Dipartimento di Economia
Facoltà di Economia
Università degli Studi di Parma
Via Kennedy, 6
43100 PARMA

M.E. Mancino[1], *S. Ogawa*[2], *S. Sanfelici*[3]

A numerical study of the smile effect in implied volatilities induced by a nonlinear feedback model

WP 1/2004

Serie: Sezione Matematica

Gennaio 2004

[1]Maria Elvira Mancino, DIMAD, Università di Firenze, Italy.
[2]Shigeyoshi Ogawa, Dept. of Mathematical Sciences, Ritsumeikan University, Japan.
[3]Simona Sanfelici, Dipartimento di Economia, Università di Parma, Italy.

1 Introduction

The Black-Scholes model requires that the volatility is constant. Nevertheless this hypothesis is recognized to cause option price distortions. A well know discrepancy between Black-Scholes predicted European option prices and market-traded option prices is expressed by the *smile* curve. Define the so-called "implied volatility" corresponding to a given option price $u(t, S_t, K, T)$ in the following way: setting $u(t, S_t, K, T) = u_{BS}(t, S_t, K, T, \sigma_{impl}(t, S_t, K, T))$, where u_{BS} is the price of an European call option in the Black-Scholes setting as function of the "unknown" volatility, then by numerical inversion of this formula the implied volatility σ_{impl} is defined. If market prices were computed by the Black-Scholes formula, then σ_{impl} would be a constant function. This is not the case: in fact it is seen that implied volatilities of market prices vary with the strike price and the time to maturity of the contract. In order to give explanation to this behavior the literature has proposed both of modelling directly volatility as a stochastic process either of relaxing the hypothesis on which Black-Scholes model is based, such as the facts that the market is complete, frictionless and that the agents act as price takers.

In this paper we follow the approach proposed by [Platen and Schweizer, 1998] which concentrates on the fact that the price is defined in equilibrium with some traders employing a trading strategy to hedge a contingent claim. The authors handle the problem by assuming that program traders define a trading strategy imposing that the asset price is described by a lognormal stochastic process, i.e., through the Black and Scholes model. They derive the SDE satisfied by the price process and show that looking at the implied volatilities it is observed a strong smile effect.

[Mancino and Ogawa, 2003] generalize the result of [Platen and Schweizer, 1998] deriving from the equilibrium condition the SDE satisfied by the price process when the Δ-hedging strategy which affects the price process is based on the true model with the feedback effects. As a consequence they get a fully nonlinear partial differential equation. The authors prove existence and uniqueness of the solution to the nonlinear partial differential equation satisfied by the price (and the hedging strategy) of the derivative. In this paper we complement the analysis of [Mancino and Ogawa, 2003] by presenting numerical results which show the impact of the hedging strategies for derivative asset analysis. In particular we obtain the smile pattern of implied volatility. The numerical results show that in our model the implied volatility smile can be reproduced as a consequence of dynamical hedging. The simulations are performed using the Infinite Elements Method [Sanfelici, 2003].

2 The model

First we recall the economic model for our empirical analysis, which has been derived in [Mancino and Ogawa, 2003]. The method consists in obtaining the asset prices dynamics implicitly from the equilibrium condition of market clearing. This approach follows from [Follmer and Schweizer, 1993]. There is only one risky asset, its price at time t is S_t. For simplicity we suppose that the riskless asset has zero interest rate, so that S_t is the discounted stock price.

Stock price is determined by the equilibrium condition

$$D(t, U_t, S_t) = constant, \qquad \text{for } t \geq 0$$

where D is a given function which can be inverted and has continuous first and second partial derivatives in U and S. Moreover $\frac{\partial D}{\partial S} \neq 0$. We specify the demand function in the following way:

