

PAPER • OPEN ACCESS

Parity dependent Josephson current through a helical Luttinger liquid

To cite this article: S Barbarino *et al* 2013 *New J. Phys.* **15** 085025

View the [article online](#) for updates and enhancements.

You may also like

- [Josephson current through interacting double quantum dots with spin-orbit coupling](#)
Stephanie Droste, Sabine Andergassen and Janine Splettstoesser
- [Proximity coupling in superconductor-graphene heterostructures](#)
Gil-Ho Lee and Hu-Jong Lee
- [Magnetoelectric effects in Josephson junctions](#)
I V Bobkova, A M Bobkov and M A Silaev

Parity dependent Josephson current through a helical Luttinger liquid

S Barbarino^{1,3}, R Fazio¹, M Sassetti² and F Taddei¹

¹ NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56126 Pisa, Italy

² Dipartimento di Fisica, Università di Genova and CNR-SPIN, I-14146 Genova, Italy

E-mail: simone.barbarino@sns.it

New Journal of Physics **15** (2013) 085025 (18pp)

Received 27 May 2013

Published 23 August 2013

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/15/8/085025

Abstract. We consider a superconductor–two dimensional topological insulator–superconductor junction and study how the 2π - and 4π -periodic Josephson currents are affected by the electron–electron interaction. In the long-junction limit the supercurrent can be evaluated by modeling the system as a helical Luttinger liquid coupled to superconducting reservoirs. After having introduced bosonization in the presence of the parity constraint we turn to consider the limit of perfect and poor interfaces. For transparent interfaces, where perfect Andreev reflections occur at the boundaries, the Josephson current is marginally affected by the interaction. However, if strong magnetic scatterers are present in the weak link, the situation changes dramatically. Here Coulomb interaction plays a crucial role both in low and high temperature regimes. Furthermore, a phase-shift of Josephson current can be induced by changing the direction of the magnetization of the impurity.

³ Author to whom any correspondence should be addressed.



Content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](https://creativecommons.org/licenses/by/3.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Contents

1. Introduction	2
2. Model of the superconductor–two dimensional topological insulator–superconductor system	3
2.1. Bosonization	5
2.2. Fermion parity	6
3. The Josephson current	7
3.1. Transparent interfaces	7
3.2. One impurity	8
3.3. Two impurities	12
4. Conclusions	14
Acknowledgments	14
Appendix A	15
Appendix B	16
References	18

1. Introduction

Two-dimensional topological insulators (2DTI) are characterized by a gapped bulk spectrum and gapless edge states which are robust against time-reversal invariant perturbations [1, 2]. Originally predicted for HgTe/CdTe quantum wells in [3, 4], and observed in [5], these systems have attracted a great deal of attention in the last few years because of their peculiar electronic transport properties. Edge states possess a helical nature, namely electrons have spin direction and momentum locked to each other and constitute Kramer partners.

Central for the present paper is the study of hybrid 2DTI–superconductor (S) systems, a topic which has lately gained increased interest both theoretically and experimentally. A comprehensive description of the activity in this field can be found in [6]. The proximity effect into a 2DTI has been largely investigated (see e.g. [7–10]). On the experimental side, Andreev reflection at S–2DTI interfaces has been recently observed by Du and co-workers [11]. The, not yet observed, Josephson effect through a topological insulator is expected to show spectacular features. Indeed in 2009 Fu and Kane [12] considered a Josephson junction with two s-wave superconductors connected by a weak link of length L , obtained by a single edge of a 2DTI. In the short-junction regime (i.e. when $L \ll \xi$, with ξ the Bardeen–Cooper–Schrieffer (BCS) coherence length), they showed that the S–2DTI–S junction exhibits a fractional Josephson effect [13–16] in which the current phase relation has a 4π -periodicity, rather than the standard 2π -periodicity, if the fermion parity (parity of the number of electrons in the system) is preserved. This phenomenon is related to the presence of Majorana fermions [14] at the S–2DTI interfaces. More recently, Beenakker *et al* [17] addressed the long-junction regime ($L \gg \xi$) showing that the amplitude of the 4π -periodic critical current, at zero temperature, is doubled with respect to the 2π -periodic one. The ac Josephson effect in S–2DTI–S has also been considered [18].

In all papers mentioned above, the effect of the electron–electron interaction on the parity dependent Josephson current (JC) was neglected. The aim of this paper is to study the 2π - and

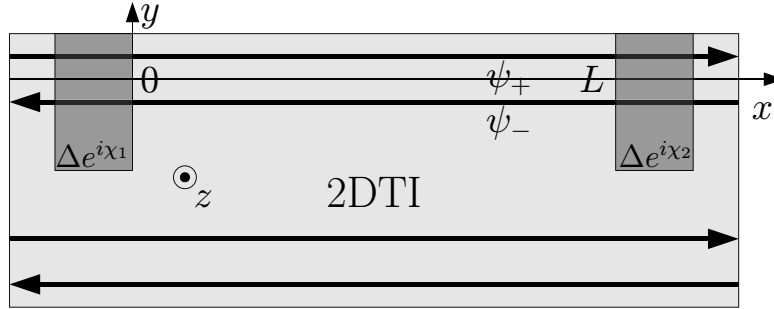


Figure 1. Helical states of a single edge of a 2DTI sandwiched between two s-wave superconductors (shaded areas).

4π -periodic Josephson effect in the long-junction regime for a S–2DTI–S system taking into account the Coulomb interaction within the framework of the helical Luttinger liquids [19, 20]. Firstly we consider the case of transparent S–2DTI interfaces and then we address the presence of magnetic impurities in the weak link [21–24]. Non-magnetic impurities cannot induce elastic back scattering in a helical liquid [25]⁴. We present analytical results for both high and low temperature regimes. If no impurities are present, at low temperature the JC exhibits a saw-tooth behavior, the 4π -periodic critical current is doubled with respect to the 2π -periodic one as in the non-interacting case. At high temperatures both the 2π - and 4π -periodic currents are sinusoidal and the 2π -periodic current is suppressed with respect to the 4π -periodic one. Our results agree with a recent paper by Crépin and Trauzettel [26]. If point-like magnetic impurities are present within the weak link the situation changes significantly. A single impurity with magnetization along the z -direction, which is the spin quantization axis of the helical states, induces a phase shift of the JC with respect to the transparent regime. Both the 2π - and 4π -periodic critical currents are not affected by such a barrier. Otherwise, when the impurity magnetization lies in an arbitrary direction of the xy -plane, the current phase relation is always sinusoidal and the critical current depends on fermion parity and on the Luttinger interaction parameter. The previous results can be generalized if two barriers are present at the 2DTI–S interfaces.

The paper is organized as follows. In section 2 we give a brief introduction to the bosonization technique with Andreev boundary conditions and discuss how the fermion parity can be implemented within the bosonization formalism. In section 3 we discuss the 2π - and 4π -periodic JC for different regimes of the system. In section 4 we summarize our results together with the conclusions of our work.

2. Model of the superconductor–two dimensional topological insulator–superconductor system

The setup we consider is depicted in figure 1. A single edge state of a 2DTI is sandwiched between two s-wave superconducting terminals. The pair potential profile is

$$\Delta(x) = \Delta e^{i\chi_1} \mathcal{H}(-x) + \Delta e^{i\chi_2} \mathcal{H}(x - L) \quad (1)$$

⁴ Non-magnetic impurities can lead to back-scattering in a helical liquid in the presence of interaction if inelastic effects are accounted for [25]. In the present work we consider elastic scattering therefore we are confined to only discuss the case of magnetic scatterers.

with χ_1, χ_2 the macroscopic phases of the superconductors, Δ the modulus of the superconductive gap and L the distance between the two superconductors. We ignore self-consistency in determining the order parameter, which is why we can safely model the space profile of Δ using the step function $\mathcal{H}(x)$.

The free Hamiltonian of the edge of the topological insulator is ($\hbar = 1$)

$$H_F = -iv_F \int_0^L dx (\psi_+^\dagger \partial_x \psi_+ - \psi_-^\dagger \partial_x \psi_-) \quad (2)$$

The operator $\psi_{+/-}$ annihilates right/left moving spin up/down electrons and v_F is the Fermi velocity. Short-range interactions between two electrons in the weak link can be analyzed in the so called *g*-ology [27, 28] framework. In a helical Luttinger liquid, one has a forward scattering term $H_{S4} = g_4/2 \int_0^L dx (\psi_+^\dagger \psi_+ \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \psi_-^\dagger \psi_-)$ and a dispersive scattering term $H_{S2} = g_2 \int_0^L dx \psi_+^\dagger \psi_+ \psi_-^\dagger \psi_-$. We have here neglected umklapp terms, which are not relevant in the renormalization-group sense [19] if interactions are not too strong.

