THE EQUIVARIANT CONCORDANCE GROUP IS NOT ABELIAN

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Abstract. We prove that the group $\widetilde{\mathcal{C}}$ of equivariant concordance of directed strongly invertible knots, defined by Sakuma [Sak86], is not abelian. We do so by exhibiting an infinite family of nontrivial commutators.

1. Introduction

A knot $K \subset S^3$ is said to be *strongly invertible* if there is an orientation preserving smooth involution ρ of S^3 such that $\rho(K) = K$ and ρ reverses the orientation on K. By introducing the notion of a *direction* on a strongly invertible knot, Sakuma [Sak86] was able to define unambiguously an operation of *equivariant connected sum* between *directed strongly invertible knots*. Moreover, he proved that this operation is not commutative, which is in stark contrast with the usual connected sum of oriented knots. In the same paper, Sakuma defined the *equivariant concordance group* $\widetilde{\mathcal{C}}$ as the quotient of the set of directed strongly invertible knots under the equivalence relation of *equivariant concordance*, with the group operation naturally induced by the equivariant connected sum.

Since the equivariant connected sum is not commutative, it is a natural question whether the equivariant concordance group $\widetilde{\mathcal{C}}$ is abelian. Alfieri and Boyle [AB15] speculate that $\widetilde{\mathcal{C}}$ contains a copy of $\mathbb{Z} * \mathbb{Z}$, and hence is nonabelian. Recently Dai, Mallick and Stoffregen [DMS75] define a homomorphism

$$h_{\tau,\iota} : \widetilde{\mathcal{C}} \longrightarrow \mathfrak{K}_{\tau,\iota},$$

where $\mathfrak{K}_{\tau,\iota}$ is a (potentially) nonabelian group defined using the action induced by the strong inversion of a knot K on the knot Floer complex $\mathcal{CFK}(K)$. Since the group $\mathfrak{K}_{\tau,\iota}$ is not (a priori) commutative, the authors observe that $h_{\tau,\iota}$ could in principle lead to a negative answer to the open question of whether $\widetilde{\mathcal{C}}$ is abelian.

In this paper we present a family of examples which answer the question in the negative, showing that the equivariant concordance group $\widetilde{\mathcal{C}}$ is indeed not abelian. First of all, we recall some definitions and facts (see [BI13] for the details). Given a knot K with a strong inversion ρ we say that $\mathrm{Fix}(\rho)$ is the *axis* of the inversion. By the solution of the Smith conjecture [BM84], the axis is an unknotted S^1 which meets K in two points. A *directed strongly invertible knot* K is a knot with a strong inversion ρ and the additional data of the choice of one of the components of $\mathrm{Fix}(\rho) \setminus K$ (a *half-axis*) and an orientation on $\mathrm{Fix}(\rho)$. Note that the points in $K \cap \mathrm{Fix}(\rho)$ have a natural order: the initial point of the chosen half-axis is the first fixed point while the other end is the second fixed point. Two directed strongly invertible knots K and J, with strong inversions ρ_K and ρ_J, are *equivariantly isotopic* if there exists an orientation preserving diffeomorphism φ of S^3 such that $\varphi(K) = J$, $\varphi \circ \rho_K = \rho_J \circ \varphi$ and φ preserves the chosen oriented half-axis. Given two directed strongly invertible knots K and J, we denote their equivariant connected sum by $K\widetilde{\#}J$. The directed strongly invertible knot $K\widetilde{\#}J$ is obtained by cutting K at its second fixed point and J at its first fixed point, gluing the two knots and axes equivariantly, in a way that is compatible with the orientations on the axes, and choosing the half-axis of $K\widetilde{\#}J$ to be the union of the half-axes of K and J. The inverse K^{-1} of a directed strongly invertible knot K in $\widetilde{\mathcal{C}}$ is represented by the mirror of K with the same strong inversion and chosen half-axis, but with the opposite orientation on the axis of the strong inversion.

Let now p be an odd integer and define K_p to be the directed strongly invertible knot given by the torus knot $T_{2,p}$ with the orientation on the axis of the strong inversion described in Figure 1 and chosen half-axis given by the solid one in the figure. Hence first fixed point of K_p is the one on the left in Figure 1, while the second one is on the right.

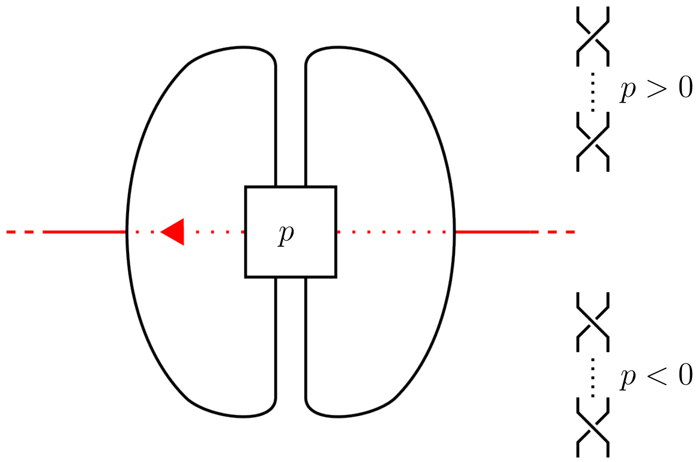

Figure 1. The directed strongly invertible knot K_p. The strong inversion is the π-rotation around the horizontal axis (colored red).

