RECTIFIABILITY OF FLAT CHAINS IN BANACH SPACES WITH COEFFICIENTS IN $\mathbf{Z}_p$

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1. Introduction

Aim of this paper is a finer analysis of the group of flat chains with coefficients in $\mathbf{Z}_p$ introduced in [7], by taking quotients of the group of integer rectifiable currents, along the lines of [27, 15]. We investigate the typical questions of the theory of currents, namely rectifiability of the measure-theoretic support and boundary rectifiability. Our main result can also be interpreted as a closure theorem for the class of integer rectifiable currents with respect to a (much) weaker convergence, induced by flat distance mod(p), and with respect to weaker mass bounds. A crucial tool in many proofs is the isoperimetric inequality proved in [7] with universal constants.

In order to illustrate our results we start with a few basic definitions. Let us denote by $\mathcal{I}_k(E)$ the class of integer rectifiable currents with finite mass in a metric space E and let us given for granted the concepts of boundary ∂, mass $\mathbf{M}$, push-forward in the more general context of currents (see [4] and the short appendix of [7]). We denote by $\mathscr{F}_k(E)$ the currents that can by written as $R + \partial S$ with $R \in \mathcal{I}_k(E)$ and $S \in \mathcal{I}_{k+1}(E)$. It is obviously an additive Abelian group and

$$T \in \mathscr{F}_k(E) \qquad \Longrightarrow \qquad \partial T \in \mathscr{F}_{k-1}(E). \tag{1.1}$$

$\mathscr{F}_k(E)$ is a metric space when endowed with the the distance $d(T_1, T_2) = \mathscr{F}(T_1 - T_2)$, where

$$\mathscr{F}(T) := \inf \left\{\mathbf{M}(R) + \mathbf{M}(S) : \ R \in \mathcal{I}_k(E),\ S \in \mathcal{I}_{k+1}(E),\ T = R + \partial S\right\}.$$

The subadditivity of $\mathscr{F}$, namely $\mathscr{F}(nT) \leq n\mathscr{F}(T)$, ensures that d is a distance, and the completeness of the groups $\mathcal{I}_k(E)$, when endowed with the mass norm, ensures that $\mathscr{F}_k(E)$ is complete. Also, the boundary rectifiability theorem in $\mathcal{I}_k(E)$ yields

$$\{T \in \mathscr{F}_k(E) : \ \mathbf{M}(T) < \infty\} = \mathcal{I}_k(E). \tag{1.2}$$

For $T \in \mathscr{F}_k(E)$ we define:

$$\mathscr{F}_p(T) := \inf \left\{\mathscr{F}(T - pQ) : \ Q \in \mathscr{F}_k(E)\right\}.$$

The definition of $\mathscr{F}$ gives

$$\mathscr{F}_p(T) = \inf \left\{\mathbf{M}(R) + \mathbf{M}(S) : \ T = R + \partial S + pQ,\ R \in \mathcal{I}_k(E),\ S \in \mathcal{I}_{k+1}(E),\ Q \in \mathscr{F}_k(E)\right\}.$$

Obviously $\mathscr{F}_p(T) \le \mathscr{F}(T)$ and therefore we can introduce an equivalence relation mod(p) in $\mathscr{F}_k(E)$, compatible with the group structure, by saying that $T = \tilde{T}$ mod(p) if $\mathscr{F}_p(T - \tilde{T}) = 0$. Our main object of investigation will be the spaces

$$\mathscr{F}_{p,k}(E) := \{[T] : \ T \in \mathscr{F}_k(E)\} .$$

The equivalence classes are closed in $\mathscr{F}_k(E)$ and (see (2.3) in the next section) the boundary operator can be defined also in the quotient spaces $\mathscr{F}_{p,k}(E)$ in such a way that

$$\partial[T] = [\partial T] \in \mathscr{F}_{p,k-1}(E) \qquad \forall T \in \mathscr{F}_k(E).$$

In $\mathscr{F}_k(E)$ one can also define a (relaxed) notion of p-mass $\mathbf{M}_p$ by

$$\mathbf{M}_p(T) := \inf \left\{ \liminf_{h\to\infty} \mathbf{M}(T_h) : T_h \in \mathscr{F}_k(E), \ \mathscr{F}_p(T_h - T) \to 0 \right\} . \tag{1.3}$$

Since $\mathbf{M}_p(T) = \mathbf{M}_p(T')$ if $T = T'$ mod(p) the definition obviously extends to the quotient spaces $\mathscr{F}_{p,k}(E)$. As in the standard theory of currents, a local variant of this definition provides a σ-additive Borel measure, that we shall denote by $\|T\|_p$, whose total mass is $\mathbf{M}_p(T)$.

From now on, we shall assume that E is a compact convex subspace of a Banach space F and a Lipschitz retract of it. In addition, we shall assume that F satisfies a strong finite-dimensional approximation property (precisely stated in Definition 7.1) that covers, for instance, all Hilbert spaces.

We can now state the main result of this paper.

Theorem 1.1 (Rectifiability of flat chains mod(p))**.** *If $T \in \mathscr{F}_k(E)$ has finite $\mathbf{M}_p$ mass, then $\|T\|_p$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set with finite $\mathscr{H}^k$-measure.*

We don't know whether the result is true without the finite-dimensional approximation assumption, unless $k = 0, 1$. In general, without this assumption, we are able to prove rectifiability only of a the "slice mass" $\|T\|_p^*$ (see Definition 3.8 and (3.8)) built using the 0-dimensional slices of the flat chain, and the validity in general spaces of the equality $\|T\|_p = \|T\|_p^*$ is still an open problem.

Since ∂ maps $\mathscr{F}_k(E)$ into $\mathscr{F}_{k-1}(E)$, the next result is a direct consequence of Theorem 1.1.

Corollary 1.2 (Boundary rectifiability)**.** *If $T \in \mathscr{F}_k(E)$ and if ∂T has finite $\mathbf{M}_p$-mass, then $\|\partial T\|_p$ is concentrated on a countably $\mathscr{H}^{k-1}$-rectifiable set with finite $\mathscr{H}^{k-1}$-measure.*

Notice that in Corollary 1.2 finiteness of mass of T is not needed. As a corollary we obtain an extension mod(p) of (1.2), namely flat chains mod(p) with finite $\mathbf{M}_p$ mass coincide with equivalence classes of integer rectifiable currents. These classes have been considered in [7] in connection with isoperimetric and filling radius inequalities.

Corollary 1.3. *If $T \in \mathscr{F}_k(E)$ has finite $\mathbf{M}_p$ mass, then there exists $S \in \mathcal{I}_k(E)$ with $S = T$ mod(p). In addition, S can be chosen so that $\mathbf{M}_p(T) = \mathbf{M}(S)$.*

We give a detailed proof of Corollary 1.3 at the end of the paper. We obtain also, as a byproduct, the following closure theorem for $\mathcal{I}_k(E)$: in comparison with the results in [4] the $\mathscr{F}_p$ convergence (instead of the weak convergence in the duality with all Lipschitz differential forms), and the bounds only on the $\mathbf{M}_p$ mass (instead of the stronger mass bounds) are considered. Obviously the result can also be stated as a closure theorem in $\mathscr{F}_{p,k}(E)$.

Corollary 1.4 (Closure theorem). *Assume that $(T_n) \subset \mathcal{I}_k(E)$ satisfies $\mathscr{F}_p(T_n - T) \to 0$ for some $T \in \mathscr{F}_k(E)$. If $\sup_n \mathbf{M}_p(T_n) < \infty$, then there exists $S \in \mathcal{I}_k(E)$ with $S = T$* $\operatorname{mod}(p)$.

We conclude the introduction with a short plan of the paper. In Section 2 we recall the basic results we need on flat chains and flat chains $\operatorname{mod}(p)$, borrowing some results from [7]. In Section 3 we study more in detail the slicing operator and the measure $\|T\|_p$. The main result is that a flat chain with finite mass and boundary with finite mass is uniquely determined by its slices. In this section we don't rely, as in [25] on the use of the deformation theorem of [26], not available in our context. We heavily use, instead, the isoperimetric inequality: in turn, this inequality (derived as well in [25] as a consequence of the deformation theorem) is proved in [7] without using the deformation theorem. In Section 4 we make a finer analysis of 1-dimensional flat chains $\operatorname{mod}(p)$ and we provide a direct proof of their rectifiability; this is a crucial ingredient to estabilish the rectifiability of the slice mass $\|T\|_p^*$ of higher dimensional flat chains, following basically the procedure in [25]. This procedure is implemented in Section 5 and Section 6 and leads to the proof that $\|T\|_p^*$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set: the main difference with respect to [25] consists in the fact that the whole family of 1-Lipschitz projections, instead of the projections on the coordinate planes typical of the Euclidean case, has to be considered. In this respect, notice that still a BV estimate analogous to the one in [3] is available in this setting, see Remark 3.5, and it is likely that also some adaptations of the ideas in [3] might provide a different proof of the rectifiability of $\|T\|_p^*$. Finally, in Section 7 we complete our analysis getting a concentration set with finite $\mathscr{H}^k$-measure and proving the equality $\|T\|_p = \|T\|_p^*$ in the class of spaces having the finite-dimensional approximation property.

2. Notation and basic results on flat chains

We use the standard notation $B_r(x)$ for the open balls in E, $\mathrm{Lip}(E)$ for the space of Lipschitz real-valued functions and $\mathrm{Lip}_b(E)$ for bounded Lipschitz functions. Now we recall a few basic facts on flat chains and flat chains $\operatorname{mod}(p)$ mostly estabilished in [7].

Throughout the paper we assume that E is a compact convex subset of a Banach space. Denoting by $\mathbf{I}_k(E)$ the space

$$\mathbf{I}_k(E) := \{T \in \mathcal{I}_k(E) : \ \partial T \in \mathcal{I}_{k-1}(E)\},$$

this assumption ensures the density of $\mathbf{I}_k(E)$ in $\mathscr{F}_k(E)$ (see [7]), and this gives the possibility to extend the restriction and slicing operators from $\mathbf{I}_k(E)$ to $\mathscr{F}_k(E)$.

2.1. **Inequalities.** Notice that

$$\mathscr{F}(\partial T) \le \mathscr{F}(T), \qquad \forall T \in \mathscr{F}_k(E). \tag{2.1}$$

In addition, since $\partial(\varphi_\sharp S) = \varphi_\sharp(\partial S)$, the inequality $\mathbf{M}(\varphi_\sharp R) \le [\mathrm{Lip}(\varphi)]^k \mathbf{M}(R)$ for R k-dimensional gives

$$\mathscr{F}(\varphi_\sharp T) \le [\mathrm{Lip}(\varphi)]^k \mathscr{F}(T) \quad \text{for all } T \in \mathscr{F}_k(E),\ \varphi \in \mathrm{Lip}(E, \mathbf{R}^k). \tag{2.2}$$

In addition, (2.1) together with (1.1) give

$$\mathscr{F}_p(\partial T) \le \mathscr{F}_p(T), \qquad \forall T \in \mathscr{F}_k(E), \tag{2.3}$$

while (2.2) gives

$$\mathscr{F}_p(\varphi_\sharp T) \le [\mathrm{Lip}(\varphi)]^k \mathscr{F}_p(T) \tag{2.4}$$

for all $T \in \mathscr{F}_k(E)$, $\varphi \in \mathrm{Lip}(E, \mathbf{R}^k)$. In particular, the push-forward operator can be defined in the quotient spaces in such a way as to commute with the equivalence relation mod(p). Using (2.3) and the inequalities $\mathscr{F}_p \le \mathbf{M}_p \le \mathbf{M}$ it is also easy to check that

$$\mathscr{F}_p(T) = \inf\left\{\mathbf{M}_p(R) + \mathbf{M}_p(S) :\ R \in \mathcal{I}_k(E),\ S \in \mathcal{I}_{k+1}(E)\right\}. \tag{2.5}$$

2.2. **The restriction operator.** Let $u \in \mathrm{Lip}(E)$. In [7] it is proved that the limit

$$\lim_{h\to\infty} T_h \llcorner \{u < r\} \tag{2.6}$$

exists in $\mathscr{F}_k(E)$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$ whenever T_h have finite mass and $\sum_h \mathscr{F}(T_h - T) < \infty$. By construction the operator $T \mapsto T\llcorner\{u < r\}$ is additive and this definition is independent, up to Lebesgue negligible sets, on the chosen approximating sequence (T_h), provided the "fast convergence" condition $\sum_h \mathscr{F}(T_h - T) < \infty$ holds. The construction provides also the inequality

$$\int_m^{*\ell} \mathscr{F}(T\llcorner\{u < r\})\, dr \le (\ell - m + \mathrm{Lip}(u))\mathscr{F}(T) \qquad \forall m,\, \ell \in \mathbf{R},\ m \le \ell, \tag{2.7}$$

where $\int^*$ denotes the outer integral. It follows immediately from the additivity of the restriction operator that

$$\int_m^{*\ell} \mathscr{F}_p(T\llcorner\{u < r\})\, dr \le (\ell - m + \mathrm{Lip}(u))\mathscr{F}_p(T) \qquad \forall m,\, \ell \in \mathbf{R},\ m \le \ell, \tag{2.8}$$

so that the restriction operator can be defined in the quotient spaces $\mathscr{F}_{p,k}(E)$ in such a way that

$$[T]\llcorner\{u < r\} = [T\llcorner\{u < r\}] \qquad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbf{R}. \tag{2.9}$$

2.3. **$\mathbf{M}_p$-mass and $\|T\|_p$-measure.** Recall that the mass measure $\|T\|$ of $T \in \mathcal{I}_k(E)$ is the finite nonnegative Borel measure characterized by $\|T\|(\{u < r\}) = \mathbf{M}(T \llcorner \{u < r\})$ for all $u \in \mathrm{Lip}(E)$ and $r \in \mathbf{R}$. In [7, Theorem 7.1] the authors proved the existence, for all $T \in \mathscr{F}_k(E)$ of finite $\mathbf{M}_p$-mass, of a finite nonnegative Borel measure $\|T\|_p$ satisfying

$$\mathbf{M}_p(T \llcorner \{u < r\}) = \|T\|_p(\{u < r\}) \qquad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbf{R}$$

for all $u \in \mathrm{Lip}(E)$. In addition, since $\|T\|_p$ arises in the proof of that result as the weak limit of $\|T_n\|$, where $T_n \in \mathscr{F}_k(E)$ are such that $\mathscr{F}_p(T_n - T) \to 0$ and $\mathbf{M}(T_n) \to \mathbf{M}_p(T)$, we can pass to the limit as $n \to \infty$ in the inequalities

$$\mathscr{F}(T_n \llcorner \{u < s\} - T_n \llcorner \{u < r\}) \leq \|T_n\|(\{r \leq u < s\}) \qquad r < s,$$

taking into account that (2.8) gives $\mathscr{F}_p(T_n \llcorner \{u < r\} - T \llcorner \{u < r\}) \to 0$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$, to obtain

$$\mathscr{F}_p(T \llcorner \{u < s\} - T \llcorner \{u < r\}) \leq \mathrm{Lip}(u)\|T\|_p(\{r \leq u \leq s\}) \qquad \forall r,\, s \in \mathbf{R} \setminus N,\ r \leq s \tag{2.10}$$

with N Lebesgue negligible (possibly dependent on T and u).

