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Regularity properties of k-Brjuno and Wilton functions

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Abstract. We study functions related to the classical Brjuno function, namely k-Brjuno functions and the Wilton function. Both appear in the study of boundary regularity properties of (quasi) modular forms and their integrals. We consider various possible versions of them, based on the α-continued fraction developments. We study their BMO regularity properties and their behaviour near rational numbers of their finite truncations. We then complexify the functional equations which they fulfill and we construct analytic extensions of the k-Brjuno and Wilton functions to the upper half-plane. We study their boundary behaviour using an extension of the continued fraction algorithm to the complex plane. We also prove that the harmonic conjugate of the real k-Brjuno function is continuous at all irrational numbers and has a decreasing jump of π/q^k at rational points p/q.

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1. Introduction

Let G denote *the Gauss map* defined on the interval $[0, 1)$ by $G(0) = 0$ and $G(x) = \left\{\frac{1}{x}\right\}$ otherwise, where $\{x\} = x - \lfloor x \rfloor$ and let

$$\beta_j(x) = \prod_{i=0}^{j} G^i(x)$$

for $j \geq 0$ with the convention $\beta_{-1}(x) = 1$, where G^j denotes the jth iteration of G, not the power of G.

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1.1. Brjuno function

In 1988, Yoccoz introduced the following function - now called *Brjuno function* - defined for irrational numbers $x \in [0,1] \setminus \mathbb{Q}$ as

$$B_1(x) = \sum_{n=0}^{\infty} \beta_{n-1}(x) \log\left(\frac{1}{G^n(x)}\right),$$

see [25,27,37], see Fig. 1 for its graph.

Let $\frac{p_n(x)}{q_n(x)}$ denote the nth convergent of x with respect to its continued fraction expansion. The series $B_1(x)$ converges if and only if

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n(x)} < \infty.$$

This condition is called *Brjuno condition* and was introduced by Brjuno in the study of certain problems in dynamical systems, see [8,9]. The points of convergence are called *Brjuno numbers*. The importance of Brjuno numbers comes from the study of analytic small divisors problems in dimension one. Indeed, extending previous fundamental work by Siegel [34], Brjuno proved that all germs of holomorphic diffeomorphisms of one complex variable with an indifferent fixed point with linear part $e^{2\pi i x}$ are linearisable if x is a Brjuno number. Conversely, in 1988 Yoccoz [37,38] proved that this condition is also necessary. Similar results hold for the local conjugacy of analytic diffeomorphisms of the circle [39] and for some complex area-preserving maps [11,23]. Moreover, the sum of the Brjuno function and the logarithm of the radius of convergence of the quadratic polynomial has a continuous extension to the real line [2] and this interpolation is conjectured to be Hölder continuous with exponent 1/2 [25]. This has been recently proved for high-type irrational numbers [10]. The Brjuno condition has been of interest in various contexts. For instance, it is conjectured that it is optimal for the existence of real analytic invariant circles in the standard family [20–22,28]. See also [4,15] and the references therein for related results.

Furthermore, the Brjuno function is $\mathbb{Z}$–periodic and satisfies the functional equation

$$B_1(x) = -\log(x) + xB_1\left(\frac{1}{x}\right) \tag{1.1}$$

for $x \in (0,1)$. The second author together with Moussa and Yoccoz investigated the regularity properties of B_1 in [25] and later constructed an analytic extension of B_1 to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \colon \mathrm{Im} z > 0\}$ [26]. Let T denote the linear operator

$$Tf(x) = xf\left(\frac{1}{x}\right) \tag{1.2}$$

acting, for example, on measurable $\mathbb{Z}$–periodic functions on $\mathbb{R}$. Then in all L^p spaces, the Brjuno function is the solution of the linear equation

$$[(1-T)B_1](x) = -\log x\,. \tag{1.3}$$

By exploiting the fact that the operator T as in (1.2) acting on L^p spaces has spectral radius strictly smaller than 1, one can indeed obtain (1.1) by a Neumann series for $(1-T)^{-1}$, see [25] for details.

Local properties of the Brjuno function have been recently investigated by Balazard and Martin [5] and its multifractal spectrum was determined by Jaffard and Martin in [18]. Our results open the path to a possible investigation of the multifractal properties of the k-Brjuno and the Wilton functions as in [18]. The investigation of the local regularity properties of the Brjuno and Wilton functions made by [18] should also be possible for the k-Brjuno and the Wilton functions.

1.2. k-Brjuno functions

We might generalise the Brjuno condition by looking at irrational numbers x fulfilling

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n^k(x)} < \infty, \tag{1.4}$$

for some $k \geq 0$. We will call this condition *k-Brjuno condition*; it has found applications in a range of different settings of which we will give an incomplete list in the following.

If for $k \geq 2$ even, we let E_k be the Eisenstein series of weight k defined in the upper half-plane $\mathbb{H}$, then its Fourier expansion is given by

$$E_k(z) = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

where b_k is the kth Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. For all $k \geq 4$, E_k is modular of weight k under the action of $\mathrm{SL}_2(\mathbb{Z})$, and E_2 is quasi-modular of weight 2 under the action of $\mathrm{SL}_2(\mathbb{Z})$, see for example [40]. The function E_2 can be viewed as a modular (or Eichler) integral on $\mathrm{SL}_2(\mathbb{Z})$ of weight 2 with the rational period function $-\frac{2\pi i}{z}$, see for example [19].

For $k \geq 2$ even and $z \in \mathbb{H}$, denote $\varphi_k(z) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} e^{2\pi i n z}$. We have that

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z} = \left(\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{k+1} \varphi_k(z),$$

and

$$\varphi_k(z) = \frac{B_k(2\pi i)^{k+1}}{k!2k} \int_{i\infty}^{z} (z-t)^k (E_k(t)-1)dt.$$

Consider the imaginary part of φ_k

$$F_k(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} \sin(2\pi n x) \quad \text{for } x \in \mathbb{R}.$$

Analytic properties, differentiability and the Hölder regularity exponent, of the function F_k (and the real part of φ_k) were studied by the third author. It has been proved that for F_k the differentiability is related to a condition resembling the Brjuno condition. Considering the special case $k = 2$, the third author proved that if $\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n^2(x)} < \infty$ and $\lim_{n\to\infty} \frac{\log(q_{n+4}(x))}{q_n^2(x)} = 0$, then F_2 is differentiable at $x \in \mathbb{R} \setminus \mathbb{Q}$, whereas if $\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n^2(x)}$ diverges, then F_2 is not differentiable at $x \in \mathbb{R} \setminus \mathbb{Q}$. It has been conjectured that for all $k \in \mathbb{N}$ even, F_k is differentiable at $x \in \mathbb{R} \setminus \mathbb{Q}$ if and only if it fulfills the k-Brjuno condition, see [30,31].

Moreover, Rivoal and Seuret [32] were looking at the sum $\mathsf{F}_s(x) = \sum_{n=1}^{\infty} \frac{e^{i\pi n^2 x}}{n^s}$ with $s \in (1/2, 1]$ for which it was proven by Hardy and Littlewood that it converges almost surely on $[-1, 1]$ but not everywhere. For the case $s = 1$ Rivoal and Seuret showed that this sum as well as a generalisation of it converges absolutely if $\sum_{n=0}^{\infty} \frac{\log(q_{n+1}(x))}{q_n^{1/2}(x)} < \infty$, i.e. if the 1/2-Brjuno condition holds.

In addition to appearing in the regularity theory of the boundary behaviour of modular integrals and theta series, the k-Brjuno condition is used in the KAM theory of Gevrey flows. For example the α-Brjuno-Rüssmann condition introduced in [3] as well as the s-Brjuno arithmetical condition introduced in [12] for one frequency systems turn out to be equivalent to the $1/\alpha$- or $1/s$-Brjuno condition.

The occurrence of a condition of this type motivates the following definition.

For $k \in \mathbb{N}$, let

$$B_k(x) = \sum_{n=0}^{\infty} (\beta_{n-1}(x))^k \log\left(\frac{1}{G^n(x)}\right) \tag{1.5}$$

be called *k-Brjuno function.*

We will see in Proposition 2.8 that the convergence of $B_k(x)$ is equivalent to x fulfilling the k-Brjuno condition. It is worth mentioning that a generalisation of the k-Brjuno function was also studied in [32] which also converges if and only if its argument fulfills the k-Brjuno condition.

From this equation, we already get the implicit definition in terms of an analogue of the functional equation of the Brjuno function: if $x \in (0,1)$

$$B_k(x) = -\log(x) + x^k \cdot B_k(G(x)),$$

see [25,27]. The k–Brjuno function converges at an irrational x if and only if (1.4) holds, see also Proposition 2.5 for an even stronger statement about the relation between the k-Brjuno function and (1.4). Obviously, for $k = 1$, the function in (1.5) gives the Brjuno function introduced before. We also remark that the Diophantine condition introduced in [32, Thm. 1] which ensures the convergence of $\mathsf{F}_s(x)$ with $s \in (1/2, 1)$ can also be obtained by means of a generalization of (1.5): we will show this in Appendix A.

1.3. α-continued fractions

Instead of considering the k-Brjuno function with respect to the Gauss map as in (1.5), it is also possible to use α-continued fractions instead. The classical Brjuno function associated to α-continued fractions was already investigated in [25]. Let $\alpha \in \left[\frac{1}{2}, 1\right]$ and let $A_\alpha : (0, \alpha) \to [0, \alpha]$ be the transformation of *the α-continued fractions* given by

$$A_\alpha(x) = \left| \frac{1}{x} - \left\lfloor \frac{1}{x} - \alpha + 1 \right\rfloor \right|. \tag{1.6}$$

For $\alpha = 1$, we obtain the Gauss map associated to the regular continued fraction transformation and for $\alpha = 1/2$, we obtain the transformation associated to the nearest integer continued fractions. Nakada [29] was the first to consider all these types of continued fractions as a one-parameter family, however, he considered the 'unfolded' version of the α-continued fraction which is defined by the Gauss map $\widetilde{A}_\alpha(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} - \alpha + 1 \right\rfloor$. The version as in (1.6) was considered a little later in [35].

Further, let

$$B_{k,\alpha}(x) = \sum_{n=0}^{\infty} \left(\beta_{n-1}^{(\alpha)}(x)\right)^k \log\left(\frac{1}{A_\alpha^n(x)}\right),$$

where $\beta_j^{(\alpha)}(x) = \prod_{i=0}^{j} A_\alpha^i(x)$ for $j \geq 0$ and $\beta_{-1}^{(\alpha)}(x) = 1$, be the generalisation of (1.5) in the sense that we consider the k-Brjuno function not only for the Gauss transformation but also for other α-continued fraction transformations A_α with $\alpha \in [1/2, 1]$.

For given $\alpha \in [1/2, 1]$, any $x \in (0,1]$ has the α-continued fraction expansion given by

$$x = \cfrac{1}{a_1 + \cfrac{\epsilon_1}{\ddots + \cfrac{\epsilon_{j-1}}{a_j + \ddots}}},$$

where $a_j := a_j^{(\alpha)}(x) := \left\lfloor \frac{1}{A_\alpha^{j-1}(x)} - \alpha + 1 \right\rfloor$ and $\epsilon_j := \epsilon_j^{(\alpha)}(x)$ is the sign of $\frac{1}{A_\alpha^{j-1}(x)} - a_j$.

Related to (1.2) we define an operator $T_{k,\alpha}$ acting on $\mathbb{Z}$–periodic measurable functions f such that $f(-x) = f(x)$ for a.e. $x \in (0, 1-\alpha)$ as

$$T_{k,\alpha} f(x) = x^k f\left(\frac{1}{x}\right), \quad x \in (0, \alpha). \tag{1.7}$$

It is the operator $T_\nu^{(\alpha)}$ that was introduced in [25], where ν corresponds to the exponent k in (1.7). It is understood that the function $T_{k,\alpha} f$ is completed outside the interval $(0, \alpha)$ by imposing on $T_{k,\alpha} f$ the same parity and periodicity conditions as those imposed on f.

Then we have

$$[(1 - T_{k,\alpha}) B_{k,\alpha}](x) = -\log x, \quad x \in (0, \alpha) \tag{1.8}$$

which follows by a simple calculation.

1.4. Wilton function

Next, we consider the related concept of the *Wilton function* which is given by

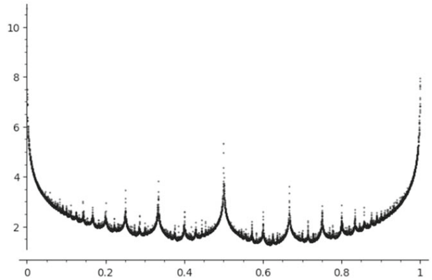
$$W(x) = \sum_{n=0}^{\infty} (-1)^n \beta_{n-1}(x) \log\left(\frac{1}{G^n(x)}\right),$$

namely by the alternate sign version of the Brjuno function series (1.1) (see Fig. 1 for its graph).

(see Fig. 1 for its graph). It converges if and only if it fulfills the *Wilton condition*

$$\left| \sum_{n=0}^{\infty} (-1)^n \frac{\log(q_{n+1}(x))}{q_n(x)} \right| < \infty,$$

see [6, Prop. 7] and Remark 2.7 for an even stronger connection between the Wilton function and the Wilton condition. The points of convergence are called *Wilton numbers* and appear in the work of Wilton, see [36]. Clearly, all Brjuno



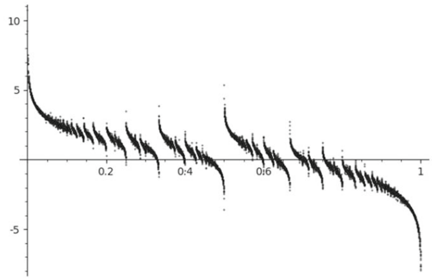


FIGURE 1. Numerical computation of the Brjuno function B (left) and of the Wilton function W (right) when $\alpha = 1$. The asymmetric logarithmic singularities at rational points provide an intuitive justification for W not belonging to the BMO space, see Sect. 2

numbers are Wilton, but not vice versa (it is not difficult to build counterexamples by using the continued fraction).

The function W satisfies the functional equation for $x \in (0,1)$ being a Wilton number:

$$W(x) = \log\left(\frac{1}{x}\right) - xW(G(x)),$$

which by using the same linear operator T as in (1.2) can be written as

$$[(1+T)W](x) = -\log x. \tag{1.9}$$

We can extend the Wilton function in the same way as we did for the k-Brjuno functions, i.e. replacing the Gauss map with the transformation of the α-continued fractions: for $\alpha \in \left[\frac{1}{2}, 1\right)$ and for all irrational x we define

$$W_\alpha(x) = \sum_{n=0}^{\infty} (-1)^n \beta_{n-1}^{(\alpha)}(x) \log\left(\frac{1}{A_\alpha^n(x)}\right).$$

We then have

$$[(1-S_\alpha)W_\alpha](x) = -\log x, \quad x \in (0,\alpha),$$

where the operator $S_\alpha = -T_{1,\alpha}$. Also in this case it is understood that S_α acts on $\mathbb{Z}$–periodic measurable functions f such that $f(-x) = f(x)$ for a.e. $x \in (0, 1-\alpha)$.