$$D(t, U_t, S_t) = U_t + \gamma \log(\frac{S_t}{S_0}) + F(\Delta(t, S_t))$$

where γ is a constant $\neq 0$, U_t represents a random error term

$$U_t = \rho W_t + mt, \tag{1}$$

where W_t is a standard one-dimensional Brownian motion, $m \in \mathbb{R}$ and $\rho \neq 0$, and the term $F(\Delta(t, S_t))$ represents the hedging strategy component. $\Delta(t, S_t)$ is the "sensitive" called *delta*, which is defined in the following way: let $\tilde{P}$ be the risk neutral probability measure, given the price of the derivative whose payoff is $f(S(1))$

$$u(t, x) = \tilde{E}[f(S(1))|S_t = x],$$

the Δ is defined as

$$\Delta(t, x) = \frac{\partial u}{\partial x}(t, x).$$

Our model is related to [Platen and Schweizer, 1998]. The relevant difference relies in the fact that their analysis concerns the case where u is the price for an

European call option, when the hedger is using a geometric Brownian motion for S_t, that is the Δ-hedging strategy considered is obtained as if the hedgers worked within the Black-Scholes world with a priori fixed constant volatility. Our case is the limiting case of [Platen and Schweizer, 1998] in the following sense: their procedure derive a model for the price $S^{(1)}$ from a model $S^{(0)}$ with constant volatility, where the second model takes into account the hedging effects induced by the old model. Clearly these procedure can be iterated obtaining a price process $S^{(k)}$ from a model $S^{(k-1)}$ increasing the accuracy. Our model corresponds to the fixed point of these sequence of processes $S^{(k)}$.

The stochastic differential equation satisfied by S_t is derived from the equilibrium condition in [Mancino and Ogawa, 2003] and when F is a linear function, $F(\Delta) = \alpha\Delta + \beta$ for positive constants α and β, the authors find that S_t satisfies

$$dS_t = \tilde{\mu}(t, S_t, u_{xx})dt + \tilde{\sigma}(t, S_t, u_{xx})dW_t \tag{2}$$

where

$$\tilde{\sigma}(t, S_t, u_{xx}) = -\frac{\rho S_t}{\gamma + S_t \alpha u_{xx}}$$

and

$$\tilde{\mu}(t, S_t, u_{xx}) = \frac{\tilde{\sigma}(t, S_t, u_{xx})}{(\gamma + S_t \alpha u_{xx})^2}[m\rho(\gamma + S_t \alpha u_{xx}) - \frac{1}{2}\rho\gamma].$$

Note that the drift and the diffusion coefficients depend on the second derivative of u. The price u of the derivative security satisfies the nonlinear Cauchy problem:

$$\begin{cases} \partial_t u + \frac{1}{2}(\frac{\rho x}{\gamma + \alpha x u_{xx}})^2 u_{xx} = 0 \\ u(1, x) = f(x). \end{cases} \tag{3}$$

The parameter α controls the size of the perturbation induced by the hedging strategy; in fact if $\alpha = 0$ the asset price simply follows a Black-Scholes model, with appropriate constant coefficients.

The proof of existence and uniqueness of the Cauchy problem (3) in a suitable functions space is given in [Mancino and Ogawa, 2003] for the following choice of the constants: $\gamma < 0$, $\alpha > 0$, $\rho \neq 0$ and for $\lambda := \frac{\alpha}{-\gamma}$ smaller than a particular $\overline{\lambda}$ depending on f. The result is proved applying the theory in [Ladyzenskaja, Solonnikov and Ural'ceva, 1968].

3 The Numerical method

The numerical computation of the solution to (3) is performed using the *Infinite Element Method* (IEM). This method has been described and analyzed in

[Sanfelici, 2003] for the Black-Scholes model. It is a very simple and efficient modification of the more common Finite Element Method (FEM). It keeps the best features of FEM, i.e. bandedness, easiness of programming, accuracy and it is an efficient automatic procedure for recovering the correct far-field behavior of the option price and greeks and for avoiding *ad hoc* boundary condition specification for PDEs defined on unbounded domains. Moreover, the effective rate of convergence on compact sets of the IEM is the same as for FEM.

