dimension d only. □

Proposition 6.1 *Let (u_n, v_n) be a sequence of solutions to (6.13) in the sense of Definition 6.5. Let*

$$\Omega_n := \{u_n > 0\} = \{v_n > 0\}$$

with $0 \in \partial\Omega_n$ for every $n \in \mathbb{N}$. Suppose that there is a sequence $\epsilon_n \to 0$ such that

$$(x_d - \epsilon_n)_+ \leq u_n(x) \leq (x_d + \epsilon_n)_+ \qquad \textit{and} \qquad (x_d - \epsilon_n)_+ \leq v_n(x) \leq (x_d + \epsilon_n)_+$$

for every $x \in B_1$. Then there are continuous functions

$$\widetilde{u}_\infty : \overline{B_1^+} \to \mathbb{R} \qquad \text{and} \qquad \widetilde{v}_\infty : \overline{B_1^+} \to \mathbb{R},$$

with $B_1^+ = B_1 \cap \{x_d > 0\}$, such that the following properties hold:

(i) The graphs over $\overline{\Omega_n}$ of

$$\widetilde{u}_n = \frac{u_n - x_d}{\epsilon_n} \qquad \text{and} \qquad \widetilde{v}_n = \frac{v_n - x_d}{\epsilon_n},$$

converge in the Hausdorff distance respectively to the graphs of u_∞ and v_∞ over $\overline{B_1^+}$.

(ii) The graph over $\overline{\Omega_n}$ of

$$\widetilde{w}_n = \frac{\sqrt{u_n v_n} - x_d}{\epsilon_n},$$

converge in the Hausdorff distance to the graph of $\frac{1}{2}(u_\infty + v_\infty)$ over $\overline{B_1^+}$.

Proof. The proof of (i) follows from Theorem 6.4 exactly as in [68], while to prove (ii) we first notice that, for any $\delta > 0$ fixed, the sequences $\widetilde{u}_n$ and $\widetilde{v}_n$ converge uniformly on $B_1 \cap \{x_d > \delta\}$ to the functions $\widetilde{u}_\infty$ and $\widetilde{v}_\infty$. In particular, this implies that

$$\frac{\sqrt{u_n v_n} - x_d}{\epsilon_n} = \frac{\sqrt{(x_d + \epsilon_n \widetilde{u}_n)(x_d + \epsilon_n \widetilde{v}_n)} - x_d}{\epsilon_n} = \frac{1}{2}\left(\widetilde{u}_n + \widetilde{v}_n\right) + o(\epsilon_n),$$

which proves the claim on every $B_1 \cap \{x_d > \delta\}$. Now, in order to have the convergence of the graphs over the whole B_1^+, we notice that by (6.20) the oscillation of $\sqrt{u_n v_n} - x_d$ decays when passing from B_1 to a smaller ball B_ρ. Using again the argument from [68], we get that the graphs of

$$\widetilde{w}_n = \frac{\sqrt{u_n v_n} - x_d}{\epsilon_n},$$

Hausdorff-converge to the graph of a Hölder-continuous function

$$\widetilde{w}_\infty : \bar{B}_1^+ \to \mathbb{R}.$$

Now, since $\widetilde{w}_\infty = \frac{1}{2}\left(\widetilde{u}_\infty + \widetilde{v}_\infty\right)$ on each set $B_1 \cap \{x_d > \delta\}$, we get that $\widetilde{w}_\infty = \frac{1}{2}\left(\widetilde{u}_\infty + \widetilde{v}_\infty\right)$ on $\bar{B}_1^+$, and this concludes the proof. □

6.4 Improvement of flatness

This section's goal is to prove our main result, the *improvement of flatness*, from which the $C^{1,\alpha}$-regularity of the free boundary will follow by standard arguments. Notice that, in view of the invariance of (6.13) under suitable multiplication (see **Step 1** of Theorem 6.4), we can express the flatness conditions of Definition 6.2 without loss of generality with

$$\alpha = \beta = 1.$$

The main result of this section is the following:

Theorem 6.5 *Let (u, v) be solutions to (6.13) in the sense of Definition 6.5, set*

$$\Omega := \{u > 0\} = \{v > 0\},$$

and suppose that $0 \in \partial\Omega$. Then there are constants $\epsilon_0 > 0$ and $C > 0$ such that the following holds:

Improvement of flatness. *If* (u, v) *is a couple of solutions satisfying*

$$(x_d - \epsilon)_+ \le u(x) \le (x_d + \epsilon)_+ \quad \text{and} \quad (x_d - \epsilon)_+ \le v(x) \le (x_d + \epsilon)_+ \quad \text{in } B_1,$$

for some $\epsilon < \epsilon_0$*, then there are a unit vector* $\nu \in \mathbb{R}^d$ *with* $|\nu - e_d| \le C\epsilon$ *and a radius* $\rho \in (0, 1)$ *such that*

$$\tilde{\alpha}\left(x \cdot \nu - \frac{\epsilon}{2}\right)_+ \le u_\rho(x) \le \tilde{\alpha}\left(x \cdot \nu + \frac{\epsilon}{2}\right)_+ \quad \text{and} \quad \tilde{\beta}\left(x \cdot \nu - \frac{\epsilon}{2}\right)_+ \le v_\rho(x) \le \tilde{\beta}\left(x \cdot \nu + \frac{\epsilon}{2}\right)_+$$

for all $x \in B_\rho$*, where* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *are positive constants that satisfy*

$$\tilde{\alpha}\tilde{\beta} = 1 \qquad \text{and} \qquad |1 - \tilde{\alpha}|, |1 - \tilde{\beta}| \le C\epsilon.$$

We postpone the construction of the limiting problem arising from the linearization to Lemma 6.3 and Lemma 6.4, and we directly prove the improvement-of-flatness result.

Proof. We argue by contradiction. Let (u_n, v_n) be a sequence of solutions such that

$$(x_d - \epsilon_n)_+ \le u_n(x) \le (x_d + \epsilon_n)_+ \qquad \text{and} \qquad (x_d - \epsilon_n)_+ \le v_n(x) \le (x_d + \epsilon_n)_+,$$

where ϵ_n is an infinitesimal sequence. Let $\Omega := \{u_n > 0\} = \{v_n > 0\}$ and consider

$$\widetilde{u}_n = \frac{u_n - x_d}{\epsilon_n} \qquad \text{and} \qquad \widetilde{v}_n = \frac{v_n - x_d}{\epsilon_n} \tag{6.25}$$

on $\bar{\Omega}_n$. By the compactness result of Proposition 6.1, we get that $(\widetilde{u}_n, \widetilde{v}_n)$ converges, up to a subsequences, to a couple of continuous functions

$$\widetilde{u}_\infty : B_1 \cap \{x_d \ge 0\} \to \mathbb{R} \qquad \text{and} \qquad \widetilde{v}_\infty : B_1 \cap \{x_d \ge 0\} \to \mathbb{R}. \tag{6.26}$$

By Lemma 6.3, the functions

$$M := \frac{1}{2}(\widetilde{u}_\infty + \widetilde{v}_\infty) \qquad \text{and} \qquad D := \frac{1}{2}(\widetilde{u}_\infty - \widetilde{v}_\infty)$$

are classic solutions of the following PDEs:

$$\begin{cases} \Delta M = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ \partial_{x_d} M = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases} \quad \text{and} \quad \begin{cases} \Delta D = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ D = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases}$$

Therefore, by the regularity result Lemma 6.4, we get

$$|\widetilde{u}_\infty(x) - x \cdot \nabla \widetilde{u}_\infty(0)| \le C_d \rho^2 \qquad \text{and} \qquad |\widetilde{v}_\infty(x) - x \cdot \nabla \widetilde{v}_\infty(0)| \le C_d \rho^2$$

for every $x \in B_\rho \cap \{x_d \ge 0\}$, which can be rewritten as follows:

$$\begin{cases} x \cdot \nabla \widetilde{u}_\infty(0) - C_d \rho^2 \le \widetilde{u}_\infty(x) \le x \cdot \nabla \widetilde{u}_\infty(0) + C_d \rho^2 \\ x \cdot \nabla \widetilde{v}_\infty(0) - C_d \rho^2 \le \widetilde{v}_\infty(x) \le x \cdot \nabla \widetilde{v}_\infty(0) + C_d \rho^2 \end{cases} \quad \text{for every } x \in B_\rho \cap \{x_d \ge 0\}.$$

This implies that, if n is large enough, then

$$\begin{cases} x \cdot \nabla \widetilde{u}_\infty(0) - 2C_d \rho^2 \le \widetilde{u}_n(x) \le x \cdot \nabla \widetilde{u}_\infty(0) + 2C_d \rho^2 \\ x \cdot \nabla \widetilde{v}_\infty(0) - 2C_d \rho^2 \le \widetilde{v}_n(x) \le x \cdot \nabla \widetilde{v}_\infty(0) + 2C_d \rho^2 \end{cases} \quad \text{for every } x \in B_\rho \cap \bar{\Omega}_n,$$

which, by the definition of $\widetilde{u}_n$ and $\widetilde{v}_n$, can also be written as

$$x \cdot (e_d + \epsilon_n \nabla \widetilde{u}_\infty(0)) - \epsilon_n 2C_d\rho \leq (u_n)_\rho(x) \leq x \cdot (e_d + \epsilon_n \nabla \widetilde{u}_\infty(0)) + \epsilon_n 2C_d\rho$$
$$x \cdot (e_d + \epsilon_n \nabla \widetilde{v}_\infty(0)) - \epsilon_n 2C_d\rho \leq (v_n)_\rho(x) \leq x \cdot (e_d + \epsilon_n \nabla \widetilde{v}_\infty(0)) + \epsilon_n 2C_d\rho$$

for every $x \in B_1 \cap \left(\frac{1}{\rho}\Omega\right)$. To simplify the notations, we introduce the quantities

$$V := \nabla M(0) \qquad \text{and} \qquad c := \frac{\partial D}{\partial x_d}(0),$$

so the inequalities above can be rewritten as follows:

$$x \cdot (e_d + \epsilon_n V + c\epsilon_n e_d) - \epsilon_n 2C_d\rho \leq (u_n)_\rho(x) \leq x \cdot (e_d + \epsilon_n V + c\epsilon_n e_d) + \epsilon_n 2C_d\rho,$$
$$x \cdot (e_d + \epsilon_n V - c\epsilon_n e_d) - \epsilon_n 2C_d\rho \leq (v_n)_\rho(x) \leq x \cdot (e_d + \epsilon_n V - c\epsilon_n e_d) + \epsilon_n 2C_d\rho.$$

Now, since by Lemma 6.3 V and e_d are orthogonal, a simple computation shows that

$$|e_d(1 \pm c\epsilon_n) + \epsilon_n V| = \sqrt{1 \pm 2c\epsilon_n + \epsilon_n^2(c^2 + |V|^2)} = 1 \pm c\epsilon_n + O(\epsilon_n^2),$$

so, fixing $\rho > 0$ small enough and taking ϵ_n sufficiently small with respect to ρ, we get

$$x \cdot \frac{e_d + \epsilon_n V}{|e_d + \epsilon_n V|} - \frac{1}{2}\epsilon_n \leq \frac{1}{1 + c\epsilon_n}(u_n)_\rho(x) \leq x \cdot \frac{e_d + \epsilon_n V}{|e_d + \epsilon_n V|} + \frac{1}{2}\epsilon_n$$
$$x \cdot \frac{e_d + \epsilon_n V}{|e_d + \epsilon_n V|} - \frac{1}{2}\epsilon_n \leq (1 + c\epsilon_n)(v_n)_\rho(x) \leq x \cdot \frac{e_d + \epsilon_n V}{|e_d + \epsilon_n V|} + \frac{1}{2}\epsilon_n$$

for every $x \in B_1 \cap \left(\frac{1}{\rho}\Omega\right)$. Finally, the contradiction follows by taking

$$\nu = \frac{e_d + \epsilon_n V}{|e_d + \epsilon_n V|}, \qquad \tilde{\alpha} = 1 + c\epsilon_n \qquad \text{and} \qquad \tilde{\beta} = \tilde{\alpha}^{-1}.$$

□

Using the same notations of the proof of Theorem 6.5, we introduce the limiting problem arising from the linearization near flat free boundary points.

Lemma 6.3 (The linearized problem) *Let* $\widetilde{u}_\infty$ *and* $\widetilde{v}_\infty$ *be as in* (6.26) *and set*

$$M := \frac{1}{2}\left(\widetilde{u}_\infty + \widetilde{v}_\infty\right) \qquad \text{and} \qquad D := \frac{1}{2}\left(\widetilde{u}_\infty - \widetilde{v}_\infty\right).$$

Then, M and D are classical solutions of

$$\begin{cases} \Delta M = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ \partial_{x_d} M = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta D = 0 & \text{in } B_1 \cap \{x_d > 0\}, \\ D = 0 & \text{on } B_1 \cap \{x_d = 0\}. \end{cases} \tag{6.27}$$

Proof. We divide the proof into several steps.

Step 1. *Equations in* $\{x_d > 0\}$.

First we notice that the equation

$$\Delta M = \Delta D = 0 \qquad \text{in } B_1 \cap \{x_d > 0\}$$

follows from the fact that on every compact subset of $B_1 \cap \{x_d > 0\}$, the functions $\widetilde{u}_n$ and $\widetilde{v}_n$ given by (6.25) are harmonic and converge uniformly to $\widetilde{u}_\infty$ and $\widetilde{v}_\infty$ respectively.

Step 2. *Boundary condition for D.*

To prove the boundary condition

$$D = 0 \qquad \text{on } B_1 \cap \{x_d = 0\},$$

we notice that the graphs of $\widetilde{u}_n$ and $\widetilde{v}_n$ over $\partial\Omega_n$ are both given by the graph of $-(1/\epsilon_n)x_d$. Thus, by the Hausdorff convergence of the graphs, we get that

$$u_\infty = v_\infty \qquad \text{on } B_1 \cap \{x_d = 0\}.$$

The other boundary condition we need to prove is that

$$\frac{\partial M}{\partial x_d} = 0 \qquad \text{on } B_1 \cap \{x_d = 0\}$$

is satisfied in the viscosity sense. Nonetheless, notice that the fact that M is a classical solution of the linearized problem follows by [68, Lemma 2.6]).

Step 3. *The boundary condition for M from below.*

Suppose that a quadratic polynomial P touches M strictly from below at a point $x_0 \in \{x_d = 0\}$. We want to show that $\partial_d P(x_0) \leq 0$ so, arguing by contradiction, let us assume that

$$\frac{\partial P}{\partial x_d}(x_0) > 0, \tag{6.28}$$

and notice that we can also assume that $\Delta P > 0$ in a neighborhood of x_0. Let

$$\widetilde{w}_n := \frac{\sqrt{u_n v_n} - x_d}{\epsilon_n} \ : \ \overline{\Omega}_n \to \mathbb{R}.$$

By Proposition 6.1, we have that the sequence of graphs of $\widetilde{w}_n$ over $\overline{\Omega}_n$ converges in the Hausdorff sense to the graph of M over $B_1 \cap \{x_d \geq 0\}$. In particular, this means that the graph of P touches from below the graph of $\widetilde{w}_n$ at some point $x_n \in \overline{\Omega}_n$ and, since $\widetilde{w}_n$ is superharmonic in Ω_n (see Remark 6.10), we have that $x_n \in \partial\Omega$. As a consequence, we get

$$P(x) \leq \frac{\sqrt{u_n(x)v_n(x)} - x_d}{\epsilon_n} \qquad \text{for every } x \in \overline{\Omega}_n,$$

with an equality when $x = x_n$, which can be rewritten as

$$\epsilon_n P(x) + x_d \leq \sqrt{u_n(x)v_n(x)} \qquad \text{for every } x \in \overline{\Omega}_n.$$

Setting

$$\varphi(x) := \epsilon_n P(x) + x_d,$$

we can easily verify that φ_+ touches $\sqrt{u_n v_n}$ from below at x_n, and thus by Lemma 6.1 we find that

$$1 + \epsilon \frac{\partial P}{\partial x_d}(x_n) = \partial_{x_d}\varphi(x_n) \leq |\nabla\varphi(x_n)| \leq 1,$$

which is a contradiction with (6.28).

Step 4. *The boundary condition for M from above.*

Suppose that a quadratic polynomial P touches M strictly from above at a point $y_0 \in \{x_d = 0\}$. We want to prove that $\partial_d P(y_0) \geq 0$ so, arguing by contradiction, we assume that

$$\frac{\partial P}{\partial x_d}(y_0) < 0,$$

and we notice that we can also require $\Delta P < 0$ in a neighborhood of y_0. By the Hausdorff convergence of the graphs, the graph of P touches from above the graph of

$$\frac{1}{2}(\widetilde{u}_n + \widetilde{v}_n)$$

at a point $y_n \in \overline{\Omega}_n$. Since $\widetilde{u}_n$ and $\widetilde{v}_n$ are harmonic in Ω_n, we have $y_n \in \partial\Omega_n$. Consequently,

$$P(x) \geq \frac{1}{2}\left(\frac{u_n(x) - x_d}{\epsilon_n} + \frac{v_n(x) - x_d}{\epsilon_n}\right) \qquad \text{for every} x \in \overline{\Omega}_n,$$

with an equality when $x = y_n$, which can be rewritten as

$$x_d + \epsilon_n P(x) \geq \frac{u_n(x) + v_n(x)}{2} \qquad \text{for every } x \in \overline{\Omega}_n.$$

Let

$$\psi(x) := x_d + \epsilon_n P(x),$$

and notice that ψ_+ touches $\frac{1}{2}(u_n(x) + v_n(x))$ from above at y_n. Thus, by Lemma 6.1, we get

$$1 \leq |\nabla\psi(y_n)|^2 = 1 + 2\epsilon_n \frac{\partial P}{\partial x_d}(y_n) + \epsilon_n^2 |\nabla P(y_n)|^2,$$

which can be rewritten as

$$\frac{\partial P}{\partial x_d}(y_n) + \frac{\epsilon_n}{2}|\nabla P(y_n)|^2 \geq 0.$$

Passing to the limit as $n \to \infty$, we deduce that $\partial_{x_d} P(y_0) \geq 0$, and this is a contradiction. □

Remark 6.11 For the vectorial Bernoulli problem, in [75, 115] the authors proved that the linearized problem arising by the improvement of the flatness technique is a system of decoupled equations in which the first component satisfies a Neumann problem, while the others have Dirichlet boundary conditions. On the contrary, in our case, the nonlinear formulation of the free boundary condition (6.3) requires considering suitable linear combinations of the solutions u, v to detect the problem solved by the limits u_∞, v_∞.

