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Non-uniqueness of weak solutions for a logarithmically supercritical hyperdissipative Navier-Stokes system

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ABSTRACT

Existence of non-unique solutions of finite kinetic energy for the three dimensional Navier-Stokes equations is proved in the slightly supercritical hyper-dissipative setting introduced by Tao [20]. The result is based on the convex integration techniques of Buckmaster and Vicol [3], and extends Luo and Titi [16] in the slightly supercritical setting. To reach the threshold identified by Tao, we introduce the impulsed Beltrami flows, a variant of the intermittent Beltrami flows of Buckmaster and Vicol.

1. Introduction

In this work we consider the following hyperdissipative Navier-Stokes system on the 3-dimensional torus

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$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nu(-\Delta)^{\theta,\beta} u + \nabla p = 0, \\ \operatorname{div}(u) = 0, \end{cases} \tag{1.1}$$

with periodic boundary conditions and zero spatial mean, where β and θ are non-negative parameters, and the dissipative term is defined via Fourier transform $\mathcal{F}$ by

$$\mathcal{F}[(-\Delta)^{\theta,\beta} u](k) = \mathbf{1}_{\{k \neq 0\}}(k) \frac{|k|^{2\theta}}{\log^\beta(10 + |k|)} \mathcal{F}(u)(k), \qquad \text{for all } k \in \mathbb{Z}^3.$$

Well-posedness of a similar system has been widely studied in the last century. In particular, it is well known that global existence and uniqueness of sufficiently smooth solutions holds when $\beta = 0$ and $\theta \geq \frac{5}{4}$ [15,10].

On the critical threshold $\theta = \frac{5}{4}$, the same result has been proved by Tao [20] and Barbato et al. [2] when the dissipative term is slightly weakened by a Fourier multiplier that mimics a logarithmic behavior raised to a power $\beta \leq 1$.

When $\theta < \frac{5}{4}$ it is well known that there are various open problems, such as uniqueness of Leray-Hopf solutions for classical NSE, i.e. $\theta = 1$ and $\beta = 0$. It is relevant to indicate that a first answer to this problem has been obtained by Albritton et al. [1]. They show the non-uniqueness of Leray-Hopf solutions if the 3-dimensional NSE is perturbed by a certain forcing term.

Even in the unforced setting, various negative results appeared during the last decade. In particular, Luo et al. [16] and Buckmaster et al. [5] proved non-uniqueness of global distributional solutions, less regular than Leray weak solutions [13], by means of the convex integration theory [3,4,6,9,17]. The convex integration machinery has been used by Li et al. [14] in order to prove non-uniqueness in suitable functional spaces also above the Lions exponent, i.e. $\theta \geq \frac{5}{4}$ and $\beta = 0$, but non-uniqueness is proved in the space $C_t L^p$ for all $1 \leq p < 2$, thus in larger spaces than the energy space (and in general below the Ladyzenskaja-Prodi-Serrin threshold).

In this paper, we work at the critical threshold $\theta = \frac{5}{4}$, whose relevance can be also highlighted by making some scaling considerations, see for instance [11] for a more complete discussion on the topic. Under the assumption $\beta = 0$, we recall that a functional space $\mathbf{X}$ is said to be a critical space if $\|u\|_{\mathbf{X}}$ is invariant under the following scaling transformation:

$$u(t,x) \to \lambda^{2\theta-1} u(\lambda^{2\theta} t, \lambda x), \qquad p(t,x) \to \lambda^{4\theta-2} p(\lambda^{2\theta} t, \lambda x).$$

Since its norm represents the kinetic energy, one important functional space is $\mathbf{X} = C_t L^2_x$, which is a critical space only if $\theta = \frac{5}{4}$.

In this work we show that when $\theta = \frac{5}{4}$ the constraint on β identified by Tao is sharp. In particular, the condition $\beta > 1$, i.e. the dissipation is weakened by a sufficiently strong logarithmic corrector, allows to prove a non-uniqueness result for distributional solutions to Eq. (1.1) that are in the relevant functional space $C_t L^2_x$. This non-uniqueness result in a logarithmic supercritical case is obtained by stretching as much as possible the convex integration scheme proposed in [3,16]. In particular from a technical point of view, the result is achieved by proposing a new modification of the classical Beltrami flows, which allows reaching the sharp assumption on β, and by exploiting estimates for smoothly projected Fourier multipliers, which allows to exploit the logarithmic corrector.

The paper is organized as follows. In Section 2 we state and prove the main result by means of the iteration lemma. In Section 3 we introduce and study various Fourier multipliers. Section 4 is devoted to the proof of the iteration lemma, namely the inductive step that solves Eq. (1.1) in the limit.

1.1. Notations

To ease the notations, for any $\alpha \geq 0$ and $p \in [1, \infty]$ we denote

$$C_t L^p_x = C([0,\infty), L^p(\mathbf{T}^3, \mathbb{R}^3)), \qquad L^\infty_t L^p_x = L^\infty([0,\infty), L^p(\mathbf{T}^3, \mathbb{R}^3)),$$

where $\mathbf{T}^3$ denotes the 3-dimensional torus. We denote by $C_w(\mathbb{R}, L^2(\mathbf{T}^3, \mathbb{R}^3))$ the space of weakly continuous functions with values in the Banach space $L^2(\mathbf{T}^3, \mathbb{R}^3)$, while $C_c^\infty(\mathbb{R}, C^\infty(\mathbf{T}^3, \mathbb{R}^3))$ is the space of smooth compactly supported and spatially periodic functions. In addition, if $v \in C_c^\infty(\mathbb{R}, C^\infty(\mathbf{T}^3, \mathbb{R}^3))$ we denote its time support by $\mathrm{Supp}_t(v)$. For any sub-interval $I \subset \mathbb{R}_+$, let $\mathbf{P}_I$ be the operator defined by

$$\mathbf{P}_I g(x) = \sum_{|k| \in I} \mathcal{F}(g)(k) e^{ik\cdot x}.$$

We denote by P_H the Leray-Helmholtz projector, namely the projector onto the space of divergence-free vector fields. For any $A \subset \mathbb{R}$ and $\delta > 0$, let $N_\delta(A)$ be the δ-neighborhood of A, namely

$$N_\delta(A) = \{z \in \mathbb{R} : \text{ there is } y \in A \text{ s.t. } |z - y| < \delta\}.$$

If $x, y \in \mathbb{R}^3$, then $x \times y$ will denote the standard vector cross-product between x and y. Given $a, b \in \mathbb{R}$, we use the notations $a \lesssim b$ and $a \approx b$ if there exist $C, C_1, C_2 > 0$ such that $a \leq Cb$ and $C_1 a \leq b \leq C_2 a$, respectively. Moreover, if C depends on an additional parameter k, then we use $a \lesssim_k b$.

2. Main result

We first recall the definition of weak solution to Eq. (1.1).

Definition 2.1. A vector field $v \in C_w(\mathbb{R}, L^2(\mathbf{T}^3, \mathbb{R}^3))$ is a weak solution to Eq. (1.1) if it is satisfied in a distributional sense, namely,

$$\int_{\mathbb{R}} \int_{\mathbf{T}^3} (\partial_t \phi) \cdot v \, dx \, dt = \int_{\mathbb{R}} \int_{\mathbf{T}^3} v \cdot (v \cdot \nabla)\phi - \big(\nu(-\Delta)^{\theta,\beta}\phi\big) \cdot v \, dx \, dt,$$

for all $\phi \in C_c^\infty(\mathbb{R}, C^\infty(\mathbf{T}^3, \mathbb{R}^3))$ such that $\operatorname{div}(\phi(t,\cdot)) = 0$ for all $t \in \mathbb{R}$.

Theorem 2.2. *For $\theta = \frac{5}{4}$ and $\beta > 1$, consider a smooth divergence-free vector field $u : [0,\infty) \times \mathbf{T}^3 \to \mathbb{R}^3$, with compact support in time and zero spatial-mean,*

$$\int_{\mathbf{T}^3} u(t,x)dx = 0.$$

Then, for all $\epsilon > 0$, there exists a weak solution $v \in C(\mathbb{R}, L^2(\mathbf{T}^3, \mathbb{R}^3))$ of Eq. (1.1) with compact support in time and such that

$$\||\nabla|^{\frac{3}{2}} T_M(v-u)\|_{L_t^\infty L_x^1} \le \epsilon,$$

where T_M and $|\nabla|^{\frac{3}{2}}$ are the operators corresponding to the Fourier multipliers $M(k) = \mathbf{1}_{\{k\neq 0\}}(k)\log^{-\beta}(|k|+10)$ and $m'(k) = |k|^{\frac{3}{2}}$ (see Definition 3.1). In particular, there are infinitely many weak solutions with zero initial condition.

The result is shown by using the classical iterative procedure of the convex integration theory, i.e. following the groundbreaking work of Buckmaster and Vicol [3]. Therefore, the proof is strongly based on the following lemma.

