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On the Kolmogorov equation associated with Volterra equations and fractional Brownian motion ☆

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ABSTRACT

We consider a Volterra convolution equation in $\mathbb{R}^d$ perturbed with an additive fractional Brownian motion of Riemann–Liouville type with Hurst parameter $H \in (0,1)$. We show that its solution solves an infinite–dimensional stochastic differential equation (SDE) in the Hilbert space of square–integrable functions. Such an equation motivates our study of an unconventional class of SDEs requiring an original extension of the drift operator and its Fréchet differentials. We prove that these infinite–dimensional SDEs generate a Markov stochastic flow which is twice Fréchet differentiable with respect to the initial data. This stochastic flow is then employed to solve, in the classical sense of infinite–dimensional calculus, the path–dependent Kolmogorov equation corresponding to the SDEs. In particular, we associate a time–dependent infinitesimal generator with the fractional Brownian motion. In the final section, we show some obstructions in the analysis of the mild formulation of the Kolmogorov equation for SDEs driven by the same infinite–dimensional noise. This problem,

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which is relevant to the theory of regularization by noise, remains open for future research.

1. Introduction

Consider the stochastic integral equation in $\mathbb{R}^d$ with additive noise

$$X_t = x_0 + \int_0^t k_1(t-s)\, b(s, X_s)\, \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, \mathrm{d}W_s, \tag{1}$$

where $x_0 \in \mathbb{R}^d$, $\alpha \in (\frac{1}{2}, 1)$, $W = (W_t)_{t \geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable vector field and k_1 is a locally square–integrable, $\mathbb{R}$-valued kernel that is continuous in $(0, \infty)$. This equation belongs to the class of stochastic Volterra equations (SVE), which is characterized by a wide and continuously expanding body of literature, see for instance [1,2,8,30,33,38]. The additive noise driving the SVE (1) is a fractional Brownian motion (henceforth, fBM) of Riemann–Liouville type, with Hurst parameter $H = \alpha - \frac{1}{2} \in (0, \frac{1}{2})$. Our motivation for studying this random perturbation stems from its relevance in mathematical finance, particularly in the field of rough volatility models, see [9,21,27].

Inspired by [24], where the authors introduce a Banach space framework for path–dependent stochastic differential equations (SDEs), we reformulate the SVE (1) as an infinite–dimensional SDE in a separable Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ of the form

$$w_t = \phi + \int_0^t B(s, w_s)\, \mathrm{d}s + \int_0^t \sigma(s)\, \mathrm{d}W_s, \quad \phi \in H, \tag{2}$$

where $\sigma \colon [0,T] \to \mathcal{L}(\mathbb{R}^d; H)$ and the operator B has an unconventional structure. More precisely, since the kernel k_1 in (1) is only square–integrable and may have a singularity at 0, lifting (1) to the infinite–dimensional SDE (2) in H requires an H-valued drift B defined in $[0,T] \times \Lambda$, where Λ is a dense subspace of H. Thus, B maps between two different spaces in its domain and codomain, unlike standard frameworks in the literature (see, for instance, the aforementioned [24]).

This motivates our study of a novel class of infinite–dimensional SDEs of the form

$$w_t = \phi + \int_0^t \overline{B}(s, w_t)\, \mathrm{d}s + \int_0^t \sigma(s)\, \mathrm{d}W_s, \quad \phi \in H, \tag{3}$$

and of the associated stochastic flow's regularity. Notably, these SDEs require an extension of the drift operator and its Fréchet differentials, which explains the notation $\overline{B}$ in (3).

Given $\Phi\colon H \to \mathbb{R}$, we then study the following backward Kolmogorov equation associated with (3):

$$\begin{cases} \partial_t u(t,\phi) + \mathcal{A}_t u(t,\phi) = 0, & t \in [0,T], \, \phi \in H, \\ u(T,\phi) = \Phi(\phi), \end{cases}$$

which will be interpreted in integral form, see (78). Here $\mathcal{A}_t$, the time–dependent infinitesimal generator, is given by

$$\mathcal{A}_t u(t,\phi) = \frac{1}{2}\operatorname{Tr}\left(D^2 u(t,\phi)\,\sigma(t)\sigma(t)^*\right) + \left\langle \overline{B}(t,\phi), \nabla u(t,\phi)\right\rangle_H .$$

As in [19, Chapter 9], the approach that we adopt for the existence of classical solutions of the Kolmogorov equation is based on a careful analysis of (3) and on the formula

$$u(t,\phi) = \mathbb{E}\left[\Phi\left(w_T^{t,\phi}\right)\right], \tag{4}$$

where $w_t^{t_0,\phi}$, $t \in [t_0,T]$, is the solution of an analogue of (3) starting at time t_0 instead of 0.

It is worth noting that we use classical tools of infinite–dimensional calculus, such as the Fréchet derivative, when analyzing the Kolmogorov equation. This is a novelty compared to other studies addressing path–dependent PDEs related to SVEs, particularly [43] (see also [4] for a similar subject). In a sense, then, we unify the study of SVEs and fBM of Riemann–Liouville type to other infinite–dimensional systems. However, the assumptions imposed on the drift coefficient are not entirely classical, resulting in an innovative abstract formulation of the problem. Consequently, the analysis developed here is only analogous to the classical one, not included into it.

A recent paper that conducts an extensive study of the relation between path-dependent PDEs and forward-backward systems of stochastic Volterra equations (FBSVEs) is [44]. In particular, Theorems 3.2 and 3.4 in that work establish solutions to path-dependent PDEs via Feynman–Kac formulas like (4), based on associated FBSVEs. There are two major differences between the current paper and [44]. First, [44] employs the functional Itô formula from [43] as a fundamental tool. As a result, solutions to path–dependent PDEs are not intended in the classical sense of infinite–dimensional calculus. Second, the assumptions imposed in [44] on the coefficients of the SVEs (see [44, Assumption 2.1]) do not allow for singular kernels in the drift or diffusion components. In particular, the framework in [44] does not cover the case of an additive fBM of Riemann–Liouville type with a Hurst index in $(0, \frac{1}{2})$. As mentioned above, these two points serve as the primary motivations for the present paper and justify the original theoretical analysis and abstract formulation we develop. On the other hand, the coefficients of the SVEs considered in [44] have a more general structure than those in (1). In particular, the diffusion coefficient in the stochastic integral can depend on the solution process X. We believe that the approach proposed in this work can be extended to study such equations with multiplicative noise, and we leave this research direction for a future project.

Regarding forward–backward stochastic systems in the Volterra framework, we also refer the reader to the paper [45]. In this work, the authors consider a decoupled system where the forward component is an SDE and the backward component a BSVE. They provide PDE representation formulas for the backward component in terms of the forward SDE. We note that in [45], the generator of the BSVE is assumed to be continuous and therefore cannot exhibit explosions to infinity. Such singular behavior for the generator is not allowed in [44], either.

A more direct approach to the Kolmogorov equation would be also of great interest for two reasons. Firstly, it would contribute to complete the comparison with the classical theory developed for other classes of problems, see [19]. Secondly, it could be used to study regularization by noise phenomena for SDEs driven by fractional Brownian motion, which are investigated in literature using different techniques, see, e.g., [25,26,30,33–35]. In fact, studying the Kolmogorov equation in mild form might allow to prove weak uniqueness of solutions of the underlying SDE when the drift is not smooth, see [5,22,31,37,40,48]; and in case of additional regularity and bounds on the solution of the Kolmogorov equation, it may even lead to strong uniqueness results, for all or only almost all initial conditions with respect to a measure, depending on the regularity, see [3,14,15,22,42] and [16,17], respectively. In an attempt to develop such a direct approach, we have identified obstructions that we report in Section 5, so this problem remains open.

The paper is structured as follows. In Section 2 we investigate the connection between the SVE (1) and the infinite–dimensional SDE (2), see Proposition 1. Here we specify the Hilbert space H considered in our study, as well as an H-valued drift B (see (9)), which is defined in a dense subspace Λ of H. Due to the particular structure of B, we also introduce another infinite–dimensional reformulation for the SVE (1) (see (17) in Proposition 2), which is at the core of our analysis. In Section 3, we study the reformulation given by (17) in an abstract setting with a general drift B (not necessarily defined as in (9)), focusing on the regularity of its solution with respect to the initial data, see Subsections 3.1-3.2. This step requires to extend the drift B to a functional $\overline{B}$ defined on the entire space H, a peculiar feature of this paper. The abstract reformulation of (17) is given in (20) and coincides with (3). The related backward Kolmogorov equation in integral form is then investigated in Section 4. In Section 5 we discuss the mild formulation of the Kolmogorov equation and its importance for the theory of regularization by noise, see Subsection 5.2. The challenges that we previously mentioned regarding the analysis of such a mild formulation are explained in Subsection 5.1. Finally, in Appendix A we study the regularity of the solution of the Kolmogorov equation constructed as in (4).

2. Infinite–dimensional reformulations for the stochastic Volterra equation

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a complete filtered probability space, with expectation denoted by $\mathbb{E}$, where the filtration $\mathbb{F}= (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. Fix $d \in \mathbb{N}$ and consider an $\mathbb{R}^d$-valued standard Brownian motion $W = (W_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. In what follows, we denote by $k_2 \colon (0, \infty) \to (0, \infty)$ the fractional kernel which controls the noise in the SVE (1), namely

$$k_2(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, \quad t > 0, \text{ for some } \alpha \in \Big(\frac{1}{2}, 1\Big). \tag{5}$$

We note that the arguments and results of this paper continue to hold even when $\alpha \in [1, \frac{3}{2})$, i.e., when the fBM governing (1) has Hurst parameter in $[\frac{1}{2}, 1)$, exhibiting smoother trajectories and longer memory (see also Remark 6).

Fix $T > 0$. Suppose that the measurable vector field $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ in (1) satisfies, for some $L > 0$,

$$|b\,(t, x)| \leq L\,(1 + |x|)\,, \qquad |b\,(t, x) - b\,(t, y)| \leq L\,|x - y|\,,$$

for every $t \in [0, T]$ and $x, y \in \mathbb{R}^d$.

Definition 1. An $\mathbb{R}^d$-valued stochastic process $X = (X)_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a (strong) solution of the SVE (1) if it is $\mathbb{F}$-adapted, has continuous trajectories and satisfies the integral equation (1) for every $t \in [0, T]$, $\mathbb{P}$-a.s.

Existence and pathwise uniqueness of strong solutions of (1) have been studied in literature under additional requirements on k_1. For instance, thanks to the uniform-in-time linear growth and Lipschitz continuity of b, by [2, Theorem 3.3], (1) is well-posed when

$$\int_0^h |k_1(t)|^2 \mathrm{d}t = \mathrm{O}(h^\gamma) \quad \text{and} \quad \int_0^T |k_1(t+h) - k_1(t)|^2 \mathrm{d}t = \mathrm{O}(h^\gamma), \qquad \text{for some } \gamma \in (0, 2].$$

We refer the reader to [46,49] for existence and uniqueness results for non-convolution stochastic Volterra equations.

Let H be the Hilbert space $L^2\left(0, T; \mathbb{R}^d\right)$ and denote by $\langle \cdot, \cdot \rangle_H$ the usual inner product. Denoting by $\mathcal{L}(\mathbb{R}^d; H)$ the space of linear and bounded operators from $\mathbb{R}^d$ to H, define $\sigma \colon [0, T] \to \mathcal{L}(\mathbb{R}^d; H)$ by

$$[\sigma\,(t)\,x]\,(\xi) = k_2\,(\xi - t)\,1_{\{t<\xi\}} x, \quad x \in \mathbb{R}^d,\, t, \xi \in [0, T]\,. \tag{6}$$

For every $q \geq 2$, we denote by $\mathcal{H}^q$ the space $L^q\big(\Omega; H\big)$, endowed with the usual norm $\|\cdot\|_{\mathcal{H}^q}$, and by $\mathcal{H}^q_t \subset \mathcal{H}^q$ the subspace of $\mathcal{F}_t$-measurable functions, $t \in [0, T]$. Notice that

$$\|\sigma(t)\|_{\mathrm{HS}}^2 \leq d\,\|k_2\|_2^2\,, \quad t \in [0,T],$$

where $\|\cdot\|_{\mathrm{HS}}$ represents the Hilbert–Schmidt norm and $\|\cdot\|_2$ the norm in $L^2(0,T;\mathbb{R})$. As a consequence, since $\int_0^T \|\sigma(s)\|_{\mathrm{HS}}^2\,\mathrm{d}s < \infty$, we can construct the stochastic integral

$$\Sigma_{s,t} = \int_s^t \sigma\left(r\right)\mathrm{d}W_r \in \mathcal{H}_t^q, \quad 0 \leq s \leq t \leq T. \tag{7}$$

By [19, Theorem 4.36], there exists a constant $C_{d,q} > 0$ such that

$$\|\Sigma_{s,t}\|_{\mathcal{H}^q} \leq C_{d,q}\,\|k_2\|_2\,\sqrt{t-s}, \quad 0 \leq s \leq t \leq T. \tag{8}$$

Let Λ be the space $C\left([0,T]\,;\mathbb{R}^d\right)$, and define $B : [0,T] \times \Lambda \to H$ by

$$\left[B\left(t,w\right)\right]\left(\xi\right) = k_1\left(\xi - t\right) 1_{\{t<\xi\}} b\left(t, w\left(t\right)\right), \quad t,\xi \in \left[0,T\right], \tag{9}$$

where we recall that k_1 is a locally square–integrable, $\mathbb{R}$-valued kernel that is continuous in $(0,\infty)$, see Introduction 1. In the sequel, a stochastic process taking values in Λ or H will be denoted by, e.g., $(w_t)_{t\in[0,T]}$, namely with the time variable as a subscript. Then, for a fixed $t_0 \in [0,T]$, w_{t_0} is a random function, denoted by $w_{t_0}\left(\xi\right)$, $\xi \in [0,T]$.

We now establish a connection between the SVE (1) and (2). More precisely, consider the infinite–dimensional SDE

$$w_t = x_0 + \int_0^t B\left(s, w_s\right)\mathrm{d}s + \int_0^t \sigma\left(s\right)\,\mathrm{d}W_s \quad \text{in } H,$$
$$\text{where } \sigma \text{ and } B \text{ are given in (6) and (9), respectively.} \tag{10}$$

Note that the equality in (10) is understood in H, i.e., it holds for a.e. $\xi \in [0,T]$. A solution to (10) is defined as follows.

Definition 2. A stochastic process $w = (w)_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a solution of (10) if it is $\mathbb{F}$-adapted, takes values in the space Λ and satisfies the identity (10) $\mathbb{P}$-a.s., for every $t \in [0,T]$.

In the following proposition, we construct a solution of (10) starting from a solution to (1) (see Definition 1). We also demonstrate the opposite direction, namely how to obtain a solution to (1) from a solution of (10) which is jointly measurable in $[0,T] \times \Omega$.

Proposition 1. *Consider the operators σ and B defined in (6) and (9), respectively. Let $X = (X_t)_{t\in[0,T]}$ be a solution of (1) according to Definition 1. For every $t \in [0,T]$, define the $\mathbb{R}^d$-valued stochastic process $\theta^{(t)} = (\theta^{(t)}_\xi)_{\xi\in[t,T]}$ by*

$$\theta_{\xi}^{(t)} = x_0 + \int_0^t k_1(\xi - s)\, b(s, X_s)\,\mathrm{d}s + \int_0^t k_2(\xi - s)\,\mathrm{d}W_s, \qquad \xi \in [t, T].$$

Define the Λ*-valued stochastic process* $(w_t)_{t\in[0,T]}$ *by setting, for each* $t \in [0,T]$,

$$w_t(\xi) = \begin{cases} X_\xi, & \xi \le t, \\ \theta_\xi^{(t)}, & \xi > t. \end{cases} \tag{11}$$

Then $(w_t)_{t\in[0,T]}$ *is a solution of* (10) *in the sense of Definition 2.*

Viceversa, if $(w_t)_{t\in[0,T]}$ *is a measurable solution of* (10)*, namely the map* $(t,\omega) \mapsto [w_t(\omega)](\cdot) \in \Lambda$ *is measurable, then* $X = (X_t)_{t\in[0,T]}$ *defined by* $X_t = w_T(t)$, $t \in [0,T]$, *is a solution of* (1).

Proof. Fix $t \in [0,T]$. Note that, by the Kolmogorov–Chentsov continuity criterion, there exists a continuous version of the stochastic process $(\int_0^t k_2(\xi - s)\,\mathrm{d}W_s)_{\xi\in[t,T]}$. Hence, also employing the dominated convergence theorem, we deduce that the process $\theta^{(t)}$ has continuous trajectories $\theta_{\cdot}^{(t)}$ in $[t,T]$. It follows that w_t defined in (11) takes values in Λ.

In addition, by [19, Proposition 3.18], we observe that w_t is an $\mathcal{F}_t$-measurable Λ-valued random variable, because X is continuous and $\mathbb{F}$-adapted, $\theta_{\cdot}^{(t)}$ is continuous and $\theta_\xi^{(t)}$ is $\mathcal{F}_t$-measurable for every $\xi \in [t,T]$. Thus, the Λ-valued stochastic process $(w_t)_{t\in[0,T]}$ is $\mathbb{F}$-adapted.

We now want to prove that w_t satisfies (10). By (1) and the definition of $\theta^{(t)}$, we have, $\mathbb{P}$-a.s.,

$$\begin{aligned} w_t(\xi) &= X_\xi 1_{\{\xi\le t\}} + \theta_\xi^{(t)} 1_{\{\xi>t\}} = x_0 + \int_0^{t\wedge\xi} k_1(\xi - s)\, b(s, X_s)\,\mathrm{d}s + \int_0^{t\wedge\xi} k_2(\xi - s)\,\mathrm{d}W_s \\ &= x_0 + \int_0^t k_1(\xi - s)\, 1_{\{\xi>s\}} b(s, X_s)\,\mathrm{d}s + \int_0^t k_2(\xi - s)\, 1_{\{\xi>s\}}\mathrm{d}W_s, \quad \xi \in [0,T]. \end{aligned} \tag{12}$$

We focus on the integral in $\mathrm{d}W$, with the aim of understanding its relation with $\Sigma_{0,t} = \int_0^t \sigma(s)\mathrm{d}W_s$, see (7). By (6) and [19, Proposition 4.30],

$$\left\langle \int_0^t \sigma(s)\,\mathrm{d}W_s, h \right\rangle_H = \int_0^t \left(\int_0^T k_2(\xi - s) 1_{\{\xi>s\}} h(\xi)\,\mathrm{d}\xi \right)^{\top} \mathrm{d}W_s, \quad \mathbb{P}\text{−a.s., for every } h \in H.$$

Moreover, an application of the stochastic Fubini's theorem yields

$$\left\langle \int_0^t k_2(\cdot - s) 1_{\{\cdot>s\}}\mathrm{d}W_s,\, h \right\rangle_H = \int_0^T \left(\int_0^t k_2(\xi - s) 1_{\{\xi>s\}}\,\mathrm{d}W_s \right)^{\top} h(\xi)\,\mathrm{d}\xi$$

$$= \int_0^t \left(\int_0^T k_2(\xi - s) 1_{\{\xi > s\}} h(\xi) \, \mathrm{d}\xi \right)^\top \mathrm{d}W_s,$$

which holds $\mathbb{P}$-a.s., for every $h \in H$. Considering that H is separable, combining the two previous equations we deduce that

$$\left\langle \int_0^t \sigma(s) \, \mathrm{d}W_s, h \right\rangle_H = \left\langle \int_0^t k_2(\cdot - s) 1_{\{\cdot > s\}} \mathrm{d}W_s, \, h \right\rangle_H, \quad h \in H, \, \mathbb{P} - \text{a.s.},$$

which in turn implies that

$$\left(\int_0^t \sigma(s) \, \mathrm{d}W_s \right)(\xi) = \int_0^t k_2(\xi - s) 1_{\{\xi > s\}} \mathrm{d}W_s, \quad \text{for a.e. } \xi \in [0, T], \, \mathbb{P} - \text{a.s.} \tag{13}$$

Recalling the definition of B in (9) and (11), standard properties of Bochner's integral yield

$$\begin{aligned} \left(\int_0^t B(s, w_s) \, \mathrm{d}s \right)(\xi) &= \int_0^t k_1(\xi - s) \, 1_{\{\xi > s\}} b(s, w_s(s)) \, \mathrm{d}s \\ &= \int_0^t k_1(\xi - s) \, 1_{\{\xi > s\}} b(s, X_s) \, \mathrm{d}s, \quad \text{for a.e. } \xi \in [0, T]. \end{aligned} \tag{14}$$

Going back to (12), we conclude that $(w_t)_{t \in [0,T]}$ is a solution of (10) according to Definition 2.

Viceversa, consider a Λ-valued $\mathbb{F}$-adapted measurable solution $w = (w_t)_{t \in [0,T]}$ of (10), and define the continuous $\mathbb{R}^d$-valued process $X = (X_t)_{t \in [0,T]}$ by $X_t = w_T(t)$, $t \in [0, T]$. From (13) and (14) we infer that, for every $t \in [0, T]$,

$$w_t(\xi) = w_T(\xi), \quad \xi \in [0, t], \, \mathbb{P} - \text{a.s.},$$

whence

$$w_t(t) = w_T(t) = X_t, \quad \mathbb{P} - \text{a.s.}, \, t \in [0, T]. \tag{15}$$

Given that the filtration $\mathbb{F}$ is complete and w is $\mathbb{F}$-adapted, this equality shows that X is $\mathbb{F}$-adapted, as well. Furthermore, since $w \colon [0, T] \times \Omega \to \Lambda$ is measurable, (15) also yields

$$w_t(t) = X_t, \quad \text{for a.e. } t \in [0, T], \, \mathbb{P} - \text{a.s.} \tag{16}$$

We now notice that, by the first equality in (14) and (16),

$$\left(\int_0^T B(s,w_s)\,\mathrm{d}s\right)(\xi)=\int_0^\xi k_1\left(\xi-s\right)b\left(s,X_s\right)\mathrm{d}s,\quad \text{for a.e. } \xi\in[0,T],\ \mathbb{P}-\text{a.s.},$$

where the right–hand side is a continuous function in $\xi \in [0,T]$ by the linear growth of b, the continuity of the trajectories of X and [29, Theorem 2.2, Chapter 2]. Therefore, also recalling (13), by (10),

$$X_\xi = x_0+\int_0^\xi k_1\left(\xi-s\right)b\left(s,X_s\right)\mathrm{d}s+\int_0^\xi k_2(\xi-s)\,\mathrm{d}W_s,\quad \text{for a.e. } \xi\in[0,T],\ \mathbb{P}-\text{a.s.}$$

In fact, since this equation involves continuous processes, the equality holds for every $\xi \in [0,T]$, which proves that X is a solution of (1) in the sense of Definition 1. The proof is then complete. □

The previous proposition gives us the classical infinite–dimensional reformulation of the SVE (1), quoted by Equation (2) in Introduction 1. Notice, however, that the drift B in (10) has an unconventional structure. More precisely, as already mentioned in Introduction 1, the issue with the expression of B in (9) is that it is meaningful only for continuous functions, as it involves a punctual evaluation, but takes values in H, due to the possible singularity of k_1 at 0. As a result, for the procedure carried out in Section 3 (see also Remark 2), it turns out that a second reformulation is more convenient.

Proposition 2. *Consider the operators σ and B defined in (6) and (9), respectively. Let $(X_t)_{t\in[0,T]}$ be a solution of (1) according to Definition 1 and $\theta_\xi^{(t)}$, $w_t(\xi)$ be defined as in Proposition 1. Then, for every $t \in [0,T]$, the following identity holds in H:*

$$w_t = x_0+\int_0^t B\left(s,w_t\right)\mathrm{d}s+\int_0^t \sigma(s)\,\mathrm{d}W_s,\quad \mathbb{P}-a.s. \tag{17}$$

Viceversa, if $(w_t)_{t\in[0,T]}$ is a Λ-valued $\mathbb{F}$-adapted process satisfying (17) for every $t \in [0,T]$, then $X=(X_t)_{t\in[0,T]}$ defined by $X_t = w_T(t)$, $t\in[0,T]$, is a solution of (1).

Notice that, unlike in (10), the time index of w in the Bochner integral in (17) is fixed at t. This is important for the abstract study carried out in Section 3, which requires an extension of the drift operator, see also Remark 2.

Proof. Take a solution $X=(X_t)_{t\in[0,T]}$ of (1) and define $(w_t)_{t\in[0,T]}$ as in (11). Observing that, for a.e. $\xi\in[0,T]$,

$$\int_0^t k_1\left(\xi-s\right)1_{\{\xi>s\}}b\left(s,X_s\right)\mathrm{d}s=\int_0^t B\left(s,w_t\right)(\xi)\,\mathrm{d}s=\left(\int_0^t B\left(s,w_t\right)\mathrm{d}s\right)(\xi), \tag{18}$$

the fact that $(w_t)_{t\in[0,T]}$ solves (17) is proved in the same way as in Proposition 1.