For the sake of completeness, we sketch the proof of the decay for the solutions of the linearized problem (6.27), which we used in the proof of Theorem 6.5.

Lemma 6.4 (First and second order estimates) *Let $\widetilde{u}_\infty$ and $\widetilde{v}_\infty$ be as in Lemma 6.3. Then $\widetilde{u}_\infty$ and $\widetilde{v}_\infty$ are C^∞ in $B_1 \cap \{x_d \geq 0\}$ and we have the estimates*

$$\|\nabla\widetilde{u}_\infty\|_{L^\infty(B_{1/2}\cap\{x_d\geq 0\})} + \|\nabla\widetilde{v}_\infty\|_{L^\infty(B_{1/2}\cap\{x_d\geq 0\})} \leq C_d\ ,$$

and

$$\begin{cases} |\widetilde{u}_\infty(x) - x \cdot \nabla\widetilde{u}_\infty(0)| \leq C_d|x|^2 \\ |\widetilde{v}_\infty(x) - x \cdot \nabla\widetilde{v}_\infty(0)| \leq C_d|x|^2 \end{cases} \qquad \text{for every } x \in B_{1/2} \cap \{x_d \geq 0\}\ . \tag{6.29}$$

Moreover, we have

$$\nabla\widetilde{u}_\infty(0) = \nabla\widetilde{v}_\infty(0) + e_d \frac{\partial(\widetilde{u}_\infty - \widetilde{v}_\infty)}{\partial x_d}(0). \tag{6.30}$$

Proof. Let M and D be as in Lemma 6.3. Then both can be extended to (respectively, an even and an odd) harmonic functions in the ball B_1 and we can use the classical gradient and second-order estimates (see, e.g., [145, Lemma 7.17]) for a harmonic function $h : B_R \to \mathbb{R}$, that is,

$$\|\nabla h\|_{L^\infty(B_{R/2})} \le \frac{C_d}{R}\|h\|_{L^\infty(B_R)},$$

and

$$|h(x) - h(0) - x \cdot \nabla h(0)| \le \frac{C_d}{R^2}|x|^2\|h\|_{L^\infty(B_R)} \qquad \text{for every } x \in B_{R/2},$$

where C_d is a dimensional constant. Now, since $\widetilde{u}_\infty(0) = \widetilde{v}_\infty(0) = 0$ and

$$|\widetilde{u}_\infty| \le 1 \qquad \text{and} \qquad |\widetilde{v}_\infty| \le 1 \quad \text{in } B_1 \cap \{x_d \ge 0\},$$

and since $M = \widetilde{u}_\infty + \widetilde{v}_\infty$ and $D = \widetilde{u}_\infty - \widetilde{v}_\infty$, we get that

$$\begin{cases} |D(x) - x \cdot \nabla D(0)| \le C_d|x|^2 \\ |M(x) - x \cdot \nabla M(0)| \le C_d|x|^2 \end{cases} \qquad \text{for every } x \in B_{1/2} \cap \{x_d \ge 0\},$$

which gives (6.29). Finally, (6.30) follows from the fact that $D \equiv 0$ on $\{x_d = 0\}$. □

7 Regularity of the minima of integral shape functionals: the non-degenerate case

In this chapter, we give a brief overview of [37], which aims to prove some regularity properties of the problem studied in Chapter 4 when $p = 2$ (see Theorem 4.2 for a precise statement). Since the work is still ongoing, we will not go over all the proofs but, instead, focus on explaining how the results obtained in Chapter 5 and Chapter 6 play a crucial role here.

7.1 Formulation of the problem

For any set of finite measure $\Omega \subset \mathbb{R}^d$, which will be the control variable, we consider the solution u_Ω, which will be the corresponding state variable, of the PDE

$$\begin{cases} -\Delta u_\Omega = f & \text{in } \Omega\,, \\ u_\Omega = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{7.1}$$

where f is a prescribed right-hand side. We define the cost functional J as

$$J(u, B_1) := \int_{B_1} -g(x)u\,dx + |B_1 \cap \{u > 0\}|,$$

where the function $g : B_1 \to \mathbb{R}$ is given. In this chapter, we consider the shape optimization problem

$$\min\left\{J(u_\Omega, B_1) \;:\; \Omega \in \mathcal{A}\right\}, \tag{7.2}$$

where the admissible class of sets $\mathcal{A}$ is defined by

$$\mathcal{A} := \left\{\Omega \subset B_1 \;:\; \Omega \text{ quasi-open},\ |\Omega| \le 1\right\}.$$

Our main result is Theorem 4.2, which we now restate slightly differently to make the remaining of this chapter easier to follow.

Theorem 7.1 *Let $d \ge 2$ and $\Omega \subset \mathbb{R}^d$ be a solution to (7.2). Suppose that the following conditions hold:*

(a) $f, g \in C^2_c(B_1)$;
(b) $f \ge 0$ in $\overline{B}_1$, and $u_\Omega > 0$ in Ω;
(c) there are constants $C_1, C_2 > 0$ such that

$$C_1 g \le f \le C_2 g \qquad \text{in } \overline{B}_1. \tag{7.3}$$

Then there is a closed set $S \subseteq \partial\Omega \cap B_1$ such that:

(i) $\partial\Omega \cap B_1 \setminus S \in C^{1,\alpha}$ for some $\alpha \in (0, 1]$;
(ii) S is empty if $d \le 4$ and $\dim_{\mathcal{H}}(S) \le d - 5$, if $d \ge 5$.

Remark 7.1 In the case $f = g$, a set Ω which is optimal in B_1, takes the form $\Omega = \{u_\Omega > 0\}$, where u_Ω solves the following minimization problem:

$$\min\left\{\frac{1}{2}\int_{B_1} |\nabla u|^2\,dx - \int_\Omega f(x)u\,dx + |B_1 \cap \{u > 0\}| \;:\; u \in H^1_0(B_1)\right\}. \tag{7.4}$$

Thus, the above result follows from the classical regularity theory for the one-phase Alt-Caffarelli problem; see, for example, the papers [4], [147], [68], [49, 50], [104], [72] and [95].

When $f \neq g$, the problem cannot be reduced to a variational problem for an integral functional with variables u and v. In this case, u and v are formally the solutions of the free boundary system

$$\begin{cases} -\Delta u = f & \text{in } \Omega \cap B_1, \\ -\Delta v = g & \text{in } \Omega \cap B_1, \\ u = v = 0 \quad \text{and} \quad |\nabla u||\nabla v| = 1 & \text{on } B_1 \cap \partial\Omega. \end{cases}$$

For viscosity solutions, the regularity of the flat free boundaries was proved in [130], while the fact that the state functions on the optimal sets are viscosity solutions follows from the boundary Harnack inequality [131]. The main novelty of our work is the analysis of the dimension of the singular set, which we achieve by developing a new theory for stable solutions to the Alt-Caffarelli problem.

Remark 7.2 A set Ω, which is optimal in B_1, is a stable critical point of the shape functional

$$\Omega \longmapsto J(u_\Omega),$$

with respect to inner variations, in the sense that

$$\frac{d}{dt}\Big|_{t=0} J\left(u_{\Omega_t}, B_1\right) = 0 \qquad \text{and} \qquad \frac{d^2}{dt^2}\Big|_{t=0} J\left(u_{\Omega_t}, B_1\right) \geq 0$$

for every vector field $\xi \in C_c^\infty(B_1; \mathbb{R}^d)$, where $\Omega_t := (\mathrm{Id} + t\xi)(\Omega)$.

We prove in Section 7.4 that this notion is stable under blow-up limits and that Federer's dimension reduction principle holds for this class of solutions. Next, we show that the blow-up limits of our solutions are global one-homogeneous (Subsection 7.4.2) minimizers of the one-phase Alt-Caffarelli problem and that our notion of stability implies the stability inequality of Caffarelli-Jerison-Kenig on the limit cones (Subsection 7.6.2). Thus, the bound on the critical dimension (which is the dimension where singular cones appear for the first time) follows by the results of Jerison-Savin [104] and De Silva-Jerison [72] for the classical one-phase problem.

7.2 On the minimality condition

In this section, we discuss the existence of minimizers for the problem (7.2). More precisely, in Subsection 7.2.1, we provide two alternative optimality conditions satisfied by the shape optimizer Ω and the corresponding state variable u_Ω. These conditions will be essential in studying the local properties of Ω in Section 7.3. Moreover, in Subsection 7.2.2, we compute some variations along inner perturbations to deduce the optimizer's stationary and stability conditions.

Let us recall few notations associated to the state equation (7.1). The state variable u_Ω can be obtained by minimizing the functional

$$E_f(u, B_1) := \frac{1}{2}\int_{B_1} |\nabla u|^2\, dx - \int_{B_1} f(x) u\, dx$$

among all functions $u \in H_0^1(\Omega)$ such that $-\Delta u_\Omega \leq f$ in $\mathbb{R}^d$ in the sense of distributions; thus, we have

$$E_f(u_\Omega, B_1) = -\frac{1}{2}\int_{B_1} f u_\Omega\, dx \qquad \text{and} \qquad \|u_\Omega\|_{L^\infty(\Omega)} \leq \frac{|\Omega|^{2/d}}{2d\, |\Omega|^{2/d}} \|f\|_{L^\infty}. \tag{7.5}$$

Proposition 7.1 *Suppose that $f, g \in L^2(B_1)$ satisfy the following assumptions:*

(a) $f, g \geq 0$ in $\overline{B_1}$;

(b) there is a constant $C_2 > 0$ such that $f(x) \leq C_2 g(x)$ for every $x \in \overline{B}_1$.

Then the minimization problem (7.2) admits a solution $\Omega \in \mathcal{A}$ which is given by

$$\Omega = \{u_\Omega > 0\}.$$

Moreover, the set Ω is open.

Proof. The existence of solutions in the admissible class follows immediately from assumption (a), as already pointed out in [45]. Indeed, by the maximum principle, the map

$$\mathcal{A} \ni \Omega \longmapsto u_\Omega \in H^1_0(\Omega)$$

is monotonically increasing, and the cost functional J is decreasing with respect to Ω, so applying [32, Theorem 2.5] gives the existence of a quasi-open solution Ω.

To prove that Ω is open, we combine (b) with Theorem 4.1 (more precisely, Remark 4.5). Indeed, since f is non-negative, by the maximum principle we have that

$$\Omega = \{u_\Omega > 0\},$$

and so the continuity of u_Ω implies the claimed result. □

Now, to exploit the results obtained in Chapter 5 and Chapter 6, as suggested in [45] we introduce the state function associated to the cost term g. More precisely, if we denote by v_Ω the solution to

$$\begin{cases} -\Delta v_\Omega = g & \text{in } \Omega, \\ v_\Omega = 0 & \text{on } \partial\Omega, \end{cases}$$

integrating by parts the functional J yields

$$\begin{aligned} J(u_\Omega, B_1) &= \int_{B_1} \left(\nabla u_\Omega \nabla v_\Omega - g(x)u_\Omega - f(x)v_\Omega\right)\,dx + |B_1 \cap \{u_\Omega > 0\}| \\ &= -\int_{B_1} f(x)v_\Omega\,dx + |B_1 \cap \{v_\Omega > 0\}| \end{aligned}$$

for every $\Omega \in \mathcal{A}$, from which it follows that the minimization problems below are equivalent:

$$\min_{\Omega\in\mathcal{A}} \left\{\int_{B_1} \left(-g(x)u_\Omega + \mathbb{1}_{\{u_\Omega>0\}}(x)\right)\,dx\right\} = \min_{\Omega\in\mathcal{A}} \left\{\int_{B_1} \left(-f(x)v_\Omega + \mathbb{1}_{\{v_\Omega>0\}}(x)\right)\,dx\right\}. \tag{7.6}$$

Moreover, it is easy to verify that, if the assumption (7.3) holds in $\overline{B}_1$, then

$$C_1 v_\Omega \le u_\Omega \le C_2 v_\Omega \qquad \text{in } \overline{B}_1.$$

7.2.1 Almost-minimality conditions

To prove that u_Ω and v_Ω are Lipschitz-continuous when Ω is optimal, we need to introduce the following two almost-minimality conditions formulated in terms of the state variables:

Proposition 7.2 *Let Ω be an optimal set in B_1 and let u_Ω be the corresponding state variable satisfying (7.1). If the inequality (7.3) holds, then u_Ω satisfies the following properties:*

(a) ***Outward optimality.*** *For every $\omega \in \mathcal{A}$ such that $\Omega \subseteq \omega$, we have*

$$E_f(u_\Omega, B_1) + \frac{C_2}{2}|B_1 \cap \Omega| \le E_f(\phi, B_1) + \frac{C_2}{2}|B_1 \cap \omega| \qquad \textit{for every } \phi \in H^1_0(\omega).$$

Moreover, for every $x_0 \in B_1$, $r > 0$ such that $\Omega \cup B_r(x_0) \subset B_1$, we have

$$E_f(u_\Omega, B_r(x_0)) \leq E_f(\phi, B_r(x_0)) + \frac{C_2}{2}\omega_d r^d, \tag{7.7}$$

for every $\phi \in H^1(B_r(x_0))$ such that $\phi - u_\Omega \in H^1_0(B_r(x_0))$.

(b) ***Inward optimality.*** *For every $\omega \in \mathcal{A}$ such that $\omega \subseteq \Omega$, we have*

$$E_f(u_\Omega, B_1) + \frac{C_1}{2}|B_1 \cap \Omega| \leq E_f(\phi, B_1) + \frac{C_1}{2}|B_1 \cap \omega| \qquad \text{for every } \phi \in H^1_0(\omega).$$

*Since both optimality conditions are satisfied, we say that u_Ω is an **almost-minimizer of** E_f **in** B_1.*

Proof. Let $\omega \in \mathcal{A}$ and let u_ω be the associated state variable. If we assume that $\Omega \subseteq \omega$, then

$$f \geq 0 \implies u_\Omega \leq u_\omega \qquad \text{in } B_1$$

follows from the maximum principle. Moreover, the optimality condition $J(u_\Omega, B_1) \leq J(u_\omega, B_1)$ implies

$$\int_{B_1} g(u_\omega - u_\Omega)\,dx \leq |B_1 \cap \omega| - |B_1 \cap \Omega|,$$

so, using the right-hand side of (7.3), we get

$$E_f(u_\Omega, B_1) + \frac{C_2}{2}|B_1 \cap \Omega| \leq E_f(u_\omega, B_1) + \frac{C_2}{2}|B_1 \cap \omega|.$$

The outward optimality follows immediately by noticing that

$$E_f(u_\omega, B_1) + \frac{C_2}{2}|B_1 \cap \{u_\omega > 0\}| \leq E_f(\phi, B_1) + \frac{C_2}{2}|B_1 \cap \omega|,$$

as a consequence of the fact that u_ω minimizes E_f in $H^1_0(\omega)$. In particular, if we take $x_0 \in B_1$ and $r > 0$ such that $\omega := \Omega \cup B_r(x_0) \subset B_1$, then the localized version (7.7) follows by choosing $\phi = u_\Omega + \varphi$ in the inequality above, where $\varphi \in H^1_0(B_r(x_0))$.
Finally, the inward optimality condition follows in a similar way using the left-hand side of (7.3). □

7.2.2 Stationary and stability condition under inner variations

In order to characterize the blow-up limit of the state variables, it is more convenient to formulate the cost functional J of (7.2) using both state functions as follows:

$$J(u_\Omega, B_1) = \int_{B_1} \left(\nabla u_\Omega \nabla v_\Omega - g u_\Omega - f v_\Omega\right)\,dx + |B_1 \cap \Omega|.$$

Moreover, for every $D \subseteq \mathbb{R}^d$ we introduce the Alt-Caffarelli functional (see [4]):

$$J_O(u, d) = \int_D |\nabla u|^2\,dx + |D \cap \{u > 0\}|, \tag{7.8}$$

defined for every $u \in H^1(D)$ which is non-negative in D. This functional will arise in the blow-up analysis and will play a crucial role in studying the singular party of the free boundary.

The first step of this section is to compute the stationary condition associated to J with respect to internal perturbations obtained with compactly supported smooth vector fields.

