RESEARCH ARTICLE | JANUARY 07 2020

Energy conditional measures and 2D turbulence

Franco Flandoli ; Dejun Luo

J. Math. Phys. 61, 013101 (2020)
https://doi.org/10.1063/1.5099359

Journal of
Mathematical Physics

ARTICLE

scitation.org/journal/jmp

Energy conditional measures and 2D turbulence

Cite as: J. Math. Phys. 61, 013101 (2020); doi: 10.1063/1.5099359
Submitted: 9 April 2019 • Accepted: 5 December 2019 •
Published Online: 7 January 2020

View Online Export Citation CrossMark

Franco Flandoli[1,a] and Dejun Luo[2,b]

AFFILIATIONS

[1] Scuola Normale Superiore of Pisa, Piazza dei Cavalieri 7, 56124 Pisa, Italy

[2] Key Laboratory of RCSDS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China and School of Mathematical Sciences, University of the Chinese Academy of Sciences, Beijing, China

[a)] **Email:** franco.flandoli@sns.it
[b)] **Email:** luodj@amss.ac.cn

ABSTRACT

We show that the invariant measures of point vortices, when conditioning the Hamiltonian to a finite interval, converge weakly to the enstrophy measure by conditioning the renormalized energy to the same interval. We also prove the existence of solutions to 2D Euler equations having the energy conditional measure as an invariant measure. Some heuristic discussions and numerical simulations are presented in Sec. VI.

I. INTRODUCTION

Invariant measures of 2D Euler equations may be candidates for the description of turbulence but which ones are more appropriate for this purpose is still an open problem. Some invariant measures, such as those obtained by tightness of the family of invariant measures for Navier-Stokes equations with vanishing viscosity, are presumably relevant, but they are also implicit, and it is not clear how to compute for relevant quantities such as the energy spectrum. Others have an explicit form, for instance, of Gibbs type, but it is not clear yet how relevant they are for turbulence. Among the latter ones, we focus on two known measures and introduce new related invariant measures of microcanonical type. In Sec. VI, we show their limitations in capturing turbulence features and discuss potential modifications with better properties, however not amenable of a rigorous presentation yet.

The two already known invariant measures that constitute the starting point of our investigation are the so-called enstrophy measure and a uniform measure on point vortices. The enstrophy measure is of Gibbs type, associated with the invariant for 2D Euler dynamics called enstrophy. Since this invariant is a non-negative quadratic form, the enstrophy measure can be defined as a Gaussian measure. More information and a precise definition are given below. The uniform measure on point vortices will be defined now, and the scaling limit relating this uniform measure to the enstrophy measure, which is a known result, will be recalled; then, we introduce the microcanonical modifications and describe our results. Let us start by recalling the point vortex dynamics associated with the 2D Euler equations.

The vorticity formulation of the 2D Euler equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ reads as

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \omega|_{t=0} = \omega_0, \tag{1.1}$$

where u is the velocity field and $\omega = \nabla^\perp \cdot u = \partial_2 u_1 - \partial_1 u_2$ is the vorticity field. We are interested in the singular case in which ω_0 is a finite sum of Dirac delta masses on $\mathbb{T}^2$:

$$\omega_0 = \sum_{i=1}^N \xi_i \delta_{X_0^i}.$$

The points $X_0^i \in \mathbb{T}^2$ are called point vortices with the corresponding intensities $\xi_i \in \mathbb{R}_* := \mathbb{R}\backslash\{0\}$, $1 \le i \le N$. In this case, the Euler equation (1.1) can be interpreted as the following interacting particle system (cf. Ref. 20, Chap. 4):

$$\frac{\mathrm{d}X_t^i}{\mathrm{d}t} = \sum_{j\neq i} \xi_j K\big(X_t^i - X_t^j\big), \quad X^i|_{t=0} = X_0^i,\ 1 \le i \le N. \tag{1.2}$$

Here, K is the Biot-Savart kernel on $\mathbb{T}^2$ given by $K = \nabla^\perp G = (\partial_2 G, -\partial_1 G)$, with G being the Green function on $\mathbb{T}^2$. This is a Hamiltonian system with the Hamiltonian

$$\mathcal{H}_N((\xi_1, x_1), \dots, (\xi_N, x_N)) = -\frac{1}{2N} \sum_{1\le i\neq j\le N} \xi_i \xi_j G(x_i - x_j). \tag{1.3}$$

The coefficient $-1/(2N)$ here is chosen so that $\mathcal{H}_N$ converges weakly to the renormalized energy of the white noise (see Sec. II A for its definition).

Since the kernel K is divergence free, it is easy to check that, for any fixed intensities $(\xi_1, \dots, \xi_N) \in \mathbb{R}_*^N$, the Lebesgue measure $\mathrm{d}x_1 \dots \mathrm{d}x_N$ on $\mathbb{T}^{2N} = (\mathbb{T}^2)^N$ is invariant for the system (1.2). If we also randomize the intensities, a measure like

$$\lambda_N(\mathrm{d}x_1, \dots, \mathrm{d}x_N, \mathrm{d}\xi_1, \dots, \mathrm{d}\xi_N) = \mathrm{d}x_1 \dots \mathrm{d}x_N \mathcal{N}(\mathrm{d}\xi_1) \dots \mathcal{N}(\mathrm{d}\xi_N) \tag{1.4}$$

is invariant too, where $\mathcal{N}$ denotes the standard Gaussian measure on $\mathbb{R}$. This follows from the fact that λ_N is a product measure of $\mathrm{d}x_1 \dots \mathrm{d}x_N$ and $\mathcal{N}(\mathrm{d}\xi_1) \dots \mathcal{N}(\mathrm{d}\xi_N)$. We can rewrite the Hamiltonian $\mathcal{H}_N$ as a functional of the scaled point vortices

$$\omega_N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{x_i}. \tag{1.5}$$

Indeed, if we regard G as a function on $\mathbb{T}^2 \times \mathbb{T}^2$ by setting $G(x, y) = G(x - y)$ and $G(x, x) = 0$ for all $x, y \in \mathbb{T}^2$, then

$$\mathcal{H}_N = -\frac{1}{2} \langle \omega_N \otimes \omega_N, G\rangle. \tag{1.6}$$

Let μ_N be the law of ω_N on $H^{-1-}(\mathbb{T}^2)$ under the measure λ_N, where $H^{-1-}(\mathbb{T}^2)$ is the intersection of all the Sobolev spaces $H^s(\mathbb{T}^2)$ of order less than -1. It is proved in Ref. 10, Proposition 21 that the sequence of measures μ_N converges weakly to the enstrophy measure μ, which has the heuristic expression

$$\mu(\mathrm{d}\omega) = \frac{1}{Z} \exp\bigg(-\frac{1}{2} \int_{\mathbb{T}^2} \omega^2(x)\, \mathrm{d}x\bigg) \mathrm{d}\omega.$$

The quantity $\int_{\mathbb{T}^2} \omega^2(x)\, \mathrm{d}x$ is called the enstrophy of a vorticity field $\omega \in L^2(\mathbb{T}^2)$. Thus, μ has the identity operator on $L^2(\mathbb{T}^2)$ as the covariance operator, which coincides with that of the white noises. This implies that μ is the law of white noise on $\mathbb{T}^2$. Recall that the measure μ is supported by $H^{-1-}(\mathbb{T}^2)$.

Let ω be a white noise on $\mathbb{T}^2$, defined on some probability space $(\Theta, \mathcal{F}, \mathbb{P})$ and taking values in $H^{-1-}(\mathbb{T}^2)$. In this paper, we use Θ, instead of Ω, to denote the probability space since ω is commonly used in fluid mechanics to denote the vorticity field (here, it is a white noise). Let $:\mathcal{H}: = :\ \mathcal{H}(\omega):$ be the renormalized energy of ω (see Sec. II B for its definition); we can also regard $:\mathcal{H}:$ as a random variable defined on the probability space $\big(H^{-1-}(\mathbb{T}^2), \mathcal{B}\big(H^{-1-}(\mathbb{T}^2)\big), \mu\big)$, where $\mathcal{B}\big(H^{-1-}(\mathbb{T}^2)\big)$ is the collection of Borel measurable sets. Fix $a, b \in \mathbb{R}$ with $a < b$; from Proposition 2.4, we always have

$$\mu(\{:\mathcal{H}: \in [a, b]\}) > 0.$$

Therefore, the conditional measure

$$\mu^{a,b}(A) = \frac{\mu(A \cap \{:\mathcal{H}: \in [a, b]\})}{\mu(\{:\mathcal{H}: \in [a, b]\})}, \quad A \in \mathcal{B}\big(H^{-1-}(\mathbb{T}^2)\big) \tag{1.7}$$

is well defined. On the other hand, we shall prove in Proposition 3.1 that $\lim_{N\to\infty} \mu_N(\{\mathcal{H}_N \in [a, b]\}) = \mu(\{:\mathcal{H}: \in [a, b]\}) > 0$; hence, we can define in the same way the energy conditional measures for the point vortices,

$$\mu_N^{a,b}(A) = \frac{\mu_N(A \cap \{\mathcal{H}_N \in [a,b]\})}{\mu_N(\{\mathcal{H}_N \in [a,b]\})}, \quad A \in \mathcal{B}\big(H^{-1-}(\mathbb{T}^2)\big). \tag{1.8}$$

The main result in the present paper is

Theorem 1.1. *The family* $\{\mu_N^{a,b}\}_{N\geq 1}$ *of energy conditional measures converges weakly to* $\mu^{a,b}$.

This result shows the convergence of a class of microcanonical measures. We mention that, when the intensities $\{\xi_i\}_{i\geq 1}$ are i.i.d. centered Bernoulli random variables, Benfatto *et al.*[3] proved that the canonical Gibbs measures of the point vortices, with appropriately regularized Green functions, converge to the Gaussian measure $\mu_{\beta,\gamma}(\mathrm{d}\omega) = \mathrm{e}^{-\beta H - \gamma E}\mathrm{d}\omega$ ($\beta, \gamma > 0$ and H and E are the energy and enstrophy functionals, respectively), which are invariant for the 2D Euler flow. In the recent work,[15] an analogous result was proved without smoothing the Green function; see Ref. 13 for a related result concerning the generalized inviscid surface quasigeostrophic equations.

It is worth mentioning that the general principle of equivalence of ensembles does not necessarily hold for the 2D Euler flows. As discussed in Ref. 16, p. 110, Sec. 25 (see also Ref. 5, Sec. 3.2), the canonical ensemble is the natural ensemble for a physical system in contact with a thermal bath, with which it can exchange energy; on the other hand, when the physical system can be considered as being isolated (or the characteristic time for energy exchanges with the environment is much longer than the characteristic time for relaxation toward equilibrium), then the microcanonical ensemble is the relevant one. It seems difficult to couple the 2D Euler flows with a thermal bath, thus the appropriate statistical ensemble is the microcanonical measure. Moreover, the microcanonical and canonical ensembles are often nonequivalent for 2D Euler flows since the interactions in the Euler dynamics are of long range. In the mean field regime of the Onsager theory, they are equivalent, see Refs. 6 and 9, but in the regime studied here, the infinite particle limit is not the canonical Gibbs measure associated with the renormalized energy. See Refs. 5 and 24 for more discussions on nonequivalence of ensembles.

The results presented here are meant to be fragments of a more general investigation on invariant measures of 2D Euler equations, in the attempt to capture some features of the inverse cascade turbulence. The Onsager theory, extremely relevant for the explanation of large scale coherent vortex structures, does not provide a description of the inverse stationary turbulence; but unfortunately, the regime studied here is not the correct description either. In a sense, the Onsager theory and the regime considered here are two extremes, both with relevant features, but turbulence is somewhat in between. In Sec. VI, we discuss this issue.

The above theorem will be proved in Sec. III. To this end, we first make some necessary preparations in Sec. II, including the definitions of $\langle \omega \otimes \omega, G\rangle$ and of the renormalized energy $: \mathcal{H} : = : \mathcal{H}(\omega) :$ for a white noise ω; the relation between them will be clarified in Sec. II C. We study in Sec. IV the limiting behavior of the correlation functions of the energy conditional measures on the "flat space" $(\mathbb{R} \times \mathbb{T}^2)^N$, following some arguments in Ref. 17, Sec. 5.4 (see also Refs. 6, 7, and 21 for related results). Based on the results in Ref. 10, we prove in Sec. V the existence of solutions to the 2D Euler equations having the energy conditional measure $\mu^{a,b}$ as an invariant measure. Finally, we present in Sec. VI some heuristic discussions together with numerical simulations of the spectrum functions for point vortices, illustrating the relevance of our results to 2D turbulence.

II. PRELIMINARY RESULTS ON THE RENORMALIZED ENERGY

In this section, we make some preparations regarding the renormalized energy of a white noise ω on $\mathbb{T}^2$. We shall regard $\mathbb{T}^2$ as $[-1/2, 1/2]^2$ endowed with the periodic boundary condition. First, we follow the idea in Ref. 10, Sec. 2.4 to define the quantity $\langle \omega \otimes \omega, G\rangle$, where G is the Green function on $\mathbb{T}^2$. Second, we recall the definition of the renormalized energy $: \mathcal{H} :$ using the Galerkin approximation; based on the series expansion of $: \mathcal{H} :$, we are able to show that its law has full support on the real line. Finally, we study the relation between $\langle \omega \otimes \omega, G\rangle$ and $: \mathcal{H} :$; see Theorem 2.8.