Lemma 2.3 *(Iteration Lemma). Fix $\theta = \frac{5}{4}$ and $\beta > 1$. Let (v_q, p_q, R_q) be a smooth solution to the approximate system*

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nu(-\Delta)^{\frac{5}{4},\beta} u + \nabla p = \operatorname{div}(R), \\ \operatorname{div}(u) = 0, \\ \operatorname{Tr}(R(t,x)) = 0, \qquad R(t,x) = R(t,x)^T \quad \text{for all } t, x, \end{cases} \tag{2.1}$$

such that

$$\|R_q\|_{L_t^\infty L_x^1} \le \delta_{q+1}.$$

Then for every $\delta_{q+2} > 0$, there exists a smooth solution $(v_{q+1}, p_{q+1}, R_{q+1})$ of Eq. (2.1) such that

$$\begin{gathered} \|R_{q+1}\|_{L_t^\infty L_x^1} \le \delta_{q+2}, \\ \operatorname{Supp}_t(v_{q+1}) \cup \operatorname{Supp}_t(R_{q+1}) \subset N_{\delta_{q+1}}(\operatorname{Supp}_t(v_q) \cup \operatorname{Supp}_t(R_q)). \end{gathered} \tag{2.2}$$

Moreover, the following estimates hold

$$\begin{gathered} \|v_{q+1} - v_q\|_{L_t^\infty L_x^2} \le C\delta_{q+1}^{\frac{1}{2}}, \\ \||\nabla|^{\frac{3}{2}} T_M(v_{q+1} - v_q)\|_{L_t^\infty L_x^1} \le \delta_{q+2}. \end{gathered} \tag{2.3}$$

Remark 2.4 *(On the parameter β).* As already mentioned in the introduction, Tao [20] and Barbato et al. [2] proved the existence and uniqueness of smooth solutions to an equation similar to Eq. (1.1), where the dissipative term $D(u)$ is defined via Fourier transform as

$$\mathcal{F}[D(u)](k) = \mathbf{1}_{\{k\neq 0\}}(k)\frac{|k|^{2\theta}}{g(|k|)}\mathcal{F}(u)(k), \qquad \text{for all } k \in \mathbb{Z}^3,$$

for any positive, non-decreasing function g such that $\int_1^\infty \frac{1}{sg(s)}\,ds = \infty$. Even if their result covers a larger family of dissipative terms, the most interesting one is clearly given by $g(s) = \log^\beta(10+|k|)$ with $\beta \le 1$.

Therefore, apart from this structural difference, our result shows the sharpness of the condition $\beta \le 1$. Namely, that weakening the dissipative term by any stronger logarithmic correction produces the existence of infinitely many distributional solutions.

Assuming the previous Lemma, the proof of Theorem 2.2 basically follows the lines of Luo and Titi [16]. We report it for completeness.

Proof of Theorem 2.2. Consider the triple

$$v_0 = u, \qquad p_0 = -\frac{1}{3}|v_0|^2,$$
$$R_0 = \mathcal{R}(\partial_t v_0 + \nu(-\Delta)^{\frac{5}{4},\beta} v_0) + v_0 \otimes v_0 + p_0 I,$$

where $\mathcal{R}$ is the anti-divergence operator, defined below in Definition 4.11.

Given $\delta_1 = \|R_0\|_{L_t^\infty L_x^1}$ and $\delta_q = 2^{-q}\epsilon$, consider the sequence $(v_q, p_q, R_q)_{q\in\mathbb{N}}$ coming from Lemma 2.3. Then,

$$\sum_q \|v_{q+1} - v_q\|_{L_t^\infty L_x^2} \lesssim \sum_q \delta_{q+1}^{\frac{1}{2}} < \infty,$$
$$\|R_q\|_{L_t^\infty L_x^1} \to 0 \qquad \text{as } q \to \infty.$$

Therefore, $(v_q)_q$ converges strongly to a weak solution $v \in C_T L_x^2$. Moreover,

$$\||\nabla|^{\frac{3}{2}} T_M(v-u)\|_{L_t^\infty L_x^1} \le \sum \||\nabla|^{\frac{3}{2}} T_M(v_{q+1} - v_q)\|_{L_t^\infty L_x^1} \le \sum_q \delta_{q+1} \le \epsilon,$$
$$\mathrm{Supp}_t(v) \subset \cup_q \mathrm{Supp}_t(v_q) \subset N_{\sum_q \delta_{q+1}}(\mathrm{Supp}_t(u)) \subset N_{\delta_1+\epsilon}\,\mathrm{Supp}_t(u).$$

Finally, infinitely many weak solutions with zero initial condition can be obtained by considering $u(t,x) = \phi(t)\sum_{|k|\le N} a_k e^{ik\cdot x}$ with $a_0 = 0$, $a_k^* = a_{-k}$, $a_k \cdot k = 0$ and $\phi \in C^\infty(\mathbb{R}_+)$, with $\phi(0) = 0$. $\square$

3. Preliminaries on Fourier multipliers

In this section we mainly prove that the operators associated with various Fourier multipliers are bounded from $L^1(\mathbf{T}^d)$ to itself. We first recall the definition of the operator corresponding to a Fourier multiplier.

Definition 3.1. Given a family $m = (m(k))_{k\in\mathbb{Z}^d}$, its corresponding operator T_m is defined by

$$T_m(f)(x) = \sum_{k\in\mathbb{Z}^d} m(k)\mathcal{F}(f)(k)e^{ik\cdot x}, \qquad \text{for } f \in C^\infty(\mathbf{T^d}),$$

where $\mathcal{F}(f)(k)$ is the k^{th} Fourier coefficient of f.

Before stating the main results of this section, we introduce the averaged Fourier projection.

Definition 3.2. Consider $L > 0$ and $\theta \in C_c^\infty(B_2(0))$ such that $\theta(x) \equiv 1$ on $B_1(0)$. If $f \in C^\infty(\mathbf{T}^d)$, then its averaged Fourier projection is defined by

$$\mathbf{P}^A_{\le L}(f)(x) = \sum_{k\in\mathbb{Z}^d} \theta\left(k/L\right)\mathcal{F}(f)(k)e^{ik\cdot x}. \tag{3.1}$$

The following lemma is a straightforward adaptation of [7, Lemma 2.1] to our framework.

Lemma 3.3. *Let $m = (m(k))_{k\in\mathbb{Z}^d}$ be a bounded multiplier and $M \in C(\mathbb{R}^d \setminus \{0\})$ such that*

1. *$M(k) = m(k)$ for all $k \in \mathbb{Z}^d \setminus \{0\}$,*
2. *M is locally continuous at any $k \in \mathbb{Z}^d \setminus \{0\}$,*
3. *For all $N \in \mathbb{N} \setminus \{0\}$ there exists $C(2^N) > 0$ s.t.*

$$\sup_{\substack{2^{N-3}\le|\xi|\le 2^{N+3},\\ |\alpha|\le d+1}} 2^{N|\alpha|}|\partial_\alpha M(\xi)| \lesssim_d C(2^N).$$

Then for all $f \in L^1(\mathbf{T}^d)$,

$$\|T_m(\mathbf{P}^A_{\le 2^N} - \mathbf{P}^A_{\le 2^{N-1}})f\|_{L^1} \lesssim_d C(2^N)\|f\|_{L^1}. \tag{3.2}$$

Proof. If there exists a Fourier multiplier on $\mathbb{R}^d$ that coincides with $m(k)[\theta(k/2^N) - \theta(k/2^{N-1})]$ on $\mathbb{Z}^d$ and that satisfies the required inequality from $L^1(\mathbb{R}^d)$ to itself, then the result holds (Theorem 3.8, Chapter 7 in [19]). The existence of such a multiplier is due to the assumptions on M and Lemma 2.1 in [7]. □

Proposition 3.4. *Assume that $m = (m(k))_{k\in\mathbb{Z}^d}$ satisfies the assumptions of the previous lemma. Moreover, suppose that*

$$\sum_{N=1}^{\infty} C(2^N) < \infty.$$

Then,

1. *For all $f \in L^1(\mathbf{T}^d)$*

$$\|T_m(I - \mathbf{P}^A_{\leq 2^{N-1}})f\|_{L^1} \lesssim_d \|f\|_{L^1} \sum_{j=N}^{\infty} C(2^j). \tag{3.3}$$

2. *T_m is bounded from $L^1(\mathbf{T}^d)$ to itself.*

Proof. 1. We can assume $f \in L^2(\mathbf{T}^d)$, up to using a density argument. Denote the operator $(I - \mathbf{P}^A_{\leq 2^{N-1}})$ by $\mathbf{P}^A_{>2^{N-1}}$. Notice that

$$\begin{aligned}(T_m\mathbf{P}^A_{>2^{N-1}}f)(x) &= \sum_{k\in\mathbb{Z}^d} [1 - \theta(k/2^{N-1})]m(k)\mathcal{F}(f)(k)e^{ikx} \\ &= \sum_{k\in\mathbb{Z}^d}\sum_{j=N}^{M} [\theta(k/2^j) - \theta(k/2^{j-1})]m(k)\mathcal{F}(f)(k)e^{ikx} \\ &+ \sum_{k\in\mathbb{Z}^d} [1 - \theta(k/2^M)]m(k)\mathcal{F}(f)(k)e^{ikx} =: I_1 + I_2.\end{aligned}$$

Let us estimate the two terms separately. The previous lemma implies that

$$\|I_1\|_{L^1} \lesssim_d \|f\|_{L^1} \sum_{j=N}^{M} C(2^j) \lesssim_d \|f\|_{L^1} \sum_{j=N}^{\infty} C(2^j).$$

While,

$$\|I_2\|_{L^1} \lesssim_d \|I_2\|_{L^2} \lesssim \|f\|_{L^2} \sup_{|k|\geq 2^{M-1}} |m(k)|,$$

which goes to 0 as long as M diverges.

