

Symplectic Rank of Non-Gaussian Quantum States

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Non-Gaussianity is a key resource for achieving quantum advantages in bosonic platforms. Here, we investigate the *symplectic rank*: a non-Gaussianity monotone that satisfies remarkable operational and resource-theoretic properties. Mathematically, the symplectic rank of a pure state is the number of symplectic eigenvalues of the covariance matrix that are strictly larger than those of the vacuum. Operationally, it (i) is easy to compute, (ii) emerges as the smallest number of modes onto which all the non-Gaussianity can be compressed via Gaussian unitaries, (iii) lower bounds the non-Gaussian gate complexity of state preparation independently of the gate set, (iv) governs the sample complexity of quantum tomography, and (v) bounds the computational complexity of bosonic circuits. Crucially, the symplectic rank is nonincreasing under postselected Gaussian operations, leading to new no-go theorems for Gaussian conversion. Remarkably, this allows us to show that the resource theory of non-Gaussianity is irreversible under exact Gaussian operations. Finally, we show that the symplectic rank is a robust non-Gaussian measure, explaining how to witness it in experiments and how to exploit it to meaningfully benchmark different bosonic platforms. In doing so, we derive lower bounds on the trace distance (resp. total variation distance) between arbitrary states (resp. classical probability distributions) in terms of the norm distance between their covariance matrices, which may be of independent interest.

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I. INTRODUCTION

Quantum computers are expected to provide considerable advantages over their classical counterparts [1–3]. This promise has sparked a technological race to build the first universal quantum computer, involving several candidate physical platforms such as neutral atoms [4], quantum dots [5], ion traps [6], nitrogen vacancy centers [7], photonics [8], and superconducting circuits [9], among others [10]. To date, it remains to be seen which architecture will come out ahead, and it is thus crucial to develop tools to compare quantum computers that are based on radically different physical platforms [11,12]. Among these, bosonic platforms have in particular emerged as promising physical systems for realizing quantum computers, both from theoretical [13–16] and experimental standpoints: for

instance, photonic processors feature deterministic generation of large-scale entangled states [17] and long coherence times [18], while bosonic modes of superconducting cavities provide flexible quantum state engineering [19] as well as remarkable error-correction capabilities [20].

Yet, not all bosonic quantum computations are equally powerful: Gaussian computations—a well-studied subclass of bosonic computations—can be simulated efficiently on a classical computer [21], thus identifying non-Gaussian quantum states and operations as necessary ingredients for universal quantum computations [22]. As it turns out, non-Gaussian resources are also instrumental for a variety of other quantum information processing tasks [23–25], including entanglement distillation [26–28], quantum error-correction [29], quantum metrology [30], quantum communication [31–33], and learning of quantum states [34,35]. The importance of non-Gaussian resources for quantum information processing has prompted the introduction of several measures aiming to capture the nonclassical capabilities of quantum devices based on their non-Gaussianity [36–47]. In particular, in the context of quantum computing, the negative volume of the Wigner function [39], the stellar rank [40,41] and the Gaussian rank [42,43] have been shown to provide non-Gaussian measures of the cost of simulating quantum computations on a classical computer [42–46]. While these

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non-Gaussianity measures connect to specific operational contexts, their practical applicability remains constrained. The stellar rank, for example, precisely quantifies resource requirements for state preparation through photon addition/subtraction operations [40,41]—a particularly natural framework for photonic platforms [48]. However, its inherent tie to this specific operational paradigm limits its broader applicability. The fundamental task-dependence of these measures creates significant obstacles when attempting cross-platform comparisons of non-Gaussian capabilities. Some metrics become computationally prohibitive for certain states, while others remain confined to narrow operational contexts that do not readily translate between different physical implementations. As a result, the field currently lacks a unified measure capable of comprehensively capturing non-Gaussian state complexity in a platform-agnostic manner. This situation raises a critical question in quantum technology development: How can we systematically assess and compare the non-classical capabilities of bosonic quantum computers built on fundamentally different physical platforms with distinct native operations? The current limitations in comparative metrics underscore the urgent need for more versatile and computationally accessible measures of non-Gaussianity that could enable meaningful cross-platform evaluations.

We propose to address this challenge by using the *symplectic rank*. The symplectic rank has originally been introduced for Gaussian quantum states in Ref. [49], where it is defined as the number of symplectic eigenvalues of the covariance matrix that are strictly larger than those of the vacuum. We extend its definition to arbitrary non-Gaussian quantum states and show that it provides a powerful measure for characterizing bosonic devices that is efficiently computable for pure states.

We prove that the symplectic rank of a quantum state has an operational interpretation: it quantifies how effectively one can perform non-Gaussian compression [34], i.e., the process of using only Gaussian transformations to squeeze all of a state’s non-Gaussianity into as few modes as possible. Consequently, the symplectic rank lower bounds the non-Gaussian gate complexity of a bosonic quantum state, independently of the type of non-Gaussianity required to engineer it. Moreover, we prove that energy-bounded bosonic quantum computations with low symplectic rank can be simulated efficiently by classical computers, making it a natural measure for benchmarking quantum computers that are based on different bosonic platforms. We also show that the symplectic rank governs the sample complexity of quantum tomography of bosonic states.

Beyond its operational interpretations for quantum information processing tasks, we show that the symplectic rank satisfies a number of desirable resource-theoretic properties, among which: it has a natural definition based

on the symplectic eigenvalues of the covariance matrix, making it easy to compute and to witness in experiments; it is a non-Gaussianity monotone and can thus be used to bound the efficiency of Gaussian protocols. For example, this property allows us to prove that no Gaussian operation can deterministically convert a single-mode non-Gaussian pure state into two non-Gaussian pure states, establishing that spreading non-Gaussianity via Gaussian operations is impossible. Furthermore, we use it to prove the irreversibility of the resource theory of non-Gaussianity for exact conversion.

Moreover, we show that the symplectic rank is robust, i.e., it cannot decrease under small perturbations. In doing so, we introduce a novel perturbation bound on the covariance matrix of general (possibly non-Gaussian) states, which may be of independent interest. Specifically, our bound addresses the seemingly simple yet fundamental question: if two states are ε -close (e.g., in trace distance [50]), how far apart are their covariance matrices? This advances the rapidly growing literature on perturbation bounds for continuous-variable quantum states [34,51–54], where such bounds had previously been derived only for the specific case of Gaussian states.

Finally, to reflect practical hardware limitations, we extend the symplectic rank to the approximate setting. We show that the resulting approximate symplectic rank is a monotone under Gaussian operations, that it can be computed numerically via optimizations over Gaussian unitaries, and that it provides a robust and operational measure of non-Gaussianity, which allows the symplectic rank to be witnessed experimentally.

At a high level, compared to other non-Gaussianity measures such as the negative volume of the Wigner function [39] and stellar rank [40], the symplectic rank is a fundamentally multimode measure, which characterizes how non-Gaussianity may be compressed to a small number of modes, in a way that is agnostic to the type of non-Gaussianity.

II. OVERVIEW

This section provides a nontechnical summary of our key findings regarding the symplectic rank, while formal statements, proofs, and additional results are deferred to Secs. V–X. We refer to Sec. IV for background material and choice of conventions.

A. The symplectic rank

The notion of symplectic rank was first introduced in Ref. [49] as a concise way to characterize properties of Gaussian states. In this section, we show that the symplectic rank not only characterizes Gaussian states but also provides a measure of non-Gaussianity with remarkable properties. Let us begin by defining the symplectic rank [49].

Definition 1 (Symplectic rank of pure states). Let ψ be a pure state with a well-defined covariance matrix $V(\psi)$. The symplectic rank of ψ , denoted as $\mathfrak{s}(\psi)$, is the number of symplectic eigenvalues of $V(\psi)$ that are strictly greater than those of the vacuum.

This definition is motivated by the fact that a pure state ψ is Gaussian if and only if all its symplectic eigenvalues are exactly equal to those of the vacuum, which is equivalent to having a vanishing symplectic rank. Throughout the rest of this paper, we use the convention $\hbar = 1$ and define the covariance matrix such that the symplectic eigenvalues of the vacuum are all equal to one.

Notably, the symplectic rank has an operational interpretation in terms of its relation to the concept of *t-compressibility*, originally introduced in [55,56] and later extended in [34,57,58]. By definition, a state is *t-compressible* if a Gaussian unitary operation can transform it into the tensor product of a *t*-mode state and the vacuum state on all other modes. We obtain the following result:

Theorem 1 (Structure theorem). The symplectic rank $\mathfrak{s}(\psi)$ of a pure state ψ is equal to the smallest *t* for which ψ is *t-compressible*. Specifically, there exists a Gaussian unitary operation G and a $\mathfrak{s}(\psi)$ -mode state ϕ such that

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-\mathfrak{s}(\psi))}), \quad (1)$$

where *n* denotes the total number of modes. Moreover, the compressed state ϕ has mean photon number upper bounded by that of ψ .

Sketch of the proof. The first part of the result follows from the techniques in Ref. [34], and we recall the general idea behind the proof hereafter. To begin, consider a pure state ψ with a given symplectic rank $\mathfrak{s}(\psi)$. By the definition of symplectic eigenvalues, there exists a symplectic transformation, i.e., a Gaussian unitary operation that maps the covariance matrix $V(\psi)$ into a diagonal matrix. The entries of this matrix are the symplectic eigenvalues of $V(\psi)$. Notably, ψ has $n - \mathfrak{s}(\psi)$ symplectic eigenvalues equal to one. The only state with such eigenvalues is the vacuum state over $n - \mathfrak{s}(\psi)$ modes. From there, it is easy to understand the general idea of the first part of the proof. The part of the proof regarding the mean photon number is more technical and can be found in Theorem 7. ■

This structure theorem gives an operational meaning to the symplectic rank: it represents the smallest number of modes onto which all the non-Gaussianity of the state can be compressed by acting with a Gaussian unitary operation (see Fig. 1). Moreover, Theorem 1 allows us to extend the definition of symplectic rank to all pure states, even when their covariance matrix is not well-defined: the symplectic

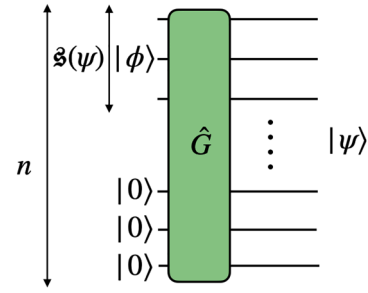


FIG. 1. Operational interpretation of the symplectic rank (Theorem 1). Given a pure state ψ over *n* modes, its symplectic rank $\mathfrak{s}(\psi)$ is the minimum number of modes *t* for which ψ can be written as $|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-t)})$, where ϕ is a non-Gaussian state over *t* modes and G is a Gaussian unitary.

rank $\mathfrak{s}(\psi)$ can be defined as the smallest *t* for which ψ is *t-compressible*. This definition can be further generalized to arbitrary mixed states through a convex roof construction, a widely used technique in the quantum resource theory literature [59]. Note that the following generalization to mixed states is different from the one employed for Gaussian states in Ref. [49].

Definition 2 (Symplectic rank of mixed states). The symplectic rank of a mixed state ρ , denoted as $\mathfrak{s}(\rho)$, is defined as

$$\mathfrak{s}(\rho) := \min_{\substack{\rho = \sum_i p_i \psi_i \\ p_i > 0}} \max_i \mathfrak{s}(\psi_i), \quad (2)$$

where the minimum is taken over all the possible decompositions of ρ as a convex combination of pure states.

The symplectic rank satisfies the following key properties: (i) it vanishes if and only if the state is a convex combination of Gaussian states (as it can be easily proven by exploiting that any Gaussian state is a convex combination of pure Gaussian states [60]); (ii) it is additive under the tensor product of pure states and subadditive under the tensor product of mixed states (as it easily follows from the definition of symplectic rank for pure states); and, most importantly, (iii) it is a quantum non-Gaussianity monotone under postselected Gaussian operations (i.e., tensoring with a Gaussian state, applying a Gaussian unitary operation, performing classical mixing, performing a heterodyne measurement with the ability to postselect on the outcome, and taking a partial trace [28]):

Theorem 2 (Monotonicity of the symplectic rank). The symplectic rank is a quantum non-Gaussianity monotone. That is, for all states ρ and all postselected Gaussian operations $\mathcal{G}$, the symplectic rank of $\mathcal{G}(\rho)$ is not larger than the one of ρ ,

$$\mathfrak{s}(\mathcal{G}(\rho)) \leq \mathfrak{s}(\rho). \quad (3)$$

The proof can be found in Theorem 8.

B. Probabilistic Gaussian conversion

The monotonicity of the symplectic rank establishes a powerful tool for proving no-go theorems in Gaussian protocols for non-Gaussian state conversion [36,61]. In particular, due to the additivity of the symplectic rank for pure states, it directly implies that no Gaussian protocol can exactly convert a single-mode non-Gaussian pure state to a tensor product of two non-Gaussian pure states, establishing that separately spreading non-Gaussianity with Gaussian operations is impossible, even allowing postselection. More generally, it is impossible to exactly map n single-mode non-Gaussian pure states to m non-Gaussian pure states via Gaussian operations whenever $m > n$. Remarkably, this impossibility persists even if the n input states are very far from the set of Gaussian states and the m output states are extremely close to the set of Gaussian states. This is summarized by the following result:

Corollary 1 (Impossibility of separately spreading non-Gaussianity via Gaussian operations). Let $n < m$, and let ρ be a state over n modes and $\phi_1, \dots, \phi_m$ be pure single-mode non-Gaussian states. Then, no postselected Gaussian protocol can map ρ to $\otimes_{i=1}^m \phi_i$.

Once the impossibility of converting certain states into others is understood, a natural question to explore is the irreversibility of resource theories. Irreversibility is closely related to the resources required to convert a state ρ into a state σ , and then to revert σ back to ρ . Informally, a resource theory is considered reversible when the resources employed in both processes are equal (we refer to Sec. VB2 for a formal definition). For example, in entanglement theory, reversibility holds for pure states [62] but fails for mixed states [63]. In this paper, we investigate the irreversibility of non-Gaussianity under exact asymptotic conversions, specifically considering the asymptotic rate of exactly converting n copies of ρ into m copies of a target state σ using postselected Gaussian operations. In this scenario, the following result holds:

Theorem 3 (Irreversibility of the resource theory of non-Gaussianity). The resource theory of quantum non-Gaussianity is irreversible under postselected Gaussian protocols in the exact conversion setup.

Sketch of the proof. The proof follows directly by considering the transformation between two Fock states $|1\rangle^{\otimes 2}$ and $|2\rangle$. It leverages the fact that the conversion rate can be bounded as a function of monotones of quantum non-Gaussianity, specifically the symplectic rank and stellar rank. A proof can be found in Theorem 9. ■

C. Complexity of tomography and simulation

We have seen that the symplectic rank is a measure of non-Gaussianity which has relevant applications in the resource theory of quantum non-Gaussianity. Here, we show that the symplectic rank has also applications in fundamental quantum information processing tasks, for which it can be interpreted as a measure of complexity. Specifically, we demonstrate how the symplectic rank relates to the difficulty of performing *quantum state tomography* and *classical simulation*.

Theorem 4 (Tomography based on the symplectic rank). Let ψ be an unknown n -mode pure state, which is promised to have a symplectic rank of at most s and second moment of the photon number upper bounded by E . Learning a classical description of a state that is ε -close in trace distance to ψ with high probability requires a number of state copies N satisfying the following upper and lower bounds:

$$\tilde{\Omega} \left(\left(\frac{E}{12s\varepsilon} \right)^s \right) \leq N \leq \tilde{O} \left(\left(\frac{22E}{s\varepsilon^2} \right)^s \right), \quad (4)$$

where the tilde notation abbreviates poly and log factors in all parameters.

The proof of this result is directly based on Theorems S74 and S80 of Ref. [34]. We provide a method to improve the energy dependence of the upper bound, compared to the results in Ref. [34]. Since the energy is an extensive quantity, which usually grows linearly with the number of modes, we can expect that $E > s$. Hence, (4) establishes that the number of state copies N must scale exponentially with the symplectic rank, as it is both upper and lower bounded by quantities scaling exponentially in s . So, the key takeaway is that the sample complexity of quantum tomography for bosonic states grows exponentially with the symplectic rank. A detailed proof of the theorem can be found in Theorem 13.

Similarly, we show that the symplectic rank can also be used to quantitatively bound the computational complexity of simulating bosonic computations:

Theorem 5 (Classical simulation based on the symplectic rank). Let ψ be an n -mode pure state with symplectic rank s and mean photon number less than E . There is a classical randomized algorithm which takes as input parameters $\varepsilon > 0$, the description of a Gaussian state $|G\rangle$, and the decomposition of ψ from Theorem 1, and outputs an estimate of the overlap $|\langle G|\psi|G\rangle|^2$ up to additive precision ε with high probability, in time T satisfying

$$T \leq \tilde{O} \left(\left(\frac{6E}{s\varepsilon^2} \right)^{3s} \right), \quad (5)$$

where the tilde notation abbreviates poly and log factors in all parameters.

Formal version and proof of Theorem 5 can be found in Theorem 14.

This result establishes high symplectic rank as a necessary—although not sufficient—ingredient for quantum computational advantage with bosonic systems: when ψ is the output of a bosonic circuit, it implies that the statistics of any Gaussian measurement at the output of the circuit can be estimated efficiently by a classical computer, whenever the symplectic rank is low enough, assuming that the decomposition of the state from Theorem 1 is available. As it turns out, this decomposition can often be computed efficiently given the description of a bosonic circuit that prepares ψ , and we give two illustrative examples below.

First, consider an n -mode state ψ prepared from the vacuum by a circuit consisting of alternating Gaussian unitary gates and s cubic phase gates. This class of circuits is natural when computing with superconducting modes, for instance for performing fault-tolerant computations in the Gottesman-Kitaev-Preskill encoding [20,64]. We have $\mathfrak{s}(\psi) \leq s$, and applying Theorem 14 allows us to efficiently estimate the statistics of any Gaussian measurement on the output states of such circuits whenever the number of non-Gaussian layers satisfies $s = \mathcal{O}(1)$ (see Theorem 17). This complements recent algorithms for estimating photon number statistics [65,66] and computing quadrature expectation values [67] at the output of such circuits.

Second, consider an n -mode state ψ prepared from the vacuum by a circuit consisting of alternating Gaussian unitary gates and s applications of creation and/or annihilation operators (i.e., operators of the form $a^{\dagger k} a^l$). This class of circuits is natural in quantum optics, where heralded photon additions and/or subtractions are commonly used to engineer non-Gaussianity [68,69]. As before, $\mathfrak{s}(\psi) \leq s$, and we can efficiently estimate Gaussian statistics of such states whenever the number of non-Gaussian layers satisfies $s = \mathcal{O}(1)$. We further refine this result to obtain a classical algorithm for strongly simulating any Gaussian measurement at the output of such circuits (see the Supplemental Material [70]) that is efficient when $s = \mathcal{O}(1)$. This improves upon a previous classical simulation algorithm [71], valid for single-photon additions and subtractions only. Those examples also illustrate that when a state is prepared using s non-Gaussian resources, the symplectic rank can only be lower than or equal to this number. Consequently, since any state can be seen as a doped Gaussian state, the symplectic rank offers a lower bound to the number of gates required to construct the state. Notably, the result is independent from the architecture used to inject non-Gaussianity. For more details see Theorem 15.

D. Robustness to small perturbations

The pursuit of precise non-Gaussian measures must be balanced against the pragmatic reality that experimental realizations can only achieve finite accuracy. To that end, we show that the symplectic rank is robust, in the sense that it cannot decrease under small perturbations of the state. Specifically, we prove that any pure state admits a ball around it of states with nonsmaller symplectic rank, establishing also a simple estimate for the minimal radius of such a ball with respect to trace distance. This result makes the symplectic rank relevant as a practical measure of non-Gaussianity, as it remains unaffected by small perturbations induced by noise inherent in any quantum device. This is formalized by the following theorem:

Theorem 6 (Slight perturbations cannot decrease the symplectic rank). Given a pure state ψ such that its second moment of the energy is bounded, then there exists an ε -ball (with respect to the trace distance) around it such that any energy-constrained state inside the ball has symplectic rank at least as large as that of ψ .

Sketch of the proof. The main idea behind the proof is that a variation in the symplectic rank of a pure state is intrinsically connected to a perturbation in the symplectic eigenvalues of its covariance matrix. To formalize such an idea, we introduce novel lower bounds on the trace distance between two states in terms of the norm distance between their covariance matrices, as reported in Theorem 23. A proof of the theorem can be found in Theorem 12. ■

A consequence of this result is that we can derive witnesses for the symplectic rank, as we explain in the following sections.

E. The approximate symplectic rank

The robustness of the symplectic rank motivates the introduction of an approximate measure based on the symplectic rank, which quantifies how the symplectic rank decreases as one gets further away from the state. The fidelity is often much easier to compute than the trace distance, which is why we use this measure to define the notion of approximate symplectic rank and symplectic fidelities, similarly to what is done in Ref. [72]:

Definition 3 (ε -approximate symplectic rank). Let ρ be a quantum state over n modes, and let $\varepsilon > 0$. The ε -approximate symplectic rank $\mathfrak{s}_\varepsilon(\rho)$ of ρ is the smallest symplectic rank among all states τ that are ε -close to ρ —where “ ε -close” means their fidelity with ρ is at least $1 - \varepsilon$.

Interestingly, we prove that the ε -approximate symplectic rank is a monotone under Gaussian operations, as shown in Theorem 19.

Definition 4 (k -symplectic fidelity). Given a quantum state ρ over n modes and $0 \leq k \leq n$, its k -symplectic fidelity $f_k^s(\rho)$ is defined as the highest fidelity between ρ and all quantum states whose symplectic rank is less than or equal to k .

In particular, f_0^s is the fidelity with the closest convex mixture of Gaussian states. Note also that the symplectic fidelity $f_k^s(\rho)$ is equal to 1 whenever $k \geq s(\rho)$. In general, symplectic fidelities and approximate symplectic rank are related as follows:

Lemma 1 (Equivalence between approximate symplectic rank and symplectic fidelities). Let ρ be a state over n modes. For all $k \leq n$ and all $0 \leq \varepsilon \leq 1$,

$$f_k^s(\rho) \geq 1 - \varepsilon \Leftrightarrow \forall \varepsilon' > \varepsilon, s_{\varepsilon'}(\rho) \leq k. \quad (6)$$

A proof can be found in Lemma 16.

We also show that the symplectic fidelities can be obtained via an optimization over Gaussian unitary operators for pure states:

Lemma 2 (Computing symplectic fidelities). Let ψ be a pure state over n modes. For all $k \leq n$, its symplectic fidelities can be expressed as

$$f_k^s(\psi) = \sup_G \text{Tr}[(\mathbb{1}_k \otimes |0\rangle\langle 0|^{\otimes(n-k)})G|\psi\rangle\langle\psi|G^\dagger], \quad (7)$$

where the supremum is over an arbitrary n -mode Gaussian unitary G .

Sketch of the proof. First, since we are focusing on pure states, the expression for the fidelity simplifies significantly. Second, we may use the fact that pure states with a symplectic rank less than or equal to s are equivalent to s -compressible states to complete the proof. For the proof, refer to Lemma 17. ■

Together with Lemma 1, this result drastically reduces the complexity of computing the approximate symplectic rank, as compared to Definition 3, since it can be deduced from the list of symplectic fidelities. While the optimization above is still nontrivial when the number of modes increases, Lemma 2 implies that the approximate symplectic rank can be computed efficiently for continuous-variable quantum systems over a small number of modes.

F. Witnessing the symplectic rank

The robustness of the symplectic rank under small perturbations indicates that it may be possible to witness it in experimental scenarios, in which quantum states are inevitably noisy and information reconstruction suffers from statistical errors.

This intuition can be formalized using symplectic fidelities, leading to an explicit and practical protocol for witnessing the symplectic rank of some unknown experimental (mixed) state ρ as follows: (i) pick a pure target state ψ and compute its symplectic fidelities $f_k^s(\psi)$; (ii) measure several copies of ρ and witness its fidelity with ψ using the samples; (iii) compare the estimated lower bound on the fidelity with the symplectic fidelities—if it is above $f_k^s(\psi)$ for some k , this implies $s(\rho) > k$, thus witnessing the symplectic rank of the experimental state.

This protocol relies on two subroutines: computing the symplectic fidelities of a target pure state of high symplectic rank and estimating a lower bound on the fidelity of an experimental state with that target state based on measurement samples. The former can be achieved via Lemma 2, while the latter can be done efficiently for target pure states that are tensor products [Lemma 2 of Ref. [73]] of arbitrary symplectic rank, as well as for a large class of t -doped states for $t = \mathcal{O}(\log n)$ [Protocol 4 of Ref. [74]], of symplectic rank less than or equal to t .

We refer to Sec. VIIC for an illustration with the specific example of the target state $|1\rangle^{\otimes 2}$: using Lemma 2, we obtain $f_0^s(|1\rangle^{\otimes 2}) = \frac{1}{4}$ and $f_1^s(|1\rangle^{\otimes 2}) = \frac{1}{2}$. In particular, this implies that preparing experimentally a (mixed) state ρ and measuring a fidelity with the state $|1\rangle^{\otimes 2}$ greater than 0.25 (resp. 0.5) implies that the state ρ has symplectic rank at least 1 (resp. 2).

G. Approximate Gaussian conversion

Like the symplectic rank, the ε -approximate symplectic rank is also a monotone under Gaussian operations (without postselection) for all ε , as we show in Theorem 19. This implies that the approximate symplectic rank may be used to give bounds on deterministic Gaussian protocols for non-Gaussian state conversion [36]: if $\mathcal{G}(\rho^{\otimes k}) = \sigma^{\otimes m}$ for some Gaussian protocol $\mathcal{G}$, then $s_\varepsilon(\sigma^{\otimes m}) \leq s_\varepsilon(\rho^{\otimes k})$.

More importantly, since we are dealing with an approximate measure, we can extend these bounds to the experimentally relevant case of approximate Gaussian conversion, similar to the approximate stellar rank [72]: if $\mathcal{G}(\rho^{\otimes k})$ is δ -close in trace distance to $\sigma^{\otimes m}$, then $s_{\varepsilon+\sqrt{2\delta}}(\sigma^{\otimes m}) \leq s_\varepsilon(\rho^{\otimes k})$, for all $\varepsilon \geq 0$.

Using Lemma 1, these bounds can equivalently be phrased in terms of symplectic fidelities as $f_n^s(\rho^{\otimes k}) \leq f_n^s(\sigma^{\otimes m}) + \sqrt{2\delta}$.

Finally, we show that for $\varepsilon = 0$, the previous bound $s_{\sqrt{2\delta}}(\sigma^{\otimes m}) \leq s(\rho^{\otimes k})$ also applies to the case of approximate and probabilistic Gaussian conversion, i.e., including postselection on Gaussian measurement outcomes.

Combined with Lemma 2, these bounds provide strong constraints on the possibilities of non-Gaussian state engineering using Gaussian protocols [36,61,75–84].

III. DISCUSSION

Our work demonstrates that it is possible to define a meaningful measure of non-Gaussianity based on properties of the covariance matrix, often perceived as characterizing Gaussian features of a quantum state. Doing so, we have introduced a novel measure for non-Gaussian states, the symplectic rank, and have established it as a non-Gaussianity monotone, leading to strong constraints on the Gaussian conversion of non-Gaussian states. What is more, we have shown that the symplectic rank is a multifaceted measure connected to the concepts of non-Gaussian compression and non-Gaussian gate complexity, as well as to the complexity of tomography of bosonic quantum states and of classical simulation of bosonic computations. Additionally, we have shown that the symplectic rank is robust to small perturbations, leading to the notions of approximate symplectic rank and symplectic fidelities, ultimately showing that the symplectic rank can be effectively witnessed in small-scale experiments.

Our findings open several research directions. From an experimental standpoint, the hardware-agnostic nature of the symplectic rank makes it a compelling candidate for benchmarking and comparing the non-Gaussianity of various bosonic platforms—an area that has so far lacked such a tool. We expect that the methods introduced here will lead to multiple experiments witnessing the symplectic rank, based on very different hardware and native non-Gaussian interactions, such as photonics devices with photon addition or subtraction [68], or superconducting devices with SNAP gates [85,86]. We also expect that, due to its intrinsic simplicity compared to other measures of non-Gaussianity, the symplectic rank can provide a useful tool for characterizing exotic quantum states.

From a theoretical standpoint, the symplectic rank is set to play a central role in advancing our understanding of non-Gaussianity in the multimode setting, particularly through its operational interpretation and associated conversion bounds. In particular, we expect that these results can be useful to extend already known no-go results on conversion between continuous variable states [87,88].

In particular, future research could explore the similarities between the symplectic rank and the stabilizer nullity in discrete-variable systems [89] to gain new insights into the symplectic rank and non-Gaussianity, as well as their interplay with entanglement. Comparative studies of the conversion bounds derived here with those based on Wigner negativity or approximate stellar rank would also be especially valuable for assessing bosonic information processing protocols, while combining approximate symplectic rank and approximate stellar rank could settle the open question of the irreversibility of the resource theory of non-Gaussianity in the approximate setting. We leave these questions for future work.

IV. PRELIMINARIES ON CONTINUOUS-VARIABLE QUANTUM SYSTEMS

In this section we introduce the relevant preliminaries of continuous-variable quantum systems; for more details, we refer to Ref. [60]. First, we start by defining what a continuous-variable quantum system is, by introducing the Hilbert space of n modes. We then define key concepts such as first moments, covariance matrices, energy operator, and photon number operator. Next, we discuss Gaussian states and Gaussian unitary operations. Finally, we introduce coherent states and the stellar representation of continuous-variable quantum states.

A. The Hilbert space of n modes

A continuous-variable system is represented by the Hilbert space $L^2(\mathbb{R}^n)$ of all square-integrable complex-valued functions over $\mathbb{R}^n$, where n represents the number of modes [60]. A state and unitary acting on such a Hilbert space are dubbed n -mode state and n -mode unitary, respectively. The *quadrature operator vector*, denoted as $\hat{\mathbf{R}}$, is defined as

$$\hat{\mathbf{R}} := (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n), \quad (8)$$

where $\hat{x}_j$ and $\hat{p}_j$ are the well-known *position* and *momentum* operators of the j th mode [60]. The position operator $\hat{x}_j$ and the momentum operator $\hat{p}_j$ of the j th mode can be written in terms of the well-known *annihilation operator* a_j and *creation operator* $a_j^\dagger$ as

$$\begin{aligned} \hat{x}_j &:= \frac{a_j + a_j^\dagger}{\sqrt{2}}, \\ \hat{p}_j &:= \frac{a_j - a_j^\dagger}{i\sqrt{2}}, \end{aligned} \quad (9)$$

with the convention $\hbar = 1$. The vector of quadrature operators satisfies the following relation, known as *canonical commutation relation*:

$$[\hat{\mathbf{R}}, \hat{\mathbf{R}}^\top] = i\Omega \mathbb{1}, \quad (10)$$

where the symbol $(\cdot)^\top$ denotes the transpose operation and where

$$\Omega = \bigoplus_{i=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (11)$$

is the so-called symplectic form. The Hilbert space $L^2(\mathbb{R}^n)$ can be seen as the span of the so-called *multimode Fock states* $\{|\mathbf{k}\rangle\}_{\mathbf{k} \in \mathbb{N}^n}$ [60],

$$L^2(\mathbb{R}^n) := \text{span} \{ |\mathbf{k}\rangle : \mathbf{k} \in \mathbb{N}^n \}, \quad (12)$$

where $|\mathbf{k}\rangle := |k_1\rangle \otimes \dots \otimes |k_n\rangle$, with $|k\rangle$ denoting the single-mode k th *Fock state* [60]. The latter represents the

state with k photons (or more generally, bosons), and it can be obtained by applying the creation operator on the vacuum state as

$$|k\rangle := \frac{(a^\dagger)^k}{\sqrt{k!}} |0\rangle. \quad (13)$$

The Fock states $|0\rangle$ and $|1\rangle$ are called the *vacuum state* and *single-photon state*, respectively [60]. Via Eq. (12), an n -mode continuous-variable quantum system can be viewed as an n -qudit system with $d = \infty$.

B. First moments, covariance matrices, and symplectic matrices

The *first moment* $\mathbf{m}(\rho)$ and the *covariance matrix* $V(\rho)$ of a quantum state ρ are defined as [60]

$$\mathbf{m}(\rho) := \text{Tr}[\hat{\mathbf{R}} \rho], \quad (14)$$

$$V(\rho) := \text{Tr} \left[\left\{ (\hat{\mathbf{R}} - \mathbf{m}(\rho)), (\hat{\mathbf{R}} - \mathbf{m}(\rho))^\top \right\} \rho \right], \quad (15)$$

where $\{A, B\} := AB + BA$ is the anticommutator. Note that the definition of the covariance matrix differs by a factor $\frac{1}{2}$ compared to Ref. [90], such that the covariance matrix of the vacuum is the identity matrix, while $\hbar = 1$. Any covariance matrix V is strictly positive and satisfies the so-called uncertainty relation $V + i\Omega \geq 0$. As such, any covariance matrix V can be written in the so-called *Williamson decomposition* as follows: there exists a *symplectic* matrix S and n real numbers $d_1 \geq d_2 \geq \dots \geq d_n$ —called the *symplectic eigenvalues* of V —such that

$$V = SDS^\top, \quad (16)$$

where $D := \text{diag}(d_1, d_1, \dots, d_n, d_n)$. Importantly, the symplectic eigenvalues of a covariance matrix are unique and always larger than or equal to one. By definition, a matrix S is said to be symplectic if it satisfies $S\Omega S^\top = \Omega$, where Ω is the symplectic form defined in Eq. (11). Symplectic matrices satisfy the following properties:

- (a) The product of symplectic matrices is symplectic;
- (b) The transpose of a symplectic matrix is symplectic;
- (c) The inverse of symplectic matrix is symplectic;
- (d) Any symplectic matrix S can be decomposed in the so-called *Euler decomposition* as follows: there exist suitable symplectic orthogonal matrices O_1, O_2 , and suitable real numbers $z_1, z_1, \dots, z_n \geq 1$ such that

$$S = O_1 Z O_2, \quad (17)$$

where Z is a diagonal matrix—known as *squeezing matrix*—defined by

$$Z := \bigoplus_{j=1}^n \begin{pmatrix} z_j & 0 \\ 0 & z_j^{-1} \end{pmatrix}. \quad (18)$$

C. Energy operator and photon number operator

The *energy operator*, denoted as $\hat{E}$, is defined as

$$\hat{E} := \frac{1}{2} \hat{\mathbf{R}}^\top \hat{\mathbf{R}} = \frac{1}{2} \sum_{i=1}^n (\hat{x}_i^2 + \hat{p}_i^2). \quad (19)$$

Moreover, the *total photon number operator*, denoted as $\hat{N}$, is defined as $\hat{N} := \hat{E} - \frac{n}{2} \mathbb{1}$ and can be written in terms of the annihilation and creation operators as

$$\hat{N} = \sum_{i=1}^n a_i^\dagger a_i. \quad (20)$$

The energy operator and the total photon number operator can be diagonalized in the basis of multimode Fock states $\{|k\rangle\}_{k \in \mathbb{N}^n}$ as follows [60]:

$$\begin{aligned} \hat{N} &= \sum_{k \in \mathbb{N}^n} \left(\sum_{i=1}^n k_i \right) |k\rangle \langle k|, \\ \hat{E} &= \sum_{k \in \mathbb{N}^n} \left(\sum_{i=1}^n k_i + \frac{n}{2} \right) |k\rangle \langle k|. \end{aligned} \quad (21)$$

By definition, the *mean energy* of a state ρ is given by $\text{Tr}[\hat{E} \rho]$, while its *mean photon number* is given by $\text{Tr}[\hat{N} \rho]$. From these definitions, it easily follows that the mean energy and the mean photon number can be expressed in terms of the first moment and the covariance matrix as

$$\begin{aligned} \text{Tr}[\hat{E} \rho] &= \frac{\text{Tr} V(\rho)}{4} + \frac{\|\mathbf{m}(\rho)\|^2}{2}, \\ \text{Tr}[\hat{N} \rho] &= \frac{\text{Tr}[V(\rho) - \mathbb{1}]}{4} + \frac{\|\mathbf{m}(\rho)\|^2}{2}. \end{aligned} \quad (22)$$

Note that the first moment and the covariance matrix of a state can admit infinite elements, i.e., they can be ill-defined. The following lemma guarantees that the first moment and the covariance matrix are well-defined as long as the mean energy of the state is finite.

Lemma 3 (Finite mean energy implies well-defined first moment and covariance matrix). Let ρ be a quantum state with finite mean energy. Then, all the elements of the covariance matrix $V(\rho)$ and first moment $\mathbf{m}(\rho)$ are finite.

Proof. The general formula for the mean energy in Eq. (22) directly implies that all the elements of the first

moment are finite if the state has finite mean energy. It suffices to prove that the operator norm of the covariance matrix can be upper bounded in terms of the mean energy. This follows by the following chain of inequalities:

$$\begin{aligned} \|V(\rho)\|_\infty &\leq \text{Tr}[V(\rho)] = 4\text{Tr}[\hat{E}\rho] - 2\|\mathbf{m}(\rho)\|^2 \\ &\leq 4\text{Tr}[\hat{E}\rho], \end{aligned} \quad (23)$$

where the first inequality follows from the fact that the covariance matrix is positive, and in the equality we exploited the general formula for the mean energy in Eq. (22). ■

1. Energy constraints

Continuous-variable quantum states are often assumed to satisfy an *energy constraint*. The most common choice is a bound on the *mean* energy, i.e., a promise that a state ρ satisfies $\text{Tr}[\rho\hat{E}] \leq E$. Equivalently, since the energy operator differs from the total photon number operator only by an offset, i.e., $\hat{E} = \hat{N} + (n/2)\mathbb{1}$, any mean-energy constraint can be rewritten (up to a trivial reparametrization of the bound) as a constraint on the mean photon number $\text{Tr}[\rho\hat{N}]$.