Using this fact, for chains T with finite $\mathbf{M}_p$-mass we can give a meaning to the restriction $[T] \llcorner C$, for all fixed closed set $C \subset E$, as follows: let π be the distance function from C, and let N be as in (2.10) with $u = \pi$. If $r_i \downarrow 0$ and $r_i \notin N$ then $T \llcorner \{\pi < r_i\}$ is a Cauchy sequence with respect to $\mathscr{F}_p$ and we denote by $[T] \llcorner C \in \mathscr{F}_{p,k}(E)$ its limit. The lower semicontinuity of $\mathbf{M}_p$ provides also the inequality

$$\mathbf{M}_p([T] \llcorner C) \leq \|T\|_p(C).$$

An analogous procedure (considering the sets $\{d(\cdot, E \setminus A) > r_i\}$, with $r_i \downarrow 0$) provides the restriction to open sets $[T] \llcorner A$, satisfying $[T] \llcorner A + [T] \llcorner (E \setminus A) = [T]$ and $\mathbf{M}_p([T] \llcorner A) \leq \|T\|_p(A)$. Since $\mathbf{M}_p$ is subadditive and $[T] = [T] \llcorner A + [T] \llcorner C$, with $C = E \setminus A$, it turns out that both inequalities are equalities:

$$\mathbf{M}_p([T] \llcorner C) = \|T\|_p(C), \qquad \mathbf{M}_p([T] \llcorner A) = \|T\|_p(A). \tag{2.11}$$

By (2.10) it follows also that

$$\mathscr{F}_p([T] \llcorner \{u < s\} - [T] \llcorner \{u < r\}) \leq \mathrm{Lip}(u)\|T\|_p(\{r \leq u < s\}) \qquad \forall r,\, s \in \mathbf{R},\ r < s, \tag{2.12}$$

so that $r \mapsto [T] \llcorner \{u < r\}$ is left continuous, as a map from $\mathbf{R}$ to $\mathscr{F}_{p,k}(E)$, and continuous out of a countable set (contained in the set of r's satisfying $\|T\|_p(\{u = r\}) > 0$).

2.4. **Cone construction.** Given $x \in E$ and $S \in \mathbf{I}_k(E)$, the cone construction in [4, Proposition 10.2] provides a current in $\mathbf{I}_{k+1}(E)$, that we shall denote by $\{x\} \times S$, supported on the union of the segments joining x to points in the support of S, and satisfying

$$\partial(\{x\} \times S) = S - \{x\} \times \partial S. \tag{2.13}$$

In addition we have the inequality

$$\mathbf{M}(\{x\} \times S) \leq r\mathbf{M}(S) \tag{2.14}$$

where r is the radius of the smallest closed ball $\overline{B}_r(x)$ containing the support of S. Since for $R \in \mathbf{I}_k(E)$ and $S \in \mathbf{I}_{k+1}(E)$ we have

$$\{x\} \times (R + \partial S) = \{x\} \times (R + S) - \partial(\{x\} \times S),$$

we immediately get $\mathscr{F}(\{x\} \times T) \leq 2\mathrm{diam}(E)\mathscr{F}(T)$ for $T \in \mathbf{I}_k(E)$. By density and continuity the cone construction uniquely extends to all $\mathscr{F}_k(E)$ and still satisfies (2.13). In addition, since $\mathbf{I}_k(E)$ is dense in mass norm in $\mathcal{I}_k(E)$, and the approximation can easily be done in such a way as to retain the bounds on the support (see [7]), we conclude that (2.14) holds when $S \in \mathcal{I}_k(E)$. In this case, it is proved in [4, Proposition 10.2] that $\{x\} \times S \in \mathcal{I}_{k+1}(E)$, so that

$$\mathscr{F}_p(\{x\} \times T) \leq 2\mathrm{diam}(E)\mathscr{F}_p(T). \tag{2.15}$$

We will also need the inequality

$$\mathbf{M}_p(\{x\} \times T) \leq r\mathbf{M}_p(T) \tag{2.16}$$

for all $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass, whose measure $\|T\|_p$ is supported in $\overline{B}_r(x)$. We sketch its simple proof, based on (2.14) and on the definition of $\mathbf{M}_p$: let $T_h \in \mathscr{F}_k(E)$ with $\mathbf{M}(T_h) \to \mathbf{M}_p(T)$ and $\mathscr{F}_p(T_h - T) \to 0$ and $r' > r$. We know that $\|T_h\|(\{d(\cdot, x) > (r + r')/2\}) \to 0$, hence we can replace T_h by its image $\tilde{T}_h$ under the 2-Lipschitz radial retraction of E onto the ball $\overline{B}_{(r+r')/2}(x)$ to obtain $\tilde{T}_h$ supported on the ball, still converging to T in $\mathscr{F}_p$ distance and with $\mathbf{M}(\tilde{T}_h) \to \mathbf{M}_p(T)$. The inequality (2.15) yields the $\mathscr{F}_p$ convergence of $\{x\} \times \tilde{T}_h$ to $\{x\} \times T$; then, passing to the limit in (2.14) gives $\mathbf{M}_p(\{x\} \times T) \leq \frac{1}{2}(r + r')\mathbf{M}_p(T)$. Eventually we can let $r' \downarrow r$ to obtain (2.16).

2.5. **Isoperimetric inequality.** The next result is proved in [7, Corollary 8.7], adapting the technique in [20, 21].

Proposition 2.1 (Isoperimetric inequality in $\mathscr{F}_{p,k}(E)$)**.** *For $k \geq 1$ there exist constants δ_k such that, if $[L] \in \mathscr{F}_{p,k}(E)$ is a non zero current with $\partial[L] = 0$ and bounded support then*

$$\inf\left\{\frac{\mathbf{M}_p([T])}{\left[\mathbf{M}_p([L])\right]^{(k+1)/k}} :\ [T] \in \mathscr{F}_{p,k+1}(E),\ \partial[T] = [L]\right\} \leq \delta_k.$$

2.6. **Slice operators.** Having defined the restriction to the sets $\{u < r\}$, $u \in \mathrm{Lip}(E)$, the slice operator $T \in \mathscr{F}_k(E) \mapsto \langle T, u, r\rangle \in \mathscr{F}_{k-1}(E)$ is defined by

$$T \mapsto \langle T, u, r\rangle := \partial(T \llcorner \{u < r\}) - (\partial T) \llcorner \{u < r\}$$

whenever the right hand side makes sense (for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$). Since $\partial^2 = 0$ we have

$$\partial\langle T, u, r\rangle = -\langle \partial T, u, r\rangle \qquad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbf{R}.$$

By (2.7) it follows that

$$\int_m^{*\ell} \mathscr{F}(\langle T, u, r\rangle)\, dr \leq 2(\ell - m + \mathrm{Lip}(u))\mathscr{F}(T) \qquad \forall m,\ \ell \in \mathbf{R},\ m \leq \ell, \tag{2.17}$$

$$\int_m^{*\ell} \mathscr{F}_p(\langle T, u, r\rangle)\, dr \le 2(\ell - m + \mathrm{Lip}(u))\mathscr{F}_p(T) \qquad \forall m,\, \ell \in \mathbf{R},\ m \le \ell. \tag{2.18}$$

2.7. **0-dimensional chains.** It is not hard to show (see [7, Proposition 8.4, Theorem 8.5] for a detailed proof) that, for $T \in \mathscr{F}_0(E)$, we have the representation

$$\|T\|_p = \sum_{i=1}^{N} m_i \delta_{x_i} \tag{2.19}$$

with $1 \le m_i \le p/2$ and $x_i \in E$ distinct.

2.8. **Euclidean currents** $\mathrm{mod}(p)$ **and flat chains with coefficients in $\mathbf{Z}_p$.** In Euclidean spaces $\mathbf{R}^n$, a more general theory of currents with coefficients in a normed Abelian group G has been developed by White in [25], [26] on the basis of Fleming's paper [17]. The basic idea of [17] is to consider the abstract completion of the class of *weakly polyhedral chains* with respect to a flat distance. These objects are described by finite sums

$$\sum_i g_i [\![S_i]\!],$$

where $g_i \in G$ and S_i are k-dimensional polyhedra, i.e. S_i is contained in a k-plane and ∂S_i is contained in finitely many $(k-1)$-planes (we use the adjective *weakly* to avoid a potential confusion with the smaller class of polyhedral currents of the deformation theorem: for these S_i are k-cells of a standard cubical decomposition of $\mathbf{R}^n$). The family of weakly polyhedral chains with coefficients in $\mathbf{Z}_p$ has an obvious additive structure inherited from G and a boundary operator in this class can be easily defined. The mass $\mathbf{M}_G(T)$ of a weakly polyhedral chain T can be defined by minimizing $\sum_i \|g_i\| \mathscr{H}^k(S_i)$ among all possible decompositions of T, and a flat distance is defined as follows:

$$\mathscr{F}_G^P(T) := \inf\left\{\mathbf{M}_G(R) + \mathbf{M}_G(\partial S) :\ R,\, S \text{ weakly polyhedral}\right\}.$$

In the particular case $G = \mathbf{Z}_p$ we can obviously think weakly polyhedral chains with coefficients in $\mathbf{Z}_p$ as currents with coefficients in $\mathbf{Z}_p$ and the flat distance $\mathscr{F}_p^P = \mathscr{F}_G^P$ reads as follows:

$$\mathscr{F}_p^P(T) := \inf\left\{\mathbf{M}_p(R) + \mathbf{M}_p(\partial S) :\ R,\, S \text{ weakly polyhedral}\right\}. \tag{2.20}$$

Obviously $\mathscr{F}_p^P(T) \ge \mathscr{F}_p(T)$ (because a larger class of currents is considered in (2.5)), but Proposition 8.3 shows that the two flat distances are equivalent in the class of weakly polyhedral chains. A direct consequence of the equivalence of the two flat distances is the following result, showing that currents $\mathrm{mod}(p)$ are in canonical 1-1 correspondence with flat chains with coefficients in $\mathbf{Z}_p$ and that the equivalence of the flat distances persists.

Proposition 2.2. *If (T_i) is a Cauchy sequence with respect to $\mathscr{F}_p^P$, with T_i k-dimensional and weakly polyhedral, then $\mathscr{F}_p(T_i - T) \to 0$ for some $T \in \mathscr{F}_k(E)$ with $\mathscr{F}_p(T) \le \lim_i \mathscr{F}_p^P(T_i)$. Conversely, if $T \in \mathscr{F}_k(E)$ then there exist T_i k-dimensional and weakly polyhedral with $T = \lim_i T_i$ with respect to $\mathscr{F}_p$; moreover (T_i) is a Cauchy sequence with* *respect to* $\mathscr{F}_p^P$ *and* $\lim_i \mathscr{F}_p^P(T_i) \le C\mathscr{F}_p(T)$. *The constant* $C = C(n,k)$ *is given by Proposition 8.3.*

3. $\mathbf{M}_p$ mass and slice mass

In this section we introduce another notion of p mass, the so-called slice mass based on the 0-dimensional slices of the flat chain, and we compare it with $\|T\|_p$.

We begin with some technical results stating more precise properties of the slice operator. The first one concerns the inequality

$$\int_{\mathbf{R}} \|\langle T,\pi,r\rangle\|_p(E)\,dr \le \|T\|_p(E) \qquad \forall T \in \mathscr{F}_k(E) \tag{3.1}$$

and all $\pi \in \mathrm{Lip}_1(E)$. Let $(T_h) \subset \mathbf{I}_k(E)$ be satisfying $\sum_h \mathscr{F}_p(T_h - T) < \infty$ and $\mathbf{M}(T_h) \to \mathbf{M}_p(T)$. We know from (2.3) that $\sum_h \mathscr{F}_p(\partial T_h - \partial T) < \infty$, hence

$$\lim_{h\to\infty} T_h \llcorner \{\pi < r\} = T \llcorner \{\pi < r\}, \qquad \lim_{h\to\infty} (\partial T_h)\llcorner\{\pi < r\} = (\partial T)\llcorner\{\pi < r\}$$

with respect to the $\mathscr{F}_p$ distance for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$. It follows that $\mathscr{F}_p(\langle T_h - T,\pi,r\rangle) \to 0$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$.

First, let us check the measurability of $r \mapsto \|\langle T,\pi,r\rangle\|_p(E)$. Since $\mathbf{M}_p$ is lower semicontinuous in $\mathscr{F}_k(E)$ we can find a nondecreasing sequence of $\mathscr{F}_p$-continuous functions $G_i : \mathscr{F}_k(E) \to [0,+\infty)$ with $G_i(T) \uparrow \mathbf{M}_p(T)$ for all $T \in \mathscr{F}_k(E)$; taking into account that

$$\mathbf{M}_p(\langle T,\pi,r\rangle) = \lim_{i\to\infty}\lim_{h\to\infty} G_i(\langle T_h,\pi,r\rangle)$$

for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$, we need only to check the measurability of $r \mapsto G_i(\langle S,\pi,r\rangle)$ for $S \in \mathcal{I}_k(E)$, which is achieved in Lemma 8.1. The inequality (3.1) is known for $T \in \mathcal{I}_k(E)$ and for the $\mathbf{M}$ mass, see [4, Theorem 5.7]. Then, lower semicontinuity of $\mathbf{M}_p$ and Fatou's lemma give

$$\begin{aligned}\int_{\mathbf{R}} \mathbf{M}_p(\langle T,\pi,r\rangle)\,dr &\le \int_{\mathbf{R}} \liminf_{h\to\infty} \mathbf{M}_p(\langle T_h,\pi,r\rangle)\,dr \le \liminf_{h\to\infty} \int_{\mathbf{R}} \mathbf{M}_p(\langle T_h,\pi,r\rangle)\,dr \\ &\le \liminf_{h\to\infty} \int_{\mathbf{R}} \mathbf{M}(\langle T_h,\pi,r\rangle)\,dr \le \liminf_{h\to\infty} \mathbf{M}(T_h) = \mathbf{M}_p(T).\end{aligned}$$

Lemma 3.1 (Slice and restriction commute)**.** *Let* $T \in \mathscr{F}_k(E)$, $\pi \in \mathrm{Lip}(E)$ *and* $u \in \mathrm{Lip}(E)$. *Then*

$$\langle T,\pi,r\rangle \llcorner \{u < s\} = \langle T\llcorner\{u < s\},\pi,r\rangle \qquad \text{for } \mathscr{L}^2\text{-a.e. } (r,s) \in \mathbf{R}^2. \tag{3.2}$$

Proof. The identity (3.2) is known when $T \in \mathcal{I}_k(E)$. Indeed (see [4, Theorem 5.7]), the slices R_r of $R \in \mathcal{I}_k(E)$ are uniquely determined, up to Lebesgue negligible sets, by the following two properties:

(a) R_r is concentrated on $\pi^{-1}(r)$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$;

(b) $\int_{\mathbf{R}} \phi(r) R_r\,dr = R\llcorner(\phi\circ\pi)d\pi$ for all $\phi : \mathbf{R} \to \mathbf{R}$ bounded Borel.

It is then immediate to check that, for s fixed, the currents in the left hand side of (3.2) fulfil (a) and (b) with $R = T \llcorner \{u < s\}$, therefore they coincide with $\langle R, \pi, r\rangle$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$.