Lemma 7.1 *Let $\Omega \subset B_1$ be open and let $f \in C^2_c(B_1)$ such that $f \geq 0$ in $\overline{B}_1$. Let $\xi \in C^\infty_c(B_1;\mathbb{R}^d)$ be a compactly supported vector field and denote by Φ_t the corresponding diffeomorphism*

$$\Phi_t(x) = x + t\xi(x) \qquad \text{in } B_1.$$

Then, if we set $\Omega_t := \Phi_t(\Omega)$, the map $t \mapsto u_t := u_{\Omega_t} \circ \Phi_t$ is C^2-differentiable and satisfies

$$u_t = u_\Omega + t\ \delta u + t^2\ \delta^2 u + o(t^2) \qquad \text{as } t \to 0^+, \tag{7.9}$$

where δu and $\delta^2 u$ are, respectively, the solutions to

$$\begin{cases} -\Delta(\delta u) = \operatorname{div}(\delta A \nabla u_\Omega) + \delta f \\ \delta u \in H^1_0(\Omega) \end{cases} \tag{7.10}$$

and

$$\begin{cases} -\Delta(\delta^2 u) = \operatorname{div}\big(\delta A \nabla(\delta u)) + \operatorname{div}((\delta^2 A)\nabla u\big) + \delta^2 f \\ \delta^2 u \in H^1_0(\Omega) \end{cases} \tag{7.11}$$

with coefficients given by

$$\begin{aligned}
&\delta A = -D\xi - (D\xi)^T + \operatorname{div}\xi\ \operatorname{Id} \\
&\delta^2 A = (D\xi)^T D\xi - \big((D\xi)^T\big)^2 - (D\xi)^2 - \operatorname{div}\xi\ \big(D\xi + (D\xi)^T\big) + \frac{(\operatorname{div}\xi)^2 - \operatorname{Tr}((D\xi)^2)}{2}\ \operatorname{Id}, \\
&\delta f = \operatorname{div}(f\xi), \\
&\delta^2 f = \frac{1}{2}\xi \cdot (D^2 f)\xi + f\frac{(\operatorname{div}\xi)^2 - \operatorname{Tr}((D\xi)^2)}{2} + \operatorname{div}\xi(\nabla f \cdot \xi).
\end{aligned} \tag{7.12}$$

Proof. First, notice that $u_t \in H^1_0(\Omega)$ is the unique solution of the PDE

$$-\operatorname{div}(A_t \nabla u_t) = f_t \qquad \text{in } \Omega, \tag{7.13}$$

where A_t and f_t are defined as follows:

$$f_t := f(\Phi_t)|\det(D\Phi_t)| \qquad \text{and} \qquad A_t := (D\Phi_t)^{-1}(D\Phi_t)^{-T}|\det(D\Phi_t)|.$$

To prove the C^2-diffentiability of the problem with respect to $t > 0$, we follow the strategy proposed in [100, Proposition 5.3.7]. Indeed, since $f \in C^2_c(B_1)$ and $\xi \in C^\infty_c(B_1;\mathbb{R}^d)$, we have

$$\begin{aligned}
&t \in \mathbb{R} \longmapsto f_t \in L^2(B_1) \subset L^2(\mathbb{R}^d) \subset H^{-1}(\Omega) &&\text{is } C^2 \\
&t \in \mathbb{R} \longmapsto A_t \in L^\infty(\mathbb{R}^d;\mathbb{R}^{d\times d}) &&\text{is } C^\infty,
\end{aligned}$$

while the map

$$L^\infty(\mathbb{R}^d;\mathbb{R}^{d\times d}) \times H^1_0(\Omega) \ni (A, w) \longmapsto -\operatorname{div}(A\nabla w) \in H^{-1}(\Omega)$$

is smooth since it is bilinear and continuous. Now, consider the operator

$$F : \mathbb{R} \times H^1_0(\Omega) \ni (s, w) \longmapsto -\operatorname{div}(A_s \nabla w) - f_s \in H^{-1}(\Omega)$$

and notice that $F(t, u_t) = 0$ for every $t > 0$. Since

$$D_w F(0, u_0)[\varphi] = -\Delta\varphi \qquad \text{for every } \varphi \in H^1_0(\Omega)$$

is an isomorphism from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$, by the implicit function theorem there exists a map

$t \mapsto w(t)$ of class C^2 such that $w(0) = u_0$ and

$$F(t, w(t)) = 0 \qquad \text{in a neighborhood of } t = 0.$$

However, the solution to (7.13) is unique, so $u_t = w(t)$ is C^2-differentiable as claimed. Thus, there are functions $\delta u, \delta^2 u \in H^1_0(\Omega)$ such that

$$u_t = u_\Omega + t\,\delta u + t^2\,\delta^2 u + o(t^2) \qquad \text{as } t \to 0,$$

which proves (7.9). On the other hand, by differentiating the terms in (7.13), we get

$$A_t = \mathrm{Id} + t\delta A + t^2\delta^2 A + \mathcal{O}(t^3), \qquad \text{and} \qquad f_t = f + t\delta f + t^2\delta^2 f + o(t^2), \tag{7.14}$$

where δA, δf, $\delta^2 A$ and $\delta^2 f$ are given by (7.12). Therefore, by substituting the previous expansions into (7.13) and differentiating with respect to t, we obtain

$$\begin{aligned} -\Delta(\delta u) &= \mathrm{div}(\delta A \nabla u_\Omega) + \delta f \\ -\Delta(\delta^2 u) &= \mathrm{div}(\delta A \nabla(\delta u)) + \mathrm{div}(\delta^2 A \nabla u) + \delta^2 f \end{aligned}$$

weakly in $H^1_0(\Omega)$, which completes the proof of (7.10) and (7.11). □

In general, it is not possible to ensure C^2-differentiability of the map $t \mapsto u_{\Omega_t}$ without assuming higher regularity of the domain Ω (see in [100, Section 5.3.5]). Nonetheless, we can slightly improve the previous result by proving C^1-differentiability of $t \mapsto u_{\Omega_t}$.

Corollary 7.1 *Let $\Omega \subset B_1$ be open and $f \in C^2_c(B_1)$ such that $f \geq 0$ in $\overline{B}_1$. Let $\xi \in C^\infty_c(B_1; \mathbb{R}^d)$ be a compactly supported smooth vector field and denote by Φ_t the corresponding diffeomorphism*

$$\Phi_t(x) = x + t\xi(x) \qquad \textit{for every } x \in B_1.$$

Then, denoting $\Omega_t = \Phi_t(\Omega)$, the map $t \mapsto u_{\Omega_t}$ is C^1-differentiable and satisfies

$$u_{\Omega_t} = u_\Omega + tu' + o(t) \qquad \textit{as } t \to 0,$$

where $u' \in H^1(\Omega)$ is a solution to

$$u' + \nabla u \cdot \xi \in H^1_0(\Omega) \qquad \textit{and} \qquad -\Delta u' = 0 \quad \textit{in } \Omega.$$

Proof. The same argument used to prove works [100, Theorem 5.3.2]. First, arguing as in Lemma 7.1, we deduce that $t \mapsto u_t := u_{\Omega_t} \circ \Phi_t$ is C^1-differentiable; moreover, we have

$$u_{\Omega_t} = u_t \circ \Psi_t \in H^1_0(\Omega),$$

where $\Psi_t := (\Phi_t)^{-1}$ is also a diffeomorphism. Thus, by applying [100, Lemma 5.3.3] (with $p = 2$ and $g(t) = u_t$), we get that $t \mapsto u_{\Omega_t}$ is C^1-differentiable and, by [100, eq. 5.36], we get

$$u' + \nabla u \cdot \xi = \delta u \qquad \text{in } H^1_0(\Omega).$$

Finally, since we have

$$\Delta(\nabla u \cdot \xi) = -\,\mathrm{div}(\delta A \nabla u_\Omega) - \delta f \qquad \text{in } \Omega,$$

it follows that u' is harmonic in Ω, concluding the proof. □

We are now ready to compute the first variation of the function J along inner smooth perturbations with compact support.

Lemma 7.2 *Let* $\Omega \subset B_1$, $\xi \in C_c^\infty(B_1;\mathbb{R}^d)$, Φ_t *and* Ω_t *be as above. Then*

$$\frac{\partial}{\partial t}\Big|_{t=0} J(u_{\Omega_t}, B_1) = \int_\Omega \left[\operatorname{div}\xi\,(\nabla u_\Omega \cdot \nabla v_\Omega + 1) - \nabla u_\Omega \cdot \left((D\xi)^T + (D\xi)\right)\nabla v_\Omega\right] dx$$
$$\cdots - \int_\Omega \left[u_\Omega \operatorname{div}(g\xi) + v_\Omega \operatorname{div}(f\xi)\right] dx. \tag{7.15}$$

Moreover, if $\operatorname{spt}\xi \cap \partial\Omega$ *is* C^2*-regular in* B_1*, then*

$$\partial J(\Omega, B_1)[\xi] := \frac{\partial}{\partial t}\Big|_{t=0} J(u_\Omega, B_1) = \int_{\partial\Omega} (\nu\cdot\xi)\,(1 - |\nabla u_\Omega||\nabla v_\Omega|)\, d\sigma,$$

where ν *is the outer normal to* $\partial\Omega$.

Proof. By applying Lemma 7.1 to both $t \mapsto u_t$ and $t \mapsto v_t$, we get

$$u_t = u_\Omega + t\delta u + o(t) \qquad \text{and} \qquad v_t = v_\Omega + t\delta v + o(t)$$

as $t \longrightarrow 0$, where $\delta u, \delta v \in H_0^1(\Omega)$ satisfy, respectively,

$$-\Delta(\delta u) = \operatorname{div}(\delta A \nabla u_\Omega) + \delta f \qquad \text{and} \qquad -\Delta(\delta v) = \operatorname{div}(\delta A \nabla v_\Omega) + \delta g. \tag{7.16}$$

Therefore, if we combine (7.12), (7.14) and (7.16), and plug them into J, we get

$$\begin{aligned}
J(u_{\Omega_t}, B_1) &= \int_{B_1} \left[\nabla u_{\Omega_t}\cdot\nabla v_{\Omega_t} - g u_{\Omega_t} - f v_{\Omega_t}\right] dx + |B_1 \cap \Omega_t| \\
&= \int_{B_1} \left[\nabla u_t \cdot A_t \nabla v_t - g_t u_t - f_t v_t + \mathbb{1}_\Omega(z)|\det(D\Phi_t)|\right] dz \\
&= J(u_\Omega, B_1) + t\int_{B_1} \left[\nabla\delta u\cdot\nabla v_\Omega + \nabla u_\Omega\cdot\nabla\delta v - g\delta u - f\delta v\right] dx \\
&\qquad \cdots + t\int_{B_1}\left[\nabla u_\Omega\cdot\delta A\nabla v_\Omega - u_\Omega\delta g - v_\Omega\delta f + \mathbb{1}_\Omega \operatorname{div}\xi\right] dx + o(t) \\
&= J(u_\Omega, B_1) + t\int_\Omega \left[\nabla u_\Omega\cdot\delta A\nabla v_\Omega - u_\Omega\delta g - v_\Omega\delta f + \operatorname{div}\xi\right] dx + o(t),
\end{aligned}$$

where the first equality follows from the change of variables $x = \Phi_t(z)$, while the last follows from an integration by parts. Thus, by differentiating in t at $t = 0$ and substituting (7.12), we obtain (7.15), concluding the first part of the proof.

Now assume $\partial\Omega \cap \operatorname{spt}\xi \in C^2$. Since for $u, v \in H^1(\Omega)$ we have

$$\begin{aligned}
\operatorname{div}\xi\,(\nabla u_\Omega\cdot\nabla v_\Omega) - \nabla u_\Omega\cdot((D\xi)^T + (D\xi))\nabla v_\Omega &= \operatorname{div}\left(\xi(\nabla u_\Omega\cdot\nabla v_\Omega) - (\nabla u_\Omega\cdot\xi)\nabla v_\Omega - (\nabla v_\Omega\cdot\xi)\nabla u_\Omega\right) \\
&\qquad \cdots + (\nabla v_\Omega\cdot\xi)\Delta u_\Omega + (\nabla u\cdot\xi)\Delta v_\Omega,
\end{aligned}$$

if we integrate by parts, we get

$$\begin{aligned}
\partial J(\Omega, B_1)[\xi] &= \int_\Omega \operatorname{div}\left(\xi((\nabla u_\Omega\cdot\nabla v_\Omega) + 1) - (\nabla u_\Omega\cdot\xi)\nabla v_\Omega - (\nabla v_\Omega\cdot\xi)\nabla u_\Omega\right) dx \\
&\qquad \cdots - \int_\Omega \left[(\nabla u_\Omega\cdot\xi)g + u_\Omega\operatorname{div}(g\xi) + (\nabla v_\Omega\cdot\xi)f + v_\Omega\operatorname{div}(f\xi)\right] dx \\
&= \int_{\partial\Omega} \left[(\nu\cdot\xi)((\nabla u\cdot\nabla v) + 1) - (\nabla u\cdot\xi)(\nu\cdot\nabla v) - (\nabla v\cdot\xi)(\nu\cdot\nabla u)\right] d\sigma \\
&\qquad \cdots - \int_\Omega \left[\operatorname{div}(g u_\Omega\xi) + \operatorname{div}(f v_\Omega\xi)\right] dx.
\end{aligned}$$

Since u_Ω and v_Ω are positive on Ω and zero on $\partial\Omega$, we have

$$\nabla u_\Omega = -\nu|\nabla u_\Omega| \qquad \text{and} \qquad \nabla v_\Omega = -\nu|\nabla v_\Omega|,$$

from which it follows that

$$\begin{aligned}\partial J(\Omega, B_1)[\xi] &= \int_{\partial\Omega} [(\nu\cdot\xi)(|\nabla u_\Omega||\nabla v_\Omega| + 1) - |\nabla u_\Omega|(\nu\cdot\xi)|\nabla v_\Omega| - |\nabla v_\Omega|(\nu\cdot\xi)|\nabla u_\Omega|]\, d\sigma \\ &= \int_{\partial\Omega} (\nu\cdot\xi)(1 - |\nabla u_\Omega||\nabla v_\Omega|)\, d\sigma,\end{aligned}$$

and this concludes the proof. □

The following result is an immediate consequence of the previous lemma; indeed, if we take as Ω a minimizer of (7.2), then

$$\Omega \text{ optimal for } J \implies \partial J(\Omega, B_1)[\xi] = 0 \text{ for every } \xi \in C_c^\infty(B_1;\mathbb{R}^d).$$

Corollary 7.2 *Let Ω be a solution to (7.2) and u_Ω, v_Ω the associated state variables. Then*

$$\int_\Omega \left[\operatorname{div}\xi\,(\nabla u_\Omega\cdot\nabla v_\Omega + 1) - \nabla u_\Omega\cdot((D\xi)^T + (D\xi))\nabla v_\Omega - u_\Omega \operatorname{div}(g\xi) - v_\Omega\operatorname{div}(f\xi)\right]\, dx = 0$$

for every $\xi \in C_c^\infty(B_1;\mathbb{R}^d)$. Moreover, if $\partial\Omega\cap B_1$ is C^2-regular, we get that

$$|\nabla u_\Omega||\nabla v_\Omega| = 1 \qquad \text{on } \partial\Omega\cap B_1.$$

Finally, in the following result, we compute the second variation of the functional J along inner perturbations with compact support.

Lemma 7.3 *Let $\Omega\subset B_1$, $\xi\in C_c^\infty(B_1;\mathbb{R}^d)$, Φ_t and Ω_t be as above. Then*

$$\begin{aligned}\partial^2 J(\Omega, B_1)[\xi] := \frac{\partial^2}{\partial t^2}\Big|_{t=0} J(u_{\Omega_t}, B_1) &= \int_\Omega \nabla u\cdot(\delta^2 A)\nabla v - \nabla\delta u\cdot\nabla\delta v - (\delta^2 f)v - (\delta^2 g)u\, dx \\ &\quad \cdots + \int_\Omega \frac{1}{2}\Big(\big(\operatorname{div}\xi\big)^2 - \operatorname{Tr}\big((D\xi)^2\big)\Big)\, dx,\end{aligned} \tag{7.17}$$

where $\delta^2 A, \delta^2 f$ and $\delta^2 g$ are given by (7.12) and $\delta u, \delta v \in H_0^1(\Omega)$ are defined in Lemma 7.1. Moreover, if $\operatorname{spt}\xi\cap\partial\Omega$ is C^2-regular in B_1, then

$$\partial^2 J(\Omega, B_1)[\xi] = \int_{\partial\Omega} (\nu\cdot\xi)\,(1 - |\nabla u_\Omega||\nabla v_\Omega|)\, d\sigma,$$

where ν is the outer normal to $\partial\Omega$.

Proof. By applying Lemma 7.1 to both

$$t\longmapsto u_t := u_{\Omega_t}\cdot\Phi_t \qquad \text{and} \qquad t\longmapsto v_t := v_{\Omega_t}\cdot\Phi_t,$$

we deduce the C^2-rectifiability and the expansions

$$u_t = u_\Omega + t\,\delta u + t^2\,\delta^2 u + o(t^2) \qquad \text{and} \qquad v_t = v_\Omega + t\,\delta v + t^2\,\delta^2 v + o(t^2)$$

as $t \to 0$, where $\delta u, \delta v$ satisfy (7.16) and $\delta^2 u, \delta^2 v \in H^1_0(\Omega)$ such that

$$-\Delta(\delta^2 u) = \operatorname{div}(\delta A \nabla(\delta u)) + \operatorname{div}\left(\delta^2 A \nabla u\right) + \delta^2 f,$$
$$-\Delta(\delta^2 v) = \operatorname{div}(\delta A \nabla(\delta v)) + \operatorname{div}\left(\delta^2 A \nabla v\right) + \delta^2 g.$$

Therefore, arguing exactly as in Lemma 7.2, we get

$$J(u_{\Omega_t}, B_1) = J(u_\Omega, B_1) + t\, \partial J(\Omega, B_1)[\xi] + t^2\, \partial^2 J(\Omega, B_1)[\xi] + o(t^2),$$

where $\partial J(\Omega, B_1)[\xi]$ is defined by (7.15), and

$$\begin{aligned}
\partial^2 J(\Omega, B_1)[\xi] = &\int_\Omega \left[\nabla(\delta^2 u)\cdot\nabla v + \nabla u \cdot (\delta^2 A)\nabla v + \nabla u \cdot \nabla(\delta^2 v)\right]\, dx \\
&\cdots + \int_\Omega \left[\nabla(\delta u)\cdot\nabla(\delta v) + \nabla(\delta u)\cdot(\delta A)\nabla v + \nabla u \cdot (\delta A)\nabla \delta v\right]\, dx \\
&\cdots - \int_\Omega \left[f(\delta^2 v) + (\delta f)(\delta v) + (\delta^2 f) v + g(\delta^2 u) + (\delta g)(\delta u) + (\delta^2 g) u\right]\, dx \\
&\cdots + \int_\Omega \frac{1}{2}\left[(\operatorname{div}\xi)^2 - \operatorname{Tr}\left((D\xi)^2\right)\right]\, dx.
\end{aligned}$$

If we now test the equations for δu and δv in (7.16) respectively with δv and δu, we get

$$\int_\Omega \nabla(\delta u)\cdot\nabla(\delta v)\, dx = \int_\Omega \left[-\nabla(\delta v)\cdot(\delta A)\nabla u + (\delta v)(\delta f)\right]\, dx = \int_\Omega \left[-\nabla(\delta u)\cdot(\delta A)\nabla v + (\delta u)(\delta g)\right]\, dx.$$

Then, by exploiting the equations $-\Delta u_\Omega = f$ and $-\Delta v_\Omega = g$, we infer that

$$\begin{aligned}
\partial^2 J(\Omega, B_1)[\xi] = &\int_\Omega \left[\nabla u \cdot (\delta^2 A)\nabla v - \nabla \delta u \cdot \nabla \delta v - (\delta^2 f) v - (\delta^2 g) u\right]\, dx \\
&\cdots + \int_\Omega \frac{1}{2}\left[(\operatorname{div}\xi)^2 - \operatorname{Tr}\left((D\xi)^2\right)\right]\, dx,
\end{aligned}$$

completing the proof of (7.17). The second assertion, on the other hand, follows from a standard computation exactly as in Lemma 7.2. □

We conclude with the following result, in which we give an equivalent formulation of the second variation along inner perturbations in the case of C^2-regular domains.

Corollary 7.3 *Let Ω and $\xi \in C^\infty_c(B_1;\mathbb{R}^d)$ be as above and assume that $\operatorname{spt}\xi \cap \partial\Omega$ is C^2-regular. Then, given $u', v' \in H^1(\Omega)$ such that*

$$\begin{cases} \Delta u' = 0 & \text{in } \Omega \\ u' = |\nabla u_\Omega|(\xi\cdot\nu) & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta v' = 0 & \text{in } \Omega \\ v' = |\nabla v_\Omega|(\xi\cdot\nu) & \text{on } \partial\Omega \end{cases}$$

the second variation (7.17) can be written as follows:

$$\partial^2 J(\Omega, B_1)[\xi] = 2\int_\Omega \nabla u'\cdot\nabla v'\, dx + \int_{\partial\Omega} (\xi\cdot\nu)^2\left[(1 + |\nabla u_\Omega||\nabla v_\Omega|)H + g|\nabla u_\Omega| + f|\nabla v_\Omega|\right]\, d\sigma,$$

where ν is the outer normal to $\partial\Omega$ and H denotes the mean curvature of $\partial\Omega$.