Viceversa, if $w = (w_t)_{t\in[0,T]}$ is a Λ-valued $\mathbb{F}$-adapted process which solves (17), then, for every $t \in [0,T]$, $\mathbb{P}$-a.s., by (9), (13), (18) and the Lipschitz continuity of b, we have

$$|w_T(\xi) - w_t(\xi)| \le L \int_0^{\xi} |k_1(\xi - s)||w_T(s) - w_t(s)|\,\mathrm{d}s, \quad \text{for a.e. } \xi \in [0,t].$$

Since this equality involves continuous functions, we infer that, $\mathbb{P}$-a.s.,

$$\left(1 - L\int_0^{\bar{t}} |k_1(s)|\mathrm{d}s\right) \sup_{\xi\in[0,\bar{t}]} |w_T(\xi) - w_t(\xi)| \le 0, \quad \text{for every } \bar{t} \in (0,t].$$

Thus, choosing $\bar{t}$ sufficiently small, the previous inequality yields $w_T(\xi) = w_t(\xi)$, $\xi \in [0,\bar{t}]$, $\mathbb{P}$-a.s. A standard argument by steps then enables us to conclude that

$$w_T(\xi) = w_t(\xi), \quad \xi \in [0,t],\ \mathbb{P} - \text{a.s.}$$

If we define the continuous $\mathbb{R}^d$-valued process $X = (X_t)_{t\in[0,T]}$ by $X_t = w_T(t)$, $t \in [0,T]$, then, recalling that w is $\mathbb{F}$-adapted and $\mathbb{F}$ is complete, the previous equality shows that X is $\mathbb{F}$-adapted, too. The fact that X solves (1) is an immediate consequence of (9), (13), (17) and (18), completing the proof. □

Motivated by the infinite–dimensional reformulation of Proposition 2, in Section 3 we focus on studying Equation (17) in an abstract setting, where $B\colon [0,T] \times \Lambda \to H$ is not necessarily given by (9). Our aim is to investigate the property of its solutions and the associated Kolmogorov equation, which is the subject of Section 4. However, the implementation of this plan is challenging, due to the particular structure of the drift $B\colon [0,T]\times\Lambda \to H$ which, unlike the classical case, maps between two different functional spaces in its domain and codomain (Λ and H, respectively). This requires an abstract formulation of the problem that, to the best of our knowledge, is not covered by the existing literature.

3. Abstract formulation and differentiability of the stochastic flow

In this section, we introduce and study an abstract formulation for Equation (17), quoted by (3) in Introduction 1, with a particular attention devoted to the differentiability of its solution with respect to the initial data, see Subsections 3.1-3.2. Here, we consider a more general drift operator B than in Section 2, meaning that B is not necessarily defined as in (9). In our reasoning, we introduce an extension of B, denoted by $\overline{B}$, which is a characterizing and original feature of the approach that we propose.

For every $k, p \in \mathbb{N}$, we denote by $\|\cdot\|_p$ the usual norm on the Banach space $L^p\big(0,T;\mathbb{R}^k\big)$. We denote by

$$H_\Box \text{ the Hilbert space } L^2\big(\,(0,T)\times(0,T)\,;\mathbb{R}^d\big) \text{ endowed with the norm } \|\cdot\|_{2,\Box}.$$

Recall $H = L^2\big(0,T;\mathbb{R}^d\big)$ and $\Lambda = C\left([0,T];\mathbb{R}^d\right)$. For every $w \in \Lambda$, we consider a map $B(w): [0,T]\times[0,T] \to \mathbb{R}^d$ subject to the next requirement.

Assumption 1. The function $B\colon \Lambda \to H_\Box$ satisfies

$$\|B(w_1)\|_{2,\Box} \le C_0\left(1+\|w_1\|_2\right), \qquad \|B(w_1)-B(w_2)\|_{2,\Box} \le C_0\,\|w_1-w_2\|_2\,, \tag{19}$$

for every w_1, $w_2 \in \Lambda$, for some constant $C_0 = C_0(d,T) > 0$.

Moreover, given $w \in \Lambda$ and $0 < t \le T$, for a.e. $r \in (0,t)$ the function $B(w)(r,\cdot) \in H$ is of Volterra–type, namely $B(w)(r,\xi) = 0$ for a.e. $\xi \in (0,r)$, and depends on w only via its restriction $w|_{(0,t)}$ to $(0,t)$.

In the sequel, we are going to progressively introduce stricter hypotheses on the drift map B (see, in particular, Assumptions 2-3), which will allow us to prove the main result on the Kolmogorov equation, see Theorem 9 in Section 4. In Example 1, we show a function B, obtained by choosing b in (9) with an affine structure, that satisfies these requirements.

Under Assumption 1, we can invoke the theorem of extension of uniformly continuous functions (see, e.g., [39, Exercise 13, Chapter 7]) to uniquely define a continuous map $\overline{B}\colon H \to H_\Box$ such that $\overline{B}\big|_\Lambda = B$. Note that $\overline{B}$ satisfies (19) for every w_1, $w_2 \in H$. Given $w \in H$ and $r \in (0,T)$, we are going to write $\overline{B}(r,w) = \overline{B}(w)(r,\cdot) \in H$: these maps are well defined for a.e. $r \in (0,T)$.

For a fixed $0 < t \le T$, we remark that also $\overline{B}(r,w)$ is of Volterra–type in the sense of Assumption 1 for a.e. $r \in (0,t)$, and that it depends on w only via $w|_{(0,t)}$. For these reasons, in the sequel we will refer to Assumption 1 while talking about $\overline{B}$.

Recall the spaces $\mathcal{H}^q = L^q\big(\Omega;H\big)$, $q \ge 2$, and the subspaces $\mathcal{H}^q_t \subset \mathcal{H}^q$ of $\mathcal{F}_t$-measurable functions introduced in Section 2, as well as the random variables $\Sigma_{s,t} \in \mathcal{H}^q_t$ in (7). For every $0 \le s \le t \le T$ and $\phi \in \mathcal{H}^q$, we are interested in the equation

$$w = \phi + \int_s^t \overline{B}(r,w)\,\mathrm{d}r + \int_s^t \sigma(r)\,\mathrm{d}W_r, \quad \mathbb{P}-\text{a.s.}, \tag{20}$$

where $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^d;H)$ is defined in (6). Note that (20) is an equality between H-valued random variables, satisfied up to a $\mathbb{P}$-null set. The well–posedness of (20) in $\mathcal{H}^q$ is given by the next result.

Theorem 3. *Under Assumption 1, for every $q \ge 2$, $\phi \in \mathcal{H}^q$ and $s,t \in [0,T]$, with $s \le t$, there exists a unique solution $w^{s,\phi}_t \in \mathcal{H}^q$ of (20). In particular, if $\phi \in \mathcal{H}^q_s$ then $w^{s,\phi}_t \in \mathcal{H}^q_t$.*

Furthermore, the following cocycle property holds in $\mathcal{H}^q$*:*

$$w_t^{s,\phi} = w_t^{u,w_u^{s,\phi}}, \quad 0 \le s < u < t \le T,\, \phi \in \mathcal{H}^q. \tag{21}$$

Proof. Fix $q \geq 2$, $0 \leq s \leq t \leq T$ and $\phi \in \mathcal{H}_s^q$. Consider $N = N(d,T) \in \mathbb{N}$ so big that $C_0\sqrt{T/N} < 1$, where $C_0 = C_0(d,T)$ is the constant in (19). Let us introduce an equispaced partition $\{t_k\}_{k=0}^N$ of $[s,t]$ where $t_0 = s$ and $t_N = t$: its mesh $\Delta \le T/N$. Define the mapping $\Gamma_{t_0}^{t_1}\colon \mathcal{H}_{t_1}^q \to \mathcal{H}_{t_1}^q$ by

$$\Gamma_{t_0}^{t_1} w = \phi + \int_{t_0}^{t_1} \overline{B}(r,w)\,\mathrm{d}r + \int_{t_0}^{t_1} \sigma(r)\,\mathrm{d}W_r, \quad w \in \mathcal{H}_{t_1}^q. \tag{22}$$

Under Assumption 1, $\Gamma_{t_0}^{t_1}$ is well defined. Indeed, for every $w \in \mathcal{H}_{t_1}^q$,

$$\begin{aligned}
\left\|\Gamma_{t_0}^{t_1} w\right\|_{\mathcal{H}^q}^q &= \mathbb{E}\left[\left\|\Gamma_{t_0}^{t_1} w\right\|_2^q\right] \\
&\le 3^{q-1}\mathbb{E}\left[\|\phi\|_2^q + \left(\int_{t_0}^{t_1}\left(\int_0^T \left|\overline{B}(w)(r,\xi)\right|^2 \mathrm{d}\xi\right)^{\frac12} \mathrm{d}r\right)^q + \left\|\int_{t_0}^{t_1}\sigma(r)\,\mathrm{d}W_r\right\|_2^q\right] \\
&\le 3^{q-1}\mathbb{E}\left[\|\phi\|_2^q + C_0^q\Delta^{\frac q2}\left(1+\|w\|_2\right)^q + \left\|\int_{t_0}^{t_1}\sigma(r)\,\mathrm{d}W_r\right\|_2^q\right] < \infty,
\end{aligned}$$

where we use Bochner's theorem (see, e.g., [47, Corollary 1, Section 5, Chapter V]) in the first inequality, and the first bound in (19), coupled with Jensen's inequality, in the second one. Analogously, using the second inequality in (19), we write

$$\begin{aligned}
\left\|\Gamma_{t_0}^{t_1} w_1 - \Gamma_{t_0}^{t_1} w_2\right\|_{\mathcal{H}^q} &\le \mathbb{E}\left[\left(\int_{t_0}^{t_1}\left(\int_0^T \left|\overline{B}(w_1) - \overline{B}(w_2)\right|^2 (r,\xi)\,\mathrm{d}\xi\right)^{\frac12} \mathrm{d}r\right)^q\right]^{\frac1q} \\
&\le C_0\sqrt{\Delta}\,\|w_1 - w_2\|_{\mathcal{H}^q}, \quad w_1,\, w_2 \in \mathcal{H}_{t_1}^q.
\end{aligned} \tag{23}$$

Hence, for our choice of $N \in \mathbb{N}$, the map $\Gamma_{t_0}^{t_1}$ is a contraction in $\mathcal{H}_{t_1}^q$, whose unique fixed point is $\overline{w}_1$. Noting that $\overline{w}_1$ is the unique solution of (20) with t_1 instead of t, we denote it by $w_{t_1}^{s,\phi}$.

Since the relation between constants in (23), which is necessary to make $\Gamma_{t_0}^{t_1}$ a contraction, does not depend on the initial condition, under Assumption 1 the previous argument can be iterated to construct the solution $w_t^{s,\phi}$ of (20). More precisely, define the map $\Gamma_{t_1}^{t_2}\colon \mathcal{H}_{t_2}^q \to \mathcal{H}_{t_2}^q$ by

$$\Gamma_{t_1}^{t_2} w = \overline{w}_1 + \int_{t_1}^{t_2} \overline{B}(r,w)\,\mathrm{d}r + \int_{t_1}^{t_2} \sigma(r)\,\mathrm{d}W_r, \quad w \in \mathcal{H}_{t_2}^q.$$

Computations similar to those above show that $\Gamma_{t_1}^{t_2}$ is well defined. Moreover,

$$\left\|\Gamma_{t_1}^{t_2} w_1 - \Gamma_{t_1}^{t_2} w_2\right\|_{\mathcal{H}^q} \leq C_0 \sqrt{\Delta} \left\|w_1 - w_2\right\|_{\mathcal{H}^q}, \quad w_1,\, w_2 \in \mathcal{H}_{t_2}^q.$$

Thus, $\Gamma_{t_1}^{t_2}$ is a contraction in $\mathcal{H}_{t_2}^q$, whose unique fixed point is $\overline{w}_2 = w_{t_2}^{t_1, w_{t_1}^{s,\phi}}$. Now, by the Volterra–type property of $\overline{B}$ and σ, together with the standard features of the Bochner's and stochastic integrals (see (13)), we infer that

$$\left(\int_{t_1}^{t_2} \overline{B}\left(r, \overline{w}_2\right) \mathrm{d}r\right)(\xi) = \left(\int_{t_1}^{t_2} \sigma\left(r\right) \mathrm{d}W_r\right)(\xi) = 0, \quad \text{for a.e. } \xi \in (0, t_1),\ \mathbb{P}-\text{a.s.}, \tag{24}$$

whence

$$\overline{w}_2\big|_{(0,t_1)} = \overline{w}_1\big|_{(0,t_1)}, \quad \mathbb{P}-\text{a.s.}$$

Furthermore, $\mathbb{P}$-a.s., for a.e. $r \in (s, t_1)$, $\overline{B}\left(r, \overline{w}_1\right)$ depends on $\overline{w}_1$ only via $\overline{w}_1\big|_{(0,r)}$, which yields

$$\overline{B}\left(r, \overline{w}_1\right) = \overline{B}\left(r, \overline{w}_2\right), \quad \text{for a.e. } r \in (s, t_1),\ \mathbb{P}-\text{a.s.} \tag{25}$$

Therefore, recalling (22),

$$\begin{aligned} \overline{w}_2 &= \phi + \int_s^{t_1} \overline{B}\left(r, \overline{w}_1\right) \mathrm{d}r + \int_{t_1}^{t_2} \overline{B}\left(r, \overline{w}_2\right) \mathrm{d}r + \int_s^{t_2} \sigma\left(r\right) \mathrm{d}W_r \\ &= \phi + \int_s^{t_2} \overline{B}\left(r, \overline{w}_2\right) \mathrm{d}r + \int_s^{t_2} \sigma\left(r\right) \mathrm{d}W_r. \end{aligned} \tag{26}$$

This shows that $\overline{w}_2$ is a solution of (20) with t_2 instead of t.

To prove that $\overline{w}_2$ is in fact the unique solution of this equation, we consider another random variable $\widetilde{w} \in \mathcal{H}_{t_2}^q$ satisfying (26). Then, relying on the same properties of $\overline{B}$ and σ as those used above, we deduce that

$$1_{(0,t_1)} \widetilde{w} = 1_{(0,t_1)} \left(\phi + \int_s^{t_1} \overline{B}\left(r, 1_{(0,t_1)} \widetilde{w}\right) \mathrm{d}r + \int_s^{t_1} \sigma\left(r\right) \mathrm{d}W_r\right). \tag{27}$$

Moreover, we observe that also $1_{(0,t_1)} \overline{w}_1 \in \mathcal{H}^q$ satisfies (27). Therefore, using Bochner's theorem and Jensen's inequality, by Assumption 1 we can compute

$$\begin{aligned}\left\|1_{(0,t_1)}\left(\overline{w}_1-\widetilde{w}\right)\right\|_{\mathcal{H}^q}^q &\leq \mathbb{E}\left[\left\|\int_s^{t_1}\left(\overline{B}\left(r,1_{(0,t_1)}\overline{w}_1\right)-\overline{B}\left(r,1_{(0,t_1)}\widetilde{w}\right)\right)\mathrm{d}r\right\|_2^q\right]\\ &\leq \mathbb{E}\left[\left(\int_s^{t_1}\left\|\overline{B}\left(r,1_{(0,t_1)}\overline{w}_1\right)-\overline{B}\left(r,1_{(0,t_1)}\widetilde{w}\right)\right\|_2\mathrm{d}r\right)^q\right]\\ &\leq \Delta^{\frac{q}{2}}\mathbb{E}\left[\left\|\overline{B}\left(1_{(0,t_1)}\overline{w}_1\right)-\overline{B}\left(1_{(0,t_1)}\widetilde{w}\right)\right\|_{2,\square}^q\right]\\ &\leq \Delta^{\frac{q}{2}}C_0^q\left\|1_{(0,t_1)}\left(\overline{w}_1-\widetilde{w}\right)\right\|_{\mathcal{H}^q}^q,\end{aligned}$$

which allow us to conclude, recalling that $\sqrt{\Delta}C_0<1$,

$$1_{(0,t_1)}\widetilde{w}=1_{(0,t_1)}\overline{w}_1,\quad \mathbb{P}-\text{a.s.}$$

Going back to (26), by (22) and the previous equality we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}\widetilde{w}&=\phi+\int_s^{t_1}\overline{B}\left(r,\overline{w}_1\right)\mathrm{d}r+\int_s^{t_1}\sigma\left(r\right)\mathrm{d}W_r+\int_{t_1}^{t_2}\overline{B}\left(r,\widetilde{w}\right)\mathrm{d}r+\Sigma_{t_1,t_2}\\ &=\overline{w}_1+\int_{t_1}^{t_2}\overline{B}\left(r,\widetilde{w}\right)\mathrm{d}r+\Sigma_{t_1,t_2}.\end{aligned}$$

It follows that $\widetilde{w}$ is a fixed point of the map $\Gamma_{t_1}^{t_2}$ in $\mathcal{H}_{t_2}^q$: by uniqueness, we obtain $\widetilde{w}=\overline{w}_2$. Hence $\overline{w}_2$ is the unique solution of (20) with t_2 instead of t, which we denote by $w_{t_2}^{s,\phi}$.

This argument by steps can be repeated to cover the whole interval $[s,t]$. In this way, we obtain the unique solution $w_t^{s,\phi}$ of (20) in $\mathcal{H}_t^q$. The same procedure also works when the initial condition $\phi\in\mathcal{H}^q$, i.e., when ϕ is not necessarily $\mathcal{F}_s$-measurable. In such a case, it provides a unique solution $w_t^{s,\phi}\in\mathcal{H}^q$.

The cocycle property in (21) follows by a similar reasoning. Indeed, if we fix $u\in(s,t)$, then by the Volterra–type property of $\overline{B}$ and σ (cf. (24)) we have

$$w_t^{u,w_u^{s,\phi}}\Big|_{(0,u)}=w_u^{s,\phi}\big|_{(0,u)},\quad \mathbb{P}-\text{a.s.}\tag{28}$$

Invoking again Assumption 1 as in (25),

$$\begin{aligned}w_t^{u,w_u^{s,\phi}}&=\phi+\int_s^u\overline{B}\left(r,w_u^{s,\phi}\right)\mathrm{d}r+\int_u^t\overline{B}\left(r,w_t^{u,w_u^{s,\phi}}\right)\mathrm{d}r+\int_s^t\sigma\left(r\right)\mathrm{d}W_r\\ &=\phi+\int_s^t\overline{B}\left(r,w_t^{u,w_u^{s,\phi}}\right)\mathrm{d}r+\int_s^t\sigma\left(r\right)\mathrm{d}W_r,\end{aligned}$$

hence the equality in (21) is inferred by the uniqueness of the solution of (20). The proof is now complete. □

Remark 1. The cocycle property in (21) (see also (28)) yields $w_t^{s,\phi}(\xi) = w_u^{s,\phi}(\xi)$ for a.e. $\xi \in (0,u)$, $\mathbb{P}$-a.s., for every $0 \le s \le u \le t \le T$ and $\phi \in \mathcal{H}^q$, $q \ge 2$.

Remark 2. The extended drift $\overline{B}\colon H \to H_\square$ enables us to introduce the abstract reformulation (20) of (17), see also (3) in Introduction 1. It cannot be used, however, to formulate (10) in an abstract setting, because an equation of the form

$$w_t = \phi + \int_0^t \overline{B}(s, w_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}W_s, \quad 0 \le t \le T,\, \phi \in \mathcal{H}^q,$$

is not well–posed. Indeed, given an H-valued process $w = (w_t)_{t\in[0,T]}$, one cannot deduce that $\overline{B}(s, w_s)$ belongs to H for a.e. $s \in (0,T)$. Hence the Bochner integral term in the equation above is not well–defined. This explains the advantage, for the abstract study conducted in Sections 3 and 4, of considering (17) instead of (10) as the infinite–dimensional reformulation of (1).

Remark 3. For every $p \in (2, (1-\alpha)^{-1})$, the fractional kernel k_2 in (5) belongs to the space $L^p\big(0,T;\mathbb{R}\big)$.

As a consequence, according to [36, Lemma 8.27, Theorem 8.29], the stochastic integral $\Sigma_{s,t}$ in (7) belongs to the space

$$\mathcal{L}_t^p = \left(L_t^p(\Omega; L^p), \|\cdot\|_{\mathcal{L}^p}\right), \quad \text{where} \quad L^p = L^p\big(0,T;\mathbb{R}^d\big).$$

As before, the subscript t in the previous expression indicates a space of $\mathcal{F}_t$-measurable random variables. Moreover, the following inequality holds (cf. (8)):

$$\|\Sigma_{s,t}\|_{\mathcal{L}^p} \le C_{d,p}\, \|k_2\|_p \sqrt{t-s}, \quad \text{for some } C_{d,p} > 0. \tag{29}$$

We denote by

$L_\square^p$ the Banach space $L^p\big((0,T)\times(0,T);\mathbb{R}^d\big)$, endowed with the norm $\|\cdot\|_{p,\square}$.

In addition to Assumption 1, suppose that $B\colon \Lambda \to L_\square^p$ and that it satisfies

$$\|B(w_1)\|_{p,\square} \le C_{0,p}\left(1 + \|w_1\|_p\right), \qquad \|B(w_1) - B(w_2)\|_{p,\square} \le C_{0,p}\,\|w_1 - w_2\|_p, \tag{30}$$

for every w_1, $w_2 \in \Lambda$, for some constant $C_{0,p} = C_{0,p}(d,T) > 0$. Note that $\overline{B}\colon H \to H_\square$ satisfies (30) for every w_1, $w_2 \in L^p$.

In this framework, one can argue as in the proof of Theorem 3 to infer that, for every $\phi \in \mathcal{L}_s^p$, there exists a unique solution $w_t^{s,\phi}$ of (20) belonging to the space $\mathcal{L}_t^p$.

The following corollary to Theorem 3 gives a Lipschitz–type dependence of the solution $w_t^{s,\phi}$ of (20) on the initial condition ϕ, which combined with (21) allows to prove the $\mathbb{F}$-Markov property of the process $(w_t^{s,\phi})_{t\in[s,T]}$.

Corollary 4. *Let $q \geq 2$. Under Assumption 1, there exists a constant $C_1 = C_1\left(d,q,T\right) > 0$ such that, for every $0 \leq s < t \leq T$,*

$$\left\| w_t^{s,\phi} - w_t^{s,\psi} \right\|_{\mathcal{H}^q} \leq C_1 \left\| \phi - \psi \right\|_{\mathcal{H}^q}, \quad \phi,\, \psi \in \mathcal{H}^q. \tag{31}$$

In addition, for all $s \in [0,T]$ and $\phi \in \mathcal{H}_s^q$, the process $(w_t^{s,\phi})_{t\in[s,T]}$ is $\mathbb{F}$-Markov, and

$$\mathbb{E}\left[\Phi\left(w_u^{s,\phi}\right)\Big|\mathcal{F}_t\right] = \mathbb{E}\left[\Phi\left(w_u^{t,\psi}\right)\right]\big|_{\psi = w_t^{s,\phi}}, \quad \mathbb{P}-a.s.,\ s \leq t \leq u \leq T,\ \Phi \in \mathcal{B}_b(H), \tag{32}$$

where $\mathcal{B}_b(H)$ denotes the space of bounded Borel measurable functions from H to $\mathbb{R}$.

Proof. Fix $q \geq 2$, $0 \leq s < t \leq T$ and consider $N = N\left(d,T\right) \in \mathbb{N}$ so big that $2C_0\sqrt{T/N} < 2^{1/q}$, where $C_0 = C_0\left(d,T\right)$ is the constant in (19). Moreover, take an equispaced partition $\{t_k\}_{k=0}^N$ of $[s,t]$ where $t_0 = s$ and $t_N = t$. By (19)-(20), for every $\phi,\, \psi \in \mathcal{H}^q$,

$$\begin{aligned}\left\| w_{t_1}^{s,\phi} - w_{t_1}^{s,\psi} \right\|_2^q &\leq 2^{q-1}\left\|\phi - \psi\right\|_2^q + 2^{q-1}\left(\frac{T}{N}\right)^{\frac{q}{2}} \left\| \overline{B}\left(w_{t_1}^{s,\phi}\right) - \overline{B}\left(w_{t_1}^{s,\psi}\right)\right\|_{2,\square}^q \\ &\leq 2^{q-1}\left\|\phi - \psi\right\|_2^q + 2^{q-1}C_0^q\left(\frac{T}{N}\right)^{\frac{q}{2}} \left\| w_{t_1}^{s,\phi} - w_{t_1}^{s,\psi} \right\|_2^q, \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

hence

$$\left\| w_{t_1}^{s,\phi} - w_{t_1}^{s,\psi} \right\|_2^q \leq 2^{q-1}\left(1 - 2^{q-1}C_0^q\left(\frac{T}{N}\right)^{\frac{q}{2}}\right)^{-1} \left\|\phi - \psi\right\|_2^q, \quad \mathbb{P}-\text{a.s.}$$

Thus, by the cocycle property in (21), for every $\phi,\, \psi \in \mathcal{H}^q$,

$$\begin{aligned}\left\| w_t^{s,\phi} - w_t^{s,\psi} \right\|_2^q &= \left\| w_{t_N}^{t_{N-1},w_{t_{N-1}}^{s,\phi}} - w_{t_N}^{t_{N-1},w_{t_{N-1}}^{s,\psi}} \right\|_2^q \\ &\leq 2^{q-1}\left(1 - 2^{q-1}C_0^q\left(\frac{T}{N}\right)^{\frac{q}{2}}\right)^{-1} \left\| w_{t_{N-1}}^{t_{N-2},w_{t_{N-2}}^{s,\phi}} - w_{t_{N-1}}^{t_{N-2},w_{t_{N-2}}^{s,\psi}} \right\|_2^q \\ &\leq 2^{N(q-1)}\left(1 - 2^{q-1}C_0^q\left(\frac{T}{N}\right)^{\frac{q}{2}}\right)^{-N} \left\|\phi - \psi\right\|_2^q, \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

which shows (31) upon taking expectations and q-th root, as desired.