2. Let $h \in \mathbb{N}$, then it follows directly from the result already proved and the following equality

$$(T_m f)(x) = \sum_{k\in\mathbb{Z}^d} [1 - \theta(k/2^h)]m(k)\mathcal{F}(f)(k)e^{ikx} + \sum_{k\in\mathbb{Z}^d} \theta(k/2^h)m(k)\mathcal{F}(f)(k)e^{ikx}. \qquad \square$$

Corollary 3.5. *Consider $\gamma_1, \gamma_2, \ldots, \gamma_d \in \mathbb{N}$ and $\beta > 1$.*

1. *The multiplier* $m(k) = \mathbf{1}_{\{k\neq 0\}}(k)\frac{\prod_{i=1}^d k_i^{\gamma_i}}{|k|^{\sum_{i=1}^d \gamma_i}}\log^{-\beta}(|k|+10)$ *is bounded from* $L^1(\mathbf{T}^d)$ *to itself. Moreover,*

$$\|T_m(I-\mathbf{P}^A_{\leq 2^{N-1}})f\|_{L^1} \lesssim_d \|f\|_{L^1}\log^{1-\beta}(2^N). \tag{3.4}$$

2. *The multiplier* $m(k) = \mathbf{1}_{\{k\neq 0\}}(k)\frac{\prod_{i=1}^d k_i^{\gamma_i}}{|k|^{1+\sum_{i=1}^d \gamma_i}}$ *is bounded from* $L^1(\mathbf{T}^d)$ *to itself. Moreover,*

$$\|T_m(I-\mathbf{P}^A_{\leq 2^{N-1}})f\|_{L^1} \lesssim_d \|f\|_{L^1}2^{-N}. \tag{3.5}$$

Remark 3.6. The projectors $\mathbf{P}^A_{\leq L}$ defined above are the counterpart, on the torus, of Littlewood-Paley projections used when working on Fourier multipliers in $\mathbb{R}^d$ (see for instance [8]). We wish to remark that the seemingly similar projectors defined by

$$\mathbf{P}_{\leq L}f(x) = \sum_{|k|\leq L}\mathcal{F}(f)(k)e^{ikx},$$

produce dramatically different results as multipliers from L^1 to itself. Indeed, $\mathbf{P}_{\leq L}$ produce estimates which are 'morally' worse than those in Proposition 3.4, see a comparison of these projectors in Proposition A.1. We believe that this comparison, apart from being interesting in itself, provides a complete understanding on why our non-uniqueness result and our new building blocks require just $\beta > 1$ with $M(k) = \log^{-\beta}(|k|+10)$.

Remark 3.7. In an earlier version of this paper the term $M(k)$ was replaced by $\log^{-\beta}(|k|_1+10)$. Again, a seemingly similar term produces a dramatically different result in our borderline setting. Indeed, Proposition 3.4 does not apply (because of lack of differentiability) and (worse) a stronger assumption on β is required, as shown in Proposition A.2, see also Remark A.3.

4. Proof of the iteration lemma

This section is devoted to the proof of Lemma 2.3, namely we modify a solution (v_q, p_q, R_q) to Eq. (2.1) in order to obtain a new solution $(v_{q+1}, p_{q+1}, R_{q+1})$ with $\|R_{q+1}\|_{L^\infty_t L^1_x}$ sufficiently small.

4.1. Impulsed Beltrami flows

In this section we build the impulsed Beltrami flows, a modification of the intermittent Beltrami flows used by Buckmaster and Vicol [3], which will have a crucial role in the construction of v_{q+1}.

Consider $\Lambda = \Lambda^+ \cup \Lambda^- \subset \mathbb{Q}^3$, where

$$\Lambda^+ = \Big\{\frac{1}{5}(3e_1 \pm 4e_2), \frac{1}{5}(3e_2 \pm 4e_3), \frac{1}{5}(3e_3 \pm 4e_1)\Big\},$$

and $\Lambda^- := -\Lambda^+$. Here e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^3$. Then, by construction, Λ is a finite symmetric subset of $\mathbb{Q}^3$, with vectors of unit length and such that

$$5\Lambda \subset \mathbb{Z}^3, \qquad \min_{\xi,\xi'\in\Lambda, \xi+\xi'\neq 0} |\xi+\xi'| \geq \frac{1}{5}.$$

Moreover, the following geometric lemma holds:

Proposition 4.1. *([3,4]) Let $B_1(I_3)$ be the ball of symmetric 3×3 matrices centered at I_3 with radius 1. Then, there exist $\epsilon_\Lambda > 0$ and a family $(\gamma_\xi)_{\xi\in\Lambda} \subset C^\infty(B_{\epsilon_\Lambda}(I_3))$ such that*

1. *$\gamma_\xi = \gamma_{-\xi}$;*
2. *for each $R \in B_{\epsilon_\Lambda}(I_3)$,*

$$R = \sum_\xi \frac{1}{2}\gamma_\xi(R)^2 (I_3 - \xi\otimes\xi). \tag{4.1}$$

Now, for any $\xi \in \Lambda$, consider $A_\xi \in \mathbb{Q}^3$ such that

$$A_\xi \cdot \xi = 0, \qquad A_{-\xi} = A_\xi, \qquad |A_\xi| = 1,$$

and define $B_\xi = \frac{1}{\sqrt{2}}(A_\xi + i\xi\times A_\xi)$.

Therefore for any $(a_\xi)_{\xi\in\Lambda} \subset \mathbb{C}$ such that $a_{-\xi} = \overline{a_\xi}$, we can define $W_\xi = B_\xi e^{i\lambda x\cdot\xi}$ and consider the classical Beltrami flows $W = \sum_\xi a_\xi W_\xi$ introduced in [3,4], which have various interesting properties such as

$$\fint_{\mathbf{T}^3} W\otimes W dx = \sum_\xi \frac{|a_\xi|^2}{2}(I_3 - \xi\otimes\xi).$$

The impulsed Beltrami flows are obtained by modifying the classical Beltrami flows with suitable spatial and temporal intermittency. In particular, given $r \in \mathbb{N}$, consider the Fejer kernel

$$F(y) = \frac{1}{r+1}\left(\frac{\sin((r+1)\frac{y}{2})}{\sin\frac{y}{2}}\right)^2 = \sum_{j=-r}^{r}\left(1 - \frac{|j|}{r+1}\right)e^{ijy},$$

for all $y \in \mathbf{T}$, and define the $3D$-Fejer kernel as

$$F_r(x) = (2\pi)^{\frac{3}{2}}\prod_{i=1}^{3} F(x_i). \tag{4.2}$$

Let us summarise some properties of the $1D$-Fejer kernels, which can be easily extended to $3D$-Fejer kernels using Fubini-Tonelli theorem. We point out the crucial fact that, unlike the Dirichlet kernels, the L^1 estimate does not contain logarithmic corrections.

Proposition 4.2. *Let $r \in 2\mathbb{N}^*$ and $n \in \mathbb{N}$, then*

$$\|F\|_{L^1} = 1, \qquad \|F\|_{L^2} \approx r^{1/2}, \qquad \|F\|_{L^\infty} \lesssim r,$$
$$\|\tfrac{d^n}{dy^n} F\|_{L^1} \lesssim r^n, \qquad \|\tfrac{d^n}{dy^n} F\|_{L^2} \lesssim r^{n+\frac{1}{2}}, \qquad \|\tfrac{d^n}{dy^n} F\|_{L^\infty} \lesssim r^{n+1}.$$

Proof. The proof of the first three inequalities can be found in [8]. Consider then the case $n > 0$. The L^2 and L^∞ norms can be easily estimated, respectively, by Fourier coefficients and by the formula

$$\sup_{[-\pi,\pi]} \Big| \frac{d^n}{dy^n} F(y) \Big| \lesssim \sum_{j=1}^{r} (1 - \frac{j}{r+1}) j^n.$$

Instead, the estimate of the L^1 norm follows from the equalities:

$$\frac{d}{dy} F(y) = \frac{2}{r+1} \frac{\sin((r+1)\frac{y}{2})}{\sin \frac{y}{2}} \frac{d}{dy} \left[\frac{\sin((r+1)\frac{y}{2})}{\sin \frac{y}{2}} \right] = \frac{2}{r+1} D_k(y) \frac{d}{dy} D_k(y),$$
$$\frac{d^n}{dy^n} F(y) = \frac{1}{r+1} \sum_{j=0}^{n-1} C(n,j) \Big(\frac{d^j}{dy^j} D_k(y) \Big) \Big(\frac{d^{n-j}}{dy^{n-j}} D_k(y) \Big),$$

where $C(n,j)$ are constants depending on n and j, $k = \frac{r}{2}$ and D_k is the Dirichlet kernel, i.e.

$$D_k(y) = \frac{\sin((k+\frac{1}{2})y)}{\sin(\frac{y}{2})} = \sum_{j=-k}^{k} e^{ijy}.$$

Indeed, the L^1 norm of the derivatives can be estimated by means of the previous equalities, Hölder inequality and since $\|\frac{d^n}{dy^n} D_k(y)\|_{L^2} \approx k^n k^{1/2}$. □

For any $\xi \in \Lambda$, its corresponding impulsed Beltrami flow is defined by

$$\mathbf{W}_\xi = \eta_\xi W_\xi, \tag{4.3}$$

where for every $x \in \mathbf{T}^3$, $t \geq 0$ and $\xi \in \Lambda^+$,

$$\begin{aligned} \eta_\xi(x,t) &= \frac{1}{\|F\|_{L^2}^3} F_r(\lambda\sigma(x \cdot \xi + \mu t, x \cdot A_\xi, x \cdot \xi \times A_\xi)), \\ \eta_\xi &= \eta_{-\xi}, \qquad \text{for every } \xi \in \Lambda^-. \end{aligned} \tag{4.4}$$

The parameters λ, σ, μ and r will be chosen later. In particular, they must satisfy the following conditions,

$$\begin{aligned} &\lambda\sigma r > 1, \qquad \sigma r < \frac{1}{2}, \qquad \sigma\lambda \in 5\mathbb{N}^*, \\ &r^{\frac{3}{2}} < \mu < \lambda^2, \qquad \lambda \in 40\mathbb{N}^*, \qquad r \in 2\mathbb{N}^*, \end{aligned} \tag{4.5}$$

in order to control the $L^\infty_t L^1_x$ norm of the stress tensor R_{q+1} by means of spatial and temporal intermittency, see Section 4.3.