Point-like magnetic barriers in a generic point $0 \leq \bar{x} \leq L$ of the weak link are described by the Hamiltonian

$$H_M = \Psi(\bar{x}) \vec{M} \cdot \vec{\sigma} \Psi^\dagger(\bar{x}), \quad (3)$$

where $\vec{M} = (M_x, M_y, M_z)$ is the magnetization vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices acting on the spinor space $\Psi = (e^{ik_F x} \psi_+, e^{-ik_F x} \psi_-)$, with k_F the Fermi momentum. By expanding the scalar product in equation (3), one obtains three terms: $H_{M_x} = M_x (e^{-2ik_F x} \psi_+^\dagger \psi_- + e^{2ik_F x} \psi_-^\dagger \psi_+)$, $H_{M_y} = iM_y (e^{-2ik_F x} \psi_+^\dagger \psi_- - e^{2ik_F x} \psi_-^\dagger \psi_+)$ and $H_{M_z} = M_z (\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-)$.

Finally the coupling of the edge to the superconducting electrodes, in the limit in which the superconducting gap is the largest energy scale in the problem, can be introduced through simple boundary conditions for the edge fermion field. Electrons impinging at the normal-superconductor (in this case the normal part is the edge of the topological insulator) interface are retro-reflected as holes, this is the well known Andreev reflection. By solving the Bogoliubov–de Gennes equations, one can find the boundary conditions for fermionic operators, which are essentially determined by Andreev reflections. In the limit $\Delta \rightarrow +\infty$, fermionic boundary conditions take a simple form because they are independent on the energy of the excitations (electrons can only be Andreev reflected and normal reflections do not occur), one finds [29] (at the two interfaces)

$$\psi_+(x=0) = -i e^{i\chi_1} \psi_-^\dagger(x=0), \quad (4)$$

$$\psi_+(x=L) = +i e^{i\chi_2} \psi_-^\dagger(x=L). \quad (5)$$

As shown in [29] such conditions, known as Andreev boundary conditions, are equivalent to twisted periodic boundary conditions for ψ_- on an interval of length twice the length of the original system

$$\psi_-(x+2L) = e^{i(\pi+\chi_2-\chi_1)} \psi_-(x) \quad (6)$$

supplemented by the connection between ψ_+ and ψ_- following from the chiral symmetry:

$$\psi_+(x) = -i e^{i\chi_1} \psi_-^\dagger(-x). \quad (7)$$

Boundary conditions (6) and (7) can be conveniently done in the bosonization language, which we introduce in the following sub-section.

2.1. Bosonization

Fermionic operators can be put into the form $\psi_{\pm}(x) = \exp[\pm i\Phi_{\pm}(x)]/\sqrt{2\pi a}$ in terms of the bosonic fields $\Phi_{\pm}(x)$ [27]. The Andreev boundary conditions (6) and (7) are automatically satisfied if the boson fields are chosen to be [29]

$$\begin{cases} \Phi_+(x) = \varphi - \left(N + \frac{\chi}{\pi}\right) \frac{\pi x}{2L} + \rho(-x), \\ \Phi_-(x) = \varphi + \left(N + \frac{\chi}{\pi}\right) \frac{\pi x}{2L} + \rho(x) \end{cases} \quad (8)$$

with

$$\rho(x) = i \sum_{q>0} \sqrt{\frac{\pi}{qL}} e^{-\frac{aq}{2}} (e^{-iqx} a_q^{\dagger} - e^{iqx} a_q). \quad (9)$$

In the previous equations $a \rightarrow 0^+$ is a convergence factor for the theory, $\chi \equiv \chi_2 - \chi_1$; φ and N are conjugated zero-mode operators and the bosonic operators satisfy $[a_q, a_{q'}^{\dagger}] = \delta_{q,q'}$ with $q = \pi n/L$, $n \in \mathbb{Z}$. Note that Φ_+ and Φ_- obey canonical commutation relations and $[N, \varphi] = 2i$, the eigenvalues of N are even, $N = 2k$, $k \in \mathbb{Z}$ as implied by the boundary condition (6).

It is convenient to define the bosonic fields $\Phi = (\Phi_- + \Phi_+)/2$ and $\Theta = (\Phi_- - \Phi_+)/2$, thus fermion operators take the form $\psi_{\pm}(x) = \exp[-i\Theta(x) \pm i\Phi(x)]/\sqrt{2\pi a}$ where

$$\begin{cases} \Phi(x) = \varphi + i \sum_{q>0} \sqrt{\frac{\pi}{qL}} e^{-\frac{aq}{2}} \cos qx (a_q^{\dagger} - a_q) \equiv \varphi + \phi(x), \\ \Theta(x) = \left(N + \frac{\chi}{\pi}\right) \frac{\pi x}{2L} + \sum_{q>0} \sqrt{\frac{\pi}{qL}} e^{-\frac{aq}{2}} \sin qx (a_q^{\dagger} + a_q) \equiv \left(N + \frac{\chi}{\pi}\right) \frac{\pi x}{2L} + \theta(x). \end{cases} \quad (10)$$

In terms of the bosonic operators (10) the Hamiltonian of the system $H = H_F + H_{S2} + H_{S4}$ (the superconducting electrodes enter only via the boundary conditions) is the sum of a zero mode term H_0 and of a bosonic term H_B :

$$H \equiv H_0 + H_B = \frac{ug\pi}{8L} \left(N + \frac{\chi}{\pi}\right)^2 + u \sum_{q>0} q a_q^{\dagger} a_q, \quad (11)$$

here $u = (1/2\pi) ((2\pi v_F + g_4)^2 - g_2^2)^{\frac{1}{2}}$ is the renormalized Fermi velocity and $g = ((2\pi v_F + g_4 - g_2)/(2\pi v_F + g_4 + g_2))^{\frac{1}{2}}$ is the Luttinger parameter which is related to the attractive ($g > 1$) or repulsive ($g < 1$) nature of the interaction.

In the bosonized form the Hamiltonian (3) associated to the scattering from the magnetic impurities takes the form

$$H_{Mx} = \frac{M_x}{\pi a} \cos(\varphi + \phi_{\bar{x}} + k_F \bar{x}), \quad (12)$$

$$H_{My} = \frac{M_y}{\pi a} \sin(\varphi + \phi_{\bar{x}} + k_F \bar{x}), \quad (13)$$

$$H_{Mz} = M_z \partial_x \theta|_{x=\bar{x}}. \quad (14)$$

We have defined $\phi_{\bar{x}} \equiv \phi(\bar{x})$.

2.2. Fermion parity

In order to see how fermion parity is implemented in the bosonization language let us start by noting that the superconductive phase difference χ is defined up to multiples of $2m\pi$ with $m \in \mathbb{Z}$. Consequently if we substitute $\chi \rightarrow \chi + 2n\pi$ ($n \in \mathbb{Z}$) in Φ_+ and $\chi \rightarrow \chi + 2m\pi$ ($m \in \mathbb{Z}$) in Φ_- , equations (8) become

$$\begin{cases} \Phi_+(x) = \varphi - \left(N + \frac{\chi}{\pi}\right) \frac{\pi x}{2L} + \rho(-x) - \frac{n\pi x}{L}, \\ \Phi_-(x) = \varphi + \left(N + \frac{\chi}{\pi}\right) \frac{\pi x}{2L} + \rho(x) + \frac{m\pi x}{L}. \end{cases} \quad (15)$$