Proof. First, recall that, under our regularity assumption, in Lemma 7.2 we proved that

$$\partial J(\Omega, B_1)[\xi] = \int_{\partial\Omega} (\nu\cdot\xi)(1 - |\nabla u_\Omega||\nabla v_\Omega|)\, d\sigma = \int_\Omega \operatorname{div}\left((1 - \nabla u_\Omega\cdot\nabla v_\Omega)\xi\right)\, dx,$$

where ν is the outer normal to $\partial\Omega$. Since $\partial\Omega$ is C^2-smooth and $f \in C^2_c(B_1)$, we have

$$t \longmapsto (1 - \nabla u_{\Omega_t} \cdot \nabla v_{\Omega_t}) \in C^1\left([0,T);\ L^1(\mathbb{R}^d)\right) \cap C^0\left([0,T);\ W^{1,1}(\mathbb{R}^d)\right),$$

so we can compute the second variation of J at Ω as follows:

$$\partial^2 J(\Omega, B_1)[\xi] = \frac{\partial}{\partial t}\bigg|_{t=0} \int_{\Omega_t} \operatorname{div}\left((1 - \nabla u_{\Omega_t} \cdot \nabla v_{\Omega_t})\xi \circ \Psi_t\right) dx,$$

where $\Psi_t := (\Phi_t)^{-1}$ is a diffeomorphism for $t > 0$ small enough. To compute this, taking into account that $\nabla u_\Omega = -|\nabla u_\Omega|\nu$ and $\nabla v_\Omega = -|\nabla v_\Omega|\nu$ on $\partial\Omega$, we consider $u', v' \in H^1(\Omega)$ satisfying

$$\begin{cases} \Delta u' = 0 & \text{in } \Omega \\ u' = |\nabla u_\Omega|(\xi \cdot \nu) & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta v' = 0 & \text{in } \Omega \\ v' = |\nabla v_\Omega|(\xi \cdot \nu) & \text{on } \partial\Omega \end{cases}$$

exactly as in Corollary 7.1. Then, we can expand u_{Ω_t} and v_{Ω_t} for $t \to 0$ as

$$u_{\Omega_t} = u_\Omega + tu' + o(t) \qquad \text{and} \qquad v_{\Omega_t} = v_\Omega + tv' + o(t),$$

and, since $(\xi \circ \varphi_t)'|_{t=0} = -D\xi^T\xi$, it follows that

$$\begin{aligned} \partial^2 J(\Omega, B_1)[\xi] = &- \int_\Omega \operatorname{div}\left((\nabla u_\Omega \cdot \nabla v' + \nabla u' \cdot \nabla v_\Omega)\xi\right) dx \\ &\cdots + \int_{\partial\Omega} \left[\operatorname{div}\left((1 - \nabla u_\Omega \cdot \nabla v_\Omega)\xi\right)(\xi \cdot \nu) - (1 - \nabla u_\Omega \cdot \nabla v_\Omega)(\nu \cdot D\xi^T\nu)(\xi \cdot \nu)\right] d\sigma. \end{aligned}$$

To conclude, notice that

$$\operatorname{div} \xi - (\nu \cdot D\xi^T\nu) = H(\xi \cdot \nu),$$

where H is the mean curvature of $\partial\Omega$, and

$$\nabla(\nabla u_\Omega \cdot \nabla v_\Omega) \cdot \nu = (\partial_\nu v_\Omega)(\partial^2_\nu u_\Omega) + (\partial_\nu u_\Omega)(\partial^2_\nu v_\Omega) = -2H|\nabla v_\Omega||\nabla v_\Omega| + g|\nabla u_\Omega| + f|\nabla v_\Omega|$$

on $\partial\Omega$. It follows that

$$\begin{aligned} \partial^2 J(\Omega, B_1)[\xi] &= \int_{\partial\Omega} (u'\partial_\nu v' + v'\partial_\nu u')\, dx + \int_{\partial\Omega} (\xi \cdot \nu)^2 \left[(1 + |\nabla u_\Omega||\nabla v_\Omega|)H + g|\nabla u_\Omega| + f|\nabla v_\Omega|\right] d\sigma \\ &= 2\int_\Omega \nabla u' \cdot \nabla v'\, dx + \int_{\partial\Omega} (\xi \cdot \nu)^2 \left[(1 + |\nabla u_\Omega||\nabla v_\Omega|)H + g|\nabla u_\Omega| + f|\nabla v_\Omega|\right] d\sigma, \end{aligned}$$

and this concludes the proof of the alternative formulation. □

To conclude this section, we give the first and second variations of the one-phase functional (7.8), which are obtained through the same computations as above.

Lemma 7.4 *Let $\xi \in C^\infty_c(B_1; \mathbb{R}^d)$ and Φ_t be as above. If u_t is the solution to*

$$\begin{cases} \Delta u_t = 0 & \text{in } D \cap \Phi_t(\{u > 0\}) \\ u_t = 0 & \text{on } D \cap \partial\Phi_t(\{u > 0\}) \\ u_t = u & \text{on } \partial D \cap \Phi_t(\{u > 0\}), \end{cases}$$

then the first variation is given by

$$\frac{\partial}{\partial t}\bigg|_{t=0} J_O(u_t, B_1) = \int_{B_1} \left[\nabla u \cdot \delta A \nabla u + \mathbb{1}_{\{u>0\}} \operatorname{div} \xi\right] dx,$$

and the second variation by

$$\frac{\partial^2}{\partial t^2}\bigg|_{t=0} J_O(u_t, B_1) = \int_{B_1} \left[\nabla u \cdot (\delta^2 A)\nabla u - |\nabla \delta u|^2 + \frac{1}{2}\left((\operatorname{div} \xi)^2 - \operatorname{Tr}\left((D\xi)^2 \right) \right) \mathbb{1}_{\{u>0\}} \right] dx,$$

where δA *and* $\delta^2 A$ *are defined in* (7.12) *and* $\delta u \in H^1_0(\{u > 0\})$ *is the solution to*

$$-\Delta(\delta u) = \operatorname{div}(\delta A \nabla u) \qquad \text{in } D \cap \{u > 0\}.$$

For the remainder of this chapter, we will use the notations $\partial J_O(u, B_1)[\xi]$ and $\partial^2 J_O(u, B_1)[\xi]$ to identify, respectively, the first and second variations of J_O at u under the inner perturbation ξ.

7.3 Lipschitz regularity and non-degeneracy of state variables

This section aims to study the local properties of shape optimizers. More precisely, we prove that the state variables u_Ω and v_Ω (associated to an optimal shape) are Lipschitz-continuous and non-degenerate, and we obtain, as a consequence, a density estimate for optimal shapes.

Lemma 7.5 *Suppose that* u_Ω *satisfies the outward optimality of Proposition 7.2. Then* u_Ω *is locally Lipschitz in* B_1*, and the Lipschitz constant depends on* d, $\|f\|_{L^\infty(\mathbb{R}^d)}$ *and* C_2.

Proof. Let $x_0 \in B_1$ and $r > 0$. If $\omega := \Omega \cap B_r(x_0)$ and $\phi \in H^1_0(\omega)$ such that $v := \phi - u_\Omega \in H^1_0(B_r(x_0))$, then, by the outward optimality of Proposition 7.2, up to taking r smaller, we get

$$E_f(u_\Omega, B_r(x_0)) \le E_f(v, B_r(x_0)) + \frac{C_2}{2}\omega_d r^d. \tag{7.18}$$

Moreover, by (7.7) the validity of the condition (7.18) can be extended to every ball in B_1. Thus, since u_Ω is a local quasi-minimizer in the sense of [26, Definition 3.1], by applying [26, Theorem 3.3] we immediately deduce the thesis. □

The next result asserts that state functions satisfying the inward optimality condition are non-degenerate. We achieve this by adapting the proof of [4, Lemma 4.4] to our framework.

Lemma 7.6 *Suppose that* u_Ω *satisfies the inward optimality of Proposition 7.2. Then there are* $C_0, r_0 > 0$*, depending only on* d, $\|f\|_{L^\infty}$ *and* C_1*, such that, for every* $x_0 \in \overline{\{u_\Omega > 0\}} \cap B_1$ *and* $r \in (0, r_0]$*, we have*

$$\|u\|_{L^\infty(B_{2r})} \le C_0 r \implies u \equiv 0 \text{ on } B_r(x_0).$$

Remark 7.3 In [95, Lemma 2.8], the same problem in the case of local minimizers of (7.4) has already been addressed. For completeness, we sketch the proof and focus on the role of inward optimality.

Proof. First, notice that it is not restrictive to assume that $x_0 = 0 \in \partial\{u_\Omega > 0\}$. For $r > 0$, denote by φ_1 and ϕ_1 the solutions to

$$\begin{cases} -\Delta\varphi_1 = 0 & \text{in } B_2 \setminus B_1 \\ \varphi_1 = 0 & \text{in } B_1 \\ \varphi_1 = 1 & \text{on } \partial B_2 \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta\phi_1 = 1 & \text{in } B_2 \setminus B_1 \\ \phi_1 = 0 & \text{in } B_1 \\ \phi_1 = 0 & \text{on } \partial B_2 \end{cases}$$

and set

$$\eta_r(x) := \|u_\Omega\|_{L^\infty(B_{2r})}\varphi_1\left(\frac{x}{r}\right) + r^2\|f\|_{L^\infty(\mathbb{R}^d)}\phi_1\left(\frac{x}{r}\right).$$

Notice that $-\Delta\eta_r = \|f\|_{L^\infty}$, η_r is radially symmetric in $\overline{B}_{2r}\setminus B_r$, and

$$\|\nabla\eta_r\|_{L^\infty(\partial B_r)} \le C_d\left(\frac{\|u_\Omega\|_{L^\infty(B_{2r})}}{r} + r\|f\|_{L^\infty(\mathbb{R}^d)}\right) \le C_d(C_0 + r_0\|f\|_{L^\infty(\mathbb{R}^d)}).$$

Therefore, if we consider the competitor $\tilde u \in H^1_{\mathrm{loc}}(B_1)$ defined by

$$\tilde u(x) := \begin{cases} u_\Omega(x), & \text{if } B_1\setminus B_{2r},\\ \min\{u_\Omega(x), \eta_r(x)\} & \text{in } B_{2r}\setminus B_r,\\ 0 & \text{in } B_r,\end{cases}$$

then $\{\tilde u > 0\} = \{u_\Omega > 0\}$ in $B_{2r}\setminus B_r$ and, more generally, we have

$$\{\tilde u > 0\} \subseteq \{u_\Omega > 0\}.$$

Therefore, since $\tilde u \ge 0$ in B_1 and $\tilde u = u_\Omega$ in $B_1 \setminus B_{2r}$, we can apply the inward optimality of Proposition 7.2 to obtain the inequality

$$E_f(u_\Omega, B_{2r}) + \frac{C_1}{2}|B_{2r}\cap\{u_\Omega>0\}| \le E_f(\tilde u, B_{2r}) + \frac{C_1}{2}|B_{2r}\cap\{\tilde u>0\}|,$$

which, taking into account the definition of $\tilde u$, can be rewritten as follows:

$$E_f(u_\Omega, B_r) + \frac{C_1}{2}|B_r\cap\{u_\Omega>0\}| \le E_f(\tilde u, B_{2r}\setminus B_r) - E_f(u_\Omega, B_{2r}\setminus B_r).$$

It follows that

$$\begin{aligned} E_f(u_\Omega, B_r) + \frac{C_1}{2}|B_r\cap\{u_\Omega>0\}| &\le \frac12\int_{B_{2r}\setminus B_r}\left[|\nabla\tilde u|^2 - |\nabla u_\Omega|^2\right]dx - \int_{B_{2r}\setminus B_r} f(\tilde u - u_\Omega)\,dx\\ &\le \int_{\{u_\Omega\neq\tilde u\}\cap(B_{2r}\setminus B_r)}\left[\nabla\tilde u\cdot\nabla(\tilde u - u_\Omega) - f(\tilde u - u_\Omega)\right]dx,\end{aligned}$$

where $\{u_\Omega \neq \tilde u\} = \{u_\Omega > \eta_r\}$ by construction. Since $-\Delta\eta_r - f \ge 0$ in $B_{2r}\setminus B_r$, we deduce that

$$\begin{aligned} E_f(u_\Omega, B_r) + \frac{C_1}{2}|B_r\cap\{u_\Omega>0\}| &\le \int_{\{u_\Omega>\eta_r\}\cap(B_{2r}\setminus B_r)}(-\Delta\eta_r - f)(\eta_r - u_\Omega)\,dx + \int_{\partial B_r} u_\Omega|\nabla\eta_r|\,d\sigma\\ &\le \|\nabla\eta_r\|_{L^\infty(\partial B_r)}\int_{\partial B_r} u_\Omega\,d\sigma,\end{aligned}$$

which, in turn, implies that

$$\int_{B_r}|\nabla u_\Omega|^2\,dx + C_1|B_r\cap\{u_\Omega>0\}| \le C_d\left(C_0 + r_0\|f\|_{L^\infty(\mathbb{R}^d)}\right)\left(\int_{\partial B_r}u_\Omega\,d\sigma + \frac1r\int_{B_r}u_\Omega\,dx\right). \tag{7.19}$$

On the other hand, by the $W^{1,1}$-trace inequality (see, e.g., [129]), we have

$$\begin{aligned}\int_{\partial B_r}u_\Omega\,d\sigma + \frac1r\int_{B_r}u_\Omega\,dx &\le C_d\left(\int_{B_r}|\nabla u_\Omega|\,dx + \frac1r\int_{B_r}u_\Omega\,dx\right)\\ &\le C_d\int_{B_r}|\nabla u_\Omega|^2\,dx + C_d\left(1 + \frac1r\|u\|_{L^\infty(B_{2r})}\right)|B_r\cap\{u_\Omega>0\}|,\end{aligned}$$

which means that

$$1 \le C_d\left(C_0 + r_0\|f\|_{L^\infty(\mathbb{R}^d)}\right)\max\{1, C_1^{-1}\}\,(1 + C_0)\,. \tag{7.20}$$

By choosing $C_0, r_0 > 0$ small enough for the right-hand side in (7.20) to be smaller than 1, we get that the left-hand side of (7.19) is equal to zero, and so $u_\Omega \equiv 0$ in B_r. In conclusion, notice that, by (7.20), the constants C_0 and r_0 depend on d, $\|f\|_{L^\infty}$ and C_1 only. □

The next result summarizes the content of this section so far, which is the Lipschitz-continuity and non-degeneracy (close to the boundary) of the state variables u_Ω and v_Ω for Ω optimal.

Proposition 7.3 *Let u_Ω and v_Ω be the state variables associated to the optimal set Ω in B_1 and suppose there are constants $C_1, C_2 > 0$ such that (7.3) holds. Then*

(i) u_Ω and v_Ω are locally Lipschitz-continuous in B_1, and the Lipschitz constants depends on d, $\|f\|_{L^\infty(\mathbb{R}^d)}$, C_1 and C_2;

(ii) if $x_0 \in \partial\Omega$, for every $r \in (0, (1-|x_0|)/2)$, we have

$$\sup_{x\in B_r(x_0)} u_\Omega(x) \geq \tilde{C}r \qquad \text{and} \qquad \sup_{x\in B_r(x_0)} v_\Omega(x) \geq \tilde{C}r, \tag{7.21}$$

where $\tilde{C} > 0$ is a constant that depends only on d, $\|f\|_{L^\infty(\mathbb{R}^d)}$, C_1 and C_2.

Proof. It is enough to combine Proposition 7.2 with Lemma 7.5 and Lemma 7.6 to conclude. □

We now show a density estimate that follow directly from Proposition 7.3.

Proposition 7.4 *Let Ω be a solution to (7.2) in B_1 and u_Ω be the associated state variable. Then there are $\epsilon_0, r_0 > 0$ such that we have*

$$\epsilon_0\omega_d r^d \leq |B_r(x_0) \cap \{u_\Omega > 0\}| \leq (1-\epsilon_0)\omega_d r^d$$

for every $x_0 \in \partial\Omega$ and every $r \leq r_0$, with ϵ_0, r_0 depending only on d, $\|f\|_{L^\infty}$, C_1 and C_2.

Proof. Suppose $x_0 = 0 \in \partial\Omega$. The proof of the lower bound follows from Proposition 7.3 since

- on the one hand, for r small enough there exists $x_r \in B_r \cap \{u_\Omega > 0\}$ such that
$$u_\Omega(x_r) \geq \tilde{C}r;$$
- on the other hand, the function u_Ω is Lipschitz-continuous and so, if we set
$$C_0 := \min\left\{1, \frac{C}{[u_\Omega]_{C^{0,1}}}\right\},$$
then we get $u_\Omega > 0$ in $B_{C_0 r}(x_r)$, which proves the lower bound. The quantity $[u_\Omega]_{C^{0,1}}$ denotes the Lipschitz seminorm, which is defined as
$$[u_\Omega]_{C^{0,1}} := \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|}.$$

For the upper bound on the density, let us consider the solution h to the problem

$$\begin{cases} -\Delta h = \|f\|_{L^\infty} & \text{in } B_r \\ h = u_\Omega & \text{on } B_1 \setminus B_r. \end{cases}$$

By the maximum principle, since $-\Delta(h - u_\Omega) \geq 0$ in B_1, we have $u_\Omega \leq h$ in B_r, and so

$$\{u_\Omega > 0\} \cap B_r \subset \{h > 0\} \cap B_r.$$

Now, the outward optimality of Proposition 7.2 yields

$$\begin{aligned}\frac{C_2}{2}|B_r \cap \{u_\Omega = 0\}| &\geq E_f(u_\Omega, B_r) - E_f(h, B_r)\\ &\geq \frac{1}{2}\int_{B_r}\left[|\nabla(u_\Omega - h)|^2 + 2\nabla h\cdot\nabla(u_\Omega - h)\right]dx - \int_{B_r} f(u_\Omega - h)\,dx\\ &\geq \frac{1}{2}\int_{B_r}\left[|\nabla(u_\Omega - h)|^2 + 2(-\Delta h - f)(u_\Omega - h)\right]dx\\ &\geq \frac{1}{2}\int_{B_r}|\nabla(u_\Omega - h)|^2\,dx.\end{aligned}$$

Next, using the classical Poincaré inequality (4.12) and the Cauchy-Schwarz inequality, we obtain

$$\int_{B_r}|\nabla(u_\Omega - h)|^2\,dx \geq \frac{C_d}{|B_r|}\left(\frac{1}{r}\int_{B_r}(h - u_\Omega)\,dx\right)^2.$$

Moreover, the classical Harnack inequality gives

$$\tilde{C}r \leq \sup_{y\in B_{r/2}} u_\Omega(y) \leq \sup_{y\in B_{r/2}} h(y) \leq C_d\left[h(x) + r^2\|f\|_{L^\infty(\mathbb{R}^d)}\right]$$

for every $x \in B_{r/2}$; thus, by taking $r_0 > 0$ such that $C_d r_0\|f\|_{L^\infty(\mathbb{R}^d)} \leq \tilde{C}$, we get

$$h \geq \frac{1}{2}\frac{\tilde{C}}{C_d}r := \bar{C}r \qquad \text{in } B_{r/2}.$$

To estimate the function u_Ω, on the other hand, we exploit the fact that u_Ω Lipschitz-continuous and $u_\Omega(0) = 0$, obtaining

$$u_\Omega \leq L\epsilon r \qquad \text{in } B_{\epsilon r},$$

for some $L > 0$ that depends on d, $\|f\|_{L^\infty}$, C_1 and C_2 only. Finally, by choosing $\epsilon > 0$ small enough, for example $\bar{C} \geq 2L\epsilon$, we get

$$\int_{B_r}(h - u_\Omega)\,dx \geq \int_{B_{\epsilon r}}(h - u_\Omega)\,dx \geq L\epsilon|B_{\epsilon r}|,$$

and this concludes the proof. □

To conclude this section, we prove a technical result which will play a crucial role in the proof of the existence of a homogeneous blow-up limit.