While finiteness of the mean energy is enough to guarantee that the covariance matrix of ρ is finite (as we saw above in Lemma 3), it is not sufficient for the continuity properties needed in several analysis of quantum information with continuous-variable systems. For example, in tomography (see Sec. VII) and in the perturbation analysis of the symplectic rank (see Sec. VI), we will need that “if two states are close in trace distance, then their covariance matrices are close (in a suitable norm).” Such an implication cannot be derived from a mean-energy constraint alone; it becomes possible only under control of the *second moment* of the energy, i.e., a bound on $\text{Tr}[\rho\hat{E}^2]$ (or equivalently on $\text{Tr}[\rho\hat{N}^2]$). Note that a second-moment constraint is strictly stronger than a first-moment one: by Jensen’s inequality we always have $\text{Tr}[\rho\hat{E}] \leq \sqrt{\text{Tr}[\rho\hat{E}^2]}$.

Let us now give an explicit example showing that a finite mean photon number does not suffice to guarantee that a small perturbation in trace distance induces only a small perturbation of the covariance matrix. Consider the vacuum state $|0\rangle\langle 0|$ and, for $k \geq 2$, the single-mode states $\rho_k := (1 - (1/k))|0\rangle\langle 0| + (1/k)|k\rangle\langle k|$. A direct computation yields $\|\rho_k - |0\rangle\langle 0|\|_1 = 2/k \rightarrow 0$ as $k \rightarrow \infty$, while the mean photon number remains uniformly bounded: $\text{Tr}[\rho_k \hat{N}] = 1$ and $\text{Tr}[|0\rangle\langle 0| \hat{N}] = 0$. Despite the fact that the two states become closer and closer in trace distance, the corresponding covariance matrices do *not* approach one another: since for a Fock state $|k\rangle$ one has $V(|k\rangle\langle k|) = (2k + 1)\mathbb{1}$, it follows that $V(\rho_k) = (1 - (1/k))\mathbb{1} + (1/k)(2k + 1)\mathbb{1} = 3\mathbb{1}$, and hence $V(\rho_k) - V(|0\rangle\langle 0|) = 2\mathbb{1}$ for all $k \geq 2$. The reason behind this behavior is that the second moment diverges, since $\text{Tr}[\rho_k \hat{N}^2] =$

$k \rightarrow \infty$. This example illustrates the general phenomenon that, without a finite second-moment constraint, small trace distance does not guarantee a small distance between covariance matrices. In contrast, in Sec. IX we show that under a finite second-moment energy constraint, closeness in trace distance does imply closeness of covariance matrices.

D. Gaussian states and Gaussian unitary operations

Let us proceed with the definition of Gaussian unitary operations and states [60]. By definition, a *Gaussian unitary operation* G is a unitary operator that can be prepared by means of evolutions induced by quadratic Hamiltonians in $\hat{\mathbf{R}}$. The most general form of a Gaussian unitary operation is as follows [60]:

$$G = U_S D_{\mathbf{r}}, \quad (24)$$

where

$$D_{\mathbf{r}} := e^{-i\mathbf{r}^T \Omega \hat{\mathbf{R}}} \quad (25)$$

represents the *displacement operator* [60], characterized by the displacement vector $\mathbf{r} \in \mathbb{R}^{2n}$, while U_S corresponds to the *symplectic Gaussian unitary* [60], associated with a *symplectic matrix* S . The displacement operator $D_{\mathbf{r}}$, associated with a vector $\mathbf{r} \in \mathbb{R}^{2n}$, acts on the quadrature operator vector as

$$D_{\mathbf{r}}^\dagger \hat{\mathbf{R}} D_{\mathbf{r}} = \hat{\mathbf{R}} + \mathbf{r}\mathbb{1}, \quad (26)$$

while it acts at the level of first moments and covariance matrices as

$$\begin{aligned} \mathbf{m}(D_{\mathbf{r}}\rho D_{\mathbf{r}}^\dagger) &= \mathbf{m}(\rho) + \mathbf{r}, \\ V(D_{\mathbf{r}}\rho D_{\mathbf{r}}^\dagger) &= V(\rho), \end{aligned} \quad (27)$$

for any state ρ . The above definition in Eq. (25) of the displacement operator $D_{\mathbf{r}}$ is with respect to the so-called *real notation*. It is sometimes useful to parametrize displacement operators with respect to the *complex notation*. That is, one may define the displacement operator, associated with the complex vector $\boldsymbol{\beta} := (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{C}^n$, as

$$D(\boldsymbol{\beta}) := e^{i(\mathbf{a}^\dagger \boldsymbol{\beta} - \boldsymbol{\beta}^\dagger \mathbf{a})} = \bigotimes_{j=1}^n e^{i(\beta_j a_j^\dagger - \beta_j^* a_j)}. \quad (28)$$

To switch from one definition to the other, one may use the following change of variables:

$$\begin{aligned} r_{2j-1} &:= \sqrt{2} \text{Re}(\beta_j), \\ r_{2j} &:= \sqrt{2} \text{Im}(\beta_j), \end{aligned} \quad (29)$$

for all $j = 1, \dots, n$, which guarantees that $D(\boldsymbol{\beta}) = D_{\mathbf{r}}$.

For a given symplectic matrix S , the symplectic Gaussian unitary U_S acts on the quadrature operator vector as

$$U_S^\dagger \hat{\mathbf{R}} U_S = S \hat{\mathbf{R}}, \quad (30)$$

while it acts at the level of first moments and covariance matrices as

$$\begin{aligned} \mathbf{m}(U_S \rho U_S^\dagger) &= S \mathbf{m}(\rho), \\ V(U_S \rho U_S^\dagger) &= S V(\rho) S^\top, \end{aligned} \quad (31)$$

for any state ρ . Moreover, given two symplectic matrices S_1 and S_2 , the associated symplectic Gaussian unitary satisfies $U_{S_1 S_2} = U_{S_1} U_{S_2}$. Hence, the Euler decomposition reported in Eq. (17) establishes that any symplectic Gaussian unitary U_S can be decomposed as

$$U_S = U_{O_2} U_Z U_{O_1}, \quad (32)$$

where O_1, O_2 are orthogonal symplectic matrices and Z is the squeezing matrix in Eq. (18). By definition, a *passive Gaussian unitary* is a Gaussian unitary operation associated with an orthogonal symplectic matrix O . Importantly, any passive Gaussian unitary U_O preserves the energy (and in particular the photon number operator) [60], i.e., $U_O^\dagger \hat{E} U_O = \hat{E}$. Notably, the set of $2n \times 2n$ orthogonal symplectic matrices is in one-to-one correspondence with the set of $n \times n$ unitary matrices. In particular, any $n \times n$ unitary matrix U defines a suitable n -mode passive Gaussian unitary, denoted as G_U , which acts on the vector of annihilation operators $\mathbf{a} := (a_1, a_2, \dots, a_n)$ as

$$G_U^\dagger \mathbf{a} G_U = U^* \mathbf{a}. \quad (33)$$

Given a squeezing matrix $Z = \text{diag}(z_1, z_1^{-1}, \dots, z_n, z_n^{-1})$ as in Eq. (18), the Gaussian unitary operation G_Z is called a *squeezing unitary*. Specifically, it can be shown that G_Z can be expressed at the level of the Hilbert space as [60]

$$G_Z = \bigotimes_{i=1}^n e^{\frac{1}{2} \ln(z_i)^2 (a_i^{\dagger 2} - a_i^2)}. \quad (34)$$

It is sometimes useful to denote the squeezing unitary as $S(\xi)$, where $\xi := (\xi_1, \dots, \xi_n)$ and $\xi_i := \ln z_i$, so that

$$S(\xi) = \bigotimes_{i=1}^n e^{\frac{1}{2} \xi_i (a_i^{\dagger 2} - a_i^2)}. \quad (35)$$

By putting everything together, Eq. (24) and the Euler decomposition in Eq. (32) implies that any Gaussian unitary operation G can always be expressed as the composition between a displacement operator $D(\beta)$, a passive

Gaussian unitary operation G_{U_1} , a squeezing unitary operation $S(\xi)$, and another passive Gaussian unitary operation G_{U_2} ,

$$G = G_{U_2} S(\xi) G_{U_1} D(\beta). \quad (36)$$

Moreover, since Eq. (33) implies that

$$G_U D(\beta) G_U^\dagger = D(U^\top \beta), \quad (37)$$

it follows that Eq. (36) can also be expressed as

$$G = G_{U_2} S(\xi) D(U_1^\top \beta) G_{U_1}. \quad (38)$$

Consequently, we conclude that the following lemma holds.

Lemma 4 (Parametrization of Gaussian unitary operations). An arbitrary n -mode Gaussian unitary G can be parametrized as follows:

$$G = G_U S(\xi) D(\beta) G_V, \quad (39)$$

where G_U, G_V are passive Gaussian unitary operations associated with some unitary matrices $U, V \in \mathbb{C}^{n \times n}$, $S(\xi)$ is the n -mode squeezing unitary operation of squeezing vector $\xi \in \mathbb{R}^n$ defined in Eq. (35), and $D(\beta)$ is the displacement operator of displacement vector $\beta \in \mathbb{C}^n$ defined in Eq. (28).

Let us now proceed with the definition of Gaussian states. By definition, a *Gaussian state* is a tensor product of Gibbs state of Hamiltonians that are quadratic in the quadrature operator vector $\hat{\mathbf{R}}$. A fundamental example of Gaussian state is the single-mode vacuum state $|0\rangle$. Any n -mode pure Gaussian state ψ can be prepared by applying a Gaussian unitary operation G to the vacuum, i.e., $|\psi\rangle = G |0\rangle^{\otimes n}$. By exploiting the parametrization of Gaussian unitary operations presented in Lemma 4 and the fact that passive Gaussian unitary operations preserves the vacuum [60], it follows that following lemma holds.

Lemma 5 (Parametrization of pure Gaussian states). An arbitrary n -mode pure Gaussian state ψ can be parametrized as follows:

$$|\psi\rangle = G_U S(\xi) D(\beta) |0\rangle^{\otimes n}, \quad (40)$$

where G_U is the passive Gaussian unitary associated with some unitary matrix $U, V \in \mathbb{C}^{n \times n}$, $S(\xi)$ is the n -mode squeezing unitary of squeezing vector $\xi \in \mathbb{R}_+^n$ defined in Eq. (35), and $D(\beta)$ is the displacement operator of displacement vector $\beta \in \mathbb{C}^n$ defined in Eq. (28).

It is well known that a Gaussian state is uniquely identified by its first moment $\mathbf{m}$ and its covariance matrix V .

That is, for any $\mathbf{m} \in \mathbb{R}^{2n}$ and any $V \in \mathbb{R}^{2n \times 2n}$ such that $V + i\Omega \geq 0$ there exists a Gaussian state with first moment $\mathbf{m}$ and covariance matrix V [60]. Moreover, any (possibly mixed) Gaussian state satisfies the following decomposition—known as *normal mode decomposition* [60].

Lemma 6 (*Normal mode decomposition of a Gaussian state* [60]). Let ρ be an n -mode Gaussian state with first moment $\mathbf{m}(\rho)$ and covariance matrix $V(\rho)$. Let

$$V(\rho) = SDS^\top \quad (41)$$

be Williamson's decomposition of V as in Eq. (16), where $D := (d_1, d_1, \dots, d_n, d_n)$ is the diagonal matrix of symplectic eigenvalues. Then, the Gaussian state ρ is unitarily equivalent—via Gaussian unitary operations—to a tensor product of thermal states, i.e.,

$$\rho = D_{\mathbf{m}(\rho)} U_S \left(\tau_{\frac{d_1-1}{2}} \otimes \dots \otimes \tau_{\frac{d_n-1}{2}} \right) U_S^\dagger D_{\mathbf{m}(\rho)}^\dagger, \quad (42)$$

where:

- (a) $D_{\mathbf{m}(\rho)}$ is the displacement operator with amplitude equal to the first moment $\mathbf{m}(\rho)$;
- (b) U_S is the symplectic Gaussian unitary associated with the symplectic matrix S that puts the covariance matrix $V(\rho)$ in Williamson's decomposition;
- (c) For any $\nu \geq 0$, the state τ_ν is the single-mode thermal state of mean photon number ν , defined as

$$\tau_\nu := \frac{1}{\nu + 1} \sum_{n=0}^{\infty} \left(\frac{\nu}{\nu + 1} \right)^n |n\rangle\langle n|, \quad (43)$$

with $|n\rangle$ being the n th Fock state.

As a consequence of the above decomposition, a Gaussian state is pure if and only if its symplectic eigenvalues are all equal to one. Additionally, it is also worth to mention the fact that a pure state is Gaussian if and only if all its symplectic eigenvalues are equal to one.

E. Coherent states and stellar functions

Given a complex vector $\mathbf{z} \in \mathbb{C}^n$, the n -mode *coherent state* of amplitude $\mathbf{z}$ is defined as

$$|\mathbf{z}\rangle := D(\mathbf{z}) |0\rangle^{\otimes n}, \quad (44)$$

where $D(\mathbf{z})$ is the displacement operator defined in Eq. (28). Notably, coherent states form an overcomplete basis of the n -mode Hilbert space, i.e., [60]

$$\int_{\mathbb{C}^n} \frac{d^{2n}\mathbf{z}}{\pi^n} |\mathbf{z}\rangle\langle\mathbf{z}| = \mathbb{1}. \quad (45)$$

Let us proceed with the definition of the *stellar function* [40,41].

Definition 5 (*Stellar function*). The stellar function $F_{|\psi\rangle}^* : \mathbb{C}^n \mapsto \mathbb{C}$ of a (possibly unnormalized) n -mode state ψ is defined as

$$F_{|\psi\rangle}^*(\mathbf{z}) := e^{\frac{1}{2}\|\mathbf{z}\|^2} \langle \mathbf{z}^* | \psi \rangle \quad \forall \mathbf{z} \in \mathbb{C}^n, \quad (46)$$

where $|\mathbf{z}\rangle$ denotes a coherent state of amplitude $\mathbf{z}$, and $\|\mathbf{z}\|^2 = |z_1|^2 + \dots + |z_n|^2$.

As a consequence of Eq. (45), any vector ψ can be retrieved by its stellar function via the following formula:

$$|\psi\rangle = \int_{\mathbb{C}^n} \frac{d^{2n}\mathbf{z}}{\pi^n} e^{-\frac{1}{2}\|\mathbf{z}\|^2} F_{|\psi\rangle}^*(\mathbf{z}) |\mathbf{z}^*\rangle. \quad (47)$$

In particular, stellar functions and pure states are in one-to-one correspondence. In addition, Eq. (45) also implies that any stellar function satisfies the following normalization condition:

$$\int_{\mathbb{C}^n} \frac{d^{2n}\mathbf{z}}{\pi^n} e^{-\|\mathbf{z}\|^2} |F_{|\psi\rangle}^*(\mathbf{z})|^2 = \langle \psi | \psi \rangle. \quad (48)$$

Let us now understand how stellar functions transform under passive Gaussian unitary operations. By exploiting Eq. (37), given a unitary matrix $U \in \mathbb{C}^{n \times n}$, the n -mode passive Gaussian unitary operation G_U acts at the level of stellar functions as follows:

$$F_{G_U|\psi\rangle}^*(\mathbf{z}) = F_{|\psi\rangle}^*(U\mathbf{z}) \quad \forall \mathbf{z} \in \mathbb{C}^n. \quad (49)$$

Recall that Lemma 5 establishes that an arbitrary pure state can be parametrized in terms of a unitary matrix $U \in \mathbb{C}^{n \times n}$, a real vector $\boldsymbol{\xi} \in \mathbb{R}_+^n$, and a complex vector $\boldsymbol{\beta} \in \mathbb{C}^n$. The following lemma establishes a closed formula for the stellar function of a Gaussian state in terms of its parameters U , $\boldsymbol{\xi}$, and $\boldsymbol{\beta}$.

Lemma 7 (*Stellar function of a Gaussian state* [40,41]). Let ψ be an n -mode pure Gaussian state, as parametrized as in Lemma 5 in terms of a unitary matrix $U \in \mathbb{C}^{n \times n}$, a real vector $\boldsymbol{\xi} \in \mathbb{R}_+^n$, and a complex vector $\boldsymbol{\beta} \in \mathbb{C}^n$. Then, the stellar function of ψ is given by

$$F_{|\psi\rangle}^*(\mathbf{z}) = \frac{1}{\mathcal{N}} e^{\frac{1}{2}\mathbf{z}^\top A \mathbf{z} + \mathbf{w}^\top \mathbf{z} + C} \quad \forall \mathbf{z} \in \mathbb{C}^n, \quad (50)$$

where we defined

$$\begin{aligned} A &:= U^\top \text{Diag}(\tanh \xi_1, \dots, \tanh \xi_n) U, \\ \mathbf{w} &:= U^\top \mathbf{b}, \\ C &:= \sum_{j=1}^n c_j, \\ \mathcal{N} &:= \sqrt{\cosh \xi_1 \cdots \cosh \xi_n}, \end{aligned} \quad (51)$$

and

$$\begin{aligned} b_j &:= \frac{\beta_j}{\cosh \xi_j}, \\ c_j &:= -\frac{1}{2}(\tanh \xi_j)\beta_j^2 - \frac{1}{2}|\beta_j|^2, \end{aligned} \quad (52)$$

for all $j \in \{1, \dots, n\}$.

V. THE SYMPLECTIC RANK

A key property of pure Gaussian states is that all the symplectic eigenvalues of their covariance matrices are equal to one [60]. Conversely, if a pure state has a symplectic eigenvalue strictly greater than one, it cannot be Gaussian [60]. This suggests that, for pure states, the number of symplectic eigenvalues strictly larger than one may play a role in quantifying the non-Gaussianity of the state. This intuition motivates the definition of *symplectic rank*, which may be defined as the number of symplectic eigenvalues of the covariance matrix that are strictly greater than one.

However, this definition would be ill-defined for states whose covariance matrices are ill-defined, such as those with infinite energy. In the forthcoming Theorem 7, we establish a deeper connection between this definition—valid only for states with a well-defined covariance matrix—and the concept of *t-compressibility* introduced in [34]. The latter concept applies to all states, even those with an ill-defined covariance matrix. Consequently, in the forthcoming definition, we provide a general definition of symplectic rank based on the concept of *t-compressibility*.

Definition 6 (Symplectic rank). The symplectic rank of an n -mode pure state ψ , denoted as $\mathfrak{s}(\psi)$, is defined as the minimum t for which there exists a Gaussian unitary operation G and a t -mode pure state ϕ such that

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-t)}). \quad (53)$$

Moreover, the symplectic rank of a mixed state ρ , denoted as $\mathfrak{s}(\rho)$, is defined as

$$\mathfrak{s}(\rho) := \min_{\rho = \sum_i p_i \psi_i} \max_i \mathfrak{s}(\psi_i), \quad (54)$$

where the minimum is taken over all the possible decompositions of ρ as a convex combination of pure states.

Note that we can define the symplectic rank of mixed states as a min max rather than an infsup, because the symplectic rank of an n -mode state is integer-valued and bounded between 0 and n .

In other words, the symplectic rank of a pure state ψ is the minimum t for which the state is t -compressible [34].

The forthcoming Theorem 7 gives a characterization of the symplectic rank of a pure state in terms of the symplectic eigenvalues of its covariance matrix [60]. In short, the symplectic rank of a pure state is the number of symplectic eigenvalues of its covariance matrix that are strictly larger than one. Moreover, this lemma establishes also that the compressed state ϕ in Definition 54 can be chosen so that its mean photon number is bounded by the one of the original state ψ . Additionally, the Gaussian unitary operation G in Definition 54 can be chosen so that it maps the Gaussification ρ_ψ of ψ (i.e., ρ_ψ is the Gaussian state with same first moment and same covariance matrix as ψ) to its normal mode decomposition (see Lemma 6) as

$$\rho_\psi = G(\tau \otimes |0\rangle\langle 0|^{\otimes(n-t)}) G^\dagger, \quad (55)$$

where τ is a tensor product of suitable t thermal states. We are now ready to present Theorem 7.

Theorem 7 (Characterization of the symplectic rank). Let ψ be a pure state with a well-defined covariance matrix $V(\psi)$, i.e., such that all the elements of $V(\psi)$ are finite. The following facts hold:

- (a) The symplectic rank $\mathfrak{s}(\psi)$ is equal to the number of symplectic eigenvalues of $V(\psi)$ which are strictly larger than one.
- (b) The state ψ can be written as

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-\mathfrak{s}(\psi))}), \quad (56)$$

where, by denoting as $V(\psi) = SDS^\top$ the Williamson decomposition of $V(\psi)$ as in Eq. (16), we have that: (i) ϕ is a state over $\mathfrak{s}(\psi)$ modes with zero first moment and covariance matrix given by $V(\phi) = \text{diag}(d_1, d_1, \dots, d_{\mathfrak{s}(\psi)}, d_{\mathfrak{s}(\psi)})$, with $d_1, \dots, d_{\mathfrak{s}(\psi)}$ being the symplectic eigenvalues of $V(\psi)$ that are strictly larger than one; and (ii) G is a Gaussian unitary operation of the form $G = D_{\mathbf{m}(\psi)} U_S$, with U_S being the symplectic Gaussian unitary associated with S , and $D_{\mathbf{m}(\psi)}$ denoting the displacement operator with amplitude given by the first moment of ψ .

- (c) The compressed state ϕ in Eq. (56) has mean total photon number upper bounded by the one of ψ ,

$$N(\phi) \leq N(\psi), \quad (57)$$

where $N(*)$ denotes the mean total photon number of the state $*$.

Proof. The first and second claims are direct consequences of Lemma S70 of Ref. [34]. Hence, it suffices to prove the third claim. To do this, let us first state the following result, which follows by taking $k = n$ and $M = \mathbb{1}$

in the statement of Lemma 5.1 of Ref. [91] (see also Lemma 1 of Ref. [92]): For any covariance matrix V it holds that

$$\text{Tr}[D] \leq \text{Tr}[V], \quad (58)$$

where D denotes the matrix of symplectic eigenvalues of V as in Eq. (16). Hence, note that

$$\begin{aligned} N(\phi) &\stackrel{(i)}{=} \frac{\text{Tr}[V(\phi)] - 2\mathfrak{s}(\psi)}{4} + \frac{\|\mathbf{m}(\phi)\|^2}{2} \\ &\stackrel{(ii)}{=} \frac{\text{Tr}[D] - 2n}{4} + \frac{\|\mathbf{m}(\phi)\|^2}{2} \\ &= \frac{\text{Tr}[D] - 2n}{4} \\ &\stackrel{(iii)}{\leq} \frac{\text{Tr}[SDS^\top] - 2n}{4} \\ &= \frac{\text{Tr}[V(\psi)] - 2n}{4} \\ &\leq \frac{\text{Tr}[V(\psi)] - 2n}{4} + \frac{\|\mathbf{m}(\psi)\|^2}{2} \\ &\stackrel{(iv)}{=} N(\psi), \end{aligned} \quad (59)$$

where in (i) we exploited the general formula for the mean photon number of a state in Eq. (22); in (ii) we exploited that $\text{Tr}[D] = 2 \sum_{i=1}^{\mathfrak{s}(\psi)} d_i + 2(n - \mathfrak{s}(\psi)) = \text{Tr}[V(\phi)] + 2(n - \mathfrak{s}(\psi))$; in (iii) we used Eq. (58); and in (iv) we used again the general formula for the mean photon number of a state. ■

Note that the only assumption of the above Theorem 7 is that the covariance matrix of the state is well defined. This assumption is always satisfied by states with finite mean photon number, as established in Lemma 3. Consequently, for states with finite energy, the symplectic rank is always equal to the number of symplectic eigenvalues which are strictly larger than one.

Moreover, the symplectic rank of a state vanishes if and only if the state is a convex combination of Gaussian states, as proven in the following lemma.

Lemma 8 (Faithfulness of the symplectic rank). Let ρ be a quantum state. It holds that $\mathfrak{s}(\rho) = 0$ if and only if ρ is a convex combination of Gaussian states.

Proof. The nontrivial direction of the proof is to show that if ρ is a convex combination of Gaussian states, then $\mathfrak{s}(\rho) = 0$. To establish this, it suffices to note that any convex combination of (possibly mixed) Gaussian states can always be expressed as a convex combination of pure Gaussian states. This follows from the fact that any mixed Gaussian state can be decomposed as a convex combination of pure Gaussian states [Lemma S4 of Ref. [53]]. ■

Additionally, the forthcoming result establishes that the symplectic rank is additive under the tensor product of pure states and subadditive under the tensor product of mixed states.

Lemma 9 (Subadditivity under tensor product). Let ρ be a quantum state. Then, it holds that

$$\begin{aligned} \mathfrak{s}(\rho \otimes \psi) &= \mathfrak{s}(\rho) + \mathfrak{s}(\psi), \\ \mathfrak{s}(\rho \otimes \sigma) &\leq \mathfrak{s}(\rho) + \mathfrak{s}(\sigma), \end{aligned} \quad (60)$$

for all pure states ψ and mixed states σ .

Proof. By exploiting the definition of symplectic rank for pure states, one can easily observe that $\mathfrak{s}(\phi \otimes \psi) = \mathfrak{s}(\phi) + \mathfrak{s}(\psi)$ for all pure states ϕ, ψ . Consequently, the proof of $\mathfrak{s}(\rho \otimes \psi) = \mathfrak{s}(\rho) + \mathfrak{s}(\psi)$ follows from the fact that any decomposition of $\rho \otimes \psi$ as a convex combination of pure states is of the form $\rho \otimes \psi = \sum_i p_i \phi_i \otimes \psi$, with $\rho = \sum_i p_i \phi_i$.

Let us now show that $\mathfrak{s}(\rho \otimes \sigma) \leq \mathfrak{s}(\rho) + \mathfrak{s}(\sigma)$. Let us consider optimal decompositions of ρ and σ as convex combinations of pure states, i.e., $\rho = \sum_i p_i \phi_i$ and $\sigma = \sum_j q_j \psi_j$ such that $\mathfrak{s}(\rho) = \max_i \mathfrak{s}(\phi_i)$ and $\mathfrak{s}(\sigma) = \max_j \mathfrak{s}(\psi_j)$. Then, $\sum_{i,j} p_i q_j \phi_i \otimes \psi_j$ constitutes a decomposition of $\rho \otimes \sigma$ as a convex combination of pure states, and hence

$$\begin{aligned} \mathfrak{s}(\rho \otimes \sigma) &\leq \max_{i,j} \mathfrak{s}(\phi_i \otimes \psi_j) = \max_{i,j} [\mathfrak{s}(\phi_i) + \mathfrak{s}(\psi_j)] \\ &= \mathfrak{s}(\rho) + \mathfrak{s}(\sigma), \end{aligned} \quad (61)$$

which concludes the proof. ■

Most importantly, the symplectic rank is also a *non-Gaussianity monotone*, as proven in the following section.

A. The symplectic rank is nonincreasing under postselected Gaussian operations

In this section, we first prove our main result: the symplectic rank is nonincreasing under *postselected Gaussian operations* [28]. Then, we show its fundamental consequences: (i) powerful no-go theorems on the task of state conversion under Gaussian protocols, and (ii) irreversibility of the resource theory of non-Gaussianity.

Definition 7 (Postselected Gaussian operation). A postselected Gaussian operation is a composition of the following five building blocks:

- (a) Tensoring with a Gaussian state;
- (b) Applying a Gaussian unitary operation;
- (c) Performing (non-Gaussian) classical mixing;
- (d) Performing an heterodyne measurement, and postselecting on the outcome;

(e) Taking a partial trace.

Let us remark that an arbitrary composition of the above five operations include any protocol one can make with linear optics. For example, any Gaussian measurement can be implemented by performing a Gaussian unitary operation followed by a heterodyne detection [28,71]. The ability to postselect on measurement outcomes also includes (and subsumes) the ability to condition quantum operations on the outcomes of measurement (i.e., feed-forward, or adaptive protocols). Moreover, note that any Gaussian channel can be implemented by partial traces, Gaussian unitary operations, and tensoring with the vacuum state via the Gaussian Stinespring dilation [60].

Let us state our main result.

Theorem 8 (Monotonicity of the symplectic rank). The symplectic rank is nonincreasing under postselected Gaussian operations:

- (a) **Tensoring with Gaussian states:** For any state ρ and any Gaussian state σ_G , it holds that $\mathfrak{s}(\rho \otimes \sigma_G) \leq \mathfrak{s}(\rho)$.
- (b) **Gaussian unitary operations:** For any state ρ and any Gaussian unitary operation G , it holds that $\mathfrak{s}(G\rho G^\dagger) \leq \mathfrak{s}(\rho)$.
- (c) **Classical mixing:** For any ensemble of states $\{p_i, \rho_i\}_i$ it holds that

$$\mathfrak{s}\left(\sum_i p_i \rho_i\right) \leq \max_i \mathfrak{s}(\rho_i). \quad (62)$$

- (d) **Heterodyne measurement with postselection:** The symplectic rank of any postoutcome state, after an heterodyne measurement on a subset of modes, is not larger than the symplectic rank of the original state. That is, for any bipartite state ρ_{AB} and any coherent state $|\alpha\rangle_A$ on A (where α represents the outcome of the measurement performed on the system A), it holds that

$$\mathfrak{s}\left(\frac{\langle \alpha |_A \rho_{AB} | \alpha \rangle_A}{\text{Tr} \langle \alpha |_A \rho_{AB} | \alpha \rangle_A}\right) \leq \mathfrak{s}(\rho). \quad (63)$$

- (e) **Partial traces:** For any bipartite state ρ_{AB} , it holds that $\mathfrak{s}(\text{Tr}_A \rho_{AB}) \leq \mathfrak{s}(\rho_{AB})$.

Before proving Theorem 8, let us show some preliminary lemmas.

Lemma 10 (Projecting a two-mode pure Gaussian state on a single-mode vacuum state). Let G_{12} be a two-mode

Gaussian unitary such that

$$\frac{\langle 0 |_1 G_{12} | 0 \rangle^{\otimes 2}}{\|\langle 0 |_1 G_{12} | 0 \rangle^{\otimes 2}\|} = |0\rangle_2. \quad (64)$$

Then, there exists $\mathbf{y} \in \mathbb{C}^2$ such that

$$\frac{\langle 0 |_1 G_{12} | \alpha \rangle}{\|\langle 0 |_1 G_{12} | \alpha \rangle\|} = e^{i\theta(\alpha)} |\mathbf{y}^\top \alpha\rangle \quad \forall \alpha \in \mathbb{C}^2, \quad (65)$$

where $e^{i\theta(\alpha)}$ is a suitable phase.

Proof. By exploiting Eq. (39), we can write

$$G_{12} = G_U S(\xi) D(\beta) G_V, \quad (66)$$

where G_U, G_V are passive Gaussian unitary operations associated with suitable unitary matrices $U, V \in \mathbb{C}^{2 \times 2}$, $S(\xi)$ is the 2-mode squeezing unitary of squeezing vector $\xi \in \mathbb{R}^2$, and $D(\beta)$ is the displacement operator of displacement vector $\beta \in \mathbb{C}^2$ [41,60]. Then, it holds that

$$\begin{aligned} G_{12} | \alpha \rangle &= G_U S(\xi) D(\beta) G_V D(\alpha) | 0 \rangle^{\otimes 2} \\ &= G_U S(\xi) D(\beta) D(V^\top \alpha) | 0 \rangle^{\otimes 2} \\ &= e^{i\phi(\alpha)} G_U S(\xi) D(\beta + V^\top \alpha) | 0 \rangle^{\otimes 2}, \end{aligned} \quad (67)$$

where in the second line we used Eq. (37), and in the last line we exploited the composition rule between displacement operators [60], with $e^{i\phi(\alpha)}$ being a suitable phase. Thanks to Lemma 7, the stellar function of $G_U S(\xi) D(\beta + V^\top \alpha) | 0 \rangle^{\otimes 2}$ reads

$$F_{G_U S(\xi) D(\beta + V^\top \alpha) | 0 \rangle^{\otimes 2}}^*(\mathbf{z}) = \frac{1}{\mathcal{N}} e^{\frac{1}{2} \mathbf{z}^\top A \mathbf{z} + \mathbf{w}^\top \mathbf{z} + \mathbf{v}^\top \mathbf{z} + \tilde{C}} \quad \forall \mathbf{z} \in \mathbb{C}^2, \quad (68)$$

where we defined

$$\begin{aligned} A &:= U^\top \text{Diag}(\tanh \xi_1, \tanh \xi_2) U, \\ \mathbf{w} &:= U^\top \mathbf{b}, \\ \mathbf{v} &:= U^\top \mathbf{a}, \\ \tilde{C} &:= \tilde{c}_1 + \tilde{c}_2, \\ \mathcal{N} &:= \sqrt{\cosh \xi_1 \cosh \xi_2}, \end{aligned} \quad (69)$$

and

$$\begin{aligned} b_j &:= \frac{\beta_j}{\cosh \xi_j}, \\ a_j &:= \frac{(V^\top \alpha)_j}{\cosh \xi_j}, \\ \tilde{c}_j &:= -\frac{1}{2} (\tanh \xi_j) (\beta_j + (V^\top \alpha)_j)^2 - \frac{1}{2} |\beta_j + (V^\top \alpha)_j|^2, \end{aligned} \quad (70)$$

for all $j \in \{1, 2\}$. Consequently, by defining

$$\begin{aligned} C &:= c_1 + c_2, \\ c_j &:= -\frac{1}{2}(\tanh \xi_j) \beta_j^2 - \frac{1}{2}|\beta_j|^2 \quad j \in \{1, 2\}, \end{aligned} \quad (71)$$

we obtain that

$$\begin{aligned} F_{GUS(\xi)D(\beta+V\tau\alpha)|0\rangle^{\otimes 2}}^*(z) &= e^{\tilde{C}-C+v^\tau z} \frac{1}{\mathcal{N}} e^{\frac{1}{2}z^\tau A z + w^\tau z + C} \\ &= e^{\tilde{C}-C+v^\tau z} F_{GUS(\xi)D(\beta)|0\rangle^{\otimes 2}}^*(z). \end{aligned} \quad (72)$$

Hence, it holds that

$$\begin{aligned} F_{|0\rangle_1 GUS(\xi)D(\beta+V\tau\alpha)|0\rangle^{\otimes 2}}^*(z) &= F_{GUS(\xi)D(\beta+V\tau\alpha)|0\rangle^{\otimes 2}}^*(0, z) \\ &= e^{\tilde{C}-C+v_2 z} F_{GUS(\xi)D(\beta)|0\rangle^{\otimes 2}}^*(0, z) \\ &= e^{\tilde{C}-C+v_2 z} F_{|0\rangle_1 GUS(\xi)D(\beta)|0\rangle^{\otimes 2}}^*(z) \\ &\stackrel{(i)}{=} \|\langle 0|_1 G_{12} |0\rangle^{\otimes 2}\| e^{\tilde{C}-C+v_2 z} F_{|0\rangle}^*(z) \\ &\stackrel{(ii)}{=} \|\langle 0|_1 G_{12} |0\rangle^{\otimes 2}\| e^{\tilde{C}-C+v_2 z} \\ &\stackrel{(iii)}{=} \|\langle 0|_1 G_{12} |0\rangle^{\otimes 2}\| e^{\tilde{C}-C-C'} F_{|v_2\rangle}^*(z). \end{aligned} \quad (73)$$

Here, in (i), we used Eq. (66) and the fact that passive Gaussian unitary operations preserve the vacuum to conclude that

$$\begin{aligned} \langle 0|_1 GUS(\xi)D(\beta) |0\rangle^{\otimes 2} &= \langle 0|_1 GUS(\xi)D(\beta) G_V |0\rangle^{\otimes 2} \\ &= \langle 0|_1 G_{12} |0\rangle^{\otimes 2} \\ &= \|\langle 0|_1 G_{12} |0\rangle^{\otimes 2}\| |0\rangle; \end{aligned} \quad (74)$$

in (ii), we exploited that the stellar function of the vacuum is one; in (iii), we identified the stellar function of the coherent state $|v_2\rangle$ by defining $C' := -\frac{1}{2}|v_2|^2$. Taking advantage of the normalization condition of the stellar function in Eq. (48), Eq. (73) implies that

$$\frac{\langle 0|_1 GUS(\xi)D(\beta+V\tau\alpha) |0\rangle^{\otimes 2}}{\|\langle 0|_1 GUS(\xi)D(\beta+V\tau\alpha) |0\rangle^{\otimes 2}\|} = e^{i\varphi(\alpha)} |v_2\rangle, \quad (75)$$

where $e^{i\varphi(\alpha)}$ is a suitable phase. By exploiting the definition of v_2 in Eq. (69), we have that

$$v_2 = U_{12} \frac{(V^\tau \alpha)_1}{\cosh \xi_1} + U_{22} \frac{(V^\tau \alpha)_2}{\cosh \xi_2}. \quad (76)$$

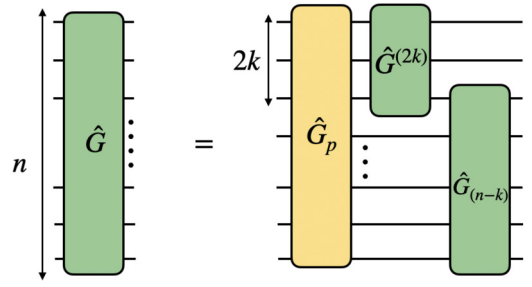


FIG. 2. Schematic representation of the block decomposition of Gaussian unitaries in Lemma 11. Specifically, any Gaussian unitary G acting on n modes can be decomposed, for all $k \leq n/2$, into the composition of: a global passive Gaussian unitary G_p , a Gaussian unitary $G^{(2k)}$ acting on the first $2k$ modes, and a Gaussian unitary $G_{(n-k)}$ acting on the last $n-k$ modes.