Let now $(T_h) \subset \mathcal{I}_k(E)$ with $\sum_h \mathscr{F}(T_h - T) < \infty$ and let us consider the identities

$$\langle T_h, \pi, r\rangle \llcorner \{u < s\} = \langle T_h \llcorner \{u < s\}, \pi, r\rangle \qquad \text{for } \mathscr{L}^2\text{-a.e. } (r, s) \in \mathbf{R}^2. \tag{3.3}$$

We know that $\sum_h \mathscr{F}(T_h \llcorner \{u < s\} - T \llcorner \{u < s\}) < \infty$ for $\mathscr{L}^1$-a.e. $s \in \mathbf{R}$; for any s for which this property holds, we have that the right hand sides in (3.3) converge to $\langle T \llcorner \{u < s\}, \pi, r\rangle$ with respect to $\mathscr{F}$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$; on the other hand, we know also that $\sum_h \mathscr{F}(\langle T_h, \pi, r\rangle - \langle T, \pi, r\rangle) < \infty$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$; for any r for which this property holds the left hand sides in (3.3) converge with respect to $\mathscr{F}$ to $\langle T_h, \pi, r\rangle \llcorner \{u < s\}$ for $\mathscr{L}^1$-a.e. $s \in \mathbf{R}$.

Therefore, passing to the limit as $h \to \infty$ in (3.3), using Fubini's theorem, we conclude. □

We can now consider the local version of (3.1).

Lemma 3.2. *For all $T \in \mathscr{F}_k(E)$, $\pi \in \mathrm{Lip}_1(E)$ and $B \subset E$ Borel the function $r \mapsto \|\langle T, \pi, r\rangle\|_p(B)$ is Lebesgue measurable and*

$$\int_{\mathbf{R}} \|\langle T, \pi, r\rangle)\|_p(B)\, dr \le \|T\|_p(B). \tag{3.4}$$

Furthermore, the support of $\|\langle T, \pi, r\rangle\|_p$ is contained in $\pi^{-1}(r) \cap \operatorname{supp} \|T\|_p$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$.

Proof. We consider a closed set $C \subset E$ and the sets $C_s := \{u < s\}$, $s > 0$, where $u := d(\cdot, C)$. Thanks to the commutativity of slice and restriction, for $\mathscr{L}^1$-a.e. $s > 0$ we have $\langle T, \pi, r\rangle \llcorner \{u < s\} = \langle T \llcorner \{u < s\}, \pi, r\rangle$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$. Also, Fubini's theorem ensures that

$$\|\langle T, \pi, r\rangle\|_p(C_s) = \mathbf{M}_p(\langle T, \pi, r\rangle \llcorner \{u < s\}) \qquad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbf{R}$$

for $\mathscr{L}^1$-a.e. $s > 0$. Therefore, for any s satisfying both conditions we conclude that $\|\langle T, \pi, r\rangle\|_p(C_s) = \mathbf{M}_p(\langle T \llcorner \{u < s\}, \pi, r\rangle)$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$. Since we already proved that $r \mapsto \mathbf{M}_p(\langle T \llcorner \{u < s\}, \pi, r\rangle)$ is Lebesgue measurable, this proves that the map $r \mapsto \|\langle T, \pi, r\rangle\|_p(C_s)$ is Lebesgue measurable. Letting $s \downarrow 0$ the same is true for the map $r \mapsto \|\langle T, \pi, r\rangle\|_p(C)$. The same argument allows to prove (3.4) from (3.1).

The class of Borel sets B such that $r \mapsto \|\langle T, \pi, r\rangle\|_p(B)$ is Lebesgue measurable contains the closed sets and satisfies the stability assumptions of Dynkin's lemma, therefore it coincides with the whole Borel σ-algebra. Finally, by monotone approximation (3.4) extends from closed sets to open sets; if B is Borel, by considering a nonincreasing sequence of open sets (A_h) such that $\|T\|_p(A_h) \downarrow \|T\|_p(B)$ we extend the validity of (3.4) from open sets to Borel sets. Eventually, choosing $A = E \setminus \operatorname{supp} \|T\|_p$ yields that $\|\langle T, \pi, r\rangle\|_p(A) = 0$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$. □

Being defined on the whole of $\mathscr{F}_k(E)$ the slicing operator can be obviously iterated, leading to the next definition.

Definition 3.3 (Iterated slices). *For $T \in \mathscr{F}_k(E)$, $2 \le m \le k$ and $\pi = (\pi_1, \dots, \pi_m) \in \mathrm{Lip}(E, \mathbf{R}^m)$ we define recursively the slices $\langle T, \pi, x \rangle$, $x \in \mathbf{R}^m$, by*

$$\langle T, \pi, x \rangle := \langle \langle T, (\pi_1, \dots, \pi_{m-1}), (x_1, \dots, x_{m-1}) \rangle, \pi_m, x_m \rangle.$$

Notice that the slices above are defined, as in the codimension 1 case, for $\mathscr{L}^m$-a.e. $x \in \mathbf{R}^m$, they are given by

$$\langle T, \pi, x \rangle = \lim_{h \to \infty} \langle T_h, \pi, x \rangle \qquad \text{in } \mathscr{F}_{k-m}(E) \tag{3.5}$$

whenever $T_h \in \mathbf{I}_k(E)$ and $\sum_h \mathscr{F}(T_h - T) < \infty$ and the definition is independent of T_h, up to $\mathscr{L}^k$-negligible sets. Moreover, a straightforward induction argument based on Lemma 3.1 gives

$$\langle T, \pi, x \rangle \llcorner \{u < s\} = \langle T \llcorner \{u < s\}, \pi, x \rangle \qquad \text{for } \mathscr{L}^{m+1}\text{-a.e. } (x, s) \in \mathbf{R}^{m+1} \tag{3.6}$$

for all $u \in \mathrm{Lip}(E)$.

Using (3.5), (3.6) and Lemma 8.1 as in the proof of (3.1) and Lemma 3.2 we obtain:

Lemma 3.4. *For all $T \in \mathscr{F}_k(E)$, $1 \le m \le k$, $\pi \in \left[\mathrm{Lip}_1(E)\right]^m$ and $B \subset E$ Borel the function $x \mapsto \|\langle T, \pi, x \rangle\|_p(B)$ is Lebesgue measurable and*

$$\int_{\mathbf{R}^m} \|\langle T, \pi, x \rangle\|_p(B)\, dx \le \|T\|_p(B). \tag{3.7}$$

Furthermore, the support of $\|\langle T, \pi, x \rangle\|_p$ is contained in $\pi^{-1}(x) \cap \operatorname{supp} \|T\|_p$ for $\mathscr{L}^m$-a.e. $x \in \mathbf{R}^m$.

Remark 3.5 (BV regularity of slices). A direct consequence of (2.10) is that, for all $T \in \mathscr{F}_k(E)$ with $\mathbf{M}_p(T)$ finite, $s \mapsto T \llcorner \{\pi < s\}$ has bounded variation in $\mathbf{R} \setminus N$ with respect to $\mathscr{F}_p$. Since N is Lebesgue negligible it follows that $T \llcorner \{\pi < s\}$ has essential bounded variation, and its total variation measure does not exceed $\mathrm{Lip}(\pi)\|T\|_p$. The same is true for the slice map $r \mapsto \langle T, \pi, r \rangle$ of currents T having finite $\mathbf{M}_p$ mass and boundary with finite $\mathbf{M}_p$ mass, and the total variation measure does not exceed $\mathrm{Lip}(\pi)(\|T\|_p + \|\partial T\|_p)$. For higher dimensional slices, we can combine (3.7) and the characterization of metric BV functions in terms of restrictions to lines (see [15, 4.5.9] or [1] for the case of real-valued maps and [2] for the case of metric space valued maps) to obtain that, for all $\pi \in \left[\mathrm{Lip}(E)\right]^m$, $1 \le m \le k$, we have $\langle T, \pi, x \rangle \in MBV(\mathbf{R}^k, \mathscr{F}_{p,k-m}(E))$ and its total variation measure $\|D\langle T, \pi, x \rangle\|$ does not exceed

$$\prod_{i=1}^m \mathrm{Lip}(\pi_i)\big(\|T\|_p + \|\partial T\|_p\big).$$

Motivated by Lemma 3.4, for $1 \le m \le k$, $\pi \in \left[\mathrm{Lip}_1(E)\right]^m$ and $B \subset E$ Borel we define

$$\|T \llcorner d\pi\|_p(B) := \int_{\mathbf{R}^m} \|\langle T, \pi, x \rangle\|_p(B)\, dx \tag{3.8}$$

(the notation is reminiscent of the *real* flat chain $T \llcorner d\pi = \int \langle T, \pi, x\rangle\, dx$). Notice that $\|T \llcorner d\pi\|_p$ is a σ-additive Borel measure less than $\|T\|_p$.

We shall also need the fact that $\|T\|_p$ has no atomic part:

Lemma 3.6. *The measure $\|T\|_p$ has no atom for all $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass.*

Proof. Writing $T = R + \partial S$ with $R \in \mathcal{I}_k(E)$ and $S \in \mathcal{I}_{k+1}(E)$, and noticing that $\|T\|_p \le \|R\|_p + \|\partial S\|_p \le \|R\| + \|\partial S\|_p$, since $\|R\|$ has no atom we can assume with no loss of generality that $T = \partial S$. Fix $x \in E$, $\varepsilon > 0$ and $\bar r > 0$ so small that $\|S\|(B_{2\bar r}(x)) < \varepsilon$. Now, notice that

$$\begin{aligned} T \llcorner \{d(x,\cdot) < s\} &= \partial(S \llcorner \{d(x,\cdot) < s\}) - \langle S, d(x,\cdot), s\rangle \\ &= \partial(S \llcorner \{d(x,\cdot) < s\} - \{x\} \times \langle S, d(x,\cdot), s\rangle) + \{x\} \times \langle T, d(x,\cdot), s\rangle \end{aligned}$$

for $\mathscr{L}^1$-a.e. $s > 0$. Let now $r \le \bar r$; since for $s < 2r$ (2.14) and (2.16) give

$$\mathbf{M}(\{x\}\times\langle S, d(x,\cdot), s\rangle) \le 2r\mathbf{M}(\langle S, d(x,\cdot), s\rangle), \quad \mathbf{M}_p(\{x\}\times\langle T, d(x,\cdot), s\rangle) \le 2r\mathbf{M}_p(\langle T, d(x,\cdot), s\rangle),$$

we can choose $s \in (r, 2r)$ and average to get

$$\begin{aligned} \frac{1}{r}\int_r^{*2r} \mathscr{F}_p(T \llcorner \{d(x,\cdot) < s\})\, ds &\le \varepsilon + 2\int_r^{2r} \mathbf{M}(\langle S, d(x,\cdot), s\rangle) + \mathbf{M}_p(\langle T, d(x,\cdot), s\rangle)\, ds \\ &\le \varepsilon + 2\|S\|(B_{2r}(x) \setminus \{x\}) + 2\|T\|_p(B_{2r}(x) \setminus \{x\}) < 2\varepsilon \end{aligned}$$

for $r \le \bar r$ small enough. Since $\varepsilon > 0$ is arbitrary, it follows that we can find $(s_j) \downarrow 0$ such that $T \llcorner \{d(x,\cdot) < s_j\} \to 0 \bmod(p)$ and

$$\mathbf{M}_p(T - T \llcorner \{d(x,\cdot) < s_j\}) = \|T\|_p(\{d(x,\cdot) \ge s_j\}) \le \mathbf{M}_p(T) - \|T\|_p(\{x\}).$$

Then, the lower semicontinuity of $\mathbf{M}_p$ gives that $\|T\|_p(\{x\}) = 0$. □

In the next theorem and in the sequel we will often deal with exceptional sets depending on the slicing map π. For this reason it will be convenient to restrict these maps to a sufficiently rich but countable set: we fix a set $\mathcal{D} \subset \mathrm{Lip}_1(E)$ countable and dense in $\mathrm{Lip}_1(E)$ with respect to the sup norm.

The next important result shows that currents with finite $\mathbf{M}_p$ mass and boundary with finite $\mathbf{M}_p$ mass are uniquely determined by their 0-dimensional slices. We don't know whether the result is true for all flat chains with finite $\mathbf{M}_p$ mass: we shall prove this fact in a more restrictive class of spaces in Section 7.

Theorem 3.7. *Let $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass and boundary with finite $\mathbf{M}_p$ mass. Assume that, for some $m \in [1, k]$ the following property holds:*

$$\text{for all } \pi \in [\mathcal{D}]^m,\ \langle T, \pi, x\rangle = 0 \bmod(p) \text{ for } \mathscr{L}^m\text{-a.e. } x \in \mathbf{R}^m.$$

Then $T = 0 \bmod(p)$.

Proof. We argue by induction on m and we consider first the case $m = 1$. In the proof of the case $m = 1$ we consider first the case when $\partial T = 0 \bmod(p)$, then the general case.
Step 1. Assume $\partial T = 0 \bmod(p)$. According to the Lyapunov theorem the range of a finite nonnegative measure with no atom is a closed interval. Hence, thanks to Lemma 3.6,

for any $\varepsilon > 0$ we can find a finite Borel partition $B_1, \ldots, B_N$ of E with $\|T\|_p(B_i) < \varepsilon$; also, we can find compact sets $K_i \subset B_i$ such that $\sum_i \|T\|_p(E \setminus \cup_i K_i) \le \varepsilon$. Since the sets K_i are pairwise disjoint, we can find $\delta > 0$ and $\phi_i \in \mathcal{D}$ such that, for $r \in (\delta, 2\delta)$, the open sets $A_i := \{\phi_i < r\}$ are pairwise disjoint, contain K_i and satisfy $\|T\|_p(A_i) \le \varepsilon$ (just choose ϕ_i very close to $d(\cdot, K_i)$ and 2δ less than the least distance between the K_i). In addition, for $\mathscr{L}^1$-a.e. $r \in (\delta, 2\delta)$ the following property is fulfilled:

$$\partial(T \llcorner A_i) = \langle T, d(\cdot, K_i), r\rangle + (\partial T) \llcorner A_i = 0 \mod(p) \qquad \text{for all } i = 1, \ldots, N.$$

Now we choose $r \in (\delta, 2\delta)$ for which all the properties above hold and we apply the isoperimetric inequality in $\mathscr{F}_{p,k}(E)$ to obtain $S_i \in \mathscr{F}_{k+1}(E)$ with $\partial S_i = T \llcorner A_i$ and

$$\mathbf{M}_p(S_i) \le \gamma_k \big(\mathbf{M}_p(T \llcorner A_i)\big)^{1+1/k} \le \varepsilon^{1/k} \|T\|_p(A_i).$$

By applying the cone construction to the cycle $T - \sum_i T \llcorner A_i \mod(p)$, whose $\mathbf{M}_p$ mass is less than $\varepsilon^{1/k}$, we obtain one more S_0 whose boundary mod(p) is $T - \sum_i T \llcorner A_i$ with mass less than $2\mathrm{diam}(E)\varepsilon$. It follows that

$$\partial \sum_{i=0}^{N} [S_i] = [T], \qquad \mathbf{M}_p(\sum_{i=0}^{N} S_i) \le 2\mathrm{diam}(E)\varepsilon + \gamma_k \varepsilon^{1/k} \mathbf{M}_p(T).$$

Since $\mathscr{F}_p(T) \le \mathscr{F}_p(\sum_0^N S_i) \le \mathbf{M}_p(\sum_0^N S_i)$ and ε is arbitrary, this proves that $[T] = 0$.
Step 2. The case $k = 1$ is covered by Corollary 4.2 in Section 4: it shows the existence of $T' \in \mathcal{I}_1(E)$ with $T' = T \mod(p)$, so that the slices of T' vanish mod(p) and therefore the multiplicity of T' is 0 mod(p). In the case $k > 1$ we can use the commutativity of slice and restriction to show that the slices of ∂T vanish mod(p), so that we can apply Step 1 to the cycle ∂T to obtain $\partial T = 0 \mod(p)$. Hence by applying Step 1 again we obtain that $T = 0 \mod(p)$.