A. Definition of $\langle \omega \otimes \omega, G\rangle$ for a white noise ω

In this part, we follow the approach in Ref. 10, Sec. 2.4 (see also Ref. 12, Sec. 2.2) to define the quantity $\langle \omega \otimes \omega, G\rangle$ when ω is a white noise on $\mathbb{T}^2$, defined on some probability space $(\Theta, \mathcal{F}, \mathbb{P})$. We recall that, by definition, ω is a centered Gaussian random variable taking values in the space of distributions $C^\infty(\mathbb{T}^2)'$ such that for any $\phi, \psi \in C^\infty(\mathbb{T}^2)$, one has

$$\mathbb{E}[\langle \omega, \phi\rangle \langle \omega, \psi\rangle] = \langle \phi, \psi\rangle.$$

Here, $\mathbb{E}$ is the mathematical expectation with respect to the probability measure $\mathbb{P}$, and $\langle .,. \rangle$ denotes the duality between distributions and smooth functions or the inner product in $L^2(\mathbb{T}^2)$ when both objects are functions. Using the Fourier basis on $\mathbb{T}^2$, it is not difficult to show that the law μ of ω is supported by $H^{-1-}(\mathbb{T}^2)$. The results below are proved in Ref. 10, Corollary 6, and we only provide abbreviated arguments here.

Lemma 2.1. Let $\omega : \Theta \to H^{-1-}(\mathbb{T}^2)$ be a white noise.

(i) *If $f \in H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$, then for every $p \geq 1$, there is constant $C_p > 0$ such that*

$$\mathbb{E}[|\langle \omega \otimes \omega, f\rangle|^p] \leq C_p \|f\|_\infty^p.$$

(ii) *We have $\mathbb{E}\langle \omega \otimes \omega, f\rangle = \int_{\mathbb{T}^2} f(x,x)\,\mathrm{d}x$.*

(iii) *If f is symmetric, then*

$$\mathbb{E}\Big[\big|\langle \omega \otimes \omega, f\rangle - \mathbb{E}\langle \omega \otimes \omega, f\rangle\big|^2\Big] = 2\int_{\mathbb{T}^2\times\mathbb{T}^2} f(x,y)^2\,\mathrm{d}x\mathrm{d}y.$$

Proof. We regularize the white noise ω by a mollifier $\theta_\varepsilon(x) = \varepsilon^{-2}\theta(\varepsilon^{-1}x)$, where $\theta \in C^\infty(\mathbb{T}^2, \mathbb{R}_+)$ is symmetric and has a small support around $x = 0$. Let $\omega_\varepsilon(x) = \langle \omega, \theta_\varepsilon(x - .)\rangle$, $x \in \mathbb{T}^2$; then, $\{\omega_\varepsilon(x)\}_{x\in\mathbb{T}^2}$ is a centered Gaussian family, satisfying

$$\mathbb{E}[\omega_\varepsilon(x)\omega_\varepsilon(y)] = \int_{\mathbb{T}^2} \theta_\varepsilon(x-z)\theta_\varepsilon(y-z)\,\mathrm{d}z = (\theta_\varepsilon * \theta_\varepsilon)(x-y) =: \delta^\varepsilon_{x-y}.$$

Here, for $x \in \mathbb{T}^2$, we denote $(\theta_\varepsilon * \theta_\varepsilon)(x)$ by δ^ε_x since it is an approximation of the Dirac delta distribution.

Using the Gaussian properties of the family $\{\omega_\varepsilon(x)\}_{x\in\mathbb{T}^2}$, one can prove the following results (the details can be found in Ref. 10, Lemma 5):

(i′) for every $p \geq 1$, there is constant $C_p > 0$ such that, for all $\varepsilon > 0$,

$$\mathbb{E}[|\langle \omega_\varepsilon \otimes \omega_\varepsilon, f\rangle|^p] \leq C_p\|f\|_\infty^p;$$

(ii′) $\mathbb{E}[\langle \omega_\varepsilon \otimes \omega_\varepsilon, f\rangle] = \int_{\mathbb{T}^2\times\mathbb{T}^2} \delta^\varepsilon_{x-y} f(x,y)\,\mathrm{d}x\mathrm{d}y$;

(iii′) if f is symmetric, then

$$\mathbb{E}\Big[\big|\langle \omega_\varepsilon \otimes \omega_\varepsilon, f\rangle - \mathbb{E}\langle \omega_\varepsilon \otimes \omega_\varepsilon, f\rangle\big|^2\Big] = 2\int_{(\mathbb{T}^2)^4} \delta^\varepsilon_{x_1-x_2}\delta^\varepsilon_{y_1-y_2} f(x_1,y_1)f(x_2,y_2)\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2.$$

Now, note that the function f belongs to $H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$; thus, it is continuous on $\mathbb{T}^2 \times \mathbb{T}^2$ by the Sobolev embedding theorem. Moreover, (i′) implies that the family $\{|\langle \omega_\varepsilon \otimes \omega_\varepsilon, f\rangle|\}_{\varepsilon>0}$ is uniformly integrable; thus, we can let ε tend to 0 in the above results to get the desired assertions. □

Based on these facts, we can give a definition of $\langle \omega \otimes \omega, G\rangle$ when ω is a white noise on $\mathbb{T}^2$.

Proposition 2.2. Let $\omega : \Theta \to H^{-1-}(\mathbb{T}^2)$ be a white noise. Assume that $G_n \in H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$ are symmetric and approximate G in the following sense:

$$\lim_{n\to\infty} \int_{\mathbb{T}^2\times\mathbb{T}^2} (G_n - G)^2(x,y)\,\mathrm{d}x\mathrm{d}y = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{T}^2} G_n(x,x)\,\mathrm{d}x = 0.$$

Then, the sequence of random variables $\langle \omega \otimes \omega, G_n\rangle$ is a Cauchy sequence in mean square. We denote its limit by $\langle \omega \otimes \omega, G\rangle$.

Moreover, the limit is the same if G_n is replaced by $\tilde{G}_n$ with the same properties and such that $\lim_{n\to\infty}\int_{\mathbb{T}^2\times\mathbb{T}^2}(G_n - \tilde{G}_n)^2(x,y)\,\mathrm{d}x\mathrm{d}y = 0$.

Proof. The proofs are the same as those of Ref. 10, Theorem 8; we recall them here for completeness. Since $\lim_{n\to\infty}\int_{\mathbb{T}^2} G_n(x,x)\,\mathrm{d}x = 0$, it is equivalent to show that $\langle \omega \otimes \omega, G_n\rangle - \int_{\mathbb{T}^2} G_n(x,x)\,\mathrm{d}x$ is a Cauchy sequence in mean square. We have

$$\begin{aligned}
&\mathbb{E}\left[\left|\langle \omega \otimes \omega, G_n\rangle - \int_{\mathbb{T}^2} G_n(x,x)\,\mathrm{d}x - \langle \omega \otimes \omega, G_m\rangle + \int_{\mathbb{T}^2} G_m(x,x)\,\mathrm{d}x\right|^2\right] \\
&= \mathbb{E}\left[\left|\langle \omega \otimes \omega, G_n - G_m\rangle - \int_{\mathbb{T}^2}(G_n - G_m)(x,x)\,\mathrm{d}x\right|^2\right] \\
&= 2\int_{\mathbb{T}^2\times\mathbb{T}^2}(G_n - G_m)^2(x,y)\,\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where the last equality follows from (ii) and (iii) of Lemma 2.1. This implies the Cauchy property, and thus, $\langle \omega \otimes \omega, G\rangle$ is well defined. The invariance property is proved similarly. □

Here is an example of the approximating functions G_n. Let $\chi : \mathbb{T}^2 = [-1/2, 1/2]^2 \to [0,1]$ be a smooth and symmetric function with support in a small ball $B(0,r)$ and equal to 1 in $B(0,r/2)$. For any $n \geq 1$, set $\chi_n(x) = \chi(nx)$, $x \in \mathbb{T}^2$. Define

$$G_n(x) = \begin{cases} G(x)(1-\chi_n(x)), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We regard G_n as a function on $\mathbb{T}^2 \times \mathbb{T}^2$ by setting $G_n(x,y) = G_n(x-y)$. Since $G_n(x,x) \equiv 0$, we have the following estimate [cf. Lemma 2.1(iii)]:

$$\mathbb{E}\Big[\big(\langle \omega \otimes \omega, G_n\rangle - \langle \omega \otimes \omega, G\rangle\big)^2\Big] \leq 2\int_{\mathbb{T}^2\times\mathbb{T}^2} (G_n - G)^2(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{2.1}$$

B. Definition of the renormalized energy : $\mathcal{H}$:

In this subsection, we recall the definition of the renormalized energy : $\mathcal{H}$: via the Galerkin approximation. To this end, let $\{e_k\}_{k\in\mathbb{Z}_0^2}$ be defined as

$$e_k(x) = \sqrt{2}\begin{cases} \cos(2\pi k\cdot x), & k \in \mathbb{Z}_+^2, \\ \sin(2\pi k\cdot x), & k \in \mathbb{Z}_-^2, \end{cases} \tag{2.2}$$

where $\mathbb{Z}_0^2 = \mathbb{Z}^2\backslash\{0\}$ and $\mathbb{Z}_+^2 = \big\{k \in \mathbb{Z}_0^2 : (k_1 > 0) \text{ or } (k_1 = 0,\ k_2 > 0)\big\}$ and $\mathbb{Z}_-^2 = -\mathbb{Z}_+^2$. This family of functions is an orthonormal basis of square integrable functions on $\mathbb{T}^2$ with vanishing mean. Let ω be a white noise on $\mathbb{T}^2$, then the random series

$$\omega = \sum_{k\in\mathbb{Z}_0^2} \langle \omega, e_k\rangle e_k,$$

converges in mean square in $H^{-1-\delta}(\mathbb{T}^2)$ for any $\delta > 0$. For $N \geq 1$, define $\Lambda_N = \{k \in \mathbb{Z}_0^2 : |k| \leq N\}$ and

$$\overline{\omega}_N = \sum_{k\in\Lambda_N} \langle \omega, e_k\rangle e_k, \quad u_N = K * \overline{\omega}_N,$$

where K is the Biot-Savart kernel:

$$K(x) = \nabla^\perp G(x) = -\frac{\mathrm{i}}{2\pi}\sum_{k\in\mathbb{Z}_0^2} \frac{k^\perp}{|k|^2} \mathrm{e}^{2\pi \mathrm{i} k\cdot x},$$

with $\nabla^\perp = (\partial_2, -\partial_1)$ and $k^\perp = (k_2, -k_1)$. Set

$$\mathcal{E}_N = \frac{1}{2}\int_{\mathbb{T}^2} |u_N(x)|^2\,\mathrm{d}x, \quad \tilde{\mathcal{E}}_N = \mathcal{E}_N - \mathbb{E}\mathcal{E}_N.$$

The following result is well known [(see, e.g., Ref. 1, p. 593) or (Ref. 2, Proposition 2.5)].

Proposition 2.3. The sequence $\{\tilde{\mathcal{E}}_N\}_{N\geq 1}$ *is Cauchy in* $L^2(\Theta, \mathbb{P})$. *Denote its limit by* : $\mathcal{H}$: *and call it the renormalized energy; one has*

$$:\mathcal{H}: = \frac{1}{8\pi^2}\sum_{k\in\mathbb{Z}_0^2} \frac{1}{|k|^2}\big(\langle \omega, e_k\rangle^2 - 1\big)$$

and

$$\mathbb{E}\big(|:\mathcal{H}:|^2\big) = \frac{1}{32\pi^4}\sum_{k\in\mathbb{Z}_0^2} \frac{1}{|k|^4}.$$

Proof. We give the detailed computations since we need the exact coefficients. Note that

$$K(x) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}_0^2} \frac{l^\perp}{|l|^2} \sin(2\pi l \cdot x).$$

Therefore, if $k \in \mathbb{Z}_+^2$, then

$$\begin{aligned}(K * e_k)(x) &= \frac{1}{2\pi} \sum_{l \in \mathbb{Z}_0^2} \frac{l^\perp}{|l|^2} \int_{\mathbb{T}^2} \sin(2\pi l \cdot (x-y)) \sqrt{2} \cos(2\pi k \cdot y)\, dy \\ &= \frac{\sqrt{2}}{2\pi} \frac{k^\perp}{|k|^2} \sin(2\pi k \cdot x) = -\frac{1}{2\pi} \frac{k^\perp}{|k|^2} e_{-k}(x),\end{aligned}$$

where the second equality is due to the fact that the integral vanishes unless $l = \pm k$. Similarly, if $k \in \mathbb{Z}_-^2$, then

$$(K * e_k)(x) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}_0^2} \frac{l^\perp}{|l|^2} \int_{\mathbb{T}^2} \sin(2\pi l \cdot (x-y)) \sqrt{2} \sin(2\pi k \cdot y)\, dy = -\frac{1}{2\pi} \frac{k^\perp}{|k|^2} e_{-k}(x).$$

Thus,

$$u_N = K * \overline{\omega}_N = -\frac{1}{2\pi} \sum_{k \in \Lambda_N} \frac{k^\perp}{|k|^2} \langle \omega, e_k \rangle e_{-k}.$$

As a result,

$$\mathcal{E}_N = \frac{1}{2} \int_{\mathbb{T}^2} |u_N(x)|^2\, dx = \frac{1}{8\pi^2} \sum_{k,l \in \Lambda_N} \frac{k \cdot l}{|k|^2 |l|^2} \langle \omega, e_k \rangle \langle \omega, e_l \rangle \langle e_{-k}, e_{-l} \rangle = \frac{1}{8\pi^2} \sum_{k \in \Lambda_N} \frac{1}{|k|^2} \langle \omega, e_k \rangle^2.$$

Consequently,

$$\tilde{\mathcal{E}}_N = \frac{1}{8\pi^2} \sum_{k \in \Lambda_N} \frac{1}{|k|^2} (\langle \omega, e_k \rangle^2 - 1). \tag{2.3}$$