Hence, we can identify a complex vector $y \in \mathbb{C}^2$ such that $v_2 = y^\tau \alpha$, so that

$$\frac{\langle 0|_1 GUS(\xi)D(\beta+V\tau\alpha) |0\rangle^{\otimes 2}}{\|\langle 0|_1 GUS(\xi)D(\beta+V\tau\alpha) |0\rangle^{\otimes 2}\|} = e^{i\varphi(\alpha)} |y^\tau \alpha\rangle. \quad (77)$$

By substituting in Eq. (67), we obtain that

$$\frac{\langle 0|_1 G_{12} |\alpha\rangle}{\|\langle 0|_1 G_{12} |\alpha\rangle\|} = e^{i\theta(\alpha)} |y^\tau \alpha\rangle, \quad (78)$$

where $e^{i\theta(\alpha)}$ is a suitable phase. ■

Now, we introduce a general decomposition for Gaussian unitary operations, which is crucial for the proof of the monotonicity of the symplectic rank. See Fig. 2 for a schematic representation of such a decomposition.

Lemma 11 (Block decomposition of Gaussian unitary operations). Let G be an n -mode Gaussian unitary. Moreover, let $k \leq n/2$ be an integer. Then, there exists an n -mode passive Gaussian unitary G_p , a $(2k)$ -mode Gaussian unitary $G^{(2k)}$ acting only on the first $2k$ modes, and an $(n-k)$ -mode Gaussian unitary $G_{(n-k)}$ acting only on the last $n-k$ modes such that

$$G = (\mathbb{1}_k \otimes G_{(n-k)}) (G^{(2k)} \otimes \mathbb{1}_{n-2k}) G_p. \quad (79)$$

Moreover, $G^{(2k)}$ satisfies

$$\frac{\langle 0|^{\otimes k} G^{(2k)} |0\rangle^{\otimes 2k}}{\|\langle 0|^{\otimes k} G^{(2k)} |0\rangle^{\otimes 2k}\|} = |0\rangle^{\otimes k}. \quad (80)$$

Here, we provide two different proofs of the decomposition in Eq. (79). The first proof is conceptually simpler, but does not allow us to prove the additional requirement in Eq. (80). In contrast, the second proof is more technical,

but it allows one to completely prove the above Lemma 11 by establishing both the decomposition in Eq. (79) and the additional requirement in Eq. (80).

Simple proof of Eq. (79). We follow a similar reasoning as in Lemma 10 of Ref. [93]. By Schmidt decomposition, two complementary reduced states of a bipartite pure state have the same nonzero spectrum. Moreover, the spectrum of a k -mode Gaussian state is given by the spectrum of the tensor product of k suitable thermal states, thanks to the normal mode decomposition in Lemma 6. Hence, since $n - k \geq k$, the reduced state on the last $n - k$ modes of the pure Gaussian state $G|0\rangle^{\otimes n}$ is unitarily equivalent—through a suitable Gaussian unitary $(n - k)$ mode Gaussian unitary $G_{(n-k)}$ —to the tensor product of k suitable thermal states and the $(n - 2k)$ -mode vacuum state. Consequently, the state $G_{(n-k)}^\dagger G|0\rangle^{\otimes n}$ is of the form

$$G_{(n-k)}^\dagger G|0\rangle^{\otimes n} = |\phi_G\rangle \otimes |0\rangle^{\otimes(n-2k)}, \quad (81)$$

where ϕ_G is a suitable $2k$ -mode Gaussian pure state. In particular, there exists a suitable $2k$ -mode Gaussian unitary $G^{(2k)}$ such that $|\phi_G\rangle = G^{(2k)}|0\rangle^{\otimes 2k}$. Hence, it follows

$$(G^{(2k)})^\dagger G_{(n-k)}^\dagger G|0\rangle^{\otimes n} = |0\rangle^{\otimes n}, \quad (82)$$

meaning that the Gaussian unitary operation $(G^{(2k)})^\dagger G_{(n-k)}^\dagger G$ preserves the vacuum state. Moreover, any Gaussian unitary operation that preserves the vacuum must be passive, as it easily follows from Eq. (31). Consequently, $(G^{(2k)})^\dagger G_{(n-k)}^\dagger G$ is equal to a suitable passive Gaussian unitary G_p , which implies

$$G = (\mathbb{1}_k \otimes G_{(n-k)}) (G^{(2k)} \otimes \mathbb{1}_{n-2k}) G_p. \quad (83)$$

■

Proof of Lemma 11. Since the state proportional to $\langle 0|^{\otimes k} G|0\rangle^{\otimes n}$ is a $(n - k)$ -mode Gaussian state, it follows that there exists an $(n - k)$ -mode Gaussian unitary $\tilde{G}_{(n-k)}$ such that

$$\frac{\langle 0|^{\otimes k} G|0\rangle^{\otimes n}}{\mathcal{N}} = \tilde{G}_{(n-k)} |0\rangle^{\otimes(n-k)}, \quad (84)$$

where we defined

$$\mathcal{N} := \|\langle 0|^{\otimes k} G|0\rangle^{\otimes n}\|. \quad (85)$$

In particular, we have that

$$\frac{\langle 0|^{\otimes k} (\mathbb{1}_k \otimes \tilde{G}_{(n-k)}^\dagger) G|0\rangle^{\otimes n}}{\mathcal{N}} = |0\rangle^{\otimes(n-k)}. \quad (86)$$

Hence, at the level of stellar functions, we have that

$$\begin{aligned} F_{(\mathbb{1}_k \otimes \tilde{G}_{(n-k)}^\dagger) G|0\rangle^{\otimes n}}^*(0, \dots, 0, z_{k+1}, \dots, z_n) \\ = \mathcal{N} F_{|0\rangle^{\otimes(n-k)}}^*(z_{k+1}, \dots, z_n) = \mathcal{N}, \end{aligned} \quad (87)$$

for any $z_{k+1}, \dots, z_n \in \mathbb{C}$. Consequently, Lemma 7 implies that there exist two matrices $A_1 \in \mathbb{C}^{k \times k}$, $A_2 \in \mathbb{C}^{k \times (n-k)}$ and a vector $\mathbf{w} \in \mathbb{C}^k$ such that

$$\begin{aligned} F_{(\mathbb{1}_k \otimes \tilde{G}_{(n-k)}^\dagger) G|0\rangle^{\otimes n}}^*(\mathbf{z}) \\ = \mathcal{N} \exp\left(\frac{1}{2}(\mathbf{z}_A)^\top A_1 \mathbf{z}_A + (\mathbf{z}_A)^\top A_2 \mathbf{z}_B + \mathbf{w}^\top \mathbf{z}_A\right) \\ \times \forall \mathbf{z} \in \mathbb{C}^n, \end{aligned} \quad (88)$$

where $\mathbf{z}_A := (z_1, \dots, z_k)$ and $\mathbf{z}_B := (z_{k+1}, \dots, z_n)$, so that $\mathbf{z} = (\mathbf{z}_A, \mathbf{z}_B)$. By exploiting the singular value decomposition of A_2 , there exist unitary matrices $U_1 \in \mathbb{C}^{k \times k}$, $U_2 \in \mathbb{C}^{(n-k) \times (n-k)}$, and a rectangular matrix $\Sigma \in \mathbb{R}^{k \times (n-k)}$ having nonzero elements only on the main diagonal, so that $A_2 = U_1 \Sigma U_2$. Moreover, let $\tilde{\Sigma}$ the $k \times k$ diagonal matrix such that $\Sigma = (\tilde{\Sigma} \ 0_{k \times (n-2k)})$, where $0_{k \times (n-2k)}$ denotes the $k \times (n - 2k)$ zero matrix. Hence, by exploiting Eq. (49), it follows that for any $\mathbf{z} \in \mathbb{C}^n$ it holds that

$$\begin{aligned} F_{(\mathbb{1}_k \otimes G_{U_2}^\dagger)(\mathbb{1}_k \otimes \tilde{G}_{(n-k)}^\dagger) G|0\rangle^{\otimes n}}^*(\mathbf{z}) \\ = \mathcal{N} \exp\left(\frac{1}{2}(\mathbf{z}_A)^\top A_1 \mathbf{z}_A + (\mathbf{z}_A)^\top U_1 \Sigma \mathbf{z}_B + \mathbf{w}^\top \mathbf{z}_A\right), \\ = \mathcal{N} \exp\left(\frac{1}{2}(\mathbf{z}_A)^\top A_1 \mathbf{z}_A + (\mathbf{z}_A)^\top U_1 \tilde{\Sigma} \mathbf{z}_C + \mathbf{w}^\top \mathbf{z}_A\right), \end{aligned} \quad (89)$$

where we defined $\mathbf{z}_C = (z_1, \dots, z_{2k})$. This implies that there exists a $2k$ -mode Gaussian unitary $G^{(2k)}$ such that

$$\begin{aligned} (\mathbb{1}_k \otimes G_{U_2}^\dagger) (\mathbb{1}_k \otimes \tilde{G}_{(n-k)}^\dagger) G|0\rangle^{\otimes n} \\ = (G^{(2k)} \otimes \mathbb{1}_{n-2k}) |0\rangle^{\otimes n}. \end{aligned} \quad (90)$$

Consequently, since any Gaussian unitary operation that preserves the vacuum state must be passive (this can be proven by considering the action of such a transformation at the level of covariance matrices), it follows that there exists a passive Gaussian unitary G_p such that

$$(G^{(2k)^\dagger} \otimes \mathbb{1}_{n-2k}) (\mathbb{1}_k \otimes G_{U_2}^\dagger \tilde{G}_{(n-k)}^\dagger) G = G_p, \quad (91)$$

or, equivalently,

$$G = (\mathbb{1}_k \otimes G_{(n-k)}) (G^{(2k)} \otimes \mathbb{1}_{n-2k}) G_p, \quad (92)$$

where we defined $G_{(n-k)} := \tilde{G}_{(n-k)} G_{U_2}$. This proves the first part of the thesis.

Moreover, by denoting as $\langle 0|_{[2k+1,n]}$ the vacuum state in the modes $2k+1, 2k+2, \dots, n$, we have that

$$\begin{aligned}
& \frac{\langle 0|^{\otimes k} G^{(2k)} |0\rangle^{\otimes 2k}}{\| \langle 0|^{\otimes k} G^{(2k)} |0\rangle^{\otimes 2k} \|} \\
& \stackrel{(i)}{=} \langle 0|_{[2k+1,n]} \frac{\langle 0|^{\otimes k} (G^{(2k)} \otimes \mathbb{1}_{n-2k}) |0\rangle^{\otimes n}}{\| \langle 0|^{\otimes k} G |0\rangle^{\otimes n} \|} \\
& \stackrel{(ii)}{=} \langle 0|_{[2k+1,n]} \frac{\langle 0|^{\otimes k} \left(\mathbb{1}_k \otimes G_{U_2}^\dagger \tilde{G}_{(n-k)}^\dagger \right) G |0\rangle^{\otimes n}}{\| \langle 0|^{\otimes k} G |0\rangle^{\otimes n} \|} \\
& = \langle 0|_{[2k+1,n]} G_{U_2}^\dagger \tilde{G}_{(n-k)}^\dagger \frac{\langle 0|^{\otimes k} G |0\rangle^{\otimes n}}{\| \langle 0|^{\otimes k} G |0\rangle^{\otimes n} \|} \\
& \stackrel{(iii)}{=} \langle 0|_{[2k+1,n]} G_{U_2}^\dagger \tilde{G}_{(n-k)}^\dagger \tilde{G}_{(n-k)} |0\rangle^{\otimes (n-k)} \\
& = \langle 0|_{[2k+1,n]} G_{U_2}^\dagger |0\rangle^{\otimes (n-k)} \\
& \stackrel{(iv)}{=} \langle 0|_{[2k+1,n]} |0\rangle^{\otimes (n-k)} \\
& = |0\rangle^{\otimes k}, \tag{93}
\end{aligned}$$

where in (i) we used Eq. (92) and the fact that G_p is passive; in (ii) we exploited Eq. (90), (iii) follows from Eq. (84), and in (iv) we exploited the fact that $G_{U_2}^\dagger$ is passive. ■

The decomposition proven in Lemma 11 leads to the following remarkable consequence.

Corollary 2. Let m an even natural number. Any bipartite (possibly mixed) Gaussian state over $m+m$ modes can be disentangled by applying a suitable $(\frac{3}{2}m)$ -mode Gaussian unitary.

Proof. Thanks to the normal mode decomposition in Lemma 6, any $(2m)$ -mode (possibly mixed) Gaussian state ρ is unitarily equivalent—via a Gaussian unitary operation G —to a state τ which is a tensor product of $2m$ thermal states,

$$\rho = G\tau G^\dagger. \tag{94}$$

Moreover, it is well known that a multimode thermal state can be written as a convex combination of coherent states [60]. That is, there exists a (Gaussian) probability distribution $P(\cdot)$ over $\mathbb{C}^{2m}$ such that

$$\tau = \int_{\mathbb{C}^{2m}} d^{2m} \mathbf{z} P(\mathbf{z}) |\mathbf{z}\rangle\langle\mathbf{z}|. \tag{95}$$

Consequently, we have that

$$\rho = \int_{\mathbb{C}^{2m}} d^{2m} \mathbf{z} P(\mathbf{z}) G |\mathbf{z}\rangle\langle\mathbf{z}| G^\dagger. \tag{96}$$

By using Lemma 11 with $n = 2m$ and $k = m/2$, we can decompose G as in Eq. (79) to deduce that

$$\begin{aligned}
\rho &= \int_{\mathbb{C}^{2m}} d^{2m} \mathbf{z} P(\mathbf{z}) \left(\mathbb{1}_{\frac{m}{2}} \otimes G_{(\frac{3}{2}m)} \right) (G^{(m)} \otimes \mathbb{1}_m) G_p |\mathbf{z}\rangle\langle\mathbf{z}| \\
&\quad \times G_p^\dagger (G^{(m)} \otimes \mathbb{1}_m)^\dagger \left(\mathbb{1}_{\frac{m}{2}} \otimes G_{(\frac{3}{2}m)} \right)^\dagger, \tag{97}
\end{aligned}$$

where $G_{(\frac{3}{2}m)}$ and $G^{(m)}$ are suitable $(\frac{3}{2}m)$ -mode and m -mode Gaussian unitary operations, respectively, and G_p is a suitable passive Gaussian unitary operation. By using Eq. (37), it easily follows that coherent states are mapped into coherent states via passive Gaussian unitary operations. Specifically, there exists a unitary matrix U such that $G_p |\mathbf{z}\rangle = |U\mathbf{z}\rangle$ for all $\mathbf{z} \in \mathbb{C}^{2m}$ and, in particular, it holds that

$$\begin{aligned}
& \left(\mathbb{1}_{\frac{m}{2}} \otimes G_{(\frac{3}{2}m)} \right)^\dagger \rho \left(\mathbb{1}_{\frac{m}{2}} \otimes G_{(\frac{3}{2}m)} \right) \\
&= \int_{\mathbb{C}^{2m}} d^{2m} \mathbf{z} P(\mathbf{z}) (G^{(m)} \otimes \mathbb{1}_m) |U\mathbf{z}\rangle\langle U\mathbf{z}| (G^{(m)} \otimes \mathbb{1}_m)^\dagger. \tag{98}
\end{aligned}$$

Since coherent states are product states, it follows that for all $\mathbf{z} \in \mathbb{C}^{2m}$ the state $(G^{(m)} \otimes \mathbb{1}_m) |U\mathbf{z}\rangle$ is product among the bipartition of the first m modes and the last m modes. Consequently, we have proven that the state

$$\left(\mathbb{1}_{\frac{m}{2}} \otimes G_{(\frac{3}{2}m)} \right)^\dagger \rho \left(\mathbb{1}_{\frac{m}{2}} \otimes G_{(\frac{3}{2}m)} \right) \tag{99}$$

can be written as a convex combination of product states over $m+m$ modes, i.e., the state is separable, which concludes the proof. ■

Remarkably, this corollary allows one to *gain 25% of modes*: although *a priori* one would use a $(2m)$ -mode unitary operation to disentangle a bipartite entangled Gaussian state over $m+m$ modes, our result establishes that just $\frac{3}{2}m$ modes are sufficient.

We are now ready to prove Theorem 8.

Proof of Theorem 8. In order to prove that the symplectic rank is nonincreasing under Gaussian operations, it suffices to prove that it is nonincreasing under each of the five operations listed in the statement of the theorem.

Tensoring with a Gaussian state. Recall that the symplectic rank is subadditive with respect to the tensor product, as it easily follows from the definition of symplectic rank. Hence, since the symplectic rank of Gaussian states is zero, it follows that the symplectic rank is nonincreasing when tensoring with a Gaussian state.

Applying a Gaussian unitary operation. Observe that any t -compressible state remains t -compressible under

Gaussian unitary operations. Since by Theorem 7 the symplectic rank of a pure state is the smallest t for which the state is t -compressible, it follows that it is nonincreasing under Gaussian unitary operations (and in fact, invariant by considering the inverse Gaussian unitary operation). As a consequence, the same statement is immediate for mixed states.

Classical mixing. Recall that the symplectic rank of a mixed state ρ is defined as in Eq. (54). Given two states with optimal decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ and $\sigma = \sum_j q_j |\phi_j\rangle\langle\phi_j|$, the symplectic rank of their mixture thus is directly less than or equal to the maximal symplectic rank of each state. Hence, the symplectic rank is nonincreasing under classical mixing.

Heterodyne measurement with postselection. Let ρ_{AB} be a bipartite state and let $|\alpha\rangle_A$ be a coherent state on A of amplitude α . We need to show that

$$\mathfrak{s}\left(\frac{\langle\alpha|_A \rho_{AB} |\alpha\rangle_A}{\text{Tr}\langle\alpha|_A \rho_{AB} |\alpha\rangle_A}\right) \leq \mathfrak{s}(\rho_{AB}). \quad (100)$$

Note that it suffices to prove the statement for the particular case in which ρ_{AB} is pure. Indeed, assume that the statement holds for pure states, and let ρ_{AB} be a mixed state. Consider the optimal decomposition $\rho_{AB} = \sum_i p_i \psi_i$ saturating the definition of symplectic rank, i.e., such that

$$\mathfrak{s}(\rho_{AB}) = \max_i \mathfrak{s}(\psi_i). \quad (101)$$

Then, we have that

$$\begin{aligned} & \mathfrak{s}\left(\frac{\langle\alpha|_A \rho_{AB} |\alpha\rangle_A}{\text{Tr}\langle\alpha|_A \rho_{AB} |\alpha\rangle_A}\right) \\ &= \mathfrak{s}\left(\sum_i p_i \frac{\text{Tr}\langle\alpha|_A \psi_i |\alpha\rangle_A}{\text{Tr}\langle\alpha|_A \rho_{AB} |\alpha\rangle_A} \frac{\langle\alpha|_A \psi_i |\alpha\rangle_A}{\text{Tr}\langle\alpha|_A \psi_i |\alpha\rangle_A}\right) \\ &\leq \max_i \mathfrak{s}\left(\frac{\langle\alpha|_A \psi_i |\alpha\rangle_A}{\text{Tr}\langle\alpha|_A \psi_i |\alpha\rangle_A}\right) \\ &\leq \max_i \mathfrak{s}(\psi_i) \\ &= \mathfrak{s}(\rho_{AB}). \end{aligned} \quad (102)$$

Here, in the first inequality we exploited the definition of symplectic rank for mixed states, together with the facts that $\{p_i(\langle\alpha|_A \psi_i |\alpha\rangle_A / \text{Tr}\langle\alpha|_A \rho_{AB} |\alpha\rangle_A)\}_i$ is a probability distribution and $\{\langle\alpha|_A \psi_i |\alpha\rangle_A / \text{Tr}\langle\alpha|_A \psi_i |\alpha\rangle_A\}_i$ are

pure states; and, in the second inequality, we exploited that Eq. (100) holds for pure states.

Hence, it suffices to show that

$$\mathfrak{s}\left(\frac{\langle\alpha|_A \psi_{AB} |\alpha\rangle_A}{\text{Tr}\langle\alpha|_A \psi_{AB} |\alpha\rangle_A}\right) \leq \mathfrak{s}(\psi_{AB}) \quad (103)$$

for any pure state ψ_{AB} and any coherent state $|\alpha\rangle_A$. Moreover, note that it suffices to prove the statement in the particular case in which A is a single-mode subsystem. Furthermore, since $|\alpha\rangle_A = D(\alpha)|0\rangle_A$ and since the symplectic rank is invariant under Gaussian unitary operations, it suffices to prove the statement for $\alpha = 0$. We give a proof hereafter.

Let ψ an n -mode state with symplectic rank s . Then, by definition, there exists a Gaussian unitary operation G and an s -mode state ϕ such that

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-s)}). \quad (104)$$

In order to prove that the symplectic rank is nonincreasing under heterodyne measurement, it suffices to show that there exists a Gaussian unitary operation G' and an s -mode state ϕ' such that

$$\frac{\langle 0|_1 |\psi\rangle}{\|\langle 0|_1 |\psi\rangle\|} = G'(|\phi'\rangle \otimes |0\rangle^{\otimes(n-s-1)}). \quad (105)$$

Note that the statement is trivial when $s \in \{0, n-1, n\}$ (since the symplectic rank is always less than or equal to the number of modes). Hence, let us assume that $s \notin \{0, n-1, n\}$.

Using Lemma 11 with $k = 1$, it follows that there exists a passive Gaussian unitary operation G_p , a two-mode Gaussian unitary operation $G^{(2)}$ acting on the first two modes, and an $(n-1)$ -mode Gaussian unitary operation acting on the last $(n-1)$ modes such that

$$G = (\mathbb{1}_1 \otimes G_{(n-1)}) (G^{(2)} \otimes \mathbb{1}_{n-2}) G_p, \quad (106)$$

where $G^{(2)}$ satisfies $\langle 0|_1 G^{(2)} |0\rangle^{\otimes 2} \propto |0\rangle_2$. Let U be the 2×2 unitary matrix associated with the passive Gaussian unitary G_p such that the action of G_p on a coherent state $|\mathbf{k}\rangle$ satisfies

$$G_p |\mathbf{k}\rangle = |U\mathbf{k}\rangle \quad \forall \mathbf{k} \in \mathbb{C}^n. \quad (107)$$

Then, we have that

$$\begin{aligned} \frac{\langle 0|_1 |\psi\rangle}{\|\langle 0|_1 |\psi\rangle\|} &= \frac{1}{\|\langle 0|_1 |\psi\rangle\|} \langle 0|_1 G(|\phi\rangle \otimes |0\rangle^{\otimes(n-s)}) \\ &= \frac{1}{\|\langle 0|_1 |\psi\rangle\|} G_{(n-1)} \langle 0|_1 (G^{(2)} \otimes \mathbb{1}_{n-2}) G_p (|\phi\rangle \otimes |0\rangle^{\otimes(n-s)}) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(i)}{=} G_{(n-1)} \int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} \langle 0|_1 (G^{(2)} \otimes \mathbb{1}_{n-2}) G_p (|\alpha\rangle \otimes |0\rangle^{\otimes(n-s)}) \\
&\stackrel{(ii)}{=} G_{(n-1)} \int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} \langle 0|_1 (G^{(2)} \otimes \mathbb{1}_{n-2}) \bigotimes_{i=1}^n \left| \sum_{j=1}^s U_{ij} \alpha_j \right\rangle_i \\
&= G_{(n-1)} \int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} \left(\langle 0|_1 G^{(2)} \left| \sum_{j=1}^s U_{1j} \alpha_j \right\rangle_1 \otimes \left| \sum_{j=1}^s U_{2j} \alpha_j \right\rangle_2 \right) \bigotimes_{i=3}^n \left| \sum_{j=1}^s U_{ij} \alpha_j \right\rangle_i \\
&\stackrel{(iii)}{=} G_{(n-1)} \int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} e^{i\phi(\alpha)} \mathcal{N}(\alpha) \left| w_1 \sum_{j=1}^s U_{1j} \alpha_j + w_2 \sum_{j=1}^s U_{2j} \alpha_j \right\rangle_2 \bigotimes_{i=3}^n \left| \sum_{j=1}^s U_{ij} \alpha_j \right\rangle_i \\
&\stackrel{(iv)}{=} G_{(n-1)} \int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} e^{i\phi(\alpha)} \mathcal{N}(\alpha) |A\alpha\rangle \\
&\stackrel{(v)}{=} G_{(n-1)} \int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} e^{i\phi(\alpha)} \mathcal{N}(\alpha) |U_1 \Sigma U_2 \alpha\rangle \\
&\stackrel{(vi)}{=} G_{(n-1)} G_{U_1} \int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} e^{i\phi(\alpha)} \mathcal{N}(\alpha) |\Sigma U_2 \alpha\rangle \\
&\stackrel{(vii)}{=} G_{(n-1)} G_{U_1} \left(\int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} e^{i\phi(\alpha)} \mathcal{N}(\alpha) |\bar{\Sigma} U_2 \alpha\rangle \otimes |0\rangle^{\otimes(n-s-1)} \right) \\
&\stackrel{(viii)}{=} G' (|\phi'\rangle \otimes |0\rangle^{\otimes(n-s-1)}). \tag{108}
\end{aligned}$$

Here, in (i), we exploited Eq. (45). In (ii), we exploited Eq. (107). In (iii), we defined

$$\mathcal{N}(\alpha) := \left\| \langle 0|_1 G^{(2)} \left| \sum_{j=1}^s U_{1j} \alpha_j \right\rangle_1 \otimes \left| \sum_{j=1}^s U_{2j} \alpha_j \right\rangle_2 \right\|, \tag{109}$$

and we exploited Lemma 10, which guarantees the existence of the complex numbers $w_1, w_2 \in \mathbb{C}$ presented in (iii). In (iv), we identified a matrix $A \in \mathbb{C}^{(n-1) \times s}$, which are defined in terms of w_1, w_2 , and U . In (v), we employed the singular value decomposition of A as $A = U_1 \Sigma U_2$, where $U_1 \in \mathbb{C}^{(n-1) \times (n-1)}$, $U_2 \in \mathbb{C}^{s \times s}$ are unitary operations, and $\Sigma \in \mathbb{R}^{(n-1) \times s}$ is a rectangular diagonal matrix with its $s \times s$ upper block being the only nonzero block. In (vi), we identified a passive Gaussian unitary operation G_{U_1} that acts on coherent states as follows:

$$G_{U_1} |k\rangle = |U_1 k\rangle \quad k \in \mathbb{C}^n. \tag{110}$$

In (vii), we denoted as $\bar{\Sigma}$ the $s \times s$ upper-block of Σ . Finally, in (viii), we defined $G' := G_{(n-1)} G_{U_1}$ and we introduced the following s -mode pure state:

$$|\phi'\rangle := \int_{\mathbb{C}^s} \frac{d^s \alpha}{\pi^s} \frac{\langle \alpha | \phi | \alpha | \phi \rangle}{\| \langle 0|_1 | \psi \rangle \|} e^{i\phi(\alpha)} \mathcal{N}(\alpha) |\bar{\Sigma} U_2 \alpha\rangle. \tag{111}$$

Hence, we conclude that the state $\langle 0|_1 | \psi \rangle / \| \langle 0|_1 | \psi \rangle \|$ is s -compressible, and thus its symplectic rank is less than or equal to s .

Taking a partial trace. Given a bipartite state ρ_{AB} , let us show that

$$\mathfrak{s}(\text{Tr}_A \rho_{AB}) \leq \mathfrak{s}(\rho_{AB}). \tag{112}$$

To this end, by denoting as $|A|$ the number of modes of A , we observe that

$$\begin{aligned}
\text{Tr}_A \rho_{AB} &\stackrel{(ix)}{=} \int_{\mathbb{C}^{|A|}} \frac{d^{|A|} \alpha}{\pi^{|A|}} \langle \alpha |_A \rho_{AB} | \alpha \rangle_A \\
&\stackrel{(x)}{=} \int_{\mathcal{S}_\rho} \frac{d^{|A|} \alpha}{\pi^{|A|}} \text{Tr}[\langle \alpha |_A \rho_{AB} | \alpha \rangle_A] \frac{\langle \alpha |_A \rho_{AB} | \alpha \rangle_A}{\text{Tr}[\langle \alpha |_A \rho_{AB} | \alpha \rangle_A]}. \tag{113}
\end{aligned}$$

Here, in (ix), we exploited Eq. (45) and, in (x), we defined

$$\mathcal{S}_\rho := \{ \alpha \in \mathbb{C}^{|A|} : \text{Tr}[\langle \alpha |_A \rho_{AB} | \alpha \rangle_A] \neq 0 \} \tag{114}$$

and we exploited that if $\text{Tr}[\langle \alpha |_A \rho_{AB} | \alpha \rangle_A] = 0$ then $\langle \alpha |_A \rho_{AB} | \alpha \rangle_A = 0$. Hence, we obtain that

$$\begin{aligned} \mathfrak{s}(\text{Tr}_A \rho_{AB}) &\stackrel{\text{(xi)}}{\leq} \sup_{\alpha \in \mathcal{S}_\rho} \mathfrak{s} \left(\frac{\langle \alpha |_A \rho_{AB} | \alpha \rangle_A}{\text{Tr}[\langle \alpha |_A \rho_{AB} | \alpha \rangle_A]} \right) \\ &\stackrel{\text{(xii)}}{\leq} \sup_{\alpha \in \mathcal{S}_\rho} \mathfrak{s}(\rho_{AB}) = \mathfrak{s}(\rho_{AB}), \end{aligned} \quad (115)$$

where in (xi) we exploited that the symplectic rank is non-increasing under classical mixing, and in (xii) we used that the symplectic rank is nonincreasing under heterodyne measurement with postselection. This concludes the proof. $\blacksquare$

1. Comment on monotonicity under homodyne postselection

In Definition 7 we take heterodyne measurements as a basic building block of Gaussian operations rather than homodyne measurements. However, homodyne detection can be viewed as a “limit” of admissible Gaussian measurements in the sense of Definition 7: one can approximate a quadrature measurement by first applying a squeezing unitary with very large (but finite) squeezing, then performing a heterodyne measurement, and finally rescaling the classical outcome. In this way, “homodyne” is recovered as the limiting case of a family of perfectly well defined Gaussian measurements that are already contained in our framework.

At the same time, the usual “ideal” description of homodyne in terms of the (generalized) quadrature eigenstates $\{|x\rangle\}_{x \in \mathbb{R}}$ is not mathematically rigorous as a postselected operation on arbitrary states. In particular, conditioning on a *single* outcome x is only distributionally defined and, in general, does not produce a proper postselected map (for instance, it may send a normalized input state to a non-normalizable postmeasurement vector). A fully rigorous definition of ideal homodyne postselection can be given in terms of *coarse graining*: namely, one conditions on a measurable set $A \subset \mathbb{R}$ of nonzero measure, rather than on a single point. The corresponding projector is

$$\Pi_A := \int_{x \in A} |x\rangle \langle x| \, dx,$$

and for a two-mode state ρ_{12} the probability that the homodyne outcome on mode 1 lies in A is

$$p(A) = \text{Tr}[(\Pi_A \otimes \mathbb{1}_2) \rho_{12}],$$

while the corresponding postselected state is

$$\rho_{12}|_A = \frac{(\Pi_A \otimes \mathbb{1}_2) \rho_{12} (\Pi_A \otimes \mathbb{1}_2)}{\text{Tr}[(\Pi_A \otimes \mathbb{1}_2) \rho_{12}]}.$$

With this definition, homodyne detection is perfectly well-defined mathematically, and postselected outputs are

proper states. However, in this fully rigorous setting, postselecting on coarse-grained homodyne outcomes can map Gaussian states to non-Gaussian conditional states; hence such postselected operations do not belong to the class of Gaussian operations as axiomatized in this manuscript.

B. Consequences of the monotonicity of the symplectic rank

In this section, we explore some consequences of the monotonicity of the symplectic rank under postselected Gaussian operations.

1. No-go theorems on state conversion under postselected Gaussian operations

Since they can only decrease under free operations, resource monotones typically imply no-go theorems for the task of state conversion under free operations. In this section, we consider the specific case of symplectic rank as a resource monotone for non-Gaussianity.

Consider the task of converting n copies of an initial state ρ into m copies of a target state σ via postselected free operations. In general, if the maximal probability of converting $\rho^{\otimes n}$ into $\sigma^{\otimes m}$ via postselected free operations is strictly larger than zero, then the conversion is *possible* and we call it *probabilistic*. Additionally, if such a maximal probability is exactly one, we say that the conversion is *deterministic*. We stress that deterministic transformations are a subset of probabilistic ones.

In the context of non-Gaussianity, the exact conversion $\rho^{\otimes n} \rightarrow \sigma^{\otimes m}$ is said to be *possible probabilistically* if there exists a postselected Gaussian operation Φ_G such that

$$\Phi_G(\rho^{\otimes n}) = \sigma^{\otimes m}. \quad (116)$$

As a consequence of the monotonicity of the symplectic rank, if the exact conversion $\rho^{\otimes n} \rightarrow \sigma^{\otimes m}$ is possible probabilistically, then the symplectic rank of $\rho^{\otimes n}$ is larger or equal to the symplectic rank of $\sigma^{\otimes m}$,

$$\mathfrak{s}(\rho^{\otimes n}) \geq \mathfrak{s}(\sigma^{\otimes m}). \quad (117)$$

In other words, if $\mathfrak{s}(\rho^{\otimes n}) < \mathfrak{s}(\sigma^{\otimes m})$, then it is *impossible* to convert—even probabilistically—the initial state $\rho^{\otimes n}$ into the target state $\sigma^{\otimes m}$.

Let us give some examples demonstrating how the monotonicity of the symplectic rank can be leveraged to establish strong no-go results for state conversion under Gaussian protocols. First, no probabilistic Gaussian protocol can exactly convert a single-mode non-Gaussian pure state to a tensor product of two non-Gaussian pure states, even when allowing postselection. More generally, the following holds:

Corollary 3 (Impossibility of separately spreading non-Gaussianity via postselected Gaussian operations). Let

$n < m$, and let ρ be a state over n modes and $\phi_1, \dots, \phi_m$ be pure single-mode non-Gaussian states. Then, no postselected Gaussian protocol can exactly map ρ to $\otimes_{i=1}^m \phi_i$.

Proof. This follows directly by applying: (i) the monotonicity of the symplectic rank from Theorem 8; (ii) the fact that the symplectic rank of a tensor product of m single-mode non-Gaussian pure states is exactly m ; and (iii) the fact that the symplectic rank of a state over n modes is less than or equal to n . ■

Remarkably, this impossibility persists even if the n initial states are very distant from the set of Gaussian states and the m target states are extremely close to the set of Gaussian states. These kinds of no-go results cannot be detected by other commonly used quantum non-Gaussianity monotones, such as Wigner logarithmic negativity (WLN) [36,39] or stellar rank [40,41,72]. For instance, given a single-mode state ψ_{in} with an extremely large WLN (or stellar rank) and a non-Gaussian state ϕ_{out} with very small WLN (or stellar rank), Corollary 3 implies that the probabilistic conversion $\psi_{\text{in}} \rightarrow \psi_{\text{out}}^{\otimes 2}$ is impossible. This is remarkable because the non-Gaussianity of the target state $\psi_{\text{out}}^{\otimes 2}$ —as measured by the WLN (or stellar rank)—can be arbitrarily greater than the one of the initial state ψ_{in} . As a concrete example, both WLN and stellar rank do not preclude the conversion of the Fock state $|4\rangle$ (WLN ≈ 0.78 , stellar rank = 4) into the tensor product of two single-photon states $|1\rangle \otimes |1\rangle$ (WLN ≈ 0.71 , stellar rank = 2), since we are moving from a state with higher WLN and stellar rank to one with lower WLN and stellar rank. However, as stated before, this is not true when considering the symplectic rank: $|4\rangle$ has a symplectic rank $\mathfrak{s}(|4\rangle) = 1$ and $|1\rangle \otimes |1\rangle$ has a symplectic rank $\mathfrak{s}(|1\rangle \otimes |1\rangle) = 2$. This presents an opposite picture to that given by the WLN and stellar rank, where now the state with a higher non-Gaussianity is $|1\rangle \otimes |1\rangle$, while the state with a lower non-Gaussianity is $|4\rangle$, establishing that the exact conversion $|4\rangle \rightarrow |1\rangle \otimes |1\rangle$ is impossible (not only deterministically but also probabilistically).

To emphasize the relevance of this result, let us recall that one of the purposes of resource theories aim to determine which noisy states can be converted into a less noisy but more resourceful states. In this context, our result helps identify which states can transform to each other, and if a protocol is to consider when designing a distillation protocol of non-Gaussianity.

2. The resource theory of non-Gaussianity is irreversible under exact postselected Gaussian operations

In this subsection, we show that the resource theory of non-Gaussianity is *irreversible* under exact postselected Gaussian operations. Investigating the reversibility of a resource theory is a fundamental question since the early days of quantum information science [63,94,95].

For example, the pioneering paper [62] showed that the resource theory of entanglement is reversible for pure states, whereas for mixed states, this is no longer the case [63].

Let us start by giving some preliminaries following the presentation of Ref. [94]. We consider the conversion of n copies of ρ into m copies of σ . To analyze this problem, it is useful to define the *exact asymptotic distillable rate* and the *exact cost rate*.