The Wilton function and its primitive have been studied recently by Balazard and Martin in terms of its convergence properties [7] and in the context of the Nyman and Beurling criterion, see [5,6] and [1]. It would also be possible to define k-Wilton functions generalising the Wilton function in the same way as we generalised the Brjuno function to k-Brjuno functions, i.e. studying

$W_k(x) = \sum_{n=0}^{\infty}(-1)^n(\beta_{n-1}(x))^k \log\left(\frac{1}{G^n(x)}\right)$. However, we will not explicitly do the calculations for these kinds of functions.

1.5. BMO properties and complex k-Brjuno and Wilton functions.

In [25] the real regularity properties of the Brjuno function were systematically investigated by exploiting the associated functional equation, culminating with the proof that the Brjuno function belongs to the space BMO of functions with Bounded Mean Oscillation (we refer to [13, 14] for its definition and properties). Here (see Sect. 2) we extend those results to the k-Brjuno function, which turns out to belong to BMO, and to the Wilton function, which does not. Moreover, we consider their generalisations obtained by replacing the Gauss map with α-continued fractions [29], $\alpha \in \left[\frac{1}{2}, 1\right]$ and we prove that for all α the associated k-Brjuno functions belong to BMO and that the same holds for the associated Wilton functions when α is restricted to the interval $\left[\frac{1}{2}, \frac{\sqrt{5}-1}{2}\right]$.

By Fefferman's duality theorem, see for example [33, p. 39], one can add an L^∞ function to a BMO function and obtain that the Hilbert transform of the sum will be an essentially bounded function. One is then led to look for a periodic holomorphic function defined on the upper half-plane whose imaginary part, when looked upon $\mathbb{R}$, is the original function. As in [26] we make this construction for the k-Brjuno functions and the Wilton function. The associated *complex k-Brjuno functions* $\mathcal{B}_k$ and *complex Wilton function* $\mathcal{W}$ and their properties are summarized in the following Theorem:

Theorem 1.1. *(1) The complex k-Brjuno function $\mathcal{B}_k : \mathbb{H} \to \mathbb{C}$ is given by the series*

$$\begin{aligned}\mathcal{B}_k(z) = &-\sum_{p/q\in\mathbb{Q}} \det\begin{pmatrix} p' & p \\ q' & q\end{pmatrix}^{k+1} \\ &\cdot\left\{\frac{1}{\pi}\left[(p'-q'z)^k \mathrm{Li}_2\left(\frac{p'-q'z}{qz-p}\right) - (q''z-p'')^k \mathrm{Li}_2\left(\frac{p''-q''z}{qz-p}\right)\right]\right. \\ &+\sum_{n=0}^{k}\frac{\det\begin{pmatrix} p' & p \\ q' & q\end{pmatrix}^n}{n\pi q^n}\left[(p'-q'z)^{k-n}\left(\log\left(1+\frac{q'}{q}\right)-\sum_{i=1}^{n-1}\frac{1}{i}\left(\frac{1}{(1+q'/q)^i}-1\right)\right)\right. \\ &\left.\left.-(q''z-p'')^{k-n}\left(\log\left(1+\frac{q''}{q}\right)-\sum_{i=1}^{n-1}\frac{1}{i}\left(\frac{1}{(1+q''/q)^i}-1\right)\right)\right]\right\}, \end{aligned} \tag{1.10}$$

where $[\frac{p'}{q'}, \frac{p''}{q''}]$ is the Farey interval such that $\frac{p}{q} = \frac{p'+p''}{q'+q''}$ (with the convention $p' = p-1$, $q' = 1$, $p'' = 1$, $q'' = 0$ if $q = 1$) and $\mathrm{Li}_2(z)$ is the dilogarithm. The complex Wilton function $\mathcal{W} : \mathbb{H} \to \mathbb{C}$ is given by the *series*

$$\mathcal{W}(z) = -\frac{1}{\pi} \sum_{p/q\in\mathbb{Q}} \left\{ \det \begin{pmatrix} p' & p \\ q' & q \end{pmatrix} (p' - q'z) \left[\mathrm{Li}_2 \left(\frac{p' - q'z}{qz - p} \right) - \mathrm{Li}_2 \left(-\frac{q'}{q} \right) \right] \right.$$
$$+ \det \begin{pmatrix} p'' & p \\ q'' & q \end{pmatrix} (p'' - q''z) \left[\mathrm{Li}_2 \left(\frac{p'' - q''z}{qz - p} \right) - \mathrm{Li}_2 \left(-\frac{q''}{q} \right) \right]$$
$$\left. + \frac{1}{q} \log \left(\frac{q^2}{(q+q')(q+q'')} \right) \right\}. \tag{1.11}$$

(2) *The real part of $\mathcal{B}_k$ is bounded on the upper half-plane and its non–tangential limit on $\mathbb{R}$ is continuous at all irrational points and has a decreasing jump of $\frac{\pi}{q^k}$ at each rational number $\frac{p}{q}$.*

(3) *As one approaches the boundary the imaginary part of $\mathcal{W}$ behaves as follows:*
   (a) *if $x \in \mathbb{R}$ is a Wilton number, then* $\mathrm{Im}\mathcal{W}(x + w)$ *converges to* $W(x)$ *as $w \to 0$ in any domain with a finite order of tangency to the real axis;*
   (b) *if $x \in \mathbb{R}$ is Diophantine, then in both cases one can allow domains with infinite order of tangency to the real line.*

As the Wilton function is not BMO, it also becomes clear by Fefferman's theorem that its harmonic conjugate can not be an L^∞ function.

1.6. Structure of the paper

The paper is organised as follows. In Sect. 2 we state and prove the BMO-properties of the k-Brjuno and Wilton functions and in Sect. 3 we give statements about the truncated real k-Brjuno functions and the truncated real Wilton function which corresponds to the finite part of the sum at rational numbers.

The subsequent sections deal with a complexification of the system. In Sect. 4 we introduce a complex continued fraction algorithm. In Sect. 5 we extend the operators $T_k := T_{k,1}$ and $S := S_1$ to the complex plane. The main findings concerning these operators are given in Sect. 6. Particularly, Proposition 6.9 indicates how T_k behaves. The analogous behaviour of S can then immediately be deduced by using the fact that $S = -T_1$.

Finally, with the help of these operators, we define the complex k-Brjuno and complex Wilton functions in Sect. 7.

Those proofs which are very similar to the ones in [26] are given in an appendix.

2. BMO properties of the real k-Brjuno and real Wilton functions

In this section, we study the bounded mean oscillation (BMO) properties of the real k-Brjuno and Wilton functions - both with respect to different transformations A_α with $\alpha \in [1/2, 1]$. Before stating the main results of this section, we will first recall the definition of a BMO function.

Let $L^1_{\text{loc}}(\mathbb{R})$ be the space of the locally integrable functions on $\mathbb{R}$. Recall that the mean value of a function $f \in L^1_{\text{loc}}(\mathbb{R})$ on an interval I is defined as

$$f_I = \frac{1}{|I|} \int_I f(x) dx.$$

For an interval U, we say that a function $f \in \text{BMO}(U)$ if

$$\|f\|_{*,U} := \sup_{I \subset U} \frac{1}{|I|} \int_I |f(x) - f_I| dx < \infty. \tag{2.1}$$

For further properties of the BMO space, see for example [25, Appendix] and the monographies [13, 14].

In the following, we will state the main properties of this section which show that the BMO properties fundamentally differ between the k-Brjuno functions and the Wilton function. We first give the statement for k-Brjuno functions.

Proposition 2.1. *For all $k \in \mathbb{R}_{>0}$ and all $\alpha \in [1/2, 1]$, the k-Brjuno function $B_{k,\alpha}$ is a* BMO *function.*

In contrast, for the Wilton function, we have the following statement:

Theorem 2.2. *The Wilton function $W = W_1$ is not a* BMO *function.*

On the other hand, we define for the following $g := \frac{\sqrt{5}-1}{2}$ and have:

Theorem 2.3. *For all $\alpha \in \left[\frac{1}{2}, g\right]$, the function W_α is a* BMO *function.*

Before we start with the proofs of the statements above, we first want to give some remarks about them: Proposition 2.1 is an extension of [25, Thm. 3.2] from the classical Brjuno function to k-Brjuno functions.

As a comparison to Fig. 1 showing $B_{1,1}$ and W_1, in Fig. 2, some numerical simulations of W_α with different values of α are shown: the numerical evidence supports the conjecture that also for $\alpha \in (g, 1)$ the function W_α is a BMO function. Indeed, the graph for $\alpha = e - 2$ and $\alpha = 0.9$ (and nearby values, not displayed in Fig. 1) suggest that the function W_α exhibits a mix of *symmetric* logarithmic singularities at rational points and jumps, a singular behaviour which is compatible with belonging to BMO. However, unfortunately, the results from Theorem 2.3 can not immediately be transferred to $\alpha \in (g, 1)$, see Remark 2.10 for an explanation of what difficulties occur.

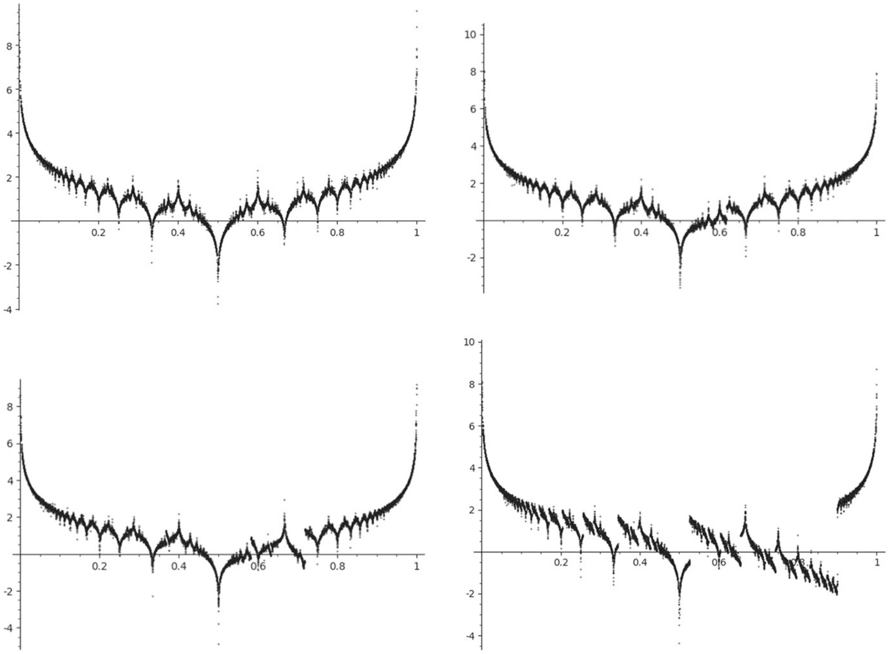


FIGURE 2. Numerical computations of W_α for $\alpha = 1/2$ (upper left), $\alpha = \frac{\sqrt{5}-1}{2}$ (upper right), $\alpha = e - 2$ (lower left) and $\alpha = 0.9$ (lower right)

2.1. Proof that real k-Brjuno functions are BMO functions

The main idea of the proof is to prove the statement for $B_{k,1/2}$, see Proposition 2.4, and to show then that $B_{k,\alpha}$ differs from $B_{k,1}$ only by an L^∞ function which follows from Propositions 2.5 and 2.8. As the proof follows from very similar arguments as those in [25, Thm. 3.2], we will only describe shortly the necessary changes in the proofs.

We start by introducing

$$\begin{aligned} X_* &= \{f \in \mathrm{BMO}(\mathbb{R}) : f(x+1) = f(x) \\ &\qquad \text{for all } x \in \mathbb{R},\ f(-x) = f(x) \text{ for all } x \in [0, 1/2]\} \end{aligned} \tag{2.2}$$

endowed with a norm which is the sum of the BMO seminorm $\|\cdot\|_{*,[0,1/2]}$ as in (2.1) and of the L^2 norm on the interval $(0, 1/2)$ (w.r.t. the $A_{1/2}$-invariant probability measure). Then one has:

Proposition 2.4. [25, Thm. 3.3] *For all $k \in \mathbb{R}_{>0}$, the operator $T_{k,1/2}$ as in (1.7) is a bounded linear operator from X_* to X_* whose spectral radius is bounded above by $(\sqrt{2}-1)^k$.*

To proceed, we prove an analog of [25, Prop. 2.3, eq. (iv)].

Proposition 2.5. *For all $k \in \mathbb{R}_{>0}$, there exists a constant $C_{1,k} > 0$ such that for all $\alpha \in [1/2, 1]$ and $x \in \mathbb{R}\backslash\mathbb{Q}$, one has*

$$\left| B_{k,\alpha}(x) - \sum_{j=0}^{\infty} \frac{\log q_{j+1}^{(\alpha)}(x)}{\left(q_j^{(\alpha)}(x)\right)^k} \right| < C_{1,k}.$$

Before we start with the proof, we recall the following property.

Remark 2.6. Let $k \in \mathbb{R}_{>0}$. As in [25, Remark 1.7], by using $\log q_j^{(\alpha)} \lesssim_k (q_j^{(\alpha)})^{k/2}$ for $k \geq 0$, we can show that for all $\alpha \in [1/2, 1]$, there are constants $c_{1,k}$ and $c_{2,k}$ such that

$$\sum_{j=0}^{\infty} \frac{\log q_j^{(\alpha)}}{(q_j^{(\alpha)})^k} \lesssim_k \sum_{j=0}^{\infty} \frac{1}{(q_j^{(\alpha)})^{k/2}} \leq c_{1,k} \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{\log 2}{(q_j^{(\alpha)})^k} \leq c_{2,k}.$$

Note that $c_{1,k}$ and $c_{2,k} \to \infty$ as $k \to 0$.