The proof of the induction step $m \mapsto m + 1$ is not difficult: let us fix $\pi \in \mathcal{D}$ and let us consider the codimension 1 slices $\langle T, \pi, t\rangle$, $\pi \in \mathcal{D}$; by assumption, for all $q \in [\mathcal{D}]^m$, the m-codimensional slices of $\langle T, \pi, t\rangle$ given by $\langle\langle T, \pi, t\rangle, q, z\rangle = 0$ vanish mod(p) for $\mathscr{L}^{m+1}$-a.e. (t, z); since $\mathcal{D}$ is countable we can find a $\mathscr{L}^1$-negligible set N such that, for $t \notin N$ and for all $q \in [\mathcal{D}]^m$ the slices vanish mod(p) for $\mathscr{L}^m$-a.e. $z \in \mathbf{R}^m$. The induction assumption then gives $\langle T, \pi, t\rangle = 0 \mod(p)$ for all $t \in \mathbf{R} \setminus N$. Eventually the first step of the induction allows to conclude that $T = 0 \mod(p)$. □

Definition 3.8 (Slice $\mathbf{M}_p$ mass $\|T\|_p^*$)**.** *We define $\|T\|_p^*$ as the least upper bound, in the lattice of nonnegative finite measures in E, of the measures $\|T \llcorner d\pi\|_p$, when π runs in $\big[\mathrm{Lip}_1(E)\big]^k$.*

Thanks to Theorem 3.7 we know that $\|T\|_p^*$ provides a reasonable notion of p-mass, since $\|T\|_p^* = 0$ implies $T = 0 \mod(p)$, at least for flat chains T whose finite $\mathbf{M}_p$ mass and boundary with finite $\mathbf{M}_p$ mass. In addition, (3.7) with $m = k$ gives the inequality

$$\|T\|_p^* \le \|T\|_p,$$

so that $\|T\|_p^*$ is well defined. We don't know, however, whether equality holds in general, or whether (in case equality fails), an isoperimetric inequality holds for $\|T\|_p^*$. In Section 7 we shall prove that the two notions of p-mass coincide in a suitable class of spaces E.

Corollary 3.9. *Let $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass. Then $\|T\|_p^*(B) = 0$ whenever B is a $\mathscr{H}^k$-negligible set.*

Proof. We fix $\pi \in \left[\mathrm{Lip}_1(E)\right]^k$ and we notice that, by the coarea inequality [15, Theorem 2.10.25], $\mathscr{H}^0(B \cap \pi^{-1}(x)) = 0$ (i.e. $B \cap \pi^{-1}(x)$ is empty) for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$. Also, Lemma 3.4 gives that $\|\langle T, \pi, x\rangle\|_p$ is concentrated on $\pi^{-1}(x)$ for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$. Then, $\|\langle T, \pi, x\rangle\|_p(B) = 0$ for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$, so that $\|T \llcorner d\pi\|_p(B) = 0$. Since π is arbitrary we conclude that $\|T\|_p^*(B) = 0$. □

4. Rectifiability in the case $k = 1$

Our goal in this section is to prove the rectifiability of 1-dimensional flat chains. We shall actually prove a more precise version of Corollary 1.3 when $\partial T = 0 \bmod(p)$, namely the existence of a cycle $T' \in \mathbf{I}_1(E)$ in the equivalence class of T.

Theorem 4.1. *If $T \in \mathscr{F}_1(E)$ has finite $\mathbf{M}_p$ mass and $\partial T = 0 \bmod(p)$ then there exists $T' \in \mathbf{I}_1(E)$ with $\partial T' = 0$ and $T' = T \bmod(p)$.*

Writing any $T \in \mathscr{F}_1(E)$ with finite $\mathbf{M}_p$ mass as $R + \partial S$ with $R \in \mathcal{I}_1(E)$ and $S \in \mathcal{I}_2(E)$ we can apply Theorem 4.1 to ∂S to obtain the 1-dimensional version of Corollary 1.3:

Corollary 4.2. *For all $T \in \mathscr{F}_1(E)$ with finite $\mathbf{M}_p$ mass there exists $T' \in \mathcal{I}_1(E)$ with $T' = T \bmod(p)$.*

The proof of Theorem 4.1 follows by the construction of a sequence $(T_n) \subset \mathbf{I}_1(E)$ of cycles satisfying

$$\mathbf{M}(T_n) \leq C \quad \text{and} \quad \mathscr{F}_p(T - T_n) \leq \frac{1}{n} \tag{4.1}$$

for all $n \in \mathbf{N}$ and for a constant C independent of n. Since E is a compact subset of a Banach space, we can then use the closure and compactness theorems in [4] to conclude that a subsequence (T_{n_j}) converges weakly (i.e. in the duality with all Lipschitz forms) to a cycle $T' \in \mathbf{I}_1(E)$. Since E is furthermore convex by [22] we infer that T_{n_j} converge in the flat norm to T'. It follows that $T = T' \bmod(p)$ because

$$\mathscr{F}_p(T - T') \leq \mathscr{F}_p(T - T_{n_j}) + \mathscr{F}(T_{n_j} - T') \to 0 \qquad \text{as } j \to \infty.$$

In order to construct a sequence (T_n) of integral cycles satisfying (4.1) we proceed as follows. First we build, in Lemma 4.3 below, approximating cycles $T_\varepsilon \in \mathbf{I}_1(E)$ whose boundary belongs to $p \cdot \mathbf{I}_0(E)$. Then, these cycles are in turn approximated by finite sums S of Lipschitz images of intervals, retaining the same boundary. Eventually a combinatorial argument provides a cycle S' in the same equivalence class mod(p) of S with mass controlled by the mass of S.

Lemma 4.3. *Let $[T]$ be as in the statement of Theorem 1.1. Then for every $\varepsilon > 0$ there exists $T_\varepsilon \in \mathbf{I}_1(E)$ such that $\partial T_\varepsilon \in p \cdot \mathbf{I}_0(E)$ and*

$$\mathbf{M}(T_\varepsilon) \leq \mathbf{M}_p(T) + \varepsilon \quad \textit{and} \quad \mathscr{F}_p(T - T_\varepsilon) < \varepsilon.$$

Proof. Let $\varepsilon \in (0,1)$ and choose $T' \in \mathbf{I}_1(E)$ satisfying $\mathscr{F}_p(T - T') < \varepsilon/3$ and

$$\mathbf{M}(T') \leq \mathbf{M}_p(T) + \frac{\varepsilon}{3}.$$

Write $T = T' + R + \partial S + p \cdot Q$ with $R \in \mathscr{I}_1(E)$, $S \in \mathscr{I}_2(E)$, $Q \in \mathbf{I}_1(E)$ and

$$\mathbf{M}(R) + \mathbf{M}(S) \leq \frac{\varepsilon}{3}.$$

Since $\mathscr{F}_p(\partial T) = 0$ we can write $\partial T = Z + \partial U + p \cdot W$ with $Z \in \mathbf{I}_0(E)$, $U \in \mathscr{I}_1(E)$, $W \in \mathbf{I}_0(E)$ and

$$\mathbf{M}(Z) + \mathbf{M}(U) \leq \frac{\varepsilon}{3}.$$

From this and the choice of ε it follows that $Z = 0$ and thus

$$\partial T' + \partial R + p \cdot \partial Q = \partial U + p \cdot W.$$

Set $T_\varepsilon := T' + R - U$. It is clear that $T_\varepsilon \in \mathscr{I}_1(E)$ and that

$$\partial T_\varepsilon = \partial T' + \partial R - \partial U = p \cdot (W - \partial Q) \in p \cdot \mathbf{I}_0(E),$$

so that $T_\varepsilon \in \mathbf{I}_1(E)$. Furthermore, we obtain

$$\mathbf{M}(T_\varepsilon) \leq \mathbf{M}(T') + \mathbf{M}(R) + \mathbf{M}(U) \leq \mathbf{M}_p(T) + \varepsilon$$

and

$$T - T_\varepsilon = T - T' - R + U = \partial S + U + p \cdot Q,$$

from which it follows that

$$\mathscr{F}_p(T - T_\varepsilon) \leq \mathbf{M}(U) + \mathbf{M}(S) \leq \frac{2\varepsilon}{3}.$$

This concludes the proof. □

The following gives an almost optimal representation of currents in $\mathbf{I}_1(E)$ as a superposition of curves. For a related result see [23], for the optimal result in $\mathbf{R}^n$ see [15, 4.2.25] (we shall actually use this result in the proof).

Lemma 4.4. *Let E be a length space and let $\tilde{T} \in \mathbf{I}_1(E)$. Then for every $\delta > 0$ there exist finitely many $(1+\delta)$-Lipschitz curves $c_i : [0, a_i] \to E$, $i = 1, \dots, n$, with image in $\overline{B}(\operatorname{supp}\tilde{T}, \delta\mathbf{M}(\tilde{T}))$ and such that $\partial\tilde{T} = \sum([\![c_i(a_i)]\!] - [\![c_i(0)]\!])$,*

$$\mathbf{M}(\partial\tilde{T}) = \sum_{i=1}^{n} \mathbf{M}([\![c_i(a_i)]\!] - [\![c_i(0)]\!]),$$

$$\mathbf{M}\left(\tilde{T} - \sum_{i=1}^{n} c_{i\sharp}[\![\chi_{[0,a_i]}]\!]\right) \leq \delta\mathbf{M}(\tilde{T}),$$

$$\sum_{i=1}^{n} a_i \leq (1+\delta)\mathbf{M}(\tilde{T}).$$

Proof. Let $\delta' > 0$ be small enough, to be determined later. Using Lemma 4 and Theorem 7 of [19] one easily shows that the existence of finitely many $(1+\delta')$-biLipschitz maps $\varphi_i : K_i \to E$, $i = 1, \dots, n$, where the sets $K_i \subset \mathbf{R}$ are compact and such that $\varphi_i(K_i) \cap \varphi_j(K_j) = \emptyset$ if $i \neq j$, and

$$\|\tilde{T}\| \left(X \setminus \cup \varphi_i(K_i)\right) \leq \delta' \mathbf{M}(\tilde{T}), \tag{4.2}$$

see also [4, Lemma 4.1]. By McShane's extension theorem there exists a $(1+\delta')$-Lipschitz extension $\overline{\eta}_i : E \to \mathbf{R}$ of φ_i^{-1} for each $i = 1, \dots, n$. Now, write $\partial\tilde{T}$ as $\partial\tilde{T} = \sum_{i=1}^{k}([\![x_i]\!] - [\![y_i]\!])$ with $x_i \neq y_j$ for all i, j (so that $\{x_1, \dots, x_k\}$ is the support of the positive part of $\partial\tilde{T}$ and $\{y_1, \dots, y_n\}$ is the support of the negative part). Note that $2k = \mathbf{M}(\partial\tilde{T})$. Set $\Omega := \bigcup \varphi_i(K_i) \cup \{x_1, \dots, x_k, y_1, \dots, y_k\}$ and let $\{z_1, \dots, z_m\} \subset \bigcup \varphi_i(K_i)$ be a finite and ν-dense set for Ω, where $\nu > 0$ is such that

$$2\nu \leq \frac{\delta'}{1+\delta'} \operatorname{dist}(\varphi_i(K_i), \varphi_j(K_j)) \quad \text{whenever } i \neq j. \tag{4.3}$$

We set $N := n + m + 2k$ and define a map $\Psi : E \to \ell_N^\infty$ by

$$\Psi(x) := \left(\overline{\eta}_1(x), \dots, \overline{\eta}_n(x), d(x, z_1), \dots, d(x, z_m), d(x, x_1), d(x, y_1), \dots, d(x, x_k), d(x, y_k)\right).$$

Note that Ψ is $(1+\delta')$-Lipschitz and $(1+\delta')$-biLipschitz on Ω. Indeed, it is clear that Ψ is $(1+\delta')$-Lipschitz and that the restriction $\Psi|_{\varphi_i(K_i)}$ is $(1+\delta')$-biLipschitz for every i. Moreover, for $x \in \varphi_i(K_i)$ and $x' \in \varphi_j(K_j)$ with $i \neq j$ there exists $z \in \varphi_i(K_i)$ with $d(x, z) \leq \nu$ and hence

$$d(x, x') \leq d(x, z) + d(z, x') \leq 2d(x, z) + d(z, x') - d(z, x) \leq \|\Psi(x') - \Psi(x)\|_\infty + 2\nu,$$

from which the biLipschitz property on $\cup\varphi_i(K_i)$ follows together with (4.3). The other cases are trivial. By [15, 4.2.25] there exist at most countably many Lipschitz curves $\varrho_j : [0, a_j] \to \ell_N^\infty$ which are parametrized by arc-length, one-to-one on $[0, a_j)$ and which satisfy $\Psi_\sharp \tilde{T} = \sum_{j=1}^\infty \varrho_{j\sharp}[\![\chi_{[0,a_j]}]\!]$ and

$$\mathbf{M}(\Psi_\sharp \tilde{T}) = \sum_{j=1}^{\infty} \mathbf{M}(\varrho_{j\sharp}[\![\chi_{[0,a_j]}]\!]) = \sum_{j=1}^{\infty} a_j \tag{4.4}$$

and

$$2k = \sum_{j=1}^{\infty} \mathbf{M}([\![\varrho_j(a_j)]\!] - [\![\varrho_j(0)]\!]).$$