Definition 8 (Exact distillable rate and Exact cost rate). Let us consider a resource theory and let $\mathcal{F}$ be the set of free operations [94]. Let ρ and σ be two quantum states. The exact distillable rate of ρ into σ , denoted as $\text{Distill}_{\mathcal{F}}(\rho \rightarrow \sigma)$, is defined as

$$\text{Distill}_{\mathcal{F}}(\rho \rightarrow \sigma) := \sup_{m,n \in \mathbb{N}} \left\{ \frac{m}{n} : \exists \Lambda \in \mathcal{F} : \Lambda(\rho^{\otimes n}) = \sigma^{\otimes m} \right\}. \quad (118)$$

Analogously, the exact cost rate of ρ into σ , denoted as $\text{Cost}_{\mathcal{F}}(\rho \rightarrow \sigma)$, is defined as

$$\text{Cost}_{\mathcal{F}}(\rho \rightarrow \sigma) = \inf_{m,n \in \mathbb{N}} \left\{ \frac{n}{m} : \exists \Lambda \in \mathcal{F} : \Lambda(\rho^{\otimes n}) = \sigma^{\otimes m} \right\}. \quad (119)$$

Note that the two definitions are not independent of each other, and it is possible to show that [94]

$$\text{Distill}_{\mathcal{F}}(\rho \rightarrow \sigma) = \frac{1}{\text{Cost}_{\mathcal{F}}(\rho \rightarrow \sigma)}, \quad (120)$$

for all states ρ and σ . We can now define the notion of *reversibility* and *irreversibility* of a resource theory under *exact* free operations.

Definition 9 (Reversibility and irreversibility of a resource theory under exact free operations). A resource theory is said to be reversible if for all states ρ and σ it holds that

$$\text{Distill}_{\mathcal{F}}(\rho \rightarrow \sigma) = \text{Cost}_{\mathcal{F}}(\sigma \rightarrow \rho), \quad (121)$$

where $\mathcal{F}$ denotes the set of free operations. A resource theory is said to be irreversible if it is not reversible.

We stress that here we are considering the *exact setting*, where free operations map $\rho^{\otimes n}$ into *exactly* $\sigma^{\otimes m}$, which differs from the *approximate setting*, where free operations map $\rho^{\otimes n}$ into *approximately* $\sigma^{\otimes m}$ with an error that vanishes in the asymptotic limit of many copies [94].

Let us give some examples of reversibility and irreversibility regarding the celebrated entanglement theory [94]: (i) the resource theory of entanglement is irreversible

under exact local operations and classical communication (LOCCs) [94]; (ii) the resource theory of entanglement is reversible under exact postselected LOCCs for pure states, where both the exact distillable entanglement and the exact entanglement cost are given by the logarithm of the Schmidt rank [94]; (iii) the resource theory of entanglement is reversible under (approximate) postselected LOCCs for pure states [62], where both the distillable entanglement and exact entanglement cost are given by the celebrated entanglement entropy [62]; (iv) the resource theory of entanglement is irreversible under (approximate) postselected LOCCs for mixed states [63].

Hereafter, we consider the set of free operations $\mathcal{F}$ to be the set $\mathcal{G}$ of all postselected Gaussian operations, as introduced in Definition 7. In the following theorem, we prove that the resource theory of non-Gaussianity is irreversible under such a set of operations $\mathcal{G}$ in the exact setting scenario.

Theorem 9 (Irreversibility of the resource theory of non-Gaussianity). The resource theory of non-Gaussianity is irreversible under postselected Gaussian protocols in the exact setting. That is, there exist two states ρ and σ such that the exact distillable rate and exact cost rate satisfy

$$\text{Distill}_{\mathcal{G}}(\rho \rightarrow \sigma) \neq \text{Cost}_{\mathcal{G}}(\sigma \rightarrow \rho). \quad (122)$$

Proof. Let us fix two states ρ and σ , and two natural numbers $n, m \in \mathbb{N}$. Let us assume that there exists a postselected Gaussian operation $\Lambda \in \mathcal{G}$ such that $\Lambda(\rho^{\otimes n}) = \sigma^{\otimes m}$. Then, for any postselected non-Gaussianity monotone $M(\cdot)$ (i.e., a real functional over the set of states which is nonincreasing under postselected Gaussian operations), it must hold that

$$M(\rho^{\otimes n}) \geq M(\sigma^{\otimes m}). \quad (123)$$

Additionally, if the non-Gaussianity monotone $M(\cdot)$ also satisfies that

$$\begin{aligned} M(\rho^{\otimes n}) &= nM(\rho), \\ M(\sigma^{\otimes m}) &= mM(\sigma), \end{aligned} \quad (124)$$

then it follows that

$$\begin{aligned} \frac{m}{n} &\leq \frac{M(\rho)}{M(\sigma)}, \\ \frac{n}{m} &\geq \frac{M(\sigma)}{M(\rho)}. \end{aligned} \quad (125)$$

In particular, these two relations imply that

$$\begin{aligned} \text{Distill}_{\mathcal{G}}(\rho \rightarrow \sigma) &\leq \frac{M(\rho)}{M(\sigma)}, \\ \text{Cost}_{\mathcal{G}}(\sigma \rightarrow \rho) &\geq \frac{M(\rho)}{M(\sigma)}. \end{aligned} \quad (126)$$

respectively. In particular, by optimizing over the set of non-Gaussianity monotones $M(\cdot)$ which satisfy the additivity property in Eq. (124), we obtain that

$$\begin{aligned} \text{Distill}_{\mathcal{G}}(\rho \rightarrow \sigma) &\leq \inf_M \frac{M(\rho)}{M(\sigma)}, \\ \text{Cost}_{\mathcal{G}}(\sigma \rightarrow \rho) &\geq \sup_M \frac{M(\rho)}{M(\sigma)}. \end{aligned} \quad (127)$$

Now, let us consider the states $\rho = |2\rangle$ and $\sigma = |1\rangle^{\otimes 2}$, and let us consider the symplectic rank $\mathfrak{s}(\cdot)$ and the *stellar rank* [72], denoted as $r^*(\cdot)$. The latter two quantities are nonincreasing under postselected Gaussian operations, and they are also additive under tensor product for pure states [41]. Note that the monotonicity of the stellar rank under postselected Gaussian measurements was not explicitly noted previously, but it is a direct consequence of the proof of Corollary 4 of Ref. [41]. Consequently, Eq. (127) implies that

$$\begin{aligned} \text{Distill}_{\mathcal{G}}(|2\rangle \rightarrow |1\rangle^{\otimes 2}) &\leq \frac{\mathfrak{s}(|2\rangle)}{\mathfrak{s}(|1\rangle^{\otimes 2})}, \\ \text{Cost}_{\mathcal{G}}(|1\rangle^{\otimes 2} \rightarrow |2\rangle) &\geq \frac{r^*(|2\rangle)}{r^*(|1\rangle^{\otimes 2})}. \end{aligned} \quad (128)$$

The choice of such states has been made because they have simple values for the symplectic rank and the stellar rank, specifically it holds that

$$\begin{aligned} \mathfrak{s}(|2\rangle) &= 1, \\ r^*(|2\rangle) &= 2, \\ \mathfrak{s}(|1\rangle^{\otimes 2}) &= 2, \\ r^*(|1\rangle^{\otimes 2}) &= 2. \end{aligned} \quad (129)$$

As a consequence, we obtain that

$$\begin{aligned} \text{Distill}_{\mathcal{G}}(|2\rangle \rightarrow |1\rangle^{\otimes 2}) &\leq \frac{\mathfrak{s}(|2\rangle)}{\mathfrak{s}(|1\rangle^{\otimes 2})} = \frac{1}{2}, \\ \text{Cost}_{\mathcal{G}}(|1\rangle^{\otimes 2} \rightarrow |2\rangle) &\geq \frac{r^*(|2\rangle)}{r^*(|1\rangle^{\otimes 2})} = 1, \end{aligned} \quad (130)$$

which proves that

$$\text{Distill}_{\mathcal{G}}(|2\rangle \rightarrow |1\rangle^{\otimes 2}) < \text{Cost}_{\mathcal{G}}(|1\rangle^{\otimes 2} \rightarrow |2\rangle). \quad (131)$$

This concludes the proof. $\blacksquare$

In the above theorem we have shown the irreversibility of the resource theory of non-Gaussianity by proving that there are two states ρ and σ such that the ratio between the

exact distillable rate and the exact cost rate can be smaller than $1/2$, i.e.,

$$\frac{\text{Distill}_G(\rho \rightarrow \sigma)}{\text{Cost}_G(\sigma \rightarrow \rho)} \leq \frac{1}{2}. \quad (132)$$

A natural question arises: can this ratio be arbitrarily small? I.e., is the irreversibility of the resource theory of non-Gaussianity arbitrarily strong? The following proposition shows that this is indeed the case.

Theorem 10 (The irreversibility of the resource theory of non-Gaussianity is arbitrarily strong). For all $n \in \mathbb{N}$, it holds that

$$\frac{\text{Distill}_G(|n\rangle \rightarrow |1\rangle^{\otimes n})}{\text{Cost}_G(|1\rangle^{\otimes n} \rightarrow |n\rangle)} \leq \frac{1}{n}. \quad (133)$$

In particular, it holds that

$$\lim_{n \rightarrow \infty} \frac{\text{Distill}_G(|n\rangle \rightarrow |1\rangle^{\otimes n})}{\text{Cost}_G(|1\rangle^{\otimes n} \rightarrow |n\rangle)} = 0. \quad (134)$$

Proof. Analogously to the above proof, we have

$$\text{Distill}_G(|n\rangle \rightarrow |1\rangle^{\otimes n}) \leq \frac{\mathfrak{s}(|n\rangle)}{\mathfrak{s}(|1\rangle^{\otimes n})} = \frac{1}{n}. \quad (135)$$

Moreover, it also holds that

$$\text{Cost}_G(|1\rangle^{\otimes n} \rightarrow |n\rangle) \geq \frac{r^*(|n\rangle)}{r^*(|1\rangle^{\otimes n})} = 1, \quad (136)$$

where we exploited that $r^*(|1\rangle^{\otimes n}) = n$ and $r^*(|n\rangle) = n$ [72]. This concludes the proof. ■

VI. THE SYMPLECTIC RANK CANNOT DECREASE UNDER LITTLE PERTURBATION OF THE STATE

In this section, we demonstrate that the symplectic rank of a pure state cannot decrease if one applies a slight perturbation to the state. In other words, the degree of non-Gaussianity, as quantified by the symplectic rank, remains robust under little perturbations: if a state exhibits a certain level of non-Gaussianity, then all states sufficiently close to it (while satisfying reasonable energy constraints) must retain at least the same level of non-Gaussianity. This result indicates that the symplectic rank is a meaningful measure of non-Gaussianity also from the practical point of view, as it is robust under little perturbations induced by the noise, which is intrinsic to any quantum device. This result is formalized in the forthcoming Theorem 11. Before presenting the theorem, let us introduce some useful notation.

First, in Theorem 11 we assume that the *second moment of the energy* of the state is finite—an assumption that is satisfied in any practical scenario. Let us recall that the *energy observable* is defined as $\hat{E} := \frac{1}{2} \hat{\mathbf{R}}^\top \hat{\mathbf{R}}$, where $\hat{\mathbf{R}}$ denotes the quadrature operator vector [60]. The *mean energy* of a state ρ is given by $\text{Tr}[\hat{E}\rho]$, while the *second moment of the energy* is given by $\text{Tr}[\hat{E}^2\rho]$. In the following, a state ρ is said to be *energy-constrained* if it satisfies $\text{Tr}[\hat{E}^2\rho] \leq E^2$, where E is a finite constant.

Second, let us recall the notions of trace distance and fidelity. The *trace distance* [50] is the most operationally meaningful notion of distance between quantum states, given its operational interpretation due to the Holevo-Helstrom theorem [96,97]. Mathematically, the trace distance between two states ρ_1 and ρ_2 is defined as $\frac{1}{2} \|\rho_1 - \rho_2\|_1$, where $\|\Theta\|_1 := \text{Tr}\sqrt{\Theta^\dagger\Theta}$ denotes the *trace norm* of an operator Θ . Another important metric for comparing quantum states is the *fidelity*, defined as $F(\rho_1, \rho_2) := \|\sqrt{\rho_1}\sqrt{\rho_2}\|_1^2$. The trace distance and the fidelity are related via the so-called Fuchs–van de Graaf inequalities [98] as follows:

$$1 - \sqrt{F(\rho_1, \rho_2)} \leq \frac{1}{2} \|\rho_1 - \rho_2\|_1 \leq \sqrt{1 - F(\rho_1, \rho_2)}. \quad (137)$$

The *infidelity* between two quantum states ρ_1 and ρ_2 is defined as $1 - F(\rho_1, \rho_2)$. The infidelity and the trace distance are always in $[0, 1]$, being equal to zero if and only if $\rho_1 = \rho_2$, and being equal to one if and only if ρ_1 and ρ_2 are orthogonal [50,99]. The Fuchs–van de Graaf inequalities in Eq. (137) imply that two states are very close with respect to the trace distance if and only if they are very close with respect to the infidelity. Moreover, the inequality on the left-hand side can be refined as $1 - F(\rho_1, \rho_2) \leq \frac{1}{2} \|\rho_1 - \rho_2\|_1$ when one of the two states is pure.

We are now ready to present Theorem 11, which roughly ensures that every energy-constrained pure state of symplectic rank s is surrounded by a neighborhood of energy-constrained states with a symplectic rank that is at least as large as s . See Fig. 3 for a pictorial representation of this theorem.

Theorem 11 (Slight perturbations cannot decrease the symplectic rank). Let ψ be a pure state with second moment of the energy bounded by a finite constant E , that is $\text{Tr}[\hat{E}^2\psi] \leq E^2$. Then, there exists $\varepsilon > 0$ for which any state σ such that

- (i) σ is ε -close to ψ , with respect to the trace distance (and thus also with respect to the infidelity),
- (ii) σ satisfies the energy constraint $\text{Tr}[\hat{E}^2\sigma] \leq E^2$,

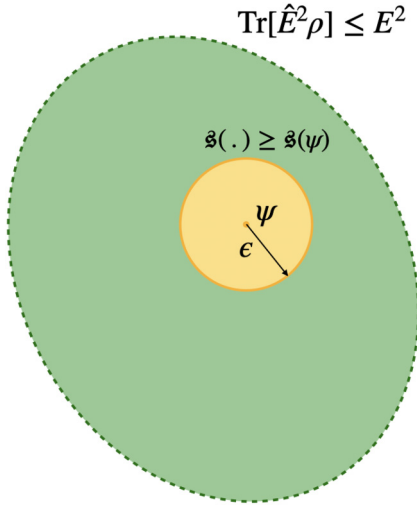


FIG. 3. This figure provides a pictorial representation of Theorem 11: any state ψ , which satisfies the energy constraint $\text{Tr}[\hat{E}^2\psi] \leq E^2$, admits an ε -ball (with respect to the trace distance) around it such that any energy-constrained state inside the ball has symplectic rank at least as large as the one of ψ . Moreover, Theorem 12 shows that one can take $\varepsilon := \frac{1}{4}((\nu - 1)/800E^2)^4$, where ν is the smallest symplectic eigenvalue of $V(\psi)$ which is strictly larger than one.

must have symplectic rank larger or equal to the one of ψ , i.e.,

$$s(\sigma) \geq s(\psi). \quad (138)$$

In other words, any pure state of symplectic rank s admits a ball around it such that all energy-constrained states inside the ball have symplectic rank $\geq s$.

Proof. First, let us observe that the concavity of the square root function implies that, for any state, the mean energy is upper bounded by the square root of the second moment of the energy. Consequently, since we are considering states with finite second moment of the energy, we deduce that also the mean energy is finite. Hence, Lemma 3 implies that we are considering states with well-defined covariance matrices.

Since the statement is trivial if $s(\psi) = 0$, let us assume that $s(\psi) \geq 1$. Let $\mathcal{S}$ be the set of energy-constrained states with symplectic rank smaller than the one of ψ . That is, $\mathcal{S}$ comprises all the states σ such that $s(\sigma) < s(\psi)$ and $\text{Tr}[\hat{E}^2\sigma] \leq E^2$. Thanks to the Fuchs–van de Graaf inequalities in Eq. (137), the trace distance is always lower bounded by the infidelity when one of the two states is pure. Consequently, in order to prove the theorem it suffices to show that the minimum infidelity between ψ and any state $\sigma \in \mathcal{S}$ is strictly positive, i.e.,

$$\inf_{\sigma \in \mathcal{S}} [1 - \langle \psi | \sigma | \psi \rangle] > 0. \quad (139)$$

To this end, let us fix $\sigma \in \mathcal{S}$. By definition of symplectic rank for mixed states in Definition 6, there exists an ensemble of pure states $\{p_i, \psi_i\}$ such that: (i) all the p_i 's are strictly positive; (ii) $\sigma = \sum_i p_i \psi_i$; and (iii) $\max_i s(\psi_i) = s(\sigma) < s(\psi)$. Moreover, the infidelity between ψ and σ can be lower bounded as

$$\begin{aligned} 1 - F(\psi, \sigma) &= 1 - \langle \psi | \sigma | \psi \rangle \\ &= \sum_i p_i (1 - |\langle \psi | \psi_i \rangle|^2) \\ &\stackrel{(i)}{=} \frac{1}{2} \sum_i p_i \|\psi - \psi_i\|_1^2 \\ &\stackrel{(ii)}{\geq} \frac{(\nu - 1)^4}{(640)^4} \sum_i \frac{p_i}{\max(E, E_i)^8} \\ &\stackrel{(iii)}{\geq} \frac{(\nu - 1)^4}{(640)^4 E^8} \left(\frac{4}{5}\right)^4 \sum_{\substack{i: \\ E_i \leq \sqrt{\frac{5}{4}}E}} p_i \\ &\stackrel{(iv)}{\geq} \frac{(\nu - 1)^4}{(640)^4 E^8} \left(\frac{4}{5}\right)^4 \frac{1}{4} \\ &= \frac{1}{4} \left(\frac{\nu - 1}{800 E^2}\right)^4. \end{aligned} \quad (140)$$

Here, in (i), we exploited the well-known formula for the trace distance between two pure states [50]. In (ii), we used the lower bound on the trace distance between two states in terms of the distance between their symplectic eigenvalues proven in Theorem 24 in Sec. IX below. Specifically, the latter theorem implies that

$$\|\psi - \psi_i\|_1 \geq \frac{(\nu - 1)^2}{(640)^2 \max(E, E_i)^4}, \quad (141)$$

where, first, we denoted as ν the $(s(\psi) - 1)$ th largest symplectic eigenvalue of $V(\psi)$, which satisfies $\nu > 1$ by definition of symplectic rank for pure states; and, second, we used that the $(s(\psi) - 1)$ th largest symplectic eigenvalue of $V(\psi_i)$ is one, as a consequence of the fact that the symplectic rank of ψ_i is strictly smaller than $s(\psi)$. In (iii), we denoted as $E_i := \sqrt{\text{Tr}[\psi_i \hat{E}^2]}$ the second moment of the energy of state ψ_i , which implies that $\sum_i p_i E_i^2 \leq E^2$. In (iv), we used that

$$\sum_{\substack{i: \\ E_i \leq \sqrt{\frac{5}{4}}E}} p_i \geq \frac{1}{4}, \quad (142)$$

which can be proven as follows. By exploiting the fact that $E_i \geq 0$, it holds that

$$E^2 \geq \sum_i p_i E_i^2 \geq \sum_{\substack{i: \\ E_i > \sqrt{\frac{5}{4}}E}} p_i E_i^2 \geq \frac{5}{4} E^2 \times \sum_{\substack{i: \\ E_i > \sqrt{\frac{5}{4}}E}} p_i = \frac{5}{4} E^2 \left(1 - \sum_{\substack{i: \\ E_i \leq \sqrt{\frac{5}{4}}E}} p_i \right), \quad (143)$$

which implies the validity of Eq. (142). In conclusion, Eq. (140) implies that

$$\inf_{\sigma \in \mathcal{S}} [1 - F(\psi, \sigma)] \geq \frac{1}{4} \left(\frac{\nu - 1}{800E^2} \right)^4, \quad (144)$$

which is strictly positive. This concludes the proof. $\blacksquare$

The above proof tells us more than what reported in the statement of Theorem 11. Indeed, the above Eq. (144) implies that the trace distance between ψ and any energy-constrained state σ with symplectic rank smaller than the one of ψ must satisfy

$$\frac{1}{2} \|\psi - \sigma\|_1 \geq \frac{1}{4} \left(\frac{\nu - 1}{800E^2} \right)^4, \quad (145)$$

where ν is the smallest symplectic eigenvalue of $V(\psi)$ that is strictly larger than one, and E is an upper bound on the second moment of the energy of ψ and σ . This result establishes a simple expression for the radius ε of the ball mentioned in the statement of Theorem 11. This is reported in the forthcoming Theorem 12, which constitutes a refinement of Theorem 11.

Theorem 12 (Expression for the radius of the ball of states with high symplectic rank). Let ψ be a non-Gaussian pure state such that its second moment of the energy is bounded by a finite constant E , that is, $\text{Tr}[\hat{E}^2 \psi] \leq E^2$. Let ν be the smallest symplectic eigenvalue of $V(\psi)$ that is strictly larger than one. Then, any state σ such that

- (i) σ is $\frac{1}{4} \left((\nu - 1)/800E^2 \right)^4$ -close to ψ with respect to the trace distance (and, in particular, with respect to the infidelity):

$$0 \leq \frac{1}{2} \|\sigma - \psi\|_1 \leq \frac{1}{4} \left(\frac{\nu - 1}{800E^2} \right)^4; \quad (146)$$

- (ii) σ satisfies the energy constraint $\text{Tr}[\hat{E}^2 \sigma] \leq E^2$;

must have symplectic rank larger or equal to the one of ψ , i.e.,

$$\mathfrak{s}(\sigma) \geq \mathfrak{s}(\psi). \quad (147)$$

In other words, any pure state of symplectic rank s is such that all the energy-constrained states inside the ball of radius $\frac{1}{4} \left((\nu - 1)/800E^2 \right)^4$ (with respect the trace distance) around it have symplectic rank $\geq s$.

VII. SYMPLECTIC RANK AS A MEASURE OF COMPLEXITY

In this section, we prove that the symplectic rank governs the *complexity* of two fundamental tasks in bosonic quantum information processing: *quantum state tomography* and *classical simulation of quantum computations*.

A. Quantum state tomography

Quantum state tomography is a fundamental task in quantum physics, aiming to obtain a classical description of an unknown quantum state from experimental data [100]. Let us briefly give some preliminaries about quantum state tomography; for more details, we refer to Ref. [34].

We start by introducing the problem of quantum state tomography. Let us consider an unknown quantum state ρ promised to be in a certain known class of states denoted as $\mathcal{S}$. Given $\varepsilon, \delta \in (0, 1)$, a tomography algorithm takes as input N copies of the unknown quantum state $\rho \in \mathcal{S}$ and outputs a classical description of a quantum state $\bar{\rho}$ which is ε -close in trace distance to ρ , with probability $\geq 1 - \delta$:

$$\Pr \left[\frac{1}{2} \|\rho - \bar{\rho}\|_1 \leq \varepsilon \right] \geq 1 - \delta, \quad (148)$$

where $\frac{1}{2} \|\rho - \bar{\rho}\|_1$ denotes the trace distance [50] between the true unknown state ρ and the estimate $\bar{\rho}$. In quantum learning theory [100], the trace distance arises as the most common notion of distance to measure errors in estimating states, as it has a strong operational meaning thanks to the Holevo-Helstrom theorem [96,97].

In order to measure the performances of quantum state tomography, one usually considers the *sample complexity*, defined as the minimum number of copies N that allows one to solve the problem of quantum state tomography. Specifically, the sample complexity is a function of the set of states $\mathcal{S}$, of the trace distance error ε , and of the failure probability δ . Moreover, another meaningful notion of complexity is the *time complexity*, defined as the minimum time required to execute the tomography algorithm, accounting for both classical and quantum steps. It is worth mentioning that the time complexity always upper bounds the sample complexity [100]. Moreover, tomography is said to be *efficient* if its sample and time complexities scale

polynomially with the number of modes n . Otherwise, tomography is said to be *inefficient*.

In Ref. [34], the problem of quantum state tomography of continuous-variable states has been extensively studied. In particular, Ref. [34] proves upper [Theorem S74 of Ref. [34]] and lower [Theorem S80 of Ref. [34]] bounds on the sample and time complexities for quantum state tomography of t -compressible states, showing together with Theorem 7 that both complexities grow exponentially with the symplectic rank. This result establishes that the symplectic rank is a fundamental measure that quantifies the hardness in the task of tomography: the higher the symplectic rank, the more challenging it becomes to perform tomography. In the following theorem, we slightly improve on the upper bound on the sample and time complexities proven in Theorem S74 of Ref. [34].

Theorem 13 (Tomography of states with a given symplectic rank). Let $\mathcal{S}$ be the set of n -mode pure states ψ which have symplectic rank at most s and satisfy the energy constraint $\text{Tr}[\hat{N}^2 \psi] \leq E^2$. Then, the sample complexity N of tomography with trace distance error ε and failure probability δ from the set $\mathcal{S}$ satisfies the following bounds:

$$\tilde{\Omega}\left(\left(\frac{E}{12s\varepsilon}\right)^s\right) \leq N \leq \tilde{O}\left(\left(\frac{22E}{s\varepsilon^2}\right)^s\right), \quad (149)$$

where tilde abbreviates poly and log factors in all parameters. Additionally, the time complexity exhibits the same behavior, thus establishing that both complexities scale exponentially with the symplectic rank s .

Proof. By Theorem 7, a pure state with a given symplectic rank s corresponds to a s -compressible state, so that one can leverage the results of Ref. [34] to obtain such bounds. The lower bound is fully proven in Theorems S80 of Ref. [34] and reads

$$N \geq \frac{1}{s g\left(\frac{1}{s}E\right)} \left[2(1-\delta) \left(\frac{E}{12s\varepsilon} - \frac{1}{s} \right)^s - (1-\delta) \log_2(32\pi) - H_2(\delta) \right], \quad (150)$$

where ε denotes the trace distance error, δ the failure probability, $g(x) := (x+1)\log_2(x+1) - x\log_2 x$ is the bosonic entropy, and $H_2(x) := -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy. Consequently, in asymptotic notation, neglecting all the polynomial and logarithmic terms, we can write

$$N \geq \tilde{\Omega}\left(\left(\frac{E}{12s\varepsilon}\right)^s\right), \quad (151)$$

which concludes the proof of the lower bound on the sample complexity.

We now outline the general idea to obtain the improved upper bound, specifically by slightly improving on the scaling with the energy parameter E compared to the upper bound of Theorem S74 of Ref. [34]. For all technical details, as well as the full proof, we refer to Theorem S74 of Ref. [34]. The proof relies on the fact that, thanks to Theorem 7, the unknown state ψ can be written as

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-s)}), \quad (152)$$

where G is a Gaussian unitary and ϕ is an s -mode state such that $N(\phi) \leq N(\psi)$, where $N(\cdot)$ denotes the mean total photon number. By exploiting that $\text{Tr}[\hat{N}^2 \psi] \leq E^2$, it thus follows that the mean total photon number of ϕ can be upper bounded as

$$N(\phi) \leq E. \quad (153)$$

Hence, in order to learn $|\psi\rangle$, it suffices to suitably learn the Gaussian unitary G and the s -mode state $|\phi\rangle$. The rough idea of such a learning algorithm, as well as of its sample complexity analysis, is as follows (see Theorem S74 of Ref. [34] for details):

- (a) First, one can estimate G by estimating the first moment and covariance matrix of ψ , which requires $\text{poly}(n, E)$ state copies;
- (b) Second, one can approximately undo G on the subsequent state copies, thus obtaining copies of a state $|\tilde{\psi}\rangle$ which is $o(\varepsilon)$ -close in trace distance to $|\phi\rangle \otimes |0\rangle^{\otimes(n-s)}$ with high probability.
- (c) Subsequently, one can perform a projective measurement to project $|\tilde{\psi}\rangle$ on the vacuum state of the last $n-s$ modes with high probability. The postoutcome state on the first s modes, denoted as $|\tilde{\phi}\rangle$, turns out to be $o(\varepsilon)$ -close in trace distance to $|\phi\rangle$ with high probability. Hence, in order to learn $|\phi\rangle$, one can perform full state tomography of the first s modes.
- (d) Finally, one can perform full state tomography of the s -mode state $|\tilde{\phi}\rangle$ using the tomography algorithm for energy-constrained states of Theorem S36 of Ref. [34]. Specifically, Theorem S36 of Ref. [34] guarantees the sample complexity of tomography of s -mode states with mean total photon number of at most N_{phot} is upper bounded by

$$\tilde{O}\left(\left(\frac{eN_{\text{phot}}}{s\varepsilon'^2}\right)^s\right), \quad (154)$$

where ε' is the trace distance error. Consequently, in order to exploit Theorem S36 of Ref. [34], we need to establish an upper bound on the mean total photon number of $|\tilde{\phi}\rangle$. In Theorem S74 of Ref. [34]

the energy of $|\tilde{\phi}\rangle$ was overestimated and upper bounded by $80E^2$ (see Eq. (397) of Ref. [34]). Here, by exploiting a completely analogous reasoning as Theorem S74 of Ref. [34] together with (153), we can easily deduce that the mean total photon number of $\tilde{\phi}$ can be upper bounded as

$$N(\tilde{\phi}) \leq 2E. \quad (155)$$

The underlying idea of (155) is that, since (i) $\tilde{\phi}$ is an approximation of ϕ , and (ii) ϕ has a mean total photon number bounded by E , the mean total photon number of $\tilde{\phi}$ can also be upper bounded by a slightly larger value than E —for instance, $2E$. Hence, by using (154) with $N_{\text{phot}} := 2E$ and $\varepsilon' := \varepsilon/2$ (this choice of ε' is due to the details of the sample complexity analysis employed in Theorem S74 of Ref. [34]), we conclude that this final step of the tomography algorithm requires a number of state copies given by

$$\tilde{\mathcal{O}}\left(\left(\frac{8eE}{s\varepsilon^2}\right)^s\right). \quad (156)$$

By using that $8e \leq 22$, this concludes the proof. $\blacksquare$

The main consequence of this result is that the sample complexity scales exponentially with the symplectic rank. This highlights the significance of the symplectic rank. Specifically, as long as the symplectic rank satisfies $s = \mathcal{O}(1)$, tomography requires only a polynomial number of copies in the number of modes. Otherwise, the number of required copies becomes super-polynomial, rendering tomography inefficient. In summary, the symplectic rank serves not only as a measure of non-Gaussianity but also as a measure of the hardness of tomography.

B. Classical simulation

Classical simulation is a fundamental tool for physicists, helping to understand and benchmark quantum systems. However, quantum mechanics is intrinsically challenging. A key difficulty arises from the fact that, for finite-dimensional systems of n qudits, one must store and manipulate d^n complex numbers. This exponential scaling imposes fundamental limitations on classical simulation. The challenge becomes even more severe when dealing with continuous-variable systems, where each mode has an infinite-dimensional Hilbert space. As a result, simulations require introducing a cut-off to make the problem tractable. In this setup, it is then fundamental to understand what can be simulated, in which way, and how many resources are

required to do so. At this end, let us first briefly introduce for the sake of completeness the notion of strong and weak simulation [101,102].

Let us first note that a quantum system and a measurement device are associated with a probability distribution, from which classical outcomes are sampled according to the Born rule. We refer to *Strong simulation* when there exists a classical algorithm capable of evaluating the output probability distribution, or any of its marginals, in polynomial time relative to the size of the quantum system. Similarly, *Weak simulation* refers to the existence of a classical algorithm that can generate samples from the output probability distribution in polynomial time relative to the size of the quantum system. These are two fundamental notions of classical simulability. It is important to note that the ability to perform strong simulation implies the ability to perform weak simulation. Additionally, strong simulation is generally a stricter requirement than what is needed for quantum computation, which is essentially equivalent to our ability to perform weak simulation. These two methods are the most relevant ways of performing classical simulation. However, for certain systems, even these approaches can be challenging. In such cases, one can look for simpler objects to estimate classically, such as performing a weaker form of simulation by estimating the overlap between two states.

With these notions in mind, we prove that the overlaps between general states with low symplectic rank and Gaussian states can be estimated efficiently on a classical computer. Then, we refine this result to obtain efficient classical simulation algorithms for specific classes of non-Gaussian circuits.

1. General-purpose algorithm

Let ψ be an n -mode pure state with symplectic rank s and mean photon number less than E . Recall that by Theorem 7, there exists a pure state ϕ over s modes with mean photon number less than E such that

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-s)}). \quad (157)$$

We now state our general classical simulation result:

Theorem 14 (Classical simulation based on the symplectic rank). Let ψ be an n -mode pure state with symplectic rank s and mean photon number less than E . There is a classical randomized algorithm which takes as input parameters $\varepsilon, \delta > 0$, the description of a Gaussian state $|G\rangle$, and the decomposition (G, ϕ) of ψ from Eq. (157) where ϕ is given in the Fock basis, and outputs an estimate of the overlap $|\langle G|\psi\rangle|^2$ up to additive precision ε with probability $1 - \delta$, in time

$$\mathcal{O}\left(n^3 + \frac{n^2 s}{\varepsilon^2} \left(\frac{6E}{s\varepsilon^2}\right)^{3s} \log\left(\frac{E}{s\varepsilon^2\delta}\right)\right). \quad (158)$$

Proof. When $s = 0$, the overlap is Gaussian and the result is trivial, so we assume $s \geq 1$ in what follows. Let $\lambda, \mu > 0$ be free parameters, to be fixed later on. By denoting as Π_N the projector on the span of the Fock states with total photon number smaller or equal to $N := \lceil E/\lambda^2 \rceil$ (we omit the ceiling function in what follows for brevity) and by denoting as $|\phi_N\rangle \propto \Pi_N |\phi\rangle$ the state obtained by truncating ϕ in Fock basis at photon number N , it holds that

$$\frac{1}{2} \|\phi - \phi_N\|_1 \stackrel{(i)}{\leq} \sqrt{\text{Tr}[(\mathbb{I} - \Pi_N)\phi]} \stackrel{(ii)}{\leq} \sqrt{\frac{\text{Tr}[\phi \hat{N}]}{N}} \leq \lambda, \quad (159)$$

where (i) follows from the gentle measurement lemma [99], in (ii) we simply used the operator inequality $(\mathbb{I} - \Pi_N) \leq \hat{N}/N$, and in (iii) we used that the mean photon number of the compressed state ϕ is bounded by E , as per (157). Consequently, we deduce that the compressed state ϕ is λ -close in trace distance to the truncated state ϕ_N . Moreover, by defining

$$|\psi_N\rangle := G(|\phi_N\rangle \otimes |0\rangle^{\otimes(n-s)}), \quad (160)$$

and by exploiting that the trace distance is invariant when tensoring with identical states and under unitary operations, we obtain that

$$\frac{1}{2} \|\psi\rangle\langle\psi| - |\psi_N\rangle\langle\psi_N|\|_1 \leq \lambda. \quad (161)$$

By the variational property of the trace distance [50], considering the binary measurement $\{|G\rangle\langle G|, I - |G\rangle\langle G|\}$, we obtain that

$$|\langle G|\psi\rangle\langle\psi|G\rangle - \langle G|\psi_N\rangle\langle\psi_N|G\rangle|^2 \leq \lambda. \quad (162)$$

Now we have

$$\begin{aligned} \langle G|\psi_N|G|\psi_N\rangle &= \langle G|G(|\phi_N\rangle \otimes |0\rangle^{\otimes(n-s)}) \\ &= \sum_{n_1+\dots+n_s \leq N} \phi_{n_1\dots n_s} \langle G'|n_1\dots n_s 0\dots 0| \\ &\quad \times G'|n_1\dots n_s 0\dots 0\rangle, \end{aligned} \quad (163)$$

where we have expanded $|\phi_N\rangle$ in Fock basis and where we have defined $|G'\rangle := G^\dagger |G\rangle$, which is a Gaussian state whose Williamson decomposition (see Lemma 5)

$$|G'\rangle = U \bigotimes_{k=1}^n (D(\alpha_i)S(\xi_i)) |0\rangle^{\otimes n} \quad (164)$$

can be computed in time $\mathcal{O}(n^3)$ from the descriptions of $|G\rangle$ and G . From Lemma 5, we recall that U represents a passive Gaussian unitary, $D(\alpha_i)$ is the displacement operator acting on the i th mode, and $S(\xi_i)$ denotes the single-mode squeezing operator on the same mode. In particular, for all $(n_1, \dots, n_s) \in \mathbb{N}^s$,

$$\begin{aligned} &\langle G'|n_1\dots n_s 0\dots 0|G'|n_1\dots n_s 0\dots 0\rangle \\ &= \left(\bigotimes_{k=1}^n \langle G_i| \right) U |n_1\dots n_s 0\dots 0\rangle, \end{aligned} \quad (165)$$

where $|G_i\rangle := D(\alpha_i)S(\xi_i)|0\rangle$ are single-mode Gaussian states and U is a passive Gaussian unitary. By Theorem 2 of Ref. [103], there is a classical randomized algorithm which estimates this quantity to additive precision ε' with probability $1 - \delta'$ in time $\mathcal{O}(n^2/\varepsilon'^2 \log(1/\delta'))$. We denote by $A_{n_1\dots n_s}$ the responding classical estimate produced by this algorithm, and we define

$$\tilde{G} := \sum_{n_1+\dots+n_s \leq N} \phi_{n_1\dots n_s} A_{n_1\dots n_s}. \quad (166)$$

Let also $M := \binom{N+s}{s}$ be the number of terms in the sum. Setting $\varepsilon' = \mu/2M$ and $\delta' = \delta/M$ we obtain

$$\begin{aligned} |\langle G|\psi_N|G|\psi_N\rangle - \tilde{G}| &= \left| \sum_{n_1+\dots+n_s \leq N} \phi_{n_1\dots n_s} (\langle G'|n_1\dots n_s 0\dots 0|G'|n_1\dots n_s 0\dots 0\rangle - A_{n_1\dots n_s}) \right| \\ &\leq \sum_{n_1+\dots+n_s \leq N} |\phi_{n_1\dots n_s}| |\langle G'|n_1\dots n_s 0\dots 0|G'|n_1\dots n_s 0\dots 0\rangle - A_{n_1\dots n_s}| \\ &\leq M\varepsilon' \\ &= \frac{1}{2}\mu, \end{aligned} \quad (167)$$

with probability $1 - \delta$, using the union bound of failure probabilities. The classical randomized algorithm outputting $\tilde{G}$ computes $|G'\rangle$ in time $\mathcal{O}(n^3)$ and then executes M times the algorithm from Theorem 2 of Ref. [103], for $\varepsilon' = \mu/2M$ and $\delta' = \delta/M$, so its running time is given by

$$\mathcal{O}\left(n^3 + \frac{n^2 M^3}{\mu^2} \log\left(\frac{M}{\delta}\right)\right), \quad (168)$$

where $M = \binom{N+s}{s}$ and $N = E/\lambda^2$. Moreover,

$$\begin{aligned} & \left| |\langle G|\psi_N|G|\psi_N\rangle|^2 - |\tilde{G}|^2 \right| \\ & \leq |\langle G|\psi_N|G|\psi_N\rangle| \left| |\langle G|\psi_N|G|\psi_N\rangle|^* - \tilde{G}^* \right| \\ & \quad + |\tilde{G}| \left| |\langle G|\psi_N|G|\psi_N\rangle - \tilde{G} \right| \\ & \leq \mu, \end{aligned} \quad (169)$$

with probability $1 - \delta$, where we used the triangle inequality in the first line, and Eq. (167) together with the facts that $|\langle G|\psi_N|G|\psi_N\rangle| \leq 1$ and $|\tilde{G}| \leq 1$ in the second line (if the latter does not hold, we can always rescale $\tilde{G}$ so that $|\tilde{G}| = 1$ without loss of generality since it can only improve the quality of the estimate). With Eq. (162) and the triangle inequality, this implies

$$\left| |\langle G|\psi|G|\psi\rangle|^2 - |\tilde{G}|^2 \right| \leq \lambda + \mu, \quad (170)$$

with probability $1 - \delta$, and the estimate is obtained in time

$$\mathcal{O}\left(n^3 + \frac{n^2 \left(\frac{E}{\lambda^2} + s\right)^3}{\mu^2} \log\left(\frac{\left(\frac{E}{\lambda^2} + s\right)}{\delta}\right)\right). \quad (171)$$

Using $\binom{m}{k} \leq (em/k)^k$ (valid for all $m \geq k > 0$), as well as $1 + (E/s\lambda^2) \leq 2E/s\lambda^2$ and $\delta \geq \delta^s$, and setting $\lambda = ((v-1)/v)\varepsilon$ and $\mu = (1/v)\varepsilon$ for some constant parameter $v > 1$, this gives a running time bounded as

$$\mathcal{O}\left(n^3 + \frac{n^2}{\varepsilon^2} \left(\frac{E}{s\varepsilon^2} \cdot \frac{2ev^2}{(v-1)^2}\right)^{3s} \log\left(\frac{E}{s\varepsilon^2\delta} \cdot \frac{2ev^2}{(v-1)^2}\right)\right). \quad (172)$$

Finally, choosing $v > 1$ large enough such that $2ev^2/(v-1)^2 \leq 6$ ($v \geq 20$ suffices), we bound the total running time of the classical simulation algorithm as

$$\mathcal{O}\left(n^3 + \frac{n^2 s}{\varepsilon^2} \left(\frac{6E}{s\varepsilon^2}\right)^{3s} \log\left(\frac{E}{s\varepsilon^2\delta}\right)\right), \quad (173)$$

while the total error in Eq. (170) is at most $\lambda + \mu = \varepsilon$, with probability $1 - \delta$. ■

Compared to existing strong simulation algorithms for bosonic computations based on stellar rank from Refs. [46, 71], Theorem 14 provides a classical algorithm for the simpler task of estimating overlaps with Gaussian states. We combine both approaches in Theorem 18 to refine the strong simulation algorithm from Ref. [71] for Gaussian circuits doped with photon additions and subtractions.