Using equation (7) with χ_1 set to zero through a proper gauge transformation, we get $n = -m$ in equations (15). We obtain the periodicity requirements

$$\begin{cases} \Phi_+(x+2L) = \Phi_+(x) - \pi \left(N + \frac{\chi}{\pi}\right) + 2m\pi, \\ \Phi_-(x+2L) = \Phi_-(x) + \pi \left(N + \frac{\chi}{\pi}\right) + 2m\pi \end{cases} \quad (16)$$

satisfied by the even eigenvalues of the operator N , $N = 2k$, $k \in \mathbb{Z}$. The field $\Phi(x)$ corresponding to the periodicity requirements (16) which obeys the condition $\Phi(x+2L) = \Phi(x) + 2m\pi$, will be called parity independent in the rest of the paper. We now assume that the superconductive phase difference χ is defined up to multiples of $4m\pi$, instead of $2m\pi$, with $m \in \mathbb{Z}$. By carrying out the same procedure used for the parity independent case we obtain the analogous of equations (16):

$$\begin{cases} \Phi_+(x+2L) = \Phi_+(x) - \pi \left(N + \frac{\chi}{\pi}\right) + 4m\pi, \\ \Phi_-(x+2L) = \Phi_-(x) + \pi \left(N + \frac{\chi}{\pi}\right) + 4m\pi \end{cases} \quad (17)$$

that lead to $\Phi^{(E)}(x+2L) = \Phi^{(E)}(x) + 4m\pi$, which will be called even parity dependent [30]. Periodicity requirements given in equations (17) are satisfied if and only if the eigenvalues of the operator N take the form $N = 4k$, $k \in \mathbb{Z}$. Furthermore, by inducing an additional shift of the superconductive phase χ of 2π in equations (17) we obtain the conditions

$$\begin{cases} \Phi_+(x+2L) = \Phi_+(x) - \pi \left(N + \frac{\chi}{\pi}\right) + (4m+2)\pi, \\ \Phi_-(x+2L) = \Phi_-(x) + \pi \left(N + \frac{\chi}{\pi}\right) + (4m+2)\pi \end{cases} \quad (18)$$

that lead to $\Phi^{(O)}(x+2L) = \Phi^{(O)}(x) + (4m+2)\pi$, which will be called odd parity dependent [30]. Periodicity requirements (18) are now satisfied if and only if the eigenvalues of the operator N take the form $N = 4k+2$, $k \in \mathbb{Z}$.

We can reach the same conclusions by studying the spectrum of the Hamiltonian (11) which is unchanged by a proper shift of the superconductive phase χ . If fermion parity is not conserved, namely the Hamiltonian is invariant with respect to a shift of the form $\chi \rightarrow \chi + 2\pi m$, one obtains that $N \rightarrow N - 2m$ and concludes the eigenvalues of N must be of the form $N = 2k$. If fermion parity is conserved $\chi \rightarrow \chi + 4\pi m$, one obtains the constraint $N \rightarrow N - 4m$ or $N \rightarrow N - (4m+2)$ and concludes that the eigenvalues must be of the form $N = 4k$ in the even case and $N = 4k+2$ in the odd one. Incidentally note that the number of fermions in the weak

link is $N/2$ [31]. It is important to stress that constraints imposed by fermion parity conservation involve only the eigenvalues of the zero mode operator N and leave the bosonic excitation modes unaffected. Summarizing, the possible eigenvalues of the operator N are $N = 2k$, $k \in \mathbb{Z}$ if parity is not conserved and $N = 4k$ (even) or $N = 4k + 2$ (odd), $k \in \mathbb{Z}$ in the opposite case.

3. The Josephson current

Equipped with the definitions given above we calculate the 2π - and 4π -periodic JC for different configurations of the system. Firstly we focus on the transparent regime where perfect Andreev reflections occur in correspondence of the S-2DTI interfaces, then we introduce magnetic barriers which induce normal reflections with spin-flip. The JC can be computed from the partition function $Z(\chi)$ as

$$I_J(\chi) = -\frac{2e}{\beta} \frac{\partial \ln Z(\chi)}{\partial \chi}, \quad (19)$$

where e is the elementary charge, $\beta = 1/T$, T the temperature and the Boltzmann constant $k_B = 1$. The partition function can be expressed in the form

$$Z(\chi) = \sum_{m=-\infty}^{\infty} \int_{\Delta\varphi=2\pi m} \mathcal{D}\varphi \int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \exp[S_0 + S_B + S_M], \quad (20)$$

where $\Delta\varphi = \varphi(\beta) - \varphi(0) = 2\pi m$ and $m \in \mathbb{Z}$; S_M is the Euclidean action corresponding to the Lagrangian of magnetic barriers, S_0 and S_B to the Lagrangian of the Luttinger Hamiltonian introduced in equation (11):

$$L_B = -\sum_{q>0} a_q^\dagger \partial_\tau a_q - u \sum_{q>0} q a_q^\dagger a_q, \quad (21)$$

$$L_0 = \frac{\partial H_0}{\partial n} n - H_0 = i \frac{\chi}{\alpha\pi} \partial_\tau \varphi - \frac{2L}{\alpha^2 \pi u g} (\partial_\tau \varphi)^2, \quad (22)$$

where

$$\partial_\tau \varphi = i \frac{\partial H_0}{\partial n} = i \frac{\alpha u g \pi}{4L} \left(\alpha n + \frac{\chi}{\pi} \right). \quad (23)$$

Here $n \equiv N/\alpha$, $n \in \mathbb{Z}$ and $\alpha = 2$ if the fermion parity is not conserved, i.e. 2π -periodic case, and $\alpha = 4$ if the fermion parity is preserved, i.e. 4π -periodic even case. In the next section we will also show how to calculate the 4π -periodic odd current. Moreover, we stress again that constraints on the eigenvalues of the zero mode operator N due to the fermion parity symmetry affect the Lagrangian L_0 , as one can see from equations (22) and (23), but not L_B .

3.1. Transparent interfaces

Firstly we calculate the 2π - and 4π -periodic supercurrent in the transparent regime for both low and high temperatures. The evaluation of the partition function is straightforward because equation (20) can be put into the form $Z(\chi) = Z_0(\chi) Z_B$, with

$$Z_0(\chi) = \sum_{m=-\infty}^{\infty} \int_{\Delta\varphi=2\pi m} \mathcal{D}\varphi \exp \left[i \frac{\chi}{\alpha\pi} \int_0^\beta d\tau \partial_\tau \varphi - \frac{2L}{\alpha^2 \pi u g} \int_0^\beta d\tau (\partial_\tau \varphi)^2 \right]. \quad (24)$$

Z_B is independent on χ and does not contribute to the JC, as one can easily verify from equation (19). In order to perform the path integral in equation (24), we parameterize $\varphi(\tau)$ as $\varphi(\tau) = 2\pi m\tau/\beta + \tilde{\varphi}(\tau)$, with $\tilde{\varphi}(0) = \tilde{\varphi}(\beta)$ obtaining

$$Z_0(\chi) = \sum_{m=-\infty}^{\infty} e^{i\frac{2m\chi}{\alpha} - \frac{8\pi m^2 L}{\alpha^2 u g \beta}} \int_{\tilde{\varphi}(0)=\tilde{\varphi}(\beta)} \mathcal{D}\tilde{\varphi} \exp \left[-\frac{L}{\alpha^2 \pi u g} \int_0^\beta d\tau (\partial_\tau \tilde{\varphi})^2 \right]. \quad (25)$$

The integral in equation (25) does not contribute to the supercurrent since it is independent on χ . Using Poisson's summation formula [32] and neglecting constants which do not contribute to the JC, one has

$$Z_0(\chi) \propto \sum_{m=-\infty}^{\infty} e^{-\frac{\beta u g \pi}{8L} (\alpha m + \frac{\chi}{\pi})^2} = \frac{2}{\alpha} \sqrt{\frac{2}{Ag}} \theta_3 \left(\frac{\chi}{\alpha}, e^{-\frac{8\pi}{A\alpha^2 g}} \right), \quad (26)$$

θ_3 is the elliptic Jacobi's function and $A \equiv \beta u/L$. In the low temperature regime $A \gg 1$, the supercurrent exhibits a saw-tooth behavior

$$I_J(\chi) = \frac{e u g}{L} \frac{\chi}{2\pi}, \quad |\chi| < \frac{\alpha}{2} \pi. \quad (27)$$

As shown by Beenakker *et al* [17] in the non-interacting case, the 4π -periodic critical current is twice the 2π -periodic one even in the presence of interactions. In the high temperature regime $A \ll 1$, the JC

$$I_J(\chi) = \frac{8e}{\alpha\beta} e^{-\frac{8\pi}{A\alpha^2 g}} \sin \frac{2\chi}{\alpha} \quad (28)$$

has a sinusoidal behavior and the ratio between the 4π -periodic critical current ($\alpha = 4$) and the 2π -periodic critical one ($\alpha = 2$) is much larger than 2, since $A \ll 1$.

Let us now focus on the role played by interactions. For a generic interaction with $g_2 \neq g_4$, one has $ug = v_F + (g_4 - g_2)/(2\pi)$, i.e. the forward scattering term and the dispersive one act differently on the critical value of the JC. Conversely, for the Coulomb interaction, one has $g_2 = g_4$, as shown in [33]. In this case, the JC is completely unaffected by the Coulomb interaction because $ug = v_F$.