After possibly reindexing the ϱ_i and the y_j we may assume without loss of generality that $\varrho_i(a_i) = \Psi(x_i)$ and $\varrho_i(0) = \Psi(y_i)$ for $i = 1, \dots, k$. It follows that $\varrho_i(a_i) = \varrho_i(0)$ for all $i \geq k+1$. Choose $M \geq k+1$ sufficiently large such that $R := \sum_{j=M+1}^\infty \varrho_{j\sharp}[\![\chi_{[0,a_j]}]\!]$ satisfies

$$\mathbf{M}(R) \leq \delta' \mathbf{M}(\tilde{T}). \tag{4.5}$$

Since E is a length space there exists a $(1+2\delta')$-Lipschitz extension $c_j : [0, a_j] \to E$ of $(\Psi|_\Omega)^{-1} \circ (\varrho_j|_{\varrho_j^{-1}(\Psi(\Omega))})$ for each $j = 1, \dots, M$, and such that $c_j(a_j) = c_j(0)$ for $j = k+1, \dots, M$. Note that $c_j(a_j) = x_j$ and $c_j(0) = y_j$ for all $j = 1, \dots, k$. We now have

$$\sum_{j=1}^{M} \varrho_{j\#}[\![\chi_{\varrho_j^{-1}(\Psi(\Omega)^c)}]\!] = \Big[\Psi_\sharp(\tilde{T} \llcorner \Omega^c) - R\Big] \llcorner \Psi(\Omega)^c,$$

from which it easily follows that

$$\begin{aligned} T' :&= \tilde{T} - \sum_{j=1}^{M} c_{j\sharp}[\![\chi_{[0,a_j]}]\!] \\ &= (\Psi|_\Omega)_\sharp^{-1}\Big[(R - \Psi_\sharp(\tilde{T} \llcorner \Omega^c)) \llcorner \Psi(\Omega)\Big] - \sum_{j=1}^{M} c_{j\sharp}[\![\chi_{\varrho_j^{-1}(\Psi(\Omega)^c)}]\!] + \tilde{T} \llcorner \Omega^c \end{aligned}$$

and, by moreover using (4.2) and (4.5),

$$A := \sum_{j=1}^{M} \mathscr{H}^1(\varrho_j^{-1}(\Psi(\Omega)^c)) = \sum_{j=1}^{M} \mathbf{M}(\varrho_{j\sharp}[\![\chi_{\varrho_j^{-1}(\Psi(\Omega)^c)}]\!]) \le \delta'(2+\delta')\mathbf{M}(\tilde{T}).$$

Using (4.2), (4.5) and the facts that c_i is $(1+2\delta')$-Lipschitz and Ψ and $(\Psi|_\Omega)^{-1}$ are $(1+\delta')$-Lipschitz, we obtain

$$\begin{aligned} \mathbf{M}(T') &\le (1+\delta')\Big[\mathbf{M}(R) + (1+\delta')\|\tilde{T}\|(\Omega^c)\Big] + (1+2\delta')A + \|\tilde{T}\|(\Omega^c) \\ &\le [5 + 8\delta' + 3\delta'^2]\delta'\mathbf{M}(\tilde{T}). \end{aligned}$$

Finally, using (4.4) and the fact that Ψ is $(1+\delta')$-Lipschitz, we estimate

$$\sum_{j=1}^{M} a_j \le \mathbf{M}(\Psi_\sharp \tilde{T}) \le (1+\delta')\mathbf{M}(\tilde{T}).$$

This proves the statement given that $\delta' > 0$ was chosen small enough. $\square$

We now apply Lemma 4.4 to $\tilde{T} := T_\varepsilon$, where T_ε is given by Lemma 4.3. Set $T'' := \sum_{i=1}^{n} c_{i\sharp}[\![\chi_{[0,a_i]}]\!]$. We obtain, in particular,

$$\partial T'' = \sum_{i=1}^{n}([\![c_i(a_i)]\!] - [\![c_i(0)]\!]) \in p \cdot \mathbf{I}_0(E). \tag{4.6}$$

To conclude the proof of Theorem 4.1 we apply the following lemma.

Lemma 4.5. *Let E be a complete metric space, $n \ge 1$ and $p \ge 2$ integers. For each $i = 1, \dots, n$, let $c_i : [0, a_i] \to E$ be a Lipschitz curve. If $S := \sum_{i=1}^{n} c_{i\sharp}[\![\chi_{[0,a_i]}]\!]$ satisfies*

$\partial S \in p \cdot \mathbf{I}_0(E)$ then there exist $1 \le i_1 < \cdots < i_k \le n$ such that the current

$$S' := S - p \cdot \sum_{j=1}^{k} c_{i_j\sharp}[\![\chi_{[0,a_{i_j}]}]\!] \tag{4.7}$$

is a cycle.

It follows in particular that $S - S' \in p \cdot \mathbf{I}_1(E)$ and that

$$\mathbf{M}(S') \le (p-1)\sum_{i=1}^{n} \mathbf{M}(c_{i\sharp}[\![\chi_{[0,a_i]}]\!]).$$

PROOF. It suffices to prove the lemma for the case that $c_i(a_i) \neq c_j(0)$ for all $1 \le i, j \le n$, since we can remove closed loops and we can concatenate c_i and c_j whenever $c_i(a_i) = c_j(0)$. Set $T_i := c_{i\sharp}[\![\chi_{[0,a_i]}]\!]$. We first establish some notation: An ordered k-uple $(\alpha_0, \dots, \alpha_k)$ with $k \ge 0$ and $\alpha_j \in \{1, \dots, n\}$ is called admissible if either $k = 0$ or $\alpha_r \neq \alpha_s$ for all $r \neq s$, $c_{\alpha_m}(a_{i_m}) = c_{\alpha_{m+1}}(a_{\alpha_{m+1}})$ for all $m < k$ even, and $c_{\alpha_m}(0) = c_{\alpha_{m+1}}(0)$ for all $m < k$ odd. A decomposition $S = S_1 + \cdots + S_\ell + Q$ is called admissible if $Q \in p \cdot \mathbf{I}_1(E)$ and every S_i is of the form

$$S_i = \sum_{m=0}^{k_i} (-1)^m T_{\alpha(i,m)},$$

where $(\alpha(i,0), \dots, \alpha(i,k_i))$ is an admissible k_i-uple for every $i \in [1, \ell]$, and if in addition the following properties hold: The sets

$$\Gamma_0 := \{\alpha(i,m) : \quad m \text{ even}, 1 \le i \le \ell\}, \qquad \Gamma_1 := \{\alpha(i,m) : \quad m \text{ odd}, 1 \le i \le \ell\}$$

are disjoint, $\Gamma_0 \cup \Gamma_1 = \{1, \dots, n\}$, every index in Γ_0 appears exactly once, every index in Γ_1 appears exactly $p-1$ times, and $\partial S_i = 0$ if and only if k_i is odd. Since the conditions imposed on admissible k_i-uples imply

$$\partial S_i = \begin{cases} [\![c_{\alpha(i,k_i)}(a_{\alpha(i,k_i)})]\!] - [\![c_{\alpha(i,0)}(0)]\!] \neq 0 & \text{if } k_i \text{ is even} \\ [\![c_{\alpha(i,k_i)}(0)]\!] - [\![c_{\alpha(i,0)}(0)]\!] & \text{if } k_i \text{ is odd} \end{cases} \tag{4.8}$$

the last requirement is equivalent to the condition $c_{\alpha(i,k_i)}(0) = c_{\alpha(i,0)}(0)$ whenever k_i is odd.

Note that, for example, the decomposition $S = S_1 + \ldots + S_n$ with $S_i := T_i$ is admissible ($Q = 0$, $k_i = 0$ for all i, $\Gamma_0 = \{1, \dots, n\}$, $\Gamma_1 = \emptyset$). Suppose now that $S = S_1 + \cdots + S_\ell + Q$ is an admissible decomposition. It is clear that if the set

$$\Lambda(S_1, \dots, S_\ell) := \{i \in \{1, \dots, \ell\} : \partial S_i \neq 0\}$$

satisfies $|\Lambda| < p$ then in fact $\Lambda = \emptyset$. Indeed, $\sum_{i \in \Lambda} \partial S_i = 0 \mod(p)$ and (4.8) together with the fact that the indices $\alpha(i, k_i)$ appear only once (because $\partial S_i \neq 0$ implies k_i even) give $\Lambda = \emptyset$. On the other hand, we claim that if $|\Lambda(S_1, \dots, S_\ell)| \ge p$ then there exists an admissible decomposition $S = S'_1 + \cdots + S'_{\ell'} + Q'$ with

$$|\Lambda(S'_1, \dots, S'_{\ell'})| < |\Lambda(S_1, \dots, S_\ell)|.$$

In order to prove the claim, recall that $\partial S_i = [\![c_{\alpha(i,k_i)}(a_{\alpha(i,k_i)})]\!] - [\![c_{\alpha(i,0)}(0)]\!]$ whenever k_i is even, and $\partial S_i = 0$ if k_i is odd. We call $[\![c_{\alpha(i,k_i)}(a_{\alpha(i,k_i)})]\!]$ the right-boundary and $[\![c_{\alpha(i,0)}(0)]\!]$ the left-boundary of S_i. We may now assume, up to a permutation of the S_i, that $\partial S_1 \neq 0$ and that the right-boundaries of $S_2, \dots, S_p$ equal the right-boundary of S_1, with $\partial S_i \neq 0$ for $i = 2, \dots, p$. We distinguish two cases:

First, suppose that $k_1 = 0$, so that $S_1 = T_{\alpha(1,0)}$. Let $r \in [1, p]$ be the number of integers $i \in [1, p]$ such that ∂S_i has the same left boundary of ∂S_1; we may assume, again up to a permutation of the $S_2, \dots, S_p$ and (in case $r < p$) of S_i for $i > p$, that the left-boundaries of $S_1, \dots, S_r$ and (in the case $r < p$) $S_{p+1}, \dots, S_{2p-r}$ are all equal, with $\partial S_i \neq 0$ for $i = p + 1, \dots, 2p - r$. We define currents S'_j by

$$S'_j := \begin{cases} S_j - T_{\alpha(1,0)} & 2 \leq j \leq r \\ S_j - T_{\alpha(1,0)} + S_{p+j-r} & r + 1 \leq j \leq p. \end{cases}$$

Clearly, this yields an admissible decomposition $S = S'_2 + \dots + S'_p + S_{2p-r+1} + \dots + S_\ell + Q'$ with $Q' = Q + pS_1$. Note that S'_j is a cycle whenever $2 \leq j \leq r$ and therefore the number of non-cycles is strictly smaller if $r \geq 2$; if $r = 1$, since some non-cycles are concatenated in groups of three, their total number is still strictly smaller in the new decomposition.

Next, suppose that $k_1 \geq 2$. Since k_1 is even the index $\alpha(1, k_1)$ is in Γ_0 and thus appears exactly once. Analogously, since $k_1 - 1$ is odd the index $\alpha(1, k_1 - 1)$ is in Γ_1 and thus appears exactly $p - 1$ times. We now construct a new decomposition in which $\alpha(1, k_1)$ appears $p-1$ times and $\alpha(1, k_1 - 1)$ only once. Let $r \in [1, p-1]$ be the number of integers $i \in [1, p]$ such that $\alpha(i, t_i) = \alpha(1, k_1 - 1)$ for some $1 \leq t_i \leq k_i$. Up to a permutation of the $S_2, \dots, S_p$ and (in case $r < p - 1$) of S_i for $i > p$ we may therefore assume that for every $i \in \{1, \dots, r\} \cup \{p + 1, \dots, 2p - r - 1\}$ we have $\alpha(i, t_i) = \alpha(1, k_1 - 1)$ for a suitable t_i. If $i > p$ and S_i is a cycle then we may furthermore assume that $t_i = k_i$. We now define the S'_j as follows: First set

$$S'_1 := \sum_{m=0}^{k_1-2} (-1)^m T_{\alpha(1,m)}.$$

For $j \in \{2, \dots, r\}$ we define a chain S'_j and a cycle S'_{l+j-1} such that $S'_j + S'_{l+j-1} = S_j - T_{\alpha(1,k_1)} + T_{\alpha(j,t_j)}$; more precisely, let S'_j be the 'part' of S_j preceding $\alpha(j, t_j)$ and S'_{l+j-1} the concatenation of the 'part' of S_j following $\alpha(j, t_j)$ with $-T_{\alpha(1,k_1)}$, thus

$$S'_j := \sum_{m=0}^{t_j-1} (-1)^m T_{\alpha(j,m)} \quad \text{and} \quad S'_{\ell+j-1} := \left[\sum_{m=t_j+1}^{k_j} (-1)^m T_{\alpha(j,m)} \right] - T_{\alpha(1,k_1)}.$$

Since the left- and right-boundaries of $T_{\alpha(1,k_1)}$ and the term in brackets in the above equation agree, $S'_{\ell+j-1}$ is a cycle. Let now $j \in \{r + 1, \dots, p - 1\}$ and observe that the right-boundaries of S_1 and S_j agree. Define

$$S'_j := S_j - T_{\alpha(1,k_1)} + \sum_{m=\bar t_j+1}^{\bar k_j} (-1)^m T_{\alpha(p-r+j,m)} \quad \text{and} \quad S'_{p-r+j} := \sum_{m=0}^{\bar t_j-1} (-1)^m T_{\alpha(p-r+j,m)},$$

where $\bar t_j = -1$ and $\bar k_j = k_{p-r+j} - 1$ if $\partial S_{p-r+j} = 0$ and $\bar t_j = t_{p-r+j}$ and $\bar k_j = k_{p-r+j}$ otherwise. In particular, if $\partial S_{p-r+j} = 0$ then $S'_{p-r+j} = 0$. Finally, set $S'_p := S_p - T_{\alpha(1,k_1)} + T_{\alpha(1,k_1-1)}$, and for $j \in \{2p-r, \dots, \ell\}$ set $S'_j := S_j$. Observe that the index $\alpha(1,k_1)$ appears exactly $p-1$ times, $\alpha(1,k_1-1)$ exactly once, and that all other indices appear exactly the same number of times as in the original admissible decomposition. We therefore obtain an admissible decomposition $S = S'_1 + \cdots + S'_{\ell+r-1} + Q$. Note that it has same number of chains with non-zero boundary, however S'_1 has two edges less than S_1, that is, $k'_1 = k_1 - 2$. Applying the same procedure finitely many times allows us to reduce the second case to the first one. This completes the proof of the claim.

We can now apply this claim finitely many times to obtain an admissible decomposition $S = S_1 + \cdots + S_\ell + Q$ in which all S_i are cycles. The current $S' := S_1 + \cdots + S_\ell$ is clearly of the form (4.7) because

$$S - \sum_{i=1}^{\ell} S_i = \sum_{i\in\Gamma_1} T_i - \sum_{i=1}^{\ell} (-1)^m \sum_{m\,=\,1,\ m \text{ odd}}^{k_i} T_{\alpha(i,m)} = p \sum_{i\in\Gamma_1} T_i$$

and thus the proof of the lemma is complete with $\{i_1, \dots, i_k\} = \Gamma_1$. □

5. Lusin approximation of Borel maps by Lipschitz maps

Let $f : A \subset \mathbf{R}^k \to E$ a Borel map. For $x \in A$ we define $\delta_x f$ as the smallest $M \ge 0$ such that

$$\lim_{r\downarrow 0} r^{-k} \mathscr{L}^k\Big(\{y \in A \cap B_r(x) :\ \frac{d(f(y), f(x))}{r} > M\}\Big) = 0.$$

This definition is a simplified version of Federer's definition of approximate upper limit of the difference quotients (we replaced $|y-x|$ by r in the denominator), but sufficient for our purposes.

Theorem 5.1. *Let $f : A \subset \mathbf{R}^k \to E$ be Borel.*

(i) *Let $k = n + m$, $x = (z, y)$, and assume that there exist Borel subsets A_1, A_2 of A such that $\delta_z(f(\cdot, y)) < \infty$ for all $(z, y) \in A_1$ and $\delta_y(f(z, \cdot)) < \infty$ for all $(z, y) \in A_2$. Then $\delta_x f < \infty$ for $\mathscr{L}^k$-a.e. $x \in A_1 \cap A_2$;*

(ii) *if $\delta_x f < \infty$ for $\mathscr{L}^k$-a.e. $x \in A$ there exists a sequence of Borel sets $B_n \subset A$ such that $\mathscr{L}^k(A \setminus \cup_n B_n) = 0$ and the restriction of f to B_n is Lipschitz for all n.*

Proof. For real-valued maps this result is basically contained in [15, Theorem 3.1.4], with slightly different definitions: here, to simplify matters as much as possible, we avoid to mention any differentiability result.