The Markov property of the process $(w_t^{s,\phi})_{t\in[s,T]}$, $\phi \in \mathcal{H}_s^q$, is a consequence of (32). In turn, the equality in (32) can be readily obtained by similar steps as in the monotone class argument of [19, Theorem 9.14], which essentially relies on the cocycle property in (21) and the Lipschitz–continuous dependence in (31). Thus, the proof is complete. □

We conclude this part with a lemma which analyzes some properties of the solution $w_t^{s,\phi} \in \mathcal{L}_t^p$ of (20) in the framework of Remark 3. Recall that $\mathcal{L}_t^p = L_t^p(\Omega; L^p)$, where $L^p = L^p\big(0,T;\mathbb{R}^d\big)$, and that $L_\square^p = L^p\big(\left(0,T\right)\times\left(0,T\right);\mathbb{R}^d\big)$.

Lemma 5. *Suppose that $B\colon \Lambda \to L_\square^p$ satisfies Assumption 1 and (30), for some $p \in \big[2, (1-\alpha)^{-1}\big)$. Then there exists a constant $C_{1,p} = C_{1,p}(\alpha, d, T) > 0$ such that*

$$\left\|w_t^{s,\phi}\right\|_{\mathcal{L}^p} \leq C_{1,p}\left(1 + \|\phi\|_p\right), \quad 0 \leq s \leq t \leq T,\ \phi \in L^p. \tag{33}$$

Furthermore, for every $\phi \in L^p$, there is a constant $C_{\phi,p} = C_{\phi,p}(\alpha, d, T) > 0$ such that

$$\left\|w_t^{s,\phi} - \phi\right\|_{\mathcal{L}^p} \leq C_{\phi,p}\sqrt{t-s}, \quad 0 \leq s \leq t \leq T. \tag{34}$$

When $p = 2$, the hypotheses of Lemma 5 reduce to Assumption 1 and $\|\cdot\|_{\mathcal{L}^p} = \|\cdot\|_{\mathcal{H}^2}$.

Proof. Fix $0 \leq s \leq t \leq T$ and $\phi \in L^p$. Recall that, under the hypotheses of the lemma, the unique solution $w_t^{s,\phi} \in \mathcal{H}$ of (20) belongs to the space $\mathcal{L}_t^p$, see Remark 3.

Consider $N = N(d,p,T) \in \mathbb{N}$ so big that $C_{0,p}(2T/N)^{1-\frac{1}{p}} < 1$, where $C_{0,p}$ is the constant appearing in (30). Take an equispaced partition $\{t_k\}_{k=0}^N$ of $[s,t]$ with $t_0 = s$ and $t_N = t$: its mesh $\Delta \leq T/N$. By (20)-(30) we have, using Bochner's theorem and Jensen's inequality,

$$\left\|w_{t_1}^{s,\phi}\right\|_{\mathcal{L}^p} \leq \|\phi\|_p + C_{0,p}(2\Delta)^{1-\frac{1}{p}}\left(1 + \left\|w_{t_1}^{s,\phi}\right\|_{\mathcal{L}^p}\right) + \left\|\int_s^{t_1} \sigma(r)\,\mathrm{d}W_r\right\|_{\mathcal{L}^p},$$

which in turn implies, by (29), for some constant $c = c(d,p,T) > 0$,

$$\left\|w_{t_1}^{s,\phi}\right\|_{\mathcal{L}^p} \leq \left(1 - C_{0,p}\left(2T/N\right)^{1-\frac{1}{p}}\right)^{-1}\left(\|\phi\|_p + c\,\|k_2\|_p + C_{0,p}\left(2T/N\right)^{1-\frac{1}{p}}\right).$$

At this point, invoking N-times the cocycle property in (21) we obtain (33).

As for (34), using (29)-(30) we compute, for some constant $C = C(d,p) > 0$, recalling the notation $\Sigma_{s,t}$ introduced in (7),

$$\begin{aligned}\left\|w_t^{s,\phi} - \phi\right\|_{\mathcal{L}^p} &\leq \mathbb{E}\left[\left(\int_s^t \left\|\overline{B}\left(r, w_t^{s,\phi}\right)\right\|_p \mathrm{d}r\right)^p\right]^{\frac{1}{p}} + \|\Sigma_{s,t}\|_{\mathcal{L}^p}\\ &\leq (t-s)^{1-\frac{1}{p}}\,\mathbb{E}\left[\int_s^t \mathrm{d}r \int_0^T \left|\overline{B}\left(w_t^{s,\phi}\right)\right|^p (r,\xi)\,\mathrm{d}\xi\right]^{\frac{1}{p}} + C\,\|k_2\|_p\sqrt{t-s}\end{aligned}$$

$$\leq \sqrt{t-s}\left(C\left\|k_2\right\|_p + 2^{1-\frac{1}{p}}T^{\frac{1}{2}-\frac{1}{p}}C_{0,p}\left(1+\left\|w_t^{s,\phi}\right\|_{\mathcal{L}^p}\right)\right).$$

Thus, by (33) the proof is complete. □

3.1. First–order differentiability in the initial data

In this subsection we focus on deterministic initial conditions for (20), i.e., $\phi \in H$. From now on, we denote the Hilbert space $\mathcal{H}^2 = L^2(\Omega; H)$ simply by $\mathcal{H}$.

In order to study the first–order Fréchet differentiability of $w_t^{s,\phi}$ in H, we require hypotheses on B which are stronger than Assumption 1. In fact, we need some conditions on the Fréchet differentiability of B in the normed space $(\Lambda, \|\cdot\|_2)$. In the sequel, we write Λ_2 for $(\Lambda, \|\cdot\|_2)$ to have a compact notation.

Assumption 2. The map $B\colon \Lambda \to H_{\square}$ satisfies Assumption 1. Moreover, B is Λ_2-Fréchet differentiable, and there exists a constant $C_0 = C_0(d, T) > 0$ such that

$$\left\|DB(w_1)(w_2)\right\|_{2,\square} \leq C_0 \left\|w_2\right\|_2, \quad w_1, w_2 \in \Lambda, \tag{35}$$

and

$$\begin{aligned}\left\|DB(w_1) - DB(w_2)\right\|_{\mathcal{L}(\Lambda_2; H_{\square})} &\leq C_0 \left\|w_1 - w_2\right\|_2^{\gamma}, \\ w_1, w_2 \in \Lambda, &\text{ for some } \gamma \in (0,1].\end{aligned} \tag{36}$$

Without loss of generality, we assume the constant C_0 in (35)-(36) to be the same as the one in (19).

Under Assumption 2, precisely by (35) and the theorem of extension of uniformly continuous functions (see, e.g., [39, Exercise 13, Chapter 7]), for every $w_1 \in \Lambda$ it is possible to extend $DB(w_1) \in \mathcal{L}(\Lambda; H_{\square})$ to an operator $\overline{DB}(w_1) \in \mathcal{L}(H; H_{\square})$ satisfying (35) for all $w_2 \in H$. Moreover, by (36),

$$\begin{aligned}\left\|\overline{DB}(w_1) - \overline{DB}(w_2)\right\|_{\mathcal{L}(H;H_{\square})} &= \left\|DB(w_1) - DB(w_2)\right\|_{\mathcal{L}(\Lambda_2;H_{\square})} \\ &\leq C_0\left\|w_1 - w_2\right\|_2^{\gamma}, \quad w_1, w_2 \in \Lambda,\end{aligned} \tag{37}$$

hence we can extend (without changing the notation)

$$\overline{DB}\colon H \to \mathcal{L}(H; H_{\square}), \text{ with } \overline{DB} \text{ satisfying (35)-(37) for every } w_1, w_2 \in H. \tag{38}$$

We want to show that $\overline{B}$ is H-Fréchet differentiable, with $D\overline{B} = \overline{DB}$. By Taylor's formula applied on B, recalling that $\overline{B}\big|_{\Lambda} = B$ and $\overline{DB}(w_1)\big|_{\Lambda} = DB(w_1)$, $w_1 \in \Lambda$, we write

$$\overline{B}(w_2) - \overline{B}(w_1) - \overline{DB}(w_1)(w_2 - w_1) = r(w_1, w_2), \quad w_1, w_2 \in \Lambda, \text{ where} \tag{39}$$

$$r(x,y) = \int_0^1 \left(\overline{DB}\,(x + h\,(y-x)) - \overline{DB}\,(x)\right)(y-x)\,\mathrm{d}h, \quad x,\, y \in H.$$

Note that $r\colon H \times H \to H_\square$ is continuous. Indeed, for every $x, y \in H$ and every sequence $H \times H \ni (x_n, y_n) \to (x, y)$ as $n \to \infty$, by Bochner's theorem and (38), after some algebraic computations we deduce that

$$\begin{aligned}
&\|r(x_n,y_n) - r(x,y)\|_{2,\square} \\
&\quad\le \frac{C_0}{\gamma+1}\|y_n - x_n\|_2^\gamma \|y_n - y + x - x_n\|_2 \\
&\qquad + C_0\left(\frac{1}{\gamma+1}\|y_n - y + x - x_n\|_2^\gamma + 2\|x - x_n\|_2^\gamma\right)\|y - x\|_2 \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

It then follows from the continuity of $\overline{B}$ in H and (38) that (39) holds for every $w_1, w_2 \in H$. Moreover, since by (38) $\|r(x,y)\|_{2,\square} \le C_0(\gamma+1)^{-1}\|y-x\|_2^{1+\gamma}$ for every $x,\, y \in H$, we conclude that

$$\overline{B}\,(w_2) - \overline{B}\,(w_1) - \overline{DB}\,(w_1)\,(w_2 - w_1) = \mathrm{o}\left(\|w_2 - w_1\|_2\right), \quad w_1, w_2 \in H.$$

Therefore $\overline{B}$ is H-Fréchet differentiable, with $D\overline{B} = \overline{DB}$.

We also notice that, for every $w_1,\, w_2 \in H$ and $0 < t \le T$,

$$D\overline{B}(w_1)\,(r, w_2) \coloneqq [D\overline{B}(w_1)(w_2)]\,(r,\cdot) \in H \text{ is of Volterra–type, for a.e. } r \in (0,t), \tag{40}$$

and that

$$D\overline{B}(w_1)\,(r, w_2) \text{ depends on } w_i \text{ only via } w_i\big|_{(0,t)},\ i = 1,2, \text{ for a.e. } r \in (0,t); \tag{41}$$

these two properties are inherited from $\overline{B}$, see Assumption 1.

The next result shows that, under Assumption 2, the solution $w_t^{s,\phi}$ of (20), considered as a map from H to $\mathcal{H}$, is H-Fréchet differentiable.

Theorem 6. *Under Assumption 2, for every $0 \le s \le t \le T$, the mapping $w_t^{s,\cdot} \in C^{1+\gamma}(H;\mathcal{H})$. In particular, for every $\phi,\, \psi \in H$, $Dw_t^{s,\phi}\psi$ is the unique solution in $\mathcal{H}$ of the following equation:*

$$Dw_t^{s,\phi}\psi = \psi + \int_s^t D\overline{B}\left(w_t^{s,\phi}\right)\left(r, Dw_t^{s,\phi}\psi\right)\mathrm{d}r. \tag{42}$$

Furthermore, there exists a constant $C_2 = C_2(d,T) > 0$ such that, for every $\phi,\, \psi,\, \eta \in H$, $\mathbb{P}$-a.s.,

$$\left\|Dw_t^{s,\phi}\eta\right\|_2 \le C_2\|\eta\|_2, \qquad \left\|Dw_t^{s,\phi}\eta - Dw_t^{s,\psi}\eta\right\|_2 \le C_2\left\|w_t^{s,\phi} - w_t^{s,\psi}\right\|_2^\gamma\|\eta\|_2. \tag{43}$$

Proof. Fix $0 \leq s \leq t \leq T$ and $\phi \in H$. Firstly, we prove the well–posedness in $\mathcal{H}$ of the equation

$$w = \psi + \int_s^t D\overline{B}\left(w_t^{s,\phi}\right)(r, w)\,\mathrm{d}r, \quad \psi \in H. \tag{44}$$

Consider $N = N(d,T) \in \mathbb{N}$ so big that $C_0\sqrt{T/N} < 1$, where $C_0 = C_0(d,T)$ is the constant in Assumptions 1-2. In addition, take an equispaced partition $\{t_k\}_{k=0}^N$ of $[s,t]$ where $t_0 = s$ and $t_N = t$: its mesh $\Delta \leq T/N$. By (38) (see also (35)) and Bochner's theorem, the following estimate holds:

$$\begin{aligned}
&\left\| \int_{t_0}^{t_1} D\overline{B}\left(w_t^{s,\phi}\right)(w_1 - w_2)(r,\cdot)\,\mathrm{d}r \right\|_{\mathcal{H}} \\
&\quad \leq \sqrt{\Delta}\,\mathbb{E}\left[\int_{t_0}^{t_1} \int_0^T \left| D\overline{B}\left(w_t^{s,\phi}\right)(w_1 - w_2) \right|^2 (r,\xi)\,\mathrm{d}\xi\,\mathrm{d}r \right]^{\frac{1}{2}} \\
&\quad \leq \sqrt{\Delta}\,\mathbb{E}\left[\left\| D\overline{B}\left(w_t^{s,\phi}\right)(w_1 - w_2) \right\|_{2,\square}^2 \right]^{\frac{1}{2}} \leq C_0\sqrt{\Delta}\,\|w_1 - w_2\|_{\mathcal{H}}, \quad w_1, w_2 \in \mathcal{H}.
\end{aligned} \tag{45}$$

Thus, employing a fixed point argument as in the proof of Theorem 3, we deduce the existence of a unique solution $\overline{w}_1^{\psi} \in \mathcal{H}$ of (44) with t_1 instead of t, for every $\psi \in H$.

We claim that the operator $Dw_{t_1}^{s,\phi} \colon H \to \mathcal{H}$ defined by $Dw_{t_1}^{s,\phi}\psi = \overline{w}_1^{\psi}$, $\psi \in H$, is the Fréchet differential of $w_{t_1}^{s,\phi}$. Indeed, the linearity of $Dw_{t_1}^{s,\phi}$ is straightforward, while the continuity is ensured by the following computation, which can be argued from (44) similarly to (45):

$$\left\| Dw_{t_1}^{s,\phi}\psi \right\|_2 \leq \left(1 - C_0\sqrt{T/N}\right)^{-1} \|\psi\|_2, \quad \mathbb{P}-\text{a.s.}, \psi \in H. \tag{46}$$

Moreover, recalling (20)-(42),

$$\begin{aligned}
&\left\| w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi} - Dw_{t_1}^{s,\phi}h \right\|_{\mathcal{H}} \\
&\quad \leq \sqrt{\Delta}\,\mathbb{E}\left[\left\| \overline{B}\left(w_{t_1}^{s,\phi+h}\right) - \overline{B}\left(w_{t_1}^{s,\phi}\right) - D\overline{B}\left(w_{t_1}^{s,\phi}\right) Dw_{t_1}^{s,\phi}h \right\|_{2,\square}^2 \right]^{\frac{1}{2}} \\
&\quad \leq \sqrt{T/N}\Bigg(\mathbb{E}\left[\left\| D\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi} - Dw_{t_1}^{s,\phi}h\right) \right\|_{2,\square}^2 \right]^{\frac{1}{2}} \\
&\quad + \mathbb{E}\left[\left\| \int_0^1 \left(D\overline{B}\left(w_{t_1}^{s,\phi} + u\left(w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi}\right)\right) - D\overline{B}\left(w_{t_1}^{s,\phi}\right)\right)\left(w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi}\right)\mathrm{d}u \right\|_{2,\square}^2 \right]^{\frac{1}{2}} \Bigg)
\end{aligned}$$

$$\leq \sqrt{T/N}C_0\left(\left\|w_{t_1}^{s,\phi+h}-w_{t_1}^{s,\phi}-Dw_{t_1}^{s,\phi}h\right\|_{\mathcal{H}}+\mathbb{E}\left[\left\|w_{t_1}^{s,\phi+h}-w_{t_1}^{s,\phi}\right\|_2^{2(1+\gamma)}\right]^{\frac{1}{2}}\right),\quad h\in H, \tag{47}$$

where we apply Taylor's formula on $\overline{B}$ for the second inequality and (38) together with Bochner's theorem for the third. Notice that $H\subset\mathcal{H}^q$ for every $q\geq 2$. Therefore, by Corollary 4 with $q=2(1+\gamma)$, from (47) we infer that

$$\begin{aligned}\left\|w_{t_1}^{s,\phi+h}-w_{t_1}^{s,\phi}-Dw_{t_1}^{s,\phi}h\right\|_{\mathcal{H}}&\leq\sqrt{T/N}C_0C_1^{1+\gamma}\left(1-\sqrt{T/N}C_0\right)^{-1}\|h\|_2^{1+\gamma}\\&=\mathrm{o}\left(\|h\|_2\right),\quad h\in H,\end{aligned}\tag{48}$$

for some constant $C_1=C_1(\gamma,d,T)>0$. This shows that $Dw_{t_1}^{s,\phi}$ is the Fréchet differential of $w_{t_1}^{s,\phi}$, as desired.

Next, consider

$$w=Dw_{t_1}^{s,\phi}\psi+\int_{t_1}^{t_2}D\overline{B}\left(w_{t_2}^{s,\phi}\right)(r,w)\,\mathrm{d}r,\quad\psi\in H: \tag{49}$$

the well–posedness of this equation in $\mathcal{H}$ can be obtained via a fixed–point argument as in the above step. We denote by $\overline{w}_2^{\psi}\in\mathcal{H}$, $\psi\in H$, the unique solution of (49).

We argue that $\overline{w}_2^{\psi}$ is the unique solution of (44) with t_2 instead of t, for every $\psi\in H$. By the Volterra–type property of $D\overline{B}$ in (40) and (49) we have, $\mathbb{P}$-a.s.,

$$\overline{w}_2^{\psi}\Big|_{(0,t_1)}=Dw_{t_1}^{s,\phi}\psi\Big|_{(0,t_1)}.$$

Furthermore, thanks to the relation $w_{t_2}^{s,\phi}=w_{t_2}^{t_1,w_{t_1}^{s,\phi}}$ in (21) and the properties of $\overline{B}$ under Assumption 1 we can write, $\mathbb{P}$-a.s.,

$$w_{t_2}^{s,\phi}\Big|_{(0,t_1)}=w_{t_1}^{s,\phi}\Big|_{(0,t_1)}, \tag{50}$$

see Remark 1. Consequently, by the property of $D\overline{B}$ in (41) and recalling that $Dw_{t_1}^{s,\phi}\psi$ satisfies (44) with t_1 instead of t, from (49) we conclude that, $\mathbb{P}$-a.s.,

$$\begin{aligned}\overline{w}_2^{\psi}&=\psi+\int_s^{t_1}D\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(r,Dw_{t_1}^{s,\phi}\psi\right)\mathrm{d}r+\int_{t_1}^{t_2}D\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(r,\overline{w}_2^{\psi}\right)\mathrm{d}r\\&=\psi+\int_s^{t_2}D\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(r,\overline{w}_2^{\psi}\right)\mathrm{d}r.\end{aligned}\tag{51}$$

Hence $\overline{w}_2^{\psi}$ solves (44) with t replaced by t_2; to prove that it is in fact the unique solution, we consider another random variable $\widetilde{w} \in \mathcal{H}$ satisfying (51). Then, by (40)-(41),

$$1_{(0,t_1)}\widetilde{w} = 1_{(0,t_1)}\left(\psi + \int_s^{t_1} D\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(r, 1_{(0,t_1)}\widetilde{w}\right)\mathrm{d}r\right). \tag{52}$$

We observe that also $1_{(0,t_1)}\overline{w}_1^{\psi} \in \mathcal{H}$ satisfies (52). Therefore, using Bochner's theorem and Jensen's inequality, by (38) we can compute

$$\begin{aligned}\left\|1_{(0,t_1)}\left(\overline{w}_1^{\psi} - \widetilde{w}\right)\right\|_{\mathcal{H}}^2 &\le \mathbb{E}\left[\left(\int_s^{t_1}\left\|D\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(r, 1_{(0,t_1)}\left(\overline{w}_1^{\psi} - \widetilde{w}\right)\right)\right\|_2 \mathrm{d}r\right)^2\right] \\ &\le \Delta\mathbb{E}\left[\left\|D\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(1_{(0,t_1)}\left(\overline{w}_1^{\psi} - \widetilde{w}\right)\right)\right\|_{2,\square}^2\right] \\ &\le \Delta C_0^2\left\|1_{(0,t_1)}\left(\overline{w}_1 - \widetilde{w}\right)\right\|_{\mathcal{H}}^2,\end{aligned} \tag{53}$$

which allow us to conclude, recalling that $\sqrt{\Delta}C_0 < 1$,

$$1_{(0,t_1)}\widetilde{w} = 1_{(0,t_1)}\overline{w}_1^{\psi}, \quad \mathbb{P} - \text{a.s.}$$

Going back to (51), by (44) and the previous equality we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}\widetilde{w} &= \psi + \int_s^{t_1} D\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(r, \overline{w}_1^{\psi}\right)\mathrm{d}r + \int_{t_1}^{t_2} D\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(r, \widetilde{w}\right)\mathrm{d}r \\ &= \overline{w}_1^{\psi} + \int_{t_1}^{t_2} D\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(r, \widetilde{w}\right)\mathrm{d}r.\end{aligned}$$

It follows that $\widetilde{w}$ satisfies (49): by uniqueness, we obtain $\widetilde{w} = \overline{w}_2^{\psi}$. Hence $\overline{w}_2^{\psi}$ is the unique solution of (44) in $\mathcal{H}$ with t_2 instead of t.

We define the operator $Dw_{t_2}^{s,\phi}\colon H \to \mathcal{H}$ by $Dw_{t_2}^{s,\phi}\psi = \overline{w}_2^{\psi}$, $\psi \in H$, and claim that it is the Fréchet differential of $w_{t_2}^{s,\phi}$. To see this, note that the linearity of $Dw_{t_2}^{s,\phi}$ is a consequence of the well–posedness of (51). As for the continuity, it is ensured by the following computations, where we use (38)-(46)-(49):

$$\begin{aligned}\left\|Dw_{t_2}^{s,\phi}\psi\right\|_2 &\le \left\|Dw_{t_1}^{s,\phi}\psi\right\|_2 + \int_{t_1}^{t_2}\left\|D\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(r, Dw_{t_2}^{s,\phi}\psi\right)\right\|_2 \mathrm{d}r \\ &\le \left(1 - C_0\sqrt{T/N}\right)^{-1}\|\psi\|_2 + \sqrt{\Delta}C_0\left\|Dw_{t_2}^{s,\phi}\psi\right\|_2, \quad \mathbb{P} - \text{a.s.}, \ \psi \in H,\end{aligned}$$

whence

$$\left\| Dw_{t_2}^{s,\phi}\psi \right\|_2 \leq \left(1 - C_0\sqrt{T/N}\right)^{-2} \|\psi\|_2 , \quad \mathbb{P}-\text{a.s.},\ \psi \in H. \tag{54}$$

Moreover, by the cocycle property in (21) and reasoning as in (47), by (20)-(49) we obtain, for some constant $c > 0$,

$$\begin{aligned} &\left\| w_{t_2}^{s,\phi+h} - w_{t_2}^{s,\phi} - Dw_{t_2}^{s,\phi}h \right\|_{\mathcal{H}} \\ &\quad = \left\| w_{t_2}^{t_1, w_{t_1}^{s,\phi+h}} - w_{t_2}^{t_1, w_{t_1}^{s,\phi}} - Dw_{t_1}^{s,\phi}h - \int\limits_{t_1}^{t_2} D\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(r, Dw_{t_2}^{s,\phi}h\right) \mathrm{d}r \right\|_{\mathcal{H}} \\ &\quad \leq \left\| w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi} - Dw_{t_1}^{s,\phi}h \right\|_{\mathcal{H}} \\ &\qquad + \left\| \int\limits_{t_1}^{t_2} \left(\overline{B}\left(w_{t_2}^{s,\phi+h}\right) - \overline{B}\left(w_{t_2}^{s,\phi}\right) - D\overline{B}\left(w_{t_2}^{s,\phi}\right) Dw_{t_2}^{s,\phi}h \right)(r,\cdot)\, \mathrm{d}r \right\|_{\mathcal{H}} \\ &\quad \leq c\, \|h\|_2^{1+\gamma} = \mathrm{o}\left(\|h\|_2\right), \quad h \in H, \end{aligned} \tag{55}$$

where we also employ (48) in the last inequality. This shows that $Dw_{t_2}^{s,\phi}$ is the Fréchet differential of $w_{t_2}^{s,\phi}$, as desired.