Note that the classical simulation algorithm from Theorem 14 takes as input the (approximate) decomposition of the state ψ as in Eq. (157). However, in practice, the state of interest is typically described as the output state of some bosonic circuit. In what follows, we show that a circuit description of the decomposition in Eq. (157) can be efficiently obtained from the circuit description of relevant examples of bosonic circuits (Theorem 16). In particular, we show that this leads to efficient classical simulation algorithms for two specific examples of families of bosonic circuits: in the first case, these are circuits composed of Gaussian gates and cubic phase gates (Theorem 17), while in the second case they are composed of Gaussian gates and creation and annihilation operators, known as photon additions/subtractions in the quantum optics literature (Theorem 18).

2. Doped Gaussian circuits

Before diving into the simulation results for specific bosonic circuits, let us start by defining the general class of (κ, s) -doped Gaussian gates. See Fig. 4 for a pictorial representation.

Definition 10 (Doped Gaussian gate). An operator C is an n -mode (κ, s) -doped Gaussian gate if it is a composition of n -mode Gaussian unitaries $G_0, \dots, G_s$ and at most s (possibly nonunitary and non-Gaussian) κ -local gates $W_1, \dots, W_s$:

$$C = G_s W_s G_{s-1} \cdots G_1 W_1 G_0. \quad (174)$$

Moreover, an operator is said to be κ -local if it is a function of at most κ operators from the set of quadrature operators.

Note that we are using a more general definition of doped Gaussian gates than the one in [34], as we are also allowing for doping gates which are not unitaries (e.g., annihilation or creation operators). Let us proceed with the definition of (κ, s) -doped Gaussian state.

Definition 11 (Doped Gaussian state). An n -mode (κ, s) -doped Gaussian state ψ is a state prepared by applying an n -mode (κ, s) -doped Gaussian gate C to the vacuum state,

$$|\psi\rangle \propto C|0\rangle^{\otimes n}. \quad (175)$$

Note that we are using the proportional symbol in (175) because the doped Gaussian gate C can be a nonunitary

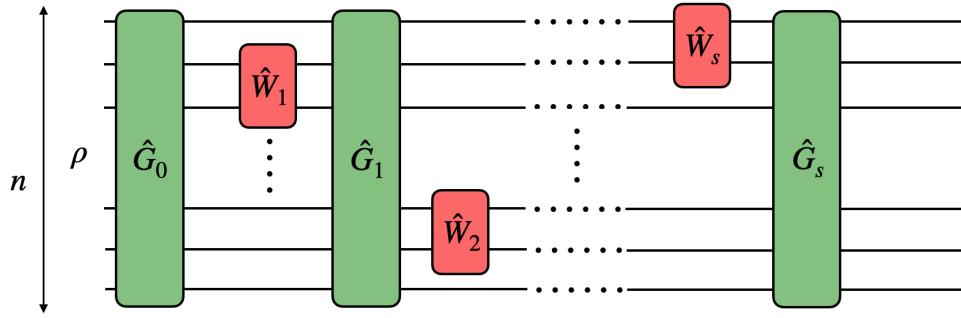


FIG. 4. Pictorial representation of a (κ, s) -doped Gaussian state. By definition, an n -mode (κ, s) -doped Gaussian state is a state prepared by applying Gaussian unitaries $G_0, \dots, G_s$ and at most s non-Gaussian κ -local gates $W_1, \dots, W_s$ to the n -mode vacuum.

operator. Let us state the following remarkable result about the symplectic rank of doped Gaussian states, which is a slight generalization of Theorem S67 of Ref. [34].

Theorem 15 (Symplectic rank of doped Gaussian states). Let $\kappa s \leq n$. Then, any n -mode (κ, s) -doped Gaussian state ψ has symplectic rank less than or equal to κs ,

$$\mathfrak{s}(\psi) \leq \kappa s. \quad (176)$$

Proof. The statement is a slight generalization of Theorem S67 of Ref. [34], which was originally stated for doped Gaussian states generated by doped Gaussian *unitary* gates. However, it is simple to observe that the same proof extends to doped Gaussian states produced via *nonunitary* doped Gaussian gates as well. This is because the proof of Theorem S67 of Ref. [34] relies solely on the fact that each doping gate admits a power series expansion in the quadrature operators. Such a proof is summarized in the proof of Theorem 16 below. ■

Thanks to Theorem 15 and to the definition of symplectic rank, for any n -mode (κ, s) -doped Gaussian state ψ with $\kappa s \leq n$ there exist an n -mode Gaussian unitary G and a (κs) -mode state ϕ such that

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-\kappa s)}). \quad (177)$$

The following theorem proves that, given an efficient circuit description of any such n -mode (κ, s) -doped Gaussian state ψ , one can *efficiently* find the Gaussian unitary G and the (κs) -mode state ϕ present in (177).

Theorem 16 (Efficient non-Gaussian compression). Let $\kappa s \leq n$. Given an efficient circuit description of an n -mode (κ, s) -doped Gaussian state ψ , i.e., with Gaussian unitary gates described by their symplectic matrix and displacement vector, and non-Gaussian gates expressed as efficiently computable functions of quadrature operators,

we can compute in time $\mathcal{O}(s^2 n^3)$ a description of a Gaussian unitary G and an efficient circuit description of a (κs) -mode state ϕ such that

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-\kappa s)}), \quad (178)$$

where ϕ is a (κ, s) -doped Gaussian state.

Proof. The proof of the decomposition in (178) is exactly the same as the one in Theorem S67 of Ref. [34], and so we refer the reader to the latter. Here, we provide a brief summary of the argument, together with an additional estimate of the computational time required to perform such a decomposition.

Let us consider the description of the (κ, s) -doped state ψ as

$$|\psi\rangle \propto C|0\rangle^{\otimes n}, \quad (179)$$

where

$$C = G_s W_s G_{s-1} \cdots G_1 W_1 G_0, \quad (180)$$

with each W_i being a non-Gaussian κ -local gate and each G_i being a Gaussian unitary. Analogously to Theorem S65 of Ref. [34], one can observe that C can be expressed as

$$C = \tilde{G}_s (\Pi_{i=1}^s \tilde{W}_i) G_{\text{passive}}, \quad (181)$$

where G_{passive} is a passive Gaussian unitary, each operator $\tilde{W}_i$ is defined as

$$\tilde{W}_i := G_{\text{passive}}^\dagger \tilde{G}_{i-1}^\dagger W_i \tilde{G}_{i-1} G_{\text{passive}}, \quad (182)$$

and each $\tilde{G}_i$ is defined by $\tilde{G}_i := G_i \cdots G_0$. Crucially, as shown in Theorem S65 of Ref. [34], G_{passive} can be constructed such that the operator $\tilde{W}_i$ acts on only the first κs modes, for all $i \in \{1, \dots, s\}$. The descriptions of each circuit element present in (180) (i.e., the descriptions of G_{passive} , $\Pi_{i=1}^s \tilde{W}_i$, and $\tilde{G}_s$) can be efficiently computed as follows.

Let us start by introducing some notation. For $s' \in \{0, 1, \dots, s\}$, let $S_{s'}$ be the symplectic matrix associated with the Gaussian unitary $\tilde{G}_{s'}$, and let $\mu(s', 1) \leq \mu(s', 2) \leq \dots \leq \mu(s', \kappa)$ be the quadratures involved in the power series that defines $W_{s'}$.

First, let us observe that given the description of $G_0, \dots, G_s$, the description of $S_0, \dots, S_s$ can be computed in time $\mathcal{O}(s^2 n^3)$. Second, by denoting as $(e_i)_{i \in [2n]}$ the canonical basis vector of $\mathbb{R}^{2n}$, one can compute the vectors

$$(v_j)_{j \in [\kappa s]} := (S_{s'-1}^T e_{\mu(s', r)})_{s' \in [s], r \in [\kappa]} \quad (183)$$

in time $\mathcal{O}(\kappa s n^2)$. As shown in Eq. (334) of Ref. [34], constructing a passive Gaussian unitary G_{passive} such that the operators $\tilde{W}_i$ in (181) act only on the first κs modes for all $i \in \{1, \dots, s\}$ is equivalent to finding a $2n \times 2n$ symplectic orthogonal matrix O_{passive} such that for all $j \in [\kappa s]$ and all $m \in \{2\kappa s + 1, \dots, 2n\}$,

$$e_m^T O_{\text{passive}}^T v_j = 0. \quad (184)$$

Following Lemma 28 of Ref. [58], computing such O_{passive} can be done in time $\mathcal{O}(n^3)$, as we explain hereafter. The $2n \times 2n$ symplectic orthogonal matrix O_{passive} is isomorphic to an $n \times n$ unitary U_{passive} through a well-defined vector space mapping. Specifically, for a $2n$ -dimensional vector $v := (v_1, \dots, v_{2n})$, there exists a bijective mapping to an n -dimensional complex vector $f(v) := (v_1 - iv_2, \dots, v_{2n-1} - iv_{2n})$. Through this bijection, the problem of finding O_{passive} maps to the problem of finding a unitary operation U_{passive} that maps the span of $\{f(v_1), \dots, f(v_{\kappa s})\}$ to the span of the first κs canonical basis vectors of this n -dimensional complex space. This is achieved using the Gram-Schmidt orthonormalization of $\{f(v_1), \dots, f(v_{\kappa s})\}$, which can be computed in time $\mathcal{O}(n\kappa^2 s^2)$ [104]. Completing this orthonormal family to a $2n \times 2n$ orthogonal matrix is then done in time $\mathcal{O}(n^3)$.

Thanks to (179), (181), and to the fact that passive Gaussian unitaries preserve the vacuum, we obtain that

$$|\psi\rangle \propto \tilde{G}_s (\Pi_{i=1}^s \tilde{W}_i) |0\rangle^{\otimes n}. \quad (185)$$

By noting that $(\Pi_{i=1}^s \tilde{W}_i)$ acts only on the first κs modes, we can define a (κs) -mode state $|\phi\rangle$ such that

$$|\phi\rangle \propto (\Pi_{i=1}^s \tilde{W}_i) |0\rangle^{\otimes (\kappa s)}. \quad (186)$$

Hence, by defining $G := \tilde{G}_s$, (185) implies that

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes (n-\kappa s)}). \quad (187)$$

In conclusion, thanks to the above estimates of the computational times, the descriptions of G and $|\phi\rangle$ can be computed in time $\mathcal{O}(s^2 n^3 + \kappa s n^2 + n^3) = \mathcal{O}(s^2 n^3)$ since $\kappa s \leq n$. ■

3. Gaussian circuits doped with cubic phase gates

In this section, we show that combining Theorems 14 and 16 allows us to estimate Gaussian statistics of the output states of Gaussian circuits doped with cubic phase gates. This result is formalized as follows:

Theorem 17 (Estimating output statistics of Gaussian circuits doped with cubic phase gates). Let $s \geq 1$, $\varepsilon, \delta > 0$ and let

$$U = G_s e^{i\gamma_s \hat{x}_1^3} G_{s-1} \dots G_1 e^{i\gamma_1 \hat{x}_1^3} G_0, \quad (188)$$

where $\hat{x}_1$ denotes the position operator on the first mode, $\gamma_1, \dots, \gamma_s$ are real parameters of at most $\mathcal{O}(\text{poly}(n))$ in magnitude, and $G_0, \dots, G_s$ are Gaussian unitaries with entries of their symplectic matrix and displacement vector at most $\mathcal{O}(\text{poly}(n))$ in magnitude. Given an n -mode Gaussian state $|G\rangle$, there is a classical randomized algorithm which outputs an estimate of the overlap $|\langle G|U|0\rangle^{\otimes n}|^2$ up to additive precision ε with probability $1 - \delta$, in time

$$\mathcal{O}\left(\frac{n^2 s}{\varepsilon^2} \left(\frac{6E}{s\varepsilon^2}\right)^{3s} \log\left(\frac{E}{s\varepsilon^2\delta}\right) + \text{poly}(n)^s\right), \quad (189)$$

where $E = \mathcal{O}(\text{poly}(n)^{2s})$. In particular, the algorithm is efficient when $s = \mathcal{O}(1)$.

Note that the cubic phase gates, denoted as $e^{i\gamma_s \hat{x}_1^3}$, act on the first mode without loss of generality, since swapping two modes is a Gaussian unitary operation. Furthermore, although the scaling $\gamma = \mathcal{O}(\text{poly}(n))$ deviates from the realistic $\mathcal{O}(1)$ behavior of cubicity parameters, Theorem 17 demonstrates that efficient classical simulation of low symplectic rank bosonic circuit remains viable in this extended regime of parameters, and even if the energy scales beyond the physical expectation of $E = \mathcal{O}(n)$.

Proof. Let $|\psi\rangle = U|0\rangle^{\otimes n}$ be the output state of the circuit. From the proof of Theorem 14, it is enough to show that an approximate decomposition $(G, \tilde{\phi})$ of ψ can be computed efficiently (where $\tilde{\phi}$ is given in Fock basis) rather than an exact decomposition (G, ϕ) , as long as $\|\phi - \tilde{\phi}\|_1 \leq \varepsilon$. We do so in four steps:

- We use Theorem 16 to show that ϕ can be efficiently written as the s -mode output state of a Gaussian circuit doped with cubic phase gates.
- We use Lemma 1 of Ref. [66] to show that ϕ can further be written as the s -mode output state of a passive Gaussian circuit doped with displacements and cubic phase gates.
- We use Lemmas 12 and 14 to show that ϕ is well approximated by a superposition of coherent states ϕ whose Fock states amplitudes can be computed in time $\mathcal{O}(\text{poly}(n)^s)$.

- (d) Finally, we obtain an energy bound on $\tilde{\phi}$ and apply Theorem 14.

Let

$$|\psi\rangle := U|0\rangle^{\otimes n}, \quad (190)$$

with

$$U = G_s e^{i\gamma_s \hat{x}_1^3} G_{s-1} \cdots G_1 e^{i\gamma_1 \hat{x}_1^3} G_0. \quad (191)$$

From Theorem 16, the state ψ can be expressed as

$$|\psi\rangle = G(|\phi\rangle \otimes |0\rangle^{\otimes(n-s)}), \quad (192)$$

in time $\mathcal{O}(\text{poly}(n))$, with

$$\begin{aligned} |\phi\rangle \otimes |0\rangle^{\otimes(n-s)} \\ = (\Pi_{i=1}^s G_{\text{passive}}^\dagger \tilde{G}_{i-1}^\dagger e^{i\gamma_i \hat{x}_1^3} \tilde{G}_{i-1} G_{\text{passive}}) |0\rangle^{\otimes n}, \end{aligned} \quad (193)$$

where G_{passive} is a passive Gaussian unitary over n modes and $\tilde{G}_{i-1} = G_{i-1} \cdots G_0$ for all $i \in \{1, \dots, s\}$, while $\tilde{G}_i G_{\text{passive}}$ is a Gaussian unitary operation over s modes.

From Lemma 1 of Ref. [66], $W_i := G_{\text{passive}}^\dagger \tilde{G}_{i-1}^\dagger e^{i\gamma_i \hat{x}_1^3} \tilde{G}_{i-1} G_{\text{passive}}$ can be put in the form

$$W_i = B_i^\dagger e^{i\gamma'_i \hat{x}_1^3} B_i, \quad (194)$$

for all i , where each B_i is combination of a passive Gaussian unitary operation and a displacement operator over s modes, whose expression can be computed efficiently, as well as the updated cubicity parameter γ'_i . This gives ϕ of the form

$$|\phi\rangle = (\Pi_{i=1}^s B_i^\dagger e^{i\gamma'_i \hat{x}_1^3} B_i) |0\rangle^{\otimes s}. \quad (195)$$

We now use the following simulation algorithm, which propagates the input vacuum through the various gates in Eq. (195): starting with the input coherent state $|0\rangle^{\otimes s}$, each displaced passive Gaussian unitary converts a tensor product of coherent states to a (possibly different) tensor product of coherent states. On the other hand, whenever we encounter the cubic phase gate, we use the following Lemma to approximate the action of the cubic phase gate on a coherent state as a superposition of coherent states:

Lemma 12 (Coherent state superposition approximation of $e^{i\gamma \hat{x}_1^3} \sum_{k=0}^{N_c} c_k |\alpha_k\rangle$). For $N_c \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $\alpha_1, \dots, \alpha_{N_c} \in \mathbb{C}^m$, let $|\psi\rangle = e^{i\gamma \hat{x}_1^3} \sum_{k=0}^{N_c} c_k |\alpha_k\rangle$. Assuming that $|\gamma|, |\alpha_{k,i}|, |c_k|, N_c = \mathcal{O}(\text{poly}(n))$, $\forall i, k$, we can compute the description of $|\phi\rangle := \sum_{i=1}^{N^2 \times N_c} C_k |\beta_k\rangle$ with $N = \mathcal{O}(\text{poly}(n))$, $|C_k| \leq \max_k |c_k|^2 \times N_c \times N^{1/2} = \mathcal{O}(\text{poly}(n))$, $|\beta_{k,i}| = \mathcal{O}(\max_k |\alpha_{k,1}|^2) = \mathcal{O}(\text{poly}(n))$ in polynomial time,

such that $\| |\psi\rangle \langle \psi| - |\phi\rangle \langle \phi| \|_1 \leq \| |\psi\rangle - |\phi\rangle \| \leq \mathcal{O}(1/\text{poly}(n))$.

The Euclidean distance between two states $|\psi\rangle$ and $|\phi\rangle$ is given by

$$\begin{aligned} \| |\psi\rangle - |\phi\rangle \| &= \sqrt{(\langle \psi| - \langle \phi|)(|\psi\rangle - |\phi\rangle)} \\ &= \sqrt{2(1 - \text{Re}(\langle \psi|\phi\rangle))}, \end{aligned} \quad (196)$$

and it upper bounds the trace distance between two states

$$\| |\psi\rangle \langle \psi| - |\phi\rangle \langle \phi| \|_1 \leq \| |\psi\rangle - |\phi\rangle \|. \quad (197)$$

Proof. The proof techniques used below are directly inspired from Lemma 2 of Ref. [66]. For brevity, we introduce the notation

$$\begin{aligned} \exp &:= \mathcal{O}(\exp(\text{poly}(n))), \\ 1/\exp &:= \mathcal{O}(1/\exp(\text{poly}(n))), \end{aligned} \quad (198)$$

and similarly for poly and $1/\text{poly}$. We write

$$|\psi\rangle = \sum_{k=1}^{N_c} c_k |\psi_{\alpha_{k,1}, \gamma}\rangle \otimes |\alpha_{k,2} \cdots \alpha_{k,m}\rangle, \quad (199)$$

where $|\psi_{\alpha_{k,1}, \gamma}\rangle := e^{i\gamma \hat{x}_1^3} |\alpha_{k,1}\rangle$. Then, we compute an approximate coherent state decomposition for $|\psi_{\alpha, \gamma}\rangle = e^{i\gamma \hat{x}_1^3} |\alpha\rangle$ (which we apply to $\alpha = \alpha_{k,1}$) in three steps:

- (a) Using the gentle measurement lemma, we truncate $|\psi_{\alpha, \gamma}\rangle$ to a state $|\psi_{\alpha, \gamma, N}\rangle$ with a finite Fock support size $N + 1$, such that

$$\| |\psi_{\alpha, \gamma}\rangle - |\psi_{\alpha, \gamma, N}\rangle \| \leq \infty/\text{poly}. \quad (200)$$

- (b) We approximate the Fock state amplitudes of $|\psi_{\alpha, \gamma, N}\rangle$ to precision $1/\exp$ in time poly to obtain $|\tilde{\psi}_{\alpha, \gamma, N}\rangle$ and ensure that

$$\| |\psi_{\alpha, \gamma, N}\rangle - |\tilde{\psi}_{\alpha, \gamma, N}\rangle \| \leq 1/\exp. \quad (201)$$

- (c) Using Lemma 5.34 of Ref. [105], given a Fock state $|n\rangle$, one can construct a coherent state decomposition $|\tilde{n}\rangle = \sum_{i=0}^n c_i |\beta_i\rangle$, with $|\beta_i| = \mathcal{O}(\sqrt{n})$ and $|c_i| = \mathcal{O}(1/\sqrt{\delta})$ that approximates $|n\rangle$ up to precision δ in trace distance. We further ensure $|\tilde{n}\rangle$ is δ -close in Euclidean distance to $|n\rangle$, as detailed below.

We first use the gentle measurement lemma to truncate $|\psi_{\alpha, \gamma}\rangle$ to a state with finite support size in Fock basis: ■

Lemma 13 (Gentle measurement lemma [99]). Given a state

$$|\psi_{\alpha,\gamma}\rangle = e^{i\gamma\hat{x}_1^3} |\alpha\rangle, \quad (202)$$

such that

$$\langle\psi_{\alpha,\gamma}|\hat{n}|\psi_{\alpha,\gamma}\rangle \leq E, \quad (203)$$

the normalized projection $|\psi_{\alpha,\gamma,N}\rangle$ of $|\psi_{\alpha,\gamma}\rangle$ on the first $N + 1$ Fock states is such that

$$\frac{1}{2} \| |\psi_{\alpha,\gamma}\rangle\langle\psi_{\alpha,\gamma}| - |\psi_{\alpha,\gamma,N}\rangle\langle\psi_{\alpha,\gamma,N}| \|_1 \leq \sqrt{\frac{E}{N}}, \quad (204)$$

where $\|\cdot\|_1$ refers to the 1-norm.

Therefore, choosing $N = 4E/\epsilon^2$, we find

$$|\psi_{\alpha,\gamma,N}\rangle = \frac{1}{\sqrt{\mathcal{N}}} \sum_{n=0}^N a_n |n\rangle, \quad (205)$$

with $a_n = \langle n | e^{i\gamma\hat{q}^3} |\alpha\rangle$ and $\mathcal{N} = \sum_{n=0}^N |a_n|^2$ such that

$$\frac{1}{2} \| |\psi_{\alpha,\gamma}\rangle\langle\psi_{\alpha,\gamma}| - |\psi_{\alpha,\gamma,N}\rangle\langle\psi_{\alpha,\gamma,N}| \| \leq \frac{\epsilon}{2}. \quad (206)$$

For $|\psi_{\alpha,\gamma}\rangle = e^{i\gamma\hat{x}_1^3} |\alpha\rangle$, E is given by

$$\begin{aligned} E &= \langle\alpha| e^{-i\gamma\hat{x}_1^3} \hat{n} e^{i\gamma\hat{x}_1^3} |\alpha\rangle = \langle\alpha| e^{-i\gamma\hat{x}_1^3} \left(\frac{1}{2}(\hat{x}_1^2 + \hat{p}_1^2) - \frac{1}{2} \right) e^{i\gamma\hat{x}_1^3} |\alpha\rangle \\ &= \frac{1}{2} \langle\alpha| \hat{x}_1^2 + (\hat{p}_1 + 3\gamma\hat{x}_1^2)^2 |\alpha\rangle - \frac{1}{2} \\ &= \frac{1}{2} \langle\alpha| \hat{x}_1^2 + \hat{p}_1^2 |\alpha\rangle - \frac{1}{2} + \frac{9}{2}\gamma^2 \langle\alpha| \hat{x}_1^4 |\alpha\rangle + 3\gamma \langle\alpha| \hat{p}_1 \hat{x}_1^2 |\alpha\rangle + 3\gamma \langle\alpha| \hat{x}_1^2 \hat{p}_1 |\alpha\rangle. \end{aligned} \quad (207)$$

Noting that with our conventions, $\hat{D}(\alpha)^\dagger \hat{x}_1 \hat{D}(\alpha) = \hat{x}_1 + \sqrt{2} \operatorname{Re}\{\alpha\}$, $\hat{D}(\alpha)^\dagger \hat{p}_1 \hat{D}(\alpha) = \hat{p}_1 + \sqrt{2} \operatorname{Im}\{\alpha\}$, we have

$$\begin{aligned} \langle\alpha| \hat{x}_1^4 |\alpha\rangle &= 3 + 12 \operatorname{Re}\{\alpha\}^2 + 4 \operatorname{Re}\{\alpha\}^4, \\ \langle\alpha| \hat{p}_1 \hat{x}_1^2 |\alpha\rangle &= 2\sqrt{2} \operatorname{Re}\{\alpha\} \langle 0 | \hat{x}_1 \hat{p}_1 | 0 \rangle + \sqrt{2} \operatorname{Im}\{\alpha\} + 2\sqrt{2} \operatorname{Re}\{\alpha\}^2 \operatorname{Im}\{\alpha\}, \\ \langle\alpha| \hat{x}_1^2 \hat{p}_1 |\alpha\rangle &= 2\sqrt{2} \operatorname{Re}\{\alpha\} \langle 0 | \hat{p}_1 \hat{x}_1 | 0 \rangle + \sqrt{2} \operatorname{Im}\{\alpha\} + 2\sqrt{2} \operatorname{Re}\{\alpha\}^2 \operatorname{Im}\{\alpha\}, \end{aligned} \quad (208)$$

so

$$E = |\alpha|^2 + \frac{9}{2}\gamma^2(3 + 12 \operatorname{Re}\{\alpha\}^2 + 4 \operatorname{Re}\{\alpha\}^4) + 6\gamma(\sqrt{2} \operatorname{Im}\{\alpha\} + 2\sqrt{2} \operatorname{Re}\{\alpha\}^2 \operatorname{Im}\{\alpha\}), \quad (209)$$

which is polynomial in α , α^* and γ . Therefore, $|\psi_{\alpha,\gamma,N}\rangle$ given by Eq. (205) with $N = 4E/\epsilon^2 + 1 = 4 \operatorname{poly}(\alpha, \gamma)/\epsilon^2 + 1 = \operatorname{poly}$, for $|\alpha|, |\gamma| = \operatorname{poly}$ and $\epsilon = 1/\operatorname{poly}$ approximates $|\psi_{\alpha,\gamma}\rangle$ up to precision $1/\operatorname{poly}$ in trace distance.

Now, following Ref. [66], we show how to compute the Fock state coefficients a_n up to precision $1/\exp$,

$$\begin{aligned}
 a_n(\alpha, \gamma) &= \langle n | e^{i\gamma \hat{x}_1^3} | \alpha \rangle \\
 &= \int_{-\infty}^{\infty} dq \langle n | q | n | q \rangle \langle q | e^{i\gamma \hat{x}_1^3} | \alpha \rangle \\
 &= \frac{1}{\sqrt{\pi 2^n n!}} \int_{-\infty}^{\infty} dq e^{i\gamma q^3} e^{-q^2/2} H_n(q) e^{-\alpha_I^2} e^{-\frac{1}{2}(q - \sqrt{2}\alpha)^2} \\
 &= \frac{e^{-\alpha_I^2 - \alpha^2}}{\sqrt{\pi 2^n n!}} \int_{-\infty}^{\infty} dq e^{i\gamma q^3} e^{-q^2/2 + \sqrt{2}q\alpha} H_n(q),
 \end{aligned} \tag{210}$$

where $|q\rangle$ represent the position eigenstates and $\alpha_I := \text{Im}\{\alpha\}$ is the imaginary part of α . Following Appendix D of Ref. [106], we perform the substitution of variables $q = yk - iy^3$, such that $y^3 = 1/3\gamma$, and obtain

$$a_n(\alpha, \gamma) = \frac{ye^{\frac{2}{3}y^6} e^{-\alpha_I^2 - \alpha^2 - i\sqrt{2}\alpha y^3}}{\sqrt{\pi 2^n n!}} \int_{-\infty}^{\infty} dk \exp \left[i(y^4 - i\sqrt{2}\alpha y)k + \frac{ik^3}{3} \right] H_n(yk - iy^3). \tag{211}$$

Using $H_n(x + y) = \sum_{j=0}^n \binom{n}{j} H_j(x) (2y)^{n-j}$,

$$\begin{aligned}
 a_n(\alpha, \gamma) &= \frac{ye^{\frac{2}{3}y^6} e^{-\alpha_I^2 - \alpha^2 - i\sqrt{2}\alpha y^3}}{\sqrt{\pi 2^n n!}} \sum_{j=0}^n \binom{n}{j} H_j(-iy^3) (2y)^{n-j} \int_{-\infty}^{\infty} dk \exp \left[i(y^4 - i\sqrt{2}\alpha y)k + \frac{ik^3}{3} \right] (k)^{n-j} \\
 &= \frac{ye^{\frac{2}{3}y^6} e^{-\alpha_I^2 - \alpha^2 - i\sqrt{2}\alpha y^3}}{\sqrt{\pi 2^n n!}} \sum_{j=0}^n \binom{n}{j} H_j(-iy^3) (2y)^{n-j} \text{Ai}^{(n-j)}(w)|_{w=y^4 - i\sqrt{2}\alpha y},
 \end{aligned} \tag{212}$$

where $\text{Ai}^{(n-j)}(w)$ is $(n-j)$ th derivative of Airy function $\text{Ai}(w)$. From Eq. (128) of Ref. [107],

$$\begin{aligned}
 \text{Ai}^{(\ell)}(w) &= \text{Ai}(0) f^{(\ell)}(w) + \text{Ai}'(0) g^{(\ell)}(w) \\
 &= \frac{\text{Ai}(0)}{\Gamma(1/3)} \sum_{n=\lceil \ell/3 \rceil}^{\infty} \frac{3^n \Gamma(n + 1/3)}{(3n - \ell)!} w^{3n - \ell} + \frac{\text{Ai}'(0)}{\Gamma(2/3)} \sum_{n=\lceil \ell/3 \rceil - 1}^{\infty} \frac{3^n \Gamma(n + 2/3)}{(3n + 1 - \ell)!} w^{3n + 1 - \ell},
 \end{aligned} \tag{213}$$

where $\text{Ai}(0) = 3^{-2/3}/\Gamma(2/3)$ and $\text{Ai}'(0) = -3^{-1/3}/\Gamma(1/3)$. If $\tilde{\text{Ai}}^{(\ell)}(w)$ is the approximated value of the ℓ th derivative by taking first d terms of the infinite series, then by Eq. (129) of Ref. [107],

$$|\text{Ai}^{(\ell)}(w) - \tilde{\text{Ai}}^{(\ell)}(w)| \leq \frac{1 + \frac{|w|}{2d}}{1 - \frac{3|w|^3}{2d}} \frac{(|w|)^{3d - \ell}}{(2d/3)^d}. \tag{214}$$

Since $|w| = |y^4 - i\sqrt{2}\alpha y| = \text{poly}$ for $\gamma, \alpha = \text{poly}$. Therefore, taking $d = \text{poly}$ with a sufficiently large polynomial, we get

$$|\text{Ai}^{(\ell)}(w) - \tilde{\text{Ai}}^{(\ell)}(w)| \leq 1/\exp. \tag{215}$$

Note that this bound is assuming that $f^{(\ell)}(w)$ and $g^{(\ell)}(w)$ can be approximated up to error $1/\exp$ in time poly . Since

$$\Gamma(n + 1/3) = (n - 2/3)(n - 5/3) \cdots \Gamma(1/3) \tag{216}$$

and $\Gamma(1/3)$ can be specified up to poly bits, i.e., up to precision error $1/\exp$ in time poly [108] and since the magnitude of each of the term in $f^{(\ell)}(w)$ is $= \exp$, $f^{(\ell)}(w)$ can be approximated up to error $1/\exp$ in time poly . A similar analysis

holds for $g^{(\ell)}(w)$. Now, setting

$$\tilde{a}_n(\alpha, \gamma) := \frac{ye^{\frac{2}{3}y^6} e^{-\alpha_I^2 - \alpha^2 - i\sqrt{2}\alpha y^3}}{\sqrt{\pi 2^n n!}} \sum_{j=0}^n \binom{n}{j} H_j(-iy^3) (2y)^{n-j} (\tilde{\text{Ai}}^{(n-j)}(w))_{w=y^4 - i\sqrt{2}\alpha y}, \quad (217)$$

with $n = \text{poly}$, all these coefficients can be computed in time poly . Moreover,

$$\begin{aligned} |a_n(\alpha, \gamma) - \tilde{a}_n(\alpha, \gamma)| &\leq \frac{ye^{\frac{2}{3}y^6} e^{-\alpha_I^2 - \alpha^2 - i\sqrt{2}\alpha y^3}}{\sqrt{\pi 2^n n!}} \\ &\times \sum_{j=0}^n \binom{n}{j} H_j(-iy^3) (2y)^{n-j} |(\text{Ai}^{(n-j)}(w))_{w=y^4 - i\sqrt{2}\alpha y} - (\tilde{\text{Ai}}^{(n-j)}(w))_{w=y^4 - i\sqrt{2}\alpha y}|. \end{aligned}$$