Lemma 7.7 *Let Ω be a solution to (7.2) in B_1 and u_Ω be the associated state variable. Then, for every $x_0 \in \partial\Omega \cap B_1$ and every $r \in (0, 1 - |x_0|)$, we have*

$$|\{0 < u_\Omega < rt\} \cap B_r(x_0)| \leq Ct|B_r| \qquad \textit{for every } t \in (0,1). \tag{7.22}$$

This estimate has already been established in [38, Theorem 1.10], but we sketch the main ideas here for completeness.

Proof. Suppose that $x_0 = 0 \in \partial\Omega$ and fix $t > 0$. If we let $\eta \in C_c^\infty(B_{2r})$ be such that $\eta \in [0,1]$ and $\eta \equiv 1$ in B_r, then we can define

$$\phi(x) := \eta(u_\Omega(x) - rt)_+ + (1-\eta)u_\Omega(x).$$

Taking $\omega = \Omega$ and ϕ as a test function in the inward optimality condition of Proposition 7.2, we get

$$E_f(u_\Omega, B_{2r}) + \frac{C_1}{2}|B_{2r} \cap \Omega| \leq E_f(\phi, B_{2r}) + \frac{C_1}{2}|B_{2r} \cap \omega|.$$

Now, consider the decomposition $\Omega = \{u_\Omega > rt\} \cup \{0 < u_\Omega \le rt\}$ and notice that

$$\phi(x) = u_\Omega(x) - rt\eta \qquad \text{in } \{u_\Omega > rt\} =: \Omega_1,$$
$$\phi(x) = (1-\eta)u_\Omega(x) \qquad \text{in } \{0 < u_\Omega \le rt\} =: \Omega_2.$$

It follows that

$$\begin{aligned} E_f(\phi, B_{2r}) &= \int_{\Omega_1} \left[\frac{1}{2}|\nabla u_\Omega|^2 + \frac{(rt)^2}{2}|\nabla\eta|^2 - rt(\nabla u_\Omega \cdot \nabla\eta) - f(u_\Omega - rt\eta)\right] dx \\ &\quad \cdots + \int_{\Omega_2} \left[\frac{(1-\eta)^2}{2}|\nabla u_\Omega|^2 + \frac{u_\Omega^2}{2}|\nabla\eta|^2 - (1-\eta)u_\Omega(\nabla u_\Omega \cdot \nabla\eta) - f u_\Omega(1-\eta)\right] dx \\ &\le E_f(u_\Omega, B_{2r}) - rt\int_{\Omega_1} \nabla u_\Omega \cdot \nabla\eta \, dx + \int_{B_{2r}} \left[\frac{(rt)^2}{2}|\nabla\eta|^2 + frt\eta\right] dx \\ &\quad \cdots + \int_{\Omega_2} \left[\frac{(1-\eta)^2 - 1}{2}|\nabla u_\Omega|^2 - (1-\eta)u_\Omega(\nabla u_\Omega \cdot \nabla\eta)\right] dx \\ &\le E_f(u_\Omega, B_{2r}) + Ct\left(r[u_\Omega]_{C^{0,1}(B_{2r})} + r\|f\|_{L^\infty(\mathbb{R}^d)} + t\right)|B_r| - \int_{\Omega_2} \frac{1}{2}|\nabla u_\Omega|^2 \, dx, \end{aligned}$$

and

$$|B_{2r} \cap \{\phi > 0\}| \le |B_r \cap \Omega_1| + |B_{2r} \setminus B_r|.$$

Therefore, we obtain the desired estimate

$$\frac{1}{2}\int_{\Omega_2} |\nabla u_\Omega|^2 \, dx + \frac{C_1}{2}|B_r \cap \Omega_2| \le Ct|B_r|$$

with $C > 0$ depending only on the Lipschitz constant of u_Ω, $\|f\|_{L^\infty(\mathbb{R}^d)}$ and d. □

7.4 Compactness and convergence of blow-up sequences

This section aims to study the compactness of blow-up sequences and the main properties of blow-up limits. Indeed, they are essential for determining the local behavior of the free boundary and for the characterization of both regular and singular strata.

Definition 7.1 *Let Ω be a solution to (7.2) and let $u := u_\Omega$ and $v := v_\Omega$. If $x_0 \in \partial\Omega$ and $r_k \searrow 0^+$ is a sequence such that $B_{r_k}(x_0) \subset B_1$ for every k, we define the associated **blow-up sequence** by*

$$u_{x_0,r_k}(x) := \frac{u(x_0 + r_k x)}{r_k} \qquad \text{and} \qquad v_{x_0,r_k}(x) := \frac{v(x_0 + r_k x)}{r_k},$$

with both functions defined in a suitable rescaling of Ω, namely

$$\Omega_k := \frac{\Omega - x_0}{r_k} = \left\{\frac{x - x_0}{r_k} \ : \ x \in \Omega\right\}.$$

Throughout this section, we will always assume (a)–(c) of Theorem 4.2. The idea is to prove the compactness of the blow-up sequences and the existence of a blow-up limit, which may not be homogeneous. Next, using the stationary conditions of Subsection 7.2.2, we establish that there is a blow-up sequence converging to a homogeneous limit at every point.

7.4.1 Compactness of blow-up sequences

By Proposition 7.3, the blow-up sequence u_{x_0,r_k} is uniformly Lipschitz-continuous and locally uniformly bounded in $\mathbb{R}^d$. Moreover, by (7.5) we have the estimate

$$\int_{B_R} |\nabla u_{x_0,r_k}|^2\,dx = \frac{1}{(r_k)^d}\int_{B_{r_kR}(x_0)} |\nabla u|^2\,dx = \frac{1}{r_k^d}\int_{B_{Rr_k}(x_0)} fu\,dx \le \frac{\|f\|^2_{L^\infty(\mathbb{R}^d)}}{2d}(\omega_d R)^d \tag{7.23}$$

for every $R \le (1-|x_0|)/r_k$, and so u_{x_0,r_k} is uniformly bounded in $H^1_{\text{loc}}(\mathbb{R}^d)$.

Remark 7.4 The estimate (7.23) holds true if we replace (u,f) with (v,g) as a consequence of the fact that, by (7.6), the two state variable are interchangeable.

Remark 7.5 Applying the Ascoli-Arzelá theorem, we deduce that, up to subsequences, the blow-up sequences converge locally uniformly on every compact set to non-trivial functions

$$u_0, v_0 \in H^1_{\text{loc}}(\mathbb{R}^d) \cap C^{0,1}_{\text{loc}}(\mathbb{R}^d)$$

such that, for every $R > 0$, the following properties hold:

- $u_{x_0,r_k} \to u_0$ and $v_{x_0,r_k} \to v$ in $C^{0,\alpha}_{\text{loc}}(\overline{B_R})$ for every $\alpha \in (0,1)$;
- $u_{x_0,r_k} \rightharpoonup u_0$ and $v_{x_0,r_k} \rightharpoonup v_0$ weakly in $H^1(B_R)$.
- the set Ω_k minimizes the functional

$$\frac{1}{2}\int_{\Omega_k} |\nabla u_{x_0,r_k}|^2\,dx - r_k^2\int_{\Omega_k} f_{x_0,r_k}u_{x_0,r_k}\,dx + |\Omega_k|$$

  in the class of admissible sets $\left\{\Omega \subset B_{1/r_k}(-x_0)\ :\ \Omega \text{ quasi open},\ |\Omega| \le {}^{1}\!/_{r_k}\right\}$.

The following proposition is a compactness result for blow-up sequences associated with almost-minimizers in the sense of Proposition 7.2. We already mentioned that the state functions of optimal sets satisfy these conditions; nevertheless, we state the result in such generality to apply the blow-up analysis in Subsection 7.4.2.

Proposition 7.5 *Let $\Omega \subset \mathbb{R}^d$ be open and $u_\Omega \in H^1_0(\Omega)$ the corresponding state variable. Assume that*

(a) u_Ω is Lipschitz-continuous in $\mathbb{R}^d$, the Lipschitz constants is universal, and

$$\int_{B_r(x_0)} |\nabla u|^2\,dx \le (\omega_d r)^d\frac{\|f\|^2_{L^\infty(\mathbb{R}^d)}}{2d} \qquad \textit{for every } x_0 \in B_1 \textit{ and } r < 1-|x_0|; \tag{7.24}$$

(b) there is $\tilde{C} > 0$ universal such that, given $x_0 \in \partial\Omega$, we have

$$\sup_{x\in B_r(x_0)} u_\Omega(x) \ge \tilde{C}r \qquad \textit{for every } r \in (0,(1-|x_0|)/2);$$

(c) for every $\phi \in H^1(B_1)$ non-negative and such that $\{\phi > 0\} \supseteq \{u_\Omega > 0\}$, we have

$$E_f(u,B_1) + \frac{C_2}{2}|B_1 \cap \Omega| \le E_f(\phi,B_1) + \frac{C_2}{2}|B_1 \cap \{\phi > 0\}|.$$

Then, given $x_0 \in \partial\Omega$ and $r_k \searrow 0^+$, the following properties hold (up to subsequences):

(1) $u_{x_0,r_k} \to u_0$ locally uniformly on every compact set and $u_0 \in H^1_{\text{loc}}(\mathbb{R}^d) \cap C^{0,1}_{\text{loc}}(\mathbb{R}^d)$;

(2) $u_{x_0,r_k} \to u_0$ strongly in $H^1_{\text{loc}}(\mathbb{R}^d)$ and u_0 satisfies (7.24);

(3) the sequence of the characteristic functions converges:

$$\mathbb{1}_{\{u_{x_0,r_k}>0\}} \longrightarrow \mathbb{1}_{\{u_0>0\}} \qquad \textit{strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^d);$$

(4) for any $R>0$, we have

$$E_0(u_0,B_R) + \frac{C_2}{2}|B_R \cap \{u_0>0\}| \le E_0(\phi,B_R) + \frac{C_2}{2}|B_R \cap \{\phi>0\}|,$$

for every $\phi \in H^1(B_R)$ such that $\phi - u_0 \in H^1_0(B_R)$ and $\{\phi>0\}\cap B_R \supseteq \{u_0>0\}\cap B_R$.

Proof. Let $u_k := u_{x_0,r_k}$, $f_k := f_{x_0,r_k}$, $g_k := g_{x_0,r_k}$ and $\Omega_k := (\Omega - x_0)/r_k$. Then, if we rescale (7.24) accordingly, we get the inequality

$$\int_{B_R} |\nabla u_k|^2\,dx \le (\omega_d R)^d \frac{\|f\|^2_{L^\infty(\mathbb{R}^d)}}{2d} \qquad \text{for every } R \le R_\Omega := \frac{1-|x_0|}{r_k}.$$

By the assumptions (a) and (b), we already know that u_k converges locally uniformly on every compact set to a non-trivial function $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^d)\cap C^{0,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\mathbb{1}_{\{u_0>0\}} \le \liminf_{k\to\infty} \mathbb{1}_{\{u_k>0\}}.$$

Now, since the sequence converges weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, we need to show that the sequence u_k converges strongly in the H^1-topology to u_0.

Step 1. *Strong convergence in the H^1-topology*

Let K be a compact set and $R>0$ a radius such that $K \subset \overline{B}_R$. Moreover, let $\eta \in C^\infty_c(B_R)$ be a smooth cut-off function satisfying the following properties:

$$\eta \in [0,1] \text{ in } B_R \qquad \text{and} \qquad \eta\big|_K \equiv 1.$$

Consider $k_0 > 0$ be such that $B_R \subset \Omega_k$ for every $k > k_0$. Then, if we test the equation (7.1) with the function $\eta^2(u_k - u_0)$, we obtain

$$\int_{B_R} \nabla u_k \cdot \nabla(\eta^2(u_k-u_0))\,dx = 0 \qquad \text{for every } k > k_0.$$

It follows that

$$\int_{B_R} |\nabla(\eta u_k)|^2\,dx \le \int_{B_R} \left[\eta^2 \nabla u_k\cdot\nabla u_0 + 2u_0\eta\nabla u_k\cdot\nabla\eta + u_0^2|\nabla\eta|^2\right]dx + \int_{B_R}(u_k^2-u_0^2)|\nabla\eta|^2\,dx,$$

which implies, by using both the weak H^1-convergence and the uniform one, that

$$\limsup_{k\to\infty}\int_{B_R}|\nabla(\eta u_k)|^2\,dx \le \int_{B_R}|\nabla(\eta u_0)|^2\,dx.$$

Therefore, the sequence ηu_k converges strongly in $H^1(B_R)$, and so $u_k \to u_0$ strongly in $H^1(K)$.

Step 2. *Convergence of the characteristic functions*

Fix $R>0$ and let $\eta \in C^\infty_c(\mathbb{R}^d)$ be a cut-off function such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on B_R. Let $k \ge k_0$ with $k_0 > 0$ be such that $B_R \subset B_{1/r_k}(-x_0)$ and consider the competitor

$$\widetilde{u}_k := \eta u_0 + (1-\eta)u_k \in H^1_{\mathrm{loc}}(\mathbb{R}^d).$$

Notice that, if we set $\widetilde{\Omega}_k = \{\widetilde{u}_k > 0\}$, since $u_0 \ge 0$ and $u_k \ge 0$ in $\mathbb{R}^d$, we have $\Omega_k \subseteq \widetilde{\Omega}_k$ and, more precisely, it is easy to verify that

$$\widetilde{u}_k = u_k \text{ in } \{\eta = 0\} \qquad \text{and} \qquad \widetilde{\Omega}_k \cap \{\eta = 0\} = \Omega_k \cap \{\eta = 0\}.$$

Then, by rescaling (c), we get

$$E_{r_k^2 f_k}(u_k, B_{1/r_k}(-x_0)) + \frac{C_2}{2}|B_{1/r_k}(-x_0) \cap \Omega_k| \le E_{r_k^2 f_k}(\phi, B_{1/r_k}(-x_0)) + \frac{C_2}{2}|B_{1/r_k}(-x_0) \cap \{\phi > 0\}|$$

for every $\phi \in H^1(B_{1/r_k}(-x_0))$ non-negative and such that $\{\phi > 0\} \supseteq \{u_k > 0\}$. Therefore, by testing the inequality taking the function $\phi = \widetilde{u}_k$, we get

$$\begin{aligned}\int_{\{\eta>0\}} \left[\frac{1}{2}|\nabla u_k|^2 - r_k^2 f_k u_k\right] dx + \frac{C_2}{2}|\Omega \cap \{\eta > 0\}| \\ \le \int_{\{\eta>0\}} \left[\frac{1}{2}|\nabla \widetilde{u}_k|^2 - r_k^2 f_k \widetilde{u}_k\right] dx + \frac{C_2}{2}|\widetilde{\Omega}_k \cap \{\eta > 0\}|.\end{aligned}$$

The second term on the right-hand side can easily be estimated by

$$|\widetilde{\Omega}_k \cap \{\eta > 0\}| \le |\Omega_0 \cap \{\eta = 1\}| + |\{0 < \eta < 1\}|,$$

from which it follows that

$$\begin{aligned}\frac{C_2}{2}\big(|\Omega_k \cap \{\eta = 1\}| - |\Omega_0 \cap \{\eta = 1\}| - |\{0 < \eta < 1\}|\big) \le \frac{1}{2}\int_{\{\eta>0\}} \left[|\nabla \widetilde{u}_k|^2 - |\nabla u_k|^2\right] dx \\ \dots + r_k^2 \int_{\{\eta>0\}} f_k |\widetilde{u}_k - u_k|\, dx,\end{aligned}$$

where, on $\{\eta > 0\}$, we have

$$\begin{aligned}|\nabla u_k|^2 - |\nabla \widetilde{u}_k|^2 = (1 - (1-\eta)^2)|\nabla u_k|^2 - \eta^2|\nabla u_0|^2 - |u_0 - u_k|^2|\nabla \eta|^2 \\ \dots - 2(u_0 - u_k)\langle \nabla \eta, \eta \nabla u_0 + (1-\eta)\nabla u_k\rangle - 2\eta(1-\eta)\langle \nabla u_0, \nabla u_k\rangle.\end{aligned}$$

Since u_k converges strongly in $H^1(B_R)$ to u_0, we get

$$\limsup_{k\to\infty} \big(|\Omega_k \cap \{\eta = 1\}| - |\Omega_0 \cap \{\eta = 1\}|\big) \le |\{0 < \eta < 1\}|.$$

Since η is arbitrary outside of the ball B_R, the right-hand side can be made as small as we need, which ultimately implies the desired equality.