From a practical point of view, the IE method performs a discretization of the unbounded domain $\mathbb{R}^+$ by means of a standard finite element decomposition of a bounded subdomain, say $[0, S_{max}]$, plus a final infinite element $[S_{max}, +\infty)$. This leads to a numerical approximation method having nearly the same computational cost as the ordinary FEM. Nevertheless, it may even reduce effective computational cost, because smaller computational domains $[0, S_{max}]$ can be considered, since no approximated boundary conditions are imposed on S_{max}.

The FE method is not based on the strong pointwise differential formulation (3), but on a weaker integral formulation requiring less regularity for the unknown function u. First we remark that (3) has *perturbed ellipticity*, as its diffusion coefficient vanishes for $x \to 0$. Consequently, the appropriate spaces for studying problem (3) are the *weighted Sobolev spaces* ([Kufner, 1980], p. 9). In particular, the weights must be conveniently chosen in order to assure integrability conditions over $[0, +\infty)$.

The Infinite Element discretization can be performed in different steps. First, we set $\tilde{w} := u_x$ and transform the problem (3) into

$$\begin{cases} \partial_t \tilde{w} + \partial_x \left(\frac{1}{2} (\frac{\rho x}{\gamma + \alpha x \tilde{w}_x})^2 \tilde{w}_x \right) = 0 \\ \tilde{w}(1, x) = f'(x). \end{cases} \tag{4}$$

After the usual change of variable $w(t, x) = \tilde{w}(1-t, x)$, we modify (4) into the initial value problem

$$\begin{cases} \partial_t w = \partial_x \left(\frac{1}{2} (\frac{\rho x}{\gamma + \alpha x w_x})^2 w_x \right) \\ w(0, x) = f'(x). \end{cases} \tag{5}$$

Then, we multiply equation (5) by a suitable decaying positive weight ω_μ in $W^{1,2}(\mathbb{R}^+)$, depending on a parameter $\mu > 0$. Since stock prices can be thought as varying in a fixed sufficiently large range $[0, S_{max}]$, we can consider weights of the form $\omega_\mu(x) = 1$ for $0 \leq x \leq S_{max}$ and $\omega_\mu(x) = (S_{max}/x)^\mu$ for $x > S_{max}$ ($\mu \geq 2$); nevertheless, a C^1 and fast decaying weight can improve the accuracy of the numerical results.

Loosely speaking, the IEM performs a *weighted residue projection* over suitable finite dimensional functional spaces $\mathcal{W}_h$. The construction of the IE space is the following [Bettess, 1992]. We consider a (not necessarily uniform) decomposition $0 = S_0 < S_1 < \ldots < S_{N_h} = S_{max}$ of the interval $[0, S_{max}]$, where $h > 0$ is the maximum length of the intervals in the decomposition. Moreover, we consider the semi-infinite interval $[S_{max}, +\infty)$. The standard FE method is applied to the bounded region $[0, S_{max}]$. Hence, the approximation spaces $\mathcal{W}_h$ are made of continuous functions whose restrictions to $[0, S_{max}]$ are piecewise polynomials of degree less then or equal to k. The *shape* functions on the infinite element are obtained by mapping it onto the reference interval $[0, 1]$ and by using the standard Lagrangian type shape functions $\hat{\varphi}_i^{(k)}(\xi)$ of degree k in the local coordinates of the reference element. If $\hat{F}$ denotes the elemental mapping from $[0, 1)$ onto $[S_{max}, +\infty)$, then the transformed shape functions in the "physical" coordinate system over the infinite element are given by $\varphi_i^{(k)}(x) = \hat{\varphi}_i^{(k)}(\hat{F}^{-1}(x))$.