Proof of Proposition 2.5. For the following calculations we drop the dependence on α and x. We obtain by analogous calculations as in [25, Prop. 2.3, eq. (iv)] that

$$\begin{aligned}
&\left| -B_{k,\alpha}(x) + \sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j^k} \right| = \left| \sum_{j=0}^{\infty} \beta_{j-1}^k \log \frac{\beta_j}{\beta_{j-1}} + \sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j^k} \right| \\
&= \left| \sum_{j=0}^{\infty} \beta_{j-1}^k \log(\beta_j q_{j+1}) - \sum_{j=0}^{\infty} \beta_{j-1}^k \log \beta_{j-1} + \sum_{j=0}^{\infty} \left(\frac{1}{q_j^k} - \beta_{j-1}^k \right) \log q_{j+1} \right| \\
&\leq \sum_{j=0}^{\infty} \left| \beta_{j-1}^k \log(\beta_j q_{j+1}) \right| + \sum_{j=0}^{\infty} \left| \beta_{j-1}^k \log \beta_{j-1} \right| + \sum_{j=0}^{\infty} \left| \left(\frac{1}{q_j^k} - \beta_{j-1}^k \right) \log q_{j+1} \right|.
\end{aligned}$$

We have

$$\sum_{j=0}^{\infty} \left| \beta_{j-1}^k \log(\beta_j q_{j+1}) \right| \leq 2^k \sum_{j=0}^{\infty} \frac{\log 2}{q_j^k} \leq 2^k c_{2,k},$$

where the last estimate follows as in the proof of [25, Prop. 2.3, eq. (iv)] and $c_{2,k}$ is given in Remark (2.6). Furthermore,

$$\sum_{j=0}^{\infty} \left| \beta_{j-1}^k \log \beta_{j-1} \right| \leq 2^k \sum_{j=0}^{\infty} \frac{\log q_j + \log 2}{q_j^k} \leq 2^k (c_{1,k} + c_{2,k})$$

which also follows as in the proof of [25, Prop. 2.3, eq. (iv)] and $c_{1,k}$ is given in Remark (2.6). Since $\frac{1}{q_j^k} = \beta_{j-1} + \epsilon_j \frac{q_{j-1}}{q_j} \beta_j$, we have

$$\sum_{j=0}^{\infty} \left| \left(\frac{1}{q_j^k} - \beta_{j-1}^k \right) \log q_{j+1} \right| \lesssim_k \sum_{j=0}^{\infty} \max \left\{ \beta_{j-1}^{k-1}, \frac{1}{q_j^{k-1}} \right\} \left| \epsilon_j \frac{q_{j-1}}{q_j} \beta_j \right| \log q_{j+1}.$$

(2.3)

If $k \geq 1$, then (2.3) is bounded above by

$$\sum_{j=0}^{\infty} \beta_j \log q_{j+1} \leq 2 \sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_{j+1}} \leq 2c_{1,k}.$$

If $0 < k < 1$, then (2.3) is bounded above by

$$\sum_{j=0}^{\infty} \max\left\{\frac{1}{\beta_{j-1}^{1-k}}, q_j^{1-k}\right\} \frac{q_{j-1}}{q_j} \beta_j \log q_{j+1} \leq \sum_{j=0}^{\infty} \max\left\{\frac{\beta_j}{\beta_{j-1}^{1-k}}, \frac{2q_{j-1}}{q_j^k q_{j+1}}\right\} \log q_{j+1}$$
$$\leq \sum_{j=0}^{\infty} \max\left\{\beta_j^k, \frac{2}{q_{j+1}^k}\right\} \log q_{j+1} \leq \sum_{j=0}^{\infty} \frac{2^k}{q_{j+1}^k} \log q_{j+1} \leq 2^k c_{1,k}.$$

Remark 2.7. With analogous methods as above using the absolute values of the sum, we also obtain the following statement: There exists a constant $C > 0$ such that for all $\alpha \in [1/2, 1]$ and $x \in \mathbb{R}\backslash\mathbb{Q}$, one has

$$\left| W_\alpha(x) - \sum_{j=0}^{\infty} (-1)^j \frac{\log q_{j+1}^{(\alpha)}(x)}{q_j^{(\alpha)}(x)} \right| < C.$$

The next proposition is an equivalent of [25, Prop. 2.4].

Proposition 2.8. *For all $k \in \mathbb{R}_{>0}$, there exists a constant $C_{2,k} > 0$ such that for all $\alpha \in [1/2, 1]$ and for all $x \in \mathbb{R}\backslash\mathbb{Q}$ one has*

$$\left| B_{k,\alpha}(x) - \sum_{j=0}^{\infty} \frac{\log q_{j+1}^{(1)}(x)}{(q_j^{(1)}(x))^k} \right| \leq C_{2,k}.$$

Proof. The proof follows completely analogously to that of [25, Prop, 2.4] with the only difference that we use Proposition 2.5 instead of [25, Prop. 2.3] and each $q_j^{(1)}$ (which is denoted by Q_j in [25]) in the denominator is replaced by $(q_j^{(1)})^k$. □

Proof of Proposition 2.1. By Proposition 2.4, $1 - T_{k,\alpha}$ is invertible on X_*. A $\mathbb{Z}$–periodic even function equal to $-\log x$ on $(0, 1/2]$ is in X_*. Thus, $B_{k,1/2}$ is BMO. Since by Proposition 2.8 for all $\alpha \in [1/2, 1]$ the k-Brjuno function $B_{k,\alpha}$ differs from $B_{k,1/2}$ only by an L^∞ function, $B_{k,\alpha}$ is a BMO function as well.

2.2. Proofs of the BMO properties of the Wilton function

Proof of Theorem 2.2. For brevity, in the following we write $O_I(f) = \frac{1}{|I|} \int_I |f(x) - f_I| dx$. Furthermore, as we only consider $\alpha = 1$, we also always write W instead of W_1. By [24, Prop. A.7], if I_1 and I_2 are two consecutive intervals, then

$$O_{I_1\cup I_2}(f) = \frac{|I_1|}{|I_1|+|I_2|}O_{I_1}(f) + \frac{|I_2|}{|I_1|+|I_2|}O_{I_2}(f) + \frac{2|I_1||I_2|}{(|I_1|+|I_2|)^2}|f_{I_1} - f_{I_2}|. \tag{2.4}$$

Let $I_n := \left[-\frac{1}{n}, \frac{1}{n}\right] = \left[-\frac{1}{n}, 0\right] \cup \left[0, \frac{1}{n}\right] =: I_n^- \cup I_n^+$. By (2.4), $O_{I_n}(f) \geq \frac{1}{2}|f_{I_n^-} - f_{I_n^+}|$ for any $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. By [6, Lem. 2], we have

$$\int_0^x W(t)\mathrm{d}t = x\log(1/x) + x + O(x^2), \tag{2.5}$$

and

$$\int_{1-x}^1 W(t)\mathrm{d}t = -x\log(1/x) - x + O(x^2\log(2/x)), \tag{2.6}$$

thus we clearly have

$$W_{I_n^+} = \log(n) + 1 + O(1/n).$$

Since W is $\mathbb{Z}$–periodic, we also have

$$W_{I_n^-} = -\log(n) - 1 + O(n^{-1}\log(2n)).$$

Thus,

$$O_{I_n}(W) \geq \log(n) + 1 + O(n^{-1}\log(2n)),$$

which completes the proof of the theorem.

The proof that k-Wilton functions are not BMO functions would follow in a similar manner with error terms $O(x^{k+1})$ and $O(x^{k+1}\log(2/x))$ in (2.5) and (2.6) instead.

For the proof of Theorem 2.3, we can not use exactly the same strategy as for the proof of Proposition 2.1. The reason is that, as we have seen in Theorem 2.2, W_1 is not a BMO function. Hence, comparing W_α for $\alpha < 1$ with W_1 can not work. Instead, the underlying idea of the proof is to use that $W_{1/2}$, the Wilton function with respect to the nearest integer continued fraction, is a BMO function and compare $W_{1/2}$ with W_α for $\alpha \in [1/2, g]$, see Proposition 2.9.

Let X_* be defined as in (2.2). Since $S_{1/2} = -T_{1,1/2}$ clearly by Proposition 2.4 we have $W_{1/2} \in X_*$. We will first show that the uniform boundedness of $W_{1/2} - W_\alpha$, for $\alpha \in [\frac{1}{2}, g]$, in the following proposition, and then we will complete the proof of Theorem 2.3.

Proposition 2.9. *For $\alpha \in [\frac{1}{2}, g]$, we have $W_{1/2} - W_\alpha \in L^\infty$, where $g = \frac{\sqrt{5}-1}{2}$.*

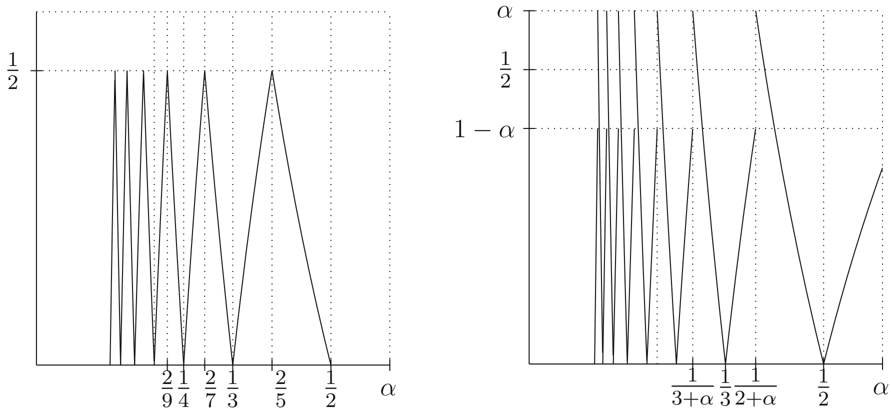


FIGURE 3. The graphs of $A_{1/2}$ and A_α for $\alpha \in [\frac{1}{2}, g]$, where $g = \frac{\sqrt{5}-1}{2}$

Proof. Let $x \in [0, \frac{1}{2}]$. Since $\alpha \in [\frac{1}{2}, g]$, we have $2 - \frac{1}{\alpha} \le 1 - g \le \frac{1}{2+\alpha} \le 1 - \alpha$. Recall that

$$A_{1/2}(x) = \begin{cases} \frac{1}{x} - k & \text{if } x \in (\frac{1}{k+1/2}, \frac{1}{k}], \\ (k+1) - \frac{1}{x} & \text{if } x \in (\frac{1}{k+1}, \frac{1}{k+1/2}], \end{cases} \qquad \text{for } k \ge 2.$$

Since $\alpha \le \frac{1}{1+\alpha}$, we have

$$A_\alpha(x) = \begin{cases} \frac{1}{x} - k & \text{if } x \in (\frac{1}{k+\alpha}, \frac{1}{k}], \\ (k+1) - \frac{1}{x} & \text{if } x \in (\frac{1}{k+1}, \frac{1}{k+\alpha}], \\ 2 - \frac{1}{x} & \text{if } x \in (\frac{1}{2}, \alpha], \end{cases} \qquad \text{for } k \ge 2.$$

See Fig. 3 for the graphs of $A_{1/2}$ and A_α for a typical $\alpha \in (1/2, g)$.

Let us denote by $(a_n^{(\alpha)}, \epsilon_n^{(\alpha)})$ the nth partial quotient of x given as a α-continued fraction, $x_n^{(\alpha)} = A_\alpha^n(x)$, where $x_0^{(\alpha)} = x$ and $p_n^{(\alpha)}/q_n^{(\alpha)}$ is its nth principal convergent.

We have $\epsilon_1^{(1/2)} \ne \epsilon_1^{(\alpha)}$ if and only if $x \in (\frac{1}{k+\alpha}, \frac{1}{k+1/2})$ for some $k \ge 2$. In this case,

$$(a_1^{(1/2)}, \epsilon_1^{(1/2)}) = (k+1, -1), \ (a_1^{(\alpha)}, \epsilon_1^{(\alpha)}) = (k, 1) \ \text{ and } \ x_1^{(1/2)} = 1 - x_1^{(\alpha)}.$$

Let

$$t_i = \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{\ddots - \cfrac{1}{3 - \cfrac{1}{2}}}}} \quad \text{and} \quad \frac{r_i}{s_i} = \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{\ddots - \cfrac{1}{3}}}},$$

where 3 in the continued fraction expansion appears i times. Note that $r_i = s_{i-1}$ and $2 - \frac{1}{1-t_i} = \frac{r_i}{s_i}$. More precisely,

$$\{t_i\}_{i\geq 0} = \left\{\frac{1}{2}, \frac{2}{5}, \frac{5}{13}, \frac{13}{34}, \frac{34}{89}, \cdots\right\} \subset \left(1-g, \frac{2}{5}\right) \cup \left\{\frac{1}{2}\right\},$$
$$\left\{\frac{r_i}{s_i}\right\}_{i\geq 0} = \left\{0, \frac{1}{3}, \frac{3}{8}, \frac{8}{21}, \frac{21}{55}, \cdots\right\} \subset \{0\} \cup \left[\frac{1}{3}, 1-g\right).$$

Then, $A_{1/2}(t_i) = t_{i-1}$, $A_\alpha(\frac{r_i}{s_i}) = A_{1/2}(\frac{r_i}{s_i}) = \frac{r_{i-1}}{s_{i-1}}$ for $i \geq 1$ and $t_i \searrow 1-g$ and $\frac{r_i}{s_i} \nearrow 1-g$.

Now, we suppose that n is the minimal index such that

$$x_1^{(1/2)} = x_1^{(\alpha)},\, x_2^{(1/2)} = x_2^{(\alpha)},\, \cdots, x_{n-1}^{(1/2)} = x_{n-1}^{(\alpha)}, \text{ but } x_n^{(1/2)} \neq x_n^{(\alpha)}.$$

Then,

$$\begin{aligned}&(a_i^{(1/2)}, \epsilon_i^{(1/2)}) = (a_i^{(\alpha)}, \epsilon_i^{(\alpha)}) \text{ and } q_i^{(1/2)} = q_i^{(\alpha)} \text{ for } 1 \leq i \leq n-1,\\ &x_n^{(1/2)} = 1 - x_n^{(\alpha)},\, a_n^{(1/2)} = a_n^{(\alpha)} + 1 \geq 3,\, \epsilon_n^{(1/2)} = -1,\, \epsilon_n^{(\alpha)} = 1 \text{and } q_n^{(1/2)}\\ &\quad = q_n^{(\alpha)} + q_{n-1}^{(1/2)}.\end{aligned}$$

Since $x_n^{(1/2)} \neq x_n^{(\alpha)}$, $x_n^{(1/2)} = 1 - x_n^{(\alpha)}$ and the domain of $A_{1/2}$ is $[0, \frac{1}{2}]$, we have

$$x_n^{(1/2)} \in \left[1-\alpha, \frac{1}{2}\right) \text{ and } x_n^{(\alpha)} \in \left(\frac{1}{2}, \alpha\right]. \tag{2.7}$$

Since $[1-\alpha, \frac{1}{2}) \subset \bigsqcup_{i=1}^{\infty}(t_i, t_{i-1}]$, there is $i \geq 1$ such that $x_n^{(1/2)} \in (t_i, t_{i-1}]$. This implies

$$x_{n+j}^{(1/2)} = 3 - \frac{1}{x_{n+j-1}^{(1/2)}} \in (t_{i-j}, t_{i-j-1}] \text{ and } (a_{n+j}^{(1/2)}, \epsilon_{n+j}^{(1/2)}) = (3, -1) \text{ for } 1 \leq j \leq i-1,$$
$$x_{n+i}^{(1/2)} = \frac{1}{x_{n+i-1}^{(1/2)}} - 2 \text{ and } (a_{n+i}^{(1/2)}, \epsilon_{n+i}^{(1/2)}) = (2, 1).$$