Next,

$$\begin{aligned}\mathbb{E}\big[(\tilde{\mathcal{E}}_N)^2\big] &= \frac{1}{64\pi^4} \sum_{k,l \in \Lambda_N} \frac{1}{|k|^2 |l|^2} \mathbb{E}\big[(\langle \omega, e_k \rangle^2 - 1)(\langle \omega, e_l \rangle^2 - 1)\big] \\ &= \frac{1}{64\pi^4} \sum_{k \in \Lambda_N} \frac{1}{|k|^4} \mathbb{E}\big[\langle \omega, e_k \rangle^4 - 1\big] = \frac{1}{32\pi^4} \sum_{k \in \Lambda_N} \frac{1}{|k|^4}\end{aligned}$$

since $\mathbb{E}\langle \omega, e_k \rangle^4 = 3$. The same calculations imply that $\{\tilde{\mathcal{E}}_N\}_{N \ge 1}$ is a Cauchy sequence in $L^2(\Theta, \mathbb{P})$ and the two desired equalities. □

As an application of the expression for the renormalized energy, we can prove

Proposition 2.4. The law of $:\mathcal{H}:$ *is supported on the whole real line.*

Proof. For any $a, b \in \mathbb{R}$, $a < b$, it suffices to show that $Z^{a,b} := \mathbb{P}(\{:\mathcal{H}:\in [a,b]\}) > 0$. Without loss of generality, assume $b - a \le 1$. We define $\delta_0 := (b-a)/5$ and the remainder

$$\mathcal{R}_N := \frac{1}{8\pi^2} \sum_{|k| > N} \frac{1}{|k|^2} (\langle \omega, e_k \rangle^2 - 1).$$

Then, $:\mathcal{H}: = \tilde{\mathcal{E}}_N + \mathcal{R}_N$ and, for all $N \ge 1$, the two random variables $\tilde{\mathcal{E}}_N$ and $\mathcal{R}_N$ are independent of one another. Moreover,

$$\{:\mathcal{H}:\in [a,b]\} \supset \big\{\tilde{\mathcal{E}}_N \in [a+\delta_0, b-\delta_0]\big\} \cap \{|\mathcal{R}_N| \le \delta_0\},$$

and therefore,

$$\mathbb{P}\big(\{: \mathcal{H} :\in [a,b]\}\big) \geq \mathbb{P}\big(\{\tilde{\mathcal{E}}_N \in [a+\delta_0, b-\delta_0]\}\big)\,\mathbb{P}\big(\{|\mathcal{R}_N| \leq \delta_0\}\big).$$

Since $\mathcal{R}_N$ tends to 0 in the norm $L^2(\Theta,\mathbb{P})$ as $N \to \infty$, we can find $N_0 \in \mathbb{Z}_+$ such that $\mathbb{P}\big(\{|\mathcal{R}_N| \leq \delta_0\}\big) \geq 1/2$ for all $N \geq N_0$. Thus, it is enough to show that

$$\mathbb{P}\big(\{\tilde{\mathcal{E}}_{N_0} \in [a+\delta_0, b-\delta_0]\}\big) > 0. \tag{2.4}$$

We define

$$L = \frac{1}{8\pi^2}\sum_{k\in\Lambda_{N_0}} \frac{1}{|k|^2}$$

and consider three different cases according to the location of the origin 0 with respect to the middle subinterval $[a+2\delta_0, a+3\delta_0]$.

(i) $a+2\delta_0 > 0$. Since $[a+3\delta_0, a+4\delta_0] \subset [a+\delta_0, b-\delta_0]$, it is sufficient to prove that

$$\mathbb{P}\big(\{\tilde{\mathcal{E}}_{N_0} \in [a+3\delta_0, a+4\delta_0]\}\big) > 0. \tag{2.5}$$

Set $c_1 := (a+3\delta_0)/L$ and $c_2 := (a+4\delta_0)/L$ which are positive constants. Recall that $\{\langle\omega, e_k\rangle\}_{k\in\mathbb{Z}_0^2}$ is a family of i.i.d. standard Gaussian random variables; we have

$$p_1 := \mathbb{P}\big(\{\langle\omega, e_k\rangle^2 \in [1+c_1, 1+c_2]\}\big) > 0.$$

The desired property (2.5) follows from the next inclusion between events:

$$\{\tilde{\mathcal{E}}_{N_0} \in [a+3\delta_0, a+4\delta_0]\} \supset \bigcap_{k\in\Lambda_{N_0}} \{\langle\omega, e_k\rangle^2 \in [1+c_1, 1+c_2]\}.$$

(ii) $a+2\delta_0 \leq 0 \leq a+3\delta_0$. In this case, we have $[-\delta_0, \delta_0] \subset [a+\delta_0, b-\delta_0]$. Similar to case (i), we deduce the desired result from the two facts as follows:

$$\{\tilde{\mathcal{E}}_{N_0} \in [-\delta_0, \delta_0]\} \supset \bigcap_{k\in\Lambda_{N_0}} \{\langle\omega, e_k\rangle^2 \in [1-\delta_0/L, 1+\delta_0/L]\}$$

and

$$p_2 := \mathbb{P}\big(\{\langle\omega, e_k\rangle^2 \in [1-\delta_0/L, 1+\delta_0/L]\}\big) > 0.$$

(iii) $a+3\delta_0 < 0$. In this case, it suffices to show that

$$\mathbb{P}\big(\{\tilde{\mathcal{E}}_{N_0} \in [a+\delta_0, a+2\delta_0]\}\big) > 0. \tag{2.6}$$

We assume N_0 is big enough such that the constant $L > -a$; then,

$$-1 < c_3 := (a+\delta_0)/L < c_4 := (a+2\delta_0)/L < 0.$$

We can get the inequality (2.6) from the facts that

$$\{\tilde{\mathcal{E}}_{N_0} \in [a+\delta_0, a+2\delta_0]\} \supset \bigcap_{k\in\Lambda_{N_0}} \{\langle\omega, e_k\rangle^2 \in [1+c_3, 1+c_4]\}$$

and

$$p_3 := \mathbb{P}\big(\{\langle\omega, e_k\rangle^2 \in [1+c_3, 1+c_4]\}\big) > 0.$$

Summarizing the above three cases, we complete the proof of (2.4). □

C. The relation between $\langle \omega \otimes \omega, G\rangle$ and $:\mathcal{H}:$

In this part, for a white noise ω, we follow the idea in Ref. 11, Sec. 4.2, to show the relation between $\langle \omega \otimes \omega, G\rangle$ and $:\mathcal{H}:$. Although we mainly work with real valued functions, we shall make use of the canonical complex orthonormal basis of $L^2(\mathbb{T}^2, \mathbb{C})$: $\tilde{e}_k(x) = e^{2\pi i k\cdot x}$, $k \in \mathbb{Z}^2, x \in \mathbb{T}^2$. Note that $\{\tilde{e}_k \otimes \tilde{e}_l\}_{k,l\in\mathbb{Z}^2}$ is an orthonormal basis of $L^2(\mathbb{T}^2 \times \mathbb{T}^2, \mathbb{C})$.

Lemma 2.5. Let ω be a white noise on $\mathbb{T}^2$. Assume $f \in C^\infty(\mathbb{T}^2 \times \mathbb{T}^2, \mathbb{R})$ is symmetric and $\int_{\mathbb{T}^2} f(x,x)\,dx = 0$. Then,

$$\langle \omega \otimes \omega, f\rangle = \sum_{k,l\in\mathbb{Z}^2} f_{k,l}\langle \omega, \tilde{e}_k\rangle\langle \omega, \tilde{e}_l\rangle \quad \text{holds in } L^2(\Theta, \mathbb{P}),$$

where

$$f_{k,l} = \langle f, \tilde{e}_k \otimes \tilde{e}_l\rangle = \int_{\mathbb{T}^2\times\mathbb{T}^2} f(x,y)\tilde{e}_k(x)\tilde{e}_l(y)\,dxdy.$$

Proof. Denote by

$$\hat{\Lambda}_N = \{k \in \mathbb{Z}^2 : |k| \le N\} = \Lambda_N \cup \{0\}. \tag{2.7}$$

Since $f \in C^\infty(\mathbb{T}^2 \times \mathbb{T}^2)$, the partial sum of the Fourier series:

$$f_N(x,y) := \sum_{k,l\in\hat{\Lambda}_N} f_{k,l}\,\tilde{e}_k(x)\tilde{e}_l(y),$$

converges to f, uniformly on $\mathbb{T}^2 \times \mathbb{T}^2$ and in $L^2(\mathbb{T}^2 \times \mathbb{T}^2)$. In particular,

$$\lim_{N\to\infty} \int_{\mathbb{T}^2} f_N(x,x)\,dx = \int_{\mathbb{T}^2} f(x,x)\,dx = 0. \tag{2.8}$$

It is obvious that $f_N(x,y)$ is smooth and symmetric. By (ii) and (iii) in Lemma 2.1,

$$\mathbb{E}\left[\left(\langle \omega \otimes \omega, f - f_N\rangle + \int_{\mathbb{T}^2} f_N(x,x)\,dx\right)^2\right] = 2\int_{\mathbb{T}^2\times\mathbb{T}^2} (f - f_N)^2(x,y)\,dxdy.$$

As a result,

$$\mathbb{E}\left[\langle \omega \otimes \omega, f - f_N\rangle^2\right] \le 4\int_{\mathbb{T}^2\times\mathbb{T}^2} (f - f_N)^2(x,y)\,dxdy + 2\left(\int_{\mathbb{T}^2} f_N(x,x)\,dx\right)^2. \tag{2.9}$$

Next, note that

$$\langle \omega \otimes \omega, f_N\rangle = \sum_{k,l\in\hat{\Lambda}_N} f_{k,l}\langle \omega, \tilde{e}_k\rangle\langle \omega, \tilde{e}_l\rangle.$$

Therefore, by (2.9),

$$\begin{aligned}&\mathbb{E}\left[\left(\langle \omega \otimes \omega, f\rangle - \sum_{k,l\in\hat{\Lambda}_N} f_{k,l}\langle \omega, \tilde{e}_k\rangle\langle \omega, \tilde{e}_l\rangle\right)^2\right]\\ &\le 4\int_{\mathbb{T}^2\times\mathbb{T}^2} (f - f_N)^2(x,y)\,dxdy + 2\left(\int_{\mathbb{T}^2} f_N(x,x)\,dx\right)^2.\end{aligned}$$

Thanks to (2.8), the desired result follows by letting $N \to \infty$. □

We need the following simple equality.

Lemma 2.6. Let $\{a_{k,l}\}_{k,l\in\hat{\Lambda}_N} \subset \mathbb{C}$ be satisfying $a_{k,l} = a_{l,k}$, $\overline{a_{k,l}} = a_{-k,-l}$. Then,

$$\mathbb{E}\left[\left|\sum_{k,l\in\hat{\Lambda}_N} a_{k,l}\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle - \sum_{k\in\hat{\Lambda}_N} a_{k,-k}\right|^2\right] = 2\sum_{k,l\in\hat{\Lambda}_N}|a_{k,l}|^2.$$

Proof. Since $\overline{\langle\omega,\tilde{e}_k\rangle} = \langle\omega,\tilde{e}_{-k}\rangle$, it is clear that $\sum_{k,l\in\hat{\Lambda}_N} a_{k,l}\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle$ is real and

$$\sum_{k\in\hat{\Lambda}_N} a_{k,-k} = \mathbb{E}\left(\sum_{k,l\in\hat{\Lambda}_N} a_{k,l}\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle\right). \tag{2.10}$$

We have

$$\left(\sum_{k,l\in\hat{\Lambda}_N} a_{k,l}\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle\right)^2 = \sum_{k,l,m,n\in\hat{\Lambda}_N} a_{k,l}a_{m,n}\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle\langle\omega,\tilde{e}_m\rangle\langle\omega,\tilde{e}_n\rangle,$$

and by the Isserlis–Wick theorem (see Ref. 23, p. 9, Proposition I.2),

$$\begin{aligned}\mathbb{E}\big(\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle\langle\omega,\tilde{e}_m\rangle\langle\omega,\tilde{e}_n\rangle\big) &= \mathbb{E}\big(\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle\big)\mathbb{E}\big(\langle\omega,\tilde{e}_m\rangle\langle\omega,\tilde{e}_n\rangle\big)\\ &\quad + \mathbb{E}\big(\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_m\rangle\big)\mathbb{E}\big(\langle\omega,\tilde{e}_l\rangle\langle\omega,\tilde{e}_n\rangle\big)\\ &\quad + \mathbb{E}\big(\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_n\rangle\big)\mathbb{E}\big(\langle\omega,\tilde{e}_l\rangle\langle\omega,\tilde{e}_m\rangle\big)\\ &= \delta_{k,-l}\delta_{m,-n} + \delta_{k,-m}\delta_{l,-n} + \delta_{k,-n}\delta_{l,-m}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(\sum_{k,l\in\hat{\Lambda}_N} a_{k,l}\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle\right)^2 &= \sum_{k,m\in\hat{\Lambda}_N} a_{k,-k}a_{m,-m} + \sum_{k,l\in\hat{\Lambda}_N} a_{k,l}a_{-k,-l} + \sum_{k,l\in\hat{\Lambda}_N} a_{k,l}a_{-l,-k}\\ &= \left(\sum_{k\in\hat{\Lambda}_N} a_{k,-k}\right)^2 + 2\sum_{k,l\in\hat{\Lambda}_N}|a_{k,l}|^2,\end{aligned}$$

where we have used the facts $a_{-l,-k} = a_{-k,-l} = \overline{a_{k,l}}$. Combining this equality with (2.10) finishes the proof. □

Recall the definition of $\hat{\Lambda}_N$ in (2.7). To simplify the notations, we introduce

$$\hat{\omega}_N = \hat{\Pi}_N\omega = \sum_{k\in\hat{\Lambda}_N}\langle\omega,\tilde{e}_k\rangle\tilde{e}_k.$$

Then,

$$\langle\hat{\omega}_N\otimes\hat{\omega}_N, G\rangle = \sum_{k,l\in\hat{\Lambda}_N}\langle G,\tilde{e}_k\otimes\tilde{e}_l\rangle\langle\omega,\tilde{e}_k\rangle\langle\omega,\tilde{e}_l\rangle \tag{2.11}$$

is the partial sum of the series.