Step 3. *Proof of property (4)*

Let $R > 0$ and $k_0 > 0$ be such that, for every $k \ge k_0$, we have $B_R \subset B_{1/r_k}(-x_0)$. Then, we consider the competitor $\phi_k \in H^1_{loc}(\mathbb{R}^d)$, defined by

$$\phi_k := u_k + \varphi$$

for some $\varphi \in H^1_0(B_R)$, and such that $\{\phi_k > 0\} \cap B_R \supseteq \Omega_k \cap B_R$. By (c), we get

$$E_{r_k^2 f_k}(u_k, B_R) + \frac{C_2}{2}|B_R \cap \Omega_k| \le E_{r_k^2 f_k}(\phi_k, B_R) + \frac{C_2}{2}|B_R \cap \{\phi_k > 0\}|$$

for every $k \ge k_0$. Thus, using the properties (2)-(3) and the uniform Lipschitz-continuity of u_k, we pass to the limit as $k \to \infty$ and (up to subsequences) get

$$E_0(u_0, B_R) + \frac{C_2}{2}|B_R \cap \Omega_0| \le E_0(\phi_0, B_R) + \frac{C_2}{2}|B_R \cap \{\phi_0 > 0\}|,$$

where $\phi_0 - u_0 = \varphi \in H^1_0(B_R)$ and $\{\phi_0 > 0\} \supseteq \{u_0 > 0\}$. □

To conclude this section, we exploit the strong compactness of Proposition 7.5 to deduce the following characterization of blow-up limits of state variables associated to **optimal** sets.

Proposition 7.6 *Let Ω be a solution to (7.2) and $(u_{x_0,r_k})_k, (v_{x_0,r_k})_k$ the blow-up sequences associated to the state variables u_Ω and v_Ω, centered at $x_0 \in \partial\Omega$ and with $r_k \searrow 0^+$. Then the following holds:*

(1) there exists $\epsilon_0 > 0$ such that

$$\epsilon_0 \omega_d r^d \le |B_r(x_0) \cap \{u_0 > 0\}| \le (1-\epsilon_0)\omega_d r^d,$$

and

$$|\{0 < u_0 < rt\} \cap B_r(x_0)| \le Ct|B_r| \tag{7.25}$$

for every $x_0 \in \partial\Omega_0$ and every $r, t > 0$, where $\Omega_0 := \{u_0 > 0\} = \{v_0 > 0\}$;

(2) the sequence of closed sets $\overline{B_R \cap \{u_{\Omega_k} > 0\}}$ and their complements in $\mathbb{R}^d$ converge in the Hausdorff sense respectively to $\overline{B_R \cap \{u_{\Omega_0} > 0\}}$ and $\mathbb{R}^d \setminus \overline{B_R \cap \{u_{\Omega_k} > 0\}}$;

(3) the blow-up limit u_0 is non-degenerate at zero, namely there exists a constant $C_0 > 0$ such that

$$\sup_{x\in B_r} u_0(x) \ge C_0 r \qquad \text{and} \qquad \sup_{x\in B_r} v_0(x) \ge C_0 r$$

for every $r > 0$.

Moreover, the blow-up limits u_0 and v_0 are harmonic in Ω_0, i.e.,

$$\begin{cases} -\Delta u_0 = 0 & \text{in } \Omega_0 \\ -\Delta v_0 = 0 & \text{in } \Omega_0, \end{cases} \tag{7.26}$$

and are stationary, in the sense that

$$\int_{\mathbb{R}^d} \left[\operatorname{div}\xi\,(\nabla u_0 \cdot \nabla v_0 + 1) - \nabla u_0 \cdot ((D\xi)^T + (D\xi))\nabla v_0\right] dx = 0 \tag{7.27}$$

for every $\xi \in C^\infty_c(\mathbb{R}^d;\, \mathbb{R}^d)$.

Proof. First, notice that by (7.23), Proposition 7.2 and Proposition 7.3, the functions u_Ω and v_Ω fulfill the assumptions of Proposition 7.5, so we can use the properties (1)–(4) in the following.

Step 1. *Density estimate (1) and Hausdorff convergence*

By Proposition 7.4, we know that for every $k > 0$ we have

$$\epsilon_0 \omega_d r^d \le |B_r(x_0) \cap \{u_k > 0\}| \le (1-\epsilon_0)\omega_d r^d, \qquad \text{for } r < {}^{r_0}/_{r_k} \text{ and } x_0 \in \partial\Omega_k. \tag{7.28}$$

Now, it is well-known that the convergence of the sequence of characteristic functions in the strong topology of L^1, which is given by property (3) of Proposition 7.5, together with (7.28), implies the Hausdorff convergence of the sets:

$$\overline{\Omega_k \cap B_R(x_0)} \longrightarrow \overline{\Omega_0 \cap B_R(x_0)} \qquad \text{locally in } \mathbb{R}^d.$$

Obviously, the same result holds for the complements. Moreover, by rescaling (7.22), for $t > 0$ we get the inequality

$$|\{0 < u_k < rt\} \cap B_r(x_0)| \le Ct|B_r| \qquad \text{for } r < {}^{r_0}/_{r_k} \text{ and } x_0 \in \partial\Omega_k,$$

which converges to (7.25) as $k \to \infty$ by Proposition 7.5.

Step 2. *Non-degeneracy of the blow-up limit* u_0

By Proposition 7.3, for every $k > 0$, the rescaled function u_k is non-degenerate in the sense that for every $x_0 \in \Omega_k$ there holds

$$\sup_{x \in B_r(x_0)} u_k(x) \geq C_0 r \qquad \text{for every } r \leq {}^1\!/_{2r_k}.$$

Notice that this inequality is obtained by applying (7.21) in $B_{r_k r}(x_0)$ for u_Ω. Finally, by the uniform convergence of u_k and **Step 1**, for every $x_0 \in \overline{\Omega_0}$ we get

$$\sup_{x \in B_r(x_0)} u_0(x) \geq C_0 r \qquad \text{for every } r > 0,$$

concluding the proof of the non-degeneracy.

Step 3. *Harmonicity and stationary condition*

If we rescale the state equations for u_Ω and v_Ω, we get

$$\begin{cases} -\Delta u_k = r_k^2 f_k & \text{in } \Omega_k, \\ -\Delta v_k = r_k^2 g_k & \text{in } \Omega_k. \end{cases} \tag{7.29}$$

Indeed, by rescaling (7.2), we have that Ω_k minimizes the functional

$$\int_{B_{1/r_k}(-x_0)} \left[-r_k^2 g_k(x) u_k \right] \, dx + |B_{1/r_k}(-x_0) \cap \{u_k > 0\}| \tag{7.30}$$

among all quasi-open subsets of $B_{1/r_k}(-x_0)$, where u_k satisfies

$$\begin{cases} -\Delta u_k = r_k^2 f_k & \text{in } \Omega_k \\ u_k = 0 & \text{on } \partial\Omega_k. \end{cases}$$

First, by combining (7.29) with the strong convergence of Proposition 7.5, we deduce the harmonicity of the blow-up limits u_0 and v_0, that is,

$$\int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla u_0 \, dx = \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla v_0 \, dx = 0 \qquad \text{for every } \varphi \in C_c^\infty(\Omega_0).$$

On the other hand, since Ω_k minimizes (7.30), by Corollary 7.2 we get

$$\int_{\Omega_k} \left[\operatorname{div} \xi \, (\nabla u_k \cdot \nabla v_k + 1) - \nabla u_k \cdot ((D\xi)^T + (D\xi)) \nabla v_k \right] \, dx = r_k^2 \int_{\Omega_k} \left[u_k \operatorname{div}(g_k \xi) + v_k \operatorname{div}(f_k \xi) \right] \, dx$$

for every $\xi \in C_c^\infty(B_{1/r_k}(-x_0);\ \mathbb{R}^d)$. Now, let $\xi \in C_c^\infty(\mathbb{R}^d;\ \mathbb{R}^d)$ and $k_0 > 0$ such that $\operatorname{spt} \xi \subset B_{1/r_k}(-x_0)$ for every $k > k_0$. Then, taking into account that

$$|\operatorname{spt}(\xi)| \leq \omega_d r_{k_0}^{-d} \qquad \text{for every } k > k_0,$$

we get

$$\left| r_k^2 \int_{\Omega_k} \left[u_k \operatorname{div}(g_k \xi) + v_k \operatorname{div}(f_k \xi) \right] \, dx \right| \leq r_k \frac{|\Omega|^{2/d}}{d\omega_d^{2/d}} \max\{\|f\|_{C^2}, \|g\|_{C^2}\}^2 \|\xi\|_{C^1} |\operatorname{spt}(\xi)|,$$

which, in turn, implies

$$\left| \int_{\operatorname{spt}\xi} \left[\operatorname{div}\xi\,(\nabla u_k \cdot \nabla v_k + 1) - \nabla u_k \cdot ((D\xi)^T + (D\xi))\nabla v_k \right] dx \right| \le r_k C(d,\Omega,f,g)\|\xi\|_{C^1} |\operatorname{spt}(\xi)|.$$

Finally, up to a subsequences, we pass to the limit as $k \to \infty$ and, by the strong H^1-convergence on every compact set of $\mathbb{R}^d$, we deduce that (7.27) holds for every $\xi \in C_c^\infty(\mathbb{R}^d;\ \mathbb{R}^d)$. □

7.4.2 Homogeneous blow-up limit

This section aims to improve Proposition 7.6 by constructing, at every $x_0 \in \partial\Omega$, a sequence of radii $r_k \searrow 0^+$ such that the corresponding blow-up limits u_0 and v_0 are 1-homogeneous and

$$u_0 = \Lambda v_0$$

for some $\Lambda > 0$. This is obtained in Proposition 7.7, but, before we can state it, we need some preliminary results that follow from the theory developed in Chapter 5.

In the lemma below, we prove that the blow-up limits of u_Ω and v_Ω obtained in Proposition 7.6 coincide up to a multiplicative Hölder-continuous function. This follows by showing that the positivity set of u_Ω satisfies the geometric conditions of Theorem 5.1, which ensure the validity of the boundary Harnack principle (see Definition 5.1 for more details).

Lemma 7.8 *Let $\Omega \subset B_1$ be a solution to (7.2) and u_0, v_0 be two blow-up limits of the corresponding state variables u_Ω and v_Ω. Then the boundary Harnack principle (B.H.P.) holds in*

$$\Omega = \{u_0 > 0\} = \{v_0 > 0\}.$$

In other words, for every $R > 0$, the ratio u_0/v_0 can be extended to a positive $C^{0,\alpha}$-regular function on $B_R \cap \overline{\Omega}_0$.

Proof. Fix $R > 0$. The result follows once we show that the set $\Omega_0 = \{u_0 > 0\} = \{v_0 > 0\}$ satisfies the assumptions (a)–(f) of Theorem 5.1 in B_R. Indeed, by Proposition 7.6 both u_0 and v_0 are

- continuous in B_R,
- positive and harmonic in $B_R \cap \Omega_0$,
- vanishing identically on $B_R \setminus \Omega_0$,

so, by the B.H.P., the ratio u_0/v_0 can be extended to a $C^{0,\alpha}$-regular function on $B_R \cap \overline{\Omega}_0$. Now, following the notations of Theorem 5.1, let $\phi := u_0 : B_R \to \mathbb{R}$ and notice that:

(a) by definition of Ω_0, we have $\phi > 0$ in Ω_0 and $\phi \equiv 0$ on $B_R \setminus \Omega_0$; moreover, using the property (1) of Proposition 7.5 and the Ascoli-Arzelá theorem, we deduce that ϕ is Lipschitz-continuous, and the Lipschitz constant depends on d, $\|f\|_{L^\infty(\mathbb{R}^d)}$, C_1 and C_2;

(b) by property (3) of Proposition 7.6, there exists $\kappa > 0$ such that

$$\phi(x) \ge \kappa d(x, B_R \setminus \Omega_0) \qquad \text{for every } x \in B_{R/2}\,;$$

(c) since $\phi \ge 0$ and $\Delta\phi = 0$ in Ω_0, we have

$$\Delta\phi \ge 0 \qquad \text{in } \mathbb{R}^d \text{ in the sense of distributions;}$$

(d) using property (1) of Proposition 7.6, there is a constant $\mu > 0$ such that, for every $x_0 \in \partial\Omega \cap B_R$, we have

$$|B_r(x_0) \setminus \Omega| \ge \mu |B_r(x_0)| \qquad \text{for every } r \in (0, R - |x_0|)\,;$$

(e) by (7.25), there is $\Lambda > 0$ such that, for every $x_0 \in \partial\Omega_0 \cap B_R$ and every $r \in (0, R - |x_0|)$, we have

$$\big|\{0 < \phi < rt\} \cap B_r(x_0)\big| \leq \Lambda t |B_r| \qquad \text{for every } t > 0\,;$$

(f) by property (3) of Proposition 7.6, there is $\eta > 0$ such that, for every $x_0 \in \partial\Omega_0 \cap B_R$ and every $r \in (0, R - |x_0|)$, we have

$$\sup_{x \in B_R(x_0)} \phi(x) \geq \eta R.$$

Since all the geometric assumptions are fulfilled, we can apply Theorem 5.1 and conclude the proof. □

In order to prove the existence of a homogeneous blow-up limit, we need to establish a *Weiss-type monotonicity formula*. As well-known in the literature, this result usually characterizes the possible blow-up limits at free boundary points.

Lemma 7.9 *Let $\Omega \subset \mathbb{R}^d$ be an unbounded open set and $u \in H^1_{\mathrm{loc}}(\Omega)$ be a continuous non-negative function such that $u \equiv 0$ in $\mathbb{R}^d \setminus \Omega$, $u > 0$ in Ω, and*

$$\int_\Omega \left[\operatorname{div}\xi\,\left(|\nabla u|^2 + 1\right) - \nabla u \cdot \left((D\xi)^T + (D\xi)\right)\nabla u\right]\,dx = 0 \qquad \text{for every } \xi \in C^\infty_c(\mathbb{R}^d;\,\mathbb{R}^d). \tag{7.31}$$

Then, for every $x_0 \in \partial\Omega$ and $r > 0$, the map

$$(0, +\infty) \ni r \longmapsto W(u_{x_0,r}) := \int_{B_1} |\nabla u_{x_0,r}|^2\,dx - \int_{\partial B_1} |u_{x_0,r}|^2\,d\sigma$$

is non-decreasing. Moreover, there holds

$$\frac{\partial}{\partial r} W(u_{x_0,r}) \geq \frac{2}{r}\int_{\partial B_1} |x \cdot \nabla u_{x_0,r} - u_{x_0,r}|^2\,d\sigma, \tag{7.32}$$

and, as a consequence, the map is constant if and only if u is 1-homogeneous in $\mathbb{R}^d$.

Proof. Let $x_0 \in \partial\Omega$ and fix $r > 0$. By [145, Lemma 9.2], we have that

$$\frac{\partial}{\partial r} W(u_{x_0,r}) = \frac{d}{r}\left[W(z_{x_0,r}) - W(u_{x_0,r})\right] + \frac{1}{r}\int_{\partial B_1} |(x - x_0) \cdot \nabla u_{x_0,r} - u_{x_0,r}|^2\,d\sigma,$$

where $z_{x_0,r}$ is the 1-homogeneous extension of $u_{x_0,r}$ in B_1, that is,

$$z_{x_0,r}(x) := |x|\; u_{x_0,r}\left(\frac{x}{|x|}\right).$$

On the other hand, since (7.31) coincides with the stationary condition associated to local minimizers of the functional

$$\mathcal{J}(u, B_1) = \int_{B_1} |\nabla u|^2\,dx + |B_1 \cap \{u > 0\}|,$$

by [145, Lemma 9.8] we get

$$W(z_{x_0,r}) - W(u_{x_0,r}) = \frac{d}{r}\int_{\partial B_1} |(x - x_0) \cdot \nabla u_{x_0,r} - u_{x_0,r}|^2\,d\sigma,$$

which implies (7.32). □

We are now ready to state the main result, which asserts that the blow-up procedure can construct two blow-up sequences associated (for the state variables) converging to homogeneous functions by iterating three times. Moreover, these two limits are equal up to a multiplicative constant and are one-homogeneous stationary solutions to the one-phase free boundary problem.

Proposition 7.7 *Let Ω be a solution to (7.2) and u_Ω, v_Ω the associated state variables. Then, at every $x_0 \in \partial\Omega$, there exists a sequence of radii $\rho_k \searrow 0^+$ such that*

(1) u_{x_0,ρ_k} and v_{x_0,ρ_k} converge, in the sense of Proposition 7.5, to some $u_{000}, v_{000} \in H^1_{loc}(\mathbb{R}^d) \cap C^{0,1}_{loc}(\mathbb{R}^d)$;

(2) there is a universal constant $\Lambda > 0$ such that $u_{000} = \Lambda v_{000}$ in $\mathbb{R}^d$;

(3) the function u_{000} is 1-homogeneous and satisfies the property (1)-(3) of Proposition 7.6.

Moreover, the blow-up limit u_{000} satisfies the PDE

$$\begin{cases} \Delta u_{000} = 0 & \text{in } \{u_{000} > 0\} \\ u_{000} = 0 & \text{on } \partial\{u_{000} > 0\}, \end{cases}$$

and the integral condition

$$\int_{\mathbb{R}^d} \left[\operatorname{div} \xi \left(|\nabla u_{000}|^2 + 1\right) - \nabla u_{000} \cdot \left((D\xi)^T + (D\xi)\right) \nabla u_{000}\right] dx = 0 \qquad \text{for every } \xi \in C^\infty_c(\mathbb{R}^d;\, \mathbb{R}^d).$$

Proof. As mentioned above, we have to apply three times the blow-up analysis to construct the desired sequence of radii; thus, we divide the proof into four steps.

Step 1. *Analysis of the first blow-up*

By Proposition 7.5 and Proposition 7.6, there exists a sequence $r_k \searrow 0^+$ such that

$$u_{x_0,r_k}(x) = \frac{u(x_0 + r_k x)}{r_k} \longrightarrow u_0 \qquad \text{and} \qquad v_{0,r_k} = \frac{v(x_0 + r_k x)}{r_k} \longrightarrow v_0,$$

where u_0 and v_0 satisfy (7.26) and (7.27). Moreover, by Lemma 7.8, there exists $w_0 \in C^{0,\alpha}(B_1 \cap \overline{\Omega}_0)$ such that $w_0 > 0$ and

$$u_0 \equiv v_0 w_0, \qquad \text{in } B_1 \cap \overline{\Omega}_0. \tag{7.33}$$

On the other hand, by (1)-(2)-(4) of Proposition 7.5 and (3) of Proposition 7.6, we know that Ω_0 and both u_0 and v_0 satisfy the assumption of Proposition 7.5, which allows to obtain the existence of blow-up limits for the functions u_0 and v_0.