On the other hand, $x_n^{(\alpha)} \in [1 - t_{i-1}, 1 - t_i)$, which implies

$$x_{n+1}^{(\alpha)} = 2 - \frac{1}{x_n^{(\alpha)}} \in \left(\frac{r_{i-1}}{s_{i-1}}, \frac{r_i}{s_i}\right] \text{ and } (a_{n+1}^{(\alpha)}, \epsilon_{n+1}^{(\alpha)}) = (2, -1),$$

$$x_{n+j}^{(\alpha)}=3-S\frac{1}{x_{n+j-1}^{(\alpha)}}\in\left(\frac{r_{i-j}}{s_{i-j}},\frac{r_{i-j+1}}{s_{i-j+1}}\right] \text{ and } (a_{n+j}^{(\alpha)},\epsilon_{n+j}^{(\alpha)})=(3,-1) \text{ for } 2\le j\le i.$$

Then,

$$q_{n+j}^{(1/2)}-q_{n+j}^{(\alpha)}=q_{n+j-1}^{(1/2)}, \text{ for } 1\le j\le i-1, \quad \text{and } q_{n+i}^{(1/2)}=q_{n+i}^{(\alpha)}.$$

In the following for $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in \mathrm{GL}_2(\mathbb{Z})$, we denote by $g.z=\frac{az+b}{cz+d}$ the Möbius transform applied on z. With this notation we have

$$x_{n+i}^{(1/2)}=\begin{pmatrix}-2&1\\1&0\end{pmatrix}\begin{pmatrix}3&-1\\1&0\end{pmatrix}^{i-1}.x_n^{(1/2)} \text{ and } x_{n+i}^{(\alpha)}=\begin{pmatrix}3&-1\\1&0\end{pmatrix}^{i-1}\begin{pmatrix}2&-1\\1&0\end{pmatrix}.x_n^{(\alpha)}.$$

Since

$$\begin{pmatrix}3&-1\\1&0\end{pmatrix}\begin{pmatrix}-1&1\\-1&2\end{pmatrix}\begin{pmatrix}3&-1\\1&0\end{pmatrix}^{-1}=\begin{pmatrix}-1&1\\-1&2\end{pmatrix},$$

$$\begin{aligned}x_{n+i}^{(1/2)}&=\begin{pmatrix}-2&1\\1&0\end{pmatrix}\begin{pmatrix}3&-1\\1&0\end{pmatrix}^{i-1}\begin{pmatrix}-1&1\\0&1\end{pmatrix}\left(\begin{pmatrix}3&-1\\1&0\end{pmatrix}^{i-1}\begin{pmatrix}2&-1\\1&0\end{pmatrix}\right)^{-1}.x_{n+i}^{(\alpha)}\\&=\begin{pmatrix}-2&1\\1&0\end{pmatrix}\begin{pmatrix}-1&1\\-1&2\end{pmatrix}.x_{n+i}^{(\alpha)}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}.x_{n+i}^{(\alpha)}.\end{aligned}$$

It means that $\frac{1}{x_{n+i}^{(1/2)}}=\frac{1}{x_{n+i}^{(\alpha)}}-1$, thus, for $0\le c_1,c_2\le 1$ and $k\in\mathbb{N}$ such that $k\ge 2$,

$$x_{n+i}^{(1/2)}\in\left(\frac{1}{k+c_1},\frac{1}{k+c_2}\right] \quad \text{if and only if} \quad x_{n+i}^{(\alpha)}\in\left(\frac{1}{k+1+c_1},\frac{1}{k+1+c_2}\right].$$

If $x_{n+i}^{(1/2)}\in\left(\frac{1}{k+1},\frac{1}{k+\alpha}\right]\cup\left(\frac{1}{k+1/2},\frac{1}{k+1}\right]$, then

$$x_{n+i+1}^{(1/2)}=x_{n+i+1}^{(\alpha)},\ a_{n+i+1}^{(1/2)}=a_{n+i+1}^{(\alpha)}-1,\ \epsilon_{n+i+1}^{(1/2)}=\epsilon_{n+i+1}^{(\alpha)} \text{ and } q_{n+i+1}^{(1/2)}=q_{n+i+1}^{(\alpha)}.$$

If $x_{n+i}^{(1/2)}\in\left(\frac{1}{k+\alpha},\frac{1}{k+1/2}\right]$, then

$$x_{n+i+1}^{(1/2)}=1-x_{n+i+1}^{(\alpha)},\ a_{n+i+1}^{(1/2)}=a_{n+i+1}^{(\alpha)},\ \epsilon_{n+i+1}^{(1/2)}=-1,\ \epsilon_{n+i+1}^{(\alpha)}=1 \text{ and}$$
$$q_{n+i+1}^{(1/2)}-q_{n+i+1}^{(\alpha)}=q_{n+i}^{(1/2)}.$$

Thus, by repeating the above process, we conclude that

$$q_j^{(1/2)}-q_j^{(\alpha)}=0 \text{ or } q_{j-1}^{(1/2)}$$

and if $q_j^{(1/2)}-q_j^{(\alpha)}=q_{j-1}^{(1/2)}$, then $(a_{j+1}^{(1/2)},\epsilon_{j+1}^{(1/2)})=(3,-1)$ or $(2,1)$.

Since $a_j^{(1/2)}\ge 2$ and $a_j^{(1/2)}=2$ implies $\epsilon_j^{(1/2)}=1$, we have

$$q_j^{(\alpha)}\ge q_j^{(1/2)}-q_{j-1}^{(1/2)}\ge q_{j-1}^{(1/2)}. \tag{2.8}$$

Hence,

$$\left|\frac{1}{q_j^{(1/2)}} - \frac{1}{q_j^{(\alpha)}}\right| = 0 \quad \text{or} \quad \left|\frac{1}{q_j^{(1/2)}} - \frac{1}{q_j^{(\alpha)}}\right| = \frac{q_{j-1}^{(1/2)}}{q_j^{(1/2)} q_j^{(\alpha)}} \le \frac{1}{q_j^{(1/2)}} \le \frac{4}{q_{j+1}^{(1/2)}}. \tag{2.9}$$

On the other hand, (2.8) also implies

$$\left|\log q_{j+1}^{(1/2)} - \log q_{j+1}^{(\alpha)}\right| = \log\left(1 + \frac{q_{j+1}^{(1/2)} - q_{j+1}^{(\alpha)}}{q_{j+1}^{(\alpha)}}\right) \le \log\left(1 + \frac{q_j^{(1/2)}}{q_{j+1}^{(\alpha)}}\right) \le \log 2.$$

Combining this with (2.9) and Remark 2.6, we obtain

$$\begin{aligned}
&\left|\sum_{j=0}^{\infty}(-1)^j \frac{\log q_{j+1}^{(1/2)}}{q_j^{(1/2)}} - \sum_{j=0}^{\infty}(-1)^j \frac{\log q_{j+1}^{(\alpha)}}{q_j^{(\alpha)}}\right| \\
&\le \sum_{j=0}^{\infty}\left|\frac{\log q_{j+1}^{(1/2)}}{q_j^{(1/2)}} - \frac{\log q_{j+1}^{(\alpha)}}{q_j^{(1/2)}}\right| + \sum_{j=0}^{\infty}\left|\frac{\log q_{j+1}^{(\alpha)}}{q_j^{(1/2)}} - \frac{\log q_{j+1}^{(\alpha)}}{q_j^{(\alpha)}}\right| \\
&\le \sum_{j=0}^{\infty}\frac{\log 2}{q_j^{(1/2)}} + \sum_{j=0}^{\infty}\frac{4\log q_{j+1}^{(1/2)}}{q_{j+1}^{(1/2)}} \le 4c_1 + c_2.
\end{aligned}$$

For $x \in (\alpha, 1]$, the values of $W_{1/2}(x)$ and $W_\alpha(x)$ are defined symmetrically as

$$W_{1/2}(x) = W_{1/2}(1-x) \text{ and } W_\alpha(x) = W_\alpha(1-x).$$

By Remark 2.7, $W_{1/2} - W_\alpha$ is bounded on $[\alpha, 1]$.

For $x \in (1/2, \alpha]$, $1 - x \in [1-\alpha, 1/2)$. Since $W_{1/2}(x) = W_{1/2}(1-x)$, we can consider that $n = 0$ as in (2.7). □

Proof of Theorem 2.3. As we mentioned before, by Proposition 2.4, we have already observed $1 - S^{(1/2)}$ is invertible in X_*, which together with the fact that the $\mathbb{Z}$–periodic even function equal to $-\log x$ on $(0, 1/2]$ is in X_* implies that $W_{1/2}$ is BMO. By Proposition 2.9, W_α is BMO for $\alpha \in [1/2, g]$.

Finally, in a remark, we give what difficulties occur if one wants to extend the results of Theorem 2.3 to $\alpha \in (g, 1)$.

Remark 2.10. For the case $\alpha > g$, we cannot directly apply the same argument as in the proof of Proposition 2.9. If $\alpha > g$, then A_α has a branch which is defined by $1/x - 1$ (see Fig. 4 for the graph of A_α) in contrast to the case of $\alpha \le g$. It causes a different behaviour of the orbits of the points under A_α. In the proof, we showed a relation between $x_n^{(1/2)}$ and $x_n^{(\alpha)}$. By following the same argument for $\alpha > g$, we only obtain a relation between $x_n^{(1/2)}$ and $x_{n+N}^{(\alpha)}$, where N depends on the number of consecutive points in the orbit of x visiting $[\frac{1}{1+\alpha}, \alpha]$.

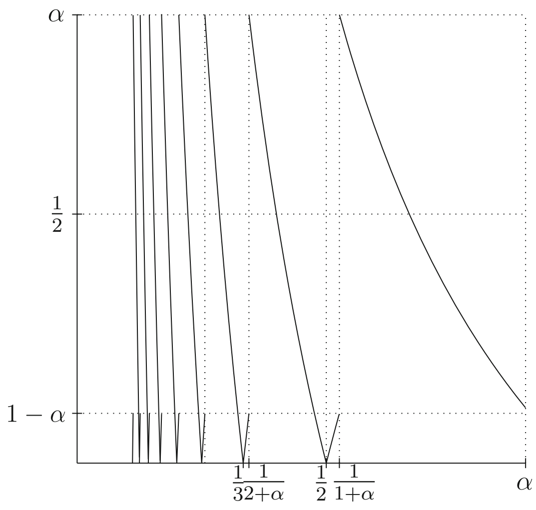


FIGURE 4. Graph of A_α when $\alpha > g$

3. Behaviour of the truncated real Brjuno function and the truncated real Wilton function

For $x \in \mathbb{R}$, recall that $\beta_{-1} = 1$ and

$$\beta_j(x) = \{x\}\, G(\{x\}) \cdots G^j(\{x\}) = |p_j(\{x\}) - q_j(\{x\})\{x\}| \quad \text{for } j \geq 0,$$

where G is the Gauss map and $p_j(\{x\})/q_j(\{x\})$ is the jth principal convergent of $\{x\}$ with respect to the regular continued fraction algorithm. Here, unlike in the previous section, we omit the α in $\beta_j^{(\alpha)}$ as we will always assume α to be one.

In this section, we are interested in comparing a finite k-Brjuno sum or finite Wilton sum to the k-Brjuno or Wilton sum of its principal convergent. To do so we first have to define the finite k-Brjuno or finite Wilton function respectively for a rational number.

Within this section, let p_j/q_j be a rational number whose continued fraction algorithm terminates after r steps, i.e. it can be written as

$$p_j/q_j = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots + \cfrac{1}{m_r}}}$$

with $m_r \geq 2$ when $q_j > 1$. (Of course, this can correspond to the rth principal convergent of a number whose continued fraction expansion starts with

$[m_0; m_1, \ldots, m_r, \ldots].$) With this, we can define the truncated real Brjuno function by

$$B_{k,\text{finite}}(p/q) = \sum_{j=0}^{r-1} \left(\beta_{j-1}(p/q - m_0)\right)^k \log\left(\frac{1}{G^j(p/q - m_0)}\right)$$

and the truncated real Wilton function by

$$W_{\text{finite}}(p/q) = \sum_{j=0}^{r-1} (-1)^j \beta_{j-1}(p/q - m_0) \log\left(\frac{1}{G^j(p/q - m_0)}\right).$$

Before stating the results of this section, we also introduce the notation $x_j = G^j(x)$ for $x \in (0,1)$. This enables us to state the next lemma which is an analog of [26, Lem. 5.20].

Lemma 3.1. *For each $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all $x \in (0,1)$ and $r \in \mathbb{N}$, we have*

$$\left| B_{k,\text{finite}}\left(\frac{p_r(x)}{q_r(x)}\right) - \sum_{j=0}^{r-1} \left(\beta_{j-1}(x)\right)^k \log\frac{1}{x_j} \right| \le C_k x_r (q_r(x))^{-1}$$

and

$$\left| W_{\text{finite}}\left(\frac{p_r(x)}{q_r(x)}\right) - \sum_{j=0}^{r-1} (-1)^j \beta_{j-1}(x) \log\frac{1}{x_j} \right| \le C_1 x_r (q_r(x))^{-1}.$$

Proof. To ease notation we write in the following p_r and q_r instead of $p_r(x)$ and $q_r(x)$ when the dependence on x is clear. If $r = 1$, then

$$\begin{aligned}
&|B_{k,\text{finite}}(p_1/q_1) - \log(1/x)| \\
&\quad = |W_{\text{finite}}(p_1/q_1) - \log(1/x)| = |\log(q_1/p_1) - \log(1/x)| \\
&\quad = |\log(m_1) - \log(m_1 + x_1)| = \log(1 + x_1/m_1) \le x_1/m_1.
\end{aligned}$$

Suppose that $r > 1$, then we have

$$\begin{aligned}
&\left| \sum_{j=0}^{r-1} \left[\left(\beta_{j-1}(p_r/q_r)\right)^k \log\frac{1}{G^j(p_r/q_r)} - \left(\beta_{j-1}(x)\right)^k \log\frac{1}{x_j} \right] \right| \\
&\le \sum_{j=0}^{r-1} \left| \left(\beta_{j-1}(p_r/q_r)\right)^k \left[\log\frac{1}{G^j(p_r/q_r)} - \log\frac{1}{x_j} \right] \right| \\
&\qquad + \sum_{j=0}^{r-1} \left| \left[\left(\beta_{j-1}(p_r/q_r)\right)^k - \left(\beta_{j-1}(x)\right)^k \right] \log\frac{1}{x_j} \right|
\end{aligned} \tag{3.1}$$

and similarly for the Wilton case if $k = 1$.