Since $|\binom{n}{j} H_j(-iy^3) (2y)^{n-j}| = \exp$, $\tilde{a}_n(\alpha, \gamma)$ approximates $a_n(\alpha, \gamma)$ to precision $1/\exp$ and can be computed in time poly (summing the $n = \text{poly}$ terms and computing the Γ function up to the required number of bits of precision). We have to compute $N = \text{poly}$ such approximate Fock state coefficients, which thus takes total time poly . We define

$$|\tilde{\psi}_{\alpha, \gamma, N}\rangle := \frac{1}{\sqrt{\mathcal{N}_N}} \sum_{n=0}^N \tilde{a}_n |n\rangle, \quad (218)$$

where $\mathcal{N}_N = \sum_{n=0}^N |\tilde{a}_n|^2$ can be computed in time poly , since $N = \text{poly}$. We have

$$\begin{aligned} \frac{1}{2} \|\psi_{\alpha, \gamma, N}\rangle\langle\psi_{\alpha, \gamma, N}| - |\tilde{\psi}_{\alpha, \gamma, N}\rangle\langle\tilde{\psi}_{\alpha, \gamma, N}|\|_1 &\leq \|\psi_{\alpha, \gamma, N}\rangle - |\tilde{\psi}_{\alpha, \gamma, N}\rangle\| = \sqrt{\sum_{n=0}^N \left| \frac{a_n - \tilde{a}_n}{\sqrt{\mathcal{N}_N}} \right|^2} + \frac{1}{\exp} \\ &\leq 1/\exp, \end{aligned} \quad (219)$$

where $\|\cdot\|_1$ denotes the trace norm and $\|\cdot\|$ the Euclidean norm. Therefore,

$$\frac{1}{2} \|\psi_{\alpha, \gamma}\rangle\langle\psi_{\alpha, \gamma}| - |\tilde{\psi}_{\alpha, \gamma, N}\rangle\langle\tilde{\psi}_{\alpha, \gamma, N}|\|_1 \leq \frac{1}{2} \|\psi_{\alpha, \gamma}\rangle\langle\psi_{\alpha, \gamma}| - |\psi_{\alpha, \gamma, N}\rangle\langle\psi_{\alpha, \gamma, N}|\|_1 \quad (220)$$

$$\begin{aligned} &+ \frac{1}{2} \|\psi_{\alpha, \gamma, N}\rangle\langle\psi_{\alpha, \gamma, N}| - |\tilde{\psi}_{\alpha, \gamma, N}\rangle\langle\tilde{\psi}_{\alpha, \gamma, N}|\|_1 \\ &\leq 1/\text{poly}. \end{aligned} \quad (221)$$

Further, since for pure states, the trace distance is given by $\sqrt{1 - |\langle\tilde{\psi}_{\alpha, \gamma, N}|\psi_{\alpha, \gamma}\rangle\langle\psi_{\alpha, \gamma}|\tilde{\psi}_{\alpha, \gamma, N}\rangle|}$, we have

$$\begin{aligned} 1 - |\langle\tilde{\psi}_{\alpha, \gamma, N}|\psi_{\alpha, \gamma}\rangle\langle\psi_{\alpha, \gamma}|\tilde{\psi}_{\alpha, \gamma, N}\rangle| &\leq 1/\text{poly}, \\ |\langle\tilde{\psi}_{\alpha, \gamma, N}|\psi_{\alpha, \gamma}\rangle\langle\psi_{\alpha, \gamma}|\tilde{\psi}_{\alpha, \gamma, N}\rangle| &\geq 1 - 1/\text{poly}. \end{aligned} \quad (222)$$

Moreover,

$$\langle\tilde{\psi}_{\alpha, \gamma, N}|\psi_{\alpha, \gamma}\rangle\langle\psi_{\alpha, \gamma}|\tilde{\psi}_{\alpha, \gamma, N}\rangle = \frac{\sum_{n=0}^N \tilde{a}_n^* a_n}{\sqrt{\sum_{n=0}^N |\tilde{a}_n|^2}} = \sqrt{\sum_{n=0}^N |\tilde{a}_n|^2} + \frac{1}{\exp}, \quad (223)$$

which is real and positive. Therefore, Eq. (222) implies

$$\text{Re}(\langle\tilde{\psi}_{\alpha, \gamma, N}|\psi_{\alpha, \gamma}\rangle\langle\psi_{\alpha, \gamma}|\tilde{\psi}_{\alpha, \gamma, N}\rangle) = |\langle\tilde{\psi}_{\alpha, \gamma, N}|\psi_{\alpha, \gamma}\rangle\langle\psi_{\alpha, \gamma}|\tilde{\psi}_{\alpha, \gamma, N}\rangle| + \frac{1}{\exp} \geq 1 - 1/\text{poly}, \quad (224)$$

and finally

$$\| |\psi_{\alpha,\gamma}\rangle - |\tilde{\psi}_{\alpha,\gamma,N}\rangle \| = \sqrt{2(1 - \text{Re}(\langle \tilde{\psi}_{\alpha,\gamma,N} | \psi_{\alpha,\gamma} \rangle))} \leq 1/\text{poly}. \quad (225)$$

Now, with

$$|\tilde{\psi}_{\alpha,\gamma,N}\rangle = \frac{1}{\sqrt{\mathcal{N}_N}} \sum_{n=0}^N \tilde{a}_n |n\rangle, \quad (226)$$

we approximate each of the Fock state $|n\rangle$ with fidelity $1 - \delta$ by a superposition of coherent states $|\tilde{n}\rangle = \sum_{k=0}^n d_k |\eta_k\rangle$ with $|\eta_k| = \mathcal{O}(\sqrt{n})$, $|c_k| = \mathcal{O}(1/\sqrt{\delta})$, $\forall k$ using Lemma 5.34 of Ref. [105]. Further, since we have analytical formulas for the overlap of Fock state with coherent state, we can compute $\langle n | \tilde{n} | n | \tilde{n} \rangle = \sum_{k=0}^n d_k \langle n | \eta_k | n | \eta_k \rangle := r e^{i\theta}$ in poly time, since $n = \text{poly}$. We add a global phase $e^{-i\theta}$ to $|\tilde{n}\rangle$ such that $\text{Re}(\langle n | \tilde{n} | n | \tilde{n} \rangle) = |\langle n | \tilde{n} | n | \tilde{n} \rangle|$, so that

$$\| |n\rangle - |\tilde{n}\rangle \| = \sqrt{2(1 - \text{Re}(\langle n | \tilde{n} | n | \tilde{n} \rangle))} = \sqrt{2(1 - |\langle n | \tilde{n} | n | \tilde{n} \rangle|)} \leq \sqrt{2\delta}, \quad (227)$$

where we have used the fact that $|\langle n | \tilde{n} | n | \tilde{n} \rangle| \geq 1 - \delta$ by construction, and $\sqrt{1-x} \geq 1-x$ for $0 \leq x \leq 1$. Then, defining

$$|\phi_{\alpha,\gamma}\rangle := \sum_{n=0}^N \tilde{a}_n |\tilde{n}\rangle, \quad (228)$$

we have that

$$\begin{aligned} \| |\tilde{\psi}_{\alpha,\gamma,N}\rangle - |\phi_{\alpha,\gamma}\rangle \| &= \left\| \sum_{n=0}^N \tilde{a}_n (|n\rangle - |\tilde{n}\rangle) \right\| \leq \sum_{n=0}^N |\tilde{a}_n| \| |n\rangle - |\tilde{n}\rangle \| \\ &\leq \left(\sum_{n=0}^N |\tilde{a}_n|^2 \right)^{1/2} \left(\sum_{n=0}^N \| |n\rangle - |\tilde{n}\rangle \|^2 \right)^{1/2} \\ &\leq N^{1/2} \sqrt{2\delta}, \end{aligned} \quad (229)$$

where the first inequality follows from the triangle inequality, second inequality follows from the Cauchy-Schwartz inequality, we use the fact that $\sum_{n=0}^N |\tilde{a}_n|^2 = 1$ (since $|\psi_N\rangle$ is normalized) and that $\| |n\rangle - |\tilde{n}\rangle \|^2 \leq 2\delta$, for all n . Further, from triangle inequality,

$$\| |\psi_{\alpha,\gamma}\rangle - |\phi_{\alpha,\gamma}\rangle \| \leq \| |\psi_{\alpha,\gamma}\rangle - |\tilde{\psi}_{\alpha,\gamma,N}\rangle \| + \| |\tilde{\psi}_{\alpha,\gamma,N}\rangle - |\phi_{\alpha,\gamma}\rangle \| \leq 1/\text{poly} + N^{1/2} \sqrt{2\delta}. \quad (230)$$

Therefore, defining

$$|\phi\rangle := \sum_{k=1}^{N_c} c_k |\phi_{\alpha_{k,1},\gamma}\rangle \otimes |\alpha_{k,2} \cdots \alpha_{k,m}\rangle, \quad (231)$$

it follows from Eq. (199) that

$$\| |\psi\rangle - |\phi\rangle \| = \left\| \sum_{k=1}^{N_c} c_k (|\psi_{\alpha_{k,1},\gamma}\rangle - |\phi_{\alpha_{k,1},\gamma}\rangle) \otimes |\alpha_{k,2} \otimes \cdots \otimes \alpha_{k,m}\rangle \right\| \quad (232)$$

$$\begin{aligned} &\leq \sum_{k=1}^{N_c} |c_k| \| |\psi_{\alpha_{k,1},\gamma}\rangle - |\phi_{\alpha_{k,1},\gamma}\rangle \| \\ &\leq \left(\sum_{k=1}^{N_c} |c_k|^2 \right)^{1/2} \left(\sum_{k=1}^{N_c} \| |\psi_{\alpha_{k,1},\gamma}\rangle - |\phi_{\alpha_{k,1},\gamma}\rangle \|^2 \right)^{1/2} \\ &\leq N_c (\max_k |c_k|) (1/\text{poly} + N^{1/2} \sqrt{2\delta}), \end{aligned} \quad (233)$$

where the chain of inequalities are similar to Eq. (229), and we have used the fact that $\|(|\psi_{\alpha_{k,1},\gamma}\rangle - |\phi_{\alpha_{k,1},\gamma}\rangle) \otimes |\alpha_{k,2} \cdots \alpha_{k,m}\rangle\| = \| |\psi_{\alpha_{k,1},\gamma}\rangle - |\phi_{\alpha_{k,1},\gamma}\rangle \|$. Therefore, given that $N_c, (\max_k |c_k|), N = \text{poly}$, choosing $\delta = 1/\text{poly}$, ensures that $|\phi\rangle$ is $1/\text{poly}$ close in Euclidean distance (hence in trace distance) to $|\psi\rangle$. Finally, we write

$$|\phi\rangle = \sum_{k=1}^{N_c} c_k |\phi_{\alpha_{k,1},\gamma}\rangle \otimes |\alpha_{k,2} \cdots \alpha_{k,m}\rangle =: \sum_{k=1}^{N_c \times N^2} C_k |\beta_k\rangle, \quad (234)$$

where $|C_k| = (\max_k |c_k|^2) N_c N^{1/2} (\max_n |a_n|) = \text{poly}$, and $|\beta_k| = \mathcal{O}(\sqrt{N}) = \mathcal{O}(\sqrt{\max_k |\alpha_k^1|^4}) = \mathcal{O}(\max_k |\alpha_k^1|^2) = \text{poly}$, for all k . ■

Repeatedly applying Lemma 12 over the s layers in Eq. (195), we obtain a superposition of coherent states $\tilde{\phi}$ which approximates ϕ . The time complexity of computing $\tilde{\phi}$ and the precision with which it approximates ϕ in trace distance are obtained using the following Lemma:

Lemma 14 (Approximation via coherent state superposition). Given an s -mode state $|\phi\rangle = (\prod_{i=1}^s B_i^\dagger e^{i\gamma'_i x_1^3} B_i) |0\rangle^{\otimes s}$ where $\gamma'_1, \dots, \gamma'_s = \mathcal{O}(\text{poly}(n)) \in \mathbb{R}$ and where $B_1, \dots, B_s$ are passive Gaussian unitary gates with entries of their symplectic matrix and displacement vector at most $\mathcal{O}(\text{poly}(n))$, we can obtain the description of a superposition of coherent states $|\tilde{\phi}\rangle = \sum_{i=1}^N c_i |\alpha_1^{(i)} \alpha_2^{(i)} \cdots \alpha_s^{(i)}\rangle$ where $N = \mathcal{O}(\text{poly}(n)^s)$ in time $\mathcal{O}(\text{poly}(n)^s)$, such that $\tilde{\phi}$ is $\mathcal{O}(s/\text{poly}(n))$ -close in trace distance to ϕ , and $|\alpha_{s'}^{(i)}| = \mathcal{O}(\text{poly}(n)^{2^s})$ and $|c_i| = \mathcal{O}(\text{poly}(n)^{2^s})$, for all $s' \in \{1, \dots, s\}$ and all $i \in \{1, \dots, N\}$.

Proof. Starting from a tensor product of coherent states (the s -mode vacuum state), whenever we encounter the displaced passive linear unitary gate, the tensor product of coherent states is converted to a (different) tensor product of coherent states analytically given the description of the displaced passive linear unitary gate. In contrast, whenever we encounter the cubic phase gate, we use Lemma 12 to approximate its action as a superposition of coherent of $\mathcal{O}(\text{poly}(n))$ coherent states in time $\mathcal{O}(\text{poly}(n))$. This approximation is $\mathcal{O}(1/\text{poly}(n))$ close to the input state to the cubic phase gate in Euclidean (and hence trace) distance. Repeating this process over s layers, we see that the output state can be approximated as a superposition of $\mathcal{O}(\text{poly}(n)^{2^s})$ coherent states in time $\mathcal{O}(s \times \text{poly}(n))$ and by the triangle inequality this state is $\mathcal{O}(s/\text{poly}(n))$ close in trace distance. Since according to Lemma 12, the magnitude of the coefficients of the coherent state and the magnitude of the coherent states themselves is squared after each term, this gives the magnitude of coherent states and their coefficients at the output ■

Therefore, for $s = \mathcal{O}(1)$, we get an inverse polynomially close description of $|\phi\rangle$ in polynomial time. Let us now bound the mean photon number of $\tilde{\phi}$. Writing $|\tilde{\phi}\rangle = \sum_{i=1}^N c_i |\alpha_1^{(i)} \cdots \alpha_m^{(i)}\rangle$ with $N = \mathcal{O}(\text{poly}(n)^{2^s})$, $|\alpha_{s'}^{(i)}| = \mathcal{O}(\text{poly}(n)^{2^s})$, and $|c_i| = \mathcal{O}(\text{poly}(n)^{2^s})$, for all $s' \in \{1, \dots, s\}$ and all $i \in \{1, \dots, N\}$ as in Lemma 14 we obtain

$$\begin{aligned} \langle \tilde{\phi} | \hat{N} | \tilde{\phi} \rangle &= \sum_{i,j=1}^N c_i c_j^* \langle \alpha_1^{(i)} \cdots \alpha_s^{(i)} | \hat{N} | \alpha_1^{(j)} \cdots \alpha_s^{(j)} \rangle \\ &= \sum_{i,j=1}^N \sum_{k=1}^s c_i \alpha_k^{(i)} c_j^* \alpha_k^{(j)*} \langle \alpha_k^{(i)} | \alpha_k^{(j)} \rangle |\alpha_k^{(i)}| |\alpha_k^{(j)}| \\ &\leq \sum_{k=1}^s \sum_{i,j=1}^N |c_i| |\alpha_k^{(i)}| |c_j| |\alpha_k^{(j)}| \\ &= \mathcal{O}(\text{poly}(n)^{2^s}). \end{aligned} \quad (235)$$

Therefore, applying Theorem 14 with the approximate decomposition (G, ϕ) and $E = \mathcal{O}(\text{poly}(n)^{2^s})$ yields the final complexity of simulation stated in Theorem 17. We note that for $s = \mathcal{O}(1)$, the simulation is efficient.

4. Gaussian circuits doped with photon additions and subtractions

When replacing cubic phase gates by applications of creation and annihilation operators (i.e., $a^{\dagger j} a^k$), a classical simulation strategy similar to that of the previous section allows to estimate output Gaussian statistics efficiently for a constant number of non-Gaussian layers. In what follows, we refine this result using stellar rank techniques from Ref. [71] to obtain a classical algorithm for *strong simulation* of Gaussian measurements that is efficient for a constant number of non-Gaussian layers. The intuition behind this improved result is that, for this class of circuits, we can take advantage of the fact that the state ϕ in the decomposition of Eq. (157) has a finite stellar rank to avoid the complexity induced by the energy truncation.

Theorem 18 (Strong simulation of Gaussian circuits doped with photon additions and subtractions). Given an input n -mode vacuum state $|0\rangle^{\otimes n}$ evolving through a (nonunitary) circuit described by

$$C = G_s(a_1^\dagger)^{j_s} (a_1)^{k_s} G_{s-1} \cdots G_1(a_1^\dagger)^{j_1} (a_1)^{k_1} G_0, \quad (236)$$

where $j_1, k_1, \dots, j_s, k_s \in \mathbb{N}$, any Gaussian measurement on $(1/\sqrt{\mathcal{N}})C|0\rangle^{\otimes n}$ (where $\mathcal{N}$ is a normalization factor) can be strongly simulated in time $\mathcal{O}((2sM+1)^{2^s} s^3 M^3 2^{2sM} + (2sM+1)^s + \text{poly}(n))$, where $M := \max(j_1, k_1, \dots, j_s, k_s)$.

Note that the creation and annihilation operators act on the first mode without loss of generality, since swapping two modes is a Gaussian unitary operation.

Proof. Given

$$C = G_s(a_1^\dagger)^{j_s}(a_1)^{k_s}G_{s-1} \cdots G_1(a_1^\dagger)^{j_1}(a_1)^{k_1}G_0, \quad (237)$$

and using Theorem 16, C can be described in time $\mathcal{O}(\text{poly}(n))$ as

$$C = G(\Pi_{i=1}^s \tilde{W}_i)G_{\text{passive}}, \quad (238)$$

where G is a Gaussian unitary operation, G_{passive} is a passive linear transformation which can be absorbed into the

input vacuum state, and $\tilde{W}_i$ is given by

$$\tilde{W}_i = G_{\text{passive}}^\dagger \tilde{G}_{i-1}^\dagger (a_1^\dagger)^{j_i} (a_1)^{k_i} \tilde{G}_{i-1} G_{\text{passive}}, \quad (239)$$

where G_{passive} is a passive Gaussian unitary over n modes and $\tilde{G}_{i-1} = G_{i-1} \cdots G_0$ for all $i \in \{1, \dots, s\}$, while $\tilde{G}_i G_{\text{passive}}$ is a Gaussian unitary operation over s modes. For brevity, we define

$$\mathcal{G}_{i-1} := \tilde{G}_{i-1} G_{\text{passive}}. \quad (240)$$

Note that the description of $\mathcal{G}_{i-1}$ can be computed in time $\mathcal{O}(sn^3) = \mathcal{O}(\text{poly}(n))$. Now,

$$\mathcal{G}_{i-1}^\dagger (a_1^\dagger)^{j_i} (a_1)^{k_i} \mathcal{G}_{i-1} = \left((\mathbf{d}_i)_1 + \sum_{l=1}^s (S_i)_{1,l} a_l^\dagger + (S_i)_{1,s+l} a_l \right)^{j_i} \left((\mathbf{d}_i)_1^* + \sum_{l=1}^s (S_i)_{1,l} a_l + (S_i)_{1,s+l} a_l^\dagger \right)^{k_i}, \quad (241)$$

where $\mathbf{d}_i$ and S_i are the displacement vector and symplectic matrix associated with $\mathcal{G}_{i-1}$, respectively. Therefore,

$$\begin{aligned} \Pi_{i=1}^s \tilde{W}_i &= \begin{pmatrix} (\mathbf{d}_s)_1 + \sum_{l=1}^s (S_s)_{1,l} a_l^\dagger + (S_s)_{1,s+l} a_l \\ \vdots \\ (\mathbf{d}_1)_1 + \sum_{l=1}^s (S_1)_{1,l} a_l^\dagger + (S_1)_{1,s+l} a_l \end{pmatrix}^{j_s} \begin{pmatrix} (\mathbf{d}_s)_1^* + \sum_{l=1}^s (S_s)_{1,l} a_l + (S_s)_{1,s+l} a_l^\dagger \\ \vdots \\ (\mathbf{d}_1)_1^* + \sum_{l=1}^s (S_1)_{1,l} a_l + (S_1)_{1,s+l} a_l^\dagger \end{pmatrix}^{k_s} \\ &\quad \vdots \\ &\quad \begin{pmatrix} (\mathbf{d}_0)_1 + \sum_{l=1}^s (S_1)_{1,l} a_l^\dagger + (S_1)_{1,s+l} a_l \end{pmatrix}^{j_1} \begin{pmatrix} (\mathbf{d}_0)_1^* + \sum_{l=1}^s (S_1)_{1,l} a_l + (S_1)_{1,s+l} a_l^\dagger \end{pmatrix}^{k_1}. \end{aligned} \quad (242)$$

Therefore, the circuit can be seen as evolving an input s -mode state $(\Pi_{i=1}^s \tilde{W}_i) |0\rangle^{\otimes n} / \sqrt{\mathcal{N}}$ of finite support in the Fock basis together with the vacuum on the rest of the $n - s$ modes, through a Gaussian unitary operation acting on n modes. Note that we have normalized the input state by adding a normalization factor $\mathcal{N}$, which can be computed by summing the moduli squared of the Fock state amplitudes of the unnormalized state $(\Pi_{i=1}^s \tilde{W}_i) |0\rangle^{\otimes n}$. This can be done in time $\mathcal{O}(s_{\text{support}})$, where s_{support} is the support size of the state, given by

$$s_{\text{support}} = \sum_{i=0}^d \binom{i+s-1}{s-1} = \binom{d+s}{s} \leq (d+1)^s, \quad (243)$$

where d is the degree of its stellar function, given by

$$d = j_1 + k_1 + \cdots + j_s + k_s. \quad (244)$$

Setting $M := \max(j_1, k_1, \dots, j_s, k_s)$, we have

$$d \leq 2sM, \quad s_{\text{support}} \leq (2sM + 1)^s. \quad (245)$$

Therefore, using Theorem 2 of Ref. [71], the heterodyne detection on $G(\Pi_{i=1}^s \tilde{W}_i) |0\rangle^{\otimes n}$ can be strongly simulated in time $\mathcal{O}((2sM + 1)^{2s} s^3 M^3 2^{2sM} + \text{poly}(n))$. Adding up the time to compute the description of $\Pi_{i=1}^s \tilde{W}_i$ from Theorem 16, the value of $\mathcal{N}$, and the description of $\mathcal{G}_0, \dots, \mathcal{G}_{s-1}$, the circuit can be simulated in $\mathcal{O}((2sM + 1)^{2s} s^3 M^3 2^{2sM} + (2sM + 1)^s + \text{poly}(n))$. ■

VIII. ε -APPROXIMATE SYMPLECTIC RANK

The symplectic rank of a pure state with a well-defined covariance matrix is the number of symplectic eigenvalues that are strictly greater than exactly one. In particular, for any $\varepsilon > 0$, a pure state whose symplectic eigenvalues are all equal to $1 + \varepsilon$ has maximum symplectic rank, regardless of how small ε is. This presents a conceptual

drawback: in practical scenarios, a pure state with all symplectic eigenvalues as close to one as, say, $1 + 10^{-100}$, would effectively behave as a Gaussian state (i.e., with zero symplectic rank), even though it formally has maximum symplectic rank. To account for this issue, we introduce in this section a new non-Gaussianity measure: the ε -approximate symplectic rank, defined as follows.

Definition 12 (ε -approximate symplectic rank). Let ρ be a quantum state and $\varepsilon \in (0, 1)$. The ε -approximate symplectic rank of ρ , denoted as $\mathfrak{s}_\varepsilon(\rho)$, is defined as the minimum symplectic rank among all the states that are ε -close to ρ in infidelity. In formula,

$$\mathfrak{s}_\varepsilon(\rho) := \min_{\substack{\tau \\ 1-F(\rho, \tau) \leq \varepsilon}} \mathfrak{s}(\tau). \quad (246)$$

Like for the symplectic rank, the fact that we can define the approximate symplectic rank as a min rather than an inf stems from the fact that it is integer-valued and bounded.

By definition of symplectic rank, it follows that for any state ρ and any $\varepsilon > 0$ there exists an $\mathfrak{s}_\varepsilon(\rho)$ -compressible state which is ε -close to ρ . Moreover, all the states that are ε -close to ρ cannot be compressed in a number of modes smaller than $\mathfrak{s}_\varepsilon(\rho)$. Additionally, in Definition 12, we used infidelity to quantify the distance between quantum states. Similar definitions can be formulated using other metrics that are nonincreasing under quantum channels, such as the trace distance.

A. Monotonicity of the approximate symplectic rank

We have seen that the symplectic rank is nonincreasing under postselected Gaussian operations. In the forthcoming Theorem 19, we show that the ε -approximate symplectic rank is nonincreasing under the smaller class of *Gaussian operations*. By definition, a Gaussian operation is a composition of the following five building blocks [28]:

- (a) tensoring with a Gaussian state;
- (b) applying a Gaussian unitary operation;
- (c) performing (non-Gaussian) classical mixing;
- (d) performing an heterodyne measurement;
- (e) taking a partial trace.

Theorem 19 (*Monotonicity of the ε -approximate symplectic rank*). For any $\varepsilon \in [0, 1]$, the ε -approximate symplectic rank is nonincreasing under Gaussian operations. That is, for any ρ and any Gaussian operation G , it holds that

$$\mathfrak{s}_\varepsilon(G(\rho)) \leq \mathfrak{s}_\varepsilon(\rho). \quad (247)$$

Proof. In Theorem 8, we have already proven that the symplectic rank is nonincreasing under postselected

Gaussian operations, and in particular under Gaussian operations. Let us generalize this result for $\varepsilon \in (0, 1]$.

We need to prove that for any $\varepsilon \in (0, 1]$, any ρ , and any Gaussian operation G , it holds that

$$\mathfrak{s}_\varepsilon(G(\rho)) \leq \mathfrak{s}_\varepsilon(\rho). \quad (248)$$

By definition of $\mathfrak{s}_\varepsilon(\rho)$, there exists τ such that $1 - F(\rho, \tau) \leq \varepsilon$ and $\mathfrak{s}_\varepsilon(\rho) = \mathfrak{s}(\tau)$. Hence, it holds that

$$\mathfrak{s}_\varepsilon(G(\rho)) \stackrel{(i)}{\leq} \mathfrak{s}(G(\tau)) \stackrel{(ii)}{\leq} \mathfrak{s}(\tau) \stackrel{(iii)}{=} \mathfrak{s}_\varepsilon(\rho). \quad (249)$$

Here, in (i), we exploited that the infidelity is nonincreasing under quantum channels [50,99] to observe that

$$1 - F(G(\rho), G(\tau)) \leq 1 - F(\rho, \tau) \leq \varepsilon, \quad (250)$$

and thus $G(\tau)$ is a feasible point in the minimum problem that defines $\mathfrak{s}_\varepsilon(G(\rho))$. In (ii), we exploited that the symplectic rank is nonincreasing under Gaussian operations, as a consequence of Theorem 8. Finally, in (iii), we used the definition of τ . ■

B. Relation with symplectic eigenvalues

We have seen that the symplectic rank of a pure state is given by the number of symplectic eigenvalues of its covariance matrix that are strictly larger than one. How this result can be generalized to the approximate setting? The forthcoming lemma provides such a generalization.

Lemma 15 (*A large number of symplectic eigenvalues far from one imply a large ε -approximate symplectic rank*). Let $\varepsilon \in [0, 1]$ and let ψ be a pure state over n modes. Let t be the number of symplectic eigenvalues of the covariance matrix of ψ that are larger than $1 + (2\varepsilon/n)$. Then, the ε -approximate symplectic rank of ρ is smaller or equal to t ,

$$\mathfrak{s}_\varepsilon(\psi) \leq t. \quad (251)$$

Proof. Let t be the number of symplectic eigenvalues of the covariance matrix of ψ that are larger than $1 + \varepsilon'$. The Williamson decomposition of the covariance matrix $V(\psi)$ is of the form

$$V(\psi) = S(D_t \oplus d_{n-t})S^\top, \quad (252)$$

where d_{n-t} is a $2(n-t)$ -dimensional diagonal matrix satisfying $\|d_{n-t} - \mathbb{1}\|_\infty \leq \varepsilon'$. Hence, it follows that there exists a Gaussian unitary operation G such that the covariance

matrix of $G^\dagger \psi G$ satisfies

$$V(G^\dagger \psi G) = D_t \oplus d_{n-t}, \quad (253)$$

and the first moment is zero

$$\mathbf{m}(G^\dagger \psi G) = 0. \quad (254)$$

In particular, the reduced state ρ on the last $n - t$ modes, defined as

$$\rho := \text{Tr}_{[t]} [G^\dagger \psi G], \quad (255)$$

satisfies $V(\rho) = d_{n-t}$ and $\mathbf{m}(\rho) = 0$. The idea now is that since d_{n-t} is close to the identity and since the only state with a covariance matrix equal to the identity is the vacuum state, the state ρ has to be close to the vacuum. By defining the t -mode state

$$|\phi\rangle := \frac{(\mathbb{1}_t \otimes \langle 0|^{\otimes(n-t)}) G^\dagger |\psi\rangle}{\sqrt{\langle \psi | G (\mathbb{1}_t \otimes |0\rangle\langle 0|^{\otimes(n-t)}) G^\dagger |\psi\rangle}}, \quad (256)$$

let us consider the t -compressible state $G(|\phi\rangle \otimes |0\rangle^{\otimes(n-t)})$. The infidelity between such a state and ψ satisfies

$$\begin{aligned} 1 - F(\psi, G(|\phi\rangle \otimes |0\rangle^{\otimes(n-t)}) G^\dagger) &= 1 - \text{Tr}[G^\dagger \psi G (|\phi\rangle \otimes |0\rangle^{\otimes(n-t)})] \\ &= 1 - \langle \psi | G (\mathbb{1}_t \otimes |0\rangle\langle 0|^{\otimes(n-t)}) G^\dagger |\psi\rangle \\ &= \text{Tr}[\rho (\mathbb{1} - |0\rangle\langle 0|^{\otimes(n-t)})] \\ &\stackrel{(i)}{\leq} \text{Tr}[\rho \hat{N}_{n-t}] \\ &\stackrel{(ii)}{=} \frac{\text{Tr}[V(\rho) - \mathbb{1}]}{4} + \frac{\|\mathbf{m}(\rho)\|}{2} \\ &= \frac{\text{Tr}[d_{n-t} - \mathbb{1}]}{4} \\ &\leq \frac{(n-t)\varepsilon'}{2} \\ &\leq \frac{n\varepsilon'}{2}. \end{aligned} \quad (257)$$

Here, in (i) we introduced the photon number operator on the last $n - t$ modes and we used the operator inequality $|0\rangle\langle 0|^{\otimes n} \geq \mathbb{1} - \hat{N}_{n-t}$, and in (ii) we exploited the general formula for the mean photon number of a state in Eq. (22). Hence, by choosing $\varepsilon' := 2\varepsilon/n$, we obtain that the t -compressible state $G(|\phi\rangle \otimes |0\rangle^{\otimes(n-t)})$ is ε -close to ψ , and in particular we conclude that $\mathfrak{s}_\varepsilon(\psi) \leq t$. ■

The above lemma establishes that if the number of symplectic eigenvalues of $V(\psi)$ that are sufficiently far away from one is t , then the approximate symplectic rank is

upper bounded by t . Can a similar lower bound exist? Or, equivalently, is it possible to express the approximate symplectic rank in terms of the number of symplectic eigenvalues that are sufficiently far away from one? Let us observe that this is not possible in general, as there exist pure quantum states which are very close but with very different symplectic eigenvalues. For example, the two states $|0\rangle$ and $\sqrt{1-\varepsilon^2}|0\rangle + \varepsilon|\lceil\varepsilon^{-1}\rceil\rangle$ are $O(\varepsilon)$ -close, but the symplectic eigenvalue associated with the former is equal to 1 while the symplectic eigenvalue associated with the latter is $2 + O(\varepsilon)$. Consequently, even if the symplectic eigenvalue of $\sqrt{1-\varepsilon^2}|0\rangle + \varepsilon|\lceil\varepsilon^{-1}\rceil\rangle$ is very far away from one, its $O(\varepsilon)$ -approximate symplectic rank is zero.

This reveals a peculiar discontinuity of symplectic eigenvalues: even if the state $\sqrt{1-\varepsilon^2}|0\rangle + \varepsilon|\lceil\varepsilon^{-1}\rceil\rangle$ converges (in trace distance) to $|0\rangle$ for $\varepsilon \rightarrow 0^+$, the symplectic eigenvalue of the former converges to 2 and the symplectic eigenvalue of the latter is exactly 1. This discontinuity arises because the second moment of the energy of the state $\sqrt{1-\varepsilon^2}|0\rangle + \varepsilon|\lceil\varepsilon^{-1}\rceil\rangle$ diverges as $\varepsilon \rightarrow 0^+$. Indeed, as a consequence of Theorem 24 in Sec. IX below, symplectic eigenvalues are continuous with respect to the state (in trace distance) for states with a *finite* second moment of the energy.

C. Symplectic fidelities

In this section, we introduce a notion of symplectic fidelities, similar to stellar fidelities for the stellar rank [72].

Definition 13 (k -symplectic fidelity). Given a quantum state ρ over n modes and $0 \leq k \leq n$, its k -symplectic fidelity is defined as

$$f_k^s(\rho) := \sup_{\substack{\tau: \\ \mathfrak{s}(\tau) \leq k}} F(\rho, \tau), \quad (258)$$

where the supremum is taken over all the states τ with symplectic rank of at most k , and F denotes the fidelity.

It is important to remark that, by definition, the k -symplectic fidelity equals 1 whenever $k \geq \mathfrak{s}(\rho)$, since we are considering the ball of states with symplectic rank less than k , which includes ρ itself.

In general, it is possible to connect symplectic fidelities and approximate symplectic rank as follows:

Lemma 16 (Equivalence between approximate symplectic rank and symplectic fidelities). Let ρ be a state over n modes. For all $k \leq n$ and all $0 \leq \varepsilon < 1$,

$$f_k^s(\rho) \geq 1 - \varepsilon \Leftrightarrow \forall \varepsilon' > \varepsilon, \mathfrak{s}_{\varepsilon'}(\rho) \leq k, \quad (259)$$

which implies, for all $1 \leq k < \mathfrak{s}(\rho)$,

$$\mathfrak{s}_\varepsilon(\rho) = \begin{cases} \mathfrak{s}(\rho) & \text{for } \varepsilon \in (0, 1 - f_{\mathfrak{s}(\rho)-1}^{\mathfrak{s}}(\rho)), \\ k & \text{for } \varepsilon \in (1 - f_k^{\mathfrak{s}}(\rho), 1 - f_{k-1}^{\mathfrak{s}}(\rho)), \\ 0 & \text{for } \varepsilon \in [1 - f_0^{\mathfrak{s}}(\rho), 1]. \end{cases} \quad (260)$$

Proof. Let us start by proving the following: for all $k \leq n$ and all $0 \leq \varepsilon \leq 1$,

$$\mathfrak{s}_\varepsilon(\rho) \leq k \Rightarrow f_k^{\mathfrak{s}}(\rho) \geq 1 - \varepsilon. \quad (261)$$

Indeed, if the approximate symplectic rank satisfies $\mathfrak{s}_\varepsilon(\rho) \leq k$, then there exists a state τ such that $1 - F(\rho, \tau) \leq \varepsilon$ and $\mathfrak{s}(\tau) \leq k$. By the definition of $\sup$, we have

$$f_k^{\mathfrak{s}}(\rho) \geq F(\rho, \tau) \geq 1 - \varepsilon. \quad (262)$$

Now by the definition of symplectic fidelities, assuming $f_k^{\mathfrak{s}}(\rho) \geq 1 - \varepsilon$ implies that for all $\varepsilon' > \varepsilon$, there exists a state τ such that $F(\rho, \tau) > 1 - \varepsilon'$ and $\mathfrak{s}(\tau) \leq k$, which also implies that the ε' -approximate symplectic rank of ρ is at most k for all $\varepsilon' > \varepsilon$. We thus obtain, for all $k \leq n$ and all $0 \leq \varepsilon \leq 1$,

$$\begin{aligned} f_k^{\mathfrak{s}}(\rho) \geq 1 - \varepsilon &\Rightarrow \forall \varepsilon' > \varepsilon, \mathfrak{s}_{\varepsilon'}(\rho) \leq k \\ &\Rightarrow \forall \varepsilon' > \varepsilon, f_k^{\mathfrak{s}}(\rho) \geq 1 - \varepsilon' \\ &\Rightarrow f_k^{\mathfrak{s}}(\rho) \geq 1 - \varepsilon, \end{aligned} \quad (263)$$

where we have used Eq. (261) in the second line. This shows that all implications are equivalences and completes the proof. ■

It is interesting to point out that if the state ρ is pure, the problem becomes simpler, as the symplectic fidelities can be computed via an optimization over Gaussian unitary operations.

Lemma 17 (Computing symplectic fidelities). Let ψ be a pure state over n modes. For all $k \leq n$, its symplectic fidelities can be expressed as

$$f_k^{\mathfrak{s}}(\psi) = \sup_G \text{Tr}[(\mathbb{1}_k \otimes |0\rangle\langle 0|^{\otimes(n-k)}) G^\dagger |\psi\rangle\langle\psi| G], \quad (264)$$

where the optimization is over all the possible Gaussian unitaries G .