Lemma 2.7. We have

$$\langle\hat{\omega}_N\otimes\hat{\omega}_N, G\rangle = -\frac{1}{4\pi^2}\sum_{k\in\Lambda_N}\frac{1}{|k|^2}\langle\omega,e_k\rangle^2.$$

Proof. Recall that

$$G(x) = -\frac{1}{4\pi^2}\sum_{k\in\mathbb{Z}_0^2}\frac{1}{|k|^2}\mathrm{e}^{2\pi\mathrm{i}k\cdot x} = -\frac{1}{4\pi^2}\sum_{k\in\mathbb{Z}_0^2}\frac{1}{|k|^2}\tilde{e}_k(x).$$

Therefore, for $l \neq 0$,

$$(G * \tilde{e}_l)(x) = -\frac{1}{4\pi^2}\frac{1}{|l|^2}\tilde{e}_l(x),$$

which implies that

$$\langle G, \tilde{e}_k \otimes \tilde{e}_l\rangle = \int_{\mathbb{T}^2} \tilde{e}_k(x)(G * \tilde{e}_l)(x)\,\mathrm{d}x = -\frac{1}{4\pi^2}\frac{1}{|l|^2}\delta_{k,-l}.$$

Hence,

$$\langle \hat{\omega}_N \otimes \hat{\omega}_N, G\rangle = -\frac{1}{4\pi^2}\sum_{k\in\Lambda_N}\frac{1}{|k|^2}\langle \omega, \tilde{e}_k\rangle\langle \omega, \tilde{e}_{-k}\rangle = -\frac{1}{4\pi^2}\sum_{k\in\Lambda_N}\frac{1}{|k|^2}|\langle \omega, \tilde{e}_k\rangle|^2.$$

The desired identity follows from $|\langle \omega, \tilde{e}_k\rangle|^2 = \frac{1}{2}(\langle \omega, e_k\rangle^2 + \langle \omega, e_{-k}\rangle^2)$ for all $k \in \Lambda_N$. □

Now, we can prove the main result of this section.

Theorem 2.8. *Let ω be a white noise on $\mathbb{T}^2$. Almost surely, it holds that*

$$\langle \omega \otimes \omega, G\rangle = -2 :\mathcal{H}: .$$

Proof. Let G_n be the smooth functions defined at the end of Sec. II A. We have

$$\begin{aligned}
&\mathbb{E}\Big[\big(\langle \omega \otimes \omega, G\rangle + 2 :\mathcal{H}:\big)^2\Big] \\
&\le 4\,\mathbb{E}\big[\langle \omega \otimes \omega, G - G_n\rangle^2\big] + 4\,\mathbb{E}\Big[\big(\langle \omega \otimes \omega, G_n\rangle - \langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n\rangle\big)^2\Big] \\
&\quad + 4\,\mathbb{E}\Big[\big(\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n\rangle + 2\,\tilde{\mathcal{E}}_N\big)^2\Big] + 16\,\mathbb{E}\Big[\big(\tilde{\mathcal{E}}_N - :\mathcal{H}:\big)^2\Big].
\end{aligned} \tag{2.12}$$

We deal with these terms one-by-one. By (2.1),

$$\mathbb{E}\big[\langle \omega \otimes \omega, G - G_n\rangle^2\big] \le 2\int_{\mathbb{T}^2\times\mathbb{T}^2}(G - G_n)^2(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{2.13}$$

Next, for any fixed $n \ge 1$, Lemma 2.5 implies

$$\mathbb{E}\Big[\big(\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n\rangle - \langle \omega \otimes \omega, G_n\rangle\big)^2\Big] \to 0 \quad \text{as } N \to \infty. \tag{2.14}$$

Moreover, by Proposition 2.3, the last term in (2.12) vanishes as $N \to \infty$.

It remains to treat the third term on the rhs of (2.12). By (2.3) and Lemma 2.7,

$$-2\,\tilde{\mathcal{E}}_N = \langle \hat{\omega}_N \otimes \hat{\omega}_N, G\rangle - \mathbb{E}\langle \hat{\omega}_N \otimes \hat{\omega}_N, G\rangle.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Big[\big(\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n\rangle + 2\,\tilde{\mathcal{E}}_N\big)^2\Big] \\
&= \mathbb{E}\Big[\big(\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n - G\rangle - \mathbb{E}\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n - G\rangle + \mathbb{E}\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n\rangle\big)^2\Big] \\
&\le 2\,\mathbb{E}\Big[\big(\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n - G\rangle - \mathbb{E}\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n - G\rangle\big)^2\Big] + 2\big[\mathbb{E}\langle \hat{\omega}_N \otimes \hat{\omega}_N, G_n\rangle\big]^2.
\end{aligned}$$

By (2.11) and Lemma 2.6,

$$\mathbb{E}\Big[\big(\langle\hat\omega_N\otimes\hat\omega_N, G_n-G\rangle-\mathbb{E}\langle\hat\omega_N\otimes\hat\omega_N, G_n-G\rangle\big)^2\Big]$$
$$=2\sum_{k,l\in\hat\Lambda_N}|\langle G_n-G,\tilde e_k\otimes\tilde e_l\rangle|^2\le 2\int_{\mathbb{T}^2\times\mathbb{T}^2}(G_n-G)^2(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Hence,

$$\mathbb{E}\Big[\big(\langle\hat\omega_N\otimes\hat\omega_N, G_n\rangle+2\tilde{\mathcal{E}}_N\big)^2\Big]\le 4\int_{\mathbb{T}^2\times\mathbb{T}^2}(G_n-G)^2(x,y)\,\mathrm{d}x\mathrm{d}y+2\big[\mathbb{E}\langle\hat\omega_N\otimes\hat\omega_N,G_n\rangle\big]^2.$$

As a result of (2.14),

$$\lim_{N\to\infty}\mathbb{E}\langle\hat\omega_N\otimes\hat\omega_N,G_n\rangle=\mathbb{E}\langle\omega\otimes\omega,G_n\rangle=\int_{\mathbb{T}^2}G_n(x,x)\,\mathrm{d}x=0,$$

where the second step is due to Lemma 2.1(ii). Thus,

$$\limsup_{N\to\infty}\mathbb{E}\Big[\big(\langle\hat\omega_N\otimes\hat\omega_N,G_n\rangle+2\,\tilde{\mathcal{E}}_N\big)^2\Big]\le 4\int_{\mathbb{T}^2\times\mathbb{T}^2}(G_n-G)^2(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Combining the above inequality with (2.12)–(2.14), letting $N\to\infty$ in (2.12) yields

$$\limsup_{N\to\infty}\mathbb{E}\Big[\big(\langle\omega\otimes\omega,G\rangle+2\,:\mathcal{H}:\big)^2\Big]\le 24\int_{\mathbb{T}^2\times\mathbb{T}^2}(G_n-G)^2(x,y)\,\mathrm{d}x\mathrm{d}y.$$

We finish the proof by sending $n\to\infty$. □

III. PROOF OF THE MAIN RESULT

In this section, we first show that the Hamiltonian $\mathcal{H}_N$ converges weakly to the renormalized energy $:\mathcal{H}:$, by making use of the weak convergence of the random point vortices ω_N to the white noise ω. Thanks to the fact that the law of $:\mathcal{H}:$ has a density, finally we are able to prove Theorem 1.1.

First of all, we prove the following intermediate result.

Proposition 3.1. The Hamiltonian $\mathcal{H}_N$ defined in (1.3) converges weakly to the renormalized energy $:\mathcal{H}:$.

By (1.6) and Theorem 2.8, it suffices to prove that $\langle\omega_N\otimes\omega_N,G\rangle$ converges weakly to $\langle\omega\otimes\omega,G\rangle$, where ω_N is the random point vortices and ω is a white noise on $\mathbb{T}^2$. This result seems to be obvious, thanks to the weak convergence of ω_N to ω; see Ref. 10, Proposition 21. However, since the idea of its proof is very helpful for understanding the arguments below Corollary 3.3, we give the details here.

The following equality will be very useful in the sequel: if $f\in L^2(\mathbb{T}^2\times\mathbb{T}^2,\mathbb{R})$ is symmetric and $f(x,x)\equiv 0$, then

$$\mathbb{E}\big[\langle\omega_N\otimes\omega_N,f\rangle^2\big]=2\int_{\mathbb{T}^2\times\mathbb{T}^2}f(x,y)^2\,\mathrm{d}x\mathrm{d}y. \tag{3.1}$$

To show this identity, we recall that the random point vortices ω_N have the form

$$\omega_N=\frac{1}{\sqrt{N}}\sum_{i=1}^N\xi_i\delta_{X_i},$$

where the random vector $\big((\xi_1,X_1),\ldots,(\xi_N,X_N)\big)$ has the law λ_N defined in (1.4); in particular, all the random variables ξ_i and X_j are independent. Note that there is a slight abuse of notation with respect to formula (1.5): there ω_N is used to denote the point vortices on the state space $H^{-1-}(\mathbb{T}^2)$, while here it is a random variable defined on some probability space Θ. We have

$$\mathbb{E}\big[\langle\omega_N\otimes\omega_N,f\rangle^2\big]=\frac{1}{N^2}\sum_{i,j,k,l=1}^N\mathbb{E}[\xi_i\xi_j\xi_k\xi_l]\,\mathbb{E}[f(X_i,X_j)f(X_k,X_l)].$$

Again by the Isserlis–Wick theorem (see Ref. 23, p. 9, Proposition I.2), it holds that

$$\mathbb{E}[\xi_i\xi_j\xi_k\xi_l] = \delta_{i,j}\delta_{k,l} + \delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}.$$

Combining this identity with the properties of the function f, we immediately get (3.1).

Recall that μ_N is the law of ω_N on $H^{-1-}(\mathbb{T}^2)$ and that the sequence $\{\mu_N\}_{N\geq 1}$ converges weakly to the enstrophy measure μ. Thus, by the Skorokhod representation theorem (see Ref. 4, p. 70, Theorem 6.7), there exists a new probability space $(\tilde\Theta, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence of random variables $\tilde\omega_N : \tilde\Theta \to H^{-1-}(\mathbb{T}^2)$, and $\tilde\omega : \tilde\Theta \to H^{-1-}(\mathbb{T}^2)$ such that

(a) $\tilde\omega_N \overset{\mathcal{L}}{\sim} \mu_N$ and $\tilde\omega \overset{\mathcal{L}}{\sim} \mu$;
(b) $\tilde{\mathbb{P}}$-a.s., $\tilde\omega_N$ converges to $\tilde\omega$ as $N \to \infty$.

In particular, $\tilde\omega$ is a white noise on $\mathbb{T}^2$.

By Ref. 4, p. 16, Theorem 2.1, it is sufficient to show that for any bounded and uniformly continuous function $g : \mathbb{R} \to \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{E}\, g\big(\langle \omega_N \otimes \omega_N, G\rangle\big) = \mathbb{E}\, g\big(\langle \omega \otimes \omega, G\rangle\big).$$

We have

$$\mathbb{E}\, g\big(\langle \omega_N \otimes \omega_N, G\rangle\big) - \mathbb{E}\, g\big(\langle \omega \otimes \omega, G\rangle\big) = \tilde{\mathbb{E}}\, g\big(\langle \tilde\omega_N \otimes \tilde\omega_N, G\rangle\big) - \tilde{\mathbb{E}}\, g\big(\langle \tilde\omega \otimes \tilde\omega, G\rangle\big).$$

From the next result, we deduce that the above quantity vanishes as $N \to \infty$.