Remark 4.3. The time intermittency has been introduced in order to estimate a specific term in $\operatorname{div}(R_{q+1})$, which is related to

$$\operatorname{div}(\mathbf{W}_\xi \otimes \mathbf{W}_{-\xi} + \mathbf{W}_{-\xi} \otimes \mathbf{W}_\xi) = \nabla \eta_\xi^2 - ((\xi \cdot \nabla)\eta_\xi^2)\xi = \nabla \eta_\xi^2 - \frac{\xi}{\mu}\partial_t \eta_\xi^2.$$

Proposition 4.4. *Assume that the conditions in Eq. (4.5) on the parameters hold, then:*

$$\mathbf{P}_{\leq 2\lambda}\eta_\xi = \eta_\xi, \qquad \mathbf{P}_{[\frac{\lambda}{4}, 3\lambda)}\mathbf{W}_\xi = \mathbf{W}_\xi, \tag{4.6}$$

$$\frac{1}{\mu}\partial_t \eta_\xi = (\xi \cdot \nabla)\eta_\xi, \qquad \text{for all } \xi \in \Lambda^+, \tag{4.7}$$

$$\mathbf{P}_{[\frac{\lambda}{10}, 4\lambda)}\mathbf{W}_\xi \otimes \mathbf{W}_{\xi'} = \mathbf{W}_\xi \otimes \mathbf{W}_{\xi'}, \qquad \text{for } \xi \neq -\xi'. \tag{4.8}$$

Moreover, for all $N, M \in \mathbb{N}$ and $p \in [1, \infty]$,

$$\|\nabla^N \partial_t^M \eta_\xi\|_{L^\infty_t L^p_x} \lesssim (\lambda\sigma r)^N (\lambda\sigma r\mu)^M r^{\frac{3}{2} - \frac{3}{p}}, \tag{4.9}$$

$$\|\nabla^N \partial_t^M \mathbf{W}_\xi\|_{L^\infty_t L^p_x} \lesssim \lambda^N (\lambda\sigma r\mu)^M r^{\frac{3}{2} - \frac{3}{p}}. \tag{4.10}$$

Proof. The equalities in Eq. (4.6), Eq. (4.7), Eq. (4.8) are true by construction. The inequalities in Eq. (4.9), Eq. (4.10) follow from Fubini-Tonelli theorem, Proposition 4.2 and the fact that the map

$$x \mapsto (\lambda\sigma(x \cdot \xi + \mu t, A_\xi \cdot x, \xi \times A_\xi \cdot x))$$

is the composition of three volume preserving maps of $\mathbf{T}^3$. $\square$

Proposition 4.5. *Consider the family of vector fields $(\mathbf{W}_\xi)_{\xi\in\Lambda}$ and a family of complex values $(a_\xi)_{\xi\in\Lambda}$ such that $\overline{a_{-\xi}} = a_\xi$. Then, $\sum_{\xi\in\Lambda} a_\xi \mathbf{W}_\xi$ is a real-valued vector field and for all $R \in B_{\epsilon_\Lambda}(I_d)$,*

$$\sum_{\xi\in\Lambda} \gamma_\xi(R)^2 \fint_{\mathbf{T}^3} \mathbf{W}_\xi \otimes \mathbf{W}_{-\xi} dx = R,$$

where ϵ and the functions $\gamma_\xi(R)$ are given in Proposition 4.1.

Proof. We have that

$$\sum_{\xi\in\Lambda}\gamma_\xi(R)^2 \fint_{\mathbf{T}^3} \mathbf{W}_\xi \otimes \mathbf{W}_{-\xi} dx = \sum_{\xi\in\Lambda}\gamma_\xi(R)^2 B_\xi \otimes B_{-\xi} \fint_{\mathbf{T}^3} \eta_\xi(x,t)^2 dx =$$
$$= \sum_{\xi\in\Lambda}\gamma_\xi(R)^2 B_\xi \otimes B_{-\xi} = \sum_{\xi\in\Lambda^+}\gamma_\xi(R)^2(I_3 - \xi\otimes\xi) =$$
$$= \frac{1}{2}\sum_{\xi\in\Lambda}\gamma_\xi(R)^2(I_3 - \xi\otimes\xi) = R,$$

where the second equality follows from the normalization introduced in Eq. (4.2) and Eq. (4.4), the fifth one follows from the symmetry of Λ and Proposition 4.1, and the last one follows from Eq. (4.1). □

Remark 4.6. The main difference between our building block and the intermittent Beltrami waves presented in [3] is that we introduce spatial intermittency by means of Fejer kernels instead of Dirichlet kernels.

This modification does not affect the most important properties of the building blocks, such as the representation formula or the active Fourier coefficients. At the same time, it allows to prove the non-uniqueness result assuming a weaker condition on the parameter β, namely the sharp requirement $\beta > 1$. Indeed, if η_ξ and $\mathbf{W}_\xi$ are replaced by η_ξ^D and $\mathbf{W}_\xi^D$, defined using Dirichlet kernels instead of Fejer ones, then the following estimates

$$\|\nabla^N \partial_t^M \eta_\xi^D\|_{L_t^\infty L_x^1} \lesssim (\lambda\sigma r)^N (\lambda\sigma r\mu)^M r^{-\frac{3}{2}} \log^3(r),$$
$$\|\nabla^N \partial_t^M \mathbf{W}_\xi^D\|_{L_t^\infty L_x^1} \lesssim \lambda^N (\lambda\sigma r\mu)^M r^{-\frac{3}{2}} \log^3(r),$$

will allow to prove the non-uniqueness result only if $\beta > \frac{17}{2}$.

Remark 4.7. Mikado flows offer another building block in the convex integration literature, see [4,5] for their most common construction. They cannot be deployed in our setting because controlling their active Fourier coefficients is less trivial. In particular, the dissipative term can be estimated as follows

$$\|\mathcal{R}(-\Delta)^{\frac{5}{4},\beta} w\|_{L_t^\infty L_x^1} \lesssim \log^{1-\beta}(\lambda r_\perp)\lambda^{\frac{3}{2}} r_\perp r_\parallel^{\frac{1}{2}},$$

where $r_\perp$ and $r_\parallel$ should satisfy certain constraints, see [4] for a full description. Then, these constraints imply that

$$r_\perp = \frac{\log^x(\lambda)}{\lambda}, \quad r_\perp = \frac{\log^y(\lambda)}{\lambda}$$

in order to cancel out $\lambda^{\frac{3}{2}}$. But the logarithmic terms, $\log^x(\lambda)$ and $\log^y(\lambda)$, cannot be kept small.

4.2. The perturbation

Following the idea proposed by Luo et al. [16], consider a cut-off function ψ that controls the time support of the perturbation, i.e. a function that has the following properties

$$\psi(t) = 1 \text{ on } \operatorname{Supp}_t(R_q), \quad \operatorname{Supp}_t(\psi) \subset N_{\delta_{q+1}}(\operatorname{Supp}_t(R_q)), \quad \text{and} \quad |\psi'(t)| \leq 2\delta_{q+1}^{-1}.$$

Given a smooth non-decreasing function such that

$$\chi(s) = \begin{cases} 1, & 0 \leq s \leq 1, \\ s, & s \geq 2, \end{cases}$$

define a function that smoothly recovers the magnitude of the tensor stress R_q,

$$\rho(t,x) := \frac{2}{\epsilon_\Lambda}\chi(\delta_{q+1}^{-1}|R_q(t,x)|)\delta_{q+1}\psi^2(t), \tag{4.11}$$

where ϵ_Λ is the value from Proposition 4.1.