Step 2. *Analysis of the second blow-up*

By Proposition 7.5, there are a sequence $R_k \searrow 0^+$ and $u_{00}, v_{00} \in H^1_{loc}(\mathbb{R}^d) \cap C^{0,1}_{loc}(\mathbb{R}^d)$ such that

$$u_{0,R_k}(x) = \frac{1}{R_k} u_0(R_k x) \longrightarrow u_{00}, \quad v_{0,R_k} = \frac{1}{R_k} v_0(R_k x) \longrightarrow v_{00}, \quad \Omega_{0,k} = \frac{1}{R_k}\Omega_0 \longrightarrow \Omega_{00}.$$

On the other hand, by rescaling (7.33), we have

$$u_{0,R_k}(x) = w(R_k x) v_{0,R_k}(x) \qquad \text{in } B_{1/R_k} \cap \overline{\Omega}_{0,k},$$

where $w > 0$ and $w \in C^{0,\alpha}(B_1 \cap \overline{\Omega}_{0,k})$. Therefore, by passing to the limit as $k \longrightarrow +\infty$, we get

$$u_{00}(x) = w(0) v_{00}(x) \qquad \text{for every } x \in B_1 \cap \Omega_{00},$$

which implies that u_{00} and v_{00} are equal up to a positive multiplicative constant. Thus, by combining it with the rescaled version of (7.26) and (7.27), we deduce that $0 \in \partial\Omega_{00}$ and

$$\begin{cases} \Delta u_{00} = 0 & \text{in } \{u_{00} > 0\} \\ u_{00} = 0 & \text{on } \partial\{u_{00} > 0\} \end{cases}$$

and

$$\int_{\mathbb{R}^d} \left[\operatorname{div} \xi \left(|\nabla u_{00}|^2 + 1\right) - \nabla u_{00} \cdot ((D\xi)^T + (D\xi))\nabla u_{00}\right] dx = 0 \qquad \text{for every } \xi \in C_c^\infty(\mathbb{R}^d;\ \mathbb{R}^d).$$

As in the previous step, by (1)-(2)-(4) of Proposition 7.5 and (3) of Proposition 7.6, we have that Ω_{00} and both u_{00} and v_{00} satisfy the assumption of Proposition 7.5.

Step 3. *Analysis of the third blow-up*

We claim that any blow-up of u_{00} at $0 \in \partial\Omega_{00}$ is one-homogeneous. Indeed, repeating the blow-up analysis, we can find $t_k \searrow 0^+$ and $u_{000} \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap C^{0,1}_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$u_{00,t_k}(x) = \frac{u_{00}(t_k x)}{t_k} \to u_{000} \qquad \text{and} \qquad \Omega_{00,k} = \frac{1}{t_k}\Omega_0 \to \Omega_{000}.$$

By Proposition 7.5, the blow-up sequence converges strongly in the H^1_{loc}-topology and $\Omega_{00,k}$ converges locally in Hausdorff in $\mathbb{R}^d$. Therefore, arguing as in Proposition 7.6, we get

$$\sup_{B_r} u_{000} \geq C_0 r \qquad \text{for every } r > 0.$$

Similarly the estimate in (1) follows by the strong L^1-convergence of (3). Now, by **Step 2**, the function u_{00} and the set Ω_{00} fulfill the assumptions of Lemma 7.9; therefore, the map

$$(0, +\infty) \ni t \longmapsto W(u_{00,t}) = \int_{B_1} |\nabla u_{00,t}|^2\, dx - \int_{\partial B_1} u_{00,t}^2\, d\sigma$$

is non-decreasing. Notice that this implies the existence of the limit

$$W = \lim_{t \to 0^+} W(u_{00,t}) \geq 0.$$

In addition, by the strong H^1-convergence, for every $s > 0$ we have

$$W = \lim_{t_k \to 0^+} W(u_{00,t_k s}) = \frac{1}{s^d}\int_{B_s} |\nabla u_{000}|^2\, dx - \frac{1}{s^{d+1}}\int_{\partial B_s} u_{000}^2\, d\sigma = W(u_{000,s}),$$

which implies $W(u_{000,s}) = W$ in $(0, +\infty)$. By (7.32), this implies that

$$0 = \frac{\partial}{\partial s} W(u_{000,s}) \geq \frac{2}{s}\int_{\partial B_1} |x \cdot \nabla u_{000,s} - u_{000,s}|^2\, d\sigma,$$

and from this we finally deduce that the blow-up limit u_{000} is one-homogeneous in $\mathbb{R}^d$.

Step 4. *Conclusion of the proof*

The idea is to now construct the sequence $(\rho_k)_k$ with a diagonal argument. Indeed, for every $t_k > 0$, we choose $R_{t_k} > 0$ and $r_{t_k} > 0$ in such a way that the following holds:

$$\left|u_{0,R_{t_k}}(t_k x) - u_{00}(t_k x)\right| \leq 2^{-k} t_k \qquad \text{and} \qquad \left|u_{x_0,r_{t_k}}(R_{t_k} t_k x) - u_0(R_{t_k} t_k x)\right| \leq 2^{-k} R_{t_k} t_k.$$

Thus, if we consider $\rho_k = t_k R_{t_k} r_{t_k}$ as before, by the triangle inequality we get

$$\begin{aligned}
\left|u_{x_0,\rho_k} - u_{000}\right| &\leq \left|u_{x_0,r_k R_k t_k} - u_{000}\right| \\
&\leq \frac{\left|u_{x_0,r_{t_k}}(R_{t_k} t_k x) - u_0(R_{t_k} t_k x)\right|}{R_{t_k} t_k} + \frac{\left|u_{0,R_{t_k}}(t_k x) - u_{00}(t_k x)\right|}{t_k} + \left|u_{00,t_k} - u_{000}\right| \\
&\leq 2^{-k+1} + \left|u_{00,t_k} - u_{000}\right|,
\end{aligned}$$

which implies the claimed result by adapting the strategy to all the topologies involved in the compactness result of Proposition 7.5. □

7.4.3 Regular and singular parts of the free boundary

In view of the previous results, we conclude this section by defining the regular and singular parts of the free boundary.

Lemma 7.10 *Let $d = 2$ and Ω be a solution to (7.2). Then, at every point $x_0 \in \partial\Omega$, there are two blow-up limits u_0 and v_0 such that*

$$u_0(x) = \alpha(x \cdot \nu)_+ \qquad \text{and} \qquad v_0(x) = \beta(x \cdot \nu)_+,$$

where $\nu \in \mathbb{R}^d$ is a unit vector and $\alpha, \beta > 0$ constants such that $\alpha \cdot \beta = 1$.

Definition 7.2 *Let Ω be a solution to (7.2) and u_Ω, v_Ω the associated state variables. We denote with* Reg$(\partial\Omega)$ *the **regular part** of $\partial\Omega$, that is the set of all $x_0 \in \partial\Omega$ such that there are two blow-up limits u_0, v_0 satisfying*

$$u_0(x) = \alpha(x \cdot \nu)_+ \qquad \text{and} \qquad v_0(x) = \beta(x \cdot \nu)_+,$$

for some unit vector $\nu \in \mathbb{R}^d$ and $\alpha, \beta > 0$ with $\alpha \cdot \beta = 1$. Consequently, the set

$$\mathrm{Sing}(\partial\Omega) = \partial\Omega \setminus \mathrm{Reg}(\partial\Omega)$$

*is called **singular part** of $\partial\Omega$.*

7.5 Proof of Theorem 4.2 (i) : $C^{1,\alpha}$-regularity of Reg$(\partial\Omega)$

The goal of this section is to prove the first assertion of Theorem 4.2, i.e., that the regular part of $\partial\Omega$ (in the sense of Definition 7.2) is $C^{1,\alpha}$-regular for some $\alpha \in (0, 1]$.

7.5.1 Viscosity formulation

Notice that, following the same arguments of [5, 130, 131], using Corollary 7.2, it is possible to derive the following boundary optimality condition for domains with smooth boundary:

$$|\nabla u_\Omega||\nabla v_\Omega| = 1 \qquad \text{on } \partial\Omega.$$

Nevertheless, in order to extend this condition to general domains, the most appropriate formulation seems to rely on the notion of viscosity solutions; see Definition 6.1 and Definition 6.5.

Proposition 7.8 *Let Ω be solution to (7.2). Then the corresponding state variables u_Ω and v_Ω satisfy*

$$|\nabla u_\Omega||\nabla v_\Omega| = 1 \qquad \text{on } \partial\Omega \cap B_1 \tag{7.34}$$

in the viscosity sense (for more details, see Theorem 6.3 and Definition 6.5 with $Q \equiv 1$).

Proof. The state variables satisfy, by definition, the equations

$$\begin{cases} -\Delta u_\Omega = f & \text{in } \Omega \cap B_1, \\ -\Delta v_\Omega = g & \text{in } \Omega \cap B_1, \end{cases}$$

in the classical sense, so we only need to focus on the boundary condition. First, by Proposition 7.7, at every $x_0 \in \partial\Omega \cap B_1$ there exists a sequence of radii $r_k \searrow 0^+$ such that

(1) u_{x_0,r_k} and v_{x_0,r_k} converge in the sense of Proposition 7.5 to some $u_0, v_0 \in H^1_{\text{loc}}(\mathbb{R}^d) \cap C^{0,1}_{\text{loc}}(\mathbb{R}^d)$;
(2) there is a universal constant $\Lambda > 0$ such that $u_0 \equiv \Lambda v_0$ in $\mathbb{R}^d$;
(3) the functions u_0 and v_0 are 1-homogeneous solutions that satisfy (1)–(3) of Proposition 7.6.

Moreover, by Proposition 7.7 we have that u_0 satisfies the integral condition

$$\int_{\mathbb{R}^d} \left[\operatorname{div}\xi \left(|\nabla u_0|^2 + 1\right) - \nabla u_0 \cdot ((D\xi)^T + (D\xi))\nabla u_0\right] dx = 0 \qquad \text{for every } \xi \in C_c^\infty(\mathbb{R}^d;\, \mathbb{R}^d), \tag{7.35}$$

and the trace φ_0 of u_0, defined on $\mathbb{S}^{d-1}$, satisfies

$$\begin{cases} -\Delta_S \varphi_0 = (d-1)\varphi_0 & \text{in } \mathbb{S}^{d-1} \cap \{u_0 > 0\} \\ \varphi_0 > 0 & \text{in } \mathbb{S}^{d-1} \cap \{u_0 > 0\} \\ \varphi_0 = 0 & \text{on } \mathbb{S}^{d-1} \cap \partial\{u_0 > 0\}, \end{cases} \tag{7.36}$$

where $-\Delta_S$ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$; see [105] for more details. On the other hand, since Ω admits a one-side tangent ball at x_0, there exists a unit vector $\nu \in \mathbb{R}^d$ such that

$$\text{either} \quad \{x \cdot \nu > 0\} \subseteq \{u_0 > 0\} \qquad \text{or} \qquad \{u_0 > 0\} \subseteq \{x \cdot \nu > 0\}.$$

In the first case, it turns out that $\{u_0 > 0\} = \{x \cdot \nu > 0\}$. In the second case, instead, since

$$\mathcal{H}^{d-1}(\{u_0 > 0\} \cap \mathbb{S}^{d-1}) \leq d\frac{\omega_d}{2},$$

we obtain the inequality

$$\dim_{\mathcal{H}}(\{u_0 > 0\} \cap \mathbb{S}^{d-1}) \geq d - 1,$$

and the equality holds if and only if $\{u_0 > 0\} = \{x \cdot \nu > 0\}$. Thus, by the uniqueness of solution to (7.36) in the half-sphere, we get that

$$u_0(x) := \alpha\,(x \cdot \nu)_+ \qquad \text{and} \qquad v_0(x) := \beta\,(x \cdot \nu)_+$$

for some $\alpha, \beta > 0$. Finally, since $\partial\{u_0 > 0\} \cap B_1$ is smooth, we can integrate by parts (7.35) as in the proof of Lemma 7.2, and deduce that

$$\int_{\partial\{u_0>0\}} (\nu \cdot \xi)(1 - |\nabla u_0||\nabla v_0|)\, d\sigma = 0 \qquad \text{for every } \xi \in C_c^\infty(\mathbb{R}^d;\, \mathbb{R}^d),$$

which, in turn, implies the thesis:

$$\alpha\beta = |\nabla u_0||\nabla v_0| = 1 \qquad \text{on } \partial\{u_0 > 0\}.$$

□

7.5.2 Proof of the $C^{1,\alpha}$-regularity

We are finally ready to prove assertion (i) of Theorem 5.1. We achieve this by proving the following lemma, which is a consequence of the epsilon-regularity theory developed in Chapter 6.

Lemma 7.11 *Let Ω be a solution to (7.2). Then* $\mathrm{Reg}(\partial\Omega)$ *is locally the graph of a* $C^{1,\alpha}$ *function.*

Proof. Let $x_0 \in \mathrm{Reg}(\partial\Omega)$ and let u_{x_0,r_k}, v_{x_0,r_k} be the blow-up sequences of Proposition 7.7. Then, by the uniform convergence on compact sets, there are $\alpha, \beta > 0$ and a unit vector $\nu \in \mathbb{R}^d$ such that

$$\alpha\beta = 1,$$

and the following holds: for every $\epsilon > 0$ there is $k_0 > 0$ such that, for every $k \geq k_0$, we have

$$\begin{cases} \|u_{x_0,r_k} - \alpha(x\cdot\nu)_+\|_{L^\infty(B_1)} \leq \epsilon \\ \|v_{x_0,r_k} - \beta(x\cdot\nu)_+\|_{L^\infty(B_1)} \leq \epsilon \end{cases}$$

and

$$\begin{cases} u_{x_0,r_k} = v_{x_0,r_k} = 0 & \text{in } \{x\cdot\nu < -\epsilon\}, \\ u_{x_0,r_k}, v_{x_0,r_k} > 0 & \text{in } \{\epsilon < x\cdot\nu\}. \end{cases}$$

Thus, we get

$$\begin{aligned} \alpha(x\cdot\nu - 2\epsilon)_+ \leq\, & u_{x_0,r_k}(x) \leq \alpha(x\cdot\nu + 2\epsilon)_+ \quad \text{for every } x \in B_1, \\ \beta(x\cdot\nu - 2\epsilon)_+ \leq\, & v_{x_0,r_k}(x) \leq \beta(x\cdot\nu + 2\epsilon)_+ \quad \text{for every } x \in B_1, \end{aligned}$$

which means that u_{x_0,r_k} and v_{x_0,r_k} are 2ϵ-flat in the direction ν (in the sense of Definition 6.2). Moreover, if we rescale the state equations, we get

$$\begin{cases} -\Delta u_{x_0,r_k} = r_k^2 f_k & \text{in } B_1 \cap \{u_{x_0,r_k} > 0\} \\ -\Delta v_{x_0,r_k} = r_k^2 g_k & \text{in } B_1 \cap \{u_{x_0,r_k} > 0\}, \end{cases}$$

where

$$\|\Delta u_{x_0,r_k}\|_{L^\infty(B_1)} + \|\Delta v_{x_0,r_k}\|_{L^\infty(B_1)} \leq r_k \left(\|f\|_{L^\infty(B_1)} + \|g\|_{L^\infty(B_1)}\right).$$

On the other hand, since both u_{x_0,r_k} and v_{x_0,r_k} satisfy (7.34) in the viscosity sense, we apply Theorem 6.3 to obtain an integer $\bar{k} > 0$ such that

$$\partial\{u_{x_0,r_k} > 0\} \cap B_{1/2} \in C^{1,\alpha} \qquad \text{for every } k \geq \bar{k}.$$

Finally, the result follows by rescaling back to the original problem. □

7.6 Proof of Theorem 4.2 (ii) : dimension of Sing($\partial\Omega$)

The goal of this section is to prove the second assertion of Theorem 4.2, which is that the Hausdorff dimension of $\mathrm{Sing}(\partial\Omega)$, where Ω is optimal, is bounded by

$$\dim_{\mathcal{H}}(\mathrm{Sing}(\partial\Omega)) \leq d - 5$$

if $d \geq 5$, and is empty if $d \leq 4$.

7.6.1 Convergence of first and second variation

The first step consists of showing that the blow-up limits given by Proposition 7.7 are stable critical points in sense of Alf-Caffarelli - see Definition 7.3 for more details -.

Lemma 7.12 *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $f \in C^2_c(\mathbb{R}^d)$ be a non-negative function. For any compactly supported smooth vector field $\xi \in C^\infty_c(\mathbb{R}^d;\, \mathbb{R}^d)$, denote by δu the solution to*

$$\begin{cases} -\Delta(\delta u) = \mathrm{div}(\delta A \nabla u_\Omega) + \delta f & \text{in } \Omega, \\ \delta u \in H^1_0(\Omega), \end{cases} \tag{7.37}$$

where δA and δf are defined as in (7.12). If $K := \mathrm{spt}\,\xi$, then

$$\|\nabla(\delta u)\|_{L^2(\Omega)} + \|\delta u\|_{L^\infty(\Omega)} \le C_d \left(\|f\|_{L^\infty} + [u_\Omega]_{C^{0,1}}\right) |K|^{1/d} \|\xi\|_{C^2}$$

for some dimensional constant $C_d > 0$.