Lemma 3.2. We have

$$\lim_{N\to\infty} \tilde{\mathbb{E}}\left|\langle \tilde\omega_N \otimes \tilde\omega_N, G\rangle - \langle \tilde\omega \otimes \tilde\omega, G\rangle\right| = 0.$$

Proof. Denote the expectation by I_N. Let G_n be the approximating functions given at the end of Sec. II A. By the triangle inequality,

$$\begin{aligned} I_N \leq{}& \tilde{\mathbb{E}}\left|\langle \tilde\omega_N \otimes \tilde\omega_N, G\rangle - \langle \tilde\omega_N \otimes \tilde\omega_N, G_n\rangle\right| + \tilde{\mathbb{E}}\left|\langle \tilde\omega_N \otimes \tilde\omega_N, G_n\rangle - \langle \tilde\omega \otimes \tilde\omega, G_n\rangle\right| \\ &+ \tilde{\mathbb{E}}\left|\langle \tilde\omega \otimes \tilde\omega, G_n\rangle - \langle \tilde\omega \otimes \tilde\omega, G\rangle\right|. \end{aligned} \tag{3.2}$$

We denote the three terms by $I_{N,i}, i = 1,2,3$. Cauchy's inequality yields

$$I_{N,1} \leq \Big(\tilde{\mathbb{E}}\left|\langle \tilde\omega_N \otimes \tilde\omega_N, G\rangle - \langle \tilde\omega_N \otimes \tilde\omega_N, G_n\rangle\right|^2\Big)^{1/2} = \bigg(2\int_{\mathbb{T}^2\times\mathbb{T}^2} (G - G_n)^2(x,y)\,\mathrm{d}x\mathrm{d}y\bigg)^{1/2},$$

where in the second step, we have used (3.1). Similarly, by (2.1),

$$I_{N,3} \leq \bigg(2\int_{\mathbb{T}^2\times\mathbb{T}^2} (G - G_n)^2(x,y)\,\mathrm{d}x\mathrm{d}y\bigg)^{1/2}.$$

Next, for any fixed $n \geq 1$, the family $\big\{\langle \tilde\omega_N \otimes \tilde\omega_N, G_n\rangle\big\}_{N\geq 1}$ is bounded in $L^2(\tilde{\mathbb{P}})$ by (3.1); hence, it is uniformly integrable. Moreover, $\tilde{\mathbb{P}}$-a.s.,

$$\langle \tilde\omega_N \otimes \tilde\omega_N, G_n\rangle \to \langle \tilde\omega \otimes \tilde\omega, G_n\rangle \quad \text{as } N \to \infty,$$

due to the a.s. convergence of $\tilde\omega_N$ to $\tilde\omega$. Therefore,

$$\lim_{N\to\infty} I_{N,2} = 0.$$

Summarizing the above discussions, we first let $N \to \infty$ and then $n \to \infty$ in (3.2) to deduce that $\lim_{N\to\infty} I_N = 0$. □

As a consequence, we can prove

Corollary 3.3. For any nontrivial interval $[a, b]$, one has

$$\lim_{N\to\infty} \mathbb{P}(\{\mathcal{H}_N \in [a,b]\}) = \mathbb{P}(\{:\mathcal{H}:\in [a,b]\}). \tag{3.3}$$

Proof. We will use the Malliavin differentiability of the renormalized energy $:\mathcal{H}:$; see, e.g., Ref. 19, Chap. 1 for the definition. By Ref. 18, Theorem 8.3 (see also Ref. 8, Theorems 3.2 and 3.3), the renormalized energy $:\mathcal{H}:$ is infinitely differentiable in the sense of Malliavin and it is nondegenerate, which implies that, as a real valued random variable, the law ν of $:\mathcal{H}:$ has a density with respect to the Lebesgue measure on $\mathbb{R}$. Thus, any interval $[a,b]$ is a ν-continuous set, that is, the boundary of $[a,b]$ (i.e., $\{a,b\}$) is ν-negligible. On the other hand, Proposition 3.1 tells us that the laws on $\mathbb{R}$ of $\mathcal{H}_N$ converge weakly to ν as $N\to\infty$. Therefore, the desired limit holds true. □

Finally, we are ready to prove the main result.

Proof of Theorem 1.1. Taking into account Proposition 2.4 and Corollary 3.3, it is sufficient to show that, for any bounded and uniformly continuous function $F: H^{-1-}(\mathbb{T}^2)\to\mathbb{R}$, one has

$$\lim_{N\to\infty} \mathbb{E}\big[F(\omega_N)\mathbf{1}_{[a,b]}(\mathcal{H}_N)\big] = \mathbb{E}\big[F(\omega)\mathbf{1}_{[a,b]}(:\mathcal{H}:)\big],$$

where ω_N and ω denote the random point vortices and the white noise, respectively.

We follow the idea of the Proof of Proposition 3.1 and use the Skorokhod representation theorem. Then, adopting the notations given there,

$$\mathbb{E}\big[F(\omega_N)\mathbf{1}_{[a,b]}(\mathcal{H}_N)\big] - \mathbb{E}\big[F(\omega)\mathbf{1}_{[a,b]}(:\mathcal{H}:)\big] = \tilde{\mathbb{E}}\big[F(\tilde\omega_N)\mathbf{1}_{[a,b]}(\tilde{\mathcal{H}}_N)\big] - \tilde{\mathbb{E}}\big[F(\tilde\omega)\mathbf{1}_{[a,b]}(:\tilde{\mathcal{H}}:)\big],$$

where the notations with a tilde denote quantities on the new probability space $(\tilde\Theta,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$. Denote by J_N the difference on the right-hand side; then,

$$|J_N| \le \tilde{\mathbb{E}}|F(\tilde\omega_N) - F(\tilde\omega)| + \|F\|_\infty \tilde{\mathbb{E}}\big|\mathbf{1}_{[a,b]}(\tilde{\mathcal{H}}_N) - \mathbf{1}_{[a,b]}(:\tilde{\mathcal{H}}:)\big|.$$

The first term tends to zero by the dominated convergence theorem and the $\tilde{\mathbb{P}}$-a.s. convergence of $\tilde\omega_N$ to $\tilde\omega$. To show that the second one also vanishes as $N\to\infty$, we take a sequence of bounded continuous functions such that

$$f_n(t) = \begin{cases} 1, & t\in[a,b],\\ 0, & t\in(-\infty,a-1/n]\cup[b+1/n,+\infty),\\ \text{linear function}, & t\in[a-1/n,a]\cup[b,b+1/n]. \end{cases}$$

Then,

$$\begin{aligned} &\tilde{\mathbb{E}}\big|\mathbf{1}_{[a,b]}(\tilde{\mathcal{H}}_N) - \mathbf{1}_{[a,b]}(:\tilde{\mathcal{H}}:)\big| \\ &\le \tilde{\mathbb{E}}\big|\mathbf{1}_{[a,b]}(\tilde{\mathcal{H}}_N) - f_n(\tilde{\mathcal{H}}_N)\big| + \tilde{\mathbb{E}}\big|f_n(\tilde{\mathcal{H}}_N) - f_n(:\tilde{\mathcal{H}}:)\big| + \tilde{\mathbb{E}}\big|f_n(:\tilde{\mathcal{H}}:) - \mathbf{1}_{[a,b]}(:\tilde{\mathcal{H}}:)\big|. \end{aligned} \tag{3.4}$$

By Lemma 3.2, we know that $\tilde{\mathcal{H}}_N = -\frac{1}{2}\langle\tilde\omega_N\otimes\tilde\omega_N, G\rangle$ converges in $L^1(\tilde{\mathbb{P}})$ to $:\tilde{\mathcal{H}}: = -\frac{1}{2}\langle\tilde\omega\otimes\tilde\omega, G\rangle$. For fixed $n\in\mathbb{N}$, f_n is the Lipschitz continuous with $\|f_n\|_{\mathrm{Lip}} = n$; therefore,

$$\lim_{N\to\infty}\tilde{\mathbb{E}}\big|f_n(\tilde{\mathcal{H}}_N) - f_n(:\tilde{\mathcal{H}}:)\big| = 0.$$

Next, let ν_N be the law of $\mathcal{H}_N$ and thus also of $\tilde{\mathcal{H}}_N$. We have

$$\tilde{\mathbb{E}}\big|\mathbf{1}_{[a,b]}(\tilde{\mathcal{H}}_N) - f_n(\tilde{\mathcal{H}}_N)\big| \le \nu_N\big([a-1/n,a]\cup[b,b+1/n]\big),$$

and hence, by Corollary 3.3,

$$\limsup_{N\to\infty} \mathbb{E}\big|\mathbf{1}_{[a,b]}(\tilde{\mathcal{H}}_N) - f_n(\tilde{\mathcal{H}}_N)\big| \le \nu\big([a-1/n,a]\cup[b,b+1/n]\big),$$

where ν is the law of $:\mathcal{H}:$ which is the same as that of $:\tilde{\mathcal{H}}:$. Finally,

$$\tilde{\mathbb{E}}\,\big|f_n(:\tilde{\mathcal{H}}:) - \mathbf{1}_{[a,b]}(:\tilde{\mathcal{H}}:)\big| \le \nu\big([a-1/n,a]\cup[b,b+1/n]\big).$$

Recall that ν is absolutely continuous with respect to the Lebesgue measure. Therefore, first letting $N\to\infty$ and then $n\to\infty$ in (3.4), we complete the proof. □

IV. TRIVIALITY OF CLUSTER POINTS

In this part, following the discussions at the end of Ref. 17, Sec. 5.4 (see also Ref. 21), we study the limit behavior of the correlation functions (i.e., marginal distributions) of the energy conditional measures $\lambda_N^{a,b}$ on the "flat space" $(\mathbb{R}\times\mathbb{T}^2)^N$. Here, for $a,b\in\mathbb{R}$, $a<b$,

$$\lambda_N^{a,b} = \frac{1}{Z_N^{a,b}}\mathbf{1}_{\{\mathcal{H}_N((\xi_1,x_1),\ldots,(\xi_N,x_N))\in[a,b]\}}\lambda_N, \tag{4.1}$$

where λ_N is defined in (1.4) and $Z_N^{a,b}$ is the normalizing constant,

$$Z_N^{a,b} = \int_{(\mathbb{R}\times\mathbb{T}^2)^N}\mathbf{1}_{\{\mathcal{H}_N((\xi_1,x_1),\ldots,(\xi_N,x_N))\in[a,b]\}}\,\mathrm{d}\lambda_N = \mathbb{P}(\{\mathcal{H}_N\in[a,b]\}).$$

Note that the measure $\mu_N^{a,b}$ defined in the Introduction is the image measure of $\lambda_N^{a,b}$ under the map $\mathcal{T}_N:(\mathbb{R}\times\mathbb{T}^2)^N\to H^{-1-}(\mathbb{T}^2)$ defined as

$$\big((\xi_1,x_1),\ldots,(\xi_N,x_N)\big)\xrightarrow{\mathcal{T}_N}\frac{1}{\sqrt{N}}\sum_{i=1}^N\xi_i\delta_{x_i}. \tag{4.2}$$

To simplify the presentation, we introduce the notations $\tilde{x}_i=(\xi_i,x_i)\in\mathbb{R}\times\mathbb{T}^2$ and $X_N=(\tilde{x}_1,\ldots,\tilde{x}_N)$. Denote by $\mathrm{d}\tilde{x}_i=\mathrm{d}x_i\mathcal{N}(\mathrm{d}\xi_i)$ and $\mathrm{d}X_N=\mathrm{d}\tilde{x}_1\ldots\mathrm{d}\tilde{x}_N$. Let

$$\rho^N(X_N) = \mathbf{1}_{\{\mathcal{H}_N(X_N)\in[a,b]\}}/Z_N^{a,b}$$

be the density function of the conditional probability measure $\lambda_N^{a,b}$ on $(\mathbb{R}\times\mathbb{T}^2)^N$. The *correlation functions* ρ_j^N $(1\le j\le N)$ are defined as follows: $\rho_N^N=\rho^N$, and for $1\le j\le N-1$,

$$\rho_j^N(\tilde{x}_1,\ldots,\tilde{x}_j) = \int_{(\mathbb{R}\times\mathbb{T}^2)^{N-j}}\rho^N(X_N)\,\mathrm{d}\tilde{x}_{j+1}\ldots\mathrm{d}\tilde{x}_N.$$

Each function is a probability density (for the first j point vortices) and is symmetric in $(\tilde{x}_1,\ldots,\tilde{x}_j)$, thanks to the symmetry of ρ^N in $(\tilde{x}_1,\ldots,\tilde{x}_N)$. To simplify the notations, we introduce

$$X_j=(\tilde{x}_1,\ldots,\tilde{x}_j),\quad X^{N-j}=(\tilde{x}_{j+1},\ldots,\tilde{x}_N),\quad 1\le j\le N-1.$$

First of all, we have the following simple result [see Ref. 17 (22) or Ref. 21, Proposition 6].

Lemma 4.1. For any $1\le j\le N-1$,

$$\int_{(\mathbb{R}\times\mathbb{T}^2)^N}\rho^N\log\rho^N\,\mathrm{d}X_N \ge \int_{(\mathbb{R}\times\mathbb{T}^2)^j}\rho_j^N\log\rho_j^N\,\mathrm{d}X_j + \int_{(\mathbb{R}\times\mathbb{T}^2)^{N-j}}\rho_{N-j}^N\log\rho_{N-j}^N\,\mathrm{d}X^{N-j}.$$

Proof. We include the proof for the reader's convenience. It is well known that $t\log t\ge t-1$ for all $t\ge 0$. Therefore,

$$\rho^N(X_N)\log\left(\frac{\rho^N(X_N)}{\rho_j^N(X_j)\rho_{N-j}^N(X^{N-j})}\right)+\rho_j^N(X_j)\rho_{N-j}^N(X^{N-j})-\rho^N(X_N)\geq 0,$$

which implies

$$\int_{(\mathbb{R}\times\mathbb{T}^2)^N}\rho^N(X_N)\log\left(\frac{\rho^N(X_N)}{\rho_j^N(X_j)\rho_{N-j}^N(X^{N-j})}\right)\mathrm{d}X_N\geq 0.$$

This is equivalent to the desired inequality. □

Next, we prove

Proposition 4.2. For any fixed $j\geq 1$, $\{\rho_j^N\}_{N\geq j}$ is weakly compact in $L^1\big((\mathbb{R}\times\mathbb{T}^2)^j,\mathrm{d}X_j\big)$.