Repeating this argument N-times, we deduce that the operator $Dw_t^{s,\phi} \colon H \to \mathcal{H}$ defined by $Dw_t^{s,\phi}\psi = \overline{w}_N^{\psi}$, where $\overline{w}_N^{\psi}$ is the unique solution of (44) in $\mathcal{H}$, for every $\psi \in H$, is the Fréchet differential of $w_t^{s,\phi}$. In particular, the first bound in (43) is true, because (cf. (46)-(54))

$$\left\| Dw_t^{s,\phi}\psi \right\|_2 \leq \left(1 - C_0\sqrt{T/N}\right)^{-N} \|\psi\|_2 =: \overline{C}\, \|\psi\|_2 , \quad \mathbb{P}-\text{a.s.},\ \phi, \psi \in H. \tag{56}$$

As regards the second inequality in (43), by (38), (42) and (56) we have, for every ϕ, ψ, $\eta \in H$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \left\| Dw_{t_1}^{s,\phi}\eta - Dw_{t_1}^{s,\psi}\eta \right\|_2 &= \left\| \int\limits_s^{t_1} \left(D\overline{B}\left(w_t^{s,\phi}\right) Dw_{t_1}^{s,\phi}\eta - D\overline{B}\left(w_t^{s,\psi}\right) Dw_{t_1}^{s,\psi}\eta \right)(r,\cdot)\, \mathrm{d}r \right\|_2 \\ &\leq \sqrt{\Delta} \left(\left\| D\overline{B}\left(w_t^{s,\phi}\right)\left(Dw_{t_1}^{s,\phi}\eta - Dw_{t_1}^{s,\psi}\eta\right) \right\|_{2,\square} \right. \\ &\qquad \left. + \left\| \left(D\overline{B}\left(w_t^{s,\phi}\right) - D\overline{B}\left(w_t^{s,\psi}\right)\right) Dw_{t_1}^{s,\psi}\eta \right\|_{2,\square} \right) \\ &\leq C_0\sqrt{T/N} \left(\left\| Dw_{t_1}^{s,\phi}\eta - Dw_{t_1}^{s,\psi}\eta \right\|_2 + \overline{C} \left\| w_t^{s,\phi} - w_t^{s,\psi} \right\|_2^{\gamma} \|\eta\|_2 \right), \end{aligned}$$

where in the first equality we also use (41) and (50) with t instead of t_2. It follows that

$$\left\|Dw_{t_1}^{s,\phi}\eta - Dw_{t_1}^{s,\psi}\eta\right\|_2 \leq \left(1 - C_0\sqrt{T/N}\right)^{-1} C_0\overline{C}\sqrt{T/N}\left\|w_t^{s,\phi} - w_t^{s,\psi}\right\|_2^{\gamma}\|\eta\|_2 .$$

By (49), we sequentially iterate this computation to obtain the second bound in (43) with

$$C_2 = \max\{\overline{C}, NC_0\overline{C}^2\sqrt{T/N}\}.$$

At this point, taking expectations and using Corollary 4 with $q = 2\gamma$ (recall that $H \subset \mathcal{H}^q$), by Jensen's inequality we infer that, for some constant $C > 0$,

$$\begin{aligned}\left\|Dw_t^{s,\phi} - Dw_t^{s,\psi}\right\|_{\mathcal{L}(H;\mathcal{H})} &= \sup_{\|\eta\|_2 \leq 1} \mathbb{E}\left[\left\|Dw_t^{s,\phi}\eta - Dw_t^{s,\psi}\eta\right\|_2^2\right]^{\frac{1}{2}} \\ &\leq C_2\mathbb{E}\left[\left\|w_t^{s,\phi} - w_t^{s,\psi}\right\|_2^{2\gamma}\right]^{\frac{1}{2}} \\ &\leq C\|\phi - \psi\|_2^{\gamma}, \quad \phi, \psi \in H.\end{aligned}$$

This shows that $Dw_t^{s,\cdot} \in C^{\gamma}(H; \mathcal{L}(H;\mathcal{H}))$, completing the proof. □

3.2. Second–order differentiability in the initial data

Recalling the normed space $\Lambda_2 = \left(\Lambda, \|\cdot\|_2\right)$, in the sequel we identify $\mathcal{L}(\Lambda_2; \mathcal{L}(\Lambda_2; H_\square))$ with the space $\mathcal{L}(\Lambda_2, \Lambda_2; H_\square)$ of bilinear forms from $\Lambda_2 \times \Lambda_2$ to $H_\square$ in the usual way.

For the purpose of investigating the second–order Fréchet differential in H of $w_t^{s,\phi}$, we need to require another condition on B.

Assumption 3. The map $B\colon \Lambda \to H_\square$ satisfies Assumption 2. Moreover, B is twice Λ_2-Fréchet differentiable, and there exists a constant $C_0 = C_0(d,T) > 0$ such that

$$\left\|D^2B(w_1)(w_2, w_3)\right\|_{2,\square} \leq C_0\|w_2\|_2\|w_3\|_2, \quad w_1, w_2, w_3 \in \Lambda, \tag{57}$$

and

$$\begin{aligned}&\left\|D^2B(w_1) - D^2B(w_2)\right\|_{\mathcal{L}(\Lambda_2,\Lambda_2;H_\square)} \leq C_0\|w_1 - w_2\|_2^{\beta}, \\ &\qquad w_1, w_2 \in \Lambda, \text{ for some } \beta \in (0,1].\end{aligned} \tag{58}$$

Once again, we can assume that the constant C_0 in (57)-(58) is the same as the one in (19) and (35)-(36).

By (57), we invoke the theorem of extension of uniformly continuous functions (see, e.g., [39, Exercise 13, Chapter 7]) to extend, for every $w_1, w_2 \in \Lambda$, the map $D^2B(w_1)(w_2, \cdot) \in \mathcal{L}(\Lambda_2; H_\square)$ to an operator $\overline{D^2B}(w_1)(w_2, \cdot) \in \mathcal{L}(H; H_\square)$ satisfying (57) for all $w_3 \in H$. It follows that, by linearity,

$$\begin{aligned}\left\|\overline{D^2B}\left(w_1\right)\left(w_2\right)-\overline{D^2B}\left(w_1\right)\left(w_3\right)\right\|_{\mathcal{L}(H;H_\square)} &= \left\|\overline{D^2B}\left(w_1\right)\left(w_2-w_3\right)\right\|_{\mathcal{L}(H;H_\square)}\\ &\le C_0\left\|w_2-w_3\right\|_2, \quad w_1,w_2,w_3\in\Lambda,\end{aligned}$$

hence we can extend (without changing notation) $\overline{D^2B}(w_1)\in\mathcal{L}(H,H;H_\square)$, for all $w_1\in\Lambda$. At this point, by (58) we infer that, for every $w_1,w_2\in\Lambda$,

$$\begin{aligned}\left\|\overline{D^2B}\left(w_1\right)-\overline{D^2B}\left(w_2\right)\right\|_{\mathcal{L}(H,H;H_\square)} &= \left\|\overline{D^2B}\left(w_1\right)-\overline{D^2B}\left(w_2\right)\right\|_{\mathcal{L}(\Lambda_2,\Lambda_2;H_\square)}\\ &\le C_0\left\|w_1-w_2\right\|_2^\beta, \end{aligned}\tag{59}$$

whence, via another extension, from now on we consider

$$\overline{D^2B}\colon H\to\mathcal{L}(H,H;H_\square)\text{ satisfying (57)-(59) for every } w_i\in H,\ i=1,2,3. \tag{60}$$

We want to show that $\overline{B}$ is twice H-Fréchet differentiable, with $D^2\overline{B}=\overline{D^2B}$. By Taylor's formula applied to DB,

$$\begin{aligned}&\left(D\overline{B}\left(w_2\right)-D\overline{B}\left(w_1\right)-\overline{D^2B}\left(w_1\right)\left(w_2-w_1\right)\right)w_3=r\left(w_1,w_2,w_3\right),\\ &\qquad w_1,w_2,w_3\in\Lambda,\ \text{where} \qquad\qquad (61)\\ &r\left(x,y,z\right)=\left(\int_0^1\left(\overline{D^2B}\left(x+h\left(y-x\right)\right)-\overline{D^2B}\left(x\right)\right)\left(y-x\right)\mathrm{d}h\right)z,\quad x,\ y,\ z\in H.\end{aligned}$$

We note that $r\colon H\times H\times H\to H_\square$ is continuous. Indeed, for every $x,y,z\in H$ and every sequence $((x_n,y_n,z_n))_n\subset H\times H\times H$ such that $(x_n,y_n,z_n)\to(x,y,z)$ as $n\to\infty$, with some algebraic computations we obtain, by (60),

$$\begin{aligned}&\left\|r\left(x_n,y_n,z_n\right)-r\left(x,y,z\right)\right\|_{2,\square}\le 2C_0\left\|y_n-x_n\right\|_2\left\|z_n-z\right\|_2\\ &\quad+C_0\left\|z\right\|_2\Bigg(2\left\|y_n-x_n+x-y\right\|_2\\ &\quad+\left(\frac{1}{\beta+1}\left\|y_n-y+x-x_n\right\|_2^\beta+2\left\|x_n-x\right\|_2^\beta\right)\left\|y-x\right\|_2\Bigg)\xrightarrow[n\to\infty]{}0.\end{aligned}$$

It then follows from the continuity of $D\overline{B}$ in H and (60) that (61) holds for every $w_1,w_2,w_3\in H$. Moreover, observing that, by (60), $\left\|r\left(x,y,\cdot\right)\right\|_{\mathcal{L}(H;H_\square)}\le C_0(\beta+1)^{-1}\left\|y-x\right\|_2^{1+\beta}$, $x,y\in H$, we conclude that

$$D\overline{B}\left(w_2\right)-D\overline{B}\left(w_1\right)-\overline{D^2B}\left(w_1\right)\left(w_2-w_1\right)=\mathrm{o}\left(\left\|w_2-w_1\right\|_2\right),\quad w_1,w_2\in H.$$

Therefore $\overline{B}$ is twice H-Fréchet differentiable, with $D^2\overline{B}=\overline{D^2B}$.

We also note that, for every w_1, w_2, $w_3 \in H$ and $0 < t \le T$,

$$D^2\overline{B}(w_1)\,(w_2, w_3)\,(r,\cdot) \in H \text{ is of Volterra–type, for a.e. } r \in (0,t), \tag{62}$$

and that

$$D^2\overline{B}(w_1)\,(w_2, w_3)\,(r,\cdot) \text{ depends on } w_i \text{ only via } w_i\big|_{(0,t)},\ i = 1,2,3, \text{ for a.e. } r \in (0,t); \tag{63}$$

these properties are inherited from $D\overline{B}$ (cf. (40)-(41) in the discussion following Assumption 2).

In conclusion, we notice that, by (60) (see also (57)),

$$\left\|D^2\overline{B}\right\|_\infty = \sup_{w\in H}\left\|D^2\overline{B}(w)\right\|_{\mathcal{L}(H,H;H_\square)} \le C_0. \tag{64}$$

As a consequence, by the mean value theorem we deduce that (37) (see also (38)) holds with $\gamma = 1$, i.e., under Assumption 3 the map $D\overline{B}\colon H \to \mathcal{L}(H;H_\square)$ is globally Lipschitz–continuous. Since $D\overline{B}$ is also bounded (see (35)-(38)), in what follows we suppose, without loss of generality, that

$$\text{under Assumption 3, } D\overline{B}\colon H \to \mathcal{L}(H;H_\square) \text{ satisfies (38) with } \gamma = \beta. \tag{65}$$

The next result shows that, in the framework of this subsection, the solution $w_t^{s,\phi}$ of (20), considered as a map from H to $\mathcal{H}$, is twice H-Fréchet differentiable.

Theorem 7. *Under Assumption 3, for every $0 \le s \le t \le T$, the mapping $w_t^{s,\cdot} \in C^{2+\beta}(H;\mathcal{H})$. In particular, for every ϕ, ψ, $\eta \in H$, $D^2 w_t^{s,\phi}(\psi,\eta)$ is the unique solution in $\mathcal{H}$ of the following equation:*

$$\begin{aligned} D^2 w_t^{s,\phi}(\psi,\eta) = \int_s^t \Big(&D^2\overline{B}\left(w_t^{s,\phi}\right)\left(Dw_t^{s,\phi}\psi, Dw_t^{s,\phi}\eta\right) \\ &+ D\overline{B}\left(w_t^{s,\phi}\right) D^2 w_t^{s,\phi}(\psi,\eta)\Big)(r,\cdot)\,\mathrm{d}r. \end{aligned} \tag{66}$$

Furthermore, there exists a constant $C_3 = C_3(d,T) > 0$ such that, for every $\phi,\psi,\eta,\theta \in H$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \left\|D^2 w_t^{s,\phi}(\eta,\theta)\right\|_2 &\le C_3\,\|\eta\|_2\,\|\theta\|_2, \\ \left\|\left(D^2 w_t^{s,\phi} - D^2 w_t^{s,\psi}\right)(\eta,\theta)\right\|_2 &\le C_3\left\|w_t^{s,\phi} - w_t^{s,\psi}\right\|_2^\beta\,\|\eta\|_2\,\|\theta\|_2. \end{aligned} \tag{67}$$

Proof. Fix $0 \le s \le t \le T$ and $\phi \in H$. We first want to prove the well–posedness in $\mathcal{H}$ of the equation

$$w = \int_s^t \left(D^2 \overline{B}\left(w_t^{s,\phi}\right)\left(Dw_t^{s,\phi}\psi, Dw_t^{s,\phi}\eta\right) + D\overline{B}\left(w_t^{s,\phi}\right) w \right)(r,\cdot)\,\mathrm{d}r, \quad \psi, \eta \in H. \tag{68}$$

Consider $N = N(d,T) \in \mathbb{N}$ so big that $C_0\sqrt{T/N} < 1$, where $C_0 = C_0(d,T)$ is the constant in Assumptions 1-2-3. In addition, take an equispaced partition $\{t_k\}_{k=0}^N$ of $[s,t]$ where $t_0 = s$ and $t_N = t$: its mesh $\Delta \leq T/N$. Under Assumption 3, the bound in (45) holds and allows to employ a fixed point argument as in the proof of Theorem 3 (see also Theorem 6) to deduce the existence of a unique solution $\overline{w}_1^{\psi,\eta} \in \mathcal{H}$ of (68) with t_1 instead of t, for every $\psi, \eta \in H$.

We claim that the operator $D^2 w_{t_1}^{s,\phi} \colon H \times H \to \mathcal{H}$ defined by $D^2 w_{t_1}^{s,\phi}(\psi,\eta) = \overline{w}_1^{\psi,\eta}$, $\psi, \eta \in H$, is the second–order Fréchet differential of $w_{t_1}^{s,\phi}$. Indeed, considering that $Dw_{t_1}^{s,\phi} \in \mathcal{L}(H;\mathcal{H})$, $D\overline{B}(w_{t_1}^{s,\phi}) \in \mathcal{L}(H;H_\square)$ and $D^2\overline{B}(w_{t_1}^{s,\phi}) \in \mathcal{L}(H,H;H_\square)$, the fact that $D^2 w_{t_1}^{s,\phi}$ is bilinear directly follows from (68). As for the boundedness, by (38)-(60) (see also (35)-(57)) and (43) we can compute, applying Bochner's theorem to (66), for some constant $C_2 = C_2(d,T) > 0$,

$$\begin{aligned} \left\| D^2 w_{t_1}^{s,\phi}(\psi,\eta) \right\|_2 &\leq C_0\sqrt{\Delta}\left(\left\| Dw_{t_1}^{s,\phi}\psi \right\|_2 \left\| Dw_{t_1}^{s,\phi}\eta \right\|_2 + \left\| D^2 w_{t_1}^{s,\phi}(\psi,\eta) \right\|_2 \right) \\ &\leq C_0\sqrt{T/N}\left(C_2 \|\psi\|_2 \|\eta\|_2 + \left\| D^2 w_{t_1}^{s,\phi}(\psi,\eta) \right\|_2 \right), \quad \mathbb{P}-\text{a.s.}, \psi,\eta \in H. \end{aligned} \tag{69}$$

Hence

$$\left\| D^2 w_{t_1}^{s,\phi}(\psi,\eta) \right\|_2 \leq \left(1 - C_0\sqrt{T/N}\right)^{-1} C_0 C_2 \sqrt{T/N}\, \|\psi\|_2 \|\eta\|_2, \quad \mathbb{P}-\text{a.s.}, \psi,\eta \in H. \tag{70}$$

We now observe that, by Taylor's formula applied to $D\overline{B}$ (cf. (61)), from (42)-(66) we have, for every $h \in H$,

$$\left\| Dw_{t_1}^{s,\phi+h} - Dw_{t_1}^{s,\phi} - D^2 w_{t_1}^{s,\phi} h \right\|_{\mathcal{L}(H;\mathcal{H})} \leq \mathbf{I}_1 + \mathbf{II}_1 + \mathbf{III}_1 + \mathbf{IV}_1, \tag{71}$$

where we set

$$\mathbf{I}_1 = \sup_{\|\eta\|_2 \leq 1} \mathbb{E}\left[\left\| \int_s^{t_1} D\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(Dw_{t_1}^{s,\phi+h}\eta - Dw_{t_1}^{s,\phi}\eta - D^2 w_{t_1}^{s,\phi}(h,\eta) \right)(r,\cdot)\,\mathrm{d}r \right\|_2^2 \right]^{\frac{1}{2}},$$

$$\mathbf{II}_1 = \sup_{\|\eta\|_2 \leq 1} \mathbb{E}\left[\left\| \int_s^{t_1} \left(D^2\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi} - Dw_{t_1}^{s,\phi}h, Dw_{t_1}^{s,\phi}\eta \right) \right)(r,\cdot)\,\mathrm{d}r \right\|_2^2 \right]^{\frac{1}{2}},$$

$$\mathbf{III}_1 = \sup_{\|\eta\|_2 \leq 1} \mathbb{E}\left[\left\| \int_s^{t_1} \left(D\overline{B}\left(w_{t_1}^{s,\phi+h}\right) - D\overline{B}\left(w_{t_1}^{s,\phi}\right) \right)\left(Dw_{t_1}^{s,\phi+h}\eta - Dw_{t_1}^{s,\phi}\eta \right)(r,\cdot)\,\mathrm{d}r \right\|_2^2 \right]^{\frac{1}{2}},$$

$$\begin{aligned}\mathbf{IV}_1 = \sup_{\|\eta\|_2\le 1} \mathbb{E}\Bigg[\Bigg\| \int_s^{t_1} \Bigg(\int_0^1 \Big(& D^2\overline{B}\left(w_{t_1}^{s,\phi} + v\left(w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi}\right)\right) \\ & -D^2\overline{B}\left(w_{t_1}^{s,\phi}\right)\Big)\left(w_{t_1}^{s,\phi+h}-w_{t_1}^{s,\phi}\right)\mathrm{d}v\Bigg) Dw_{t_1}^{s,\phi}\eta\,(r,\cdot)\,\mathrm{d}r\Bigg\|_2^2\Bigg]^{\frac{1}{2}}.\end{aligned}$$

By (38) (see, in particular, (35))

$$\begin{aligned}|\mathbf{I}_1| &\le C_0\sqrt{T/N}\sup_{\|\eta\|_2\le 1}\mathbb{E}\left[\left\|Dw_{t_1}^{s,\phi+h}\eta - Dw_{t_1}^{s,\phi}\eta - D^2 w_{t_1}^{s,\phi}(h,\eta)\right\|_2^2\right]^{\frac{1}{2}} \\ &= C_0\sqrt{T/N}\left\|Dw_{t_1}^{s,\phi+h} - Dw_{t_1}^{s,\phi} - D^2w_{t_1}^{s,\phi}h\right\|_{\mathcal{L}(H;\mathcal{H})}.\end{aligned}$$

Moreover, considering (37)-(43) (see also (65)) and Corollary 4, which we can apply with $q = 2(1+\beta)$ because $\phi, h \in H \subset \mathcal{H}^q$ (see also), for some $C_1 = C_1(\beta, d, T) > 0$ we can write

$$\begin{aligned}|\mathbf{III}_1| &\le \sqrt{\Delta}\sup_{\|\eta\|_2\le 1}\mathbb{E}\left[\left\|\left(D\overline{B}\left(w_{t_1}^{s,\phi+h}\right) - D\overline{B}\left(w_{t_1}^{s,\phi}\right)\right)\left(Dw_{t_1}^{s,\phi+h}\eta - Dw_{t_1}^{s,\phi}\eta\right)\right\|_{2,\square}^2\right]^{\frac{1}{2}} \\ &\le \left\|D^2\overline{B}\right\|_\infty C_2\sqrt{T/N}\sup_{\|\eta\|_2\le 1}\mathbb{E}\left[\left\|w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi}\right\|_2^{2(1+\beta)}\|\eta\|_2^2\right]^{\frac{1}{2}} \\ &\le C_0C_1^{1+\beta}C_2\sqrt{T/N}\,\|h\|_2^{1+\beta},\end{aligned}$$

where we also use the mean value theorem on $D\overline{B}$ and (64). As for $\mathbf{II}_1$, by (43)-(60) we compute

$$\begin{aligned}|\mathbf{II}_1| &\le \sqrt{\Delta}\sup_{\|\eta\|_2\le 1}\mathbb{E}\left[\left\|D^2\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi} - Dw_{t_1}^{s,\phi}h, Dw_{t_1}^{s,\phi}\eta\right)\right\|_{2,\square}^2\right]^{\frac{1}{2}} \\ &\le C_0C_2\sqrt{\Delta}\sup_{\|\eta\|_2\le 1}\mathbb{E}\left[\left\|w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi} - Dw_{t_1}^{s,\phi}h\right\|_2^2\|\eta\|_2^2\right]^{\frac{1}{2}} \\ &\le C_0C_2\sqrt{T/N}\left\|w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi} - Dw_{t_1}^{s,\phi}h\right\|_{\mathcal{H}} = \mathrm{o}\left(\|h\|_2\right).\end{aligned}$$

Finally, again by (43)-(60) (see also (59)) and Corollary 4, employed with $q = 2(1+\beta)$, we have

$$\begin{aligned}|\mathbf{IV}_1| \le \sqrt{\Delta}\mathbb{E}\Bigg[\Bigg(\int_0^1 \left\|D^2\overline{B}\left(w_{t_1}^{s,\phi} + v\left(w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi}\right)\right) - D^2\overline{B}\left(w_{t_1}^{s,\phi}\right)\right\|_{\mathcal{L}(H,H;H_\square)}\mathrm{d}v\Bigg)^2 & \\ \times\left\|w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi}\right\|_2^2\left\|Dw_{t_1}^{s,\phi}\eta\right\|_2^2\Bigg]^{\frac{1}{2}} &\end{aligned}$$

$$\leq C_0 C_2 \sqrt{T/N} \sup_{\|\eta\|_2 \leq 1} \mathbb{E}\left[\left\|w_{t_1}^{s,\phi+h} - w_{t_1}^{s,\phi}\right\|_2^{2(1+\beta)} \|\eta\|_2^2\right]^{\frac{1}{2}}$$

$$\leq C_0 C_1^{1+\beta} C_2 \sqrt{T/N} \, \|h\|_2^{1+\beta}.$$

Going back to (71), we conclude that

$$\left\|D w_{t_1}^{s,\phi+h} - D w_{t_1}^{s,\phi} - D^2 w_{t_1}^{s,\phi} h\right\|_{\mathcal{L}(H;\mathcal{H})} \leq \left(1 - C_0\sqrt{T/N}\right)^{-1} (\mathbf{II}_1 + \mathbf{III}_1 + \mathbf{IV}_1)$$

$$= \mathrm{o}\left(\|h\|_2\right), \quad h \in H. \tag{72}$$

This shows that $D^2 w_{t_1}^{s,\phi}$ is the second–order Fréchet differential of $w_{t_1}^{s,\phi}$, as desired.

Next, consider

$$w = D^2 w_{t_1}^{s,\phi}(\psi,\eta) + \int_{t_1}^{t_2} \left(D^2\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(D w_{t_2}^{s,\phi}\psi, D w_{t_2}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_2}^{s,\phi}\right) w\right)(r,\cdot)\,\mathrm{d}r,$$

$$\psi, \eta \in H. \tag{73}$$

Arguing as in the previous step, we infer the well–posedness of this equation in $\mathcal{H}$: we denote by $\overline{w}_2^{\psi,\eta} \in \mathcal{H}$ its unique solution, for every ψ, $\eta \in H$.