(i) By an exhaustion argument we can assume with no loss of generality that, for some constant N, $\delta_z f < N$ in A_1 and $\delta_y f < N$ in A_2. Moreover, by Egorov theorem (which allows to transform pointwise limits, in our case as $r \downarrow 0$, into uniform ones, at the expense of passing to a slightly smaller domain in measure), we can also assume that

$$\lim_{r\downarrow 0} r^{-m}\mathscr{L}^m\bigl(\{y' \in B^m_r(y) :\ \frac{d(f(z,y'),f(z,y))}{r} > N\}\bigr) = 0 \qquad \text{uniformly for } (z,y) \in A_2. \tag{5.1}$$

We are going to show that $\delta_x f \le 2N$ $\mathscr{L}^k$-a.e. in $A_1 \cap A_2$. By the triangle inequality, it suffices to show that

$$\lim_{r\downarrow 0} r^{-k}\mathscr{L}^k\bigl(\{(z',y') \in B_r((z,y)) :\ \frac{d(f(z',y),f(z,y))}{r} > N\}\bigr) = 0 \tag{5.2}$$

and

$$\lim_{r\downarrow 0} r^{-k}\mathscr{L}^k\bigl(\{(z',y') \in B_r((z,y)) :\ \frac{d(f(z',y'),f(z',y))}{r} > N\}\bigr) = 0 \tag{5.3}$$

for $\mathscr{L}^k$-a.e. $(z,y) \in A_1 \cap A_2$. The first property is clearly satisfied at all $(z,y) \in A_1$, because the sets in (5.2) are contained in

$$\{z' \in B_r(z) :\ \frac{d(f(z',y),f(z,y))}{r} > N\} \times B_r(y).$$

In order to show the second property (5.3) we can estimate the quantity therein by

$$r^{-n}\int_{B^1_r(z)} r^{-m}\mathscr{L}^m\bigl(\{y' \in B_r(y) :\ \frac{d(f(z',y'),f(z',y)}{r} > M\}\bigr)\,dz' + \frac{\mathscr{L}^m(B_r(y))\mathscr{L}^n(B^2_r(z))}{r^{m+n}},$$

where $B^1_r(z) = \{z' \in B_r(z) :\ (z',y) \in A_2\}$ and $B^2_r(z) := B_r(z) \setminus B^1_r(z)$. If we let $r \downarrow 0$, the first term gives no contribution thanks to (5.1); the second one gives no contribution as well provided that z is a density point in $\mathbf{R}^n$ for the slice $(A_2)_y := \{z' :\ (y,z') \in A_2\}$. Since, for all y, $\mathscr{L}^n$-a.e. point of $(A_2)_y$ is a density point $(A_2)_y$, by Fubini's theorem we get that $\mathscr{L}^k$-a.e. $(y,z) \in A_2$ has this property.

(ii) Let $e_0 \in E$ be fixed. Denote by C_N the subset of A where both δ_x and $d(f,e_0)$ do not exceed N. Since the union of C_N covers $\mathscr{L}^k$-almost all of A, it suffices to find a family (B_n) with the required properties covering $\mathscr{L}^k$-almost all of C_N. Let χ_k be a geometric constant defined by the property

$$\mathscr{L}^k(B_{|x_1-x_2|}(x_1) \cap B_{|x_1-x_2|}(x_2)) = \chi_k\mathscr{L}^k(B_{|x_1-x_2|}(0)).$$

We choose $B_n \subset C_N$ and $r_n > 0$ in such a way that $\mathscr{L}^k(C_N \setminus \cup_n B_n) = 0$ and, for all $x \in B_n$ and $r \in (0,r_n)$, we have

$$\mathscr{L}^k\bigl(\{y \in B_r(x) :\ \frac{d(f(y),f(x))}{r} > N+1\}\bigr) \le \frac{\chi_k}{2}\mathscr{L}^k(B_r(x)). \tag{5.4}$$

The existence of B_n is again ensured by Egorov theorem.

We now claim that the restriction of f to C_n is Lipschitz. Indeed, take x_1, $x_2 \in B_n$: if $|x_1 - x_2| \ge r_n$ we estimate $d(f(x_1),f(x_2))$ simply with $4r_n^{-1}\sup_{B_n} d(f,e_0)|x_1-x_2|$. If not, by (5.4) at $x = x_i$ with $r = |x_1 - x_2|$ and our choice of χ_k we can find $y \in B_r(x_1) \cap B_r(x_2)$ where

$$\frac{d(f(y), f(x_1))}{r} \leq N + 1 \quad \text{and} \quad \frac{d(f(y), f(x_2))}{r} \leq N + 1.$$

It follows that $d(f(x_1), f(x_2)) \leq 2(N + 1)|x_1 - x_2|$. □

Proposition 5.2. *Let $K \subset \Gamma \subset E$, with K countably $\mathscr{H}^1$-rectifiable, and let $\pi \in \mathrm{Lip}(E)$ be injective on Γ. Then $\delta(\pi|_\Gamma)^{-1}$ is finite $\mathscr{L}^1$-a.e. on $\pi(K)$.*

Proof. Assume first that $K = f(C)$ with $C \subset \mathbf{R}$ closed and $f : C \to K$ Lipschitz and invertible. The condition $\delta(\pi|_\Gamma)^{-1} < \infty$ clearly holds at all points $t = \pi(x)$ of density 1 for $\pi(K)$, with $x \in K$ satisfying

$$\liminf_{y \in K \to x} \frac{|\pi(y) - \pi(x)|}{d(y, x)} > 0.$$

Indeed, at these points $t = \pi(x)$ we have $x = (\pi|_\Gamma)^{-1}(t)$ and

$$\liminf_{s \in \pi(K) \to t} \frac{|(\pi|_\Gamma)^{-1}(s) - x|}{|s - t|} < \infty.$$

If $N \subset K$ is the set where the condition above fails, the Lipschitz function $p = \pi \circ f$ has null derivative at all points in $f^{-1}(N)$ where it is differentiable, hence $\mathscr{L}^1(p(f^{-1}(N))) = 0$. It follows that $\mathscr{L}^1(\pi(N)) = 0$.

In the general case, write $K = N \cup \cup_i K_i$ with $\mathscr{H}^1(N) = 0$ and $K_i = f_i(C_i)$ pairwise disjoint, with $C_i \subset \mathbf{R}$ closed and $f_i : C_i \to K_i$ Lipschitz and invertible. Let $B_i \subset \pi(K_i)$ be Borel sets such that the inverse g_i of $\pi|_{K_i}$ satisfies $\delta g_i < \infty$ on B_i and $\mathscr{L}^1(\pi(K_i) \setminus B_i) = 0$. Since $\mathscr{H}^1(\pi(N)) = 0$, the union $\cup_i \pi(K_i)$ covers $\mathscr{L}^1$-almost all of $\pi(K)$. Hence, it suffices to show that $\delta(\pi|_\Gamma)^{-1} < \infty$ at all points of density 1 for one of the sets B_i. This property easily follows from the definition of δ and from the fact that $(\pi|_\Gamma)^{-1}$ and g_i coincide on B_i. □

6. Countable rectifiability of $\|T\|_p^*$ in the case $k > 1$

In this section we show that the slice mass $\|T\|_p^*$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set, adapting to this context White's argument [25]; this provides a first step towards the proof of Theorem 1.1.

The next technical lemma provides a useful commutativity property of the iterated slice operator.

Lemma 6.1 (Commutativity of slices)**.** *Let $T \in \mathscr{F}_k(E)$ and $\pi = (p, q)$ with $p \in \mathrm{Lip}(E; \mathbf{R}^{m_1})$, $q \in \mathrm{Lip}(E; \mathbf{R}^{m_2})$, $m_i \geq 1$ and $m_1 + m_2 \leq k$. Then*

$$\langle\langle T, p, z\rangle, q, y\rangle = (-1)^{m_1 m_2} \langle\langle T, q, y\rangle, p, z\rangle \qquad \text{for } \mathscr{L}^{m_1+m_2}\text{-a.e. } (z, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}. \tag{6.1}$$

Proof. If $T \in \mathcal{I}_k(E)$ we know by [4, Theorem 5.7] that the slices $S_{yz} = \langle\langle T, p, z\rangle, q, y\rangle$ are characterized by the following two properties:

(a) S_{yz} is concentrated on $p^{-1}(z) \cap q^{-1}(y)$ for $\mathscr{L}^{m_1+m_2}$-a.e. (z,y);
(b) $\int \psi(y,z) S_{yz}\, dydz = T \llcorner \psi(p,q) dp \wedge dq$ as $(k - m_1 - m_2)$-dimensional currents for all bounded Borel functions ψ.

It is immediate to check that $\langle\langle T, q, y\rangle, p, z\rangle$ satisfy (a) and

$$\int \psi(y,z)\langle\langle T,q,y\rangle,p,z\rangle\, dydz = T \llcorner \psi(p,q) dq \wedge dp = (-1)^{m_1+m_2} T \llcorner \psi(p,q) dp \wedge dq,$$

hence (6.1) holds. The general case can be achieved using (3.5), choosing a sequence $(T_h) \subset \mathbf{I}_k(E)$ with $\sum_h \mathscr{F}(T_h - T) < \infty$. □

In the next proposition we consider first the rectifiability of the measures $\|T \llcorner d\pi\|_p$ for π fixed.

Proposition 6.2. *Let $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass. Then, for all $\pi \in \left[\mathrm{Lip}_1(E)\right]^k$, $\|T \llcorner d\pi\|_p$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set.*

Proof. By (2.19) we obtain that $\|\langle T, \pi, x\rangle\|_p$ consists for $\mathscr{L}^k$-a.e. x of a finite sum of Dirac masses, with weights between 1 and $p/2$. Hence, we can define

$$\Lambda(x) := \left\{y \in \pi^{-1}(x) :\ y \in \operatorname{supp} \|\langle T, \pi, x\rangle\|_p\right\}$$

and we can check that the set-valued function Λ fulfils the measurability assumption of Lemma 8.2. Indeed, for all Borel sets B

$$\left\{x \in \mathbf{R}^k :\ \Lambda(x) \cap B \neq \emptyset\right\} = \left\{x \in \mathbf{R}^k :\ \|\langle T, \pi, x\rangle\|_p(B) > 0\right\}$$

and we know that the latter set is measurable, thanks to Lemma 3.4. By Lemma 8.2 we obtain disjoint measurable sets $B_n = \{x : \operatorname{card}\Lambda(x) = n\}$ and measurable maps $f_{j_1}, \dots, f_{j_n}$ satisfying (8.4).

Obviously it suffices to show that, for n fixed and $C \subset B_n$ compact, the measure $B \mapsto \int_C \|\langle T, \pi, x\rangle\|_p(B)\, dx$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set. By a further approximation based on Lusin's theorem we can also assume that $f_{j_1}, \dots, f_{j_n}$ are continuous in C. Finally, since $f_{j_i}(x) \neq f_{j_\ell}(x)$ whenever $x \in B_n$ and $i \neq \ell$ we can also assume that the sets $K_i := f_{j_i}(C)$, $i = 1, \dots, n$, are pairwise disjoint. Notice that $\pi : K_i \to C$ is injective and its inverse is f_{j_i}.

We consider now $u_i = d(\cdot, K_i)$ and let $s > 0$ be the least distance between the sets K_i, so that for $s_i \in (0, s/2)$ the sets $\{u_i < s\}$ are pairwise disjoint; thanks to the commutativity of slice and restriction, for $\mathscr{L}^1$-a.e $s_i > 0$ we have

$$\langle T \llcorner \{u_i < s_i\}, \pi, x\rangle = \langle T, \pi, x\rangle \llcorner \{u_i < s_i\} \qquad \text{for } \mathscr{L}^k\text{-a.e. } x \tag{6.2}$$

for $i = 1, \dots, n$. Choosing $s_i \in (0, s/2)$ with this property and setting $T_i := T \llcorner \{u_i < s_i\}$, we have

$$\int_C \|\langle T, \pi, x\rangle\|_p(B)\, dx = \sum_{i=1}^n \int_C \|\langle T_i, \pi, x\rangle\|_p(B)\, dx$$

and it suffices to show that all measures $\mu_i(B) := \int_C \|\langle T_i, \pi, x\rangle\|_p(B)\,dx$ are concentrated on a countably $\mathscr{H}^k$-rectifiable set. By (6.2) it follows that $\mathscr{L}^k$-almost all measures $\|\langle T_i, \pi, x\rangle\|_p$, $x \in C$, are Dirac masses concentrated on $f_{j_i}(x)$.

We now fix i and prove that μ_i is concentrated on a countably $\mathscr{H}^k$-rectifiable set by applying Theorem 5.1(ii) to the inverse f_{j_i} of $\pi|_{K_i}$. Let us consider the sets

$$C_z := \{t \in \mathbf{R} :\ (z,t) \in C\}, \qquad K_{iz} := \{x \in K_i :\ (\pi_1, \ldots, \pi_{n-1})(x) = z\}$$

and the maps $g_z(t) := f_{j_i}(z,t) : C_z \to K_{iz}$. We claim that, for $\mathscr{L}^{k-1}$-a.e. z, $\delta_t g_z < \infty$ $\mathscr{L}^1$-a.e. in C_z. Indeed, writing $x = (z,t)$ and

$$\langle T_i, \pi, x\rangle = \langle S_z, \pi_k, t\rangle \qquad \text{with} \qquad S_z := \langle T_i, (\pi_1, \ldots, \pi_{k-1}), z\rangle$$

we know that for $\mathscr{L}^{k-1}$-a.e. z the flat chain $S_z \in \mathscr{F}_1(E)$ has finite $\mathbf{M}_p$ mass and $\|\langle S_z, \pi_k, t\rangle\|_p$ is a Dirac mass on $g_{iz}(t)$ for $\mathscr{L}^1$-a.e. $t \in C_z$.

We fix now z with these properties; since $S_z \in \mathscr{F}_1(E)$, Corollary 4.2 provides $S'_z \in \mathcal{I}_1(E)$ with $S'_z = S_z \bmod(p)$. Then, we can find countably $\mathscr{H}^1$-rectifiable set G on which S'_z is concentrated, and therefore a $\mathscr{L}^1$-negligible set N_z such that $\langle S'_z, \pi_k, t\rangle$ is concentrated on G for all $t \in \mathbf{R} \setminus N_z$. But since $\langle S'_z, \pi_k, t\rangle = \langle S_z, \pi_k, t\rangle \bmod(p)$ for $\mathscr{L}^1$-a.e. t, possibly adding to N_z another $\mathscr{L}^1$-negligible set we can assume that $\|\langle S'_z, \pi_k, t\rangle\|_p$ is a Dirac mass on $g_{iz}(t) \in G$ for all $t \in C_z \setminus N_z$. We denote by $\tilde{K}_{iz} \subset G$ the set

$$\tilde{K}_{iz} := \{g_{iz}(t) :\ t \in C_z \setminus N_z\}$$

which is countably $\mathscr{H}^1$-rectifiable as well and contained in K_{iz}. Notice also that $\mathscr{L}^1(\pi_k(K_{iz} \setminus \tilde{K}_{iz})) = 0$ because this set is contained in N_z. Since $\pi_k|_{K_{iz}}$ is injective, we can now apply Proposition 5.2 with $K = \tilde{K}_{iz}$ and $\Gamma = K_{iz}$ to obtain that $\delta((\pi_k)|_{K_{iz}})^{-1} < \infty$ $\mathscr{L}^1$-a.e. on $\pi_k(\tilde{K}_{iz})$ and therefore $\mathscr{L}^1$-a.e. on $\pi_k(K_{iz})$. But, since the inverse of $\pi|_{K_i}$ is f_{j_i}, the inverse of $(\pi_k)|_{K_{iz}}$ is g_z. It follows that $\delta g_z < \infty$ $\mathscr{L}^1$-a.e. on C_z.