Proof. For any $N\geq j$, there exist $m=m(j,N)\in\mathbb{N}$ and $r=r(j,N)\in\{0,1,\ldots,j-1\}$ such that $N=mj+r$. By Lemma 4.1,

$$\int_{(\mathbb{R}\times\mathbb{T}^2)^N}\rho^N\ \log\rho^N\ \mathrm{d}X_N\geq m\int_{(\mathbb{R}\times\mathbb{T}^2)^j}\rho_j^N\ \log\ \rho_j^N\ \mathrm{d}X_j+r\int_{\mathbb{R}\times\mathbb{T}^2}\rho_1^N\ \log\ \rho_1^N\ \mathrm{d}X_1.$$

Using the inequality $t\log t\geq t-1 (t\geq 0)$, it is clear that $\int_{\mathbb{R}\times\mathbb{T}^2}\rho_1^N\ \log\ \rho_1^N\ \mathrm{d}X_1\geq 0$. Thus,

$$\int_{(\mathbb{R}\times\mathbb{T}^2)^j}\rho_j^N\ \log\ \rho_j^N\ \mathrm{d}X_j\leq\frac{1}{m}\int_{(\mathbb{R}\times\mathbb{T}^2)^N}\rho^N\ \log\rho^N\ \mathrm{d}X_N=\frac{1}{m}\log\frac{1}{Z_N^{a,b}},$$

where the last step follows from the definition of ρ^N. Note that $\frac{1}{m}=O\big(\frac{1}{N}\big)$, thus by (3.3), the right-hand side vanishes as $N\to\infty$. In particular, we conclude that $\{\rho_j^N\ \log\ \rho_j^N\}_{N\geq j}$ is bounded in $L^1\big((\mathbb{R}\times\mathbb{T}^2)^j,\mathrm{d}X_j\big)$. The proof is complete. □

We say that a family $\{\rho_j\}_{j\geq 1}$ of probability densities is a weak cluster point of $\{\rho_j^N\}_{j\geq 1}$ if there exists a subsequence $\{N_k\}_{k\geq 1}$ of integers such that, for any $j\geq 1$, $\rho_j^{N_k}$ converges weakly to ρ_j in $L^1\big((\mathbb{R}\times\mathbb{T}^2)^j,\mathrm{d}X_j\big)$. Now, we prove the main result of this section.

Theorem 4.3. *Any weak cluster point $\{\rho_j\}_{j\geq 1}$ of $\{\rho_j^N\}_{j\geq 1}$ is trivial, that is, for any $j\geq 1$, $\rho_j=1$ almost surely on $(\mathbb{R}\times\mathbb{T}^2)^j$. Consequently, for any $j\geq 1$, the whole sequence $\{\rho_j^N\}_{N\geq j}$ converges weakly to 1.*

Proof. Fix any $\varepsilon>0$ and $j\geq 1$; let

$$\mathcal{C}_\varepsilon=\left\{u\in L^1\big((\mathbb{R}\times\mathbb{T}^2)^j,\mathrm{d}X_j\big):u\geq 0,\int_{(\mathbb{R}\times\mathbb{T}^2)^j}u\ \log\ u\,\mathrm{d}X_j\leq\varepsilon\right\}.$$

Let $\{N_k\}_{k\geq 1}$ be the subsequence such that $\rho_j^{N_k}$ converges weakly to ρ_j in $L^1\big((\mathbb{R}\times\mathbb{T}^2)^j,\mathrm{d}X_j\big)$. By the Proof of Proposition 4.2, we have $\rho_j^{N_k}\in\mathcal{C}_\varepsilon$ for all k big enough. Therefore, ρ_j is a weak cluster point of $\mathcal{C}_\varepsilon$, which is a convex subset of $L^1\big((\mathbb{R}\times\mathbb{T}^2)^j,\mathrm{d}X_j\big)$. Since the weak closure of $\mathcal{C}_\varepsilon$ coincides with the strong one, there exists a sequence of functions $\{u_n\}\subset\mathcal{C}_\varepsilon$ which converges strongly to ρ_j in $L^1\big((\mathbb{R}\times\mathbb{T}^2)^j,\mathrm{d}X_j\big)$. Along a subsequence, u_n converges to ρ_j almost everywhere. Therefore, by Fatou's lemma, we have

$$\int_{(\mathbb{R}\times\mathbb{T}^2)^j}\rho_j\ \log\rho_j\ \mathrm{d}X_j\leq\liminf_{n\to\infty}\int_{(\mathbb{R}\times\mathbb{T}^2)^j}u_n\ \log u_n\ \mathrm{d}X_j\leq\varepsilon.$$

The arbitrariness of $\varepsilon>0$ leads to $\int_{(\mathbb{R}\times\mathbb{T}^2)^j}\rho_j\ \log\rho_j\ \mathrm{d}X_j=0$, which implies $\rho_j=1$ almost surely. The last assertion follows from the weak compactness of $\{\rho_j^N\}_{N\geq j}$ and the uniqueness of the weak limit. □

The above theorem implies that, in the limit, the joint density function ρ_j of the first j point vortices is identically equal to 1; that is, the energy constraints on the vortices disappear and the vortices are mutually independent. Hence, Theorem 4.3 is a propagation-of-chaos type result: as N increases, the first few point vortices tend to be less and less correlated with each other and, in the limit, they become totally independent. The limiting behavior of the vortices is chaotic; it is not possible to deduce the information of one point vortex from those of others. Based on the propagation of chaos, some formal argument (cf. Ref. 17, pp. 15–16) yields that the weak cluster point obtained above gives a trivial solution to the mean field equation,

$$\rho(\xi, x) = \frac{1}{Z_\beta} e^{-\beta \xi U_\rho(x)}, \quad \beta \in \mathbb{R},$$

where Z_β is the normalizing constant and U_ρ is the averaged stream function,

$$U_\rho(x) = \int_{\mathbb{R}\times\mathbb{T}^2} \xi G(x, y)\rho(\xi, y)\,\mathcal{N}(\mathrm{d}\xi)\mathrm{d}y, \quad x \in \mathbb{T}^2.$$

In our case, $\rho = 1$ a.s. and $U_\rho = 0$ a.s. The corresponding free energy $F(1) = S(1) + \beta E(1) = 0$, where the entropy

$$S(\rho) = \int_{\mathbb{R}\times\mathbb{T}^2} \rho(\tilde{x}) \log \rho(\tilde{x})\,\mathrm{d}\tilde{x}$$

and the energy

$$E(\rho) = \int_{(\mathbb{R}\times\mathbb{T}^2)^2} \mathcal{H}_2(\tilde{x}_1, \tilde{x}_2)\rho(\tilde{x}_1)\rho(\tilde{x}_2)\,\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2.$$

We conclude this section by showing that, under the measure $\lambda_N^{a,b} = \rho^N(X_N)\,\mathrm{d}X_N$, the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{\tilde{x}_i}$ converges weakly to the trivial measure $\mathrm{d}\tilde{x} = \mathcal{N}(\mathrm{d}\xi)\mathrm{d}x$ on $\mathbb{R}\times\mathbb{T}^2$.

Corollary 4.4. For any $\phi \in C_b(\mathbb{R}\times\mathbb{T}^2)$,

$$\lim_{N\to\infty} \int_{(\mathbb{R}\times\mathbb{T}^2)^N} \left| \frac{1}{N}\sum_{i=1}^N \phi(\tilde{x}_i) - \int_{\mathbb{R}\times\mathbb{T}^2} \phi(\tilde{x})\,\mathrm{d}\tilde{x} \right|^2 \mathrm{d}\lambda_N^{a,b} = 0.$$

Proof. We denote the integral by I_N. Expanding the square in the integral leads to

$$\begin{aligned} I_N =& \frac{1}{N^2}\sum_{i,j=1}^N \int_{(\mathbb{R}\times\mathbb{T}^2)^N} \phi(\tilde{x}_i)\phi(\tilde{x}_j)\,\mathrm{d}\lambda_N^{a,b} + \left(\int_{\mathbb{R}\times\mathbb{T}^2} \phi(\tilde{x})\,\mathrm{d}\tilde{x}\right)^2 \\ &- \frac{2}{N}\left(\int_{\mathbb{R}\times\mathbb{T}^2} \phi(\tilde{x})\,\mathrm{d}\tilde{x}\right)\sum_{i=1}^N \int_{(\mathbb{R}\times\mathbb{T}^2)^N} \phi(\tilde{x}_i)\,\mathrm{d}\lambda_N^{a,b}. \end{aligned}$$

Note that $\lambda_N^{a,b} = \rho^N(X_N)\,\mathrm{d}X_N$. Using the marginal densities ρ_j^N, $j = 1, 2$, we have

$$\begin{aligned} I_N =& \frac{N-1}{N}\int_{(\mathbb{R}\times\mathbb{T}^2)^2} \phi(\tilde{x}_1)\phi(\tilde{x}_2)\rho_2^N(\tilde{x}_1, \tilde{x}_2)\,\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2 + \frac{1}{N}\int_{\mathbb{R}\times\mathbb{T}^2} \phi(\tilde{x}_1)^2\rho_1^N(\tilde{x}_1)\,\mathrm{d}\tilde{x}_1 \\ &+ \left(\int_{\mathbb{R}\times\mathbb{T}^2} \phi(\tilde{x})\,\mathrm{d}\tilde{x}\right)^2 - 2\left(\int_{\mathbb{R}\times\mathbb{T}^2} \phi(\tilde{x})\,\mathrm{d}\tilde{x}\right)\int_{\mathbb{R}\times\mathbb{T}^2} \phi(\tilde{x}_1)\rho_1^N(\tilde{x}_1)\,\mathrm{d}\tilde{x}_1. \end{aligned}$$

Now, we finish the proof by letting $N \to \infty$ and using Theorem 4.3. □

V. ENERGY CONDITIONAL SOLUTIONS TO 2D EULER EQUATIONS

In this part, we show the existence of solutions to 2D Euler equations whose renormalized energy is confined in an interval $[a, b]$. First, we give the precise meaning of the solution.

Definition 5.1. Let $a, b \in \mathbb{R}, a < b$ be fixed. A stochastic process $\{\omega_t\}_{t\in[0,T]}$ defined on some probability space $(\Theta, \mathcal{F}, \mathbb{P})$ with trajectories in $C([0,T], H^{-1-}(\mathbb{T}^2))$ is called an energy conditional solution of the 2D Euler equations if for any $t \in [0,T]$, ω_t has the law $\mu^{a,b}$ defined in (1.7), and for any $\phi \in C^\infty(\mathbb{T}^2)$, $\mathbb{P}$-a.s.,

$$\langle \omega_t, \phi\rangle = \langle \omega_0, \phi\rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi\rangle \, \mathrm{d}s \quad \text{for all } t \in [0,T]. \tag{5.1}$$

The above equation is called the weak vorticity formulation of the 2D Euler equation (see Ref. 22). Here, for any $\phi \in C^\infty(\mathbb{T}^2)$,

$$H_\phi(x,y) = \frac{1}{2} K(x-y)\cdot(\nabla\phi(x) - \nabla\phi(y)), \quad x, y \in \mathbb{T}^2, x \neq y,$$

in which K is the Biot–Savart kernel on $\mathbb{T}^2$. We shall set $H_\phi(x,x) = 0$ for all $x \in \mathbb{T}^2$. Note that $\mu^{a,b}$ is absolutely continuous with respect to the enstrophy measure μ, with a density function bounded by $1/Z^{a,b}$, where $Z^{a,b} = \mu(\{:\mathcal{H}:\in [a,b]\}) > 0$. Thus, in a similar way, as in Proposition 2.2, one can show that the nonlinear term $\langle \omega_s \otimes \omega_s, H_\phi\rangle$ is well defined; see Ref. 10, Theorem 10 and Definition 11 for details.

Remark 5.2. We recall that Cipriano showed in Ref. 8, Theorem 4.1 the existence of solutions to 2D Euler equations with given energy $a \in \mathbb{R}$, as long as the density function of $:\mathcal{H}:$ is positive at a. It is interesting to prove the same result by letting $b \to a$ in the above definition. The key ingredient is to show uniform estimates (independent of a and b) of the type proven in Lemma 5.6 (without the parameter N). For the moment, we do not know how to do this.

Now, we state our main result in this part.

Theorem 5.3. *There exists a probability space $(\Theta, \mathcal{F}, \mathbb{P})$ with the following properties.*

(i) *There exists a stochastic process $\omega : [0,T] \times \Theta \to H^{-1-}(\mathbb{T}^2)$ such that it is an energy conditional solution of the 2D Euler equations in the sense of Definition 5.1.*
(ii) *On $(\Theta, \mathcal{F}, \mathbb{P})$, one can define a subsequence of random point vortex systems which converges $\mathbb{P}$-a.s. in $C([0,T], H^{-1-}(\mathbb{T}^2))$ to the solution of point* (i).
(iii) *On $(\Theta, \mathcal{F}, \mathbb{P})$, one can define a subsequence of functions $\omega^{(n)}(\theta, t, x)$, $(\theta, t, x) \in \Theta \times [0,T] \times \mathbb{T}^2$, such that for $\mathbb{P}$-a.s. $\theta \in \Theta$, the functions $(t,x) \mapsto \omega^{(n)}(\theta, t, x)$ are L^∞-solutions of 2D Euler equations and converge to $\omega.(\theta)$ in $C([0,T], H^{-1-}(\mathbb{T}^2))$.*

Recalling that $\mu^{a,b} = (Z^{a,b})^{-1}\mathbf{1}_{\{:\mathcal{H}:\in[a,b]\}}\mu$, it may seem that the above result follows from Ref. 10, Theorem 25. However, the initial density function in the present case is not continuous, thus our result is not a direct consequence of Ref. 10, Theorem 25. A careful investigation of the proof in Ref. 10, Sec. 5, reveals that the continuity of the density function was only used there to show that the normalizing constants C_N tend to 1 as $N \to \infty$ (see the arguments below Ref. 10, Lemma 29). Since we have already shown in (3.3) the convergence $Z_N^{a,b} \to Z^{a,b}$, we can follow the ideas in Ref. 10 to prove Theorem 5.3. In the sequel, we introduce the main preliminaries needed in the proof.