Proof. For pure states, since the fidelity becomes a linear function and since the set of states of symplectic rank bounded by k is a convex set whose extreme points are pure states of symplectic rank bounded by k , it is sufficient to run the optimization over pure states. Moreover,

by definition of symplectic rank, a pure state τ satisfies $\mathfrak{s}(\tau) \leq k$ if and only if it can be written as

$$\tau = G(\phi \otimes |0\rangle\langle 0|^{\otimes(n-k)}) G^\dagger \quad (265)$$

for a suitable n -mode Gaussian unitary G and a suitable k -mode state ϕ . Hence, by introducing the k -mode state

$$|\phi_G\rangle := \frac{(\mathbb{1}_k \otimes |0\rangle\langle 0|^{\otimes(n-k)}) G^\dagger |\psi\rangle}{\sqrt{\text{Tr}[(\mathbb{1}_k \otimes |0\rangle\langle 0|^{\otimes(n-k)}) G^\dagger |\psi\rangle\langle\psi| G]}}, \quad (266)$$

the symplectic fidelity can be rewritten as

$$\begin{aligned} f_k^{\mathfrak{s}}(\psi) &= \sup_{G, \phi} F(\psi, G(\phi \otimes |0\rangle\langle 0|^{\otimes(n-k)}) G^\dagger) \\ &= \sup_{G, \phi} F(G^\dagger \psi G, \phi \otimes |0\rangle\langle 0|^{\otimes(n-k)}) \\ &= \sup_{G, \phi} \text{Tr}[(\mathbb{1}_k \otimes |0\rangle\langle 0|^{\otimes(n-k)}) G^\dagger |\psi\rangle\langle\psi| G] \\ &\quad \times F(\phi_G, \phi) \\ &= \sup_G \text{Tr}[(\mathbb{1}_k \otimes |0\rangle\langle 0|^{\otimes(n-k)}) G^\dagger |\psi\rangle\langle\psi| G], \end{aligned} \quad (267)$$

which concludes the proof. ■

In particular, $f_0^{\mathfrak{s}}$ is the fidelity with the closest Gaussian pure state. Together with Lemma 16, this result implies that the approximate symplectic rank can often be computed efficiently for a small number of modes, as it can be inferred from the list of symplectic fidelities, which are obtainable through the above optimization for systems with a small number of modes. Let us provide a specific example of computation of symplectic fidelities, for the two-mode state $|1\rangle^{\otimes 2}$. This state has symplectic rank equal to 2, so $f_n^{\mathfrak{s}}(|1\rangle^{\otimes 2}) = 1$ for $n \geq 2$.

For $n = 0$, Lemma 17 implies that the corresponding symplectic fidelity is equal to the fidelity with the closest Gaussian pure state, which is given by $f_0^{\mathfrak{s}}(|1\rangle^{\otimes 2}) = f_0^*(|1\rangle^{\otimes 2}) = 0.25$ [109] (an optimal approximation of symplectic rank 0 is a two-mode squeezed state).

For $n = 1$, Lemma 17 implies that the corresponding symplectic fidelity is given by

$$f_1^{\mathfrak{s}}(|1\rangle^{\otimes 2}) = \sup_G \langle 11 | G(\mathbb{1} \otimes |0\rangle\langle 0|) G^\dagger | 11 \rangle, \quad (268)$$

where the optimization is over 2-mode Gaussian unitaries. Writing (S, d) the symplectic matrix and displacement vector of a Gaussian unitary G , and computing the overlap via the trace rule for Wigner functions [90] we obtain

$$\begin{aligned} \langle 11|G(\mathbb{I} \otimes |0\rangle\langle 0|)G^\dagger|11\rangle &= \frac{2}{\pi^2} \int_{\mathbb{R}^4} dx_1 dp_1 dx_2 dp_2 (2(x_1^2 + p_1^2) - 1)(2(x_2^2 + p_2^2) - 1) \\ &\times \exp[-x_1^2 - p_1^2 - x_2^2 - p_2^2 - (d_3 + S_{31}x_1 + S_{32}p_1 + S_{33}x_2 + S_{34}p_2)^2 - (d_4 + S_{41}x_1 + S_{42}p_1 + S_{43}x_2 + S_{44}p_2)^2], \end{aligned} \quad (269)$$

where we used

$$\begin{aligned} W_{|0\rangle\langle 0|}(x, p) &= \frac{1}{\pi} e^{-x^2 - p^2}, \\ W_{|1\rangle\langle 1|}(x, p) &= \frac{1}{\pi} (2(x^2 + p^2) - 1) e^{-x^2 - p^2}, \\ W_{\mathbb{I}}(x, p) &= \frac{1}{2\pi}, \\ \text{Tr}[AB] &= (2\pi)^n \int_{\mathbb{R}^{2n}} d^n x d^n p W_A(x, p) W_B(x, p). \end{aligned} \quad (270)$$

Now setting $X = (x_1, p_1, x_2, p_2)^T$, $S_3 = (S_{31}, S_{32}, S_{33}, S_{34})$, and $S_4 = (S_{41}, S_{42}, S_{43}, S_{44})$, the exponent in Eq. (269) can be written as $-X^T A X - 2B^T X - C$, where $A := I + S_3 S_3^T + S_4 S_4^T$ (where I is the 4×4 identity matrix), $B := d_3 S_3 + d_4 S_4$ and $C := d_3^2 + d_4^2$. Completing the square, the integral rewrites as

$$\langle 11|G(\mathbb{I} \otimes |0\rangle\langle 0|)G^\dagger|11\rangle = \frac{2}{\pi^2} e^{B^T A^{-1} B - C} \int_{\mathbb{R}^4} d^4 X (2X_1^2 + 2X_2^2 - 1)(2X_3^2 + 2X_4^2 - 1) e^{-(X + A^{-1}B)^T A^{-1} (X + A^{-1}B)}. \quad (271)$$

Defining the covariance matrix $\Sigma := \frac{1}{2}A^{-1}$ and the mean vector $\mu := -A^{-1}B$, we obtain a linear combination of moments of the normal distribution $\mathcal{N}(\mu, \Sigma)$,

$$\begin{aligned} \langle 11|G(\mathbb{I} \otimes |0\rangle\langle 0|)G^\dagger|11\rangle \\ = N_0 \int_{X \leftarrow \mathcal{N}(\mu, \Sigma)} \mathbb{E} [(2X_1^2 + 2X_2^2 - 1)(2X_3^2 + 2X_4^2 - 1)], \end{aligned} \quad (272)$$

where $N_0 = 2/\pi^2 \int_{\mathbb{R}^4} d^4 X e^{-X^T A X - 2B^T X - C} d^4 \mathbf{X} = (2/\sqrt{\det A}) e^{B^T A^{-1} B - C}$ is a normalization factor. An expression amenable for numerical optimization is then obtained using Isserlis' theorem, which implies

$$\mathbb{E}[X_a^2 X_b^2] = (\Sigma_{aa} + \mu_a^2)(\Sigma_{bb} + \mu_b^2) + 2\Sigma_{ab}^2 + 4\mu_a \mu_b \Sigma_{ab}, \quad (273)$$

and

$$\mathbb{E}[X_a^2] = \Sigma_{aa} + \mu_a^2. \quad (274)$$

To perform the optimization of this expression over the set of 2-mode Gaussian unitaries, we parametrize the corresponding 4×4 symplectic matrix S via the Euler (Bloch-Messiah) decomposition [90] as

$$S = PK_1 DK_2 P, \quad (275)$$

where K_1 and K_2 are orthogonal symplectic matrices, where D is diagonal and where P is a permutation matrix,

with

$$K_1 = \begin{pmatrix} X_1 & -Y_1 \\ Y_1 & X_1 \end{pmatrix}, \quad (276)$$

$$D = \text{Diag}(e^{r_1}, e^{r_2}, e^{-r_1}, e^{-r_2}), \quad (277)$$

$$K_2 = \begin{pmatrix} X_2 & -Y_2 \\ Y_2 & X_2 \end{pmatrix}, \quad (278)$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (279)$$

Here $X_1 + iY_1$ and $X_2 + iY_2$ are 2×2 unitary matrices, which we both parametrize as

$$U(\phi_j, \psi_{1j}, \psi_{2j}, \theta_j) = e^{i\phi_j} \begin{pmatrix} e^{i\psi_{1j}} \cos \theta_j & e^{i\psi_{2j}} \sin \theta_j \\ -e^{-i\psi_{2j}} \sin \theta_j & e^{-i\psi_{1j}} \cos \theta_j \end{pmatrix}, \quad (280)$$

for $j = 1, 2$. Using this parametrization and optimizing over the ten free parameters $(r_j, \phi_j, \psi_{1j}, \psi_{2j}, \theta_j)_{j=1,2}$ (see Ref. [110] for a numerical implementation), we obtain

$f_1^s(|1\rangle^{\otimes 2}) = 0.50$ (an optimal approximation of symplectic rank 1 is the state $U_{BS}|20\rangle$, where U_{BS} is a balanced beam-splitter).

Interestingly, computing the symplectic fidelities of pure states can also inform us about the symplectic rank of some mixed states. For instance, consider the two-mode mixed quantum state $\frac{1}{2}|0\rangle\langle 0|^{\otimes 2} + \frac{1}{2}|1\rangle\langle 1|^{\otimes 2}$. This state has symplectic rank at most 2 by definition, but it also has symplectic rank at least 2, since its fidelity with the state $|1\rangle^{\otimes 2}$ is $0.5 \geq f_1^s(|1\rangle^{\otimes 2})$.

Let us emphasize some limitations of Lemma 17: while we ran extensive numerical search, the values $f_0^s(|1\rangle^{\otimes 2}) = \frac{1}{4}$ and $f_1^s(|1\rangle^{\otimes 2}) = \frac{1}{2}$ are obtained numerically via nonconvex optimizations, i.e., with no certificate of optimality. Moreover, for quantum states over more modes, we expect the optimization to be significantly more challenging. We leave a detailed study, based for instance on a direct Riemannian optimization over the set of Gaussian unitaries [111], to future work.

D. Approximate Gaussian conversion

We have seen that the approximate symplectic rank is nonincreasing under Gaussian operations. This implies that the approximate symplectic rank may also be used to give bounds on Gaussian protocols for non-Gaussian state conversion [36]: if $\mathcal{G}(\rho^{\otimes k}) = \sigma^{\otimes m}$ for some Gaussian protocol $\mathcal{G}$, then

$$\mathfrak{s}_\varepsilon(\sigma^{\otimes m}) \leq \mathfrak{s}_\varepsilon(\rho^{\otimes k}). \quad (281)$$

In this section, we show that we can extend these bounds to the case of approximate Gaussian conversion, similar to the approximate stellar rank [72]:

Theorem 20 (Gaussian conversion bounds based on the approximate symplectic rank). If $\mathcal{G}(\rho^{\otimes k})$ is δ -close in trace distance to $\sigma^{\otimes m}$, then

$$\mathfrak{s}_{\varepsilon+\sqrt{2\delta}}(\sigma^{\otimes m}) \leq \mathfrak{s}_\varepsilon(\rho^{\otimes k}), \quad (282)$$

for all $\varepsilon > 0$. For pure states $\rho = \Psi$ and $\sigma = \Phi$ this can be tightened as

$$\mathfrak{s}_{\varepsilon+\delta}(\Phi^{\otimes m}) \leq \mathfrak{s}_\varepsilon(\Psi^{\otimes k}). \quad (283)$$

These bounds can be phrased in terms of symplectic fidelities as

$$f_n^s(\rho^{\otimes k}) \leq f_n^s(\sigma^{\otimes m}) + \sqrt{2\delta}, \quad (284)$$

for all $n \in \mathbb{N}$, and, for pure states $\rho = \Psi$ and $\sigma = \Phi$,

$$f_n^s(\Psi^{\otimes k}) \leq f_n^s(\Phi^{\otimes m}) + \delta. \quad (285)$$

Moreover, when allowing for postselection, if $\mathcal{G}(\rho^{\otimes k})$ is δ -close in trace distance to $\sigma^{\otimes m}$ with any nonzero probability, then for all $\varepsilon > 0$,

$$\mathfrak{s}_{\sqrt{2\delta}}(\sigma^{\otimes m}) \leq \mathfrak{s}(\rho^{\otimes k}), \quad (286)$$

and for pure states,

$$\mathfrak{s}_\delta(\Phi^{\otimes m}) \leq \mathfrak{s}(\Psi^{\otimes k}) = k\mathfrak{s}(\Psi). \quad (287)$$

Proof. Let ω be a density operator which can be obtained from $\rho^{\otimes k}$ using Gaussian operations, and such that $\frac{1}{2}\|\omega - \sigma^{\otimes m}\|_1 \leq \delta$. We make use the following result [Lemma 5 of Ref. [112]]:

Lemma 18. For density operators ρ, σ and τ , we have

$$|F(\rho, \tau) - F(\sigma, \tau)| \leq \sqrt{\|\rho - \sigma\|_1}. \quad (288)$$

Moreover, if $\rho = \Psi$ and $\sigma = \Phi$ are pure states,

$$|F(\Psi, \tau) - F(\Phi, \tau)| \leq \frac{1}{2}\|\Psi - \Phi\|_1. \quad (289)$$

Lemma 18 implies that the set $\{\tau | F(\sigma^{\otimes m}, \tau) \geq 1 - (\varepsilon + \sqrt{2\delta})\}$ contains the set $\{\tau | F(\omega, \tau) \geq 1 - \varepsilon\}$, so

$$\begin{aligned} \mathfrak{s}_{\varepsilon+\sqrt{2\delta}}(\sigma^{\otimes m}) &= \min_{\tau: F(\sigma^{\otimes m}, \tau) \geq 1 - (\varepsilon + \sqrt{2\delta})} \mathfrak{s}(\tau) \\ &\leq \min_{\tau: F(\omega, \tau) \geq 1 - \varepsilon} \mathfrak{s}(\tau) \\ &= \mathfrak{s}_\varepsilon(\omega). \end{aligned} \quad (290)$$

By the exact Gaussian conversion bound in Eq. (281), we have $\mathfrak{s}_\varepsilon(\rho^{\otimes k}) \geq \mathfrak{s}_\varepsilon(\omega)$, which together with Eq. (290) implies the conversion bound in Eq. (282): if m copies of σ can be obtained from k copies of ρ using Gaussian operations, then for all $0 \leq \varepsilon \leq 1$,

$$\mathfrak{s}_\varepsilon(\rho^{\otimes k}) \geq \mathfrak{s}_{\varepsilon+\sqrt{2\delta}}(\sigma^{\otimes m}). \quad (291)$$

We now rephrase this bound in terms of symplectic fidelities using Lemma 16. Since the approximate symplectic rank is integer-valued, this is equivalent to

$$\mathfrak{s}_\varepsilon(\rho^{\otimes k}) \leq n \Rightarrow \mathfrak{s}_{\varepsilon+\sqrt{2\delta}}(\sigma^{\otimes m}) \leq n, \quad (292)$$

for all $n \in \mathbb{N}$, or equivalently, if $\mathcal{G}(\rho^{\otimes k})$ is δ -close in trace distance to $\sigma^{\otimes m}$, then

$$\mathfrak{s}_\varepsilon(\rho^{\otimes k}) \leq n \Rightarrow \mathfrak{s}_{\varepsilon+\sqrt{2\delta}}(\sigma^{\otimes m}) \leq n, \quad (293)$$

for all $n \in \mathbb{N}$ and all $0 \leq \varepsilon \leq 1$.

Now, by Lemma 16, for all $n \in \mathbb{N}$ and all $0 \leq \varepsilon < 1$,

$$1 - f_n^s(\rho^{\otimes k}) < \varepsilon \Rightarrow \mathfrak{s}_\varepsilon(\rho^{\otimes k}) \leq n, \quad (294)$$

and by Eq. (261), for all $n \in \mathbb{N}$ and all $0 \leq \varepsilon < 1 - \sqrt{2\delta}$,

$$\mathfrak{s}_{\varepsilon+\sqrt{2\delta}}(\sigma^{\otimes m}) \leq n \Rightarrow 1 - f_n^s(\sigma^{\otimes m}) \geq \varepsilon + \sqrt{2\delta}. \quad (295)$$

Hence, if $\mathcal{G}(\rho^{\otimes k})$ is δ -close in trace distance to $\sigma^{\otimes m}$, then for all $n \in \mathbb{N}$ and all $0 \leq \varepsilon < 1 - \sqrt{2\delta}$ the following chain of implications holds:

$$\begin{aligned} 1 - f_n^s(\rho^{\otimes k}) < \varepsilon &\Rightarrow \mathfrak{s}_\varepsilon(\rho^{\otimes k}) \leq n \\ &\Rightarrow \mathfrak{s}_{\varepsilon+\sqrt{2\delta}}(\sigma^{\otimes m}) \leq n \\ &\Rightarrow 1 - f_n^s(\sigma^{\otimes m}) \geq \varepsilon + \sqrt{2\delta}, \end{aligned} \quad (296)$$

where we used Eq. (294) in the first line, Eq. (293) in the second line, and Eq. (295) in the last line. This implies $1 - f_n^s(\rho^{\otimes k}) \geq 1 - f_n^s(\sigma^{\otimes m}) - \sqrt{2\delta}$ for all $n \in \mathbb{N}$, and thus

$$f_n^s(\rho^{\otimes k}) \leq f_n^s(\sigma^{\otimes m}) + \sqrt{2\delta}, \quad (297)$$

for all $n \in \mathbb{N}$, completing the proof of Eq. (284).

Finally, the proof of Eq. (286) is analogous, using the fact that the exact symplectic rank is nonincreasing under postselected Gaussian operations (see Theorem 8). Moreover, the proofs are identical when $\rho = \Psi$ and $\sigma = \Phi$ are pure states, by replacing $\sqrt{2\delta}$ with δ . ■

IX. PERTURBATION BOUNDS FOR CONTINUOUS-VARIABLE QUANTUM STATES

This section is devoted to establishing lower bounds on the trace distance between two (possibly non-Gaussian) states in terms of the norm distance of their first moments and covariance matrices, which we regard as new technical tools of independent interest for the field of continuous-variable quantum information. These new bounds are summarized in the forthcoming Theorem 21, which is proven below in Sec. IX C.

Theorem 21 (Lower bounds on the trace distance between non-Gaussian states). Let ρ and σ be (possibly non-Gaussian) quantum states. Let $\mathbf{m}(\rho), V(\rho)$ be the first moment and the covariance matrix of ρ , and let $\mathbf{m}(\sigma), V(\sigma)$ be the first moment and the covariance matrix of σ . Then, the trace distance between ρ and σ can be

lower bounded as follows:

$$\|\rho - \sigma\|_1 \geq \frac{\|\mathbf{m}(\sigma) - \mathbf{m}(\rho)\|^2}{16 \max\left(\text{Tr}[\hat{E}\rho], \text{Tr}[\hat{E}\sigma]\right)}, \quad (298)$$

$$\|\rho - \sigma\|_1 \geq \frac{\|V(\rho) - V(\sigma)\|_\infty^2}{1549 \max(\text{Tr}[\hat{E}^2\rho], \text{Tr}[\hat{E}^2\sigma])}. \quad (299)$$

Here, $\hat{E}$ denotes the energy operator; given a matrix V , $|V| := \sqrt{V^\dagger V}$ denotes its modulus, $\|V\|_\infty$ denotes the operator norm (i.e., the maximum singular values), and $\|V\|_1 := \text{Tr}|V|$ denotes the trace norm. Furthermore, given a vector $\mathbf{m}$, $\|\mathbf{m}\| := \sqrt{\mathbf{m}^\top \mathbf{m}}$ denotes its Euclidean norm.

The bounds stated in Theorem 21 contribute to the rapidly growing literature regarding *bosonic trace distance bounds* [34,51,52,54,113,114]. The latter constitute a technical toolbox of independent interest [51,52]. Indeed, they can be useful, e.g., in analyzing tomography of Gaussian states [34], testing properties of Gaussian systems [114], learning Gaussian processes [115], and characterizing quantum Gaussian observables [51].

A. Bosonic trace distance bounds

Let us give a brief literature review regarding bosonic trace distance bounds [34,51,52,54,113]. A central question in this line of research is how to bound the trace distance between two continuous-variable quantum states ρ and σ in terms of the distance between their first moments, $\|\mathbf{m}(\rho) - \mathbf{m}(\sigma)\|$, and their covariance matrices, $\|V(\rho) - V(\sigma)\|$. This question is of fundamental importance because, although the trace distance plays a key role in all the areas of quantum information science [116], it is generally hard to compute for continuous-variable systems. In contrast, first moments and covariance matrices are typically much easier to estimate, both analytically and experimentally. The answer to this question depends critically on the assumptions made about the states ρ and σ ; various bounds have been derived under different assumptions regarding the Gaussianity or the purity of the states:

- (a) **Upper bound on the trace distance between Gaussian states:** It is well known that Gaussian states are uniquely determined by their first moments and covariance matrices. This naturally raises the fundamental question: *If two Gaussian states have first moments and covariance matrices that are ε -close, how close are the states in trace distance?* This problem was first introduced in Ref. [34], where partial results were obtained. Later, Holevo [51] and Fanizza *et al.* [54] made further progress on this problem. However, neither the findings of Ref. [34] nor those of Holevo [51] and

Fanizza *et al.* [54] have fully resolved the question of establishing a tight upper bound on the trace distance. Remarkably, this question has been recently fully solved by Ref. [53], where the following tight bound has been established:

Let ρ be the Gaussian state with first moment $\mathbf{m}$ and covariance matrix V . Similarly, let σ be the Gaussian state with first moment $\mathbf{t}$ and covariance matrix W . Then, the following trace norm between ρ and σ satisfies the following tight upper bounds [53]:

$$\|\rho - \sigma\|_1 \leq \frac{1 + \sqrt{3}}{4} \text{Tr} \left[|V - W| \Omega^\top \left(\frac{V + W}{2} \right) \Omega \right] + \sqrt{2 \min(\|V\|_\infty, \|W\|_\infty)} \|\mathbf{m} - \mathbf{t}\|. \quad (300)$$

Moreover, note that Hölder's inequality implies that the term $\text{Tr} \left[|V - W| \Omega^\top \left(\frac{V + W}{2} \right) \Omega \right]$ in Eq. (300) can be further upper bounded either by $\max(\|V\|_\infty, \|W\|_\infty) \|V - W\|_1$, or by $\max(\text{Tr} V, \text{Tr} W) \|V - W\|_\infty$. In the fermionic setting [114, 117], a similar inequality has been established first in the particular case of pure states [117] and later in the general case of mixed states [114].

- (b) **Lower bound on the trace distance between Gaussian states:** In Eq. (S265) of Ref. [34] the following lower bound on the trace distance between Gaussian states was established:

Let ρ be the Gaussian state with first moment $\mathbf{m}$ and covariance matrix V . Similarly, let σ be the Gaussian state with first moment $\mathbf{t}$ and covariance matrix W . Then, the trace distance between ρ and σ can be lower bounded as Eq. (S265) of Ref. [34]:

$$\begin{aligned} \|\rho - \sigma\|_1 &\geq \frac{1}{100} \min \left\{ 1, \frac{\|\mathbf{m} - \mathbf{t}\|}{\sqrt{4 \min(\|V\|_\infty, \|W\|_\infty) + 1}} \right\}, \\ \|\rho - \sigma\|_1 &\geq \frac{1}{100} \min \left\{ 1, \frac{\|V - W\|_2}{4 \min(\|V\|_\infty, \|W\|_\infty) + 1} \right\}. \end{aligned} \quad (301)$$

- (c) **Upper bound on the trace distance between a non-Gaussian state and a pure Gaussian state:** In Ref. [34] the following upper bound on the trace distance between a non-Gaussian state and a pure Gaussian state was established:

Let ψ be a pure Gaussian state and let ρ be a (possibly non-Gaussian) state. Then, the trace distance between ψ and ρ can be upper bounded as [34]

$$\frac{1}{2} \|\rho - \psi\|_1 \leq \sqrt{\max \left(\text{Tr} [\hat{E} \rho], \text{Tr} [\hat{E} \sigma] \right)} \sqrt{2 \|\mathbf{m}(\rho) - \mathbf{m}(\psi)\|^2 + \|V(\rho) - V(\psi)\|_\infty}, \quad (302)$$

where $\hat{E}$ denotes the energy operator.

- (d) **Lower bound on the trace distance between non-Gaussian states:** In this work—specifically in Theorem 21—we establish lower bounds on the trace distance between two possibly non-Gaussian states ρ and σ ,

$$\begin{aligned} \|\rho - \sigma\|_1 &\geq \frac{\|\mathbf{m}(\sigma) - \mathbf{m}(\rho)\|^2}{16 \max \left(\text{Tr} [\hat{E} \rho], \text{Tr} [\hat{E} \sigma] \right)}, \\ \|\rho - \sigma\|_1 &\geq \frac{\|V(\rho) - V(\sigma)\|_\infty^2}{1549 \max(\text{Tr} [\hat{E}^2 \rho], \text{Tr} [\hat{E}^2 \sigma])}. \end{aligned} \quad (303)$$

It is worth noting that this trace distance lower bound scales quadratically with the norm distance between the covariance matrices, in contrast to the lower bound for Gaussian states in Theorem 301, which exhibits linear scaling. Nevertheless, this quadratic behavior is shown to be tight, as demonstrated in Theorem 23 below.

Another related result is presented in Ref. [113], where an **algorithm to compute the trace distance between two Gaussian states up to a fixed precision** has been developed. The construction of this algorithm relies on two key steps:

- (a) First, Ref. [113] derives a simple bound on the probability $P_{>M}$ that a given Gaussian state has more than M photons, establishing that such a probability decays exponentially in M . The result is as follows [113]:

Let ρ be an n -mode Gaussian state, and let $P_{>M}$ denote the probability that ρ has more than M photons. According to Born's rule, this probability is defined by

$$P_{>M} := \text{Tr}[(\mathbb{1} - \Pi_M)\rho], \quad (304)$$

where Π_M is the projector onto the subspace spanned by all n -mode Fock states with total photon number less than or equal to M :

$$\Pi_M := \sum_{\mathbf{k} \in \mathbb{N}^n: \sum_{i=1}^n k_i \leq M} |\mathbf{k}\rangle\langle\mathbf{k}|, \quad (305)$$

with $|\mathbf{k}\rangle = |k_1\rangle \otimes |k_2\rangle \otimes \dots \otimes |k_n\rangle$ denoting an n -mode Fock state. Then, the following bound holds:

$$P_{>M} \leq 2^n e^{1/2} e^{-M/(4\text{Tr}[\rho\hat{N}]+2)}, \quad (306)$$

where $\hat{N}$ is the photon number operator.

- (b) Second, as an application of this bound, Ref. [113] establishes an upper bound on the trace distance between the Gaussian state ρ and its finite-dimensional truncation ρ_M , defined as

$$\rho_M := \frac{\Pi_M \rho \Pi_M}{\text{Tr}[\Pi_M \rho \Pi_M]}, \quad (307)$$

where Π_M is the projector given in Eq. (305). The trace distance between ρ and ρ_M is shown to vanish exponentially in M as follows:

$$\frac{1}{2} \|\rho - \rho_M\|_1 \leq 2^{n/2} e^{1/4} e^{-M/(8\text{Tr}[\rho\hat{N}]+4)}. \quad (308)$$

These results play a key role in constructing the aforementioned algorithm that computes the trace distance between two Gaussian states up to a fixed precision [113].

B. Trace distance perturbation bounds on first moment, covariance matrices, and symplectic eigenvalues

The trace distance bounds of Theorem 21 can be equivalently seen as continuity bounds on the first moments and the covariance matrices. In other words, we are answering the following fundamental question: If two states are ε -close in trace distance, how close are their first moments and covariance matrices? More specifically, given two states ρ and σ , how to upper bound the norm distance

between their first moments $\|\mathbf{m}(\rho) - \mathbf{m}(\sigma)\|$ and their covariance matrices $\|V(\rho) - V(\sigma)\|$ in terms of the trace distance $\frac{1}{2}\|\rho - \sigma\|_1$? The forthcoming Theorem 22 and the forthcoming Theorem 23 provide such bounds for the first moments and covariance matrices, respectively. The proof of these theorems are direct consequences of the above Theorem 21.

Theorem 22 (Trace distance perturbation bounds on first moments). Let ρ and σ be two quantum states with mean energy upper bounded by a finite constant E , that is $\text{Tr}[\hat{E}\rho] \leq E$ and $\text{Tr}[\hat{E}\sigma] \leq E$. Then, the Euclidean distance between their first moments can be upper bounded in terms of the trace distance as

$$\|\mathbf{m}(\rho) - \mathbf{m}(\sigma)\| \leq 4\sqrt{E\|\rho - \sigma\|_1}, \quad (309)$$

where norm $\|\cdot\|$ is defined as $\|\mathbf{v}\| := \sqrt{\mathbf{v}^\top \mathbf{v}}$.

Theorem 23 (Trace distance perturbation bounds on covariance matrices). Let ρ and σ be two quantum states with second moment of the energy upper bounded by a finite constant E , that is $\text{Tr}[\hat{E}^2\rho] \leq E^2$ and $\text{Tr}[\hat{E}^2\sigma] \leq E^2$. Then, the operator norm distance between their covariance matrices can be upper bounded in terms of the trace distance as

$$\|V(\rho) - V(\sigma)\|_\infty \leq 40E\sqrt{\|\rho - \sigma\|_1}. \quad (310)$$

The above theorem implies that if two energy-constrained states ρ and σ are ε -close in trace distance, then their covariance matrices are at most $O(\sqrt{\varepsilon})$ far apart. Moreover, the bound in Eq. (310) is tight in the scaling with the second moment of the energy E and the trace norm $\|\rho - \sigma\|_1$. To show this, let us consider the following example.

Example 1 (Tightness in the scaling of the parameters in Theorem 23). For any $n \in \mathbb{N}$, let us consider the state

$$\rho_n := \left(1 - \frac{1}{n}\right) |0\rangle\langle 0| + \frac{1}{n} |n^2\rangle\langle n^2|, \quad (311)$$

which satisfies

$$\|V(\rho_n) - V(|0\rangle\langle 0|)\|_\infty = 2n, \quad (312)$$

$$E_n := \sqrt{\text{Tr}[\hat{E}^2\rho_n]} = \sqrt{\frac{1}{n} \left(n^2 + \frac{1}{2}\right)}, \quad (313)$$

$$\|\rho_n - |0\rangle\langle 0|\|_1 = \frac{2}{n}. \quad (314)$$

The above identities imply that

$$\frac{\|V(\rho_n) - V(|0\rangle\langle 0|)\|_\infty}{E_n \sqrt{\|\rho_n - |0\rangle\langle 0|\|_1}} \propto 1 + O(n^{-1}), \quad (315)$$

establishing the tightness of the bound of Theorem 23 in terms of the scaling of all the parameters involved.

Before proving the above Theorems 22 and 23, let us see the following interesting application, which turned out to be crucial in the analysis of the symplectic rank in Sec. VI (specifically, in the proof of Theorem 11).

Theorem 24 (Trace distance perturbation bound on symplectic eigenvalues). Given a quantum state ρ , let us denote as $v_i^\downarrow(\rho)$ the i th largest symplectic eigenvalue of its covariance matrix $V(\rho)$. Let ρ and σ be two quantum states with second moment of the energy upper bounded by a finite constant E , that is $\text{Tr}[\hat{E}^2\rho] \leq E^2$ and $\text{Tr}[\hat{E}^2\sigma] \leq E^2$. Then, it holds that

$$\max_i |v_i^\downarrow(\rho) - v_i^\downarrow(\sigma)| \leq 640E^2 \sqrt{\|\rho - \sigma\|_1}. \quad (316)$$

Before proving such a theorem, we need to establish some preliminary results. Let us start by stating the following known perturbation bound on the symplectic diagonalization.

Lemma 19 (Perturbation on symplectic diagonalization [118]). Let $V_1, V_2 \in \mathbb{R}^{2n \times 2n}$ be two strictly positive matrices with symplectic diagonalizations $V_1 = S_1 D_1 S_1^\top$ and $V_2 = S_2 D_2 S_2^\top$, where the elements on the diagonal of D_1 and D_2 are arranged in descending order. Then, the operator norm distance between D_1 and D_2 can be upper bounded in terms of the operator norm distance between V_1 and V_2 as follows:

$$\|D_1 - D_2\|_\infty \leq \sqrt{K(V_1)K(V_2)} \|V_1 - V_2\|_\infty, \quad (317)$$

where $K(V)$ denotes the condition number of the matrix V , defined as $K(V) := \|V\|_\infty \|V^{-1}\|_\infty$.

Moreover, the condition number of a covariance matrix can be upper bounded as follows.

Lemma 20 (Bounding the condition number of the covariance matrix in terms of the energy). Let ρ be a quantum state with covariance matrix $V(\rho)$. The condition number

$$K(V(\rho)) := \|V(\rho)\|_\infty \|V^{-1}(\rho)\|_\infty \quad (318)$$

can be upper bounded in terms of the second moment of the energy as

$$K(V(\rho)) \leq 16\text{Tr}[\hat{E}^2\rho]. \quad (319)$$

Proof. It holds that

$$\begin{aligned} K(V(\rho)) &\stackrel{(i)}{\leq} \|V(\rho)\|_\infty^2 \\ &\stackrel{(ii)}{\leq} 16 \left(\text{Tr}[\hat{E}\rho] \right)^2 \\ &\stackrel{(iii)}{\leq} 16\text{Tr}[\hat{E}^2\rho], \end{aligned} \quad (320)$$

where, in (i), we exploited Lemma S77 of Ref. [34]; in (ii) we leveraged Eq. (23); and, in (iii), we used the concavity of the square root function. ■

We are now ready to prove Theorem 24.

Proof of Theorem 24. It holds that

$$\begin{aligned} \max_i |v_i^\downarrow(\rho) - v_i^\downarrow(\sigma)| &\stackrel{(i)}{\leq} \sqrt{K(V(\rho))K(V(\sigma))} \|V(\rho) - V(\sigma)\|_\infty \\ &\stackrel{(ii)}{\leq} 16E^2 \|V(\rho) - V(\sigma)\|_\infty \\ &\stackrel{(iii)}{\leq} 640E^2 \sqrt{\|\rho - \sigma\|_1}, \end{aligned} \quad (321)$$

where, in (i), we applied Lemma 19; in (ii), we exploited Lemma 20, and, in (iii), we leveraged Theorem 23. ■

C. Proofs of the trace distance lower bounds

Let us now proceed with the proof of the above-stated Theorems 22 and 23. Let us start by establishing the following lower bound on the trace distance between two quantum states in terms of the difference of their expectation values of a quadrature operator $\hat{x}$ (i.e., a position operator or a momentum operator) and in terms of the difference of their expectation values of $\hat{x}^2$.

Lemma 21 (Lower bound on the trace distance in terms of the mean value of a quadrature operator). Let ρ and σ be quantum states. Let $\hat{x}$ be a quadrature operator associated with a given mode. Then, the trace norm between ρ and σ can be lower bounded in terms of the expectation value of $\hat{x}$ as

$$\|\rho - \sigma\|_1 \geq \frac{|\text{Tr}[\hat{x}(\rho - \sigma)]|^2}{8 \max(\text{Tr}[\hat{x}^2\rho], \text{Tr}[\hat{x}^2\sigma])}, \quad (322)$$

and in terms of the second moment as

$$\|\rho - \sigma\|_1 \geq \frac{|\text{Tr}[\hat{x}^2(\rho - \sigma)]|^2}{4 \max(\text{Tr}[\hat{x}^4\rho], \text{Tr}[\hat{x}^4\sigma])}. \quad (323)$$

Proof. Let us start with the proof of Eq. (322). The trace norm can be expressed as follows [119]:

$$\|\rho - \sigma\|_1 = \max_{\|\Theta\|_\infty \leq 1} |\text{Tr}[\Theta(\rho - \sigma)]|, \quad (324)$$

where the maximum is over all operators with operator norm upper bounded by one. Let us consider the quadrature operator $\hat{x}$ in spectral decomposition,

$$\hat{x} = \int_{x \in \mathbb{R}} x |x\rangle \langle x| dx, \quad (325)$$

where $|x\rangle$ denotes the (generalized) eigenvectors of $\hat{x}$ [60]. By defining for all $\bar{x} > 0$ the operator

$$\Theta_{\bar{x}} := \frac{1}{\bar{x}} \int_{|x| \leq \bar{x}} dx x |x\rangle \langle x|, \quad (326)$$

we have that $\|\Theta_{\bar{x}}\|_\infty = 1$ and thus

$$\begin{aligned} \|\rho - \sigma\|_1 &\geq |\text{Tr}[\Theta_{\bar{x}}(\rho - \sigma)]| \\ &\geq \text{Tr}[\Theta_{\bar{x}}(\rho - \sigma)] \\ &= \frac{1}{\bar{x}} \text{Tr} \left[\int_{|x| \leq \bar{x}} dx x |x\rangle \langle x| (\rho - \sigma) \right] \\ &= \frac{1}{\bar{x}} \text{Tr} [\hat{x}(\rho - \sigma)] - \frac{1}{\bar{x}} \text{Tr} \left[\int_{|x| \geq \bar{x}} dx x |x\rangle \langle x| (\rho - \sigma) \right] \\ &\geq \frac{1}{\bar{x}} \text{Tr} [\hat{x}(\rho - \sigma)] - \frac{2}{\bar{x}} \max \left(\left| \text{Tr} \left[\int_{|x| \geq \bar{x}} dx x |x\rangle \langle x| \rho \right] \right|, \left| \text{Tr} \left[\int_{|x| \geq \bar{x}} dx x |x\rangle \langle x| \sigma \right] \right| \right). \end{aligned} \quad (327)$$

Moreover, by exploiting that

$$\begin{aligned} - \int_{|x| \geq \bar{x}} dx x |x\rangle \langle x| &\leq \int_{|x| \geq \bar{x}} dx x |x\rangle \langle x| \\ &\leq \int_{|x| \geq \bar{x}} dx x |x\rangle \langle x|, \end{aligned} \quad (328)$$

it follows that

$$\begin{aligned} \left| \text{Tr} \left[\int_{|x| \geq \bar{x}} dx x |x\rangle \langle x| \rho \right] \right| &\leq \text{Tr} \left[\int_{|x| \geq \bar{x}} dx x |x\rangle \langle x| \rho \right] \\ &\leq \frac{1}{\bar{x}} \text{Tr} \left[\int_{|x| \geq \bar{x}} dx x^2 |x\rangle \langle x| \rho \right] \\ &\leq \frac{1}{\bar{x}} \text{Tr} \left[\int_{x \in \mathbb{R}} dx x^2 |x\rangle \langle x| \rho \right] \\ &= \frac{1}{\bar{x}} \text{Tr} [\hat{x}^2 \rho]. \end{aligned} \quad (329)$$

Consequently, by substituting in Eq. (327), we have that

$$\begin{aligned} \|\rho - \sigma\|_1 &\geq \frac{1}{\bar{x}} \text{Tr} [\hat{x}(\rho - \sigma)] \\ &\quad - \frac{2}{\bar{x}^2} \max (\text{Tr} [\hat{x}^2 \rho], \text{Tr} [\hat{x}^2 \sigma]). \end{aligned} \quad (330)$$

By exchanging the role of ρ and σ , we also obtain that

$$\begin{aligned} \|\rho - \sigma\|_1 &\geq -\frac{1}{\bar{x}} \text{Tr} [\hat{x}(\rho - \sigma)] \\ &\quad - \frac{2}{\bar{x}^2} \max (\text{Tr} [\hat{x}^2 \rho], \text{Tr} [\hat{x}^2 \sigma]), \end{aligned} \quad (331)$$

and thus also that

$$\begin{aligned} \|\rho - \sigma\|_1 &\geq \frac{1}{\bar{x}} |\text{Tr} [\hat{x}(\rho - \sigma)]| \\ &\quad - \frac{2}{\bar{x}^2} \max (\text{Tr} [\hat{x}^2 \rho], \text{Tr} [\hat{x}^2 \sigma]). \end{aligned}$$

By optimizing over $\bar{x}$, we get

$$\|\rho - \sigma\|_1 \geq \frac{|\text{Tr} [\hat{x}(\rho - \sigma)]|^2}{8 \max (\text{Tr} [\hat{x}^2 \rho], \text{Tr} [\hat{x}^2 \sigma])}. \quad (332)$$

This concludes the proof of Eq. (322).