Theorem 1.1. *Let p and q be odd integers such that $1 < p < q$. Then the commutator $[K_p, K_q]$ is not equivariantly slice. In particular the equivariant concordance group $\widetilde{\mathcal{C}}$ is not abelian.*

Finally, in Section 3 we give an alternate proof of the fact that $\widetilde{\mathcal{C}}$ is not commutative, by use of the results in [HKL08], and we briefly explain how to adapt the argument to show that the *cobordism group of θ-curves*, introduced by Taniyama [Tan93], is also not abelian.

2. Proof of Theorem 1.1

Boyle and Issa associate a directed strongly invertible knot K with a 2-component link $L_b(K)$, called the *butterfly link* of K (Definition 4.1 in [BI13]). The link $L_b(K)$ is obtained from K by a band move along a band parallel to the chosen half-axis of K, in such a way that the linking number between the two components of $L_b(K)$ is zero. Recall that a n-component link $L \subset S^3$ is said to be *strongly slice* if it bounds n disjoint disks properly embedded in B^4. The following result is a weaker version of [BI13, Proposition 7], where the authors actually use the fact that $L_b(K)$ is a 2-*periodic* link to prove that if K is trivial in $\widetilde{\mathcal{C}}$ then $L_b(K)$ is *equivariantly* slice.

Proposition 2.1. *Let K be a directed strongly invertible knot which is equivariantly slice. Then its butterfly link $L_b(K)$ is strongly slice.*

Hence, in order to prove Theorem 1.1 we are going to show that the butterfly link $L_b([K_p, K_q])$ is not strongly slice.

Observe that K_p^{-1}, represented in Figure 2, is equivariantly isotopic to K_{-p} through the π-rotation around the vertical axis in the figure. By performing the sequence

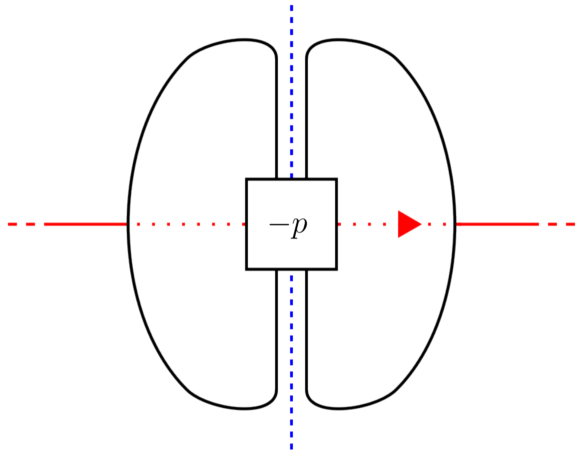

FIGURE 2. The directed strongly invertible knot K_p^{-1}. The π-rotation around the vertical axis (colored blue) shows that $K_p^{-1} = K_{-p}$.

of equivariant connected sums, we see that the directed strongly invertible knot in Figure 3 (where the chosen half-axis is the solid one) represents the commutator $[K_p, K_q] = K_p \widetilde{\#} K_q \widetilde{\#} K_p^{-1} \widetilde{\#} K_q^{-1}$. We obtain now the butterfly link $L_b([K_p, K_q])$ in

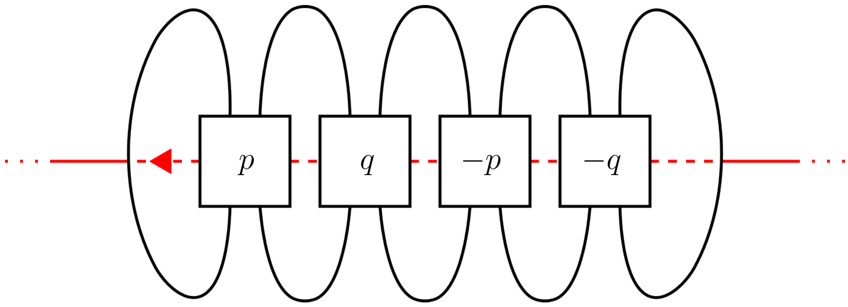

FIGURE 3. The commutator $[K_p, K_q]$.

Figure 4 by a band move along a band parallel to the chosen half-axis of $[K_p, K_q]$ (the solid one in Figure 3).

Since $L_b([K_p, K_q])$ is the pretzel link $P(p, q, -p, -q)$, we can conclude that it is not strongly slice, in application of the following result of Aceto, Kim, Park and Ray.