Given ψ, $\eta \in H$, we now show that $\overline{w}_2^{\psi,\eta}$ is the unique solution of (68) with t_2 instead of t. By the Volterra–type property of $D^2\overline{B}$ [resp., $D\overline{B}$] in (62) [resp., (40)] and (73) we have, $\mathbb{P}$-a.s.,

$$\overline{w}_2^{\psi,\eta}\Big|_{(0,t_1)} = D^2 w_{t_1}^{s,\phi}(\psi,\eta)\Big|_{(0,t_1)}.$$

Moreover, since $D w_{t_2}^{s,\phi}\psi$ satisfies (49), we infer that, $\mathbb{P}$-a.s.,

$$D w_{t_2}^{s,\phi}\psi\Big|_{(0,t_1)} = D w_{t_1}^{s,\phi}\psi\Big|_{(0,t_1)},$$

with an analogous result holding for η. Consequently, recalling also (50) and Remark 1, by the property of $D^2\overline{B}$ [resp., $D\overline{B}$] in (63) [resp., (41)], from (73) we obtain, $\mathbb{P}$-a.s.,

$$\overline{w}_2^{\psi,\eta} = \int_s^{t_1} \left(D^2\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(D w_{t_1}^{s,\phi}\psi, D w_{t_1}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_1}^{s,\phi}\right) D^2 w_{t_1}^{s,\phi}(\psi,\eta)\right)(r,\cdot)\,\mathrm{d}r$$

$$+ \int_{t_1}^{t_2} \left(D^2\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(D w_{t_2}^{s,\phi}\psi, D w_{t_2}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_2}^{s,\phi}\right)\overline{w}_2^{\psi,\eta}\right)(r,\cdot)\,\mathrm{d}r$$

$$= \int_s^{t_2} \left(D^2\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(D w_{t_2}^{s,\phi}\psi, D w_{t_2}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_2}^{s,\phi}\right)\overline{w}_2^{\psi,\eta}\right)(r,\cdot)\,\mathrm{d}r, \tag{74}$$

where we also use the fact that $D^2 w_{t_1}^{s,\phi}(\psi,\eta)$ solves (68) with t_1 instead of t. Hence $\overline{w}_2^{\psi,\eta}$ solves (68) with t replaced by t_2. In order to prove that it is in fact the unique solution of this equation, we consider another random variable $\widetilde{w} \in \mathcal{H}$ satisfying (74). Then, by (62)-(63),

$$1_{(0,t_1)}\widetilde{w} = 1_{(0,t_1)}\left(\int_s^{t_1}\left(D^2\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(Dw_{t_1}^{s,\phi}\psi, Dw_{t_1}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_1}^{s,\phi}\right)1_{(0,t_1)}\widetilde{w}\right)(r,\cdot)\,\mathrm{d}r\right). \tag{75}$$

We observe that also $1_{(0,t_1)}\overline{w}_1^{\psi,\eta} \in \mathcal{H}$ satisfies (75). Therefore we can perform the same computations as in (53) to deduce that

$$1_{(0,t_1)}\widetilde{w} = 1_{(0,t_1)}\overline{w}_1^{\psi,\eta}, \quad \mathbb{P} - \text{a.s.}$$

Going back to (74), by the previous equality we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\widetilde{w} &= \int_s^{t_1}\left(D^2\overline{B}\left(w_{t_1}^{s,\phi}\right)\left(Dw_{t_1}^{s,\phi}\psi, Dw_{t_1}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_1}^{s,\phi}\right)\widetilde{w}\right)(r,\cdot)\,\mathrm{d}r \\
&\quad + \int_{t_1}^{t_2}\left(D^2\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(Dw_{t_2}^{s,\phi}\psi, Dw_{t_2}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_2}^{s,\phi}\right)\widetilde{w}\right)(r,\cdot)\,\mathrm{d}r \\
&= \overline{w}_1^{\psi,\eta} + \int_{t_1}^{t_2}\left(D^2\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(Dw_{t_2}^{s,\phi}\psi, Dw_{t_2}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_2}^{s,\phi}\right)\widetilde{w}\right)(r,\cdot)\,\mathrm{d}r.
\end{aligned}$$

It follows that $\widetilde{w}$ satisfies (73): by uniqueness, we obtain $\widetilde{w} = \overline{w}_2^{\psi,\eta}$. Hence $\overline{w}_2^{\psi,\eta}$ is the unique solution of (68) in $\mathcal{H}$ with t_2 instead of t.

We define the operator $D^2 w_{t_2}^{s,\phi} \colon H \times H \to \mathcal{H}$ by $D^2 w_{t_2}^{s,\phi}(\psi,\eta) = \overline{w}_2^{\psi,\eta}$, $\psi,\eta \in H$, and claim that it is the second–order Fréchet differential of $w_{t_2}^{s,\phi}$. Indeed, as we have argued for $D^2 w_{t_1}^{s,\phi}$, the map $D^2 w_{t_2}^{s,\phi}$ is bilinear thanks to the well–posedness of (74). As for the boundedness, arguing as in (69), by (70)-(73) we can write, for every ψ, $\eta \in H$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\left\|Dw_{t_2}^{s,\phi}(\psi,\eta)\right\|_2 &\le \left\|D^2 w_{t_1}^{s,\phi}(\psi,\eta)\right\|_2 \\
&\quad + \int_{t_1}^{t_2}\left\|\left(D^2\overline{B}\left(w_{t_2}^{s,\phi}\right)\left(Dw_{t_2}^{s,\phi}\psi, Dw_{t_2}^{s,\phi}\eta\right) + D\overline{B}\left(w_{t_2}^{s,\phi}\right)Dw_{t_2}^{s,\phi}(\psi,\eta)\right)(r,\cdot)\right\|_2 \mathrm{d}r \\
&\le C_0C_2\left(\left(1 - C_0\sqrt{T/N}\right)^{-1}\sqrt{T/N} + \sqrt{\Delta}\right)\|\psi\|_2\,\|\eta\|_2 \\
&\quad + \sqrt{\Delta}C_0\left\|D^2 w_{t_2}^{s,\phi}(\psi,\eta)\right\|_2,
\end{aligned}$$

whence

$$\left\| Dw_{t_2}^{s,\phi}(\psi,\eta) \right\|_2 \leq 2C_0C_2 \left(1 - C_0\sqrt{T/N}\right)^{-2} \sqrt{T/N} \, \|\psi\|_2 \, \|\eta\|_2 \,, \quad \mathbb{P} - \text{a.s.}, \ \psi, \ \eta \in H. \tag{76}$$

Moreover, combining (49) with (73), we can argue as in (71) to infer that

$$\left\| Dw_{t_2}^{s,\phi+h} - Dw_{t_2}^{s,\phi} - D^2w_{t_2}^{s,\phi}h \right\|_{\mathcal{L}(H;\mathcal{H})} = \mathrm{o}\left(\|h\|_2\right), \quad h \in H,$$

which shows that $D^2w_{t_2}^{s,\phi}$ is the second–order Fréchet differential of $w_{t_2}^{s,\phi}$, as desired.

This reasoning can be repeated N-times to deduce that the operator $D^2w_t^{s,\phi} \colon H \times H \to \mathcal{H}$ defined by $D^2w_t^{s,\phi}(\psi,\eta) = \overline{w}_N^{\psi,\eta}$, where $\overline{w}_N^{\psi,\eta}$ is the unique solution of (68) in $\mathcal{H}$, for every ψ, $\eta \in H$, is the second–order Fréchet differential of $w_t^{s,\phi}$. In particular, the first bound in (67) is true, because (cf. (70)-(76))

$$\left\| D^2w_t^{s,\phi}\left(\psi,\eta\right) \right\|_2 \leq NC_0C_2\left(1 - C_0\sqrt{T/N}\right)^{-N} \sqrt{T/N}\, \|\psi\|_2 \, \|\eta\|_2 =: \widetilde{C} \, \|\psi\|_2 \, \|\eta\|_2 \,,$$
$$\mathbb{P} - \text{a.s.}, \ \phi, \psi, \eta \in H. \tag{77}$$

As for the second inequality in (67), by (38), (43), (60), (65), (66) and (77) we compute, for every $\phi, \psi, \eta, \theta \in H$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\left\| D^2w_{t_1}^{s,\phi}\left(\eta,\theta\right) - D^2w_{t_1}^{s,\psi}\left(\eta,\theta\right) \right\|_2 \\
&= \Bigg\| \int_s^{t_1} \Big(D^2\overline{B}\left(w_t^{s,\phi}\right)\left(Dw_t^{s,\phi}\eta, Dw_t^{s,\phi}\theta\right) - D^2\overline{B}\left(w_t^{s,\psi}\right)\left(Dw_t^{s,\psi}\eta, Dw_t^{s,\psi}\theta\right) \\
&\quad + D\overline{B}\left(w_t^{s,\phi}\right) D^2w_{t_1}^{s,\phi}\left(\eta,\theta\right) - D\overline{B}\left(w_t^{s,\psi}\right) D^2w_{t_1}^{s,\psi}\left(\eta,\theta\right) \Big)\left(r,\cdot\right) \mathrm{d}r \Bigg\|_2 \\
&\leq \sqrt{\Delta}\Big(\left\| \left(D^2\overline{B}\left(w_t^{s,\phi}\right) - D^2\overline{B}\left(w_t^{s,\psi}\right)\right)\left(Dw_t^{s,\phi}\eta, Dw_t^{s,\phi}\theta\right) \right\|_{2,\square} \\
&\quad + \left\| D^2\overline{B}\left(w_t^{s,\psi}\right)\left(\left(Dw_t^{s,\phi} - Dw_t^{s,\psi}\right)\eta, Dw_t^{s,\phi}\theta\right) \right\|_{2,\square} \\
&\quad + \left\| D^2\overline{B}\left(w_t^{s,\psi}\right)\left(Dw_t^{s,\psi}\eta, \left(Dw_t^{s,\phi} - Dw_t^{s,\psi}\right)\theta\right) \right\|_{2,\square} \\
&\quad + \left\| \left(D\overline{B}\left(w_t^{s,\phi}\right) - D\overline{B}\left(w_t^{s,\psi}\right)\right) D^2w_{t_1}^{s,\phi}\left(\eta,\theta\right) \right\|_{2,\square} \\
&\quad + \left\| D\overline{B}\left(w_t^{s,\psi}\right)\left(D^2w_{t_1}^{s,\phi}\left(\eta,\theta\right) - D^2w_{t_1}^{s,\psi}\left(\eta,\theta\right)\right) \right\|_{2,\square} \Big) \\
&\leq C_0\sqrt{T/N}\Big(\left(\widetilde{C} + 3C_2\right) \left\| w_t^{s,\phi} - w_t^{s,\psi} \right\|_2^{\beta} \|\eta\|_2 \, \|\theta\|_2 \\
&\quad + \left\| D^2w_{t_1}^{s,\phi}\left(\eta,\theta\right) - D^2w_{t_1}^{s,\psi}\left(\eta,\theta\right) \right\|_2 \Big),
\end{aligned}$$

whence

$$\begin{aligned}&\left\|D^2 w_{t_1}^{s,\phi}(\eta,\theta) - D^2 w_{t_1}^{s,\psi}(\eta,\theta)\right\|_2 \\ &\qquad \le \left(1 - C_0\sqrt{T/N}\right)^{-1} C_0\left(\widetilde{C} + 3C_2\right)\sqrt{T/N}\left\|w_t^{s,\phi} - w_t^{s,\psi}\right\|_2^{\beta}\|\eta\|_2\|\theta\|_2 .\end{aligned}$$

By (73), we sequentially iterate this computation to obtain the second inequality in (67) with

$$C_3 = \max\{\widetilde{C}, N\left(1 - C_0\sqrt{T/N}\right)^{-N} C_0(\widetilde{C} + 3C_2)\sqrt{T/N}\}.$$

Thus, taking expectations and using Corollary 4 with $q = 2$, by Jensen's inequality we deduce that, for some constant $c > 0$,

$$\begin{aligned}\left\|D^2 w_t^{s,\phi} - D^2 w_t^{s,\psi}\right\|_{\mathcal{L}(H,H;\mathcal{H})} &= \sup_{\|\eta\|_2,\|\theta\|_2\le 1}\mathbb{E}\left[\left\|D^2 w_t^{s,\phi}(\eta,\theta) - D^2 w_t^{s,\psi}(\eta,\theta)\right\|_2^2\right]^{\frac{1}{2}} \\ &\le C_3\mathbb{E}\left[\left\|w_t^{s,\phi} - w_t^{s,\psi}\right\|_2^{2\beta}\right]^{\frac{1}{2}} \le c\|\phi-\psi\|_2^{\beta}, \quad \phi,\psi\in H.\end{aligned}$$

This shows that $D^2 w_t^{s,\cdot} \in C^{\beta}(H;\mathcal{L}(H,H;\mathcal{H}))$, completing the proof. □

4. The Kolmogorov equation

Recall the map $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^d; H)$ defined in (6), given by

$$[\sigma(t)x](\xi) = k_2(\xi - t)1_{\{t<\xi\}}x, \quad x\in\mathbb{R}^d,\ t,\xi\in[0,T],$$

where k_2 is the fractional kernel in (5).

Given $u\colon [0,T]\times H \to \mathbb{R}$ and a terminal condition $\Phi\colon H\to\mathbb{R}$, in this section we investigate the following *Kolmogorov backward equation* in integral form:

$$\begin{aligned}u(t,\phi) = \Phi(\phi) + \int_t^T \left\langle \nabla u(r,\phi), \overline{B}(r,\phi)\right\rangle_H \mathrm{d}r + \frac{1}{2}\int_t^T \operatorname{Tr}\left(D^2 u(r,\phi)\sigma(r)\sigma(r)^*\right)\mathrm{d}r,\\ t\in[0,T],\ \phi\in H. \qquad (78)\end{aligned}$$

Our aim is to find a solution of (78) via the random variables $w_T^{t,\phi}\in\mathcal{H}$ satisfying (20) for every $t\in[0,T]$ and $\phi\in H$. This is done in Theorem 9, for which we need the following preparatory result.

Lemma 8. *There exists a constant $C_{\alpha,d} > 0$ such that*

$$\left\| \int_s^t (\sigma(t) - \sigma(r))\, \mathrm{d}W_r \right\|_{\mathcal{H}} \leq C_{\alpha,d} |t-s|^\alpha, \quad 0 \leq s \leq t \leq T. \tag{79}$$

Proof. Fix $0 \leq s \leq t \leq T$ and denote by $(e_k)_{k=1,\dots,d}$ the canonical basis of $\mathbb{R}^d$. Using straightforward substitutions, by (6) we compute, for every $k = 1, \dots, d$,

$$\begin{aligned}\|(\sigma(t) - \sigma(r))\, e_k\|_2^2 &= \int_0^T \left| k_2(\xi - t)\, 1_{\{\xi > t\}} - k_2(\xi - r)\, 1_{\{\xi > r\}} \right|^2 \mathrm{d}\xi \\ &= \int_0^{t-r} |k_2(\xi)|^2\, \mathrm{d}\xi + \int_0^{T-t} |k_2(\xi + t - r) - k_2(\xi)|^2\, \mathrm{d}\xi, \quad r \in [s,t]. \end{aligned} \tag{80}$$

Recalling that (see (5)) $k_2(u) = \frac{1}{\Gamma(\alpha)} u^{\alpha-1}$, $\alpha \in (1/2, 1)$, $u > 0$, for every $r \in [s,t]$ we have

$$\int_0^{t-r} |k_2(\xi)|^2\, \mathrm{d}\xi = \frac{1}{(\Gamma(\alpha))^2 (2\alpha - 1)} |t-r|^{2\alpha-1},$$

and, thanks to the change of variables $z = \xi/(t-r)$,

$$\int_0^{T-t} |k_2(\xi + t - r) - k_2(\xi)|^2\, \mathrm{d}\xi \leq \frac{1}{(\Gamma(\alpha))^2} \left(\int_0^\infty \left((z+1)^{\alpha-1} - z^{\alpha-1} \right)^2 \mathrm{d}z \right) |t-r|^{2\alpha-1}.$$

Therefore, combining [19, Propositions 4.20-4.22] as explained in the final paragraph of [19, Page 98] ensures that

$$\left\| \int_s^t (\sigma(t) - \sigma(r))\, \mathrm{d}W_r \right\|_{\mathcal{H}} \leq \left(\sum_{k=1}^d \int_s^t \|(\sigma(t) - \sigma(r))\, e_k\|_2^2\, \mathrm{d}r \right)^{\frac{1}{2}} \leq C_{\alpha,d} |t-s|^\alpha,$$

where

$$C_{\alpha,d} = \frac{\sqrt{d}}{\Gamma(\alpha)} \left(\frac{1}{2\alpha} \right)^{\frac{1}{2}} \left(\frac{1}{2\alpha - 1} + \int_0^\infty \left((z+1)^{\alpha-1} - z^{\alpha-1} \right)^2 \mathrm{d}z \right)^{\frac{1}{2}}.$$

This gives (79), completing the proof. □

Recall that $\mathcal{L}_t^p = L_t^p(\Omega; L^p)$, where $L^p = L^p(0,T;\mathbb{R}^d)$, and $L_\square^p = L^p\big((0,T) \times (0,T); \mathbb{R}^d\big)$, see Remark 3 and Lemma 5. We are now ready to prove the main result of the paper, which shows the connection between the solution $w_T^{t,\phi}$, $t \in [0,T]$, $\phi \in H$, of (20) and the backward Kolmogorov equation in integral form (78).

Theorem 9. *Suppose that* $B\colon \Lambda \to L^p_\square$ *satisfies Assumption 3 and* (30)*, for some* $p \in \left(2, (1-\alpha)^{-1}\right)$*. In addition, let the function* $r \mapsto B(r,\phi)$ *belong to* $C\big([0,T];H\big)$*, for every* $\phi \in \Lambda$*. Fix* $\Phi \in C_b^{2+\beta}(H)$ *and define the map* $u\colon [0,T]\times H \to \mathbb{R}$ *by*

$$u(t,\phi) = \mathbb{E}\left[\Phi\left(w_T^{t,\phi}\right)\right], \quad t \in [0,T],\, \phi \in H, \tag{81}$$

where $w_T^{t,\phi} \in \mathcal{H}$ *is the unique solution of* (20)*. Then* $u \in L^\infty\big(0,T;C_b^{2+\beta}(H)\big) \cap C([0,T]\times H;\mathbb{R})$ *and solves the Kolmogorov backward equation in integral form* (78)*.*

Proof. The fact that the function u defined in (81) belongs to $L^\infty\big(0,T;C_b^{2+\beta}(H)\big) \cap C([0,T]\times H;\mathbb{R})$ is one of the results contained in Lemma 11 (see Appendix A). Consequently, here we only focus on proving that u solves (78).

Fix $0 \le s < t \le T$ and $\phi \in \Lambda$. Since $\Lambda \subset \mathcal{H}_s^q$, $q \ge 2$, we can use (32) in Corollary 4 to write

$$u(s,\phi) = \mathbb{E}\left[\mathbb{E}\left[\Phi\left(w_T^{s,\phi}\right)\Big|\mathcal{F}_t\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\Phi\left(w_T^{t,\psi}\right)\right]\Big|_{\psi = w_t^{s,\phi}}\right] = \mathbb{E}\left[u\left(t, w_t^{s,\phi}\right)\right]. \tag{82}$$

Taylor's formula applied to the mapping $u(t,\cdot) \in C_b^{2+\beta}(H)$ yields, denoting by $h = w_t^{s,\phi} - \phi \in \mathcal{H}$,

$$u\left(t,w_t^{s,\phi}\right) - u(t,\phi) = \langle \nabla u(t,\phi), h\rangle_H + \frac{1}{2}\left\langle D^2 u(t,\phi)h, h\right\rangle_H + r_{u(t,\cdot)}\left(\phi, w_t^{s,\phi}\right), \quad \text{where}$$

$$r_{u(t,\cdot)}(x,y) = \int_0^1 (1-r)\left\langle \big(D^2u(t,x+r(y-x)) - D^2u(t,x)\big)(y-x), y-x\right\rangle_H \mathrm{d}r,$$

$$x,\, y \in H. \tag{83}$$

To keep the notation simple, in this proof we denote by $\overline{B}_{s,t}(w_t^{s,\phi}) = \int_s^t \overline{B}(r, w_t^{s,\phi})\,\mathrm{d}r \in \mathcal{H}$. Using the expression in (20) for $h = w_t^{s,\phi} - \phi$ and noticing that $\mathbb{E}[\Sigma_{s,t}] = 0 \in H$ by [19, Proposition 4.28], we take expectations in the previous equation to obtain, from (82),

$$\begin{aligned} u(s,\phi) - u(t,\phi) = &\left\langle \nabla u(t,\phi), \mathbb{E}\left[\int_s^t \overline{B}\left(r, w_t^{s,\phi}\right)\mathrm{d}r\right]\right\rangle_H \\ &+ \frac{1}{2}\mathbb{E}\left[\left\langle D^2u(t,\phi)\left(\overline{B}_{s,t}\left(w_t^{s,\phi}\right) + \Sigma_{s,t}\right), \overline{B}_{s,t}\left(w_t^{s,\phi}\right) + \Sigma_{s,t}\right\rangle_H\right] \\ &+ \mathbb{E}\left[r_{u(t,\cdot)}\left(\phi, w_t^{s,\phi}\right)\right]. \end{aligned} \tag{84}$$

For all $N \in \mathbb{N}$, consider an equispaced partition $\{t_k^{(N)}\}_{k=0}^N$ of $[s,T]$ with mesh Δ_N, where $t_0^{(N)} = s$ and $t_N^{(N)} = T$. By (84), we have

$$
\begin{aligned}
u\left(s,\phi\right) - \Phi\left(\phi\right) &= \sum_{k=1}^{N} \left(u\left(t_{k-1}^{(N)},\phi\right) - u\left(t_k^{(N)},\phi\right) \right) \\
&= \sum_{k=1}^{N} \left\langle \nabla u\left(t_k^{(N)},\phi\right), \mathbb{E}\left[\overline{B}_{t_{k-1}^{(N)},t_k^{(N)}}\left(w_{t_k^{(N)}}^{t_{k-1}^{(N)},\phi} \right) \right] \right\rangle_H \\
&\quad + \frac{1}{2}\sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k^{(N)},\phi\right)\left(\overline{B}_{t_{k-1}^{(N)},t_k^{(N)}}\left(w_{t_k^{(N)}}^{t_{k-1}^{(N)},\phi} \right) + \Sigma_{t_{k-1}^{(N)},t_k^{(N)}} \right), \right.\right. \\
&\qquad\qquad \left.\left. \overline{B}_{t_{k-1}^{(N)},t_k^{(N)}}\left(w_{t_k^{(N)}}^{t_{k-1}^{(N)},\phi} \right) + \Sigma_{t_{k-1}^{(N)},t_k^{(N)}} \right\rangle_H \right] \\
&\quad + \sum_{k=1}^{N} \mathbb{E}\left[r_{u\left(t_k^{(N)},\cdot\right)}\left(\phi, w_{t_k^{(N)}}^{t_{k-1}^{(N)},\phi} \right) \right] =: \mathbf{I}^N + \mathbf{II}^N + \mathbf{III}^N. \qquad (85)
\end{aligned}
$$

In the sequel, we omit the superscript N from the points of the partition to ease notation, i.e., we write t_k for $t_k^{(N)}$. Firstly, we analyze $\mathbf{I}^N$, which we decompose using the properties of the Bochner's integral as follows:

$$
\begin{aligned}
\mathbf{I}^N &= \sum_{k=1}^{N} \left\langle \nabla u\left(t_k,\phi\right), B\left(t_k,\phi\right)\right\rangle_H \left(t_k - t_{k-1}\right) \\
&\quad + \sum_{k=1}^{N} \mathbb{E}\left[\int_{t_{k-1}}^{t_k} \left\langle \nabla u\left(t_k,\phi\right), \overline{B}\left(r, w_{t_k}^{t_{k-1},\phi}\right) - B\left(r,\phi\right) \right\rangle_H \mathrm{d}r \right] \\
&\quad + \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \left\langle \nabla u\left(t_k,\phi\right), B\left(r,\phi\right) - B\left(t_k,\phi\right)\right\rangle_H \mathrm{d}r =: \mathbf{I}_1^N + \mathbf{I}_2^N + \mathbf{I}_3^N.
\end{aligned}
$$