Proof. First, notice that δu is also given by the unique solution to the minimization problem

$$\min_{\phi \in H^1_0(\Omega)} \left\{ \int_\Omega \frac{1}{2}|\nabla \phi|^2\,dx + \int_\Omega \nabla\phi \cdot (\delta A \nabla u_\Omega + f\xi)\,dx \right\},$$

so we consider the competitor $\phi := \min\{\delta u, t\}$ for $t > 0$ and, by minimality of δu, we have

$$\int_{\Omega_t} \frac{1}{2}|\nabla(\delta u - t)_+|^2\,dx \le \int_{\Omega_t} \nabla(\delta u - t)_+ \cdot ((\delta A)\nabla u_\Omega + f\xi)\,dx,$$

where $\Omega_t = \{\delta u > t\}$. Using Young's inequality, we get

$$\begin{aligned} \int_{\Omega_t} |\nabla(\delta u - t)_+|^2\,dx &\le C \int_{\Omega_t \cap K} |(\delta A)\nabla u_\Omega + f\xi|^2\,dx \\ &\le C\|\xi\|^2_{C^2} \int_{\Omega_t \cap K} |\nabla u_\Omega|^2\,dx + C\|f\xi\|^2_{L^\infty}|\Omega_t \cap K| \\ &\le C\left(\|f\|^2_{L^\infty} + [u_\Omega]^2_{C^{0,1}}\right)|\Omega_t \cap K|\|\xi\|^2_{C^2}, \end{aligned}$$

which, by the Sobolev embedding (Theorem 4.7), leads to the following inequality:

$$\|(\delta u - t)_+\|_{L^{2^*}(\Omega_t)} \le C\left(\|f\|_{L^\infty} + [u_\Omega]_{C^{0,1}}\right)|\Omega_t \cap K|\|\xi\|_{C^2}.$$

Moreover, for any $T > t$, we have

$$\|(\delta u - t)_+\|_{L^{2^*}(\Omega_t)} \ge (T - t)|\Omega_T \cap K|^{(d-2)/(2d)},$$

so, if we consider the sequence $t_k := (1 - 2^{-k})T$ and use the inequality above, we get

$$T 2^{-k} M_{k+1}^{(d-2)/(2d)} \le C\left(\|f\|_{L^\infty} + [u_\Omega]_{C^{0,1}}\right)\|\xi\|_{C^2}\|f\|_{L^\infty(\mathbb{R}^d)} M_k^{1/2},$$

where we use the notation $M_k := |\Omega_{t_k} \cap K|$. It follows that

$$M_{k+1} \le C\left(\frac{\|f\|_{L^\infty} + [u_\Omega]_{C^{0,1}}}{T}\|f\|_{L^\infty}\|\xi\|_{C^2}\right)^{2d/(d-2)} 2^{2dk/(d-2)} M_k^{1+2/(d-2)},$$

which means that, if we choose T such that

$$C\left(\frac{\|f\|_{L^\infty}+[u_\Omega]_{C^{0,1}}}{T}\|f\|_{L^\infty}\|\xi\|_{C^2}\right)^{2d/(d-2)}=2^d|\Omega\cap K|^{-2/(d-2)}=2^dM_0^{-2/(d-2)},$$

then it is easy to verify that

$$|\Omega_T\cap K|=\lim_{k\to\infty}M_k=0.$$

If we repeat the same argument with $\max\{\delta u,t\}$ instead of $\min\{\delta u,t\}$, we conclude that

$$\|\delta u\|_{L^\infty(\Omega)}\le C_d\left(\|f\|_{L^\infty}+[u_\Omega]_{C^{0,1}}\right)|K|^{1/d}\|f\|_{L^\infty(\mathbb{R}^d)}\|\xi\|_{C^2}$$

for some $C_d>0$. Similarly, since δu solves (7.37), by Hölder's inequality we get

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla(\delta u)|^2\,dx&\le\int_K\left|(\delta A)\nabla u_\Omega+f\xi\right|^2\,dx\\&\le\|\xi\|_{C^2}^2\int_K|\nabla u_\Omega|^2\,dx+\|f\|_{L^\infty}^2\|\xi\|_{C^2}^2|K|,\end{aligned}$$

which, using (7.5), implies the desired estimate:

$$\|\nabla\delta u\|_{L^2(\Omega)}^2\le C_d\left(\|f\|_{L^\infty}^2+[u_\Omega]_{C^{0,1}}^2\right)\|\xi\|_{C^2}^2|K|.$$

□

Lemma 7.13 *Let Ω be a solution to (7.2), let $\xi\in C_c^\infty(\mathbb{R}^d;\,\mathbb{R}^d)$ be a smooth compactly supported vector field, and fix $r>0$ and $x_0\in\partial\Omega$. If the blow-up family (u_{x_0,r_k},Ω_k) converges to a blow-up limit (u_0,Ω_0) in the sense of Proposition 7.7, then the sequence $\delta u_{x_0,r_k}$ of solutions to*

$$\begin{cases}-\Delta(\delta u_{x_0,r_k})=\operatorname{div}\left(\delta A\nabla u_{x_0,r_k}+f_{x_0,r_k}\xi\right)&\text{in }\Omega_k\\ \delta u_{x_0,r_k}\in H_0^1(\Omega_k)\end{cases}$$

converges strongly in $H^1_{\rm loc}(\mathbb{R}^d)\cap\dot H^1(\mathbb{R}^d)$ to the solution $w\in H_0^1(\Omega_0)$ to

$$-\Delta w=\operatorname{div}(\delta A\nabla u_0)\qquad\text{in }\Omega_0.$$

Proof. Let $u_k=u_{x_0,r_k}$, $f_k=r_k^2f_{x_0,r_k}$ and $K=\operatorname{spt}\xi$, and denote by $\delta u_k\in H_0^1(\Omega_k)$ the solution to

$$-\Delta(\delta u_k)=\operatorname{div}(\delta A\nabla u_k)+\operatorname{div}(f_k\xi)=0\qquad\text{in }\Omega_k.\tag{7.38}$$

In other words, the function δu_k minimizes in $H_0^1(\Omega_k)$ the functional

$$G_k(\phi)=\int_{\mathbb{R}^d}\frac12|\nabla\phi|^2\,dx+\int_{\mathbb{R}^d}\nabla\phi\cdot F_k\,dx,\qquad\text{where }F_k=\delta A\nabla u_k+f_k\xi\in L^\infty(K),$$

and, integrating by parts (7.38), we get

$$G_k(\delta u_k)=\frac12\int_K\nabla\delta u_k\cdot F_k\,dx=-\frac12\int_{\mathbb{R}^d}|\nabla\delta u_k|^2\,dx.$$

Now, by Lemma 7.12, we have the estimate

$$\begin{aligned}\|\nabla\delta u_k\|_{L^2(\Omega_k)}+\|\delta u_k\|_{L^\infty(\Omega_k)}&\le C_d\left(\|f_k\|_{L^\infty}+[u_k]_{C^{0,1}}\right)|K|^{1/d}\|\xi\|_{C^2}\\&\le C_d\left(r_k^2\|f\|_{L^\infty}+[u_\Omega]_{C^{0,1}}\right)|K|^{1/d}\|\xi\|_{C^2},\end{aligned}$$

so the sequence δu_k is uniformly bounded in $H^1_{loc}(\mathbb{R}^d)$ and, up to a subsequences, we have

$$\delta u_k \to w \in H^1_{loc}(\mathbb{R}^d) \qquad \text{weakly in } H^1_{loc}(\mathbb{R}^d) \cap \dot{H}^1(\mathbb{R}^d).$$

On the other hand, by Proposition 7.7 we have $u_k \to u_0 \in H^1_{loc}(\mathbb{R}^d) \cap C^{0,1}_{loc}(\mathbb{R}^d)$ strongly in $H^1_{loc}(\mathbb{R}^d)$ and locally uniformly in $\mathbb{R}^d$. Moreover, we have

$$\Omega_k \cap K \xrightarrow{k\to+\infty} \Omega_0 \cap K \qquad \text{in the Hausdorff distance,}$$
$$\mathbb{1}_{\Omega_k} \to \mathbb{1}_{\Omega_0} \qquad \text{in } L^1(K),$$

so, using (7.38), we can find $k_0 > 0$ such that, for every $k \geq k_0$ we have

$$\int_{\mathbb{R}^d} \left[\nabla \delta u_k \cdot \nabla \phi + \nabla \phi \cdot F_k\right] dx = 0 \qquad \text{for every } \phi \in H^1_0(\Omega_0).$$

Therefore, since $f_k \to 0$, $|\nabla u_k| \to |\nabla u_0|$ strongly in $L^2_{loc}(\mathbb{R}^d)$, we take the limit $k \to \infty$ and deduce that

$$\int_{\mathbb{R}^d} \left[\nabla w \cdot \nabla \phi + \nabla \phi \cdot \delta A \nabla u_0\right] dx = 0 \qquad \text{for every } \phi \in H^1_0(\Omega_0).$$

To conclude, we need to show the strong convergence of δu_k in $\dot{H}^1(\mathbb{R}^d)$. However, since δu_k converges to w weakly in $\dot{H}^1(\mathbb{R}^d)$, it is enough to prove the convergence of the L^2-norms:

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |\nabla(\delta u_k)|^2\,dx = \int_{\mathbb{R}^d} |\nabla w|^2\,dx.$$

Since $w \in H^1_0(\Omega)$ minimizes (in $H^1_0(\Omega)$) the functional

$$G_0(\phi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla \phi|^2\,dx + \int_{\mathbb{R}^d} \nabla \phi \cdot F_0\,dx, \qquad \text{with } F_0 = \delta A \nabla u_0,$$

by a direct computation we get

$$G_k(\delta u_k) = G_0(w) + \frac{1}{2}\int_K (w - \delta u_k)\,\mathrm{div}\,F_0\,dx + \frac{1}{2}\int_K \nabla \delta u_k \cdot (F_k - F_0)\,dx.$$

It follows that

$$|G_k(\delta u_k) - G_0(w)| \leq \frac{1}{2}\|w - \delta u_k\|_{L^2(K)}\|\,\mathrm{div}\,F_0\|_{L^2(K)} + \frac{1}{2}\|\delta u_k\|_{H^1(K)}\|F_k - F_0\|_{L^2(K)}$$

and, since δu_k is uniformly bounded in $H^1(K)$, we get that $\delta u_k \to w$ strongly in $L^2(K)$ and so the previous right hand-side goes to zero, as $k \to \infty$. □

To conclude this section, we improve Proposition 7.7 by showing that at every point of $\partial\Omega$ there exists a blow-up limit which is a one-homogeneous stable critical point of the Alt-Caffarelli functional.

Proposition 7.9 *Let Ω be a solution to (7.2) and u_Ω, v_Ω be the associated state variables. Then, at every point $x_0 \in \partial\Omega$, there exists a sequence $\rho_k \searrow 0^+$ such that*

(i) u_{x_0,ρ_k} and v_{x_0,ρ_k} converge in the sense of Proposition 7.7 to some multiple of $u_0 \in H^1_{loc}(\mathbb{R}^d) \cap C^{0,1}_{loc}(\mathbb{R}^d)$;

(ii) the function u_0 is one-homogeneous and satisfies

$$\partial J_O(u_0, B_1)[\xi] = 0 \qquad \text{and} \qquad \partial^2 J_O(u_0, B_1)[\xi] \geq 0$$

for every smooth compactly supported vector field $\xi \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^d)$. In other words, it is a stable critical point in sense of Definition 7.3.

7.6.2 Stable solutions for the Alt-Caffarelli functional

In view of what we have proved in Subsection 7.2.2, Section 7.4 and Subsection 7.6.1, we give the following definition of stable critical point in sense of Alf-Caffarelli:

Definition 7.3 *Let D be an open set and let $u \in H^1(D)$ be non-negative. We say that u is a **stable critical point** for the Alt-Caffarelli functional if, for every $\xi \in C_c^\infty(\mathbb{R}^d;\ \mathbb{R}^d)$, we have*

$$\partial J_O(u, B_1)[\xi] = 0 \qquad \text{and} \qquad \partial^2 J_O(u, B_1)[\xi] \geq 0,$$

where δA and $\delta^2 A$ are given in (7.12). For more details on the first and second variation, see Lemma 7.4

Remark 7.6 Here we consider a slightly different definition of stable critical point. Indeed, we use the diffeomorphism

$$\Phi_t(x) = x + t\xi(x),$$

while in [4] they consider $\Phi_t(x) = x + \xi(t, x)$, so they only coincide at the infinitesimal level.

Proposition 7.10 *Let D be an open set in $\mathbb{R}^d$. Suppose that $u \in H^1(D)$ be a non-negative function. Then, the following are equivalent:*

(1) u is a stable critical point for the Alt-Caffarelli functional in D;

(2) u is a stable critical point of the Alt-Caffarelli functional in every bounded open set $\Omega \subset D$.

Proposition 7.11 *Blow-up limits of stable critical points are stable critical cones.*

Theorem 7.2 *The stable critical cones with isolated singuarity in zero are stable in the sense of Jerison-Savin and Caffarelli-Jerison-Kenig.*

This result follows immediately exactly as in the proof of [50, Lemma 1].

Proposition 7.12 *There is a critical dimension d^* such that the stable critical cones for Alt-Caffarelli are smooth in $\mathbb{R}^d$ if $d < d^*$, are smooth in $\mathbb{R}^d \setminus \{0\}$ if $d = d^*$ and, if $d > d^*$, $\dim(\mathrm{Sing}(\partial\Omega)) < d - d^*$.*

Remark 7.7 By [50] and [104], we know that $d^* \geq 5$. Conversely, by [72], we have $d^* \leq 7$.

7.6.3 Analysis of the dimension of the singular set

We are finally ready to prove that the second assertion of our main result Theorem 4.2.

Lemma 7.14 (Existence of points of positive density) *Let $s > 0$ and let $K \subset \mathbb{R}^d$ be a given set. If $\mathcal{H}^s(K) > 0$, then there exists $x_0 \in K$ such that*

$$\limsup_{r\to 0} \frac{\mathcal{H}^s\left(K \cap B_r(x_0)\right)}{r^s} > 0.$$

Let (u_n, v_n, Ω_n) be a blow-up sequence as in Proposition 7.9 and denote by (u_0, v_0, Ω_0) the blow-up limits. We are now interested in the connection between

$$\mathrm{Sing}(\partial\Omega_n) \qquad \text{and} \qquad \mathrm{Sing}(\partial\Omega_0)$$

The following result, proved in [145], gives us a kind of convergence and, more importantly, the upper semicontinuity condition (7.39).

Lemma 7.15 *Let $u_n : D \to \mathbb{R}$ be a sequence of continuous non-negative functions and let $\Omega_n := \{u_n > 0\}$. Suppose that the following conditions hold:*

(i) ***Uniform ϵ-regularity.*** *There are constants $\epsilon, R > 0$ such that the following holds: if $n \in \mathbb{N}$, $x_0 \in \partial\Omega_n \cap D$ and $r \in (0, R)$ are such that $B_r(x_0) \subset D$ and*

$$\|u_n - \alpha(x - x_0) \cdot \nu\|_{L^\infty(B_r(x_0))} \le \epsilon r \qquad \text{for some } \nu \in \partial B_1,$$

then $\partial\Omega_n = \mathrm{Reg}(\partial\Omega_n)$ in $B_{r/2}(x_0)$.

(ii) ***Uniform non-degeneracy.*** *There are constants $\kappa, r_0 > 0$ for which if $n \in \mathbb{N}$, $x_0 \in \partial\Omega_n \cap D$ and $r \in (0, r_0)$ are such that $B_r(x_0) \subset D$, then*

$$\|u_n\|_{L^\infty(B_r(x_0))} \ge \kappa r.$$

(iii) ***Uniform convergence.*** *The sequence u_n converges locally uniformly in D to a function $u_0 : D \to R$.*

Then, for every compact set $K \subset D$, the following holds: if $U \subset D$ is open and contains $K \cap \mathrm{Sing}\partial\Omega_0$, then there is $n_0 \in \mathbb{N}$ such that

$$\mathrm{Sing}(\partial\Omega_n) \cap K \subset U \qquad \text{for every } n \ge n_0.$$

In particular, for every $s > 0$, we have

$$\mathcal{H}^s_\infty\left(K \cap \mathrm{Sing}(\partial\Omega_0)\right) \ge \limsup_{n\to\infty} \mathcal{H}^s_\infty\left(K \cap \mathrm{Sing}(\partial\Omega_n)\right). \tag{7.39}$$

Lemma 7.16 (Blowup of one-homogeneous functions) *Let $z : \mathbb{R}^d \to \mathbb{R}$ be a one-homogeneous locally Lipschitz continuous function. Let $0 \neq x_0 \in \partial\Omega_z$, $r_n \to 0$ and*

$$z_{r_n,x_0}(x) := \frac{1}{r_n} z(x_0 + r_n x)$$

be a blowup sequence converging locally uniformly to a function $z_0 : \mathbb{R}^d \to \mathbb{R}$. Then z_0 is invariant in the direction x_0, that is,

$$z_0(x + t x_0) = z_0(x) \quad \text{for every } x, t \in \mathbb{R}. \tag{7.40}$$

This result is crucial because, by Proposition 7.9, we have that

$$u_0 = \Lambda v_0 \qquad \text{and} \qquad u_0 \text{ is one-homogeneous,}$$

so, if we consider a further blow-up, we can apply (7.40) to deduce that u_{0x_0} is invariant in one direction; therefore, if both 0 and $x_0 \neq 0$ are singular points, then

$$[0, x_0] = \{t x_0 \ : \ t \in [0, 1]\} \subset \mathrm{Sing}(\partial\{u_\Omega > 0\}),$$

and, as we see below, this obviously plays a fundamental role in determining the Hausdorff dimension of singular points.

Lemma 7.17 (Dimension reduction) *Let u_0 be a one-homogeneous functions which is a stable critical point in the sense of Definition 7.3. Then the following holds:*

$$\mathcal{H}^{d-d^*+\epsilon}(\mathrm{Sing}(\partial\{u_0 > 0\})) = 0 \qquad \text{for every } \epsilon > 0.$$

Proof of Theorem 4.2. We argue by contradiction and assume that

$$\mathcal{H}^{d-d^*+\epsilon}(\mathrm{Sing}(\partial\Omega)) > 0$$

for some $\epsilon > 0$. We can assume, up to translation, that $0 \in \mathrm{Sing}(\partial\Omega)$. Consider now the blow-up limit u_0 given by Proposition 7.7. It is one-homogeneous and, by Lemma 7.13, we have

$$u_\Omega \text{ stable critical point} \implies u_0 \text{ stable critical point}$$

in the sense of Definition 7.3. Moreover, by Lemma 7.15 we know that

$$\mathcal{H}^{d-d^*+\epsilon}(\mathrm{Sing}(\partial\Omega)) > 0 \implies \mathcal{H}^{d-d^*+\epsilon}(\mathrm{Sing}(\partial\{u_0 > 0\})) > 0,$$

so we can apply Lemma 7.14 and find a singular point $x_0 \neq 0$ for u_0 and a sequence of radii $r_n \to 0$ such that the following holds:

$$\mathcal{H}^{d-d^*+\epsilon}\left(B_{r_n}(x_0) \cap \mathrm{Sing}(\partial\{u_0 > 0\})\right) \geq \epsilon r_n^{d-5+\epsilon}.$$

Taking the blow-up sequence $u_n := (u_0)_{r_n,x_0}$, we get that

$$\mathcal{H}^{d-d^*+\epsilon}\left(B_1 \cap \mathrm{Sing}(\partial\{u_n > 0\})\right) \geq \epsilon.$$

Moreover, since u_0 is one-homogeneous, the same is true for u_{00}, which denotes the blow-up limit of u_n; therefore, we apply Lemma 7.17 and deduce that

$$\mathcal{H}^{d-d^*+\epsilon}(B_1 \cap \mathrm{Sing}(\partial\{u_{00} > 0\})) \geq \epsilon,$$

but this is a contradiction with (7.39), concluding the proof of Theorem 4.2. □

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