Note that

$$\beta_{j-1}(p_r/q_r) = \left|q_{j-1}\frac{p_r}{q_r} - p_{j-1}\right| \tag{3.2}$$

for $r \le j-1$. Since $x_j,\ G^j(p_r/q_r) \in \left[\frac{1}{m_{j+1}+1}, \frac{1}{m_{j+1}}\right]$, we have $2^{-1} \le \frac{x_j}{G^j(p_r/q_r)} \le 2$. By using (3.2) and the fact that $\log(y_2/y_1) \le (y_2-y_1)y_1^{-1}$ for $y_1 < y_2 \in \mathbb{R}_{>0}$, we have

$$\begin{aligned}\left|\log\frac{1}{G^j(p_r/q_r)} - \log\frac{1}{x_j}\right| &\le \max\left\{G^j(p_r/q_r), x_j\right\}\left|\frac{1}{G^j(p_r/q_r)} - \frac{1}{x_j}\right| \\ &\le 2\,G^j(p_r/q_r)\,\frac{|x-p_r/q_r|}{|q_j\,\frac{p_r}{q_r} - p_j|\,|q_jx-p_j|} \\ &\le \frac{2|x-p_r/q_r|}{|q_{j-1}\,\frac{p_r}{q_r} - p_{j-1}|\,|q_jx-p_j|}.\end{aligned}$$

Furthermore, we note

$$x = \frac{p_{i-1}x_i + p_i}{q_{i-1}x_i + q_i} \qquad \text{and} \qquad x_i = -\frac{q_ix - p_i}{q_{i-1}x - p_{i-1}} \tag{3.3}$$

which imply $\prod_{i=0}^{\ell}(-x_i) = q_\ell x - p_\ell$ and

$$|p_\ell - q_\ell x| = \left|p_\ell - q_\ell\frac{p_\ell x_{\ell+1} + p_{\ell+1}}{q_\ell x_{\ell+1} + q_{\ell+1}}\right| = \frac{|p_\ell q_{\ell+1} - q_\ell p_{\ell+1}|}{|q_\ell x_{\ell+1} + q_{\ell+1}|} < \frac{1}{q_{\ell+1}}. \tag{3.4}$$

Hence, for the first summand of (3.1), from (3.2), (3.3) and (3.4), we have

$$\begin{aligned}&\sum_{j=0}^{r-1}\left|\left(q_{j-1}\frac{p_r}{q_r} - p_{j-1}\right)^k\left[\log\frac{1}{G^j(p_r/q_r)} - \log\frac{1}{x_j}\right]\right| \\ &\le \frac{2}{q_r}\sum_{j=0}^{r-1}\left|q_{j-1}\frac{p_r}{q_r} - p_{j-1}\right|^{k-1}\left|\frac{p_r - xq_r}{q_jx - p_j}\right| \le \frac{2x_r}{q_r}\sum_{j=0}^{r-1}x_{j+1}\cdots x_{r-1} \\ &\le \frac{2x_r}{q_r}\sum_{j=0}^{r-1}\left(\frac{\sqrt{5}-1}{2}\right)^{r-j-1} < 2C'x_rq_r^{-1}, \qquad \text{(by Proposition 1.4-(iv) in [25])},\end{aligned}$$

where $C' = \sum_{j=0}^{\infty}\left(\frac{\sqrt{5}-1}{2}\right)^j = \frac{\sqrt{5}+3}{2}$.

On the other hand, by setting $X_{j,r} = q_{j-1}\,\frac{p_r}{q_r} - p_{j-1}$ and $Y_{j,r} = q_{j-1}x - p_{j-1}$, noting that $|X_{j,r}| \le 1/q_{j-1}$ and $|Y_{j,r}| \le 1/q_{j-1}$ we obtain for the second term of (3.1) that

$$\sum_{j=0}^{r-1}\left|\left(q_{j-1}\,\frac{p_r}{q_r} - p_{j-1}\right)^k - (q_{j-1}x - p_{j-1})^k\right|\log\frac{1}{x_j} \tag{3.5}$$

$$
\begin{aligned}
&= \sum_{j=0}^{r-1} |X_{j,r} - Y_{j,r}| \left| X_{j,r}^{k-1} + X_{j,r}^{k-2} Y + \cdots + X_{j,r} Y_{j,r}^{k-2} + Y_{j,r}^{k-1} \right| \log \frac{1}{x_j} \\
&\le \sum_{j=0}^{r-1} \frac{k}{q_{j-1}^{k-2}} |p_r/q_r - x| \log \frac{1}{x_j} \\
&\le \sum_{j=0}^{r-1} k q_{j-1} |p_r/q_r - x| \log \frac{1}{x_j}. \qquad (3.6)
\end{aligned}
$$

Since $\log(y) < y$ and using (3.4) and (3.3), the value in (3.6) is bounded above by

$$
\frac{k}{q_r} \sum_{j=0}^{r-1} \frac{|p_r - x q_r|}{|q_{j-1} x - p_{j-1}|} \frac{1}{x_j} = \frac{k}{q_r} \sum_{j=0}^{r-1} x_{j+1} \cdots x_{r-1} x_r \le k C' x_r q_r^{-1}.
$$

By letting $C_k := 2kC'$, we complete the proof. □

4. Complex continued fractions

We consider a continued fraction on a compact subset of $\mathbb{C}$ which is a complex analog of regular continued fractions. At the beginning of this section we will first define some domains which will be important for defining our complex continued fraction algorithm which we introduce in the sequel.

Similarly as in [26], we consider the following sets:

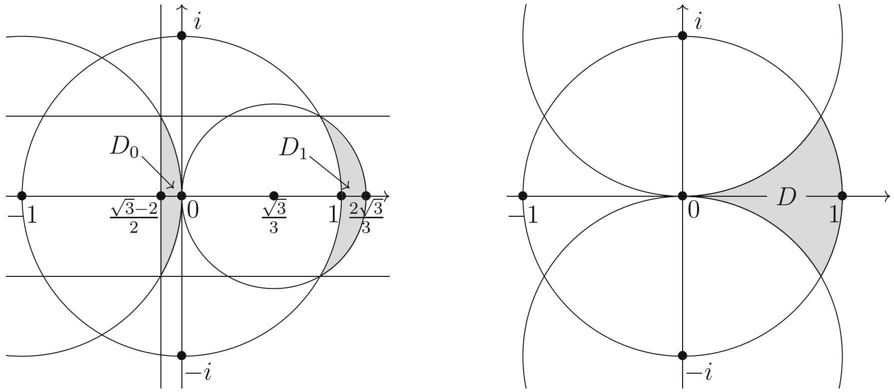
$$
D_0 = \left\{ z \in \mathbb{C} \mid |z+1| \le 1, \mathrm{re(z)} \ge \frac{\sqrt{3}}{2} - 1 \right\} \quad \text{(Figure 5a)}, \tag{4.1}
$$

$$
D_1 = \left\{ z \in \mathbb{C} \mid |z| \ge 1, \left| z - \frac{\sqrt{3}}{3} \right| \le \frac{\sqrt{3}}{3} \right\} \quad \text{(Figure 5a)}, \tag{4.2}
$$

$$
D = \{ z \in \mathbb{C} \mid |z| \le 1, |z-i| \ge 1, |z+i| \ge 1, \mathrm{re(z)} > 0 \} \quad \text{(Figures 5b and 8)}, \tag{4.3}
$$

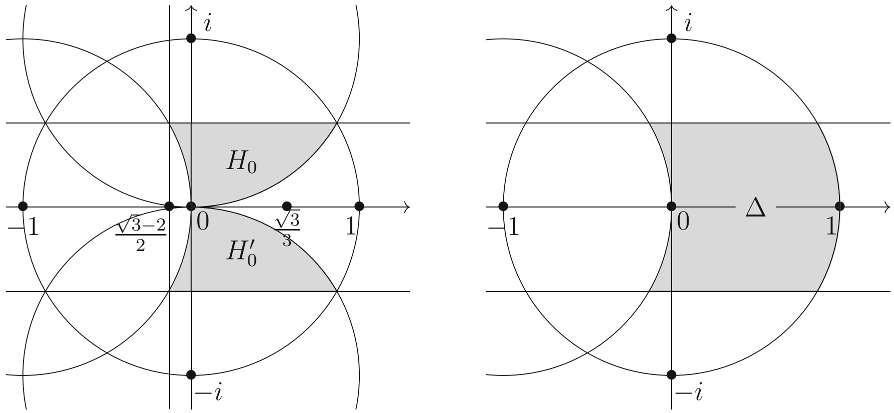
$$
H_0 = \left\{ z \in \mathbb{C} \mid |z-i| \le 1, |z+1| \ge 1, |\mathrm{Im(z)}| \le \frac{1}{2} \right\} \quad \text{(Figure 6a)}, \tag{4.4}
$$

$$
H_0' = \left\{ z \in \mathbb{C} \mid |z+i| \le 1, |z+1| \ge 1, |\mathrm{Im(z)}| \le \frac{1}{2} \right\} \quad \text{(Figure 6a)}, \tag{4.5}
$$



(a) D_0 and D_1.　　(b) D

FIGURE 5. (a)D_0 and D_1. (b)D



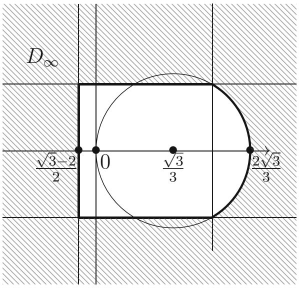
(a) H_0, H_0' and $H = H_0 \cup H_0'$.　　(b) Δ.

FIGURE 6. (a)H_0, H_0' and $H = H_0 \cup H_0'$. (b)Δ

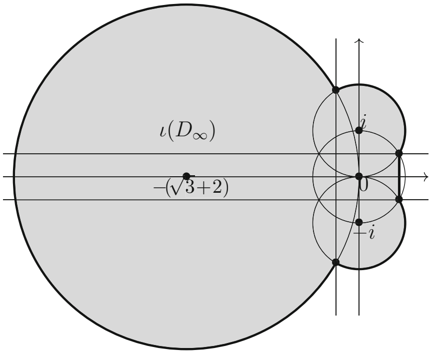
$$H = H_0 \cup H_0' \qquad \text{(Figure 6a)}, \tag{4.6}$$

$$\Delta = D \cup H_0 \cup H_0' = \left\{ z \in \mathbb{C} \mid |z| \le 1, |z+1| \ge 1, |\mathrm{Im(z)}| \le \frac{1}{2} \right\} \quad \text{(Figure 6b)}, \tag{4.7}$$

$$D_\infty = \overline{\mathbb{C}} \backslash (D_0 \cup \Delta \cup D)$$



(7a) D_∞



(7b) $\iota(D_\infty)$ where $\iota(z) = 1/z$.

FIGURE 7. (a)D_∞. (b)$\iota(D_\infty)$ where $\iota(z) = 1/z$

$$= \left\{ z \in \overline{\mathbb{C}} \mid |\mathrm{Im(z)}| > \frac{1}{2} \right\} \cup \left\{ z \in \overline{\mathbb{C}} \mid \mathrm{Re(z)} > \frac{\sqrt{3}}{2} - 1 \right\}$$
$$\cup \left\{ z \in \overline{\mathbb{C}} \mid \mathrm{Re(z)} > \frac{\sqrt{3}}{2}, \left| z - \frac{\sqrt{3}}{3} \right| > \frac{\sqrt{3}}{3} \right\} \qquad \text{(Figure7a).} \tag{4.8}$$

We define an extension of the continued fraction to the complex plane as follows. Let $z \in D$ as in Fig. 5b. Then $1/z \in \bigcup_{n\in\mathbb{N}}(n + \Delta)$, see Figure 6a. If $1/z \in n + \Delta$, then we take $m_1 = n$ and we set $z_1 := 1/z - m_1$. If $z_1 \in \Delta - D$, then we finish the process. If $z_1 \in D$, then we define m_2 by an integer n such that $1/z_1 \in n + \Delta$. Repeating this process, we obtain a continued fraction expansion $\{m_i\}_{i=1}^r$ such that

$$\frac{1}{z_i} = m_{i+1} + z_{i+1} \qquad \text{for all } 0 < i \leq r, \tag{4.9}$$

where $z_i \in D$ for $i < r$, and $z_r \in \Delta$. Let $D(m_1, \cdots, m_r)$ be the set of $z_0 \in D$ whose first r complex continued fraction entries equal $\{m_i\}_{i=1}^r$ (see Fig. 8).

Note that this continued fraction algorithm does not coincide with Hurwitz' continued fractions, which are often used on the complex plane, but which are expanded with Gaussian integers [17], see also [16]. The goal here is very different: our complex continued fraction expansion follows as closely as possible the standard continued fraction of neighboring real points. For non-real numbers the iteration of our algorithm leads to an increasing sequence of imaginary parts and it stops after finitely many steps, since "it is not possible anymore to associate a neighboring real point". We will see in the following section why this continued fraction definition makes sense.

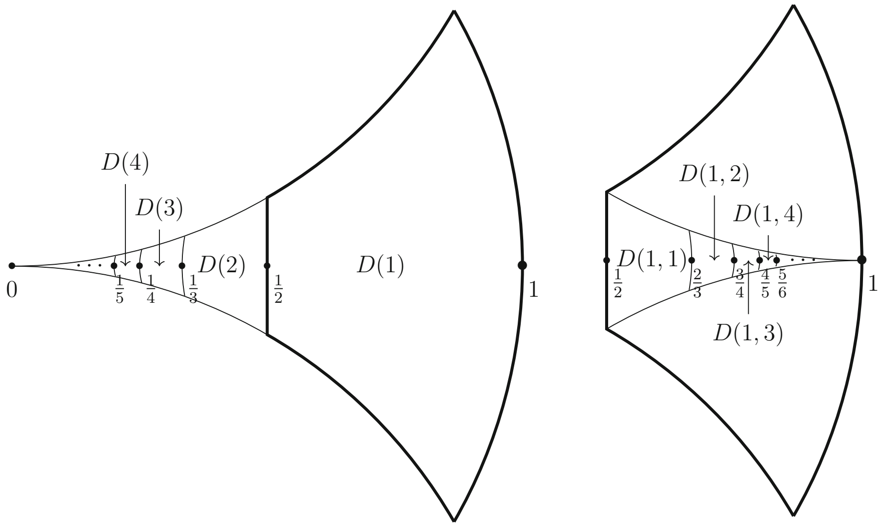


FIGURE 8. The sets $D(m_1, \cdots, m_r)$. The left figure is the partition of D by $D(n)$. The right figure is that of the sets $D(1, n)$ in $D(1)$

For $z \in \Delta$ and $i \geq 1$, let

$$\varepsilon_i = \begin{cases} 0 & \text{if } z \in D(m_1, \cdots, m_i) \text{ and } m_i = 1, \\ 1 & \text{otherwise.} \end{cases} \tag{4.10}$$

We note here that the just defined ε_i is independent of the definition of ϵ_i defined for the α-continued fractions. However, since in the following sections we only consider the regular continued fraction algorithm with the Gauss map G we do not expect it causes confusion for the reader.