After the weighted residue projection is performed, we get the following system of ordinary differential equations

$$\frac{d}{dt}(w_h(t), \varphi_i^{(k)})_\mu + \mathcal{A}_\mu^t(w_h(t), \varphi_i^{(k)}) = 0, \qquad i = 1, \ldots, N_h + k \tag{6}$$

$$w_h(0) = w_{0,h},$$

where we have set

$$\mathcal{A}_\mu^t(w, \varphi) := \int_{\mathbb{R}+} \frac{1}{2}\,\omega_\mu^2(x) \left(\frac{\rho x}{\gamma + \alpha x w_x}\right)^2 w_x \varphi_x dx + \int_{\mathbb{R}+} \omega_\mu(x)\omega_\mu'(x) \left(\frac{\rho x}{\gamma + \alpha x w_x}\right)^2 w_x \varphi\, dx$$

and $w_{0,h}$ is a suitable approximation in $\mathcal{W}_h$ of the discontinuous datum f', for instance the L_μ^2-projection of f' onto $\mathcal{W}_h$. This projection acts as a sort of smoothing and needs to be done especially in the presence of discontinuous initial data, such as for the European option case, in order to guarantee optimal order convergence. Note that the usual piecewise linear interpolation of f' is not in general convenient or even possible for discontinuous data, because it is not clear which values have to be interpolated at the discontinuity points.

The nonlinearity is dealt with using a semi-implicit time discretization approach, which consists in evaluating the nonlinear coefficient in the bilinear form $\mathcal{A}_\mu^t(\cdot, \cdot)$ at time t_n, whereas the remaining parts are considered at time t_{n+1}. Thus, after a decomposition $0 = t_0 < t_1 < \ldots < t_M = T$ of the interval $[0, T]$, with $\Delta t_n = t_{n+1} - t_n$,

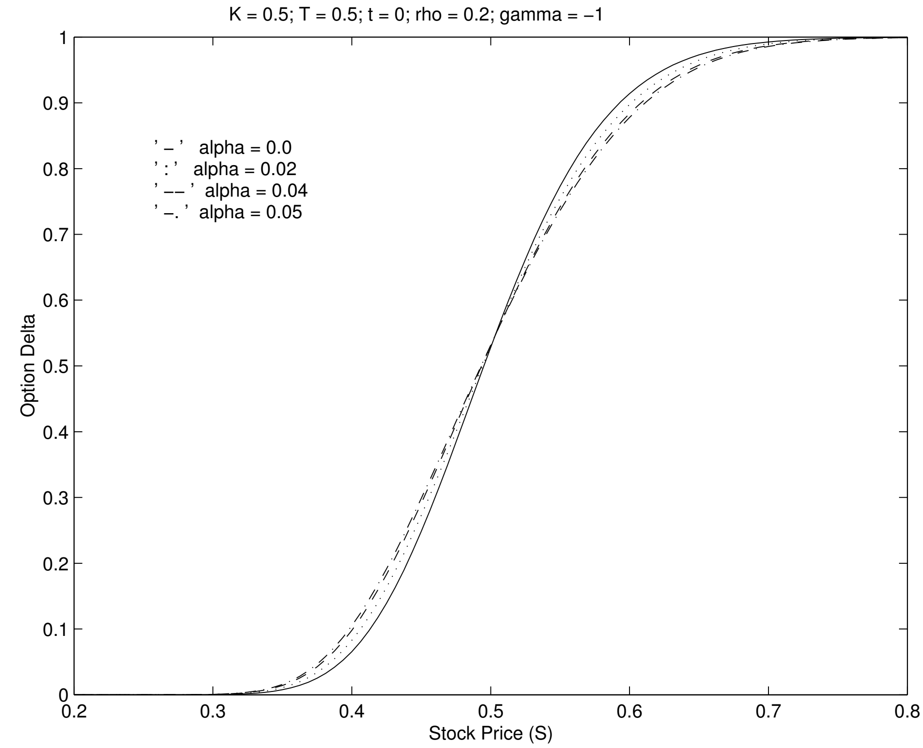

Figure 1: Option delta. Parameter values: $\rho = 0.2$, $\gamma = -1$, $K = 0.5$, $T = 0.5$, $t = 0$, $\alpha = 0,\ 0.2,\ 0.4,\ 0.5$. Discretization parameters: $\Delta t = 0.01$ and $h = 0.0063$.