Note that $\mathbf{I}_1^N \to \int_s^T \langle \nabla u(r,\phi), B(r,\phi)\rangle_H \mathrm{d}r$ as $N \to \infty$ by Lemma 11 in Appendix A. Next, Jensen's inequality, (30), (34) and the continuous immersion $L^p\big((t_{k-1},t_k) \times (0,T)\,;\mathbb{R}^d\big) \hookrightarrow L^2\big((t_{k-1},t_k) \times (0,T)\,;\mathbb{R}^d\big)$ yield, for some constant $C_{\phi,p} = C_{\phi,p}(\alpha,d,T) > 0$,

$$
\begin{aligned}
\left|\mathbf{I}_2^N\right| &\le \left\|\nabla u\right\|_\infty \sqrt{\Delta_N} \sum_{k=1}^{N} \mathbb{E}\left[\left(\int_{t_{k-1}}^{t_k} \mathrm{d}r \int_0^T \left| \overline{B}\left(w_{t_k}^{t_{k-1},\phi}\right) - B\left(\phi\right)\right|^2 \left(r,\xi\right) \mathrm{d}\xi \right)^{\frac{1}{2}} \right] \\
&\le T^{\frac{1}{2}-\frac{1}{p}} \left(\Delta_N\right)^{1-\frac{1}{p}} \left\|\nabla u\right\|_\infty \sum_{k=1}^{N} \mathbb{E}\left[\left\| \overline{B}\left(w_{t_k}^{t_{k-1},\phi}\right) - B\left(\phi\right)\right\|_{p,\square} \right]
\end{aligned}
$$

$$\leq T^{\frac{1}{2}-\frac{1}{p}} C_{0,p} \left(\Delta_N\right)^{1-\frac{1}{p}} \|\nabla u\|_\infty \sum_{k=1}^{N} \mathbb{E}\left[\left\|w_{t_k}^{t_{k-1},\phi} - \phi\right\|_p\right]$$

$$\leq T^{\frac{3}{2}-\frac{1}{p}} C_{0,p} C_{\phi,p} \|\nabla u\|_\infty \left(\Delta_N\right)^{\frac{1}{2}-\frac{1}{p}} \xrightarrow[N\to\infty]{} 0.$$

Here, we set $\|\nabla u\|_\infty = \sup_{t\in[0,T]} \sup_{\phi\in H} \|\nabla u(t,\phi)\|_2$. Regarding $\mathbf{I}_3^N$, we define the modulus of continuity of the map $B(\cdot,\phi)\colon [0,T] \to H$ by

$$\mathfrak{w}\left(B(\cdot,\phi),\delta\right) = \sup_{|u-v|\leq\delta} . \left\|B\left(u,\phi\right) - B\left(v,\phi\right)\right\|_2, \quad \delta > 0.$$

Since, by hypothesis, $B(\cdot,\phi)$ is continuous on the compact $[0,T]$, it is also uniformly continuous, hence we infer that $\left|\mathbf{I}_3^N\right| \leq T \|\nabla u\|_\infty \mathfrak{w}\left(B\left(\cdot,\phi\right),\Delta_N\right) \xrightarrow[N\to\infty]{} 0$. Therefore, we have just shown that

$$\lim_{N\to\infty} \mathbf{I}^N = \int_s^T \left\langle \nabla u(r,\phi), B(r,\phi)\right\rangle_H \mathrm{d}r. \tag{86}$$

Now we investigate $\mathbf{II}^N$, which we split as follows:

$$\begin{aligned} 2\mathbf{II}^N =& \sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k,\phi\right) \overline{B}_{t_{k-1},t_k}\left(w_{t_k}^{t_{k-1},\phi}\right), \overline{B}_{t_{k-1},t_k}\left(w_{t_k}^{t_{k-1},\phi}\right)\right\rangle_H\right] \\ &+ \sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k,\phi\right) \overline{B}_{t_{k-1},t_k}\left(w_{t_k}^{t_{k-1},\phi}\right), \Sigma_{t_{k-1},t_k}\right\rangle_H\right] \\ &+ \sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k,\phi\right) \Sigma_{t_{k-1},t_k}, \overline{B}_{t_{k-1},t_k}\left(w_{t_k}^{t_{k-1},\phi}\right)\right\rangle_H\right] \\ &+ \sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k,\phi\right) \Sigma_{t_{k-1},t_k}, \Sigma_{t_{k-1},t_k}\right\rangle_H\right] =: \mathbf{II}_1^N + \mathbf{II}_2^N + \mathbf{II}_3^N + \mathbf{II}_4^N. \end{aligned}$$

Let us set $\left\|D^2 u\right\|_\infty = \sup_{t\in[0,T]} \sup_{\phi\in H} \left\|D^2 u(t,\phi)\right\|_{\mathcal{L}(H;H)}$. By (30)-(33), arguing similarly to $\mathbf{I}_2^N$ we have, for some $c > 0$,

$$\begin{aligned} \left|\mathbf{II}_1^N\right| &\leq \left\|D^2 u\right\|_\infty \sum_{k=1}^{N} \mathbb{E}\left[\left\|\overline{B}_{t_{k-1},t_k}\left(w_{t_k}^{t_{k-1},\phi}\right)\right\|_2^2\right] \\ &\leq T^{1-\frac{2}{p}} \Delta_N^{1-\frac{2}{p}} \left\|D^2 u\right\|_\infty \sum_{k=1}^{N} \mathbb{E}\left[\left\|\overline{B}\left(w_{t_k}^{t_{k-1},\phi}\right)\right\|_{p,\square}^2\right] \left(t_k - t_{k-1}\right) \\ &\leq c \Delta_N^{1-\frac{2}{p}} \left\|D^2 u\right\|_\infty \left(1 + \|\phi\|_p^2\right). \end{aligned}$$

Moreover, by Hölder's inequality and (8), for some $\tilde{c} > 0$,

$$\begin{aligned}\left|\mathbf{II}_2^N\right| &\leq \left\|D^2 u\right\|_\infty \sum_{k=1}^{N} \left\|\overline{B}_{t_{k-1},t_k}\left(w_{t_k}^{t_{k-1},\phi}\right)\right\|_{\mathcal{H}} \left\|\Sigma_{t_{k-1},t_k}\right\|_{\mathcal{H}} \\ &\leq \left\|D^2 u\right\|_\infty \widetilde{c}\, T^{\frac{3}{2}-\frac{1}{p}} \left\|k_2\right\|_2 \Delta_N^{\frac{1}{2}-\frac{1}{p}} \left(1 + \|\phi\|_p\right).\end{aligned}$$

Since the second bound holds for $\mathbf{II}_3^N$, too, we see that $\mathbf{II}_i^N \to 0$ as $N \to \infty$, $i = 1, 2, 3$.

As for $\mathbf{II}_4^N$, we write it as the following sum:

$$\begin{aligned}\mathbf{II}_4^N =& \sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k, \phi\right) \int_{t_{k-1}}^{t_k} \sigma\left(t_k\right) \mathrm{d}W_r, \int_{t_{k-1}}^{t_k} \sigma\left(t_k\right) \mathrm{d}W_r \right\rangle_H\right] \\ &+ \sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k, \phi\right) \int_{t_{k-1}}^{t_k} \left(\sigma\left(r\right) - \sigma\left(t_k\right)\right) \mathrm{d}W_r, \int_{t_{k-1}}^{t_k} \sigma\left(t_k\right) \mathrm{d}W_r \right\rangle_H\right] \\ &+ \sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k, \phi\right) \int_{t_{k-1}}^{t_k} \sigma\left(t_k\right) \mathrm{d}W_r, \int_{t_{k-1}}^{t_k} \left(\sigma\left(r\right) - \sigma\left(t_k\right)\right) \mathrm{d}W_r \right\rangle_H\right] \\ &+ \sum_{k=1}^{N} \mathbb{E}\left[\left\langle D^2 u\left(t_k, \phi\right) \int_{t_{k-1}}^{t_k} \left(\sigma\left(r\right) - \sigma\left(t_k\right)\right) \mathrm{d}W_r, \int_{t_{k-1}}^{t_k} \left(\sigma\left(r\right) - \sigma\left(t_k\right)\right) \mathrm{d}W_r \right\rangle_H\right] \\ =:& \mathbf{II}_{4,1}^N + \mathbf{II}_{4,2}^N + \mathbf{II}_{4,3}^N + \mathbf{II}_{4,4}^N.\end{aligned}$$

By [19, Proposition 4.30], we have, for every $k = 1, \ldots, N$,

$$D^2 u\left(t_k, \phi\right) \int_{t_{k-1}}^{t_k} \sigma\left(t_k\right) \mathrm{d}W_r = \int_{t_{k-1}}^{t_k} D^2 u\left(t_k, \phi\right) \sigma\left(t_k\right) \mathrm{d}W_r, \quad \mathbb{P} - \text{a.s.},$$

whence, by [19, Corollary 4.29] and Lemma 11,

$$\begin{aligned}\mathbf{II}_{4,1}^N &= \sum_{k=1}^{N} \operatorname{Tr}\left(D^2 u\left(t_k, \phi\right) \sigma\left(t_k\right) \sigma\left(t_k\right)^*\right) \left(t_k - t_{k-1}\right) \\ &\xrightarrow[N \to \infty]{} \int_s^T \operatorname{Tr}\left(D^2 u\left(r, \phi\right) \sigma\left(r\right) \sigma\left(r\right)^*\right) \mathrm{d}r.\end{aligned}$$

Furthermore, Hölder's inequality, (79) in Lemma 8 and [19, Proposition 4.20] yield, for $i = 2, 3$, for some constants $c_1, c_2 > 0$,

$$\begin{aligned}\left|\mathbf{II}_{4,i}^{N}\right| &\le c_1 \left\|k_2\right\|_2 \left\|D^2 u\right\|_\infty \sqrt{\Delta_N} \sum_{k=1}^{N} \left\| \int_{t_{k-1}}^{t_k} \left(\sigma\left(r\right)-\sigma\left(t_k\right)\right) \mathrm{d}W_r \right\|_{\mathcal{H}} \\ &\le T c_2 \left\|k_2\right\|_2 \left\|D^2 u\right\|_\infty \Delta_N^{\alpha-\frac{1}{2}} \underset{N\to\infty}{\longrightarrow} 0.\end{aligned}$$

Analogous estimates show that $\mathbf{II}_{4,4}^{N} \to 0$ as $N \to \infty$, as well. Thus,

$$\lim_{N\to\infty} \mathbf{II}^{N} = \frac{1}{2} \int_{s}^{T} \operatorname{Tr}\left(D^2 u\left(r,\phi\right) \sigma\left(r\right) \sigma\left(r\right)^{*}\right) \mathrm{d}r. \tag{87}$$

At last we study the remainder term $\mathbf{III}^{N}$ in (85). To do this, we employ the fact that $D^2 u\left(t,\cdot\right): H \to \mathcal{L}\left(H;H\right)$ is β-Hölder continuous uniformly in time, see (102) in Lemma 11. We choose $\tilde{\beta} \in (0,\beta)$ such that $2+\tilde{\beta} < p$; by the expression of $r_{u(t_k,\cdot)}$ in (83) we deduce that

$$\begin{aligned}\left|\mathbf{III}^{N}\right| &\le \sum_{k=1}^{N} \int_{0}^{1} \mathbb{E}\Bigg[\left\| D^2 u\left(t_k, \phi + r\left(w_{t_k}^{t_{k-1},\phi} - \phi\right)\right) - D^2 u\left(t_k,\phi\right)\right\|_{\mathcal{L}(H;H)} \\ &\qquad \times \left\| w_{t_k}^{t_{k-1},\phi} - \phi \right\|_2^2 \Bigg] \mathrm{d}r \\ &\le C \sum_{k=1}^{N} \mathbb{E}\left[\left\| w_{t_k}^{t_{k-1},\phi} - \phi \right\|_2^{2+\tilde{\beta}} \right] \le C T^{\left(\frac{1}{2}-\frac{1}{p}\right)(2+\tilde{\beta})} \sum_{k=1}^{N} \mathbb{E}\left[\left\| w_{t_k}^{t_{k-1},\phi} - \phi \right\|_p^{2+\tilde{\beta}} \right] \\ &\le C \sum_{k=1}^{N} \left(t_k - t_{k-1}\right)^{1+\frac{\tilde{\beta}}{2}} \underset{N\to\infty}{\longrightarrow} 0,\end{aligned} \tag{88}$$

where in the last passage we use Lemma 5 and Jensen's inequality. Here $C > 0$ is a constant allowed to change from line to line. Combining (86), (87), (88) in (85), we obtain

$$u\left(s,\phi\right) - \Phi\left(\phi\right) = \int_{s}^{T} \left\langle \nabla u\left(r,\phi\right), B\left(r,\phi\right)\right\rangle_H \mathrm{d}r + \frac{1}{2} \int_{s}^{T} \operatorname{Tr}\left(D^2 u\left(r,\phi\right) \sigma\left(r\right) \sigma\left(r\right)^{*}\right) \mathrm{d}r,$$

i.e., (78) holds when $\phi \in \Lambda$.

We now consider the general case $\phi \in H$, which can be recovered by an approximation procedure. Indeed, given $\phi \in H$, there exists a sequence $(\phi_n)_n \subset \Lambda$ such that $\lim_{n\to\infty} \left\|\phi_n - \phi\right\|_2 = 0$. Using the extension $\overline{B}\colon H \to H_\square$ of B (see Section 3), the above argument proves that, for every $s \in [0,T)$ and $n \in \mathbb{N}$,

$$\begin{aligned} u\left(s,\phi_n\right)-\Phi\left(\phi_n\right) &= \int_s^T \left\langle \nabla u\left(r,\phi_n\right), \overline{B}\left(r,\phi_n\right)\right\rangle_H \mathrm{d}r + \frac{1}{2}\int_s^T \operatorname{Tr}\left(D^2 u\left(r,\phi_n\right)\sigma\left(r\right)\sigma\left(r\right)^*\right)\mathrm{d}r \\ &= \int_0^T \left(\int_0^T 1_{\{s<r\}} (\nabla u\left(r,\phi_n\right)^\top \overline{B}(\phi_n)\left(r,\cdot\right))(\xi)\mathrm{d}\xi \right)\mathrm{d}r \\ &\quad + \frac{1}{2}\int_s^T \operatorname{Tr}\left(D^2 u\left(r,\phi_n\right)\sigma\left(r\right)\sigma\left(r\right)^*\right)\mathrm{d}r. \end{aligned} \tag{89}$$

Since $u(s,\cdot)$ and Φ are continuous in H,

$$\lim_{n\to\infty}\left(u\left(s,\phi_n\right)-\Phi\left(\phi_n\right)\right) = u\left(s,\phi\right)-\Phi\left(\phi\right).$$

By the continuity of $\overline{B}$ in H, we know that $\overline{B}(\phi_n)\to\overline{B}(\phi)$ in $H_\square$ as $n\to\infty$. Moreover, considering that $u\in L^\infty\left(0,T;C_b^{2+\beta}\left(H\right)\right)$, the dominated convergence theorem yields $\nabla u(\cdot,\phi_n)\to\nabla u(\cdot,\phi)$ in $H_\square$ as $n\to\infty$. Hence, denoting by $\langle\cdot,\cdot\rangle_{2,\square}$ the inner product in $H_\square$,

$$\begin{aligned} &\lim_{n\to\infty}\int_0^T \left(\int_0^T 1_{\{s<r\}} (\nabla u\left(r,\phi_n\right)^\top \overline{B}(\phi_n)\left(r,\cdot\right))(\xi)\mathrm{d}\xi \right)\mathrm{d}r \\ &= \lim_{n\to\infty} \langle 1_{\{s<\cdot\}}\nabla u(\cdot,\phi_n), \overline{B}(\phi_n)\rangle_{2,\square} \\ &= \langle 1_{\{s<\cdot\}}\nabla u(\cdot,\phi), \overline{B}(\phi)\rangle_{2,\square} = \int_s^T \left\langle \nabla u\left(r,\phi\right), \overline{B}\left(r,\phi\right)\right\rangle_H \mathrm{d}r. \end{aligned}$$

As for the last term in the right–hand side of (89), the aforementioned regularity properties of u, along with [19, Corollary C.2 and Proposition C.4] and the dominated convergence theorem, yield

$$\lim_{n\to\infty}\frac{1}{2}\int_s^T \operatorname{Tr}\left(D^2 u\left(r,\phi_n\right)\sigma\left(r\right)\sigma\left(r\right)^*\right)\mathrm{d}r = \frac{1}{2}\int_s^T \operatorname{Tr}\left(D^2 u\left(r,\phi\right)\sigma\left(r\right)\sigma\left(r\right)^*\right)\mathrm{d}r.$$

The three previous equations enable us to take the limit as $n\to\infty$ in (89), whence we deduce that (78) holds for a general $\phi\in H$. The proof is now complete. □

Remark 4. Under the hypotheses of Theorem 9, for every $\phi\in H$ the function $u\left(\cdot,\phi\right):[0,T]\to\mathbb{R}$ defined in (81) is absolutely continuous on $[0,T]$, because the integrands on the right–hand side of (78) belong to $L^1(0,T)$. Thus, the fundamental theorem of calculus shows that $u\colon[0,T]\times H\to\mathbb{R}$ satisfies the following *Kolmogorov backward equation* in differential form:

$$\begin{cases} \partial_t u\left(t,\phi\right)+\left\langle \nabla u\left(t,\phi\right),\overline{B}\left(t,\phi\right)\right\rangle_H \\ \quad +\frac{1}{2}\operatorname{Tr}\left(D^2 u\left(t,\phi\right)\sigma\left(t\right)\sigma\left(t\right)^*\right)=0, \quad \text{for a.e. } t\in\left(0,T\right),\ \phi\in H, \\ u\left(T,\phi\right)=\Phi\left(\phi\right), \quad \phi\in H. \end{cases} \tag{90}$$

In fact, when $\phi\in\Lambda$, $u(\cdot,\phi)$ satisfies the Kolmogorov differential equation in (90) for every $t\in[0,T]$. Indeed, denoting by $(e_k)_{k=1,\dots,d}$ [resp., $\cdot$] the canonical basis [resp., scalar product] of $\mathbb{R}^d$, for every $t\in[0,T]$ and complete orthonormal system $(f_j)_{j\in\mathbb{N}}$ of H, which possibly depends on t, we can write

$$\begin{aligned}\operatorname{Tr}\left(D^2 u\left(t,\phi\right)\sigma\left(t\right)\sigma\left(t\right)^*\right)&=\sum_{j=1}^{\infty}\langle D^2 u\left(t,\phi\right)\sigma\left(t\right)\sigma\left(t\right)^* f_j,f_j\rangle_H\\&=\sum_{j=1}^{\infty}\sum_{k=1}^{d}(\sigma(t)^* f_j\cdot e_k)\langle D^2 u\left(t,\phi\right)\sigma\left(t\right)e_k,f_j\rangle_H\\&=\sum_{j=1}^{\infty}\sum_{k=1}^{d}\langle\sigma(t)e_k,f_j\rangle_H\langle D^2 u\left(t,\phi\right)\sigma\left(t\right)e_k,f_j\rangle_H.\end{aligned}$$

Choosing an Hilbert basis $(f_j)_{j\in\mathbb{N}}$ such that $f_j=\frac{\sigma(t)e_j}{\|\sigma(t)e_j\|_2}$ for $j=1,\dots,d$ (these vectors are orthogonal in H, see (6)), from the previous equation we deduce that

$$\begin{aligned}\operatorname{Tr}\left(D^2 u\left(t,\phi\right)\sigma\left(t\right)\sigma\left(t\right)^*\right)&=\sum_{j=1}^{d}\sum_{k=1}^{d}\delta_{jk}\left\|\sigma(t)e_k\right\|_2\left\langle D^2 u\left(t,\phi\right)\sigma\left(t\right)e_k,\frac{\sigma\left(t\right)e_j}{\left\|\sigma\left(t\right)e_j\right\|_2}\right\rangle_H\\&=\sum_{j=1}^{d}\langle D^2 u\left(t,\phi\right)\sigma\left(t\right)e_j,\sigma\left(t\right)e_j\rangle_H,\end{aligned}$$

where δ_{jk} is the Kronecker delta. Since $\sigma\colon[0,T]\to\mathcal{L}(\mathbb{R}^d;H)$ is continuous (even as a map taking values in the space of Hilbert–Schmidt operators, see the proof of Lemma 8), this expression combined with Lemma 11 implies that $t\mapsto\operatorname{Tr}\left(D^2 u\left(t,\phi\right)\sigma\left(t\right)\sigma\left(t\right)^*\right)$ is continuous on $[0,T]$. Moreover, again by Lemma 11 and the continuity of $t\mapsto B(t,\phi)$, the function $t\mapsto\langle\nabla u(t,\phi),B(t,\phi)\rangle_H$ is also continuous on $[0,T]$. Thus, from (78) and the fundamental theorem of calculus we conclude that $u(\cdot,\phi)\in C^1([0,T])$. Differentiating (78) with respect to t, we arrive at the desired Kolmogorov backward differential equation, namely

$$\partial_t u\left(t,\phi\right)+\left\langle\nabla u\left(t,\phi\right),B\left(t,\phi\right)\right\rangle_H+\frac{1}{2}\operatorname{Tr}\left(D^2 u\left(t,\phi\right)\sigma\left(t\right)\sigma\left(t\right)^*\right)=0,\quad t\in[0,T],\ \phi\in\Lambda.$$

Remark 5. The time evolution of the solution $u\colon[0,T]\times H\to\mathbb{R}$ to the Kolmogorov backward equation (78), defined in (81), depends on the initial time t of the solution $w_T^{t,\phi}$ to (20). As a result, it is not clear how to apply the classical Itô's lemma (in the spirit, for instance, of [19, Theorem 9.23]) to the dynamics (20), see also Remark 1, to prove Theorem 9.

Remark 6. All the arguments and computations leading to Theorem 9 continue to hold when the power α of the kernel k_2 in (5) varies in $[1, \frac{3}{2})$, i.e., k_2 is the continuous kernel in $\mathbb{R}_+$ given by

$$k_2(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, \quad t \geq 0, \text{ for some } \alpha \in \left[1, \frac{3}{2}\right).$$

We have however decided to present the theory in the case $\alpha \in (\frac{1}{2}, 1)$ to emphasize the fact that our approach is able to handle rough kernels with explosions at $t = 0$.

Example 1. Given two continuous maps $A\colon [0, T] \to \mathbb{R}^{d\times d}$ and $b\colon [0, T] \to \mathbb{R}^d$, define $B\colon \Lambda \to H_\square$ by (cf. (9))

$$\begin{aligned} &B(w)\colon [0, T] \times [0, T] \to \mathbb{R}^d \quad \text{such that} \\ &B(w)(t, \xi) = 1_{\{\xi>t\}} k_2(\xi - t) \, (A(t)w(t) + b(t)) \, , \, t, \xi \in [0, T], \end{aligned} \tag{91}$$

for every $w \in \Lambda$. We now show that B satisfies all the hypotheses of Theorem 9.

For every $t \in (0, T]$ and $r \in (0, t)$, from the definition in (91) it is immediate to see that $B(w)(r, \xi) = 0$, $\xi \in (0, r)$, and that $B(w)(r, \cdot)$ depends on w only via $w\big|_{(0,t)}$. Denote by $\|A\|_\infty = \sup_{t\in[0,T]} |A(t)|$ and by $\|b\|_\infty = \sup_{t\in[0,T]} |b(t)|$, where $|A(t)|$ is the operator norm in $\mathbb{R}^{d\times d}$. Computing, for every w_1, $w_2 \in \Lambda$,

$$\begin{aligned} \|B(w_1)\|_\square^2 &\leq \int_0^T \left(\int_0^T |k_2(\xi - t)|^2 \, 1_{\{\xi>t\}} \left(\|b\|_\infty + \|A\|_\infty |w_1(t)|\right)^2 \mathrm{d}\xi \right) \mathrm{d}t \\ &\leq 2T \max\left\{\|b\|_\infty^2, \|A\|_\infty^2\right\} \|k_2\|_2^2 \, (1 + \|w_1\|_2^2), \end{aligned}$$

and

$$\begin{aligned} \|B(w_2) - B(w_1)\|_\square^2 &\leq \|A\|_\infty^2 \int_0^T \left(\int_0^T |k_2(\xi - t)|^2 \, 1_{\{\xi>t\}} \, |w_2(t) - w_1(t)|^2 \, \mathrm{d}\xi \right) \mathrm{d}t \\ &\leq \|A\|_\infty^2 \, \|k_2\|_2^2 \, \|w_2 - w_1\|_2^2 \, , \end{aligned}$$

we deduce that Assumption 1 is satisfied. Since the previous computations can be repeated for every $p \in (2, (1 - \alpha)^{-1})$, then condition (30) in Remark 3 is verified, as well.

As for Assumption 2, evidently the operator $DB(w_1) \in \mathcal{L}(\Lambda_2; H_\square)$ defined by

$$[DB(w_1)(w_2)](t, \xi) = 1_{\{\xi>t\}} k_2(\xi - t) A(t) w_2(t), \quad t, \xi \in [0, T], \, w_2 \in \Lambda_2, \tag{92}$$

is the Λ_2-Fréchet differential of B in w_1, for any $w_1 \in \Lambda$. Indeed,

$$B(w_1 + h) - B(w_1) - DB(w_1)(h) = 0, \quad w_1, h \in \Lambda.$$

Moreover, from (92) we have, for every $w_1, w_2 \in \Lambda$,

$$\|DB(w_1)(w_2)\|_{\square} \leq \|A\|_\infty \|k_2\|_2 \|w_2\|_2, \qquad \|DB(w_1) - DB(w_2)\|_{\mathcal{L}(\Lambda_2; H_\square)} = 0,$$

which in particular gives (36) with $\gamma = 1$.