We define p_ℓ/q_ℓ by

$$\frac{p_\ell}{q_\ell} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_\ell}}}}$$

and $q_0 = p_{-1} = q_{-2} = 1$ and $p_0 = p_{-2} = q_{-1} = 0$. Then,

$$z_0 = \frac{p_{i-1}z_i + p_i}{q_{i-1}z_i + q_i} \quad \text{and} \quad z_i = -\frac{q_i z_0 - p_i}{q_{i-1}z_0 - p_{i-1}}. \tag{4.11}$$

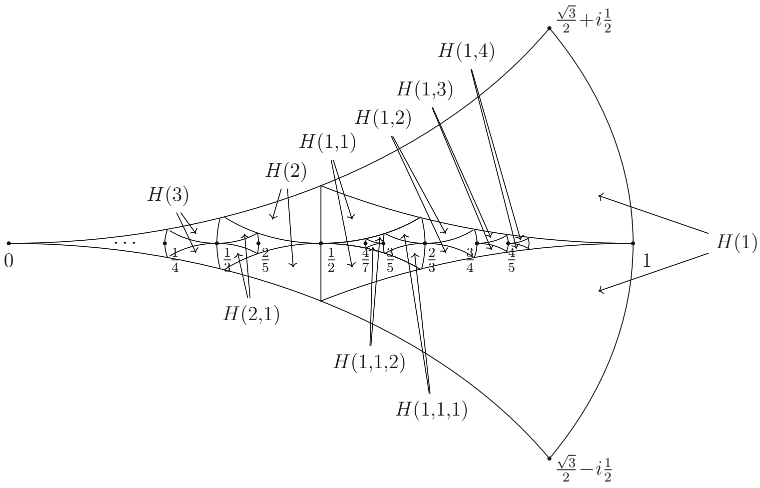


FIGURE 9. The partition of D by the sets $H(m_1, \cdots, m_r)$

By (4.11), we can easily see that

$$\prod_{i=0}^{\ell}(-z_i) = q_\ell z_0 - p_\ell. \tag{4.12}$$

We have

$$p_\ell - q_\ell z_0 = p_\ell - q_\ell \frac{p_\ell z_{\ell+1} + p_{\ell+1}}{q_\ell z_{\ell+1} + q_{\ell+1}} = \frac{p_\ell q_{\ell+1} - q_\ell p_{\ell+1}}{q_\ell z_{\ell+1} + q_{\ell+1}} = \frac{(-1)^{\ell+1}}{q_\ell z_{\ell+1} + q_{\ell+1}}. \tag{4.13}$$

Thus, we have

$$|p_\ell - q_\ell z_0| < \frac{1}{q_{\ell+1}}. \tag{4.14}$$

We set

$$H(m_1, \cdots, m_r) = D(m_1, \cdots, m_r) \setminus \operatorname{int}\left(\bigcup_{m_{r+1}\geq 1} D(m_1, \cdots, m_{r+1})\right) \tag{4.15}$$

where it is defined by H when $r = 0$, see (4.6) for the definition of H. The sets $H(m_1, \cdots,$
$m_r)$ give a partition of D as in Fig. 9. Then we have

$$\{z : |\mathrm{Im} z| \leq 1/2\} = \bigcup_{n\in\mathbb{Z}} \bigcup_{r\geq 0} \bigcup_{m_1,\cdots,m_r\geq 1} \left[H(m_1, \cdots, m_r) + n\right] \sqcup \mathbb{R}\backslash\mathbb{Q},$$

where the sets in the right-hand term have disjoint interiors. A set $H(m_1, \cdots,$
$m_r) + n$ meets $\mathbb{R}$ in a unique point which is rational.

5. Complexification of the operators T_k and S

Let J be a closed interval and $k \in \mathbb{N}$. Let $\mathcal{O}^k(\overline{\mathbb{C}} \setminus J)$ be the complex vector space of holomorphic functions in $\mathbb{C} \setminus J$, meromorphic in $\overline{\mathbb{C}} \setminus J$ with a zero at infinity of order at least k. There exists $C_{V,k} > 0$ such that for each $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}} \setminus J)$ and each neighbourhood V of J we have

$$|\varphi(z)| \leq C_{V,k}|z|^{-k} \sup_{\mathbb{C}\setminus\overline{V}} |\varphi| \quad \text{for } z \in C \setminus \overline{V}. \tag{5.1}$$

This fact can be easily proven in the following way. We obtain that the function $\tilde{\tilde{\varphi}}$ defined as $\tilde{\tilde{\varphi}}(w) = \varphi(w) \cdot w^k$ is still analytic on $\mathbb{C} \setminus \overline{V}$. By the maximum principle, we obtain that both functions $|\tilde{\tilde{\varphi}}|$ and $|\varphi|$ attain their maximum on the boundary of $\mathbb{C} \setminus \overline{V}$. Thus, we obtain for all $w \in \mathbb{C} \setminus \overline{V}$

$$|\varphi(w)| = \frac{|\tilde{\tilde{\varphi}}(w)|}{|w|^k} \leq \frac{\sup_{\mathbb{C}\setminus\overline{V}} |\tilde{\tilde{\varphi}}|}{|w|^k} = \frac{\sup_{\partial(\mathbb{C}\setminus\overline{V})} |\tilde{\tilde{\varphi}}|}{|w|^k} \leq \frac{C_{V,k} \sup_{\mathbb{C}\setminus\overline{V}} |\varphi|}{|w|^k} \tag{5.2}$$

with $C_{V,k} = \sup_{x\in\partial(\mathbb{C}\setminus\overline{V})} |x|^k$.

Let $\mathcal{O}^{-k}(\overline{\mathbb{C}} \setminus J)$ be the complex vector space of holomorphic functions in $\mathbb{C} \setminus J$, meromorphic in $\overline{\mathbb{C}} \setminus J$ with a pole at infinity of order at most k.

5.1. Hyperfunctions and extensions to spaces of complex analytic functions of the operators T_k and S

Recall that $T_k := T_{k,\alpha}$ as in (1.7) for given $\alpha \in [1/2, 1]$ and $S := S_\alpha$. We want to extend T_k and S for $\alpha = 1$ to the space of complex analytic functions.

We proceed as in [26], namely, we extend T_k and S to $A'([0, 1])$ the space of hyperfunctions with support contained in $[0, 1]$ for which we will first introduce some definitions. In Proposition 6.1, we will give the definition of these operators and prove that they are indeed well-defined. In Proposition 6.6, we will see that this definition makes sense in terms of T_k and S being a complex extension of T_k and S being studied in the previous sections.

5.1.1. Hyperfunctions. Let K be a non-empty compact set $K \subset \mathbb{R}$. Let us denote by $\mathcal{O}(K)$ the space of functions analytic in a neighbourhood V of K. *A hyperfunction with support in* K is a linear functional u on $\mathcal{O}(K)$, such that for all neighbourhoods V of K, there is a constant $C_V > 0$ such that

$$|u(\varphi)| \leq C_V \sup_V |\varphi|, \quad \forall \varphi \in \mathcal{O}(V).$$

We denote by $A'(K)$ the space of hyperfunctions with support in K.

Lemma 5.1. [26, Prop. A2.1.] *The space $A'([0, 1])$ is canonically isomorphic to $\mathcal{O}^1(\overline{\mathbb{C}} \setminus [0, 1])$ the complex vector space of holomorphic functions on $\overline{\mathbb{C}} \setminus [0, 1]$ vanishing at infinity.*

Let us explain the isomorphism in the lemma. Let

$$c_z(x) = \frac{1}{\pi(x-z)}. \tag{5.3}$$

Given $u \in A'([0,1])$, the corresponding $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$ is obtained by

$$\varphi(z) = u(c_z), \quad \forall z \in \mathbb{C} \setminus [0,1]. \tag{5.4}$$

On the other hand, for every $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$, the corresponding $u \in A'([0,1])$ is given by

$$u(\psi) = \frac{i}{2} \int_\gamma \varphi(z)\psi(z)\mathrm{d}z, \quad \forall \psi \in \mathcal{O}(V), \tag{5.5}$$

where V is a complex neighbourhood of $[0,1]$ and γ is any piecewise $\mathcal{C}^1$ path winding around $[0,1]$ in the positive direction. That u as in (5.5) and c_z as in (5.3) are inverse to each other can be easily seen by using the substitution $\xi = 1/\omega$ resulting in $\mathrm{d}\omega = -\xi^{-2}\mathrm{d}\xi$ which gives

$$\begin{aligned} u(c_z) &= \frac{i}{2}\int_\gamma \varphi(\omega)c_z(\omega)\mathrm{d}\omega = \frac{i}{2}\int_\gamma \varphi(\omega)\,\frac{1}{\pi}\,\frac{1}{\omega - z}\,\mathrm{d}\omega \\ &= \frac{i}{2\pi}\int_{1/\gamma} \frac{\varphi(1/\xi)}{(1/\xi - z)\,(-\xi^2)}\,\mathrm{d}\xi \\ &= \frac{1}{z}\int_{1/\gamma} \frac{i}{2\pi}\,\frac{\varphi(\xi^{-1})\xi^{-1}}{\xi - 1/z}\,\mathrm{d}\xi = \frac{1}{z}\,\varphi(z)\left(\frac{1}{z}\right)^{-1} = \varphi(z), \end{aligned}$$

where we denote by $1/\gamma$ the transformed path and notice that $1/\gamma$ goes in the negative direction. We remark that in (5.5) we correct the formula for u given in [26, Appendix 2] where it is given with an additional factor π. We write $u(x) = \frac{1}{2i}(\varphi(x+i0) - \varphi(x-i0))$ (see [26, Appendix 2] for details).

5.1.2. Formulas of the extensions of T_k and S. We obviously have $S = -T_1$. Hence, for the following, we will introduce the main formulas for T_k only and give explanations where differences occur for S. We also want to remark that if one wanted to study k-Wilton functions their appropriate operator would be $S_k = -T_k$ and most of the following calculations would follow analogously. For $k \in \mathbb{N}$, $m \in \mathbb{N}^*$ and $f \in L^2([0,1])$, let us consider

$$T_{k,m}f(x) = \begin{cases} x^k f\left(\frac{1}{x} - m\right), & \text{if } x \in \left[\frac{1}{m+1}, \frac{1}{m}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}$$

We then have $T_k = \sum_{m=1}^{\infty} T_{k,m}$. For $\varphi,\ \psi \in L^2([0,1])$, we define the L^2-adjoint $T_{k,m}^*$ by

$$\int_0^1 T_{k,m}\varphi(x)\psi(x)dx = \int_0^1 \varphi(x)T_{k,m}^*\psi(x)dx. \tag{5.7}$$

Therefore, we have

$$T^*_{k,m}\psi(x) = \frac{1}{(m+x)^{k+2}}\psi\left(\frac{1}{m+x}\right). \tag{5.8}$$

By a slight generalisation of [26, (1.10)], we have that if ψ is holomorphic in a neighbourhood V of $[0,1]$, then $T^*_{k,m}\psi$ is holomorphic in V and we have

$$\sup_V |T^*_{k,m}\psi| \leq \frac{1}{2m^{k+2}} \sup_V |\psi|,$$

where we may take $V = D^c_\infty$ equipped with the Poincaré metric on the hyperbolic Riemann surface $\overline{\mathbb{C}}\backslash[0,1]$. It follows that the series $\sum_{m=1}^\infty T_{k,m}u$ converges to a hyperfunction $T_k u$ in $A'([0,1])$.

We have

$$T^*_{k,m}c_z(x) = -z^k\left(c_{\frac{1}{z}-m}(x) - c_{-m}(x)\right) + \sum_{n=1}^{k}\frac{z^{k-n}}{n!}\cdot\frac{\partial^n}{\partial z^n}c_z(x)\Big|_{z=-m},$$

where c_z is as in (5.3). For $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}}\backslash[0,1])$, let u be the corresponding hyperfunction in $A'([0,1])$ as in (5.4). Then, $T_{k,m}\varphi$ is defined by $(T_{k,m}u)(c_z)$. If $z \notin [\frac{1}{m+1}, \frac{1}{m}]$, then we have

$$(T_{k,m}u)(c_z) = u(T^*_{k,m}c_z).$$

This follows from (5.7) and the fact that for $z \notin [\frac{1}{m+1}, \frac{1}{m}]$ we have that $T_{k,m}u$ and $T^*_{k,m}c_z$ are analytic. Hence, (5.7) already implies the equality.

By (5.4) and (5.5), we have

$$\begin{aligned} u(T^*_{k,m}c_z) &= -z^k\left(u(c_{\frac{1}{z}-m}(x)) - u\left(c_{-m}(x)\right)\right) + \sum_{n=1}^{k}\frac{z^{k-n}}{n!}u\left(\frac{\partial^n}{\partial z^n}c_z(x)|_{z=-m}\right) \\ &= -z^k\left(\varphi\left(\frac{1}{z}-m\right) - \varphi(-m)\right) + \sum_{n=1}^{k}\frac{z^{k-n}}{n!}\varphi^{(n)}(-m). \end{aligned}$$

For $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$, the formula for T_k is given by

$$T_k\varphi(z) = -\sum_{m=1}^{\infty} z^k\left(\varphi\left(\frac{1}{z}-m\right) - \varphi(-m)\right) + \sum_{m=1}^{\infty}\sum_{n=1}^{k}\frac{z^{k-n}}{n!}\varphi^{(n)}(-m), \tag{5.9}$$

which will be shown in Proposition 6.1. By using $S = -T_1$, we can also deduce that a natural extension of S to the space of complex analytic functions is

$$S\varphi(z) = \sum_{m=1}^{\infty} z\left(\varphi\left(\frac{1}{z}-m\right) - \varphi(-m)\right) - \sum_{m=1}^{\infty}\varphi'(-m). \tag{5.10}$$

5.1.3. Algebraic properties of the inversion of $(1 - T_k)$ and $(1 - S)$, the monoid $\mathcal{M}$ and its actions. Let us consider the monoid

$$\mathcal{M} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \mid d \geq b \geq a \geq 0 \text{ and } d \geq c \geq a \geq 0 \right\} \cup \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is generated by $g(m) := \left(\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}\right)$ for $m \in \mathbb{N}$. For $g \in \mathcal{M}$ and $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$, we define

$$\begin{aligned} L_{k,g}\varphi(z) : = \det(g)^{k+1} \Bigg[&(a-cz)^k \varphi\left(\frac{dz-b}{a-cz}\right) \\ &- \sum_{n=0}^{k} (a-cz)^{k-n} \frac{\det(g)^n}{c^n n!} \varphi^{(n)}\left(-\frac{d}{c}\right) \Bigg] \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} \overline{L}_g\varphi(z) : = \det(g)(a-cz) \Bigg(&\varphi\left(\frac{dz-b}{a-cz}\right) - \varphi\left(-\frac{d}{c}\right)\Bigg) \\ &- \frac{1}{c}\varphi'\left(-\frac{d}{c}\right) = \det(g) L_{1,g}\varphi(z). \end{aligned} \tag{5.12}$$

As we will see in Proposition 6.6, we have

$$(1 - T_k)^{-1}\varphi(z) = \sum_{j=0}^{\infty} T_k^j \varphi(z) = \sum_{g \in \mathcal{M}} L_{k,g}\varphi(z), \tag{5.13}$$

and from analogous arguments, we will see that

$$(1 - S)^{-1}\varphi(z) = \sum_{j=0}^{\infty} S^j \varphi(z) = \sum_{g \in \mathcal{M}} \overline{L}_g\varphi(z). \tag{5.14}$$