In the odd parity conserving case the partition function has the form

$$Z_0(\chi) \propto \sum_{m=-\infty}^{\infty} e^{-\frac{\beta u g \pi}{8L} (\alpha m + 2 + \frac{\chi}{\pi})^2} \Bigg|_{\alpha=4} = \frac{2}{\alpha} \sqrt{\frac{2}{Ag}} \theta_3 \left(\frac{\chi + 2\pi}{\alpha}, e^{-\frac{8\pi}{A\alpha^2 g}} \right) \Bigg|_{\alpha=4} \quad (29)$$

from which one can easily see that the corresponding current is equal to the even one translated of 2π .

3.2. One impurity

We start by considering a single magnetic impurity described by the Hamiltonian (14) in a generic point of the weak link whose magnetization has the same direction of the spin quantization axis, namely z -axis. The calculation proceeds similarly to the transparent regime, where L_0 is now given by

$$L_0 = \frac{\partial H_0}{\partial n} n - H_0 = i \left(\frac{\chi}{\alpha\pi} + \frac{2M_z}{\alpha u g} \right) \partial_\tau \varphi - \frac{2L}{\alpha^2 \pi u g} (\partial_\tau \varphi)^2 \quad (30)$$

with

$$\partial_\tau \varphi = i \frac{\partial H_0}{\partial n} = i \frac{\alpha u g \pi}{4L} \left(\alpha n + \frac{\chi}{\pi} \right) + i \frac{\pi \alpha}{2L} M_z. \quad (31)$$

From the partition function

$$Z_0(\chi) \propto \frac{2}{\alpha} \sqrt{\frac{2}{Ag}} \theta_3 \left(\frac{\chi}{\alpha} + \frac{2M_z \pi}{\alpha u g}, e^{-\frac{8\pi}{A\alpha^2 g}} \right), \quad (32)$$

we get the JC in the low temperature regime

$$I_J(\chi) = \frac{e u g}{L} \frac{\chi}{2\pi} + \frac{e M_z}{L}, \quad \left| \chi + \frac{2\pi M_z}{u g} \right| < \frac{\alpha}{2} \pi \quad (33)$$

and in the high temperature regime

$$I_J(\chi) = \frac{8e}{\alpha \beta} e^{-\frac{8\pi}{A\alpha^2 g}} \sin \left(\frac{2\chi}{\alpha} + \frac{4M_z \pi}{\alpha u g} \right). \quad (34)$$

The values of the critical currents remain unchanged with respect to the transparent regime found in section 3.1. The current-phase relation, however, exhibits a phase shift whose magnitude depends on M_z and, for the high-temperature regime, on the periodicity of the JC. Note that the supercurrent remains finite even at $\chi = 0$ because time reversal symmetry is broken by the magnetic barrier.

We now consider a magnetic impurity in $x = 0$, or equivalently in $x = L$, with magnetization lying in an arbitrary direction of the xy -plane. The corresponding Hamiltonian is the sum of the Hamiltonians given in equations (12) and (13):

$$H_M = H_{M_x} + H_{M_y} = \frac{M_x}{\pi a} \cos(\varphi + \phi_0) + \frac{M_y}{\pi a} \sin(\varphi + \phi_0) \quad (35)$$

that can be conveniently written as

$$H_M = \frac{|M|}{\pi a} \cos(\varphi + \phi_0 + \delta_0) \quad (36)$$

with $|M| = \sqrt{M_x^2 + M_y^2}$ and $\tan(\delta_0 - \pi/2) = M_y/M_x$, $-\pi \leq \delta_0 \leq \pi$; $\phi_0 = \phi(x = 0)$. The impurity potential is a relevant term in the renormalization group sense [34] if $g < 1$, then we consider the strong barrier limit, where the argument of the cos function in equation (36) is strongly pinned in the minima. The partition function of the system is given by equation (20), where $S_M = -\int_0^\beta d\tau H_M$ and S_0 consists of a linear term $S_{0,l}$ in $\partial_\tau \varphi$ and of a quadratic term $S_{0,q}$, as shown in equation (22). Consequently $Z(\chi)$ can be expressed in the form

$$Z(\chi) \propto Z_0 \left[1 + 2 \sum_{m=1}^{\infty} \cos \left(\frac{2m}{\alpha} \chi \right) \frac{Z_m}{Z_0} \right], \quad (37)$$

where

$$Z_m = \int_{\Delta\varphi=2\pi m} \mathcal{D}\varphi \int \mathcal{D}\phi_0 \exp [S_{0,q} + S_B^{\text{eff}} + S^\Lambda + S_M]. \quad (38)$$

Here

$$S_B^{\text{eff}} = -\frac{1}{2\pi g \beta} \sum_{\omega} \frac{\omega^2}{-\frac{u}{L} + \omega \coth \frac{L\omega}{u}} |\phi_0(\omega)|^2 \quad (39)$$

is an effective action obtained by integrating the degrees of freedom of S_B away from the impurity (more details are given in appendix A where the effective action is calculated for a generic point $\bar{x}$ in the weak link); $\omega = 2\pi n/\beta$, $n \in \mathbb{Z}$ are the Matsubara frequencies and

$$S^\Lambda = -\frac{M_0}{2} \int_0^\beta d\tau (\partial_\tau \phi_0)^2 \quad (40)$$

is a high-frequency cut-off action with $M_0 = 1/\Lambda \approx 1/\Delta$. We evaluate the partition function in the semiclassical limit simply by searching the stationary path of the action $S \equiv S_{0,q} + S^\Lambda + S_M$ which gives the most relevant contribution to the functional integral. Such a procedure is justified by the strong magnetic barrier limit. Then, we include the contributions of S_B^{eff} , which plays the role of a dissipative environment, by integrating out the low energy fluctuations of φ and ϕ_0 around the stationary path [35–38].

It is convenient to introduce the fields

$$\begin{cases} \phi_r = \varphi + \phi_0, \\ \phi_R = \frac{1}{m_L + M_0} (m_L \varphi - M_0 \phi_0), \end{cases} \quad (41)$$

where $m_L = 4L/(\alpha^2 \pi u g)$. The action S takes then the form

$$S = - \int_0^\beta d\tau \left(\frac{M_r}{2} (\partial_\tau \phi_r)^2 + \frac{M_R}{2} (\partial_\tau \phi_R)^2 + \frac{|M|}{\pi a} \cos(\phi_r + \delta_0) \right) \quad (42)$$

with $M_r = M_0 m_L / (m_L + M_0)$, $M_R = m_L + M_0$. The stationary requirement leads to

$$\begin{cases} \frac{\delta S}{\delta \phi_r} = -M_r \partial_\tau^2 \phi_r + \frac{|M|}{\pi a} \sin(\phi_r + \delta_0) = 0, \\ \frac{\delta S}{\delta \phi_R} = -M_R \partial_\tau^2 \phi_R = 0 \end{cases} \quad (43)$$

with the boundary conditions $\phi_r(\beta) - \phi_r(0) = 2\pi m$ and $\phi_R(\beta) - \phi_R(0) = 2\pi m m_L / (m_L + M_0)$. As $m_L \gg M_0$, one gets $\phi_R(\beta) - \phi_R(0) \approx 2\pi m$ and $\phi_R^{\text{st}} \approx \varphi^{\text{st}} = 2\pi m \tau / \beta + 2\pi l$, $l \in \mathbb{Z}$. The saddle point solution for $\phi_r(\tau)$ obeys a Sine–Gordon equation which admits the instanton solution

$$\phi_r^{\text{st}}(\tau) = 2 \arccos \left[-\tanh \left(\sqrt{\frac{|M|}{\pi a M_0}} \tau \right) \right] + 2\pi l_0 - \delta_0 \quad (44)$$

with $l_0 \in \mathbb{Z}$. The solution for $\phi_0^{\text{st}}(\tau)$ is

$$\phi_0^{\text{st}}(\tau) = \sum_{j=m}^J e_j \phi_r^{\text{st}}(\tau - \tau_j) - \frac{2\pi \tau}{\beta} m - 2\pi l \quad (45)$$

with $\sum_{j=m}^J e_j = m$, $J \in \mathbb{N}$ is the number of instantons. By calculating the saddle point action [39], one can show that the quantity Z_m/Z_0 is proportional to $\delta^J \ll 1$, where $\delta = \lambda^{-\frac{1}{2}} \exp(-2M_r \sqrt{\lambda})$, with $\lambda \equiv |M|/(\pi a M_0) = \Lambda |M|/(\pi a) \gg 1$. In this limit, we can take $m = 1$ with $J = 1$ and equation (45) can be approximated $\phi_0^{\text{st}}(\tau) = 2\pi \mathcal{H}(\tau - \tau_1) - 2\pi \tau / \beta - \delta_0 + 2\pi(l_0 - l)$ whose Fourier transform is

$$\phi_0^{\text{st}}(\omega) = \frac{2i\pi}{\omega} e^{i\omega \tau_1} + (\pi(2\beta l + \beta - 2\tau_1) - \delta_0 \beta) \delta_{\omega,0} \quad (46)$$

with $l \equiv l_0 - l$. Due to the strong impurity potential, φ and ϕ_0 can fluctuate under the condition that $\varphi + \phi_0$ is strongly pinned. Such low energy fluctuations can be taken into account by introducing the field ψ

$$\begin{cases} \varphi = \varphi^{\text{st}} + \psi, \\ \phi_0 = \phi_0^{\text{st}} - \psi. \end{cases} \quad (47)$$