Now, given $\xi \in \Lambda$, we define the magnitude of its impulsed Beltrami flow, in order to recover R_q through Proposition 4.5, as

$$a_\xi(x,t) = \rho(x,t)^{\frac{1}{2}}\gamma_\xi\left(I_3 - \frac{R_q(x,t)}{\rho(x,t)}\right). \tag{4.12}$$

Indeed, since

$$\frac{|R_q|}{\rho} \leq \frac{\epsilon_\Lambda}{2} \qquad \text{on } \operatorname{Supp}_t(R_q),$$

then

$$\sum_{\xi\in\Lambda} a_\xi^2 \fint \mathbf{W}_\xi \otimes \mathbf{W}_{-\xi} dx = \rho I_3 - R_q. \tag{4.13}$$

We can finally define the new velocity field as $v_{q+1} = v_q + w$, where

$$\begin{aligned} w &= w^{(p)} + w^{(c)} + w^{(t)}, \\ w^{(p)} &= \sum_{\xi\in\Lambda} a_\xi \mathbf{W}_\xi, \\ w^{(c)} &= \frac{1}{\lambda}\sum_{\xi\in\Lambda} \nabla(a_\xi\eta_\xi) \times W_\xi, \\ w^{(t)} &= \frac{1}{\mu}\sum_{\xi\in\Lambda^+} P_H\mathbf{P}_{\neq 0}(a_\xi^2\eta_\xi^2\xi). \end{aligned} \tag{4.14}$$

Notice that the function ψ allows to identify the temporal support of the perturbation w. Indeed,

$$\mathrm{Supp}_t(w) \subset \mathrm{Supp}_t(\rho) \subset \mathrm{Supp}_t(\psi) \subset N_{\delta_{q+1}}(\mathrm{Supp}_t\, R_q). \tag{4.15}$$

Now, let us recall various estimates regarding the perturbation w.

Lemma 4.8 *(Lemma 2, [16]). Let ρ, a_ξ be defined as in Eq. (4.11), Eq. (4.12), then*

$$\|\rho\|_{L^\infty_t L^1_x} \le C\delta_{q+1}, \qquad \|\rho^{-1}\|_{C^0(\mathrm{Supp}_t(R_q))} \lesssim \delta_{q+1}^{-1}, \qquad \|\rho\|_{C^N_{t,x}} \lesssim C(\delta_{q+1}, |R_q|_{C^N_{t,x}}),$$
$$\|a_\xi\|_{L^\infty_t L^2_x} \lesssim \delta_{q+1}^{\frac{1}{2}}, \qquad \|a_\xi\|_{C^N_{t,x}} \lesssim C(\delta_{q+1}, |R_q|_{C^N_{t,x}}).$$

Lemma 4.9 *(Lemma 2.1, [17]). Given $1 \le p \le \infty$, if $f, g \in C^\infty(\mathbf{T}^3)$, with g $(\frac{\mathbf{T}}{N})^3$-periodic and $N \in \mathbb{N}$, then,*

$$\|fg\|_{L^p} \lesssim \|f\|_{L^p}\|g\|_{L^p} + N^{-\frac{1}{p}}\|f\|_{C^1}\|g\|_{L^p}.$$

In the rest of the section we will denote by $C_a(N)$ any value that depends polynomially on $\sup_{n\le N}\|a_\xi\|_{C^n_{t,x}}$.

Lemma 4.10. *If w, $w^{(p)}$, $w^{(c)}$, $w^{(t)}$ are defined as in Eq. (4.14), then*

$$\begin{aligned}
\|w^{(p)}\|_{L^\infty_t L^2_x} &\lesssim \delta_{q+1}^{\frac{1}{2}} + C_a(1)(\lambda\sigma)^{-\frac{1}{2}},\\
\|w^{(c)}\|_{L^\infty_t L^2_x} &\lesssim C_a(1)\Big(\frac{1}{\lambda} + \sigma r\Big),\\
\|w^{(t)}\|_{L^\infty_t L^2_x} &\lesssim C_a(0)\frac{r^{\frac{3}{2}}}{\mu},\\
\|w\|_{L^\infty_t L^s_x} &\lesssim C_a(0) r^{\frac{3}{2}-\frac{3}{s}}, \qquad \textit{for } s \in (1,2].
\end{aligned}$$

Proof. Since $\mathbf{W}_\xi$ is $(\frac{\mathbf{T}}{\lambda\sigma})^3$-periodic (this follows from Eq. (4.6)), Lemma 4.9 can be applied to any term $a_\xi \mathbf{W}_\xi$. Then, the first inequality follows from the estimates shown in Lemma 4.8, and from Eq. (4.10).

The other inequalities from Proposition 4.4 and the fact that the Leray-Helmholtz projector P_H is bounded from L^q to itself for $1 < q < \infty$. □

4.3. Estimating the stress

In the previous section we have defined the perturbation w and the new velocity field v_{q+1}. Now, we complete the proof of Lemma 2.3 showing how to properly build the pressure p_{q+1} and the tensor stress R_{q+1}.

Let us first introduce an anti-divergence operator, namely a right-inverse of the divergence operator, which maps vector fields into trace-free matrix-valued functions.

Definition 4.11. Let $u \in C^\infty(\mathbf{T}^3, \mathbb{R}^3)$ be a smooth vector field. A *anti-divergence* is the linear operator defined by

$$\mathcal{R}(u) = \frac{1}{4}\big(\nabla P_H v + (\nabla P_H v)^T\big) + \frac{3}{4}\big(\nabla v + (\nabla v)^T\big) - \frac{1}{2}\operatorname{div}(v) I_d,$$

where $v \in C^\infty(\mathbf{T}^3, \mathbb{R}^3)$ is the unique solution to the problem

$$\begin{cases} \Delta v = u - \fint u \, dx, \\ \fint v \, dx = 0, \end{cases}$$

with periodic boundary conditions.

The next proposition fully justifies the name given to the operator $\mathcal{R}$.

Proposition 4.12. *Consider $u \in C^\infty(\mathbf{T}^3, \mathbb{R}^3)$, then,*

$$\operatorname{div}(\mathcal{R}(u)) = u - \fint_{\mathbf{T}^3} u, \qquad \operatorname{Tr}(\mathcal{R}(u)) = 0. \tag{4.16}$$

Moreover, for all $k \in \mathbb{Z}^3$, with $k \neq 0$,

$$\begin{aligned} [\mathcal{F}(\mathcal{R}(u))(k)_j^h]_{j,h=1}^3 = \frac{i}{2|k|^2}(\mathcal{F}(u)(k) \cdot k)\mathbf{1}_{j=h} + \frac{i\, k_h k_j}{2|k|^4}(\mathcal{F}(u)(k) \cdot k) + \\ - \frac{i k_h}{|k|^2}\mathcal{F}(u)(k)_j - \frac{i k_j}{|k|^2}\mathcal{F}(u)(k)_h, \end{aligned} \tag{4.17}$$

where $\mathcal{F}(v)(k)_j$ is the j^{th} component of the k^{th} Fourier coefficient of v.

Proof. This result follows from straightforward computations. □

Remark 4.13. Eq. (4.17) shows that $\mathcal{R}(u)$ is obtained by applying the Fourier multipliers introduced in Corollary 3.5 to the components of the vector field u.

Since (v_q, p_q, R_q) solves Eq. (2.1), then

$$\begin{aligned} &\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) + (-\Delta)^{\frac{5}{4},\beta} v_{q+1} = \\ &\quad = \underbrace{\operatorname{div}(R_q) - \nabla p_q + \operatorname{div}(w^{(p)} \otimes w^{(p)}) + \partial_t w^{(t)}}_{\text{oscillation error}} + \\ &\quad + \underbrace{\operatorname{div}(v_q \otimes w + w \otimes v_q) + (-\Delta)^{\frac{5}{4},\beta} w + \partial_t (w^{(c)} + w^{(p)})}_{\text{linear error}} + \\ &\quad + \underbrace{\operatorname{div}(w^{(p)} \otimes (w^{(c)} + w^{(t)}) + (w^{(c)} + w^{(t)}) \otimes w)}_{\text{corrector error}}. \end{aligned}$$

Now, we show that the terms on the RHS can be expressed as the gradient of a scalar function p_{q+1} and as the divergence of a matrix-valued function R_{q+1}. In addition, we shall prove that $\|R_{q+1}\|_{L_t^\infty L_x^1} \leq \delta_{q+2}$.

Before estimating $\|R_{q+1}\|_{L_t^\infty L_x^1}$, let us prove a useful lemma.

Lemma 4.14. *Consider $a, b \in C^\infty(\mathbf{T}^3)$ such that $\mathbf{P}_{\geq N} b = b$.*

1. *Let $\beta > 1$ and $N \geq 4$. Then,*

$$\|\mathcal{R}|\nabla| T_M(ab)\|_{L^1} \lesssim \log^{1-\beta}(N)\|a\|_{H^3}\|b\|_{L^1}, \tag{4.18}$$

where $|\nabla|$ is the operator associated with the multiplier $m(k) = |k|$.