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $n \in \mathbb{Z}$, let

$$L_g^{(n)}\varphi(z) := \frac{\det(g)^{n+1}}{(a-cz)^n} \varphi\left(\frac{dz-b}{a-cz}\right) \tag{5.15}$$

and

$$\overline{L}_g^{(n)}\varphi(z) := \frac{\det(g)^n}{(a-cz)^n} \varphi\left(\frac{dz-b}{a-cz}\right) = \det(g)\, L_g^{(n)}\varphi(z). \tag{5.16}$$

Analogously to [26, (2.2)], we are interested in a connection between $L_{k,g}$ and $L_g^{(k+2)}$ and between $\overline{L}_g$ and $\overline{L}_g^{(3)}$ respectively. In the first case, we obtain by induction that for $t \in \mathbb{N}^*$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} (L_g^{(n)}\varphi)^{(t)} = \det(g)^{n+1} &\sum_{\ell=0}^{t-1} (n+\ell)(n+\ell+1)\cdots(n+t-1)\binom{t}{\ell} \\ &c^{t-\ell} \det(g)^{\ell} L_g^{(n+t+\ell)}\varphi^{(\ell)} + \det(g)^{t+n+1} L_g^{(n+2t)}\varphi^{(t)}, \end{aligned}$$

where $\sum_{\ell=0}^{-1}$ denotes an empty sum. In particular, for all $k \in \mathbb{N}^*$, we have

$$(L_g^{(-k)}\varphi)^{(k+1)} = L_g^{(k+2)}\varphi^{(k+1)}. \tag{5.17}$$

5.1.4. Polynomial corrections to the actions of $\mathcal{M}$. If $\varphi \in \mathcal{O}^{-k}(\overline{\mathbb{C}}\setminus[0,1])$, then it can be uniquely written as

$$\varphi(z) = \xi_k z^k + \xi_{k-1}z^{k-1} + \cdots + \xi_0 + p_k(\varphi)(z), \tag{5.18}$$

with $p_k(\varphi) \in \mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$. Note that if $\varphi \in \mathcal{O}^{-k}(\overline{\mathbb{C}}\setminus[0,1])$, then $\varphi^{(k+1)} = p_k(\varphi)^{(k+1)}$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ and $\psi(z) = \xi_k z^k + \xi_{k-1}z^{k-1} + \cdots + \xi_0$, then

$$\begin{aligned} L_g^{(-k)}\psi(z) = \det(g)^{1-k}\big(&\xi_k(dz-b)^k \\ &+\xi_{k-1}(dz-b)^{k-1}(a-cz) + \cdots + \xi_0(a-cz)^k\big), \end{aligned} \tag{5.19}$$

which means that $p_k(L_g^{(-k)}\psi) = 0$. Thus, p_k is a projection from $\mathcal{O}^{-k}(\overline{\mathbb{C}}\setminus[0,1])$ to $\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$. We have the formula

$$L_{k,g}\varphi = p_k(L_g^{(-k)}\varphi) \tag{5.20}$$

for $g \in \mathcal{M},\ \varphi \in \mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$ which defines an action of $\mathcal{M}$ on $\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$, where p_k depends on k. To verify this formula we define $\psi(u){:=}(L_g^{(-k)}\varphi)\big(\frac{a}{c} - \frac{1}{cu}\big)$. Since $\lim_{z\to\infty} p_k(L_g^{(-k)}\varphi)(z) = 0$ is equivalent to $\lim_{u\to 0}\psi(u) = 0$, it is enough to look at a projection of ψ. We have

$$\psi(u) = \det(g)^{1-k}u^{-k}\varphi\left(\frac{u\det(g) - d}{c}\right)$$

and if we denote by $\widetilde{p}_k$ the projection to a function vanishing at 0, then we obtain

$$\begin{aligned} \widetilde{p}_k\left(\psi(u)\right) &= \psi(u) - \det(g)^{1-k}u^{-k}\sum_{n=0}^{k}\frac{\psi^{(n)}(0)\,u^n}{n!} \\ &= \psi(u) - \det(g)^{1-k}\sum_{n=0}^{k}\frac{\det(g)^n\,\varphi^{(n)}\left(-\frac{d}{c}\right)\,u^{n-k}}{c^n n!}. \end{aligned} \tag{5.21}$$

Substituting back $u = (a-cz)^{-1}$ gives (5.20). From the definition of $L_{k,g}$ as in (5.11) and (5.19), the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}^{-k}(\overline{\mathbb{C}}\setminus[0,1]) \arrow[r, "L_g^{(-k)}"] \arrow[d, "p_k"'] & \mathcal{O}^{-k}(\overline{\mathbb{C}}\setminus[0,1]) \arrow[d, "p_k"] \\ \mathcal{O}^{1}(\overline{\mathbb{C}}\setminus[0,1]) \arrow[r, "L_{k,g}"'] & \mathcal{O}^{1}(\overline{\mathbb{C}}\setminus[0,1]) \end{tikzcd}$$

By (5.11) and (5.12), the connection between $\overline{L}_g$ and $\overline{L}_g^{(-1)}$ is similar as they only differ by a factor $\det(g)$ from $L_{1,g}$ and $L_g^{(-1)}$.

By the calculations in (5.21), the $(k+1)$th derivative of the difference between $L_{k,g}\varphi$ and $L_g^{(-k)}\varphi$ is zero for $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$. Thus, it follows from (5.17) that

$$(L_{k,g}\varphi)^{(k+1)}(\xi) = L_g^{(k+2)}\varphi^{(k+1)}(\xi) = \frac{\det(g)^{k+1}}{(a-c\xi)^{k+2}}\varphi^{(k+1)}(g^{-1}.\xi), \quad (5.22)$$

where $\varphi^{(k+1)} \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus [0,1])$. By Taylor's theorem, we have for every $\omega \in \overline{\mathbb{C}}\setminus[0,1]$ that

$$L_{k,g}\varphi(z) = L_{k,g}\varphi(\omega) + (L_{k,g}\varphi)'(\omega)(z-\omega) + \cdots + \frac{(L_{k,g}\varphi)^{(k-1)}}{(k-1)!}(z-\omega)^{k-1} + \int_\omega^z \frac{(L_{k,g}\varphi)^{(k+1)}(\xi)}{k!}(z-\xi)^k d\xi.$$

When ω goes to ∞, then only the remainder term will be left since $\frac{(L_{k,g}\varphi)^{(\ell)}(\omega)}{\ell!}(z-\omega)^\ell \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$ for all $0 \le \ell \le k-1$. From (5.22), the remainder term can be written as

$$\frac{\det(g)^{k+1}}{k!}\int_\infty^z \frac{(z-\xi)^k}{(a-c\xi)^{k+2}}\varphi^{(k+1)}(g^{-1}.\xi)d\xi.$$

By a change of variable with $\xi = \xi(t) = g.\left(-\frac{d}{c} + \frac{\det(g)t}{c(a-cz)}\right)$, which means that $\frac{1}{(a-c\xi)^2}d\xi = \frac{1}{c(a-cz)}dt$ and $\xi = -\frac{(a-cz)}{ct} + \frac{a}{c}$, we then deduce that the remainder term equals

$$\frac{\det(g)^{k+1}}{k!}\int_0^1 \left(\frac{1-t}{c}\right)^k \frac{\varphi^{(k+1)}(-\frac{d}{c} + \frac{\det(g)t}{c(a-cz)})}{c(a-cz)}dt.$$

Then, for $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1]), g \in \mathcal{M}$ and $z \notin \left[\frac{b}{d}, \frac{a}{c}\right]$, we have

$$L_{k,g}\varphi(z) = \frac{c^{-(k+1)}\det(g)^{k+1}}{k!}(a-cz)^{-1}\int_0^1 (1-t)^k\varphi^{(k+1)}\left(-\frac{d}{c} + \frac{\det(g)t}{c(a-cz)}\right)dt. \quad (5.23)$$

We then have for the special case $g = g(m)$ that

$$L_{k,g(m)}\varphi(z) = -z^k\left(\varphi\left(\frac{1}{z} - m\right) - \varphi(-m)\right) + \sum_{n=1}^k \frac{z^{k-n}}{n!}\varphi^{(n)}(-m) \quad (5.24)$$

$$= \frac{(-1)^k}{z\,k!}\int_0^1 \varphi^{(k+1)}\left(\frac{t}{z} - m\right)(1-t)^k \mathrm{d}t. \quad (5.25)$$

Connected to that we note that by (5.22) $L_g^{(k+2)}$ can be represented as

$$L_g^{(k+2)}\psi(z) = \frac{\det(g)^{k+1}}{(a-cz)^{k+2}}\psi\left(\frac{dz-b}{a-cz}\right) = \left(L_{k,g}\psi^{(-(k+1))}\right)^{(k+1)}(z), \quad (5.26)$$

for $\psi \in \mathcal{O}^{-k}(\overline{\mathbb{C}} \setminus [0,1])$, where $\psi^{(-(k+1))}$ denotes the $(k+1)$th primitive (i.e. an integration between ∞ and z) of ψ. In the special case that $g = g(m)$, we have that $\det(g(m)) = -1$ and thus

$$L_{g(m)}^{(k+2)}\psi(z) = -z^{-(k+2)}\psi\left(\frac{1}{z} - m\right). \quad (5.27)$$

For $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$ we can immediately conclude from (5.12), (5.24) and (5.25) that

$$\begin{aligned} \overline{L}_{g(m)}\varphi(z) &= z\left(\varphi\left(\frac{1}{z} - m\right) - \varphi(-m)\right) - \varphi'(-m) \\ &= \frac{1}{z}\int_0^1 \varphi''\left(\frac{t}{z} - m\right)(1-t)dt. \end{aligned} \quad (5.28)$$

we restricted ourselves to $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}} \setminus [0,1])$. However, the same calculations of this section hold true if we consider instead $\gamma_1 > \gamma_0 > -1$, $I = [\gamma_0, \gamma_1]$, $z \notin \left[0, \frac{1}{\gamma_0+m}\right]$, and $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}} \setminus I)$.

6. Convergence of the sum over the monoid and boundary behaviour

In this Section we will show how to adapt the results of Sections 3 and 4 of [26] to our monoid action. Also in our case the complex continued fraction introduced in Sect. 4 will play a fundamental role.

6.1. Convergence of the sum over the monoid

From the following proposition, we will be able to deduce that T_k is given by (5.9). It is an analog of [26, Prop. 3.1].

Let $-1 < \gamma_0 < \gamma_1$, $I = [\gamma_0, \gamma_1]$, $J = \left[0, \frac{1}{(1+\gamma_0)}\right]$, and

$$U_\varepsilon := \{z \in \mathbb{C} \mid \mathrm{re(z)} \in (-\infty, \gamma_0 - \varepsilon] \cup [\gamma_1 + \varepsilon, +\infty) \text{ or } \mathrm{Im(z)} \notin (-\varepsilon, \varepsilon)\}. \quad (6.1)$$

Proposition 6.1. *We have the following statements.*

(i) For all $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus I)$, the series $\sum_{m=1}^{\infty} L_{k,g(m)}\varphi$ converges uniformly on compact subsets $K \subseteq \overline{\mathbb{C}} \setminus J$ to a function $T_k\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus J)$ and there exist $\varepsilon > 0$ and $C_{K,k} > 0$ such that $\sup_K |T_k\varphi| \le C_{K,k} \sup_{U_\varepsilon} |\varphi|$.

(ii) For all $\psi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus I)$, the series $\sum_{m=1}^{\infty} L_{g(m)}^{(k+2)}\psi$ converges uniformly on compact subsets $K \subseteq \overline{\mathbb{C}} \setminus J$ to a function in $\mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus J)$, which we denote by $T_k^{(k+2)}\psi$ and there exist $\varepsilon > 0$ and $\overline{C}_{K,k} > 0$ such that $\sup_K |T_k^{(k+2)}\psi| \le \overline{C}_{K,k} \sup_{U_\varepsilon} |\psi|$.

(iii) For all $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus I)$, we have $T_k^{(k+2)}\varphi^{(k+1)} = (T_k\varphi)^{(k+1)}$.

Remark 6.2. Analogous statements can immediately be deduced for $\overline{L}_{g(m)}$ and S by using the relations $\overline{L}_{g(m)} = -L_{1,g(m)}$ and $S = -T$.

Proof of Proposition 6.1. Let $\varepsilon > 0$, then, from (5.1), there exists $c_{1,\varepsilon,k} > 0$ such that for all $z \in U_\varepsilon$ and $\psi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus I)$ we have $|\psi(z)| \le c_{1,\varepsilon,k}|z|^{-(k+2)} \sup_{U_\varepsilon} |\psi|$.

If $K \subseteq \overline{\mathbb{C}} \setminus J$ is compact, then there exists $\varepsilon(K) \in (0, \max\{|\gamma_0|, |\gamma_1|\})$ such that $\frac{1}{z} - m \in U_{\varepsilon(K)}$, for all $z \in K$ and $m \in \mathbb{N}$, which implies that $\varepsilon(K) \le |1/z - m|$. Also, there exist $c_{2,K} > 0$ and $M_{\gamma_0} \in \mathbb{N}$ such that for all $z \in K$ and $m \ge M_{\gamma_0}$ we have that $\left|\frac{1}{z} - m\right|^{-1} \le c_{2,K}m^{-1}$. Therefore, for all $z \in K$ and $\psi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus I)$ we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \left|\psi\left(\frac{1}{z} - m\right)\right| \le \sum_{m=1}^{\infty} c_{1,\varepsilon(K)} \left|\frac{1}{z} - m\right|^{-(k+2)} \sup_{U_{\varepsilon(K)}} |\psi| \\
=&c_{1,\varepsilon(K),k} \sup_{U_{\varepsilon(K)}} |\psi| \left(\sum_{m=1}^{M_{\gamma_0}-1} \left(\left|\frac{1}{z} - m\right|^{-1}\right)^{k+2} + \sum_{m=M_{\gamma_0}}^{\infty} \left(\left|\frac{1}{z} - m\right|^{-1}\right)^{k+2}\right) \\
\le&c_{1,\varepsilon(K),k} \sup_{U_{\varepsilon(K)}} |\psi| \left(\sum_{m=1}^{M_{\gamma_0}-1} \left(\varepsilon(K)^{-1}\right)^{k+2} + \sum_{m=M_{\gamma_0}}^{\infty} \left(c_{2,K}\, m^{-1}\right)^{k+2}\right) \\
=&c_{1,\varepsilon(K),k} \sup_{U_{\varepsilon(K)}} |\psi| \left((M_{\gamma_0} - 1)\varepsilon(K)^{-(k+2)} + {c_{2,K}}^{k+2} \sum_{m=M_{\gamma_0}}^{\infty} m^{-(k+2)}\right) \\
\le&\overline{C}_{K,k} \sup_{U_{\varepsilon(K)}} |\psi|, \qquad (6.2)
\end{aligned}$$

with $\overline{C}_{K,k} = c_{2,\varepsilon(K),k}\big((M_{\gamma_0} - 1)\varepsilon(K)^{-(k+2)} + c_{2,K}^{k+2} \sum_{m=M_{\gamma_0}}^{\infty} m^{-(k+2)}\big)$ and we obtain the first part of (i). By integrating $k + 1$ times between ∞ and z and substituting $\varphi = \psi^{(-(k+1))}$, we obtain the first part of (ii). The assertion (iii) then follows immediately.