The partition function becomes

$$\frac{Z_1}{Z_0} = \delta e^{-\frac{8\pi L}{\alpha^2 u g \beta}} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau_1 \int \mathcal{D}\psi \exp[-S_{\text{ins}}] \quad (48)$$

with

$$S_{\text{ins}} = \frac{1}{2\pi g \beta} \sum_{\omega} \left(\frac{4L\omega^2}{\alpha^2 u} |\psi(\omega)|^2 + \frac{\omega^2}{-\frac{u}{L} + \omega \coth \frac{L\omega}{u}} |\phi_0^{\text{st}}(\omega) - \psi(\omega)|^2 \right) \quad (49)$$

and integration over ψ gives

$$\frac{Z_1}{Z_0} = \delta e^{-\frac{8\pi L}{\alpha^2 u g \beta}} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau_1 \exp[-\tilde{S}_{\text{ins}}] \quad (50)$$

with

$$\tilde{S}_{\text{ins}} = \frac{2\pi}{g\beta} \sum_{\omega \neq 0} \frac{1}{\epsilon \frac{u}{L} + \omega \coth \frac{L\omega}{u}}. \quad (51)$$

In the previous expression $\epsilon = \alpha^2/4 - 1$: if $\alpha = 2$, $\epsilon = 0$; if $\alpha = 4$, $\epsilon = 3$; remarkably no contribution to $\tilde{S}_{\text{ins}}$ arises from the $\omega = 0$ term. Using the definition (19) of the JC definition one finds (more details are given in appendix B.1) for the high-temperature regime

$$I_J(\chi) = \delta \frac{8e}{\alpha} e^{-\frac{8\pi L}{\alpha^2 u g \beta}} \left(\frac{2\pi}{e^\gamma \Lambda \beta} \right)^{\frac{2}{g}} \sin \frac{2\chi}{\alpha} \quad (52)$$

which is sinusoidal both in the 2π - and 4π -periodic case. Moreover, the ratio between the 4π -periodic critical current and the 2π -periodic critical one is much larger than 1. Keeping the parity fixed, the critical current is reduced by the Coulomb interaction according to the power $2/g$. In the low temperature regime one obtains

$$I_J(\chi) = \delta \frac{8e}{2} e^{-\frac{2}{g}(\gamma + 2 \ln 2)} \left(\frac{\pi u}{\Lambda L} \right)^{\frac{2}{g}} \sin \chi \quad (53)$$

for the 2π -periodic JC and

$$I_J(\chi) = \delta \frac{8e}{4} \left(\frac{\pi u}{\Lambda L} \right)^{\frac{2}{g}} \sin \frac{\chi}{2} \quad (54)$$

for the 4π -periodic JC (even). As expected, the odd 4π -periodic current can be obtained by translating the even 4π -periodic current of 2π . In the non-interacting case the 4π -periodic current is larger than the 2π -periodic one, as in the transparent regime. The sinusoidal behavior is a direct consequence of the strong barrier limit and it is not related to helical nature of the weak link: Andreev reflections at S-2DTI interfaces are strongly suppressed with respect to normal reflections induced by magnetic barriers. Interestingly, if the repulsive interaction is strong, the 4π -periodic critical current is more robust with respect to the 2π -periodic because

it lacks the exponential term $\exp[-2g^{-1}(\gamma + 2 \ln 2)]$, thus the ratio between the 4π -periodic JC and the 2π -periodic one is bigger than 2. We note that the JC is unaffected by changing the direction of the magnetization in the xy -plane because equations (52)–(54) do not depend on δ_0 .

Finally we point out that by varying the position of the barrier, keeping fixed the direction of the magnetization in the xy -plane, different power laws are obtained. For example, if the barrier is in $L/2$ of the weak link, one obtains $4/g$ instead of $2/g$ for both high and low temperature regimes, i.e. in the middle of the junction the barrier acts stronger than at the interfaces.

3.3. Two impurities

Finally we discuss the case of two magnetic impurities at the S–2DTI interfaces, namely in $x = 0$ and L . The two magnetizations, equal in magnitude, lying in the xy -plane and not collinear, are taken into account by the Hamiltonian

$$H_M = \frac{|M|}{\pi a} [\cos(\varphi + \phi_0 + \delta_1) + \cos(\varphi + \phi_L + \delta_2)] \quad (55)$$

with $\phi_0 \equiv \phi(0, \tau)$ and $\phi_L \equiv \phi(L, \tau)$, or equivalently

$$H_M = \frac{2|M|}{\pi a} \cos(\varphi + \bar{\phi} + \bar{\delta}) \cos\left(\frac{\tilde{\phi}}{2} + \frac{\tilde{\delta}}{2}\right) \quad (56)$$

with $\bar{\phi} = 1/2(\phi_L + \phi_0)$, $\tilde{\phi} = \phi_L - \phi_0$. The parameters δ_1 and δ_2 specify the angle of magnetizations with respect to the y -axis, $\bar{\delta} \equiv (\delta_1 + \delta_2)/2$ and $\tilde{\delta} \equiv \delta_2 - \delta_1$, $k_F L$ has been chosen proportional to π . The partition function takes the form

$$Z_m = \int_{\Delta\varphi=2\pi m} \mathcal{D}\varphi \int \mathcal{D}\bar{\phi} \int \mathcal{D}\tilde{\phi} \exp[S_{0,q} + S_B^{\text{eff}} + S^\Lambda + S_M], \quad (57)$$

S_B^{eff} is the effective action obtained by integrating out the bosonic modes away from $x = 0$ and L of the Hamiltonian (56) [37]:

$$S_B^{\text{eff}} = -\frac{1}{\pi g\beta} \sum_{\omega} \bar{J}^{-1}(\omega) |\bar{\phi}(\omega)|^2 - \frac{1}{4\pi g\beta} \sum_{\omega} \tilde{J}^{-1}(\omega) |\tilde{\phi}(\omega)|^2 \quad (58)$$

with

$$\bar{J}(\omega) = \frac{1}{\omega} \coth \frac{L\omega}{2u} - \frac{2u}{L\omega^2}, \quad \tilde{J}(\omega) = \frac{1}{\omega} \tanh \frac{L\omega}{2u} \quad (59)$$

and

$$S^\Lambda = - \int_0^\beta d\tau \left(\frac{\bar{M}}{2} (\partial_\tau \bar{\phi})^2 + \frac{\tilde{M}}{2} (\partial_\tau \tilde{\phi})^2 \right) \quad (60)$$

provides the high frequency cut-off ($\bar{M} = 2/\Lambda \approx 2/\Delta$ and $\tilde{M} = 1/(2\Lambda) = 1/(2\Delta)$).

We proceed as in the single impurity problem and calculate the functional integral in the semiclassical approximation. By introducing the fields

$$\begin{cases} \phi_r = \varphi + \bar{\phi}, \\ \phi_R = \frac{1}{\bar{M} + m_L} (m_L \varphi - \bar{M} \bar{\phi}) \end{cases} \quad (61)$$

with $m_L = 4L/(\alpha^2 \pi u g)$. The saddle point for $S_{0,q} + S^\Lambda + S_M$ leads to the equations

$$\begin{cases} \frac{\delta S_{\text{st}}}{\delta \phi_r} = -M_r \partial_\tau^2 \phi_r - \frac{2|M|}{\pi a} \sin(\phi_r + \bar{\delta}) \cos\left(\frac{\tilde{\phi}}{2} + \frac{\tilde{\delta}}{2}\right) = 0, \\ \frac{\delta S_{\text{st}}}{\delta \phi_R} = -M_R \partial_\tau^2 \phi_R = 0, \\ \frac{\delta S_{\text{st}}}{\delta \tilde{\phi}} = -\tilde{M} \partial_\tau^2 \tilde{\phi} - \frac{|M|}{\pi a} \cos(\phi_r + \bar{\delta}) \sin\left(\frac{\tilde{\phi}}{2} + \frac{\tilde{\delta}}{2}\right) = 0 \end{cases} \quad (62)$$