2. *Moreover,*

$$\|\mathcal{R}\mathbf{P}_{\neq 0}(ab)\|_{L^1} \lesssim \frac{1}{N}\|a\|_{H^4}\|b\|_{L^1}. \tag{4.19}$$

Proof. Define $a_1 = \mathbf{P}_{<\frac{N}{2}}(a)$ and $a_2 = \mathbf{P}_{\geq \frac{N}{2}}(a)$. The assumption on b implies that

$$ab = \mathbf{P}_{\geq \frac{N}{2}}(a_1 b) + (a_2 b).$$

Notice that Eq. (4.17) implies that the operator $\mathcal{R}|\nabla|T_M$ corresponds to the multiplier

$$\frac{m_1(k)}{|k|} M(k)|k|,$$

where m_1 is a suitable linear combination of iterated Riesz multipliers, i.e. multipliers of the form $\frac{\prod_{i=1}^d k_i^{\gamma_i}}{|k|^{\sum_{i=1}^d \gamma_i}}$. Moreover, it holds that $\left(I - \mathbf{P}^A_{\leq 2^M}\right)\mathbf{P}_{\geq N/2} = \mathbf{P}_{\geq N/2}$ for $M = \lfloor \log_2 N \rfloor - 2$, therefore

$$\begin{aligned}\|\mathcal{R}|\nabla|T_M(ab)\|_{L^1} &\lesssim \log^{1-\beta}(N)\|a_1 b\|_{L^1} + \|a_2 b\|_{L^1} \leq \\ &\leq \log^{1-\beta}(N)\|a_1\|_{L^\infty}\|b\|_{L^1} + \|a_2\|_{L^\infty}\|b\|_{L^1} \leq \\ &\leq \left(\log^{1-\beta}(N)\|a\|_{H^2} + \tfrac{1}{N}\|a\|_{H^3}\right)\|b\|_{L^1},\end{aligned}$$

where the first inequality is due to Eq. (3.4) and Corollary 3.5 and the third one can be justified by Sobolev embeddings.

The second inequality, Eq. (4.19), follows from a slight modification of the previous proof. In particular, recall that $\mathcal{R}\mathbf{P}_{\neq 0}$ can be seen as corresponding to the Fourier multiplier given by Eq. (4.17), which is equal to a linear combination of Fourier multipliers treated in Corollary 3.5. Then, the result is easily achieved by using Eq. (3.5) and Corollary 3.5. □

4.3.1. Corrector error

Given its formulation, it produces two addends of R_{q+1} whom $L^\infty_t L^1_x$ norm can be estimated thanks to Lemma 4.10. Indeed,

$$
\begin{aligned}
&\|w^{(p)} \otimes (w^{(c)} + w^{(t)}) + (w^{(c)} + w^{(t)}) \otimes w\|_{L^\infty_t L^1_x} \leq \\
&\quad \leq \|w + w^{(p)}\|_{L^\infty_t L^2_x} \cdot \|w^{(c)} + w^{(t)}\|_{L^\infty_t L^2_x} \\
&\quad \leq C_a(1)\Big(\frac{1}{\lambda} + \sigma r + \frac{r^{\frac{3}{2}}}{\mu}\Big)\Big(\frac{1}{\lambda} + \sigma r + \frac{r^{\frac{3}{2}}}{\mu} + \delta_{q+1}^{\frac{1}{2}} + (\lambda\sigma)^{-\frac{1}{2}}\Big).
\end{aligned}
$$

4.3.2. Linear error

Since $\mathcal{R}$ is the right inverse to the divergence operator (Eq. (4.16)), the terms $v_q \otimes w + w \otimes v_q$ and $\mathcal{R}((-\Delta)^{\frac{5}{4},\beta} w + \partial_t(w^{(c)} + w^{(p)}))$ are part of R_{q+1}. In addition, Lemma 4.10 implies that

$$
\|v_q \otimes w\|_{L^\infty_t L^1_x} \leq \|v_q\|_{L^\infty_t L^\infty_x} \|w\|_{L^\infty_t L^s_x} \lesssim C_a(0)\|v_q\|_{L^\infty_t L^\infty_x} r^{\frac{3}{2}-\frac{3}{s}},
$$

for $s \in (1, 2)$.

The remaining terms are estimated through Lemma 4.14.

Proposition 4.15. *The following estimates hold:*

1. $\|\mathcal{R}(\partial_t(w^{(c)} + w^{(p)}))\|_{L^\infty_t L^1_x} \lesssim C_a(6)\sigma\mu r^{-\frac{1}{2}}$,
2. $\|\mathcal{R}(-\Delta)^{\frac{5}{4},\beta}(w^{(p)} + w^{(c)})\|_{L^\infty_t L^1_x} \lesssim C_a(6)\Big(\frac{\lambda}{r}\Big)^{\frac{3}{2}} \log^{1-\beta}(\lambda)$;
3. $\|\mathcal{R}(-\Delta)^{\frac{5}{4},\beta}(w^{(t)})\|_{L^\infty_t L^1_x} \lesssim C_a(2)\frac{(\lambda\sigma r)^{\frac{3}{2}}}{\mu}$.

Proof. First of all, we have that

$$
\begin{aligned}
\partial_z(\nabla \times (a_\xi \mathbf{W}_\xi)) = (\partial_z a_\xi)(\nabla \times \mathbf{W}_\xi) + a_\xi \partial_z(\nabla \times \mathbf{W}_\xi) \\
+ \partial_z \nabla a_\xi \times \mathbf{W}_\xi + \nabla a_\xi \times \partial_z \mathbf{W}_\xi,
\end{aligned}
\tag{4.20}
$$

where ∂_z denotes both ∂_t or ∂_x. Notice that Eq. (4.6) implies that all the terms are composed by a suitable derivative of a_ξ and a $(\frac{\mathbf{T}}{\lambda/5})^3$-periodic function.

Now, to prove the first inequality, notice that

$$
w^{(p)} + w^{(c)} = \frac{1}{\lambda}\nabla \times w^{(p)} = \frac{1}{\lambda}\mathbf{P}_{\neq 0}(\nabla \times w^{(p)}),
$$

so

$$
\begin{aligned}
&\|\mathcal{R}(\partial_t(w^{(c)} + w^{(p)}))\|_{L^\infty_t L^1_x} = \frac{1}{\lambda}\|\mathcal{R}\mathbf{P}_{\neq 0}\partial_t(\nabla \times (w^{(p)}))\|_{L^\infty_t L^1_x} \lesssim \\
&\qquad \lesssim \frac{1}{\lambda}\|\mathcal{R}\mathbf{P}_{\neq 0}\partial_t(\nabla \times (a_\xi \mathbf{W}_\xi))\|_{L^\infty_t L^1_x} \lesssim \frac{1}{\lambda^2} C_a(6)\|\partial_t \nabla \mathbf{W}_\xi\|_{L^\infty_t L^1_x}
\end{aligned}
$$

where the second inequality is due to Eq. (4.19) applied to all the terms in Eq. (4.20). Now, Eq. (4.10) implies the required inequality.

We turn to the proof of the second inequality. Recall that T_M is the operator associated with the multiplier

$$M(k) = \mathbf{1}_{\{k\neq 0\}}(k)\log^{-\beta}(|k|+10).$$

Then by a classical interpolation argument, we have

$$\begin{aligned}\|\mathcal{R}(-\Delta)^{\frac{5}{4},\beta}(w^{(p)}+w^{(c)})\|_{L^\infty_t L^1_x} &= \frac{1}{\lambda}\||\nabla|^{\frac{3}{2}}(\mathcal{R}|\nabla|T_M(\nabla\times w^{(p)}))\|_{L^\infty_t L^1_x} \leq \\ &\leq \frac{1}{\lambda}\|\mathcal{R}|\nabla|T_M\nabla(\nabla\times w^{(p)}))\|^{\frac{1}{2}}_{L^\infty_t L^1_x}\|\mathcal{R}|\nabla|T_M\nabla^2(\nabla\times w^{(p)}))\|^{\frac{1}{2}}_{L^\infty_t L^1_x}.\end{aligned}$$

Now we estimate the first factor $\|\mathcal{R}|\nabla|T_M\nabla(\nabla\times w^{(p)}))\|_{L^\infty_t L^1_x}$, the second one can be estimated by a similar procedure.

Because of Eq. (4.20) we can apply Lemma 4.14

$$\begin{aligned}\|\mathcal{R}|\nabla|T_M\nabla(\nabla\times w^{(p)}))\|_{L^\infty_t L^1_x} &\lesssim \log^{1-\beta}(\lambda)C_a(5)\|\nabla^2\mathbf{W}_\xi\|_{L^\infty_t L^1_x} \\ &\lesssim C_a(5)\lambda^2 r^{-\frac{3}{2}}\log^{1-\beta}(\lambda),\end{aligned}$$

where the first inequality is due to Eq. (4.18), while Eq. (4.10) implies the second one.