we get the following totally discretized scheme

$$(w_h^{n+1} - w_h^n, \varphi_i^{(k)})_\mu + \Delta t_n\ \mathcal{A}_\mu^n(w_h^{n+1}, \varphi_i^{(k)}) = 0, \qquad i = 1, \ldots, N_h + k$$

$$w_h^0 = w_{0,h},$$

where

$$\mathcal{A}_\mu^n(w, \varphi) := \int_{\mathbb{R}^+} \frac{1}{2}\,\omega_\mu^2 \left(\frac{\rho x}{\gamma + \alpha x w_x(t_n)}\right)^2 w_x \varphi_x dx + \int_{\mathbb{R}^+} \omega_\mu \omega_\mu' \left(\frac{\rho x}{\gamma + \alpha x w_x(t_n)}\right)^2 w_x \varphi\, dx.$$

4 Numerical results

In this paper we evaluate the capability of the above model of addressing some phenomena observed in financial markets. In particular following [Platen and Schweizer, 1998] we can evaluate (i) the effect of the hedging strategies on the price process, the *hedge ratio* $u_x(t, x)$ (i.e. the delta) and the *gamma* $u_{xx}(t, x)$, (ii) the volatility effect, (iii) the smile-skewness effect.

(i) Figure 1 shows the hedge ratio for an European call option for the parameter values $\rho = 0.2$, $\gamma = -1$, $K = 0.5$, $T = 0.5$ year, $t = 0$ and for varying α. Qualitatively the curves are similar to the Black-Scholes case ($\alpha = 0$). Nevertheless, in the

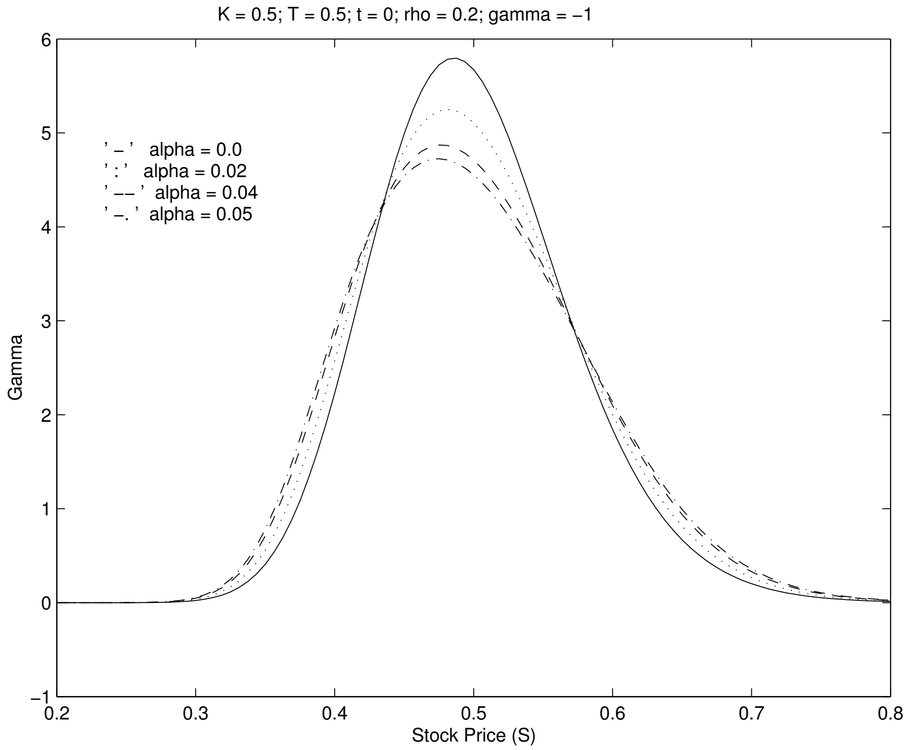

Figure 2: Option gamma. Parameter values: $\rho = 0.2$, $\gamma = -1$, $K = 0.5$, $T = 0.5$, $t = 0$, $\alpha = 0,\ 0.2,\ 0.4,\ 0.5$. Discretization parameters: $\Delta t = 0.01$ and $h = 0.0063$.

nonlinear case it is possible to point out that the hedge ratio is increasing in α when $S < K$ and decreasing when $S > K$. Figure 2 shows the behavior of the gamma for different values of α. Notice that as α increases, then the maximum of gamma decreases.