Let $N \geq 2$ be fixed, we consider the point vortex dynamics on $\mathbb{T}^2$:

$$\frac{\mathrm{d}X_t^{i,N}}{\mathrm{d}t} = \frac{1}{\sqrt{N}} \sum_{j=1}^N a_j K\big(X_t^{i,N} - X_t^{j,N}\big), \quad i = 1, \ldots, N, \tag{5.2}$$

with the vortex intensities $(a_1, \ldots, a_N) \in (\mathbb{R}\backslash\{0\})^N$ and initial positions $\big(X_0^{1,N}, \ldots, X_0^{N,N}\big) \in (\mathbb{T}^2)^N \backslash \Delta_N$, where $\Delta_N = \big\{(x_1, \ldots, x_N) \in (\mathbb{T}^2)^N : \exists\, i \neq j \text{ such that } x_i = x_j\big\}$ is the generalized diagonal. It is well known that, for $\mathrm{Leb}_{\mathbb{T}^2}^{\otimes N}$-a.e. initial condition $\big(X_0^{1,N}, \ldots, X_0^{N,N}\big) \in (\mathbb{T}^2)^N \backslash \Delta_N$, the above system of equations has a global solution, that is, the vortex points do not collapse, cf. Ref. 20, Sec. 4.4. Therefore, we can define the vorticity field

$$\omega_t^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N a_i \delta_{X_t^{i,N}}, \quad t \geq 0,$$

which satisfies, for any $\phi \in C^\infty(\mathbb{T}^2)$,

$$\langle \omega_t^N, \phi \rangle = \langle \omega_0^N, \phi \rangle + \int_0^t \langle \omega_s^N \otimes \omega_s^N, H_\phi \rangle \mathrm{d}s, \quad \text{for all } t \geq 0. \tag{5.3}$$

We mention that an interesting model involving the creation and damping of point vortices is studied in the recent work.[14]

We shall consider the point vortex dynamics with random intensities and random initial conditions. Thus, on a probability space $(\Theta, \mathcal{F}, \mathbb{P})$, let $\{\xi_i\}_{i\geq 1}$ be an i.i.d. sequence of random variables with the standard Gaussian distribution $N(0,1)$, and $\{X_0^i\}_{i\geq 1}$ be an i.i.d. sequence of random variables with uniform distribution on $\mathbb{T}^2$; assume the two families are independent. Note that the measures λ_N and μ_N defined in Sec. I are the laws of $\big((\xi_1, X_0^1), \ldots, (\xi_N, X_0^N)\big)$ and of $\omega_0^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{X_0^i}$, respectively.

We recall the following result which is the same as Ref. 10, Proposition 22.

Proposition 5.4. Consider the point vortex dynamics (5.2) with random intensities $(\xi_1, \ldots, \xi_N)$ and random initial positions $(X_0^1, \ldots, X_0^N)$ distributed as λ_N. $\mathbb{P}$-almost surely, the dynamics $\big(X_t^{1,N}, \ldots, X_t^{N,N}\big)$ is well defined in $(\mathbb{T}^2)^N \backslash \Delta_N$ for all $t \geq 0$, and $\big((\xi_1, X_t^{1,N}), \ldots, (\xi_N, X_t^{N,N})\big)$ has the invariant law λ_N. The associated measure-valued vorticity ω_t^N is a stochastic process with the invariant marginal law μ_N; moreover, $\mathbb{P}$-a.s., ω_t^N satisfies (5.3) and

$$\mathcal{H}_N\big(\omega_t^N\big) = \mathcal{H}_N\big((\xi_1, X_t^{1,N}), \ldots, (\xi_N, X_t^{N,N})\big) = \mathcal{H}_N\big(\omega_0^N\big), \quad \text{for all } t \geq 0. \tag{5.4}$$

Proof. The first assertion follows from the discussions below (5.2) and those in the paragraph containing the formula (1.4). The second assertion is a direct consequence of the first one and of the definition of ω_t^N, while (5.4) is due to the invariance of the Hamiltonian $\mathcal{H}_N$ under the point vortex dynamics (5.2). □

Next, we confine the point vortex dynamics to those initial configurations with energy belonging to the interval $[a,b]$. To this end, we introduce the conditional probability measures on $(\Theta, \mathcal{F})$,

$$\mathbb{P}_N^{a,b} = \frac{\mathbf{1}_{\{\mathcal{H}_N(\omega_0^N) \in [a,b]\}}}{\mathbb{P}(\{\mathcal{H}_N(\omega_0^N) \in [a,b]\})} \mathbb{P} = \frac{\mathbf{1}_{\{\mathcal{H}_N(\omega_0^N) \in [a,b]\}}}{Z_N^{a,b}} \mathbb{P}.$$

The measure $\lambda_N^{a,b}$ defined in (4.1) is the law on $(\mathbb{R} \times \mathbb{T}^2)^N$ of $\big((\xi_1, X_0^1), \ldots, (\xi_N, X_0^N)\big)$ under $\mathbb{P}_N^{a,b}$, and we have $\mu_N^{a,b} = (\mathcal{T}_N)_\# \lambda_N^{a,b}$, where $\mathcal{T}_N$ is defined in (4.2). From Proposition 5.4, we deduce the following result.

Proposition 5.5. Consider the point vortex dynamics (5.2) with random intensities $(\xi_1, \ldots, \xi_N)$ and random initial positions $(X_0^1, \ldots, X_0^N)$ distributed as $\lambda_N^{a,b}$. Then, $\mathbb{P}_N^{a,b}$-a.s., for all $t \geq 0$, the dynamics $\big(X_t^{1,N}, \ldots, X_t^{N,N}\big)$ is well defined in $(\mathbb{T}^2)^N \backslash \Delta_N$, and $\big((\xi_1, X_t^{1,N}), \ldots, (\xi_N, X_t^{N,N})\big)$ has the invariant distribution $\lambda_N^{a,b}$. The associated measure-valued vorticity ω_t^N is a stochastic process with the invariant marginal law $\mu_N^{a,b}$; moreover, $\mathbb{P}_N^{a,b}$-a.s., ω_t^N satisfies (5.3) and

$$\mathcal{H}_N\big(\omega_t^N\big) = \mathcal{H}_N\big(\omega_0^N\big) \in [a,b], \quad \text{for all } t \geq 0.$$

Proof. Since the conditional probability measure $\mathbb{P}_N^{a,b}$ is absolutely continuous with respect to $\mathbb{P}$, the properties that hold $\mathbb{P}$-a.s. also hold almost surely with respect to $\mathbb{P}_N^{a,b}$. It remains to show that $\lambda_N^{a,b}$ is the invariant distribution of $\big((\xi_1, X_t^{1,N}), \ldots, (\xi_N, X_t^{N,N})\big)$. Once this is proved, we deduce that ω_t^N has the invariant marginal law $\mu_N^{a,b}$ since

$$\operatorname{law}\big(\omega_t^N\big) = (\mathcal{T}_N)_\# \operatorname{law}\big((\xi_1, X_t^{1,N}), \ldots, (\xi_N, X_t^{N,N})\big) = (\mathcal{T}_N)_\# \lambda_N^{a,b} = \mu_N^{a,b}.$$

To simplify the notations, we write $\xi = (\xi_1, \ldots, \xi_N)$ and $X_t^N = \big(X_t^{1,N}, \ldots, X_t^{N,N}\big)$. For any bounded measurable function $F : (\mathbb{R} \times \mathbb{T}^2)^N \to \mathbb{R}$, by the definition of $\mathbb{P}_N^{a,b}$ and (5.4),

$$\begin{aligned}
\int_\Theta F\big(\xi, X_t^N\big)\, \mathrm{d}\mathbb{P}_N^{a,b} &= \frac{1}{Z_N^{a,b}} \int_\Theta F\big(\xi, X_t^N\big) \mathbf{1}_{\{\mathcal{H}_N(\omega_0^N) \in [a,b]\}}\, \mathrm{d}\mathbb{P} \\
&= \frac{1}{Z_N^{a,b}} \int_\Theta F\big(\xi, X_t^N\big) \mathbf{1}_{\{\mathcal{H}_N(\xi, X_t^N) \in [a,b]\}}\, \mathrm{d}\mathbb{P} \\
&= \frac{1}{Z_N^{a,b}} \int_\Theta F\big(\xi, X_0^N\big) \mathbf{1}_{\{\mathcal{H}_N(\xi, X_0^N) \in [a,b]\}}\, \mathrm{d}\mathbb{P},
\end{aligned}$$

where the last step is due to the fact that, under $\mathbb{P}$, $\big(\xi, X_t^N\big)$ has the invariant distribution λ_N. Therefore,

$$\int_\Theta F\big(\xi, X_t^N\big)\,\mathrm{d}\mathbb{P}_N^{a,b} = \int_\Theta F\big(\xi, X_0^N\big)\,\mathrm{d}\mathbb{P}_N^{a,b},$$

which implies that, under the conditional probability measure $\mathbb{P}_N^{a,b}$, $\big(\xi, X_t^N\big)$ has the same law as $\big(\xi, X_0^N\big)$, i.e., $\lambda_N^{a,b}$. □

To emphasize the dependence on the parameters a, b, we denote by $\omega_{a,b}^N(\cdot)$ the associated measure-valued vorticity field obtained in Proposition 5.5. The next lemma gives useful estimates on $\omega_{a,b}^N(\cdot)$.

Lemma 5.6. Let N_0 be large enough such that $Z_0 := \inf_{N\ge N_0} Z_N^{a,b} > 0$ and $f : \mathbb{T}^2\times\mathbb{T}^2 \to \mathbb{R}$ be symmetric, bounded, and measurable. Then, for all $p \ge 1$ and $\delta > 0$, there are constants $C_p, C_{p,\delta} > 0$ such that for all $N \ge N_0$,

$$\mathbb{E}_{\mathbb{P}_N^{a,b}}\Big[\big|\big\langle\omega_{a,b}^N(t)\otimes\omega_{a,b}^N(t), f\big\rangle\big|^p\Big] \le C_p\|f\|_\infty^p/Z_0, \quad \mathbb{E}_{\mathbb{P}_N^{a,b}}\Big[\big\|\omega_{a,b}^N(t)\big\|_{H^{-1-\delta}}^p\Big] \le C_{p,\delta}/Z_0.$$

Moreover, if $f(x,x) = 0$ for all $x \in \mathbb{T}^2$, then

$$\mathbb{E}_{\mathbb{P}_N^{a,b}}\Big[\big\langle\omega_{a,b}^N(t)\otimes\omega_{a,b}^N(t), f\big\rangle^2\Big] \le \frac{2}{Z_0}\int_{\mathbb{T}^2\times\mathbb{T}^2} f(x,y)^2\,\mathrm{d}x\mathrm{d}y.$$

Proof. First, we mention that similar results hold for the unconditioned point vortices ω_t^N under the probability measure $\mathbb{P}$. Namely, for every $p \ge 1$ and $\delta > 0$, there are positive constants $C_p, C_{p,\delta} > 0$ such that

$$\mathbb{E}_{\mathbb{P}}\Big[\big|\big\langle\omega_{a,b}^N(t)\otimes\omega_{a,b}^N(t), f\big\rangle\big|^p\Big] \le C_p\|f\|_\infty^p, \quad \mathbb{E}_{\mathbb{P}}\Big[\big\|\omega_t^N\big\|_{H^{-1-\delta}}^p\Big] \le C_{p,\delta},$$

and if $f(x,x) = 0$ for all $x \in \mathbb{T}^2$, then

$$\mathbb{E}_{\mathbb{P}}\Big[\big\langle\omega_t^N\otimes\omega_t^N, f\big\rangle^2\Big] = 2\int_{\mathbb{T}^2\times\mathbb{T}^2} f(x,y)^2\,\mathrm{d}x\mathrm{d}y.$$

The proofs of these results are based on the expression of ω_t^N; see Ref. 10, Lemma 23 for detailed computations. In particular, the first inequality can be proved using the Isserlis-Wick theorem and a combinatorial argument, while the second one follows from the fact $\|\delta_x\|_{H^{-1-\delta}} \le C_\delta$. The last equality is the same as (3.1).

Now, by observing that

$$\mathbb{E}_{\mathbb{P}_N^{a,b}}\Big[\big\langle\omega_{a,b}^N(t)\otimes\omega_{a,b}^N(t), f\big\rangle^p\Big] = \frac{1}{Z_N^{a,b}}\mathbb{E}_{\mathbb{P}}\Big[\big\langle\omega_t^N\otimes\omega_t^N, f\big\rangle^p \mathbf{1}_{\{\mathcal{H}_N(\omega_0^N)\in[a,b]\}}\Big],$$

we immediately get the first estimate. The proofs of the others are similar. □

With these preliminaries in hand, we can complete the Proof of Theorem 5.3. More precisely, let Q^N be the law of the process $\big\{\omega_{a,b}^N(t)\big\}_{t\in[0,T]}$ on $\mathcal{X} = C\big([0,T], H^{-1-}(\mathbb{T}^2)\big)$. Using Eq. (5.3) and the estimates in Lemma 5.6, we can show that the family $\{Q^N\}_{N\ge N_0}$ is tight on $\mathcal{X}$; see the beginning part of Ref. 10, Sec. 4.2 for details. By Prohorov's theorem (Ref. 4, p. 59, Theorem 5.1), there is a subsequence $\{N_k\}_{k\ge1}$ such that Q^{N_k} converges weakly to some probability measure Q on $\mathcal{X}$. Skorokhod's representation theorem implies that there exist a probability space $\big(\tilde\Theta, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\big)$ and processes $\tilde\omega^{N_k}$ and $\tilde\omega$, with trajectories in $\mathcal{X}$, such that their laws are Q^{N_k} and Q, respectively; $\tilde{\mathbb{P}}$-a.s., $\tilde\omega^{N_k}$ converges to $\tilde\omega$ in the topology of $\mathcal{X}$. Moreover, the processes $\tilde\omega^{N_k}$ can be represented as a sum of Dirac deltas,

$$\tilde\omega_t^{N_k} = \frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k}\tilde\xi_i\delta_{\tilde X_t^{i,N_k}}, \quad t \in [0,T],$$

where $\big(\big(\tilde\xi_1, \tilde X_t^{i,N_k}\big), \ldots, \big(\tilde\xi_{N_k}, \tilde X_t^{N_k,N_k}\big)\big)$ is a random vector with the invariant law $\lambda_{N_k}^{a,b}$ for all $t \in [0,T]$, and it solves the point vortex dynamics (5.2); see Ref. 10, Lemma 28 for the detailed proof.