Finally, we prove the third inequality. Following the same strategy used above, we have that

$$\begin{aligned}\|\mathcal{R}(-\Delta)^{\frac{5}{4},\beta}(w^{(t)})\|_{L^\infty_t L^1_x} &= \||\nabla|^{\frac{3}{2}}(\mathcal{R}|\nabla|T_M(w^{(t)}))\|_{L^\infty_t L^1_x} \\ &\leq \|\mathcal{R}|\nabla|T_M\nabla(w^{(t)}))\|^{\frac{1}{2}}_{L^\infty_t L^1_x}\|\mathcal{R}|\nabla|T_M\nabla^2(w^{(t)}))\|^{\frac{1}{2}}_{L^\infty_t L^1_x}.\end{aligned}$$

Let us show how we can estimate the first factor. Recall that the Leray-Helmholtz projector P_H corresponds to the multiplier $I-\frac{k\otimes k}{|k|^2}$, therefore $\mathcal{R}|\nabla|P_H$ is associated with a linear combination of iterated Riesz transforms, i.e. multipliers like $\frac{\prod_{i=1}^d k_i^{\gamma_i}}{|k|^{\sum_{i=1}^d \gamma_i}}$. So, by Corollary 3.5, we have that $\mathcal{R}|\nabla|P_H T_M$ maps L^1 into itself. Hence,

$$\begin{aligned}\|\mathcal{R}|\nabla|T_M\nabla(w^{(t)}))\|_{L^\infty_t L^1_x} &\lesssim \frac{1}{\mu}\|\mathcal{R}|\nabla|P_H T_M(\nabla(a_\xi^2)\eta_\xi^2 + a_\xi^2\nabla(\eta_\xi^2))\|_{L^\infty_t L^1_x} \\ &\lesssim \frac{1}{\mu}\|\nabla(a_\xi^2)\eta_\xi^2 + a_\xi^2\nabla(\eta_\xi^2)\|_{L^\infty_t L^1_x} \\ &\lesssim \frac{1}{\mu}C_a(1)\big(\|\eta_\xi^2\|_{L^\infty_t L^1_x} + \|\nabla\eta_\xi^2\|_{L^\infty_t L^1_x}\big) \\ &\lesssim \frac{1}{\mu}C_a(1)\lambda\sigma r,\end{aligned}$$

where the third inequality follows from Eq. (4.9). $\square$

4.3.3. Oscillation error

Finally, the oscillation error is treated as in [3]. So,

$$\begin{aligned}
&\operatorname{div}(R_q - w^{(p)} \otimes w^{(p)}) = \\
&\quad = \operatorname{div}\big(R_q - \sum_{\xi,\xi'} a_\xi a_{\xi'} \fint_{\mathbf{T}^3} \mathbf{W}_\xi \otimes \mathbf{W}_{\xi'} dx + \sum_{\xi,\xi'} a_\xi a_{\xi'} \mathbf{P}_{\neq 0} \mathbf{W}_\xi \otimes \mathbf{W}_{\xi'}\big) \\
&\quad = \operatorname{div}\big(R_q - \sum_{\xi} a_\xi^2 \fint_{\mathbf{T}^3} \mathbf{W}_\xi \otimes \mathbf{W}_{-\xi} dx + \sum_{\xi,\xi'} a_\xi a_{\xi'} \mathbf{P}_{\neq 0} \mathbf{W}_\xi \otimes \mathbf{W}_{\xi'}\big) \\
&\quad = \operatorname{div}\big(\rho I_3 + \sum_{\xi,\xi'} a_\xi a_{\xi'} \mathbf{P}_{\neq 0} \mathbf{W}_\xi \otimes \mathbf{W}_{\xi'}\big),
\end{aligned}$$

where the second equality follows from Eq. (4.8) and the third one from Eq. (4.13). Since $\operatorname{div}(\rho I_3) = \nabla \rho$, then ρ goes into p_{q+1} as well as p_q.

Now, we can treat $\operatorname{div}(\sum_{\xi,\xi'} a_\xi a_{\xi'} P_{\neq 0} \mathbf{W}_\xi \otimes \mathbf{W}_{\xi'}) =: \sum_{\xi,\xi'} E_{\xi,\xi'}$ by taking into account each term $E_{\xi,\xi'} + E_{\xi',\xi}$. In particular, since $E_{\xi,\xi'} + E_{\xi',\xi}$ has zero mean, then

$$\begin{aligned}
E_{\xi,\xi'} + E_{\xi',\xi} &= \mathbf{P}_{\neq 0}\big(\nabla(a_\xi a_{\xi'}) \cdot \mathbf{P}_{\geq \lambda\sigma/2}(\mathbf{W}_\xi \otimes \mathbf{W}_{\xi'} + \mathbf{W}_{\xi'} \otimes \mathbf{W}_\xi)\big) \\
&\quad + \mathbf{P}_{\neq 0}\big(a_\xi a_{\xi'} \operatorname{div}(\mathbf{W}_\xi \otimes \mathbf{W}_{\xi'} + \mathbf{W}_{\xi'} \otimes \mathbf{W}_\xi)\big) \\
&=: E_{\xi,\xi',1} + E_{\xi,\xi',2},
\end{aligned}$$

The term $\mathcal{R}E_{\xi,\xi',1}$ goes into R_{q+1}, Eq. (4.19) and Eq. (4.9) imply that

$$\begin{aligned}
&\|\mathcal{R}E_{\xi,\xi',1}\|_{L^\infty_t L^1_x} \lesssim \frac{1}{\lambda\sigma} C_a(6) \|\mathbf{W}_\xi \otimes \mathbf{W}_{\xi'}\|_{L^\infty_t L^1_x} \lesssim \\
&\qquad\qquad \lesssim \frac{1}{\lambda\sigma} C_a(6) \|\eta_\xi \eta_{\xi'}\|_{L^\infty_t L^1_x} \lesssim \frac{C_a(6)}{\lambda\sigma}.
\end{aligned}$$

Instead, we need to distinguish two cases in order to deal with $E_{\xi,\xi',2}$.

<u>*First case:* $\xi \neq -\xi'$</u>. If $\xi \neq -\xi'$, then $\mathbf{P}_{>\lambda/10}(\mathbf{W}_\xi \otimes \mathbf{W}_{\xi'}) = \mathbf{W}_\xi \otimes \mathbf{W}_{\xi'}$. Therefore,

$$\begin{aligned}
&a_\xi a_{\xi'} \operatorname{div}((\mathbf{W}_\xi \otimes \mathbf{W}_{\xi'} + \mathbf{W}_{\xi'} \otimes \mathbf{W}_\xi) = \\
&\quad = a_\xi a_{\xi'} \operatorname{div}(\mathbf{P}_{\geq \lambda/10}\big(\eta_\xi \eta_{\xi'}(W_\xi \otimes W_{\xi'} + W_{\xi'} \otimes W_\xi)\big) \\
&\quad = a_\xi a_{\xi'} \mathbf{P}_{\geq \lambda/10}\big(\nabla(\eta_\xi \eta_{\xi'}) \cdot (W_\xi \otimes W_{\xi'} + W_{\xi'} \otimes W_\xi)\big) + \\
&\qquad + a_\xi a_{\xi'} \mathbf{P}_{\geq \lambda/10}\big(\eta_\xi \eta_{\xi'} \nabla(W_\xi \cdot W_{\xi'})\big) \\
&\quad = a_\xi a_{\xi'} \mathbf{P}_{\geq \lambda/10}\big(\nabla(\eta_\xi \eta_{\xi'}) \cdot (W_\xi \otimes W_{\xi'} + W_{\xi'} \otimes W_\xi)\big) + \nabla(a_\xi a_{\xi'} \mathbf{W}_\xi \cdot \mathbf{W}_{\xi'}) + \\
&\qquad - \nabla(a_\xi a_{\xi'}) \mathbf{P}_{\geq \lambda/10}(\mathbf{W}_\xi \cdot \mathbf{W}_{\xi'}) - a_\xi a_{\xi'} \mathbf{P}_{\geq \lambda/10}\big(W_\xi \cdot W_{\xi'} \nabla(\eta_\xi \eta_{\xi'})\big),
\end{aligned}$$

where the second term produces a pressure, the third one can be estimated as $E_{\xi,\xi',1}$, the first and the fourth produce two new terms of R_{q+1} whose $L^\infty_t L^1_x$ norms can be estimated in the same way. Namely,

$$\|\mathcal{R}\mathbf{P}_{\neq 0}\Big(a_\xi a_{\xi'}\mathbf{P}_{\geq\lambda/10}\big(W_\xi\cdot W_{\xi'}\nabla(\eta_\xi\eta_{\xi'})\big)\Big)\|_{L^\infty_t L^1_x}\lesssim$$
$$\lesssim\frac{C_a(6)}{\lambda}\|\nabla(\eta_\xi\eta_{\xi'})\|_{L^\infty_t L^1_x}\lesssim\frac{C_a(6)}{\lambda}\|\nabla(\eta_\xi)\eta_{\xi'}\|_{L^\infty_t L^1_x}\lesssim\frac{C_a(6)}{\lambda}\lambda\sigma r,$$

where we have used Eq. (4.19).