The requirements of Assumption 3 are trivially satisfied (with $\beta = 1$) because, given the affine structure of this example, $D^2 B(w_1) = 0 \in \mathcal{L}(\Lambda_2, \Lambda_2; H_\square)$, $w_1 \in \Lambda$.

In conclusion, for every $w \in \Lambda$, the map $t \mapsto B(t, w) = B(w)(t, \cdot)$ is continuous from $[0, T]$ to H. Indeed, denoting by $\tilde{b}(t)$ the $\mathbb{R}^d$-valued continuous function $A(t)w(t) + b(t)$, by (80) and the two following equations we have, for any $r, t \in [0, T]$,

$$\begin{aligned}\|B(t, w) - B(r, w)\|_2^2 &= \int_0^T \left|k_2(\xi - t) 1_{\{\xi > t\}} \tilde{b}(t) - k_2(\xi - r) 1_{\{\xi > r\}} \tilde{b}(r)\right|^2 \mathrm{d}\xi \\ &\leq 2\|\tilde{b}\|_\infty^2 \int_0^T \left|k_2(\xi - t) 1_{\{\xi > t\}} - k_2(\xi - r) 1_{\{\xi > r\}}\right|^2 \mathrm{d}\xi \\ &\quad + 2\|k_2\|_2^2 \left|\tilde{b}(t) - \tilde{b}(r)\right|^2 \\ &\leq L\left(|t - r|^{2\alpha - 1} + \left|\tilde{b}(t) - \tilde{b}(r)\right|^2\right),\end{aligned}$$

for some constant $L > 0$.

5. The mild Kolmogorov equation

A classical approach to the study of the Kolmogorov equation is its mild formulation, see for example [18, Section 6.5] and [19, Section 9.5]. Contrary to the strategy adopted in the previous section, where we have constructed a solution to (78) via a stochastic equation (cf. Theorem 9), for the mild Kolmogorov equation we look for a *direct* solution. With the term *direct*, we mean a solution which is determined by a fixed point argument, hence which does not rely on the underlying stochastic differential equation.

In this section, we first present a formal reasoning leading to the mild form of (78), see (96). After that, in Subsection 5.1 we explain some difficulties in proving the well–posedness of such a mild formulation, which are essentially due to the structure of the noise. Since it not the purpose of this section to present a general theory with abstract hypotheses, we limit ourselves to observe that the mild Kolmogorov equation cannot be solved for a class of interesting drifts b using common techniques (cf. Lemma 10). Finally, in Subsection 5.2, we highlight the theoretical importance of the mild Kolmogorov equation. In particular, we sketch a procedure –relying on the mild form– typically used to prove uniqueness in law for a stochastic PDE under weak regularity requirements on the coefficients. We only mention that studying the relation between the transition semigroup of an SDE and the corresponding mild Kolmogorov equation can also be used for numerical applications, as recently investigated by [23] in the Brownian case and [7] in the case of isotropic, stable Lévy processes.

Let $\mathcal{C} = C_b(H;\mathbb{R})$ and consider the backward Kolmogorov equation in differential form, formally written as

$$\begin{cases} \partial_s v(s,x) + \langle b(s,x), \nabla v(s,x)\rangle_H \\ \quad +\frac{1}{2}\mathrm{Tr}\left(D^2 v(s,x)\,\sigma(s)\,\sigma(s)^\star\right) = 0, \qquad s \in [0,T),\, x \in H, \\ v(T,x) = \phi(x), \quad \phi \in \mathcal{C}. \end{cases} \tag{93}$$

Here, H and σ are those of the previous sections (see, in particular, (6)), whereas the drift $b\colon [0,T] \times H \to H$ is a bounded measurable map which could be non–smooth.

We reformulate (93) in order to study it in the space $\mathcal{C}$. Let $u(t,x) := v(T-t,x)$: u solves the forward equation

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}^0_{T-t} u(t,x) + \langle b(T-t,x), \nabla u(t,x)\rangle_H, \qquad t \in (0,T],\, x \in H, \\ u(0,x) = \phi(x), \qquad \phi \in \mathcal{C}, \end{cases} \tag{94}$$

where we denote by $\mathcal{A}^0_{T-t}$ the non–autonomous Gross Laplacian

$$\mathcal{A}^0_{T-t} f(x) = \frac{1}{2}\mathrm{Tr}\left(D^2 f(x)\,\sigma(T-t)\,\sigma(T-t)^*\right).$$

Fix $s \in [0,T]$. For every $t \in [s,T]$, we define the linear evolution operator $R_T(t,s) : \mathcal{C} \to \mathcal{C}$ by

$$\left(R_T(t,s)\,\phi\right)(x) = \mathbb{E}\left[\phi\left(x + \int_s^t \sigma(T-r)\,\mathrm{d}W_r\right)\right], \quad x \in H,\ \phi \in \mathcal{C},$$

where W is an $\mathbb{R}^d$-valued, standard Brownian motion as the one introduced in Section 2. Consider the auxiliary equation

$$\begin{cases} \partial_t z(t,x) = \mathcal{A}^0_{T-t} z(t,x), \qquad t \in (s,T],\, x \in H, \\ z(s,x) = \phi(x), \quad \phi \in \mathcal{C}; \end{cases} \tag{95}$$

if $\phi \in C^{2+\beta}_b(H)$, then Theorem 9 and Remark 4 imply that the function $(R_T(t,s)\phi)(x)$ solves this Cauchy problem for almost every $t \in (s,T)$, for every $x \in H$. At this point, we can introduce the mild formulation of the Kolmogorov equation (94):

$$u(t,x) = (R_T(t,0)\phi)(x) + \int_0^t (R_T(t,s)\langle b(T-s,\cdot), \nabla u(s,\cdot)\rangle_H)(x)\,\mathrm{d}s, \quad \phi \in \mathcal{C}. \tag{96}$$

Note that, heuristically speaking, (96) corresponds to (94). Indeed, if $u(t,x)$ solves (96), then a formal application of Leibnitz integral rule and (95) yield

$$\begin{aligned}\partial_t u(t,\cdot) &= \partial_t R_T(t,0)\phi + R_T(t,t)\langle b(T-t,\cdot), \nabla u(t,\cdot)\rangle_H \\ &\quad + \int_0^t \partial_t R_T(t,s)\langle b(T-s,\cdot), \nabla u(s,\cdot)\rangle_H \,\mathrm{d}s \\ &= \mathcal{A}^0_{T-t} R_T(t,0)\phi + \langle b(T-t,\cdot), \nabla u(t,\cdot)\rangle_H \\ &\quad + \int_0^t \mathcal{A}^0_{T-t} R_T(t,s)\langle b(T-s,\cdot), \nabla u(s,\cdot)\rangle_H \,\mathrm{d}s \\ &= \mathcal{A}^0_{T-t} u + \langle b(T-t,\cdot), \nabla u(t,\cdot)\rangle_H .\end{aligned}$$

As we have already mentioned, the aim is to prove directly, i.e., by a fixed point argument not relying on a stochastic equation, that (96) admits a solution of class, e.g., $C([0,T];\mathcal{C})$. In this regard, the regularity properties of the evolution operator $R_T(t,s)$ are paramount, hence we now discuss them.

According to [19, Proposition 4.28], the H-valued random variable $\int_s^t \sigma(T-r)\,\mathrm{d}W_r$ is Gaussian, centered, with covariance operator

$$Q_T(t,s) = \int_s^t \sigma(T-r)\sigma(T-r)^*\,\mathrm{d}r = \int_{T-t}^{T-s} \sigma(\tau)\sigma(\tau)^*\,\mathrm{d}\tau. \tag{97}$$

This covariance operator is not trivial as it would be in the case of constant σ. In fact, in such a case the Gross Laplacian operator $\mathcal{A}^0 f(x) = \frac{1}{2}\mathrm{Tr}\left(D^2 f\sigma\sigma^*\right)$ has been well studied, see for instance [10], where it is proved that $R_T(t,s)\phi$, $\phi \in \mathcal{C}$, is differentiable in the direction σ (and only in this direction). In our framework with a time–varying σ, the question of the directions of differentiability of $R_T(t,s)\phi$, $\phi \in \mathcal{C}$, is much more complex. Nevertheless, it has to be addressed, because the directional differentiability of $R_T(t,s)\phi$ is essential to solve directly (96). This may be seen in various ways, one of which is the change of variable

$$\theta_T(t,x) = \langle b(T-t,x), \nabla u(t,x)\rangle_H ,$$

that leads to the study of the equation

$$\theta_T(t,x) = \langle b(T-t,x), \nabla(R_T(t,0)\phi)(x)\rangle_H$$

$$+\int_0^t \langle b(T-t,x), \nabla (R_T(t,s)\theta_T(s,\cdot))(x)\rangle_H \,\mathrm{d}s, \quad \phi \in \mathcal{C}. \tag{98}$$

If we can prove that, for some $C, \epsilon > 0$,

$$\sup_{x\in H} \left|\langle b(T-t,x), \nabla (R_T(t,s)\psi)(x)\rangle_H\right| \le \frac{C}{|t-s|^{1-\epsilon}} \|\psi\|_\infty, \quad 0 \le s < t \le T,\ \psi \in \mathcal{C}, \tag{99}$$

then we may try to set up a fixed point argument for the θ_T-equation (98) in a suitable space of bounded, measurable functions. This would in turn give a solution for equation (96) by simply setting

$$u(t,x) = (R_T(t,0)\phi)(x) + \int_0^t (R_T(t,s)\theta_T(s,\cdot))(x)\,\mathrm{d}s.$$

Gradient estimates, regularity of transition semigroups and the corresponding regularity of solutions of the mild formulation of Kolmogorov equations have been widely explored, both in classical references [13,19] and with more advanced results in more recent literature, see for instance [6,11], also in the case of regularity only in some directions, [20,28]. We hope that progress in this theory will also allow us to understand the open case treated here, overcoming the hurdles shown in Subsection 5.1.

5.1. *The gradient estimate*

Using the Gaussian structure of the H-valued random variable

$$Z_T(t,s) = \int_s^t \sigma(T-r)\,\mathrm{d}W_r, \quad 0 \le s < t \le T,$$

and denoting by $Q_T(t,s)^{-1}$ the pseudo–inverse of $Q_T(t,s)$, one can prove –via the Cameron Martin formula (see, e.g., [19, Theorem 2.23])– that

$$\begin{aligned}&\langle b(T-t,x), \nabla (R_T(t,s)\psi)(x)\rangle_H \\ &\quad = \mathbb{E}\left[\left\langle Q_T(t,s)^{-1} b(T-t,x), Z_T(t,s)\right\rangle_H \psi(x+Z_T(t,s))\right], \quad \psi \in \mathcal{C},\end{aligned}$$

if

$$b(T-t,x) \in \mathrm{Range}\,(Q_T(t,s)).$$

This is not the most general condition to obtain the existence of such directional derivative. Indeed, we could split $Q_T(t,s)^{-1}$ and use the fact that $Q_T(t,s)^{-1/2} Z_T(t,s)$ has good properties, which reduces the problem to investigating $b\left(T-t,x\right)\in \mathrm{Range}\left(Q_T(t,s)^{1/2}\right)$. However, handling the square root is even more difficult and thus, for the time being, we analyze the more restrictive condition.

When the previous holds, arguing as in (8), for some $c>0$ we have

$$\begin{aligned}
&\sup_{x\in H}\left|\left\langle b\left(T-t,x\right),\nabla\left(R_T\left(t,s\right)\psi\right)\left(x\right)\right\rangle_H\right| \\
&\quad\le \left\|\psi\right\|_\infty \sup_{x\in H}\mathbb{E}\left[\left|\left\langle Q_T\left(t,s\right)^{-1}b\left(T-t,x\right),Z_T\left(t,s\right)\right\rangle_H\right|\right] \\
&\quad\le c\left\|\psi\right\|_\infty \left\|k_2\right\|_2\left(t-s\right)^{1/2}\sup_{x\in H}\left\|Q_T\left(t,s\right)^{-1}b\left(T-t,x\right)\right\|_2 .
\end{aligned}$$

Therefore a sufficient condition for the gradient estimate (99) is

$$\sup_{x\in H}\left\|Q_T\left(t,s\right)^{-1}b\left(T-t,x\right)\right\|_2\le \frac{C}{\left|t-s\right|^{\frac{3}{2}-\epsilon}},\quad 0\le s<t\le T,\ \text{for some } C>0.$$

For a general b, standing the potentially very strong degeneracy of $Q_T\left(t,s\right)$, we do not see any hope to prove the gradient estimate (99). A particular case that, a priori, may look promising, is when the Volterra drift is of the same kind as the noise part, namely (cf. (6))

$$\begin{gathered}
[b\left(t,x\right)](\xi)=\bar{\beta}\left(x\right)k_2\left(\xi-t\right)1_{\{t<\xi\}}=[\sigma(t)\bar{\beta}(x)]\left(\xi\right), \\
\xi\in[0,T],\ \text{for some } \bar{\beta}\in\mathcal{B}_b\left(H;\mathbb{R}^d\right).
\end{gathered}$$

In this case, since $b\left(T-t,x\right)=\sigma\left(T-t\right)\bar{\beta}\left(x\right)$, we need to prove that

$$\sigma\left(T-t\right)e_k\in \mathrm{Range}\left(Q_T\left(t,s\right)\right),\quad k=1,\dots,d, \tag{100}$$

and that

$$\begin{gathered}
\left\|Q_T\left(t,s\right)^{-1}\sigma\left(T-t\right)e_k\right\|_2\le\frac{C}{\left|t-s\right|^{\frac{3}{2}-\epsilon}}, \\
0\le s<t\le T,\ k=1,\dots,d,\ \text{for some } C>0,
\end{gathered}$$

where $(e_k)_{k=1,\dots,d}$ is the canonical basis of $\mathbb{R}^d$. Recalling that, by (97), $Q_T\left(t,s\right)=\int_{T-t}^{T-s}\sigma\left(\tau\right)\sigma\left(\tau\right)^*\mathrm{d}\tau$, apparently we could think that (100) is true. But it is not, as the necessary condition given by the next lemma shows.

Lemma 10. *Let $0\le s<t\le T$ and suppose that $f\in \mathrm{Range}\left(Q_T(t,s)\right)\subset H$, where the operator $Q_T(t,s)$ is defined in (97). Then f admits a continuous representative that is equal to 0 in $(0,T-t)$. In other words, there exists a continuous function $g\colon(0,T)\to\mathbb{R}^d$ such that $f=g$ almost everywhere in $(0,T)$ and $g=0$ in $(0,T-t)$.*

Proof. Fix $0 \leq s < t \leq T$. Consider $f \in \text{Range}\left(Q_T(t,s)\right)$, so that there exists $v \in H$ such that, by (97), $f = \int_{T-t}^{T-s} \sigma(\tau)\sigma(\tau)^* v \mathrm{d}\tau$. In particular, for every $k = 1, \dots, d$, denoting by $\cdot$ the scalar product in $\mathbb{R}^d$, by the standard properties of Bochner's integral we obtain

$$f \cdot e_k = \left(\int_{T-t}^{T-s} \left(\sigma(\tau)^* v\right) \; k_2(\cdot - \tau) 1_{\{\cdot > \tau\}} \mathrm{d}\tau \right) \cdot e_k = \int_{T-t}^{T-s} \langle \sigma(\tau) e_k, v \rangle_H \, k_2(\cdot - \tau) 1_{\{\cdot > \tau\}} \mathrm{d}\tau.$$

Furthermore, recalling (5), for a.e. $\xi \in (0,T)$ we have

$$(f \cdot e_k)(\xi) = \frac{1}{\Gamma(\alpha)} \int_{T-t}^{T-s} 1_{\{\tau < \xi\}} \langle \sigma(\tau) e_k, v \rangle_H (\xi - \tau)^{\alpha - 1} \mathrm{d}\tau.$$

We denote by g_k the function appearing on the right–hand side of the previous equation, i.e.,

$$g_k(\xi) = \frac{1}{\Gamma(\alpha)} \int_{T-t}^{T-s} 1_{\{\tau < \xi\}} \langle \sigma(\tau) e_k, v \rangle_H (\xi - \tau)^{\alpha - 1} \mathrm{d}\tau, \quad \xi \in (0,T).$$

We want to show the continuity of g_k on the interval $[T-t, T)$: this ensures that g_k is continuous on the whole $(0,T)$, since trivially $g_k = 0$ on $(0, T-t]$. We first write

$$g_k(\xi) = \int_0^{\xi} 1_{\{\tau > T-t\}} \langle \sigma(\tau) e_k, v \rangle_H (\xi - \tau)^{\alpha - 1} \mathrm{d}\tau, \quad \xi \in [T-t, T-s],$$

and notice that, as $\sigma(\cdot)e_k \in C([0,T]; H)$ (see (80) in the proof of Lemma 8), the mapping $\langle \sigma(\cdot)e_k, v\rangle_H$ is continuous on $[0,T]$. Therefore we invoke [29, Theorem 2.2 (i), Chapter 2] to conclude that g_k is continuous on $[T-t, T-s]$. Secondly, since

$$g_k(\xi) = \int_{T-t}^{T-s} \langle \sigma(\tau) e_k, v \rangle_H (\xi - \tau)^{\alpha - 1} \mathrm{d}\tau, \quad \xi \in [T-s, T),$$

the continuity of g_k on $[T-s, T)$ can be inferred employing the dominated convergence theorem. Thus, g_k is continuous on $(0,T)$. This shows that the components $f \cdot e_k$, $k = 1, \dots, d$, of the function $f\colon [0,T] \to \mathbb{R}^d$ are almost everywhere equal on $(0,T)$ to continuous functions g_k, which completes the proof. □

Remark 7. Lemma 10 prevents us from choosing another interesting drift $b(t,x)$, namely

$$[b(t,x)](\xi) = \bar{\beta}(x) 1_{(t,T)}(\xi), \quad \xi \in [0,T], \text{ for some } \bar{\beta} \in \mathcal{B}_b(H; \mathbb{R}^d).$$

5.2. Concerning regularization by noise via the Kolmogorov equation

The previous subsections discuss the open question, for the abstract evolution equation introduced in this paper, of the regularity of the transition semigroup, of interest to prove existence and regularity of solutions of the mild formulation of the Kolmogorov equation. As already mentioned, one of the main aims for such existence and regularity result is its application to regularization by noise, since it would not require so much regularity of the drift, being based on regularizing properties of the transition semigroup.

By "regularization by noise" we mean, in this subsection, proving uniqueness of solutions, thanks to the noise, even when the drift does not have usual Lipschitz continuity assumptions. For the following discussion it is important to distinguish between weak (or in law) uniqueness and strong (or pathwise) uniqueness. Our discussion will be restricted to weak uniqueness. In terms of Kolmogorov equation, weak uniqueness corresponds to gradient estimates, hence it links directly with the previous section; on the contrary, strong uniqueness is based on estimates on second derivatives, hence it is beyond the analysis of this paper. Weak uniqueness is more classically proved by Girsanov, but in infinite dimensions the assumptions for Girsanov transform are more restrictive than those required to apply Kolmogorov equation and precisely to estimate its first derivatives. In the sequel we describe the method referring ourselves to the pedagogical description of [22], but more classical and precise results on this approach can be found in [12,32,48]. In particular, the condition on (A, Q) introduced below is classical in these references.

For pedagogical reasons, we then recall in this subsection the main steps that one should implement in case of a positive solution to the problem of existence of the mild solution of the Kolmogorov equation. Details can be found in [22, Section 2.3.3] and in [48], inspired by the finite–dimensional case treated in [40].

The model studied in [22] is an abstract equation in a separable Hilbert space H of the form

$$\mathrm{d}X_t = AX_t\mathrm{d}t + b\left(t, X_t\right)\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t, \quad X_0 = x \in H, \tag{101}$$

which "a priori" is not well posed, because $b\colon [0,T] \times H \to H$ is subject to weak regularity assumptions, not including Lipschitz continuity. In [22], where the nonlinearity b is denoted by B, the case

$$b \in C\left([0,T]; C_b^{\alpha}\left(H, H\right)\right)$$

is treated under a certain assumption on the pair (A, Q) (for the details, see [22, Section 2.3.3]), that we summarize here in the form

$$\int_0^T \left\| Q_t^{-1/2} e^{tA} \right\| \mathrm{d}t < \infty,$$

where $Q_t = \int_0^t e^{sA^*} Q e^{sA} \mathrm{d}s$; see also [48]. This is weaker than the assumptions required to apply the Girsanov method, which also leads to uniqueness in law, and weaker than the conditions in [14] that allow to prove strong uniqueness.

In this framework, one can prove that the mild form of the Kolmogorov equation is well posed for a large class of initial conditions ϕ, thanks to suitable gradient estimates for the transition semigroup. The result is quantitatively stable under a suitable regularization (which, in [22], is simply the Galerkin approximation), so that the next rigorous computations can be performed for the approximating system and then sent to the limit.

The computation then is the application of Itô's formula to $u\left(t_f - t, X_t\right)$ for every $t_f \in (0, T]$, where u is the solution of the (forward) mild Kolmogorov equation with initial condition ϕ. After proving that the Itô term is a martingale, one gets

$$\mathbb{E}\left[\phi\left(X_{t_f}\right)\right] = u\left(t_f, x\right).$$

Therefore, if $X_t^{(1)}$ and $X_t^{(2)}$ are two solutions of Equation (101), they must have the same marginal law at time t_f (since the class of ϕ is large enough). The details can be found in [22, Section 2.3.3]. Passing from the same marginals to the same law on path space requires a further argument that can be found in [48] or [40,41].

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Appendix A. Regularity of the solution (81) of the Kolmogorov equation

In this appendix, we present an auxiliary lemma, namely Lemma 11, containing regularity results about the solution $u\colon [0,T] \times H \to \mathbb{R}$ of the Kolmogorov backward equation (78) defined in (81). Such a lemma plays a key role in the proof of Theorem 9.

Lemma 11. *Suppose that* $\Phi \in C_b^{2+\beta}(H)$ *and that Assumption 3 holds. Then, the map* $u\colon [0,T] \times H \to \mathbb{R}$ *defined in* (81) *belongs to* $L^\infty\left(0, T; C_b^{2+\beta}(H)\right) \cap C([0,T] \times H; \mathbb{R})$. *In particular, there exists a constant* $C_{d,T,\beta,\Phi} > 0$ *such that*

$$\left\|D^2 u(t, \phi) - D^2 u(t, \psi)\right\|_{\mathcal{L}(H;H)} \leq C_{d,T,\beta,\Phi} \left\|\phi - \psi\right\|_2^\beta, \quad \phi, \psi \in H,\, t \in [0,T]. \tag{102}$$

Furthermore, the map $(t, \phi, \psi) \mapsto \langle \nabla u(t,\phi), \psi\rangle_H$ *[resp.,* $(t, \phi, \psi, \eta) \mapsto \langle D^2 u(t,\phi)\psi, \eta\rangle_H$*] is continuous in* $[0,T] \times H \times H$ *[resp.,* $[0,T] \times H \times H \times H$*].*

Proof. We start off by proving that $u \in C([0,T] \times H; \mathbb{R})$. Consider $t \in [0,T]$, $\phi \in H$ and two sequences $(t_n)_n \subset [0,T]$ and $(\phi_n)_n \subset H$ such that $t_n \to t$ and $\phi_n \to \phi$ as $n \to \infty$.

Since $\nabla\Phi\colon H \to H$ is bounded, by the mean value theorem we compute, recalling the definition of u in (81),

$$\begin{aligned} |u(t_n,\phi_n) - u(t,\phi)| &\leq \mathbb{E}\left[\left|\Phi\left(w_T^{t_n,\phi_n}\right) - \Phi\left(w_T^{t_n,\phi}\right)\right|\right] + \mathbb{E}\left[\left|\Phi\left(w_T^{t_n,\phi}\right) - \Phi\left(w_T^{t,\phi}\right)\right|\right] \\ &\leq \left\|\nabla\Phi\right\|_\infty \left(\left\|w_T^{t_n,\phi_n} - w_T^{t_n,\phi}\right\|_{\mathcal{H}} + \left\|w_T^{t_n,\phi} - w_T^{t,\phi}\right\|_{\mathcal{H}}\right). \end{aligned} \tag{103}$$

By (31) in Corollary 4, we infer that $\lim_{n\to\infty}\|w_T^{t_n,\phi_n} - w_T^{t_n,\phi}\|_{\mathcal{H}} = 0$. As for $\|w_T^{t_n,\phi} - w_T^{t,\phi}\|_{\mathcal{H}}$, we first assume that $t_n > t$. Then, by the flow property in (21) and Corollary 4 we have, for some constants $c_1, c_2 > 0$ which might depend on ϕ,

$$\left\|w_T^{t_n,\phi} - w_T^{t,\phi}\right\|_{\mathcal{H}} = \left\|w_T^{t_n,\phi} - w_T^{t_n,w_{t_n}^{t,\phi}}\right\|_{\mathcal{H}} \leq c_1\left\|w_{t_n}^{t,\phi} - \phi\right\|_{\mathcal{H}} \leq c_2\sqrt{|t_n - t|},$$

where the last inequality is due to Lemma 5, see (34). An analogous argument shows that the previous bound holds even in the case $t_n \leq t$, therefore $\lim_{n\to\infty}\|w_T^{t_n,\phi} - w_T^{t,\phi}\|_{\mathcal{H}} = 0$. Going back to (103), we conclude that $\lim_{n\to\infty}|u(t_n,\phi_n) - u(t,\phi)| = 0$, hence $u\colon [0,T]\times H \to \mathbb{R}$ is continuous, as desired.