with the boundary conditions $\phi_R(\beta) = \phi_R(0) + 2m\pi m_L/(m_L + \bar{M})$ and $\phi_r(\beta) = \phi_r(0) + 2\pi m$, where $M_r = m_L \bar{M}/(m_L + \bar{M})$ and $M_R = m_L + \bar{M}$. In the strong barrier limit it is sufficient to consider only $m = 1$. The instanton solutions take the form

$$\begin{cases} \bar{\phi}^{\text{st}}(\tau) = \pi \mathcal{H}(\tau - \tau_1) + \pi \mathcal{H}(\tau - \tau_2) + \pi(l_1 + l_2) - \frac{2\pi\tau}{\beta} - 2\pi l - \bar{\delta}, \\ \tilde{\phi}^{\text{st}}(\tau) = 2\pi \mathcal{H}(\tau - \tau_1) - 2\pi \mathcal{H}(\tau - \tau_2) + 2\pi(l_2 - l_1) - \tilde{\delta} \end{cases} \quad (63)$$

whose Fourier transform are

$$\begin{cases} \bar{\phi}^{\text{st}}(\omega) = \frac{\pi i}{\omega} (e^{i\omega\tau_1} + e^{i\omega\tau_2}) + (\pi \bar{l}\beta + \pi\beta - \pi\tau_1 - \pi\tau_2 - \bar{\delta}\beta) \delta_{\omega,0}, \\ \tilde{\phi}^{\text{st}}(\omega) = \frac{2\pi i}{\omega} (e^{i\omega\tau_1} - e^{i\omega\tau_2}) + (2\pi \tilde{l}\beta + 2\pi\tau_2 - 2\pi\tau_1 - \tilde{\delta}\beta) \delta_{\omega,0} \end{cases} \quad (64)$$

with $\tilde{l} = l_1 + l_2 - 2l$ and $\bar{l} = l_1 - l_2$. We substitute $\bar{J}(\omega)$ and $\tilde{J}(\omega)$ in the action $S_{0,q} + S_B^{\text{eff}}$ and we take into account the low energy fluctuations of $\varphi + \bar{\phi}$ as in the single impurity problem by introducing the auxiliary fields ψ , while $\tilde{\phi}$ is kept strongly pinned. Integration over ψ gives two contributions

$$\tilde{S}_{\text{ins},\omega} = \frac{2\pi}{g\beta} \sum_{\omega \neq 0} \left(\frac{2}{\epsilon \frac{u}{L} + \omega \coth \frac{L\omega}{2u}} + \frac{\epsilon \frac{u}{L} + \frac{2\omega}{\sinh \frac{L\omega}{u}}}{\omega^2 + \epsilon \omega \frac{u}{L} \tanh \frac{L\omega}{2u}} (1 - \cos \omega\tau) \right) \quad (65)$$

with $\tau \equiv \tau_2 - \tau_1$, $\epsilon \equiv \alpha^2/2 - 2$, for $\alpha = 2$ one has $\epsilon = 0$; while $\epsilon = 6$ for $\alpha = 4$; and

$$\tilde{S}_{\text{ins},0}^{\tilde{l}} = \frac{1}{4\pi g\beta} \left(\frac{2u}{L} \right) (2\pi\tau + 2\pi\tilde{l}\beta - \tilde{\delta}\beta)^2. \quad (66)$$

Integration over τ_1 and τ_2 or equivalently over $\tau = \tau_2 - \tau_1$ and $\tau' = (\tau_1 + \tau_2)/2$ gives

$$\frac{Z_1}{Z_0} = \delta e^{-\frac{8\pi L}{\alpha^2 u g \beta}} \beta \frac{\int_{-\beta}^{\beta} d\tau \exp[-\tilde{S}_{\text{ins},\omega}] \sum_{\tilde{l}=-\infty}^{+\infty} \exp[-\tilde{S}_{\text{ins},0}^{\tilde{l}}]}{\sum_{\tilde{l}=-\infty}^{+\infty} \exp[-\tilde{S}_{\text{ins},0}^{\tilde{l}}]} \Big|_{\tau=0} \quad (67)$$

with

$$\sum_{\tilde{l}=-\infty}^{+\infty} \exp[-\tilde{S}_{\text{ins},0}^{\tilde{l}}] = \sqrt{\frac{gL}{2\beta u}} \theta_3 \left(\frac{\pi\tau}{\beta} - \frac{\tilde{\delta}}{2}, e^{-\frac{\pi g L}{2\beta u}} \right). \quad (68)$$

From equation (19), in the high-temperature regime, the supercurrent (more details on the calculations are given in appendix B.2)

$$I_J(\chi) = \delta \frac{8e}{\alpha} e^{-\frac{8\pi L}{\alpha^2 u g \beta}} \left(\frac{2\pi}{e^\gamma \Lambda \beta} \right)^{\frac{4}{g}} \beta \sin \frac{2\chi}{\alpha} \quad (69)$$

exhibits a power law in β whose exponent depends on the strength of the interactions and it is unaffected by the angle $\tilde{\delta}$ between the two magnetizations. In the opposite limit of low temperatures the 2π -periodic JC

$$I_J(\chi) = \delta \frac{8e}{2} e^{-\frac{4}{g}(\gamma+2\ln 2)} \left(\frac{2\pi u}{\Lambda L} \right)^{\frac{4}{g}} \frac{2L}{\pi u} 4^{\frac{2}{g}-1} \eta_2(\tilde{\delta}, g) \sin \chi \quad (70)$$

shows a dependance on the angle $\tilde{\delta}$ between the two magnetizations through the modulation function

$$\eta_2(\tilde{\delta}, g) = \frac{\Gamma\left(\frac{2\pi+\tilde{\delta}}{\pi g}\right) \Gamma\left(\frac{2\pi-\tilde{\delta}}{\pi g}\right)}{\Gamma\left(\frac{4}{g}\right)} = \eta_2(-\tilde{\delta}, g) \quad (71)$$

with $\Gamma(x)$ the Euler–Gamma function. The modulating function has a minimum in $\tilde{\delta} = 0$, i.e. the two magnetizations are parallel and maxima in $\tilde{\delta} = \pm\pi$, i.e. the two magnetizations are anti-parallel and exhibits a weak dependance on g .

The 4π -periodic JC is given by

$$I_J(\chi) = \delta \frac{8e}{4} \left(\frac{2\pi u}{\Lambda L} \right)^{\frac{4}{g}} \frac{gL}{2u} 4^{\frac{2}{g}} \eta_4(\tilde{\delta}, g) \sin \frac{\chi}{2}, \quad (72)$$

where the modulation function $\eta_4(\tilde{\delta}, g)$ can be expressed in terms of the Gaussian hypergeometric function as

$$\eta_4(\tilde{\delta}, g) = \sum_{s=\pm 1} \frac{{}_2F_1\left(\frac{4}{g}, \frac{5}{g} + \frac{s\tilde{\delta}}{\pi g}; 1 + \frac{5}{g} + \frac{s\tilde{\delta}}{\pi g}; -1\right)}{5\pi + s\tilde{\delta}} = \eta_4(-\tilde{\delta}, g) \quad (73)$$

which has a minimum in $\tilde{\delta} = 0$ and maxima in $\tilde{\delta} = \pm\pi$.

4. Conclusions

In this paper we studied the parity-dependent JC in a S–2DTI–S junction if $L \gg \xi$ taking into account the Coulomb interaction. For transparent S–2DTI interfaces no significant corrections arise with respect to the non-interacting case. When a single magnetic impurity whose magnetization is collinear with the spin quantization z -axis, the JC is only shifted with respect to the transparent regime. If the magnetization lies in the xy -plane, the current is sinusoidal and is strongly renormalized by the interaction. In particular, for a single barrier at the S–2DTI interface, the current is proportional to $\beta^{-\frac{2}{g}}$ in the high temperature regime and to $L^{-\frac{2}{g}}$ in the low temperature regime. The 2π -periodic critical current is more suppressed by repulsive interactions with respect to the 4π -periodic. If two impurities are present at the S–2DTI interfaces new power laws are obtained. Remarkably in the low temperature regime both the 2π - and 4π -periodic currents exhibit a dependance on the angle between the two magnetizations.

Acknowledgments

We thank B Trauzettel and F Crépin for useful comments on the manuscript. This work has been supported by the EU FP7 Programme under grant agreement no. 234970-NANOCTM, IP-SIQS

and by MIUR under project PRIN ‘Collective quantum phenomena: from strongly correlated systems to quantum simulators’.