Now, let us proceed with the proof of Eq. (323). Note that $\hat{x}^2$ can be written in spectral decomposition as

$$\hat{x}^2 = \int_{-\infty}^{\infty} x^2 |x\rangle \langle x| dx. \quad (333)$$

By defining for all $\bar{x} > 0$ the operator

$$\Theta_{\bar{x}}^2 := \frac{1}{\bar{x}^2} \int_{|x| \leq \bar{x}} x^2 |x\rangle \langle x| dx, \quad (334)$$

we have that $\|\Theta_{\bar{x}}^2\|_{\infty} = 1$ and thus

$$\begin{aligned}
 \|\rho - \sigma\|_1 &\geq \text{Tr}[\Theta_{\bar{x}}^2(\rho - \sigma)] \\
 &= \frac{1}{\bar{x}^2} \left(\text{Tr}[\hat{x}^2(\rho - \sigma)] - \text{Tr} \left[\int_{|x| \geq \bar{x}} x^2 |x\rangle \langle x| dx (\rho - \sigma) \right] \right) \\
 &\geq \frac{1}{\bar{x}^2} \left(\text{Tr}[\hat{x}^2(\rho - \sigma)] - \text{Tr} \left[\int_{|x| \geq \bar{x}} x^2 |x\rangle \langle x| dx \rho \right] \right) \\
 &\geq \frac{1}{\bar{x}^2} \left(\text{Tr}[\hat{x}^2(\rho - \sigma)] - \frac{1}{\bar{x}^2} \text{Tr} \left[\int_{|x| \geq \bar{x}} x^4 |x\rangle \langle x| dx \rho \right] \right) \\
 &\geq \frac{1}{\bar{x}^2} \text{Tr}[\hat{x}^2(\rho - \sigma)] - \frac{1}{\bar{x}^4} \text{Tr}[\hat{x}^4 \rho] \\
 &\geq \frac{1}{\bar{x}^2} \text{Tr}[\hat{x}^2(\rho - \sigma)] - \frac{1}{\bar{x}^4} \max(\text{Tr}[\hat{x}^4 \rho], \text{Tr}[\hat{x}^4 \sigma]).
 \end{aligned} \tag{335}$$

Exchanging ρ and σ , we obtain the lower bound

$$\begin{aligned}
 \|\rho - \sigma\|_1 &\geq \frac{1}{\bar{x}^2} |\text{Tr}[\hat{x}^2(\rho - \sigma)]| \\
 &\quad - \frac{1}{\bar{x}^4} \max(\text{Tr}[\hat{x}^4 \rho], \text{Tr}[\hat{x}^4 \sigma]).
 \end{aligned} \tag{336}$$

By optimizing over $\bar{x}$, we conclude that

$$\|\rho - \sigma\|_1 \geq \frac{|\text{Tr}[\hat{x}^2(\rho - \sigma)]|^2}{4 \max(\text{Tr}[\hat{x}^4 \rho], \text{Tr}[\hat{x}^4 \sigma])}. \tag{337}$$

■

The bound of the above lemma exhibits a dependence upon the expectation value of the observable $\hat{x}^4$. In the next lemma, we show how to upper bound the latter in terms of the energy.

Lemma 22 (Bounding $\hat{x}^4$ in terms of $\hat{E}^2$). Let $\hat{x}$ and $\hat{E}$ be the position operator and the energy operator, respectively, of a single-mode system. Then, the following operator inequality holds:

$$\hat{x}^4 \leq (3 + 6\sqrt{2} + 2\sqrt{6}) \hat{E}^2. \tag{338}$$

Proof. The goal is to find α such that $\hat{x}^4 \leq \alpha \hat{E}^2$. To this end, the crucial idea is to exploit the *diagonally dominant criterion*: if a Hermitian matrix $A = (a_{ij})_{i,j}$ is such that

$$a_{nn} \geq \sum_{m: m \neq n} |a_{mn}| \quad \forall n, \tag{339}$$

then necessarily A is positive semidefinite [Chap. 6 of Ref. [120]]. Hence, the operator $\alpha \hat{E}^2 - \hat{x}^4$ is positive semidefinite if for all $n \in \mathbb{N}$ it holds that

$$\sum_{j \neq n} |\langle j | (\alpha \hat{E}^2 - \hat{x}^4) | n \rangle| \leq \langle n | (\alpha \hat{E}^2 - \hat{x}^4) | n \rangle. \tag{340}$$

Since $\hat{E}^2 = \sum_{n=0}^{\infty} (n + \frac{1}{2})^2 |n\rangle \langle n|$, it suffices to choose α such that

$$\alpha \geq \sup_{n \in \mathbb{N}} f(n), \tag{341}$$

where $f : \mathbb{N} \mapsto \mathbb{R}$ is defined as

$$f(n) := \frac{1}{(n + \frac{1}{2})^2} \sum_{j=0}^{\infty} |\langle j | \hat{x}^4 | n \rangle|. \tag{342}$$

By exploiting that [60]

$$\hat{x} = \frac{a + a^\dagger}{\sqrt{2}}, \tag{343}$$

$$a |n\rangle = \sqrt{n} |n-1\rangle, \tag{344}$$

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \tag{345}$$

one obtains that

$$\begin{aligned}
 f(n) = \frac{1}{4(n + \frac{1}{2})^2} &\left(\sqrt{n(n-1)(n-2)(n-3)} + (4n-2)\sqrt{n(n-1)} + 3(n+1)^2 \right. \\
 &\left. + (4n+6)\sqrt{(n+1)(n+2)} + 3n^2 + \sqrt{(n+1)(n+2)(n+3)(n+4)} \right)
 \end{aligned} \tag{346}$$

for all $n \geq 3$. Moreover, it holds that

$$f(2) = \frac{(6\sqrt{2} + 39 + 28\sqrt{3} + 6\sqrt{10})}{4(2 + \frac{1}{2})^2}, \quad (347)$$

$$f(1) = \frac{(21 + 14\sqrt{6} + 2\sqrt{30})}{4(1 + \frac{1}{2})^2}, \quad (348)$$

$$f(0) = 3 + 6\sqrt{2} + 2\sqrt{6}. \quad (349)$$

Consequently, one can show that

$$\sup_{n \in \mathbb{N}} f(n) = f(0) = 3 + 6\sqrt{2} + 2\sqrt{6}. \quad (350)$$

Hence, it suffices to choose $\alpha = 3 + 6\sqrt{2} + 2\sqrt{6}$. This concludes the proof. ■

We are now ready to prove Eq. (298) in Theorem 21.

Proof of Eq. (298). Thanks to Lemma 21, we have that

$$\|\rho - \sigma\|_1 \geq \frac{|\text{Tr}[\hat{x}_1(\rho - \sigma)]|^2}{8 \max(\text{Tr}[\hat{x}_1^2 \rho], \text{Tr}[\hat{x}_1^2 \sigma])}, \quad (351)$$

where $\hat{x}_1$ denotes the position operator of the first mode. Since $\text{Tr}[\hat{x}_1^2 \rho] \leq 2\text{Tr}[\hat{E} \rho]$, it follows that

$$\|\rho - \sigma\|_1 \geq \frac{|\text{Tr}[\hat{x}_1(\rho - \sigma)]|^2}{16 \max(\text{Tr}[\hat{E} \rho], \text{Tr}[\hat{E} \sigma])}. \quad (352)$$

By defining

$$\mathbf{w} := \frac{\mathbf{m}(\sigma) - \mathbf{m}(\rho)}{\|\mathbf{m}(\sigma) - \mathbf{m}(\rho)\|}, \quad (353)$$

it holds that

$$\begin{aligned} \|\mathbf{m}(\sigma) - \mathbf{m}(\rho)\| &= \mathbf{w}^\top [\mathbf{m}(\sigma) - \mathbf{m}(\rho)] \\ &= \text{Tr}[(\mathbf{w}^\top \hat{\mathbf{R}})(\sigma - \rho)]. \end{aligned} \quad (354)$$

Furthermore, Lemma 13 of Ref. [121] establishes that there exists a passive Gaussian unitary operation G such that

$\mathbf{w}^\top \hat{\mathbf{R}} = G^\dagger \hat{x}_1 G$. Hence, we have that

$$\|\mathbf{m}(\sigma) - \mathbf{m}(\rho)\| = \text{Tr}[\hat{x}_1 G(\sigma - \rho)G^\dagger]. \quad (355)$$

Consequently, we get

$$\begin{aligned} \|\rho - \sigma\|_1 &= \|G(\rho - \sigma)G^\dagger\|_1 \\ &\geq \frac{|\text{Tr}[\hat{x}_1 G(\rho - \sigma)G^\dagger]|^2}{16 \max(\text{Tr}[\hat{E} G \rho G^\dagger], \text{Tr}[\hat{E} G \sigma G^\dagger])} \\ &= \frac{\|\mathbf{m}(\sigma) - \mathbf{m}(\rho)\|^2}{16 \max(\text{Tr}[\hat{E} \rho], \text{Tr}[\hat{E} \sigma])}, \end{aligned} \quad (356)$$

where the first line exploits the invariance of the trace distance under unitary operations, the second line follows from Eq. (352), and the last line exploits Eq. (355) together with the fact that passive Gaussian unitary operations are energy-preserving. ■

Before proving the trace distance perturbation bounds on the covariance matrices, we need to establish the following preliminary result:

Lemma 23 (Trace distance perturbation bounds on second moments). Let ρ and σ be quantum states. Let $T(\rho)$ be the second moment matrix defined as

$$T(\rho) := \text{Tr}[\{\hat{\mathbf{R}}, \hat{\mathbf{R}}^\top\} \rho]. \quad (357)$$

Then, the operator norm distance between the second moments $T(\rho)$ and $T(\sigma)$ can be upper bounded in terms of the trace distance as

$$\begin{aligned} \|T(\rho) - T(\sigma)\|_\infty &\leq 17 \sqrt{\max(\text{Tr}[\hat{E}^2 \rho], \text{Tr}[\hat{E}^2 \sigma])} \|\rho - \sigma\|_1. \end{aligned} \quad (358)$$

Proof. Lemma (21) establishes that

$$\|\rho - \sigma\|_1 \geq \frac{|\text{Tr}[\hat{x}_1^2(\rho - \sigma)]|^2}{4 \max(\text{Tr}[\hat{x}_1^4 \rho], \text{Tr}[\hat{x}_1^4 \sigma])}. \quad (359)$$

Moreover, thanks to Lemma 22, we have that

$$\hat{x}_1^4 \leq c \hat{E}_1^2, \quad (360)$$

where $\hat{E}_1$ denotes the energy operator of the first mode and c is defined by

$$c := 3 + 6\sqrt{2} + 2\sqrt{6}. \quad (361)$$

Moreover, note that $\hat{E}_1^2 \leq \hat{E}^2$, where $\hat{E}$ denotes the total energy operator. Consequently, by substituting in

Eq. (359), we obtain

$$\|\rho - \sigma\|_1 \geq \frac{|\text{Tr}[\hat{x}_1^2(\rho - \sigma)]|^2}{4c \max(\text{Tr}[\hat{E}^2\rho], \text{Tr}[\hat{E}^2\sigma])}. \quad (362)$$

Let $\mathbf{w}$ with $\mathbf{w}^\top \mathbf{w} = 1$ such that

$$\begin{aligned} \|T(\rho) - T(\sigma)\|_\infty &= |\mathbf{w}^\top [T(\rho) - T(\sigma)] \mathbf{w}| \\ &= 2 \left| \text{Tr}[(\rho - \sigma)(\mathbf{w}^\top \hat{\mathbf{R}})^2] \right|. \end{aligned} \quad (363)$$

By exploiting Lemma 13 of Ref. [121], there exists a passive Gaussian unitary operation G such that $\mathbf{w}^\top \hat{\mathbf{R}} = G^\dagger \hat{x}_1 G$. Consequently, we have that

$$\|T(\rho) - T(\sigma)\|_\infty = 2 \left| \text{Tr}[G(\rho - \sigma)G^\dagger \hat{x}_1^2] \right|. \quad (364)$$

Hence,

$$\begin{aligned} \|\rho - \sigma\|_1 &= \|G(\rho - \sigma)G^\dagger\|_1 \\ &\geq \frac{|\text{Tr}[\hat{x}_1^2 G(\rho - \sigma)G^\dagger]|^2}{4c \max(\text{Tr}[\hat{E}^2 G\rho G^\dagger], \text{Tr}[\hat{E}^2 G\sigma G^\dagger])} \\ &= \frac{\|T(\rho) - T(\sigma)\|_\infty^2}{16c \max(\text{Tr}[\hat{E}^2\rho], \text{Tr}[\hat{E}^2\sigma])}, \end{aligned} \quad (365)$$

where the first inequality is given by Eq. (362) and in the last inequality we employed Eq. (364) together with the fact that G is passive. Hence, by rearranging and by exploiting that $\sqrt{16c} \leq 17$, we conclude the proof. ■

We are now ready to prove Eq. (299) in the above-stated Theorem 21, which concludes the proof of the latter theorem.

Proof of Eq. (299). Note that Eq. (298) implies that

$$\|\mathbf{m}(\sigma) - \mathbf{m}(\rho)\| \leq 4 \sqrt{\max(\text{Tr}[\hat{E}\rho], \text{Tr}[\hat{E}\sigma])} \sqrt{\|\rho - \sigma\|_1}. \quad (366)$$

By the triangle inequality, for any A and B it holds that

$$\|A + B\|_\infty^2 \leq 2(\|A\|_\infty^2 + \|B\|_\infty^2). \quad (367)$$

Consequently, for any A and B it holds that

$$\|A\|_\infty^2 \geq \frac{\|A + B\|_\infty^2}{2} - \|B\|_\infty^2, \quad (368)$$

and hence

$$\|A - B\|_\infty^2 \geq \frac{\|A\|_\infty^2}{2} - \|B\|_\infty^2. \quad (369)$$

Consequently,

$$\begin{aligned} \|T(\rho) - T(\sigma)\|_\infty^2 &= \|V(\rho) + 2\mathbf{m}(\rho)\mathbf{m}(\rho)^\top - V(\sigma) - 2\mathbf{m}(\sigma)\mathbf{m}(\sigma)^\top\|_\infty^2 \\ &\geq \frac{1}{2} \|V(\rho) - V(\sigma)\|_\infty^2 - 4\|\mathbf{m}(\rho)\mathbf{m}(\rho)^\top - \mathbf{m}(\sigma)\mathbf{m}(\sigma)^\top\|_\infty^2. \end{aligned} \quad (370)$$

$$\geq \frac{1}{2} \|V(\rho) - V(\sigma)\|_\infty^2 - 4\|\mathbf{m}(\rho)\mathbf{m}(\rho)^\top - \mathbf{m}(\sigma)\mathbf{m}(\sigma)^\top\|_\infty^2. \quad (371)$$

By exploiting the triangle inequality and the fact that $\|\mathbf{v}\mathbf{w}^\top\|_\infty = \|\mathbf{v}\| \|\mathbf{w}\|$, we obtain

$$\begin{aligned} \|\mathbf{m}(\rho)\mathbf{m}(\rho)^\top - \mathbf{m}(\sigma)\mathbf{m}(\sigma)^\top\|_\infty &\leq \|\mathbf{m}(\sigma) - \mathbf{m}(\rho)\| (\|\mathbf{m}(\rho)\| + \|\mathbf{m}(\sigma)\|). \end{aligned} \quad (372)$$

By employing Cauchy-Schwarz inequality and the concavity of the square root function, the norm of the first moment of any state ρ can be upper bounded as follows:

$$\begin{aligned} \|\mathbf{m}(\rho)\| &= \sqrt{\sum_{i=1}^{2n} |\text{Tr}[R_i \rho]|^2} \leq \sqrt{\sum_{i=1}^{2n} \text{Tr}[R_i^2 \rho]} \\ &= \sqrt{2\text{Tr}[\hat{E}\rho]}. \end{aligned} \quad (373)$$

By exploiting Eqs. (372), (373), and (366), one obtains that

$$\begin{aligned} \|\mathbf{m}(\rho)\mathbf{m}(\rho)^\top - \mathbf{m}(\sigma)\mathbf{m}(\sigma)^\top\|_\infty &\leq 8\sqrt{2} \max(\text{Tr}[\hat{E}\rho], \text{Tr}[\hat{E}\sigma]) \sqrt{\|\rho - \sigma\|_1}. \end{aligned} \quad (374)$$

In particular, it holds that

$$\begin{aligned} \|\mathbf{m}(\rho)\mathbf{m}(\rho)^\top - \mathbf{m}(\sigma)\mathbf{m}(\sigma)^\top\|_\infty^2 &\leq 2^7 \left[\max(\text{Tr}[\hat{E}\rho], \text{Tr}[\hat{E}\sigma]) \right]^2 \|\rho - \sigma\|_1. \end{aligned} \quad (375)$$

Hence, by substituting in Eq. (370), we have that

$$\begin{aligned} \|T(\rho) - T(\sigma)\|_\infty^2 &\geq \frac{1}{2} \|V(\rho) - V(\sigma)\|_\infty^2 \\ &\quad - 2^9 \left[\max(\text{Tr}[\hat{E}\rho], \text{Tr}[\hat{E}\sigma]) \right]^2 \\ &\quad \times \|\rho - \sigma\|_1. \end{aligned} \quad (376)$$

By substituting in Eq. (365) and rearranging the terms, we conclude that

$$\begin{aligned}
\|\rho - \sigma\|_1 &\geq \frac{\|V(\rho) - V(\sigma)\|_\infty^2}{2^{10} \max(\text{Tr}[\hat{E}\rho], \text{Tr}[\hat{E}\sigma])^2 + 32c \max(\text{Tr}[\hat{E}^2\rho], \text{Tr}[\hat{E}^2\sigma])} \\
&\geq \frac{\|V(\rho) - V(\sigma)\|_\infty^2}{(2^{10} + 32c) \max(\text{Tr}[\hat{E}^2\rho], \text{Tr}[\hat{E}^2\sigma])},
\end{aligned} \tag{377}$$

where in the last inequality we used that $\text{Tr}[\hat{E}\rho]^2 \leq \text{Tr}[\hat{E}^2\rho]$. By rearranging and using $2^{10} + 32c \leq 1549$, we conclude the proof. ■

X. PERTURBATION BOUNDS FOR CLASSICAL PROBABILITY DISTRIBUTIONS

In this section, we apply the method introduced in Sec. IX to the setting of classical probability theory; specifically, we establish lower bounds on the total variation distance between two classical probability distributions in terms of the difference of their mean values and their covariance matrices. The inequalities derived in this section may be of independent interest for the classical probability literature. The noninterested reader can skip the present section, as it does not have any application to the analysis of the symplectic rank.

Let $p : \mathbb{R}^n \mapsto \mathbb{R}$ and $q : \mathbb{R}^n \mapsto \mathbb{R}$ be probability distributions over $\mathbb{R}^n$. The L_1 -distance between p and q is defined as

$$\|p - q\|_1 := \int_{\mathbb{R}^n} d^n \mathbf{x} |p(\mathbf{x}) - q(\mathbf{x})|. \tag{378}$$

The *total variation distance* between p and q is given by $\frac{1}{2}\|p - q\|_1$ and is a very important notion of distance

between classical probability distributions [50]. The question is: if the moments of p and q differ by ε , what can we say about the total variation distance between p and q ? Here, we provide a quantitative answer to this question, by finding lower bounds on the total variation distance between p and q in terms of the norm distance between their moments. The approach is completely analogous to the one we used in the previous section in order to find lower bounds on the trace distance between two quantum states in terms of the norm distance of the moments.

Let us start with the following.

Lemma 24 (Lower bounds on the total variation distance in terms of the difference of the mean values). Let $p : \mathbb{R}^n \mapsto \mathbb{R}$ and $q : \mathbb{R}^n \mapsto \mathbb{R}$ be probability distributions over $\mathbb{R}^n$. Then, it holds that

$$\|p - q\|_1 \geq \frac{\|\mathbb{E}_p[\mathbf{x}] - \mathbb{E}_q[\mathbf{x}]\|^2}{8 \max(\mathbb{E}_p[\|\mathbf{x}\|^2], \mathbb{E}_q[\|\mathbf{x}\|^2])}, \tag{379}$$

where $\|\mathbf{x}\| := \sqrt{\sum_{i=1}^n x_i^2}$ and where $\mathbb{E}_p$ denotes the expectation value with respect to the probability distribution p .

Proof. The proof is similar to the one of Theorem 22. Note that for any $\bar{x} \geq 0$ we have that

$$\begin{aligned}
\|p - q\|_1 &= \int_{\mathbb{R}^n} d^n \mathbf{x} |p(\mathbf{x}) - q(\mathbf{x})| \\
&\geq \frac{1}{\bar{x}} \int_{|\mathbf{x}_1| \leq \bar{x}} d^n \mathbf{x} x_1 [p(\mathbf{x}) - q(\mathbf{x})] \\
&= \frac{1}{\bar{x}} \int_{\mathbb{R}^n} d^n \mathbf{x} x_1 [p(\mathbf{x}) - q(\mathbf{x})] - \frac{1}{\bar{x}} \int_{|\mathbf{x}_1| \geq \bar{x}} d^n \mathbf{x} x_1 [p(\mathbf{x}) - q(\mathbf{x})] \\
&\geq \frac{1}{\bar{x}} (\mathbb{E}_p[x_1] - \mathbb{E}_q[x_1]) - \frac{1}{\bar{x}^2} \int_{|\mathbf{x}_1| \geq \bar{x}} d^n \mathbf{x} x_1^2 [p(\mathbf{x}) + q(\mathbf{x})] \\
&\geq \frac{1}{\bar{x}} (\mathbb{E}_p[x_1] - \mathbb{E}_q[x_1]) - \frac{2}{\bar{x}^2} \max(\mathbb{E}_p[x_1^2], \mathbb{E}_q[x_1^2]).
\end{aligned} \tag{380}$$

By exchanging the role of p and q , we obtain that

$$\|p - q\|_1 \geq \frac{1}{\bar{x}} |\mathbb{E}_p[x_1] - \mathbb{E}_q[x_1]| - \frac{2}{\bar{x}^2} \max(\mathbb{E}_p[x_1^2], \mathbb{E}_q[x_1^2]). \tag{381}$$

By optimizing over $\bar{x}$, we have that

$$\|p - q\|_1 \geq \frac{|\mathbb{E}_p[x_1] - \mathbb{E}_q[x_1]|^2}{8 \max(\mathbb{E}_p[x_1^2], \mathbb{E}_q[x_1^2])}. \quad (382)$$

In particular, the following less tight bound holds:

$$\|p - q\|_1 \geq \frac{|\mathbb{E}_p[x_1] - \mathbb{E}_q[x_1]|^2}{8 \max(\mathbb{E}_p[\|\mathbf{x}\|^2], \mathbb{E}_q[\|\mathbf{x}\|^2])}. \quad (383)$$

Let us define

$$\mathbf{w} := \frac{\mathbb{E}_p[\mathbf{x}] - \mathbb{E}_q[\mathbf{x}]}{\|\mathbb{E}_p[\mathbf{x}] - \mathbb{E}_q[\mathbf{x}]\|}, \quad (384)$$

so that

$$\|\mathbb{E}_p[\mathbf{x}] - \mathbb{E}_q[\mathbf{x}]\| = \mathbb{E}_p[\mathbf{w}^\top \mathbf{x}] - \mathbb{E}_q[\mathbf{w}^\top \mathbf{x}]. \quad (385)$$

Let O be an orthogonal matrix with first column equal to $\mathbf{w}$. Let $\tilde{p}$ and $\tilde{q}$ defined as

$$\tilde{p}(\mathbf{x}) := p(O\mathbf{x}), \quad (386)$$

$$\tilde{q}(\mathbf{x}) := q(O\mathbf{x}). \quad (387)$$

Note that

$$\mathbb{E}_{\tilde{p}}[x_1] = \mathbb{E}_p[\mathbf{w}^\top \mathbf{x}], \quad (388)$$

$$\mathbb{E}_{\tilde{q}}[x_1] = \mathbb{E}_q[\mathbf{w}^\top \mathbf{x}], \quad (389)$$

and thus we have that

$$\mathbb{E}_{\tilde{p}}[x_1] - \mathbb{E}_{\tilde{q}}[x_1] = \|\mathbb{E}_p[\mathbf{x}] - \mathbb{E}_q[\mathbf{x}]\|. \quad (390)$$

Moreover, since O is orthogonal, we have also that

$$\mathbb{E}_p[\|\mathbf{x}\|^2] = \mathbb{E}_{\tilde{p}}[\|\mathbf{x}\|^2], \quad (391)$$

$$\mathbb{E}_q[\|\mathbf{x}\|^2] = \mathbb{E}_{\tilde{q}}[\|\mathbf{x}\|^2]. \quad (392)$$

Consequently, we conclude that

$$\begin{aligned} \|p - q\|_1 &= \|\tilde{p} - \tilde{q}\|_1 \\ &\geq \frac{|\mathbb{E}_{\tilde{p}}[x_1] - \mathbb{E}_{\tilde{q}}[x_1]|^2}{8 \max(\mathbb{E}_{\tilde{p}}[\|\mathbf{x}\|^2], \mathbb{E}_{\tilde{q}}[\|\mathbf{x}\|^2])} \\ &= \frac{\|\mathbb{E}_p[\mathbf{x}] - \mathbb{E}_q[\mathbf{x}]\|^2}{8 \max(\mathbb{E}_p[\|\mathbf{x}\|^2], \mathbb{E}_q[\|\mathbf{x}\|^2])}. \end{aligned} \quad (393)$$

We also have the following.

Lemma 25 (Lower bounds on the total variation distance with the difference of the second moment matrices). Let $p : \mathbb{R}^n \mapsto \mathbb{R}$ and $q : \mathbb{R}^n \mapsto \mathbb{R}$ be probability distributions over $\mathbb{R}^n$. Then, it holds that

$$\|p - q\|_1 \geq \frac{\|\mathbb{E}_p[\mathbf{x}\mathbf{x}^\top] - \mathbb{E}_q[\mathbf{x}\mathbf{x}^\top]\|_\infty^2}{4 \max(\mathbb{E}_p[\|\mathbf{x}\|^4], \mathbb{E}_q[\|\mathbf{x}\|^4])}, \quad (394)$$

where $\|\mathbf{x}\| := \sqrt{\sum_{i=1}^n x_i^2}$.

Proof. The proof is completely analogous to the one of Lemma 24 and of Lemma 23. Namely, analogously to Eq. (382), we find

$$\|p - q\|_1 \geq \frac{|\mathbb{E}_p[x_1^2] - \mathbb{E}_q[x_1^2]|^2}{4 \max(\mathbb{E}_p[x_1^4], \mathbb{E}_q[x_1^4])}, \quad (395)$$

and thus also that

$$\|p - q\|_1 \geq \frac{|\mathbb{E}_p[x_1^2] - \mathbb{E}_q[x_1^2]|^2}{4 \max(\mathbb{E}_p[\|\mathbf{x}\|^4], \mathbb{E}_q[\|\mathbf{x}\|^4])}. \quad (396)$$

Moreover, let $\mathbf{w}$ with $\mathbf{w}^\top \mathbf{w} = 1$ such that

$$\begin{aligned} &\|\mathbb{E}_p[\mathbf{x}\mathbf{x}^\top] - \mathbb{E}_q[\mathbf{x}\mathbf{x}^\top]\|_\infty \\ &= |\mathbf{w}^\top (\mathbb{E}_p[\mathbf{x}\mathbf{x}^\top] - \mathbb{E}_q[\mathbf{x}\mathbf{x}^\top]) \mathbf{w}|, \end{aligned} \quad (397)$$

so that

$$\|\mathbb{E}_p[\mathbf{x}\mathbf{x}^\top] - \mathbb{E}_q[\mathbf{x}\mathbf{x}^\top]\|_\infty = |\mathbb{E}_p[(\mathbf{w}^\top \mathbf{x})^2] - \mathbb{E}_q[(\mathbf{w}^\top \mathbf{x})^2]|. \quad (398)$$

Let O be an orthogonal matrix with first column equal to $\mathbf{w}$. Let $\tilde{p}$ and $\tilde{q}$ defined as

$$\tilde{p}(\mathbf{x}) := p(O\mathbf{x}), \quad (399)$$

$$\tilde{q}(\mathbf{x}) := q(O\mathbf{x}). \quad (400)$$

Note that

$$\mathbb{E}_{\tilde{p}}[x_1^2] = \mathbb{E}_p[(\mathbf{w}^\top \mathbf{x})^2], \quad (401)$$

$$\mathbb{E}_{\tilde{q}}[x_1^2] = \mathbb{E}_q[(\mathbf{w}^\top \mathbf{x})^2], \quad (402)$$

and thus we have that

$$|\mathbb{E}_{\tilde{p}}[x_1^2] - \mathbb{E}_{\tilde{q}}[x_1^2]| = \|\mathbb{E}_p[\mathbf{x}\mathbf{x}^\top] - \mathbb{E}_q[\mathbf{x}\mathbf{x}^\top]\|_\infty. \quad (403)$$

■

Moreover, since O is orthogonal, we have also that

$$\mathbb{E}_p[\|\mathbf{x}\|^4] = \mathbb{E}_{\tilde{p}}[\|\mathbf{x}\|^4], \quad (404)$$

$$\mathbb{E}_q [\|\mathbf{x}\|^4] = \mathbb{E}_{\tilde{q}} [\|\mathbf{x}\|^4]. \quad (405)$$

Consequently, we conclude that

$$\begin{aligned} \|p - q\|_1 &= \|\tilde{p} - \tilde{q}\|_1 \\ &\geq \frac{|\mathbb{E}_{\tilde{p}} [x_1^2] - \mathbb{E}_{\tilde{q}} [x_1^2]|^2}{4 \max (\mathbb{E}_{\tilde{p}} [\|\mathbf{x}\|^4], \mathbb{E}_{\tilde{q}} [\|\mathbf{x}\|^4])} \\ &= \frac{\|\mathbb{E}_p [\mathbf{x}\mathbf{x}^\top] - \mathbb{E}_q [\mathbf{x}\mathbf{x}^\top]\|_\infty^2}{4 \max (\mathbb{E}_p [\|\mathbf{x}\|^4], \mathbb{E}_q [\|\mathbf{x}\|^4])}. \end{aligned} \quad (406)$$

■

Finally, we establish the following lower bound on the total variation distance in terms of the covariance matrices.

Lemma 26 (Lower bounds on the total variation distance in terms of the difference of the covariance matrices). Let

$p : \mathbb{R}^n \mapsto \mathbb{R}$ be a probability distribution over $\mathbb{R}^n$ and let $V(p)$ be its covariance matrix defined by

$$V(p) := \mathbb{E}_p [(\mathbf{x} - \mathbb{E}_p[\mathbf{x}])(\mathbf{x} - \mathbb{E}_p[\mathbf{x}])^\top]. \quad (407)$$

Let p and q be probability distributions over $\mathbb{R}^n$. Then, it holds that

$$\|p - q\|_1 \geq \frac{\|V(p) - V(q)\|_\infty^2}{72 \max (\mathbb{E}_p [\|\mathbf{x}\|^4], \mathbb{E}_q [\|\mathbf{x}\|^4])}, \quad (408)$$

where $\|\mathbf{x}\| := \sqrt{\sum_{i=1}^n x_i^2}$.

Proof. The proof is analogous to the one of Theorem 23. Namely, by using that

$$V(p) = \mathbb{E}_p [\mathbf{x}\mathbf{x}^\top] - \mathbb{E}_p[\mathbf{x}]\mathbb{E}_p[\mathbf{x}]^\top, \quad (409)$$

we have that

$$\begin{aligned} \|p - q\|_1 &\geq \frac{\|\mathbb{E}_p [\mathbf{x}\mathbf{x}^\top] - \mathbb{E}_q [\mathbf{x}\mathbf{x}^\top]\|_\infty^2}{4 \max (\mathbb{E}_p [\|\mathbf{x}\|^4], \mathbb{E}_q [\|\mathbf{x}\|^4])} \\ &= \frac{\|V(p) - V(q) + \mathbb{E}_p[\mathbf{x}]\mathbb{E}_p[\mathbf{x}]^\top - \mathbb{E}_q[\mathbf{x}]\mathbb{E}_q[\mathbf{x}]^\top\|_\infty^2}{4 \max (\mathbb{E}_p [\|\mathbf{x}\|^4], \mathbb{E}_q [\|\mathbf{x}\|^4])} \\ &\geq \frac{\frac{1}{2} \|V(p) - V(q)\|_\infty^2 - \|\mathbb{E}_p[\mathbf{x}]\mathbb{E}_p[\mathbf{x}]^\top - \mathbb{E}_q[\mathbf{x}]\mathbb{E}_q[\mathbf{x}]^\top\|_\infty^2}{4 \max (\mathbb{E}_p [\|\mathbf{x}\|^4], \mathbb{E}_q [\|\mathbf{x}\|^4])}, \end{aligned} \quad (410)$$

where in the last line we exploited that $\|A - B\|_\infty^2 \geq (\|A\|_\infty^2/2) - \|B\|_\infty^2$. Moreover, we have that

$$\begin{aligned} \|\mathbb{E}_p[\mathbf{x}]\mathbb{E}_p[\mathbf{x}]^\top - \mathbb{E}_q[\mathbf{x}]\mathbb{E}_q[\mathbf{x}]^\top\|_\infty &\leq (\|\mathbb{E}_p[\mathbf{x}]\| + \|\mathbb{E}_q[\mathbf{x}]\|) \|\mathbb{E}_p[\mathbf{x}] - \mathbb{E}_q[\mathbf{x}]\| \\ &\leq 2\sqrt{\max (\mathbb{E}_p [\|\mathbf{x}\|^2], \mathbb{E}_q [\|\mathbf{x}\|^2])} \|\mathbb{E}_p[\mathbf{x}] - \mathbb{E}_q[\mathbf{x}]\| \\ &\leq 2\sqrt{8} \max (\mathbb{E}_p [\|\mathbf{x}\|^2], \mathbb{E}_q [\|\mathbf{x}\|^2]) \sqrt{\|p - q\|_1} \\ &\leq 2\sqrt{8} \sqrt{\max (\mathbb{E}_p [\|\mathbf{x}\|^4], \mathbb{E}_q [\|\mathbf{x}\|^4])} \sqrt{\|p - q\|_1}, \end{aligned} \quad (411)$$

where in the third line we exploited Lemma 24 and the concavity of the square root. Consequently, we obtain that

$$\|p - q\|_1 \geq \frac{\|V(p) - V(q)\|_\infty^2}{8 \max (\mathbb{E}_p [\|\mathbf{x}\|^4], \mathbb{E}_q [\|\mathbf{x}\|^4])} - 8\|p - q\|_1. \quad (412)$$

Rearranging the terms, we conclude the proof. ■

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [70] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/1rtk-1jsn> for the rest of the paper is structured as follows. In Sec. IV, we review the necessary background on continuous-variable systems, with a focus on Gaussian operations, covariance matrices, and stellar functions. In Sec. V, we establish the main structural and resource-theoretic properties of the symplectic rank, including its monotonicity and implications for Gaussian conversion. In Secs. VI and VII, we prove its robustness under perturbations and its connection to tomography and classical simulation. In Sec. VIII, we introduce approximate symplectic rank and symplectic fidelities, and apply them to witnessing and approximate conversion. Finally, in Secs. IX and X, we derive perturbation bounds for the covariance matrices and first moments of continuous-variable quantum states, as well as for classical probability distributions.
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