We now prove that $u \in L^\infty\big(0,T; C_b^{2+\beta}(H)\big)$. Since $\Phi \in C_b^{2+\beta}(H)$, there exists a constant $C_\Phi > 0$ such that

$$\left\|D^2\Phi(\phi) - D^2\Phi(\psi)\right\|_{\mathcal{L}(H;H)} \leq C_\Phi \left\|\phi - \psi\right\|_2^\beta, \quad \phi, \psi \in H. \tag{104}$$

Obviously, from the boundedness of Φ we have $\|u\|_\infty = \sup_{t\in[0,T]}\sup_{\phi\in H}|u(t,\phi)| < \infty$. First, we want to show that, for every $t \in [0,T]$, $u(t,\cdot) \in C_b^1(H)$, with

$$\langle\nabla u(t,\phi), \psi\rangle_H = \mathbb{E}\left[\left\langle\nabla\Phi\left(w_T^{t,\phi}\right), Dw_T^{t,\phi}\psi\right\rangle_H\right], \quad \phi,\, \psi \in H. \tag{105}$$

To see this, by Taylor's formula applied to Φ we compute, for every ϕ, $h \in H$,

$$\begin{aligned} &\mathbb{E}\left[\left|\Phi\left(w_T^{t,\phi+h}\right) - \Phi\left(w_T^{t,\phi}\right) - \left\langle\nabla\Phi\left(w_T^{t,\phi}\right), Dw_T^{t,\phi}h\right\rangle_H\right|\right] \\ &\quad\leq \left\|\nabla\Phi\right\|_\infty \mathbb{E}\left[\left\|w_T^{t,\phi+h} - w_T^{t,\phi} - Dw_T^{t,\phi}h\right\|_2\right] \\ &\qquad + \mathbb{E}\left[\left|\int_0^1 \left\langle\nabla\Phi\left(w_T^{t,\phi} + r\left(w_T^{t,\phi+h} - w_T^{t,\phi}\right)\right) - \nabla\Phi\left(w_T^{t,\phi}\right), w_T^{t,\phi+h} - w_T^{t,\phi}\right\rangle_H \mathrm{d}r\right|\right] \\ &\quad\leq \left\|\nabla\Phi\right\|_\infty \left\|w_T^{t,\phi+h} - w_T^{t,\phi} - Dw_T^{t,\phi}h\right\|_{\mathcal{H}} + \left\|D^2\Phi\right\|_\infty \left\|w_T^{t,\phi+h} - w_T^{t,\phi}\right\|_{\mathcal{H}}^2 = \mathrm{o}\left(\|h\|_2\right). \end{aligned} \tag{106}$$

Here, for the second inequality we use the Lipschitz continuity of the map $\nabla\Phi\colon H \to H$ –guaranteed by the mean value theorem– and for the third equality we invoke Corollary 4 and Theorem 6. This shows (105), from which we deduce the continuity of the function

$\nabla u(t,\cdot)\colon H \to H$. In particular, by (43), there exists a constant $C_1 = C_1(d,T)$ such that $\|\nabla u\|_\infty \le C_1 \|\nabla\Phi\|_\infty$.

We also note that, arguing as in (106) and thanks to the estimates of $\|w_T^{t,\phi+h} - w_T^{t,\phi} - Dw_T^{t,\phi}h\|_{\mathcal{H}}$ in the proof of Theorem 6 (see, for instance, (48)-(55)), for every $M > 0$ we have

$$\sup_{t\in[0,T]}\ \sup_{\|\phi\|_2,\|\psi\|_2\le M} \mathbb{E}\left[\left|\Phi\left(w_T^{t,\phi+h\psi}\right) - \Phi\left(w_T^{t,\phi}\right) - h\left\langle\nabla\Phi\left(w_T^{t,\phi}\right), Dw_T^{t,\phi}\psi\right\rangle_H\right|\right] = \mathrm{o}\left(h\right), \quad h \in \mathbb{R}. \tag{107}$$

which gives the continuity of the map $(t,\phi,\psi) \mapsto \langle\nabla u(t,\phi),\psi\rangle_H$ in $[0,T]\times H\times H$ as $u \in C([0,T]\times H;\mathbb{R})$.

Secondly, we claim that $u(t,\cdot)$ is twice Fréchet differentiable in H, with

$$\begin{aligned}\left\langle D^2u\left(t,\phi\right)\psi,\eta\right\rangle_H = \mathbb{E}\Big[&\left\langle D^2\Phi\left(w_T^{t,\phi}\right)Dw_T^{t,\phi}\psi, Dw_T^{t,\phi}\eta\right\rangle_H \\ &+ \left\langle\nabla\Phi\left(w_T^{t,\phi}\right), D^2w_T^{t,\phi}\left(\psi,\eta\right)\right\rangle_H\Big], \quad \phi,\psi,\eta \in H.\end{aligned} \tag{108}$$

Indeed, recalling (105), an application of Taylor's formula on $\nabla\Phi$ yields

$$\begin{aligned}&\left|\left\langle\nabla u\left(t,\phi+h\right) - \nabla u\left(t,\phi\right) - D^2u\left(t,\phi\right)h,\psi\right\rangle_H\right| \\ &= \Big|\mathbb{E}\Big[\left\langle\nabla\Phi\left(w_T^{t,\phi+h}\right), Dw_T^{t,\phi+h}\psi\right\rangle_H - \left\langle\nabla\Phi\left(w_T^{t,\phi}\right), Dw_T^{t,\phi}\psi\right\rangle_H \\ &\qquad - \left\langle D^2\Phi\left(w_T^{t,\phi}\right)Dw_T^{t,\phi}h, Dw_T^{t,\phi}\psi\right\rangle_H - \left\langle\nabla\Phi\left(w_T^{t,\phi}\right), D^2w_T^{t,\phi}\left(h,\psi\right)\right\rangle_H\Big]\Big| \\ &\le \mathbb{E}\left[\left|\left\langle D^2\Phi\left(w_T^{t,\phi}\right)\left(w_T^{t,\phi+h} - w_T^{t,\phi} - Dw_T^{t,\phi}h\right), Dw_T^{t,\phi}\psi\right\rangle_H\right|\right] \\ &\quad + \mathbb{E}\left[\left|\left\langle\nabla\Phi\left(w_T^{t,\phi}\right), Dw_T^{t,\phi+h}\psi - Dw_T^{t,\phi}\psi - D^2w_T^{t,\phi}\left(h,\psi\right)\right\rangle_H\right|\right] \\ &\quad + \mathbb{E}\left[\left|\left\langle\nabla\Phi\left(w_T^{t,\phi+h}\right) - \nabla\Phi\left(w_T^{t,\phi}\right), \left(Dw_T^{t,\phi+h} - Dw_T^{t,\phi}\right)\psi\right\rangle_H\right|\right] + R_\Phi\left(\phi,\psi,h\right) \\ &=: \left(\mathbf{I}_1 + \mathbf{II}_1 + \mathbf{III}_1 + R_\Phi\right)\left(\phi,\psi,h\right),\end{aligned} \tag{109}$$

for every $\phi,\psi,h \in H$. Here, we denote by

$$\begin{aligned}R_\Phi\left(\phi,\psi,h\right) = \mathbb{E}\Bigg[\Bigg|\Bigg\langle \int_0^1 &\left(D^2\Phi\left(w_T^{t,\phi} + r\left(w_T^{t,\phi+h} - w_T^{t,\phi}\right)\right) - D^2\Phi\left(w_T^{t,\phi}\right)\right) \\ &\times \left(w_T^{t,\phi+h} - w_T^{t,\phi}\right)\mathrm{d}r, Dw_T^{t,\phi}\psi\Bigg\rangle_H\Bigg|\Bigg].\end{aligned}$$

Using (43), (104) and Corollary 4, for some constant $c_3 > 0$ we compute

$$R_\Phi(\phi,\psi,h) \le C_\Phi C_1 \mathbb{E}\left[\left\| w_T^{t,\phi+h} - w_T^{t,\phi}\right\|_2^{1+\beta}\right] \|\psi\|_2 \le C_\Phi C_1 \left\| w_T^{t,\phi+h} - w_T^{t,\phi}\right\|_{\mathcal{H}}^{1+\beta} \|\psi\|_2$$
$$\le c_3 \|\psi\|_2 \|h\|_2^{1+\beta}, \quad \phi,\psi,h \in H,$$

where we also employ Jensen's inequality noticing that $1+\beta \le 2$. Next,

$$|\mathbf{I}_1(\phi,\psi,h)| \le C_1 \left\|D^2\Phi\right\|_\infty \|\psi\|_2 \left\| w_T^{t,\phi+h} - w_T^{t,\phi} - Dw_T^{t,\phi}h\right\|_{\mathcal{H}}, \quad \phi,\psi,h\in H,$$

and

$$|\mathbf{II}_1(\phi,\psi,h)| \le \|\nabla\Phi\|_\infty \|\psi\|_2 \left\| Dw_T^{t,\phi+h} - Dw_T^{t,\phi} - D^2 w_T^{t,\phi}(h,\cdot)\right\|_{\mathcal{L}(H;\mathcal{H})}, \quad \phi,\psi,h\in H.$$

Finally, by Corollary 4 and (43) (recall that, under Assumption 3, we take $\gamma=\beta$ in (38), see (65))

$$|\mathbf{III}_1(\phi,\psi,h)| \le C_1 \left\|D^2\Phi\right\|_\infty \|\psi\|_2 \mathbb{E}\left[\left\| w_T^{t,\phi+h} - w_T^{t,\phi}\right\|_2^{1+\beta}\right] \le \tilde{c}\left\|D^2\Phi\right\|_\infty \|\psi\|_2 \|h\|_2^{1+\beta},$$
$$\phi,\psi,h\in H,$$

for some $\tilde{c}>0$. Going back to (109), by Theorem 7, the previous estimates let us write, for some constant $C>0$,

$$\begin{aligned}
&\left\|\nabla u(t,\phi+h) - \nabla u(t,\phi) - D^2u(t,\phi)h\right\|_2 \\
&\quad = \sup_{\|\psi\|_2\le 1} \left|\left\langle \nabla u(t,\phi+h) - \nabla u(t,\phi) - D^2u(t,\phi)h, \psi\right\rangle_H\right| \\
&\quad \le C\Bigg(\left\| w_T^{t,\phi+h} - w_T^{t,\phi} - Dw_T^{t,\phi}h\right\|_{\mathcal{H}} \\
&\qquad + \left\| Dw_T^{t,\phi+h} - Dw_T^{t,\phi} - D^2 w_T^{t,\phi}(h,\cdot)\right\|_{\mathcal{L}(H;\mathcal{H})} + \|h\|_2^{1+\beta}\Bigg) \\
&\quad = \mathrm{o}(\|h\|_2), \quad \phi,h\in H,
\end{aligned} \tag{110}$$

which proves (108). In particular, by (43)-(67), there is a constant $C_2 = C_2(d,T)>0$ such that

$$\left\|D^2u\right\|_\infty \le C_2\left(\left\|D^2\Phi\right\|_\infty + \|\nabla\Phi\|_\infty\right).$$

In addition, arguing as in (110) (see also (107)) and thanks to the estimates of $\big\| Dw_T^{t,\phi+h} - Dw_T^{t,\phi} - D^2 w_T^{t,\phi}(h,\cdot)\big\|_{\mathcal{L}(H;\mathcal{H})}$ in the proof of Theorem 7 (see, for instance, (72)), for every $M>0$ we have

$$\sup_{t\in[0,T]} \sup_{\|\phi\|_2,\|\psi\|_2,\|\eta\|_2\le M} \left|\left\langle \nabla u(t,\phi+h\psi) - \nabla u(t,\phi) - hD^2u(t,\phi)\psi, \eta\right\rangle_H\right|$$
$$= \mathrm{o}(h), \quad h\in\mathbb{R}.$$

Since we have proved that $\langle \nabla u(t,\phi),\psi\rangle_H$ is continuous in $[0,T]\times H\times H$, the previous equation ensures that the map $(t,\phi,\psi,\eta)\mapsto \langle D^2u(t,\phi)\psi,\eta\rangle_H$ is continuous in $[0,T]\times H\times H\times H$, as desired.

In conclusion, we prove that $u\left(t,\cdot\right)\in C_b^{2+\beta}\left(H\right)$. From (108), for every $\phi_1,\phi_2\in H$,

$$\begin{aligned}
&\left\langle\left(D^2u\left(t,\phi_1\right)-D^2u\left(t,\phi_2\right)\right)\psi,\eta\right\rangle_H\\
&=\mathbb{E}\left[\left\langle\left(D^2\Phi\left(w_T^{t,\phi_1}\right)-D^2\Phi\left(w_T^{t,\phi_2}\right)\right)Dw_T^{t,\phi_1}\psi, Dw_T^{t,\phi_1}\eta\right\rangle_H\right]\\
&\quad+\mathbb{E}\left[\left\langle D^2\Phi\left(w_T^{t,\phi_2}\right)\left(Dw_T^{t,\phi_1}-Dw_T^{t,\phi_2}\right)\psi, Dw_T^{t,\phi_1}\eta\right\rangle_H\right]\\
&\quad+\mathbb{E}\left[\left\langle D^2\Phi\left(w_T^{t,\phi_2}\right)Dw_T^{t,\phi_2}\psi, \left(Dw_T^{t,\phi_1}-Dw_T^{t,\phi_2}\right)\eta\right\rangle_H\right]\\
&\quad+\mathbb{E}\left[\left\langle \nabla\Phi\left(w_T^{t,\phi_1}\right)-\nabla\Phi\left(w_T^{t,\phi_2}\right), D^2w_T^{t,\phi_1}\left(\psi,\eta\right)\right\rangle_H\right]\\
&\quad+\mathbb{E}\left[\left\langle \nabla\Phi\left(w_T^{t,\phi_2}\right), \left(D^2w_T^{t,\phi_1}-D^2w_T^{t,\phi_2}\right)\left(\psi,\eta\right)\right\rangle_H\right]\\
&=:\left(\mathbf{I}_2+\mathbf{II}_2+\mathbf{III}_2+\mathbf{IV}_2+\mathbf{V}_2\right)\left(\phi_1,\phi_2,\psi,\eta\right),\quad \psi,\eta\in H.
\end{aligned}$$

To keep notation the short, in what follows we consider arbitrary $\psi,\eta\in H$, we do not write $(\phi_1,\phi_2,\psi,\eta)$ and we denote by $c=c(d,T,\beta)>0$ a constant that might change from line to line. Observe that, by (43)-(104), Corollary 4 and Jensen's inequality,

$$\left|\mathbf{I}_2\right|\leq c\,C_\Phi\left\|\psi\right\|_2\left\|\eta\right\|_2\left\|\phi_1-\phi_2\right\|_2^\beta.$$

Moreover, by (43) (see also (65)),

$$\left|\mathbf{II}_2\right|\leq c\left\|D^2\Phi\right\|_\infty\left\|\psi\right\|_2\left\|\eta\right\|_2\left\|\phi_1-\phi_2\right\|_2^\beta.$$

An analogous estimate holds for $|\mathbf{III}_2|$, too. As for the remaining addends, by (67) we have

$$\left|\mathbf{IV}_2\right|\leq c\left\|D^2\Phi\right\|_\infty\left\|\psi\right\|_2\left\|\eta\right\|_2\left\|\phi_1-\phi_2\right\|_2,$$

and

$$\left|\mathbf{V}_2\right|\leq c\left\|\nabla\Phi\right\|_\infty\left\|\psi\right\|_2\left\|\eta\right\|_2\left\|\phi_1-\phi_2\right\|_2^\beta.$$

Thus, the function $D^2u\left(t,\cdot\right):H\to\mathcal{L}\left(H;H\right)$ is β-Hölder continuous uniformly in time and the proof is complete. □

Data availability

No data was used for the research described in the article.

References

[1] E. Abi Jaber, C. Cuchiero, M. Larsson, S. Pulido, A weak solution theory for stochastic Volterra equations of convolution type, Ann. Appl. Probab. 31 (6) (2021) 2924–2952.

[2] E. Abi Jaber, M. Larsson, S. Pulido, Affine Volterra processes, Ann. Appl. Probab. 29 (5) (2019) 3155–3200.

[3] D. Addona, F. Masiero, E. Priola, A BSDEs approach to pathwise uniqueness for stochastic evolution equations, J. Differ. Equ. 366 (2023) 192–248.

[4] A. Barrasso, F. Russo, Gâteaux type path–dependent PDEs and BSDEs with Gaussian forward processes, Stoch. Dyn. 22 (01) (2022) 2250007.

[5] F. Bertacco, C. Orrieri, L. Scarpa, Weak uniqueness by noise for singular stochastic PDEs, preprint, arXiv:2308.01642, 2024.

[6] D.A. Bignamini, P. De Fazio, Log-Sobolev inequalities and hypercontractivity for Ornstein–Uhlenbeck evolution operators in infinite dimension, J. Evol. Equ. 24 (2024) 78.

[7] A. Bondi, Probability computation for high-dimensional semilinear SDEs driven by isotropic α-stable processes via mild Kolmogorov equations, Electron. J. Probab. 28 (2023) 1–31.

[8] A. Bondi, G. Livieri, S. Pulido, Affine Volterra processes with jumps, Stoch. Process. Appl. 168 (2024) 104264.

[9] A. Bondi, S. Pulido, S. Scotti, The rough Hawkes Heston stochastic volatility model, Math. Finance 34 (4) (2024) 1197–1241.

[10] P. Cannarsa, G. Da Prato, Infinite-dimensional elliptic equations with Hölder-continuous coefficients, Adv. Differ. Equ. 1 (3) (1996) 425–452.

[11] S. Cerrai, A. Lunardi, Smoothing effects and maximal Hölder regularity for non-autonomous Kolmogorov equations in infinite dimension, preprint, arXiv:2111.05421, 2021.

[12] A. Chojnowska-Michalik, B. Goldys, Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces, Probab. Theory Relat. Fields 102 (3) (1995) 331–356.

[13] G. Da Prato, Kolmogorov Equations for Stochastic PDEs, Springer Science & Business Media, 2004.

[14] G. Da Prato, F. Flandoli, Pathwise uniqueness for a class of SDE in Hilbert spaces and applications, J. Funct. Anal. 259 (1) (2010) 243–267.

[15] G. Da Prato, F. Flandoli, Some results for pathwise uniqueness in Hilbert spaces, Commun. Pure Appl. Anal. 13 (5) (2014) 1789–1797.

[16] G. Da Prato, F. Flandoli, E. Priola, M. Röckner, Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift, Ann. Probab. 41 (5) (2013) 3306–3344.

[17] G. Da Prato, F. Flandoli, M. Röckner, A.Y. Veretennikov, Strong uniqueness for SDEs in Hilbert spaces with nonregular drift, Ann. Probab. 44 (3) (2016) 1985–2023.

[18] G. Da Prato, J. Zabczyk, Second Order Partial Differential Equations in Hilbert Spaces, vol. 293, Cambridge University Press, 2002.

[19] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, 2014.

[20] P. De Fazio, On smoothing in non autonomous Ornstein-Uhlenbeck equations in infinite dimensions, preprint, arXiv:2212.05559, 2022.

[21] O. El Euch, M. Rosenbaum, The characteristic function of rough Heston models, Math. Finance 29 (1) (2019) 3–38.

[22] F. Flandoli, Random Perturbation of PDEs and Fluid Dynamic Models: École D'été de Probabilités de Saint-Flour XL–2010, vol. 2015, Springer Science & Business Media, 2011.

[23] F. Flandoli, D. Luo, C. Ricci, Numerical computation of probabilities for nonlinear SDEs in high dimension using Kolmogorov equation, Appl. Math. Comput. 436 (2023) 127520.

[24] F. Flandoli, G. Zanco, An infinite-dimensional approach to path-dependent Kolmogorov equations, Ann. Probab. 44 (4) (2016) 2643–2693.

[25] L. Galeati, F.A. Harang, Regularization of multiplicative SDEs through additive noise, Ann. Appl. Probab. 32 (5) (2022) 3930–3963.

[26] L. Galeati, F.A. Harang, A. Mayorcas, Distribution dependent SDEs driven by additive fractional Brownian motion, Probab. Theory Relat. Fields 185 (1–2) (2023) 251–309.

[27] J. Gatheral, P. Jusselin, M. Rosenbaum, The quadratic rough Heston model and the joint S&P 500/VIX smile calibration problem, Risk, cutting edge, https://www.risk.net/cutting-edge/banking/7530461/the-quadratic-rough-heston-model-and-the-joint-sp-500vix-smile-calibration-problem, 2020.

[28] F. Gozzi, F. Masiero, Stochastic optimal control with delay in the control I: solving the HJB equation through partial smoothing, SIAM J. Control Optim. 55 (5) (2017) 2981–3012.

[29] G. Gripenberg, S.O. Londen, O. Staffans, Volterra Integral and Functional Equations, vol. 34, Cambridge University Press, 1990.

[30] F.A. Harang, C. Ling, Regularity of local times associated with Volterra–Lévy processes and path–wise regularization of stochastic differential equations, J. Theor. Probab. 35 (3) (2022) 1706–1735.

[31] M. Kunze, On a class of martingale problems on Banach spaces, Electron. J. Probab. 18 (104) (2013) 1–30.

[32] M. Kunze, Perturbation of strong Feller semigroups and well-posedness of semilinear stochastic equations on Banach spaces, Stoch. Int. J. Probab. Stoch. Process. 85 (6) (2013) 960–986.

[33] L. Mytnik, T.S. Salisbury, Uniqueness for Volterra-type stochastic integral equations, preprint, arXiv:1502.05513, 2015.

[34] D. Nualart, Y. Ouknine, Regularization of differential equations by fractional noise, Stoch. Process. Appl. 102 (1) (2002) 103–116.

[35] D. Nualart, Y. Ouknine, Stochastic differential equations with additive fractional noise and locally unbounded drift, in: Stochastic Inequalities and Applications, Birkhäuser, Basel, 2003, pp. 353–365.

[36] S. Peszat, J. Zabczyk, Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach, vol. 113, Cambridge University Press, 2007.

[37] E. Priola, An optimal regularity result for Kolmogorov equations and weak uniqueness for some critical SPDEs, Ann. Probab. 49 (3) (2021) 1310–1346.

[38] P. Protter, Volterra equations driven by semimartingales, Ann. Probab. 13 (2) (1985) 519–530.

[39] W. Rudin, Functional Analysis, 2nd edition, McGraw-Hill, 1991.

[40] D.W. Stroock, S.S. Varadhan, Multidimensional Diffusion Processes, vol. 233, Springer Science & Business Media, 1997.

[41] D. Trevisan, Well–posedness of multidimensional diffusion processes with weakly differentiable coefficients, Electron. J. Probab. 21 (22) (2016) 1–41.

[42] A.Y. Veretennikov, On strong solutions and explicit formulas for solutions of stochastic integral equations, Math. USSR Sb. 39 (3) (1981) 387.

[43] F. Viens, J. Zhang, A martingale approach for fractional Brownian motions and related path dependent PDEs, Ann. Appl. Probab. 29 (6) (2019) 3489–3540.

[44] H. Wang, J. Yong, J. Zhang, Path dependent Feynman–Kac formula for forward backward stochastic Volterra integral equations, Ann. Inst. Henri Poincaré B, Probab. Stat. 58 (2) (2022) 603–638.

[45] T. Wang, J. Yong, Backward stochastic Volterra integral equations—representation of adapted solutions, Stoch. Process. Appl. 129 (12) (2019) 4926–4964.

[46] Z. Wang, Existence and uniqueness of solutions to stochastic Volterra equations with singular kernels and non-Lipschitz coefficients, Stat. Probab. Lett. 78 (9) (2008) 1062–1071.

[47] K. Yosida, Functional Analysis, vol. 123, Springer Science & Business Media, 2012.

[48] L. Zambotti, An analytic approach to existence and uniqueness for martingale problems in infinite dimensions, Probab. Theory Relat. Fields 118 (2) (2000) 147–168.

[49] X. Zhang, Stochastic Volterra equations in Banach spaces and stochastic partial differential equation, J. Funct. Anal. 258 (4) (2010) 1361–1425.