(ii) We suppose that the parameters are $\gamma < 0$ and $\alpha > 0$: the first one represents the way the agents react to changes in stock prices and the second one measures the feedback impact on the price per traded share. We have

$$\tilde{\sigma}(x, u_{xx}) = \frac{\frac{\rho}{-\gamma} x}{1 - x \frac{\alpha}{-\gamma} u_{xx}}, \qquad \sigma(x) = \frac{\rho}{-\gamma} x$$

where $\tilde{\sigma}$ is the modified volatility under the feedback effect and σ is the volatility of the price process without the effect of the *delta*-hedging strategy. It is easy to see that the volatility is increased if the trader uses a positive feedback strategy which calls for additional buying if the stock price rises, that is if $u_{xx} > 0$. This is our case. Therefore the volatility $\tilde{\sigma}(x, u_{xx})$ is bounded below by $\sigma(x)$; moreover $\tilde{\sigma}(x, u_{xx})$ is increasing in the parameter α, as it can be seen in Figure 3. This behavior is typical for hedging strategies for derivatives with a convex terminal payoff such as European call or put option, see [El Karoui, Jeanblanc-Piqué and Shreve, 1998].

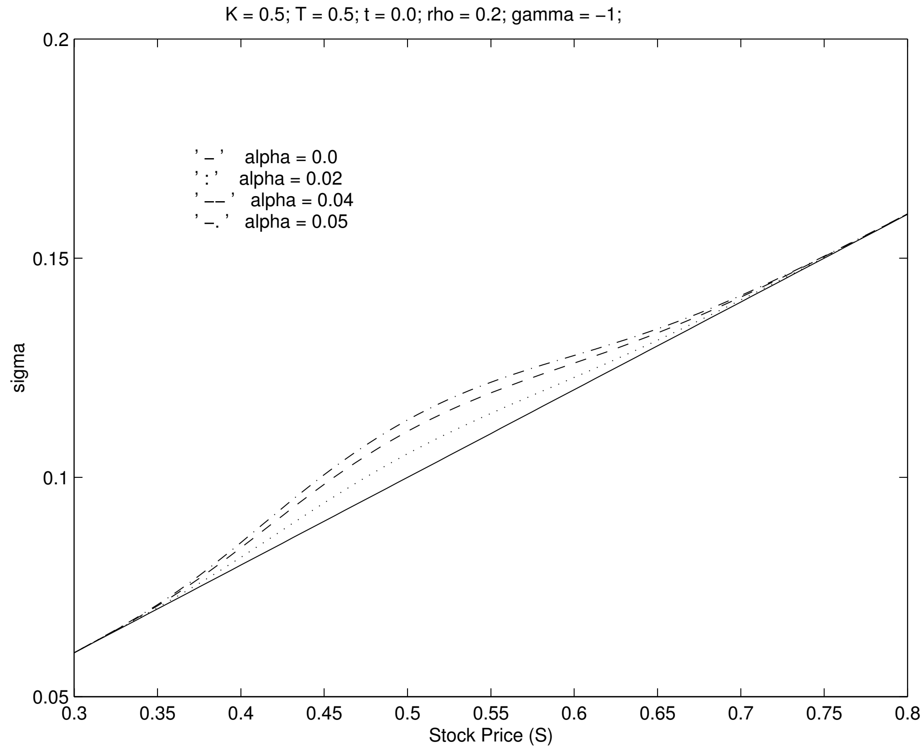

Figure 3: Modified volatility. Parameter values: $\rho = 0.2$, $\gamma = -1$, $K = 0.5$, $T = 0.5$, $t = 0$, $\alpha = 0,\ 0.2,\ 0.4,\ 0.5$. Discretization parameters: $\Delta t = 0.01$ and $h = 0.0063$.