This proves the claim. Thanks to the commutativity of slice and restriction, a similar property is fulfilled by f_{j_i} with respect to the other $(k-1)$ variables, hence Theorem 5.1(i) ensures that $\delta_x f_{j_i} < \infty$ $\mathscr{L}^k$-a.e. on C. This ensures that Theorem 5.1(ii) is applicable to f_{j_i}, and in turn the fact that μ_i is concentrated on a $\mathscr{H}^k$-rectifiable set. □

We recall that the supremum $\mathcal{M} - \sup_{i\in I} \mu_i$ of a family of measures $\{\mu_i\}_{i\in I}$ is the smallest measure greater than all μ_i; it can be constructively defined by

$$\mathcal{M} - \sup_{i\in I} \mu_i(B) := \sup \sum_{j=1}^{N} \mu_{i_j}(B_j) \tag{6.3}$$

where the supremum runs among all finite Borel partitions $B_1, \ldots, B_N$ of B, with $i_1, \ldots, i_N \in I$.

Proposition 6.3. *Let $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass. Then $\|T\|_p^*$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set.*

Proof. Let I be an index set for $\big[\mathrm{Lip}_1(E)\big]^k$, and consider for any $n \in \mathbf{N}$ a finite set $J_n \subset I$ such that

$$\|T\|_p^*(E) \le \mathcal{M} - \sup_{i \in J_n} \|T \llcorner d\pi_i\|_p(E) + 2^{-n}$$

(its existence is a direct consequence of (6.3)). Then, denoting by J the union of the sets J_n, the measure

$$\sigma := \mathcal{M} - \sup_{i \in J} \|T \llcorner d\pi_i\|_p$$

is smaller than $\|T\|_p^*$ and with the same total mass, hence it coincides with $\|T\|_p^*$. Since J is countable, a countably $\mathscr{H}^k$-rectifiable concentration set for σ can be obtained by taking the union of countably $\mathscr{H}^k$-rectifiable sets, given by Proposition 6.2, on which the measures $\|T \llcorner d\pi_i\|_p$, $i \in J$, are concentrated. □

7. Absolute continuity of $\|T\|_p$

In this section we prove the absolute continuity of $\|T\|_p$ with respect to $\|T\|_p^*$, and therefore the fact that also $\|T\|_p$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set. Then, we can prove, using the isoperimetric inequality, density lower bounds for $\|T\|_p$; these imply that the (minimal) concentration set has actually finite $\mathscr{H}^k$-measure.

The absolute continuity of $\|T\|_p$ depends on the following extension of Theorem 3.7 to all flat chains with finite $\mathbf{M}_p$ mass. We are presently able to prove this extension, relying on the finite-dimensional results in [25] (in turn based on the deformation theorem in [26]), only in a smaller class of spaces E.

Definition 7.1. *We say that a Banach space $(F, \|\cdot\|)$ has the strong finite-dimensional approximation property if there exist maps $\pi_n : F \to F$, with uniformly bounded Lipschitz constants, such that $\pi_n(F)$ is contained in a finite-dimensional subspace of F and*

$$\lim_{n\to\infty} \|x - \pi_n(x)\| = 0 \qquad \forall x \in F.$$

Obviously all spaces having a Schauder basis (and, in particular, separable Hilbert spaces) have the strong finite-dimensional approximation property and, in this case, π_n can be chosen to be linear. Unfortunately this assumption does not cover ℓ_∞ spaces, which satisfy only the weak finite-dimensional approximation property considered in [4].

We begin with a technical lemma on the commutativity of slice and push-forward; the validity of this identity for rectifiable currents is proved in [4, Lemma 5.9]; its extension to $\mathscr{F}_k(E)$ can be proved arguing as in Lemma 3.1 and in Lemma 6.1, so we omit a detailed proof.

Lemma 7.2 (Slice and push-forward commute)**.** *Let $f \in \mathrm{Lip}(E; \mathbf{R}^n)$ and $T \in \mathscr{F}_k(E)$. Let $q : \mathbf{R}^n \to \mathbf{R}^k$, $q^\perp : \mathbf{R}^n \to \mathbf{R}^{n-k}$ be respectively the projections on the first k coordinates and on the last $n-k$ coordinates. Then*

$$q^\perp_\sharp \langle f_\sharp T, q, x\rangle = (q^\perp \circ f)_\sharp \langle T, q\circ f, x\rangle \qquad \textit{for } \mathscr{L}^k\textit{-a.e. } x \in \mathbf{R}^k.$$

Then, we recall the basic result of [25], a consequence of the deformation theorem in [26]. Thanks to Proposition 2.2 we can state White's result in our language of currents mod(p), instead of flat chains with coefficients in $\mathbf{Z}_p$.

Theorem 7.3. *Let $T \in \mathscr{F}_k(\mathbf{R}^N)$. Then $T = 0 \mod(p)$ if and only if, for all orthogonal projections q on a k-dimensional subspace of $\mathbf{R}^N$, $\langle T, q, x\rangle = 0 \mod(p)$ for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$.*

Theorem 7.4. *Assume that E is a compact convex subset of a Banach space F having the strong finite-dimensional approximation property, and that E is a Lipschitz retract of F.*
Let $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass and assume that, for some $m \in [1, k]$ the following property holds:

$$\text{for all } \pi \in [\mathcal{D}]^m,\ \langle T, \pi, x\rangle = 0 \mod(p) \text{ for } \mathscr{L}^m\text{-a.e. } x \in \mathbf{R}^m.$$

Then $T = 0 \mod(p)$.

Proof. We shall directly prove the statement in the case $m = k$, which obviously implies all others, by the definition of iterated slice operator. Let $\pi_n : F \to F$ be given by the strong finite-dimensional approximation property and let $T_n = \pi_{n\sharp}T$. We shall prove in the first step that $T_n = 0 \mod(p)$, and in the second one that $T_n \to T$ in $\mathscr{F}$ distance in F. Considering the images S_n of T_n under a Lipschitz retraction of F onto E, which converge to T in $\mathscr{F}$ distance in E and are still equal to 0 mod(p), this implies that $T = 0$ mod(p).
Step 1. Since the range of π_n is finite-dimensional we can obviously think of T_n as a flat chain in a suitable Euclidean space $\mathbf{R}^N$. So, by Theorem 7.3, it suffices to show that the slices induced by orthogonal projections q on k-planes vanish. With no loss of generality we can assume that q is the orthogonal projection on the first k coordinates, and apply Lemma 7.2 with $f = \pi_n$ to obtain that

$$q^\perp_\sharp\langle T_n, q, x\rangle = 0 \mod(p)$$

for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$. But since $\|\langle T_n, q, x\rangle\|_p$ is concentrated on $\{q = x\}$, and $q^\perp : \{q = x\} \to \{q = 0\}$ is an isometry, it follows that $\langle T_n, q, x\rangle = 0 \mod(p)$ for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$.
Step 2. Let E_1 be the compact metric space $E \cup \bigcup_n \pi_n(E_n)$ and let $E_2 \subset F$ be its closed convex hull. In order to conclude, it suffices to show that $\mathscr{F}(T_n - T) \to 0$ in E_2. Taking into account subadditivity of the flat norm and (2.1), it suffices to show that $\mathscr{F}(\pi_{n\sharp}R - R) \to 0$ for all $R \in \mathcal{I}_k(E_2)$; by density in mass norm, it suffices to prove this fact for $R \in \mathbf{I}_k(E_2)$. Obviously $\pi_{n\sharp}R \to R$ weakly in E_2, i.e. in the duality with Lipschitz forms; then, it suffices to apply [22] to obtain convergence in flat norm in E_2. □

We can now prove two basic absolute continuity properties of $\|T\|_p$.

Theorem 7.5. *Let $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass. Then $\|T\|_p \ll \|T\|_p^*$. In particular $\|T\|_p$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set.*

Proof. We fix a compact set K such that $\|T\|_p^*(K) = 0$ and we have to show that $\|T\|_p(K) = 0$. Let $\pi \in [\mathcal{D}]^k$, $u = d(\cdot, K)$ and let $N \subset \mathbf{R}$ be the Lebesgue negligible set as in (2.10). If we consider any sequence $(s_j) \subset \mathbf{R} \setminus N$ with $s_j \downarrow 0$, then $T \llcorner \{u < s_j\}$ converges with respect to $\mathscr{F}_p$ to $S \in \mathscr{F}_k(E)$ with $[S] = [T] \llcorner K$. The commutativity of slice and restriction gives

$$\|\langle T \llcorner \{u < s\}, \pi, x\rangle\|_p(E) = \|\langle T, \pi, x\rangle\|_p(\{u < s\}) \qquad \text{for } \mathscr{L}^k\text{-a.e. } x \in \mathbf{R}^k$$

for $\mathscr{L}^1$-a.e. $s > 0$. Choosing $(s_j) \downarrow 0$ with this additional property, and assuming also that $\sum_j \mathscr{F}_p(T \llcorner \{u < s_j\} - S) < \infty$, from (2.18) we infer

$$\lim_{j\to\infty} \langle T \llcorner \{u < s_j\}, \pi, x\rangle = \langle S, \pi, x\rangle \qquad \text{with respect to } \mathscr{F}_p, \text{ for } \mathscr{L}^k\text{-a.e. } x \in \mathbf{R}^k.$$

Since

$$\|\langle T \llcorner \{u < s_j\}, \pi, x\rangle\|_p(E) = \|\langle T, \pi, x\rangle\|_p(\{u < s_j\}) \to 0 \qquad \text{for } \mathscr{L}^k\text{-a.e. } x \in \mathbf{R}^k$$

it follows that $\langle S, \pi, x\rangle = 0 \bmod(p)$ for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$. Then, Theorem 7.4 gives that $[T] \llcorner K = 0$. By (2.11) it follows that $\|T\|_p(K) = \mathbf{M}_p([T] \llcorner K) = 0$. □

A direct consequence of Corollary 3.9, ensuring the absolute continuity of $\|T\|_p$ with respect to $\|T\|_p^*$, is the density upper bound

$$\limsup_{r\downarrow 0} \frac{\|T\|_p(B_r(x))}{r^k} < \infty \qquad \text{for } \|T\|_p\text{-a.e. } x \in E. \tag{7.1}$$

Indeed, general covering arguments imply that the set of points where the lim sup is $+\infty$ is $\mathscr{H}^k$-negligible (see e.g. [5, Theorem 2.4.3]), and hence $\|T\|_p$-negligible.

We are now going to a density lower bound for the measure $\|T\|_p$ that gives, as a byproduct, the finiteness of the measure theoretic support of flat chains with finite $\mathbf{M}_p$ mass, completing the proof of Theorem 1.1. Since we can write $T = R + \partial S$ with $R \in \mathcal{I}_k(E)$, possibly replacing T by $T - R$ we need only to consider chains T with $\partial T = 0$.

We use a general principle, maybe first introduced by White [24], and then used in [9], [6] in different contexts: any lower semicontinuous and additive energy has the property that any object with finite energy, when seen on a sufficiently small scale, is a quasiminimizer.

Proposition 7.6. *Let $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass and $\partial T = 0 \bmod(p)$. Then, for all $\varepsilon > 0$ the following holds: for $\|T\|_p$-a.e. x there exists $r_\varepsilon(x) > 0$ such that*

$$\|T\|_p(\overline{B}_r(x)) \leq 2\|S + T\|_p(\overline{B}_r(x)) \tag{7.2}$$

whenever $r \in (0, r_\varepsilon(x))$, $\|T\|_p(B_r(x)) \geq \varepsilon r^k$, $\partial[S] = 0$ and $[S] \llcorner \big(E \setminus \overline{B}_r(x)\big) = 0$.

Proof. Assume that for some $\varepsilon > 0$ the statement fails. Then, there exists a compact set $K \subset E$ with $\|T\|_p(K) > 0$ such that, for all $x \in K$, we can find balls $\overline{B}_r(x)$ with arbitrarily small radius satisfying $\|T\|_p(\overline{B}_r(x)) \geq \varepsilon r^k$ and cycles $[S_{r,x}]$ with $\|T\|_p(\overline{B}_r(x)) \geq 2\|T + S_{r,x}\|_p(\overline{B}_r(x))$ and $[S_{r,x}] \llcorner \big(E \setminus \overline{B}_r(x)\big) = 0$.

Let $\delta > 0$. By a classical covering argument (see for instance [5, Theorem 2.2.2]), we can find a disjoint family $\{\overline{B}_{r_i}(x_i)\}_{i\in I}$ of these balls, all with radii $r_i < \delta$, whose union covers $\mathscr{H}^k$-almost all, and hence $\|T\|_p$-almost all, of K. Set now

$$[T_\delta] := [T] + \sum_{i\in I}[S_{r_i,x_i}] = [T]\llcorner\big(E\setminus \cup_{i\in I}\overline{B}_{r_i}(x_i)\big) + \sum_{i\in I}[T+S_i]\llcorner\overline{B}_{r_i}(x_i),$$

where $S_i := S_{r_i,x_i}$. By approximating with finite unions and sums, taking into account that $\mathbf{M}_p([S_i]) \le 3\|T\|_p(\overline{B}_{r_i}(x_i))/2$, it is not hard to show that $[T_\delta] \in \mathscr{F}_{p,k}(E)$ and that

$$\mathbf{M}_p([T_\delta]) \le \|T\|_p(E\setminus\cup_{i\in I}\overline{B}_{r_i}(x_i)) + \frac{1}{2}\sum_{i\in I}\|T\|_p(\overline{B}_{r_i}(x_i)) \le \|T\|_p(E\setminus K) + \frac{1}{2}\sum_{i\in I}\|T\|_p(\overline{B}_{r_i}(x_i)).$$

In addition, denoting by $[S_i']$ cycles with $\partial[S_i'] = [S_i]$ given by the isoperimetric inequality we have

$$\mathscr{F}_p([T_\delta]-[T]) \le \mathscr{F}_p(\sum_i[S_i]) \le \mathscr{F}_p(\sum_i[S_i']) \le \delta_k 3^{1+1/k}\sum_i\big[\|T\|_p(B_{r_i}(x_i))\big]^{1+1/k}.$$

Since $\|T\|_p(B_\delta(x)) \to 0$ uniformly on K as $\delta\downarrow 0$, it follows that $[T_\delta]\to[T]$ in $\mathscr{F}_{p,k}(E)$ as $\delta\downarrow 0$. Hence, letting $\delta\downarrow 0$ the lower semicontinuity of $\mathbf{M}_p$ provides the inequality $\|T\|_p(E) \le \|T\|_p(E\setminus K) + \|T\|_p(K)/2$, which implies $\|T\|_p(K) = 0$. □

Then, a general and well-known argument based on the isoperimetric inequalities and an ODE argument provides the following result:

Theorem 7.7. *Let $T\in\mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass and $\partial T = 0 \bmod(p)$. Then*

$$\liminf_{r\downarrow 0}\frac{\|T\|_p(B_r(x))}{r^k} \ge c > 0 \qquad \text{for } \|T\|_p\text{-a.e. } x \tag{7.3}$$

with $c>0$ depending only on k. As a consequence $\|T\|_p$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set with finite $\mathscr{H}^k$-measure.