Appendix A

In this appendix we calculate the effective bosonic action S_B^{eff} in the single impurity problem. Let us consider the Hamiltonian H_B defined in equation (11) which can be written as

$$H_B = \frac{u}{2} \int_0^L dx \left(\frac{1}{g} (\partial_x \phi)^2 + g (\partial_x \theta)^2 \right), \quad (\text{A.1})$$

where ϕ and θ are the bosonic modes of the fields Φ and Θ and the corresponding Lagrangian

$$L_B = \frac{\partial H_B}{\partial \partial_x \theta} \partial_x \theta - H_B = -\frac{1}{2g} \int_0^L dx \left(u (\partial_x \phi)^2 + \frac{1}{u} (\partial_x \theta)^2 \right). \quad (\text{A.2})$$

If L_M is the Lagrangian of a magnetic impurity in $\bar{x}$, the functional integral $\int \mathcal{D}\phi \exp \left[\int_0^\beta d\tau (L(\tau) + L_M(\tau)) \right]$ can be simplified by integrating out the degrees of freedom not involved by the Lagrangian L_M [34, 38]:

$$\int \mathcal{D}\phi_{\bar{x}} \mathcal{D}\lambda \mathcal{D}\phi \exp \left[\int_0^\beta d\tau (L_0(\tau) + L_M(\tau)) - i\lambda(\tau) (\phi_{\bar{x}}(\tau) - \phi(\bar{x}, \tau)) \right], \quad (\text{A.3})$$

where we have introduced the auxiliary fields $\lambda(\tau)$ and $\phi_{\bar{x}}(\tau)$ whose Fourier series are $\lambda(\tau) = 1/\beta \sum_\omega \lambda(\omega) e^{-i\omega\tau}$ and $\phi_{\bar{x}}(\tau) = 1/\beta \sum_\omega \phi_{\bar{x}}(\omega) e^{-i\omega\tau}$. Substituting in (A.3), we get

$$\int \mathcal{D}\phi_{\bar{x}} \int \mathcal{D}\lambda \exp \left[-\frac{i}{\beta} \sum_\omega \lambda(\omega) \phi_{\bar{x}}(\omega) \right] \int \mathcal{D}\eta_q \exp \left[\frac{1}{\beta} \sum_\omega \sum_{q>0} \left(\frac{1}{4\pi g} \left(uq + \frac{\omega^2}{uq} \right) |\eta_q(\omega)|^2 - \sqrt{\frac{\pi}{Lq}} \lambda(\omega) \eta_q(-\omega) \cos q\bar{x} \right) \right] \quad (\text{A.4})$$

with $\eta_q(\tau) = a_q^\dagger(\tau) - a_q(\tau)$, $q = \frac{\pi n}{L}$, $n \in \mathbb{Z}$, which can be integrated over η_q and then over the auxiliary field λ , obtaining

$$\int \mathcal{D}\phi_{\bar{x}} \exp \left[-\frac{L}{4\pi u g \beta} \sum_\omega \frac{\omega^2}{\sum_{q>0} \frac{\cos^2 q\bar{x}}{1 + \frac{u^2}{\omega^2} q^2}} |\phi_{\bar{x}}(\omega)|^2 \right]. \quad (\text{A.5})$$

As the sum on q can be exactly evaluated for an arbitrary $\bar{x}$

$$\frac{\pi}{L} \sum_{q>0} \frac{\cos^2 q\bar{x}}{1 + \frac{u^2}{\omega^2} q^2} = -\frac{\pi}{2L} + \frac{\pi\omega}{4u} \left(1 + \frac{\cosh \frac{L\omega}{u} (1 - \frac{2\bar{x}}{L})}{\cosh \frac{L\omega}{u}} \right) \coth \frac{L\omega}{u} \quad (\text{A.6})$$

we derive the effective action S_B^{eff}

$$S_B^{\text{eff}} = -\frac{1}{2\pi g \beta} \sum_\omega \frac{\omega^2}{-\frac{u}{L} + \frac{\omega}{2} \left(1 + \frac{\cosh \frac{L\omega}{u} (1 - \frac{2\bar{x}}{L})}{\cosh \frac{L\omega}{u}} \right) \coth \frac{L\omega}{u}} |\phi_{\bar{x}}(\omega)|^2 \quad (\text{A.7})$$

which exhibits a gap of the order u/L if $\omega \rightarrow 0$ as a consequence of the finite size L of the junction.

Appendix B

In this appendix some details about the calculation of the partition function Z are given.

B.1. Single impurity problem

We consider the single impurity problem.

If the adimensional parameter $A = \beta u/L$ is introduced, the action (51) takes the form

$$\tilde{S}_{\text{ins}} = \frac{4\pi}{g} \sum_{n>0}^{n_{\text{max}}} \frac{1}{\epsilon A + 2\pi n \coth \frac{2\pi n}{A}}, \quad (\text{B.1})$$

where $n_{\text{max}} = \frac{\beta\Lambda}{2\pi}$ corresponds to the cut-off frequency Λ . In the high temperature limit $A \ll 1$ and $\coth(2\pi n/A) \approx 1 + \mathcal{O}(\exp[-2\pi n/A])$, one obtains

$$\tilde{S}_{\text{ins}} \approx \frac{2}{g} \ln \left(\frac{e^\gamma \beta \Lambda}{2\pi} \right) + \mathcal{O}(A), \quad (\text{B.2})$$

where γ is the Euler–Mascheroni constant. In the low temperature limit $A \gg 1$, equation (B.1) can be written as

$$\tilde{S}_{\text{ins}} = \frac{2}{g} \int_0^{\frac{L\Lambda}{u}} dx \frac{1}{x \coth x + \epsilon} \quad (\text{B.3})$$

with $x = L\omega/u$. If $\epsilon = 0$, namely $\alpha = 2$, the integral (B.3) can be exactly evaluated

$$\tilde{S}_{\text{ins}} = \frac{2}{g} \int_0^{\frac{L\Lambda}{u}} dx \frac{\tanh x}{x} = \frac{2}{g} \ln \left(\frac{L\Lambda}{\pi u} \right) - \frac{2}{g} \psi \left(\frac{1}{2} \right), \quad (\text{B.4})$$

where ψ is the digamma function [32]. If $\epsilon = 3$, namely $\alpha = 4$, the integral (B.3) cannot be exactly evaluated, but one gets

$$\begin{aligned} \tilde{S}_{\text{ins}} - \frac{2}{g} \left[\int_0^{\frac{L\Lambda}{u}} dx \frac{\tanh x}{x} \right] &= \frac{2}{g} \left[\int_0^{+\infty} \left(-\frac{\tanh x}{x} + \frac{1}{x \coth x + 3} \right) dx \right] \\ &\approx \frac{-3.94}{g} \end{aligned} \quad (\text{B.5})$$

and the second integral can be numerically solved. Finally one gets $2g^{-1} \ln[L\Lambda/(\pi u)] - 2g^{-1} \psi(1/2) - 3.94g^{-1} \approx 2g^{-1} \ln[L\Lambda/(\pi u)]$, because $2\psi(1/2) = -2\gamma - 4 \ln 2 \approx -3.92$. The calculation of the partition function follows straightforwardly from (48) because $\tilde{S}_{\text{ins}}$ does not depend on τ_1 .

B.2. Double impurity problem

We focus on the double impurity problem. The total action $\tilde{S}_{\text{ins}}$ has two contributions given by equations (65) and (66). Firstly we consider $\tilde{S}_{\text{ins},\omega}$, namely equation (65), which can be written

as $\tilde{S}_{\text{ins},\omega} \equiv \tilde{S}_{\text{ins1},\omega} + \tilde{S}_{\text{ins2},\omega}$. In terms of the adimensional parameter A one has

$$\tilde{S}_{\text{ins1},\omega} = \frac{8\pi}{g} \sum_{n>0}^{n_{\text{max}}} \frac{1}{\epsilon A + 2\pi n \coth \frac{\pi n}{A}}, \quad (\text{B.6})$$

$$\tilde{S}_{\text{ins2},\omega} = \frac{4\pi}{g} \sum_{n>0}^{n_{\text{max}}} \frac{\frac{\epsilon A}{2\pi n} + \frac{2}{\sinh \frac{2\pi n}{A}}}{2\pi n + \epsilon A \tanh \frac{\pi n}{A}} \left(1 - \cos \frac{2\pi n \tau}{\beta}\right). \quad (\text{B.7})$$

In the high temperature limit $A \ll 1$ the dominant contribution to $\tilde{S}_{\text{ins},\omega}$ is given by equation (B.6):

$$\tilde{S}_{\text{ins1},\omega} \approx \frac{4}{g} \ln \left(\frac{e^\nu \beta \Lambda}{2\pi} \right) + \mathcal{O}(A) \quad (\text{B.8})$$

while $\tilde{S}_{\text{ins2},\omega} \propto \mathcal{O}(A)$. Moreover $\sum_{\tilde{l}=-\infty}^{+\infty} \exp[-\tilde{S}_{\text{ins},0}^{\tilde{l}}] \approx \sqrt{g/A}$ is independent on τ and $\tilde{\delta}$. In the low temperature limit $A \gg 1$, $\tilde{S}_{\text{ins1},\omega}$ can be exactly evaluated as in the single barrier case if $\epsilon = 0$ and one obtains $\tilde{S}_{\text{ins1},\omega} = 4g^{-1} \ln[L\Lambda/(2\pi u)] - 4g^{-1} \psi(1/2)$, while if $\epsilon = 6$ one has the approximated solution $\tilde{S}_{\text{ins1},\omega} = 4g^{-1} \ln[L\Lambda/(2\pi u)]$.