(iii) Finally we verify the smile-skewness effect. Define the so-called "implied volatility" corresponding to a given option price $u(t, S_t, K, T)$ computed with our model, setting

$$u(t, S_t, K, T) = u_{BS}(t, S_t, K, T, \sigma_{impl}(t, S_t, K, T)) \tag{7}$$

where u_{BS} is the price of an European call option in the Black-Scholes setting, as function of the "unknown" volatility. We solve the nonlinear equation (7) in the unknown σ_{impl} by the Newton's method, the secant method or by the bisection algorithm, which turns out to be more robust for far-out-the-money or deep-in-the-money options.

Following [Platen and Schweizer, 1998], in Figure 4 we show that for fixed t, S_t the surface defined by the mapping $(T, K) \mapsto \sigma_{impl}(t, S_t, T, K)$ exhibits a strong "smile effect". The parameter values are $\rho = 0.2$, $\gamma = 0.5$, $\alpha = 0.05$, $S_t = 100$, $t = 0$ and the contract is an European call option; the parameter T may be interpreted either as expiration date or as time to maturity and ranges between $T = 0.1$ and $T = 1$ year, while the strikes K range between $K = 80$ and $K = 130$. Although the numerical results are quite different from the ones in [Platen and Schweizer, 1998],

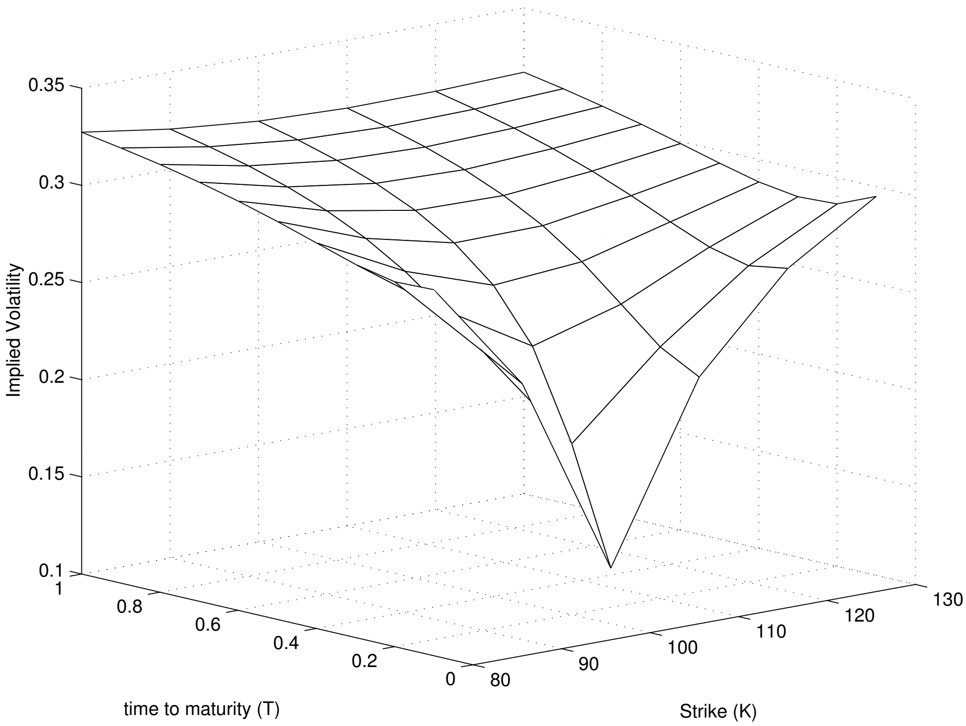

Figure 4: Implied volatility. Parameter values: $\rho = 0.2$, $\gamma = 0.5$, $\alpha = 0.05$, $S_t = 100$, $t = 0$; $T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$; $K = 80, 90, 100, 110, 120, 130$.

we observe some qualitative features in common, which are supported by empirical findings: strikes at-the-money have lower implied volatility than those in or out of the money and the difference decreases with increasing time to maturity. Moreover, in our simulations the curves $K \mapsto \sigma_{impl}(t, S_t, T, K)$, for fixed T, are not always convex and for large values of T they tend to become monotone decreasing. This is the so called "skewness effect": implied volatilities tend to rise more for decreasing than for increasing strike prices and, for large T, the implied volatility curves are not centered around $K = S_t$ but rather around a larger strike.