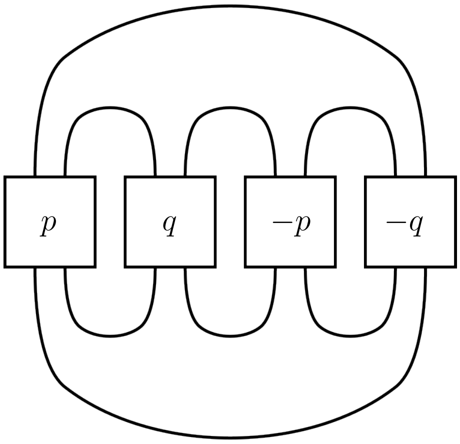

FIGURE 4. The butterfly link of $[K_p, K_q]$.

Theorem 2.2 ([AKPR21], Theorem 1.2)**.** *Let p and q be odd integers such that $1 < p < q$. Then the 2-components pretzel link $P(p, q, -p, -q)$ is not strongly slice.*

Therefore by Proposition 2.1 the commutator $[K_p, K_q]$ is not equivariantly slice. This concludes the proof of Theorem 1.1. □

3. Further remarks

3.1. **An alternate proof.** Herald, Kirk and Livingston [HKL08, Section 11] prove that the pretzel knot $P(3, 5, -3, -5, 7)$ is not topologically slice by use of the twisted Alexander polynomials. This result leads to a proof that the pretzel link $P(3, 5, -3, -5)$ is not topologically strongly slice, and in turn to an alternate proof that $[K_3, K_5]$ is nontrivial in $\widetilde{\mathcal{C}}$, by using Proposition 2.1. In fact, observe that $P(3, 5, -3, -5, 7)$ is obtained by a band move on $P(3, 5, -3, -5)$ which connects the two components of the link, as in Figure 5. This band move can be seen as a

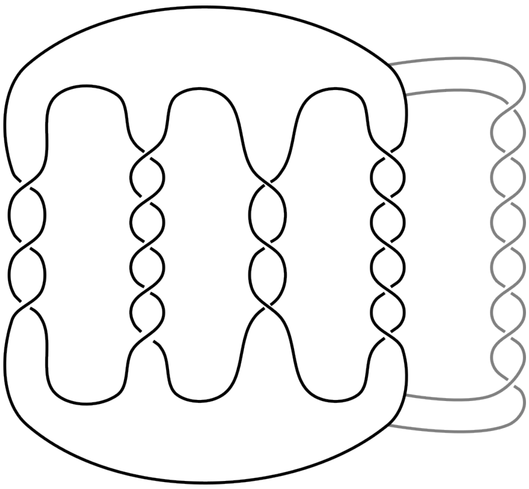

FIGURE 5. The pretzel link $P(3, 5, -3, -5)$ and the band (in grey) that gives $P(3, 5, -3, -5, 7)$.

cobordism $C \subset S^3 \times [0,1]$ of genus 0 (i.e. a pair of pants) between $P(3,5,-3,-5)$ and $P(3,5,-3,-5,7)$. If $P(3,5,-3,-5)$ bounds a pair of (locally flat) disjoint disks $D_1 \sqcup D_2$ in B^4 then $D = (D_1 \sqcup D_2) \cup C \subset B^4 \cup (S^3 \times [0,1]) \cong B^4$ would be a topological slice disk for $P(3,5,-3,-5,7)$, in contradiction with the result from [HKL08].

3.2. **The theta-curves cobordism group is not abelian.** Sakuma in [Sak86] used the relation between strongly invertible knots and θ-curves in order to prove, in particular, that the equivariant connected sum of directed strongly invertible knots is not commutative. The study of the concordance of θ-curves (in the piecewise linear category) was started by Taniyama [Tan93], who defined the *cobordism group of θ-curves* Θ. Miyazaki in [Miy94] provided a proof that the group Θ is not commutative, appealing on a result of Gilmer [Gil83]. However, Friedl [Fri04] found gaps in the proof of the result in [Gil83].

We observe that the example in Subsection 3.1 of a nontrivial commutator in $\widetilde{\mathcal{C}}$ can be adapted to provide a different proof that Θ is not abelian. Notice that the union of a directed strongly invertible knot K with its chosen half-axis produces a θ-curve $\theta(K)$, with vertices ordered as initial and final point of the chosen half-axis (as in Section 1). It is easy to check that this defines a homomorphism

$$\theta : \widetilde{\mathcal{C}} \longrightarrow \Theta.$$

Taniyama proved that a θ-curve is slice if and only if any (and hence all) of its *parallel links* is strongly slice (see Theorem 5 in [Tan93]). Given a directed strongly invertible knot K, one of the parallel links of $\theta(K)$ is easily seen to be exactly the butterfly link $L_b(K)$. Therefore, using the result from [HKL08] described in Subsection 3.1, we get that the commutator $[\theta(K_3), \theta(K_5)]$ is nontrivial in Θ, and hence that the cobordism group of θ-curves is not abelian. Note that in this case [AKPR21] cannot be applied because the results established there only hold in the smooth category.

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