Let us now consider the contributions to the partition function arising from $\tilde{S}_{\text{ins2},\omega}$. In this case one has to evaluate the integral (67) which can be cast in the form

$$e^{\frac{\beta u \tilde{\delta}^2}{2\pi g L}} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau e^{-\frac{\beta u}{2\pi g L} \left(\frac{2\pi \tau}{\beta} - \tilde{\delta}\right)^2} \exp[-\tilde{S}_{\text{ins2},\omega}] \quad (\text{B.9})$$

because $\sum_{\tilde{l}=-\infty}^{+\infty} \exp[-\tilde{S}_{\text{ins},0}^{\tilde{l}}] \approx \exp[-\frac{\beta u}{2\pi g L} (\frac{2\pi \tau}{\beta} - \tilde{\delta})^2]$.

If $\epsilon = 0$, the action $\tilde{S}_{\text{ins2},\omega}$ can be analytically evaluated [32] as

$$\tilde{S}_{\text{ins2},\omega} = \frac{4}{g} \int_0^{+\infty} dx \frac{1 - \cos \frac{xu\tau}{L}}{x \sinh x} = \frac{4}{g} \ln \cosh \left(\frac{\pi u \tau}{L} \right) \quad (\text{B.10})$$

and from equation (B.9) with $x = \pi u \tau / L$ one finally gets

$$\frac{2L}{\pi u} \int_0^{+\infty} dx \cosh \left(\frac{2x\tilde{\delta}}{\pi g} \right) \cosh^{-\frac{4}{g}} x = \frac{2L}{\pi u} 4^{\frac{2}{g}-1} \frac{\Gamma\left(\frac{2\pi+\tilde{\delta}}{\pi g}\right) \Gamma\left(\frac{2\pi-\tilde{\delta}}{\pi g}\right)}{\Gamma\left(\frac{4}{g}\right)}. \quad (\text{B.11})$$

If $\epsilon = 6$, $\tilde{S}_{\text{ins2},\omega}$ cannot be exactly evaluated. However, one can see from equation (65) that dominant corrections to $\tilde{S}_{\text{ins2},\omega}$ with respect to the case $\epsilon = 0$, are given by

$$\frac{2\epsilon u}{Lg} \int_0^{+\infty} d\omega \frac{1 - \cos \omega \tau}{\omega^2} = \frac{\epsilon u \pi |\tau|}{Lg} \quad (\text{B.12})$$

and from equation (B.9) with $x = \pi u \tau / L$, one obtains

$$\frac{2L}{\pi u} \int_0^{+\infty} dx e^{-\frac{6x}{g}} \cosh \left(\frac{2x\tilde{\delta}}{\pi g} \right) \cosh^{-\frac{4}{g}} x = \frac{4^{\frac{2}{g}} g L}{2u} \sum_{s=\pm 1} \frac{{}_2\mathcal{F}_1\left(\frac{4}{g}, \frac{5}{g} + \frac{s\tilde{\delta}}{\pi g}; 1 + \frac{5}{g} + \frac{s\tilde{\delta}}{\pi g}; -1\right)}{5\pi + s\tilde{\delta}}, \quad (\text{B.13})$$

where ${}_2\mathcal{F}_1$ is the Gaussian hypergeometric function.

References

- [1] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
- [2] Qi X L and Zhang S C 2011 *Rev. Mod. Phys.* **83** 1057
- [3] Bernevig B A and Zhang S C 2006 *Phys. Rev. Lett.* **96** 106802
- [4] Bernevig B A, Hughes T L and Zhang S C 2006 *Science* **314** 1757
- [5] König M, Wiedmann S, Brune C, Roth A, Buhmann H, Molenkamp L W, Qi X L and Zhang S C 2007 *Science* **318** 766
- [6] Tkachov G and Hankiewicz E M 2013 *Phys. Status Solidi* **250** 215
- [7] Sato K, Loss D and Tserkovnyak Y 2010 *Phys. Rev. Lett.* **105** 226401
- [8] Adroguer P, Grenier C, Carpentier D, Cayssol J, Degiovanni P and Orignac E 2010 *Phys. Rev. B* **82** 081303
- [9] Virtanen P and Recher P 2012 *Phys. Rev. B* **85** 035310
- [10] Fidkowski L, Alicea J, Lindner N H, Lutchyn R M and Fisher M P A 2012 *Phys. Rev. B* **85** 245121
- [11] Knez I, Du R R and Sullivan G 2012 *Phys. Rev. Lett.* **109** 186603
- [12] Fu L and Kane C L 2009 *Phys. Rev. B* **79** 161408
- [13] Kwon H J, Yakovenko V M and Sengupta K 2004 *Low Temp. Phys.* **30** 613
- [14] Kitaev Y A 2001 *Phys.—Usp.* **44** 131
- [15] Lutchyn R M, Sau J D and Das Sarma S 2010 *Phys. Rev. Lett.* **105** 077001
- [16] Jiang L, Pekker D, Alicea J, Refael G, Oreg Y and von Oppen F 2011 *Phys. Rev. Lett.* **107** 236401
- [17] Beenakker C W J, Pikulin D I, Hyart T, Schomerus H and Dahlhaus J P 2013 *Phys. Rev. Lett.* **110** 017003
- [18] Badiane D M, Houzet M and Meyer J S 2011 *Phys. Rev. Lett.* **107** 177002
- [19] Wu C, Bernevig B A and Zhang S C 2006 *Phys. Rev. Lett.* **96** 106401
- [20] Teo J C Y and Kane C L 2009 *Phys. Rev. B* **79** 235321
- [21] Qi X-L, Hughes T L and Zhang S C 2008 *Nature Phys.* **4** 273
- [22] Meng Q, Vishveshwara S and Hughes T L 2012 *Phys. Rev. Lett.* **109** 176803
- [23] Timm C 2012 *Phys. Rev. B* **86** 155456
- [24] Dolcetto G, Traverso Ziani N, Biggio M, Cavaliere F and Sassetti M 2013 *Phys. Rev. B* **87** 235423
- [25] Xu C and Moore J E 2006 *Phys. Rev. B* **73** 045322
- [26] Schmidt T L, Rachel S, von Oppen F and Glazman L I 2012 *Phys. Rev. Lett.* **108** 156402
- [27] Crépin F and Trauzettel B 2013 arXiv:1305.7433
- [28] Giamarchi T 2003 *Quantum Physics in One-Dimension* (Oxford: Oxford University Press)
- [29] Miranda E 2003 *Braz. J. Phys.* **33** 3
- [30] Maslov D L, Stone M, Goldbart P M and Loss D 1996 *Phys. Rev. B* **53** 1548
- [31] Fidkowski L, Lutchyn R M, Nayak C and Fisher M P A 2011 *Phys. Rev. B* **84** 195436
- [32] von Delft J and Schoeller H 1998 *Ann. Phys.* **7** 225
- [33] Gradshteyn I S and Ryzhik I M 1957 *Table of Integrals, Series and Product* (Berlin: VEB Deutsche Verlag der Wissenschaften)
- [34] Bruus H and Flensberg K 2004 *Many-Body Quantum Theory in Condensed Matter Physics* (Oxford: Oxford University Press)
- [35] Kane C L and Fisher M P A 1992 *Phys. Rev. B* **46** 15233
- [36] Takane Y and Koyama Y 1996 *Japan. J. Phys.* **65** 3630
- [37] Takane Y 1997 *Japan. J. Phys.* **66** 537
- [38] Takane Y 2002 *J. Phys. Soc. Japan* **71** 550
- [39] Furusaki A and Nagaosa N 1993 *Phys. Rev. B* **47** 4631
- [40] Schmid A 1983 *Phys. Rev. Lett.* **51** 1506