Thus, we conclude that our new pricing model has the potential to reproduce the implied volatility smile-skewness effect observed in real markets as a consequence of dynamical hedging. Our next step will be the comparative analysis of market data in order to validate our model.

References

[Bettess, 1992] Bettess, P. (1992). Infinite Elements. Penshaw Press, UK.

[Black, 1986] Black, F. (1986). "Noise". *Journal of Finance*, 41/3, 529–543.

[Brennan and Schwartz, 1989] Brennan, M. and Schwartz, E. (1989) Portfolio insurance and Financial market equilibrium. *Journal of Business*, 62:455–472.

[Cox and Huang, 1989] Cox, J. and Huang, C. (1989). Optimal consumption and portfolio Policies when asset prices Follow a Diffusion Process. *Journal of Economic Theory*, 49:33–83.

[Cvitanic and Ma, 1996] Cvitanić, J. and Ma, J. (1996). Hedging Options for a Large Investor and Forward-backward SDE's. *Annals of Applied Probability*, 6 :370–398.

[El Karoui, Jeanblanc-Piqué and Shreve, 1998] El Karoui, N., Jeanblanc-Piqué, M. and Shreve, S. (1998). Robustness of the Black and Scholes Formula. *Mathematical Finance*, 8, 93–126.

[Follmer and Schweizer, 1993] Föllmer, H. and Schweizer M. (1993). A microeconomic approach to diffusion models for stock prices. *Mathematical Finance*, 3, 1–23.

[Frey, 1998] Frey, R. (1998). Perfect option replication for a large trader. *Finance and Stochastics*, 2, 115–148.

[Frey and Stremme, 1997] Frey, R. and Stremme, A. (1997). Market volatility and feedback effects from dynamic hedging. *Mathematical Finance*, 7(4), 351–374.

[Hofmann, Platen and Schweizer, 1992] Hofmann, N., Platen, E. and Schweizer, M. (1992) Option pricing under incompleteness and stochastic volatility. Math. Finance, 2 , 153–187.

[Kufner, 1980] Kufner, A. (1980), Weighted Sobolev spaces. Teubner-Texte zur Mathematik, 31, BSB B. G. Teubner.

[Ladyzenskaja, Solonnikov and Ural'ceva, 1968] Ladyzenskaja, O.A., Solonnikov, V.A. and Ural'ceva, N.N. (1968). *Linear and quasilinear equation of parabolic type.* American Mathematical Society, Providence, Rhode Island.

[Liu and Yong, 2002] Liu, H. and Yong, J. (2002). Option Pricing with an Illiquid Underlying Asset Market. Recent Developments in Mathematical Finance, ed. J. Yong, World Scientific, Singapore, 249–262.

[Mancino and Ogawa, 2003] Mancino, M.E. and Ogawa, S. (2003). Nonlinear feedback effects by hedging strategies. *Stochastic Processes and Applications to Mathematical Finance, Proceedings of 2003 Symposium at Ritsumeikan University*, World Scientific.

[Platen and Schweizer, 1998] Platen, E. and Schweizer M. (1998). On feedback effects from hedging derivatives. *Mathematical Finance*, 8:63–72.

[Sanfelici, 2003] Sanfelici S. (2003). Galerkin infinite element approximation for pricing barrier options and options with discontinuous payoff. *To appear in Decision in Economics and Finance.*

[Sircar and Papanicolaou, 1998] Sircar, K.R. and Papanicolaou, G. (1998). General Black-Scholes models accounting for increased market volatility from hedging strategies. *Applied Mathematical Finance*, 5, 45–82.

1 del D.L. 31 agosto 1945 n. 660.

Stampato in proprio dal Dipartimento di Economia dell'Università degli Studi di Parma in Via J.F. Kennedy 6, 43100 Parma, adempiuti gli obblighi di cui all'articolo 1 del D.L. 31 agosto 1945 n. 660.