Second case: $\xi=-\xi'$. If $\xi'=-\xi$, then $E_{\xi,-\xi,2}=\mathbf{P}_{\neq 0}\big(a_\xi^2(\nabla\mathbf{P}_{\geq\lambda\sigma/2}(\eta_\xi^2)-\frac{\xi}{\mu}\partial_t\eta_\xi^2)\big)$. Now, let us recall that $\partial_t w^{(t)}$ is part of the oscillation term, i.e. we need to deal with $\sum_{\xi\in\Lambda^+}[\mathbf{P}_{\neq 0}\big(a_\xi^2(\nabla\mathbf{P}_{\geq\lambda\sigma/2}(\eta_\xi^2)-\frac{\xi}{\mu}\partial_t\eta_\xi^2)\big)+\frac{1}{\mu}\partial_t P_H\mathbf{P}_{\neq 0}(a_\xi^2\eta_\xi^2\xi)]$. Now,

$$\mathbf{P}_{\neq 0}\big(a_\xi^2(\nabla\mathbf{P}_{\geq\lambda\sigma/2}(\eta_\xi^2))\big)=\nabla(a_\xi^2\mathbf{P}_{\geq\lambda\sigma/2}(\eta_\xi^2))-\mathbf{P}_{\neq 0}(\nabla a_\xi^2\mathbf{P}_{\geq\lambda\sigma/2}(\eta_\xi^2)),$$

where the first term goes into the pressure p_{q+1}, while the second produces a term of R_{q+1} with $L^\infty_t L^1_x$ estimated using Eq. (4.19). Indeed,

$$\|\mathcal{R}\mathbf{P}_{\neq 0}(\nabla a_\xi^2\mathbf{P}_{\geq\lambda\sigma/2}(\eta_\xi^2))\|_{L^\infty_t L^1_x}\lesssim\frac{C_a(6)}{\lambda\sigma}.$$

Instead,

$$-\mathbf{P}_{\neq 0}\Big(a_\xi^2\partial_t\eta_\xi^2\frac{\xi}{\mu}\Big)+P_H\mathbf{P}_{\neq 0}\Big(a_\xi^2\eta_\xi^2\frac{\xi}{\mu}\Big)=-(I-P_H)\Big(\mathbf{P}_{\neq 0}\partial_t(a_\xi^2\eta_\xi^2)\frac{\xi}{\mu}\Big)+\mathbf{P}_{\neq 0}\Big(\frac{\xi}{\mu}\partial_t(a_\xi^2)\eta_\xi^2\Big),$$

where the first term goes into the pressure thanks to $I-P_H=\nabla\Delta^{-1}\operatorname{div}$, while the second one can be seen as part of R_{q+1}, and

$$\big\|\mathcal{R}\mathbf{P}_{\neq 0}\big(\tfrac{\xi}{\mu}\partial_t(a_\xi^2)\eta_\xi^2\big)\big\|_{L^\infty_t L^1_x}\lesssim C_a(5)\frac{1}{\mu}.$$

4.3.4. Conclusion of the proof

Summing up all the contributions to R_{q+1}, we end up with

$$\begin{aligned}\|R_{q+1}\|_{L^\infty_t L^1_x}\lesssim&C_a(1)\big(\frac{1}{\lambda}+\sigma r+\frac{r^{\frac32}}{\mu}\big)\big(\frac{1}{\lambda}+\sigma r+\frac{r^{\frac32}}{\mu}+\delta_{q+1}^{\frac12}+(\lambda\sigma)^{-\frac12}\big)\\&+C_a(0)r^{\frac32-\frac3s}+C_a(6)\sigma\mu r^{-\frac12}+C_a(6)\left(\frac{\lambda}{r}\right)^{\frac32}\log^{1-\beta}(\lambda)\\&+C_a(2)\frac{(\lambda\sigma r)^{\frac32}}{\mu}+C_a(6)\frac{1}{\lambda\sigma}+C_a(6)\sigma r+C_a(5)\frac{1}{\mu}.\end{aligned}\tag{4.21}$$

Since the values of $C_a(N)$ do not depend on the parameter λ (see Lemma 4.8), then we can choose the parameters s, σ, μ and r such that Eq. (4.5) hold and $\|R_{q+1}\|_{L^\infty_t L^1_x}$ is smaller than δ_{q+2} as long as λ is large enough. Therefore, consider

$$s=\frac{3}{2},\quad\sigma=\frac{1}{\lambda}\log^4(\log(\lambda)),\quad r=\frac{\lambda}{\log^{(\beta-1)/3}(\lambda)},\quad\mu=\lambda^{\frac32}\log^{(1-\beta)/4}(\lambda).$$

Then, by this choice, the RHS of Eq. (4.21) tends to zero as long as λ diverges to ∞. In order to conclude the proof of Lemma 2.3, we need to show that Eq. (2.2) and Eq. (2.3) are fully satisfied by our construction and our choice of the parameters. In particular, the control on the temporal support follows directly from Eq. (4.15), while the estimates on $\|v_{q+1} - v_q\|_{L_t^\infty L_x^2}$ and $\||\nabla|^{\frac{3}{2}} T_m(v_{q+1} - v_q)\|_{L_t^\infty L_x^1}$ are due to Lemma 4.10 and Proposition 4.15.

This concludes the proof of the *Iteration Lemma* (Lemma 2.3).

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Appendix A. Fourier projectors in the torus

This appendix gives some explicit estimates about the projectors used in the paper, in order to show how crucial is the choice of projectors, as already detailed in Remark 3.6 and Remark 3.7. Indeed, their norms differ of the order of logarithmic terms. This in general makes no difference, but clearly becomes of paramount importance in our setting.

We first compare the projectors $\mathbf{P}_{\leq N}$ and $\mathbf{P}^A_{\leq N}$ as bounded linear operators from $L^1(\mathbf{T})$ to itself. The following proposition shows that a sharp projection onto Fourier components produces worse estimates.

Proposition A.1. *Consider a positive integer N. Then,*

$$\|\mathbf{P}_{\leq N}\|_{\mathcal{L}(L^1,L^1)} \geq \|D_N\|_{L^1} \approx \log(N), \qquad \|\mathbf{P}^A_{\leq N}\|_{\mathcal{L}(L^1,L^1)} \lesssim 1, \tag{A.1}$$

where D_N is the Dirichlet kernel, that is $D_N(x) = \sum_{|k|\leq N} e^{ikx}$.

Proof. We can focus on $\mathbf{P}_{\leq N}$ because the estimate on $\mathbf{P}^A_{\leq N}$ follows from Bernstein inequality. Let $g_r(x) = \sum_{|k|\leq r} \left(1 - \frac{|k|}{r+1}\right) e^{ikx}$ be the Fejer kernel with parameter r and remind that $\|g_r\|_{L^1} = 1$ for all $r \in \mathbb{N}$. Then,

$$\int_{\mathbf{T}} |P_{\leq N} g_r(x)| dx = \int_{\mathbf{T}} \left| \sum_{|k| \leq N} e^{ikx} - \frac{1}{r+1} \sum_{|k| \leq N} |k| e^{ikx} \right| dx \overset{r \to \infty}{\to} \|D_N\|_{L^1}. \qquad \square$$

Logarithmic factors also appear when the projectors $\mathbf{P}_{\leq N}$ are composed with Fourier multiplier such as those in Corollary 3.5.

Proposition A.2. *Let $m(k) = \mathbf{1}_{\{k \neq 0\}}(k) \frac{1}{|k|_1}$, then T_m is bounded from $L^1(\mathbf{T}^d)$ to itself. Moreover, for all $N \in \mathbb{N}$,*

$$\|\mathbf{P}_{|k|_1 > N} T_m\|_{\mathcal{L}(L^1, L^1)} \lesssim \frac{1}{N} \log^d(N). \tag{A.2}$$

Proof. Consider $f_N(x) = \sum_{|k|_1 \leq N} \frac{1}{|k|_1} e^{ik \cdot x}$, then the following formula holds,

$$f_N(x) = \sum_{j=1}^{N-1} D_j^1(x) \Big(\frac{1}{j} - \frac{1}{j+1} \Big) + D_N^1(x) \frac{1}{N},$$

where $D_j^1(x) = \sum_{|k|_1 \leq j} e^{ik \cdot x}$. Therefore, for $N < M$,

$$\|f_N - f_M\|_{L^1} \leq \frac{1}{N} \|D_N^1\|_{L^1} + \frac{1}{M} \|D_M^1\|_{L^1} + \sum_{j=N}^{M-1} \|D_j^1\|_{L^1} \big(\frac{1}{j} - \frac{1}{j+1} \big).$$

It is known that $\|D_j^1\|_{L^1} \lesssim \log^d(j)$ (see [12]), so

$$\|f_N - f_M\|_{L^1} \leq \frac{1}{N} \log^d(N) + \frac{1}{M} \log^d(M) + \sum_{j=N}^{M-1} \frac{\log^d(j)}{j(j+1)} \lesssim \frac{1}{N} \log^d(N).$$

The sequence $(f_N)_N$ is a Cauchy sequence in L^1 and converges to f such that $\nu(A) = \int_A f(x)\, dx$ is a finite Borel measure on the torus which has $m(k)$ as Fourier coefficients. The existence of the measure ν implies that m is a $(1,1)$-multiplier and that $\|T_m\|_{\mathcal{L}(L^1, L^1)} \leq |\nu| = \|f\|_{L^1}$, where $|\nu|$ is the total variation of the measure ν (see [8]). At last, by the same argument, the estimate on $\|\mathbf{P}_{|k|_1 > N} T_m\|_{\mathcal{L}(L^1, L^1)}$ follows from

$$\|f - f_N\|_{L^1} = \lim_{M \to \infty} \|f_M - f_N\|_{L^1} \leq \frac{1}{N} \log^d(N). \qquad \square$$

Remark A.3. By repeating the same argument above, the conclusions of Proposition A.2 also hold for the multiplier $M(k) = \mathbf{1}_{k \neq 0}(k) \frac{1}{\log^\beta(|k|_1 + 10)}$, with $\beta > d$, and

$$\|\mathbf{P}_{|k|_1 > N} T_M\|_{\mathcal{L}(L^1, L^1)} \lesssim \frac{\log^d(N)}{\log^\beta(10 + N)}. \tag{A.3}$$

The appearance of a factor $\log^d(N)$ is coherent with various results obtained for L^1-convergence of double Fourier series [18]. We highlight that the multiplier of the previous result is not covered by Corollary 3.5 and cannot be studied through Proposition 3.4.

Data availability

No data was used for the research described in the article.

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