Proof. First of all, we notice that

$$\limsup_{r\downarrow 0}\frac{\|T\|_p(B_r(x))}{r^k} > 0 \qquad \text{for } \|T\|_p\text{-a.e. } x\in E.$$

Indeed, if B is the set where the lim sup above vanishes, general differentiation results (see e.g. [5, Theorem 2.4.3]) imply that $\|T\|_p(B') = 0$ for any Borel set $B'\subset B$ with $\mathscr{H}^k(B') < \infty$. Since, by Theorem 7.5, $\|T\|_p$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set, and vanishes on $\mathscr{H}^k$-negligible sets, it follows that $\|T\|_p(B) = 0$.

Let $\varepsilon > 0$ and let us denote by C_ε the set where the lim sup above is larger than 2ε; Proposition 7.6 yields, for $\|T\|_p$-a.e. $x\in C_\varepsilon$, $r_\varepsilon(x) > 0$ such that (7.2) holds whenever $r\in(0,r_\varepsilon(x))$, $\|T\|_p(B_r(x)) \ge \varepsilon r^k$ and $[S]$ is a k-cycle satisfying $[S-T]\llcorner\big(E\setminus\overline{B}_r(x)\big) = 0$. Then, the energy comparison argument and the isoperimetric inequality give, for all such points x, the differential inequality

$$\frac{d}{dr}\|T\|_p^{1/k}(B_r(x)) \ge d_k > 0$$

for $\mathscr{L}^1$-a.e. $r \in (0, r_\varepsilon(x))$ such that $\|T\|_p(B_r(x)) > \varepsilon r^k$, for some constant d_k independent of ε. Since $x \in C_\varepsilon$ we know that the condition $\|T\|_p(B_r(x)) > \varepsilon r^k$ is fulfilled by arbitrarily small radii $r \in (0, r_\varepsilon(x))$ and we claim that, provided that $\varepsilon < d_k^k$, if it holds for some r, it holds for all $r' \in (r, r_\varepsilon(x))$: indeed, if r' is the smallest $r' \in (r, r_\varepsilon(x))$ for which it fails, in the interval (r, r') the function $\|T\|_p^{1/k}(B_r(x))$ has derivative larger than d_k, while $\varepsilon^{1/k} r$ has a smaller derivative. It follows that $\|T\|_p(B_r(x)) > \varepsilon r^k$ for all $r \in (0, r_\varepsilon(x))$ and the differential inequality yields (7.3) at $\|T\|_p$-a.e. $x \in C_\varepsilon$ with $c = d_k^k$. Since $\cup_{\varepsilon>0} C_\varepsilon$ cover $\|T\|_p$-almost all of E the proof is finished. □

Finally, we complete the list of announced result with the proof of Corollary 1.3.

Proof. The statement can be easily checked for chains $T \in \mathscr{F}_k(\mathbf{R}^k)$ since $\mathscr{F}_k(\mathbf{R}^k) = \mathcal{I}_k(\mathbf{R}^k)$ (recall that $\mathbf{R}^k$ can't support a nonzero $(k+1)$-dimensional integer rectifiable current). In the general case, let $T \in \mathscr{F}_k(E)$ with finite $\mathbf{M}_p$ mass, let $S \subset E$ be a countably $\mathscr{H}^k$-rectifiable Borel set with finite $\mathscr{H}^k$-measure where $\|T\|_p$ is concentrated and let $B_i \subset \mathbf{R}^k$ be compact, $f_i : B_i \to E$ be such that $f_i(B_i)$ are pairwise disjoint, $\cup_i f_i(B_i)$ covers $\mathscr{H}^k$-almost all of S and $f_i : B_i \to f_i(B_i)$ is bi-Lipschitz, with Lipschitz constants less than 2. By McShane's extension theorem we can also assume that $f_i : E \to \mathbf{R}^k$ are globally defined and Lipschitz. Then, we can find $\theta_i \in L^1(\mathbf{R}^k, \mathbf{Z})$ with $|\theta_i| \le p/2$ and $\theta_i = 0$ out of B_i such that $(f_i)_\sharp([T] \llcorner f_i(B_i)) = [[\![\theta_i]\!]]$. Since

$$\sum_i \int_{B_i} |\theta_i| \, dx = \sum_i \mathbf{M}_p([\![\theta_i]\!]) \le 2^k \sum_i \mathbf{M}_p([T] \llcorner f_i(B_i)) \le 2^k \|T\|_p(E) < \infty$$

if follows that the current $S := \sum_i (f_i)_\sharp [\![\theta_i]\!] \in \mathcal{I}_k(E)$ is well defined and, by construction, $[S] = [T]$. In addition, since $f_i(B_i)$ are pairwise disjoint, the multiplicity of S takes values in $[-p/2, p/2]$, hence [7, Theorem 8.5] gives that $\mathbf{M}(S) = \mathbf{M}_p(S)$. □

8. Appendix

In this appendix we state some technical results.

Lemma 8.1. *Let $k \ge 1$ and $m \in [1, k]$. Let $G : \mathscr{F}_{k-m}(E) \to [0, +\infty]$ be continuous with respect to the $\mathscr{F}$ distance and let $S \in \mathcal{I}_k(E)$. Then, for all $\pi \in \left[\mathrm{Lip}(E)\right]^m$, the function $x \mapsto G(\langle S, \pi, x\rangle)$ is Lebesgue measurable.*

Proof. With no loss of generality we can assume that $\pi \in \left[\mathrm{Lip}_1(E)\right]^m$. It suffices to show that $x \mapsto \langle S, \pi, x\rangle$ is the pointwise $\mathscr{L}^m$-a.e. limit of simple maps. In order to make a diagonal argument we use, instead, local convergence in measure, so our goal is to find simple maps $f_h : \mathbf{R}^m \to \mathscr{F}_{k-m}(E)$ such that

$$\lim_{h\to\infty} \mathscr{L}^m\left(\{x \in B_R(0) : \ \mathscr{F}(\langle S, \pi, x\rangle - f_h(x)) > \varepsilon\}\right) = 0 \qquad \forall R > 0, \ \forall \varepsilon > 0.$$

Since $\mathbf{I}_k(E)$ is dense in $\mathcal{I}_k(E)$, by a first diagonal argument we can assume with no loss of generality that $S \in \mathbf{I}_k(E)$, so that $S_x := \langle S, \pi, x\rangle \in \mathbf{I}_{k-m}(E)$ for $\mathscr{L}^m$-a.e. x and

$$\int_{\mathbf{R}^m} \mathbf{M}(S_x) + \mathbf{M}(\partial S_x) \, dx \le \mathbf{M}(S) + \mathbf{M}(\partial S) < \infty.$$

In $\mathbf{I}_{k-m}(E)$ we consider the distance

$$d(T,T') := \sup\left\{|T(fdp) - T'(fdp)| :\ |f| \le 1,\ f, p_1, \ldots, p_{k-m} \in \mathrm{Lip}_1(E)\right\}.$$

Notice that $d(T,T') \le \mathbf{M}(T-T')$ and $d(\partial T, \partial T') \le \mathbf{M}(T-T')$, so that $d(T,T') \le \mathscr{F}(T-T')$ and $\mathscr{F}$ convergence is stronger than d-convergence. On the other hand, thanks to the results in [22] the two distances are equivalent in the sets $\{T \in \mathbf{I}_{k-m}(E) : \mathbf{M}(T) + \mathbf{M}(\partial T) \le M\}$, $M > 0$. Since, according to [4], S_x is an MBV map with respect to d, we can divide $\mathbf{R}^m$ in open cubes Q_h^j with sides $1/h$ and apply the Poincaré inequality for MBV maps (see [2] or [12]) in each of these cubes to find $x_j \in Q_h^j$ with

$$\int_{Q_h^j} d(S_{x_j}, S_x)\, dx \le \frac{c}{h}\mu(Q_h^j),$$

where μ is the total variation measure of $x \mapsto S_x$. It follows that the piecewise constant map g_h equal to S_{x_j} on Q_h^j satisfies $\int_{\mathbf{R}^m} d(g_h(x), S_x)\, dx \to 0$. In order to improve this convergence from d to $\mathscr{F}$ we argue as follows: we notice that

$$\mathscr{L}^m\big(\{y \in Q_h^j :\ d(S_y, S_{x_j}) > 4ch^{m-1}\mu(Q_h^j)\}\big) \le \frac{1}{4}\mathscr{L}^m(Q_h^j),$$

$$\mathscr{L}^m\Big(\{y \in Q_h^j :\ \mathbf{M}(S_y) + \mathbf{M}(\partial S_y) > 4h^m \int_{Q_h^j} \mathbf{M}(S_x) + \mathbf{M}(\partial S_x)\, dx\}\Big) \le \frac{1}{4}\mathscr{L}^m(Q_h^j) \quad (8.1)$$

and therefore we can find points y_j in the intersection of the complements of these sets. By the triangle inequality

$$d(S_{y_j}, S_x) \le 4ch^{m-1}\mu(Q_h^j) + d(S_{x_j}, S_x)$$

so that, if we denote by f_h the piecewise constant map equal to S_{y_j} on Q_h^j, we still have

$$\lim_{h\to\infty} \int_{\mathbf{R}^m} d(f_h(x), S_x)\, dx = 0. \quad (8.2)$$

In addition, taking (8.1) into account, we have also

$$\sup_{h\in\mathbf{N}} \int_{\mathbf{R}^m} \mathbf{M}(f_h(x)) + \mathbf{M}(\partial f_h(x))\, dx \le 4\int_{\mathbf{R}^m} \mathbf{M}(S_x) + \mathbf{M}(\partial S_x)\, dx < \infty. \quad (8.3)$$

By (8.2) and (8.3), taking into account that d and $\mathscr{F}$ are equivalent on the sets $\{T : \mathbf{M}(T) + \mathbf{M}(\partial T) \le M\}$ it is easy to infer the local convergence in measure of f_h to S_x with respect to $\mathscr{F}$ (given $\delta > 0$ and $R > 0$ it suffices to find M such that all sets $B_R \cap \{\mathbf{M}(f_h) + \mathbf{M}(\partial f_h) > M\}$ and $B_R \cap \{\mathbf{M}(f) + \mathbf{M}(\partial f) > M\}$ and have measure less than δ, then choose $\varepsilon > 0$ such that $d(S, S') < \varepsilon$ implies $\mathscr{F}(S - S') < \delta$ whenever $\mathbf{M}(S) + \mathbf{M}(\partial S) \le M$; eventually one can use the fact that $B_R \cap \{d(f_h, f) > \varepsilon\}$ has measure less than $\delta/3$ for h sufficiently large). □

We now state a standard result on measurable set-valued functions, see for instance [8].

Lemma 8.2. *Let us assign for all $x \in \mathbf{R}^k$ a finite set $\Lambda(x) \subset E$, and let us assume that $\{x : \Lambda(x) \cap C \neq \emptyset\}$ is Lebesgue measurable for all closed sets $C \subset E$. Then the sets*

$$B_n := \left\{x \in \mathbf{R}^k : \operatorname{card}\Lambda(x) = n\right\}$$

are Lebesgue measurable and there exist Lebesgue measurable maps $f_{j_1}, \dots, f_{j_n} : B_n \to E$ such that

$$\Lambda(x) = \{f_{j_1}(x), \dots, f_{j_n}(x)\} \qquad \text{for } \mathscr{L}^k\text{-a.e. } x \in B_n. \tag{8.4}$$

Finally, we conclude this appendix by comparing $\mathscr{F}_p$ with the "polyhedral" flat distance $\mathscr{F}_p^P$ in (2.20).

Proposition 8.3. *There exists $C = C(n,k)$ satisfying*

$$\mathscr{F}_p^P(T) \leq C\mathscr{F}_p(T) \qquad \text{for all } T \in \mathbf{I}_k(\mathbf{R}^n) \text{ weakly polyhedral.}$$

Proof. Denoting in this proof by c a generic constant depending on dimension and codimension, let us recall the Federer-Fleming deformation theorem mod(p): for $\epsilon > 0$ given, any $R \in \mathbf{I}_k(\mathbf{R}^n)$ can be written as $P + U + \partial Q$, with P polyhedral on the scale ϵ, $\mathbf{M}_p(P) \leq c(\mathbf{M}_p(R) + \epsilon\mathbf{M}_p(\partial R))$, $\mathbf{M}_p(\partial P) \leq c\mathbf{M}_p(\partial R)$, $\mathbf{M}(U) \leq c\epsilon\mathbf{M}_p(\partial R)$ and $\mathbf{M}_p(Q) \leq c\epsilon\mathbf{M}_p(R)$. The main observation is that, in the case when ∂R is weakly polyhedral, the construction (based on piecewise affine deformations of R on skeleta of lower and lower dimension, until dimension k is reached) of P, U and Q provides us with a current U which is weakly polyhedral as well. Indeed, U corresponds to the k-surface swapt by ∂R during the deformation.

Now, assume that $T = R + \partial S$ with $R \in \mathbf{I}_k(\mathbf{R}^n)$ and $S \in \mathbf{I}_{k+1}(\mathbf{R}^n)$ and let us write $R = P + U + \partial Q$ as above. Since $\partial R = \partial T$ is weakly polyhedral, it follows that U is weakly polyhedral as well. Now we write $T = P + U + \partial(S + Q)$ and apply the deformation theorem again to $S + Q$ to obtain $S + Q = P' + U' + \partial Q'$. Again, since $\partial(S + Q) = \partial(T - P - U)$ is weakly polyhedral, we know that U' is weakly polyhedral. Now we have $T = (P + U) + \partial(P' + U')$ where $P + U$ and $P' + U'$ are both weakly polyhedral, so that $\mathscr{F}_p^P(T) \leq \mathbf{M}_p(P + U) + \mathbf{M}_p(P' + U')$. We have also

$$\mathbf{M}_p(P) \leq c(\mathbf{M}(R) + \epsilon\mathbf{M}_p(\partial R)) = c\mathbf{M}_p(R) + c\epsilon\mathbf{M}_p(\partial T), \quad \mathbf{M}_p(U) \leq c\epsilon\mathbf{M}_p(\partial R) = c\epsilon\mathbf{M}_p(\partial T).$$

Analogously we have $\mathbf{M}_p(P') + \mathbf{M}_p(U') \leq c\mathbf{M}_p(R) + c\mathbf{M}_p(S) + c\epsilon(\mathbf{M}_p(T) + \mathbf{M}_p(\partial T))$ and, since $\epsilon > 0$ is arbitrary, we conclude. □

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