Next, we prove the law of $\tilde\omega_t$ is the energy conditional measure $\mu^{a,b}$ for all $t \in [0,T]$. For any $F \in C_b\big(H^{-1-}(\mathbb{T}^2)\big)$, since $\tilde\omega^{N_k}$ converges $\tilde{\mathbb{P}}$-a.s. to $\tilde\omega$ in the topology of $\mathcal{X}$,

$$\int_{\tilde{\Theta}} F(\tilde{\omega}_t)\, d\tilde{\mathbb{P}} = \lim_{k\to\infty} \int_{\tilde{\Theta}} F\big(\tilde{\omega}_t^{N_k}\big)\, d\tilde{\mathbb{P}} = \lim_{k\to\infty} \int_{H^{-1-}(\mathbb{T}^2)} F(\omega)\, d\mu_{N_k}^{a,b}(\omega) = \int_{H^{-1-}(\mathbb{T}^2)} F(\omega)\, d\mu^{a,b}(\omega),$$

where in the last step, we have used the weak convergence of $\mu_N^{a,b}$ to $\mu^{a,b}$ proved in Sec. III.

Finally, using again the estimates in Lemma 5.6 and repeating the arguments below (Ref. 10, Lemma 28), we can show that $\{\tilde{\omega}_t\}_{t\in[0,T]}$ satisfies the weak vorticity formulation (5.1) of the 2D Euler equation. Summarizing the above discussions, we have proved the first two assertions of Theorem 5.3. The last assertion is proved in the same way as the end of Ref. 10, Sec. 4.2.

VI. STRUCTURES AND INTERMEDIATE REGIMES

In the classical Onsager theory, the microcanonical measure is defined as the uniform measure on configurations $(x_1,\ldots,x_N)$ such that

$$\sum_{i\neq j} \xi_i\xi_j \log \frac{1}{|x_i - x_j|} \sim N^2 a, \tag{6.1}$$

for some value of $a > 0$ (in this section, we heuristically write a instead of $[a, b]$ since, for $a > 0$, it is the value of a which plays a practical role, independently of b). For typical configurations $(x_1,\ldots,x_N)$, when N is large, the empirical measure

$$\frac{1}{N}\sum_{i=1}^{N} \xi_i \delta_{x_i}$$

is close to the solutions of a certain mean field equation (Onsager theory). There is a natural explanation, for $a \gg 0$: in order to have (6.1), we need roughly N^2 terms in the sum $\sum_{i\neq j}$ with value $\xi_i\xi_j \log \frac{1}{|x_i-x_j|}$ close to a (this argument is very rough). The "only" way to reach such result is to group positive vortices together, all very close to each other, and similarly for the negative ones, with the two clusters not so close to each other: roughly $(N/2)^2$ terms will be positive and close to a (those corresponding to positive pairs), other $(N/2)^2$ terms will be positive as well and close to a (those corresponding to negative pairs), and the remaining pairs, composed of vortices of opposite signs, have small values of $\xi_i\xi_j \log \frac{1}{|x_i-x_j|}$ because the two points belong to clusters which are relatively far from each other.

In our "white noise" model, the microcanonical measure corresponds to the constraint

$$\sum_{i\neq j} \xi_i\xi_j \log \frac{1}{|x_i - x_j|} \sim Na. \tag{6.2}$$

The typical configurations $(x_1,\ldots,x_N)$, for large N, have the renormalized empirical measure

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \xi_i \delta_{x_i}$$

close to white noise conditioned to renormalized energy equal to a. In Fig. 1, we show the histogram of the interaction energy of 200 point vortices (its features do not change by increasing the number of vortices). It shows the typical values of "a" in formula (6.2). They are very small, and the corresponding configurations are quite disordered, as opposed to the structures of the Onsager theory and consistently with the white noise limit. The theoretical energy spectrum of the *free* white noise ensemble (not constrained by the energy) decays as k^{-1}, as oppose to the predicted decay $k^{-5/3}$ of the inverse stationary 2D turbulence. The question then is the decay of the spectrum for the microcanonical ensemble, especially for large values of a, when we expect some degree of clustering of the vortices and then, potentially, the emergence of a more interesting spectrum.

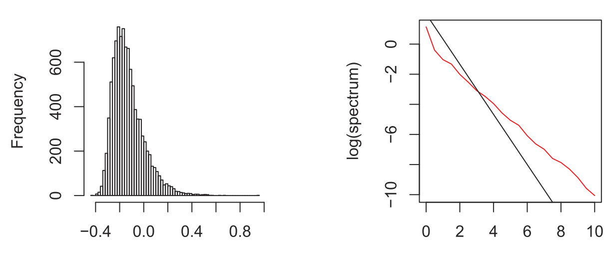

FIG. 1. Left: histogram of the interaction energies of 200 uniformly distributed point vortices over 10 000 samples. Right: the curve is the spectrum function computed from the 11 samples with largest interaction energies, while the straight line shows the reference slope −5/3.

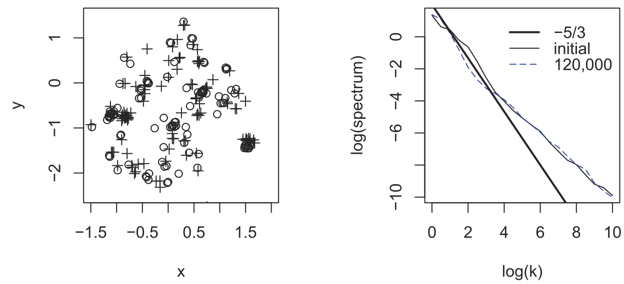

FIG. 2. Left: the configuration of (most of) vortices after 120 000 steps of evolution. Right: spectrum functions before and after running dynamics.

It is very difficult to compute theoretically the spectrum function of the microcanonical measure (1.8); thus, we do some numerical simulations. We generate 10 000 samples of uniformly distributed point vortices; each sample consists of 200 vortices, half of which have intensity $1/\sqrt{200}$ and the rest $-1/\sqrt{200}$. We single out the 11 samples which have the largest interaction energies ($a = 0.51$ in this case) and compute their average spectrum function. The results are shown in Fig. 1. It shows that the slope of the spectrum is still very close to -1 like in the case of the free ensemble, far from $-5/3$.

Deviations of the spectrum slope from the flat value -1 are due to the clustering of point vortex configurations. To prove this claim numerically, we proceed as follows: we produce artificially an initial condition with small clusters and then let it evolve by point vortex dynamics. We do not have a theorem of convergence to equilibrium but hope that after some time, the configuration is more typical for the microcanonical ensemble. Precisely, we generate a point vortex configuration which, apart from some uniformly distributed point vortices, contains small clusters with 2, 4, and 8 vortices (these numbers are chosen for convenience). The clusters have uniformly distributed centers, and their diameters are of the order 0.01. To get a smoother spectrum function, we produce 10 such samples (with average energy 1.364 966) and compute the averaged spectrum function, which is shown by the thin solid line on the right of Fig. 2. We see that it is close to the line with slope $-5/3$ in a certain range of $\log(k)$. We take these special configurations as initial conditions and run the dynamics (use Heun's method, cf. Ref. 25, p. 266), with a small time step $h = 0.0001$. In Fig. 2, we show the vortex distribution of one of the samples after 120 000 steps of evolution: + and ∘ represent vortices of positive and negative intensity, respectively. The graph of the final spectrum function is shown by the dashed line on the right of Fig. 2, which, on the range $\log(k) \in [1, 3]$, has the approximative slope -1.775. Compared to the cases considered in Fig. 1, here we find a slope considerably different from -1 and in the direction of $-5/3$.

The question then is how to obtain spontaneously some degree of local clusterization from an invariant measure and, in particular, from a microcanonical ensemble. Compared to turbulence, it seems that the two regimes of the Onsager theory and the conditional white noise are two "extremes." Turbulence is in the middle: typical configurations are not so uniformly distributed as in the white noise case, and they have locally a great degree of clustering, but only locally, at small scales, not globally as the two big clusters of the Onsager case. Thus, in the turbulence regime, we expect that each vortex interacts neither with all those of the same sign [as in (6.1)] nor only with very few of the same sign [as in (6.2)] but with an intermediate amount.

A natural microcanonical condition is therefore

$$\sum_{i\neq j} \xi_i \xi_j \log \frac{1}{|x_i - x_j|} \sim c(N)e$$

for some

$$N \ll c(N) \ll N^2.$$

The mathematical question then is whether it is possible to study the limit as $N \to \infty$ of this intermediate regime. For finite N, the microcanonical measure with normalizing constant $c(N)$ is invariant for Euler dynamics, but we do not know a corresponding invariant measure obtained as $N \to \infty$. We leave this question open but hope the clarifications of this work help address the question.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referee for valuable comments on an earlier version of the paper. They are also grateful to the referee and Nicola De Mitri for their help in improving the presentation of the paper. The second author thanks the financial support of the grant "Stochastic models with spatial structure" at the Scuola Normale Superiore di Pisa, the National Natural Science Foundation of China (Grant Nos. 11571347 and 11688101), and the Youth Innovation Promotion Association, CAS (Grant No. 2017003).

REFERENCES

[1] S. Albeverio, M. Ribeiro de Faria, and R. Høegh-Krohn, "Stationary measures for the periodic Euler flow in two dimensions," J. Stat. Phys. **20**, 585–595 (1979).
[2] S. Albeverio and B. Ferrario, "Some methods of infinite dimensional analysis in hydrodynamics: An introduction," in *SPDE in Hydrodynamic: Recent Progress and Prospects*, CIME Lectures, edited by G. Da Prato and M. Röckner (Springer-Verlag, Berlin, 2008).

[3] G. Benfatto, P. Picco, and M. Pulvirenti, "On the invariant measures for the two-dimensional Euler flow," J. Stat. Phys. **46**(3-4), 729–742 (1987).
[4] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley Series in Probability and Statistics (A Wiley-Interscience Publication; John Wiley & Sons, Inc., New York, 1999).
[5] F. Bouchet and A. Venaille, "Statistical mechanics of two-dimensional and geophysical flows," Phys. Rep. **515**(5), 227–295 (2012).
[6] E. Caglioti, P. L. Lions, C. Marchioro, and M. Pulvirenti, "A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description," Commun. Math. Phys. **143**(3), 501–525 (1992).
[7] E. Caglioti, P. L. Lions, C. Marchioro, and M. Pulvirenti, "A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description. Part II," Commun. Math. Phys. **174**(2), 229–260 (1995).
[8] F. Cipriano, "The two-dimensional Euler equation: A statistical study," Commun. Math. Phys. **201**(1), 139–154 (1999).
[9] G. L. Eyink and H. Spohn, "Negative-temperature states and large-scale, long-lived vortices in two-dimensional turbulence," J. Stat. Phys. **70**(3-4), 833–886 (1993).
[10] F. Flandoli, "Weak vorticity formulation of 2D Euler equations with white noise initial condition," Commun. Partial Differ. Equations **43**, 1102–1149 (2018).
[11] F. Flandoli and D. Luo, "Convergence of transport noise to Ornstein–Uhlenbeck for 2D Euler equations under the enstrophy measure," Ann. Probab. (to be published).
[12] F. Flandoli and M. Saal, "mSQG equations in distributional spaces and point vortex approximation," e-print arXiv:1812.05361v1.
[13] C. Geldhauser and M. Romito, "Limit theorems and fluctuations for point vortices of generalized Euler equations," e-print arXiv:1810.12706.
[14] F. Grotto, "Stationary solutions of damped stochastic 2-dimensional Euler's equation," e-print arXiv:1901.06744.
[15] F. Grotto and M. Romito, "A central limit theorem for Gibbsian invariant measures of 2D Euler equation," e-print arXiv:1904.01871v1.
[16] A. I. Khinchin, *Mathematical Foundations of Statistical Mechanics*, translated by G. Gamow (Dover Publications, Inc., New York, NY, 1949).
[17] P. L. Lions, *On Euler Equations and Statistical Physics*, [Galileo Chair] Scuola Normale Superiore (Classe di Scienze, Pisa, 1998).
[18] P. Malliavin, "Universal Wiener space," in *Barcelona Seminar on Stochastic Analysis: St. Feliu de Guxols, 1991*, Progress in Probability Vol. 32 (Birkhäuser, Basel, 1993), pp. 77–102.
[19] D. Nualart, *The Malliavin Calculus and Related Topics*, 2nd ed. (Springer-Verlag Berlin Heidelberg, 2006).
[20] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Volume 96 of Applied Mathematical Sciences (Springer-Verlag, New York, 1994).
[21] C. Neri, "Statistical mechanics of the N-point vortex system with random intensities on a bounded domain," Ann. Inst. H. Poincaré Anal. Non Linéaire **21**(3), 381–399 (2004).
[22] S. Schochet, "The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation," Commun. Partial Differ. Equations **20**(5-6), 1077–1104 (1995).
[23] B. Simon, *The $P(\phi)_2$ Euclidean (quantum) Field Theory*, Princeton Series in Physics (Princeton University Press, Princeton, NJ, 1974).
[24] H. Touchette, R. S. Ellis, and B. Turkington, "An introduction to the thermodynamic and macrostate levels of nonequivalent ensembles," Physica A **340**(1-3), 138–146 (2004).
[25] W. Y. Yang, W. Cao, T.-S. Chung, and J. Morris, *Applied Numerical Methods Using MATLAB* (Wiley-Interscience; John Wiley & Sons, Hoboken, NJ, 2005).