Hence, by (5.22), (5.27) and (6.2) we have for all $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus I)$ and all $z \in K$ that

$$|(T_k\varphi)^{(k+1)}(z)| = |(T_k^{(k+2)}\varphi^{(k+1)})(z)| \leq \sum_{m=1}^{\infty} \left| L_{g(m)}^{(k+2)} \varphi^{(k+1)}(z) \right|$$

$$\leq |z|^{-(k+2)} \cdot \sum_{k=1}^{\infty} \left| \varphi^{(k+1)} \left(\frac{1}{z} - m \right) \right| \leq \overline{C}_{K,k} |z|^{-(k+2)} \sup_{U_{\varepsilon(K)}} |\varphi^{(k+1)}|,$$

implying (ii). We conclude (i) by using Cauchy's formula.

In particular, we immediately obtain from (5.24) that

$$T_k\varphi(z) = -\sum_{m=1}^{\infty} z^k \left(\varphi\left(\frac{1}{z} - m\right) - \varphi(-m) \right) + \sum_{m=1}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \varphi^{(n)}(-m)$$

proving (5.9).

Let D_∞ be given as in (4.8) and for $\rho > 0$ let

$$V_\rho(D_\infty) = \{z \in \overline{\mathbb{C}} \setminus [0,1] \mid d_{\text{hyper}}(z, D_\infty) < \rho\},$$

where d_{hyper} denotes the Poincaré metric on $\overline{\mathbb{C}} \setminus [0,1]$.

To define the complex k-Brjuno functions, we deal with $(1-T_k)^{-1}$ as in the real case, see (1.8) and also (7.9). The following proposition is an analogous property to [25, Thm. 2.6] for the real Brjuno function and it is a generalisation of [26, Prop. 3.3] for the case of $k = 1$, which guarantees that $(1 - T_k)^{-1} = \sum_{r=1}^{\infty} T_k^r$ converges.

However, before we state this proposition we first give an estimate of the derivatives of φ using Cauchy's integral formula. For $\varphi \in \mathcal{O}^n(\overline{\mathbb{C}} \setminus [0,1])$, let $\tilde{\tilde{\varphi}}(z) = \varphi(z) \cdot z^n$. Then we have

$$\varphi^{(j)}(z) = \sum_{i=0}^{j} \binom{j}{i} \frac{(-1)^i (n+i-1)!}{(n-1)!} \cdot \frac{\tilde{\tilde{\varphi}}^{(j-i)}(z)}{z^{n+i}}.$$

By Cauchy's integral formula, for a circle $\{\omega : |\omega - z| = R\}$ contained in $V_\rho(D_\infty)$ whose center is given by $z \in V_\rho(D_\infty)$, we have

$$|\tilde{\tilde{\varphi}}^{(\ell)}(z)| = \left| \frac{\ell!}{2\pi i} \int_{|\omega - z| = R} \frac{\tilde{\tilde{\varphi}}(\omega)}{(\omega - z)^{\ell+1}} d\omega \right| \leq \frac{\ell! \sup_{V_\rho(D_\infty)} |\tilde{\tilde{\varphi}}|}{R^\ell}.$$

Since we have

$$\sup_{V_\rho(D_\infty)} |\tilde{\tilde{\varphi}}| \leq \sup_{V_\rho(D_\infty)} |\varphi| \cdot \sup_{z \in \partial V_\rho(D_\infty)} |z|^n \leq \sup_{V_\rho(D_\infty)} |\varphi| \cdot \sup_{z \in \partial D_\infty} |z|^n,$$

we obtain

$$|\varphi^{(j)}(z)| \leq c'_{n,j} \frac{j! \sup_{V_\rho(D_\infty)} |\varphi|}{R^{n+j}}, \quad \text{where } c'_{n,j} := \sup_{z \in \partial D_\infty} |z|^n \cdot \sum_{i=0}^{j} \binom{n+i-1}{i}. \tag{6.3}$$

Proposition 6.3. *Let $\rho \geq 0$. The following statements hold.*

(i) For all $k \in \mathbb{N}$, there exists $C_{\rho,k} > 0$ such that, for all $r \geq 0$ and $\psi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus [0,1])$, we have

$$\sup_{z \in V_\rho(D_\infty)} |((T_k^{(k+2)})^r \psi)(z)| \leq C_{\rho,k} \left(\frac{\sqrt{5}-1}{2}\right)^{rk} \sup_{z \in V_\rho(D_\infty)} |\psi(z)|.$$

(ii) For all $k \in \mathbb{N}$, there exists $\overline{C}_{\rho,k}$ such that, for all $r \geq 0$ and $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$, we have

$$\sup_{z \in V_\rho(D_\infty)} |(T_k^r \varphi)(z)| \leq \overline{C}_{\rho,k} \left(\frac{\sqrt{5}-1}{2}\right)^{rk} \sup_{z \in V_\rho(D_\infty)} |\varphi(z)|.$$

Similarly to the last proposition, we remark also here that analogous statements can immediately be deduced for $\overline{L}_{g(m)}$ and S by using the relations $\overline{L}_{g(m)} = -L_{1,g(m)}$ and $S = -T$.

The proof of the above proposition is given in the Appendix.

In order to state the next proposition, we define

$$Z = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \mid n \in \mathbb{Z} \right\}. \tag{6.4}$$

From now on, we consider $L_{k,g}$ and $\overline{L}_g$ for $g \in \mathrm{GL}_2(\mathbb{Z})$. For $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$, since $\varphi_0^{(n)} \in \mathcal{O}^{n+1}(\overline{\mathbb{C}} \setminus [0,1])$, we have $\lim_{c \to 0} \varphi_0^{(n)}(-d/c)/c^n = 0$ for $n \geq 0$. Thus, we define $L_{k,g}$ when $c = 0$ by $L_{k,g}\varphi(z) = \det(g)^{k+1} a^k \varphi\left(\frac{dz-b}{a}\right)$. Especially, for $g \in Z$, we have

$$L_{k,\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)} \varphi(z) = \overline{L}_{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)} \varphi(z) = \varphi(z-n).$$

For $H \subset \mathrm{GL}_2(\mathbb{Z})$, let us denote

$$\sum\nolimits_H^{(k+2)} \psi := \sum_{g \in H} L_g^{(k+2)} \psi \quad \text{and} \quad \sum\nolimits_{H,k} \varphi := \sum_{g \in H} L_{k,g} \varphi,$$

which are uniformly summable on compact subsets of $\overline{\mathbb{C}} \setminus [0,1]$.

The following three propositions which give a generalisation of [26, Coro. 3.6] will show us a relation between $\sum_{\mathcal{M},k}$ and T_k. They state that $\sum_{\mathcal{M},k} \varphi$ and $\sum_{Z \cdot \mathcal{M},k} \varphi$ are well-defined on any compact set of $\overline{\mathbb{C}} \setminus [0,1]$ for $\varphi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus [0,1])$ and $\sum_{\mathcal{M},k}$ is the inverse of $(1 - T_k)$.

Proposition 6.4. *Let $\psi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus [0,1])$. We have*

$$\sum\nolimits_{\mathcal{M}}^{(k+2)} \psi = \sum_{r=0}^{\infty} (T_k^{(k+2)})^r \psi. \tag{6.5}$$

Let $K \subset \overline{\mathbb{C}} \setminus [0,1]$ be compact. Then, for all $k \in \mathbb{N}$, there exist $\varepsilon(K) > 0$ and $C_{K,k}$ such that

$$\sup_K \left| \sum\nolimits_{\mathcal{M}}^{(k+2)} \psi \right| \leq C_{K,k} \sup_{U_{\varepsilon(K)}} |\psi|, \tag{6.6}$$

where $U_{\varepsilon(K)}$ is as in (6.1) with $\gamma_0 = 0$ and $\gamma_1 = 1$.

Furthermore, $\sum_{Z\cdot\mathcal{M}}^{(k+2)} \psi$ is uniformly summable on all domains of the form $\{|\mathrm{Re}(z)| < A, |\mathrm{Im}(z)| \geq \delta\}$, for some $A, \delta > 0$. It is holomorphic in $\mathbb{C} \setminus \mathbb{R}$, $\mathbb{Z}$-periodic, bounded in the neighbourhood of $\pm i\infty$, and

$$\sum\nolimits_{Z\cdot\mathcal{M}}^{(k+2)} \psi = \sum\nolimits_{Z}^{(k+2)} \sum\nolimits_{\mathcal{M}}^{(k+2)} \psi. \tag{6.7}$$

Proof. Equation (6.5) follows from (B.2), and equation (6.6) follows directly from Proposition 6.3 taking ε and ρ such that $K \subset V_\rho(D_\infty) \subset U_{\varepsilon(K)} \subset \overline{\mathbb{C}} \backslash [0,1]$.

Furthermore, making use of (5.26), we have for $g' = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in Z$ that $L_{g'}^{(k+2)} \psi(z) = \psi(z-n)$. Then, we have for all $g' \in Z$ and $g \in \mathcal{M}$ that $L_{g'\cdot g}^{(k+2)} \psi = L_{g'}^{(k+2)} \left(L_g^{(k+2)} \psi \right)$ giving (6.7) in the case at least one side is uniformly summable which we will show in the following.

We have by Proposition 6.1 that $(T_k^{(k+2)})^r \psi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \backslash [0,1])$ for any $r \geq 0$. Furthermore, by the above consideration, we have that $\sum_Z^{(k+2)} \sum_{\mathcal{M}}^{(k+2)} \psi(z) = \sum_{n=1}^{\infty} \left(\sum_{\mathcal{M}}^{(k+2)} \psi(z-n) \right)$. Since $\sum_{n=1}^{\infty} f(z-n) \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus [0,1])$ holds if $f \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus [0,1])$, it follows that $\sum_Z^{(k+2)} \sum_{\mathcal{M}}^{(k+2)} \psi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus [0,1])$. □

Proposition 6.5. *Let $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$. We have*

$$\sum\nolimits_{\mathcal{M},k} \varphi = \sum_{r=0}^{\infty} T_k^r \varphi. \tag{6.8}$$

Furthermore, $\sum_{Z\cdot\mathcal{M},k} \varphi$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$, $\mathbb{Z}$-periodic, vanishing at $\pm i\infty$ and we have

$$\left(\sum_{\mathcal{M},k} \varphi \right)^{(k+1)} = \sum\nolimits_{\mathcal{M}}^{(k+2)} \varphi^{(k+1)} \quad \text{and} \quad \left(\sum_{Z\cdot\mathcal{M},k} \varphi \right)^{(k+1)} = \sum\nolimits_{Z\cdot\mathcal{M}}^{(k+2)} \varphi^{(k+1)}. \tag{6.9}$$

Proof. By the definition of $\sum_{H,k}$, $\sum_H^{(k+2)}$ and (5.22), the equations in (6.9) hold. Equation (6.8) follows from integrating the expression in Proposition 6.4 with $\psi = \varphi^{(k+1)}$, combining with 6.1-(iii) and the first equation of (6.9). The remaining statements follow immediately from the properties of φ and the uniform summability. □

Proposition 6.6. *We have*

$$(1-T_k)\sum\nolimits_{\mathcal{M},k} = \sum\nolimits_{\mathcal{M},k}(1-T_k) = \mathrm{id}, \quad \textit{and}$$
$$(1-T_k^{(k+2)})\sum\nolimits_{\mathcal{M}}^{(k+2)} = \sum\nolimits_{\mathcal{M}}^{(k+2)}(1-T_k^{(k+2)}) = \mathrm{id}.$$

Proof. It follows from Propositions 6.4 and 6.5. □

For $H \subset \mathrm{GL}_2(\mathbb{Z})$, we denote by

$$\overline{\sum}\nolimits_{H}^{(3)}\psi := \sum_{g\in H}\overline{L}_g^{(3)}\psi \quad \text{and} \quad \overline{\sum}\nolimits_{H}\varphi := \sum_{g\in H}\overline{L}_g\varphi, \tag{6.10}$$

which are uniformly summable on compact subsets of $\mathbb{C}\backslash[0,1]$. As an analog of the statements above and to [26, Coro. 3.6] we obtain the same statements as in Propositions 6.4, 6.5 and 6.6 with $k=1$ where we have to replace T_1 by S, $\sum$ by $\overline{\sum}$ and $L_{1,g}$ and $L_{1,g}^{(3)}$ by $\overline{L}_g$ and $\overline{L}_g^{(3)}$. In particular, it shows that (6.10) is summable when $H=\mathcal{M}$ or $H = Z\cdot\mathcal{M}$.

6.2. Boundary behaviour of the sums $\sum_{\mathcal{M},k}\varphi$ and $\overline{\sum}_{\mathcal{M}}\varphi$ over the monoid

We will consider the boundary behaviour of $\sum_{\mathcal{M},k}\varphi$ for $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}}\setminus[0,1])$. The following proposition is a generalisation of [26, Prop. 4.1] and explains the behaviour of $\sum_{\mathcal{M},k}\varphi$ near 0.

Proposition 6.7. *Let $-1<\gamma_0<\gamma_1$ and $I=[\gamma_0,\gamma_1]$. Also let*

$$U = \{z\in\mathbb{C} \mid \textit{re}(z) \in (-\infty,\gamma_0-1]\cup[\gamma_1+1,+\infty) \ \textit{or}\ \mathrm{Im}(z)\notin(-1/2,1/2)\}. \tag{6.11}$$

There exists $C_{I,k}>0$ such that, for all $\varphi\in\mathcal{O}^k(\overline{\mathbb{C}}\backslash I)$ and all $z\in D_0\cup H_0\cup H_0'$, we have

$$\left|T_k\varphi(z) - \sum_{m=1}^{\infty}\sum_{n=1}^{k}\frac{z^{k-n}}{n!}\varphi^{(n)}(-m)\right| \le C_{I,k}|z|^k\log(1+|z|^{-1})\sup_U|\varphi|. \tag{6.12}$$

Moreover, there exists $C_k>0$ such that, for all $\varphi\in\mathcal{O}^k(\overline{\mathbb{C}}\setminus[0,1])$ and all $z\in D_0\cup H_0\cup H_0'$, we have

$$\left|\sum_{\mathcal{M},k}\varphi(z) - \varphi(z) - \sum_{m=1}^{\infty}\sum_{n=1}^{k}\frac{z^{k-n}}{n!}\left(\sum_{\mathcal{M},k}\varphi\right)^{(n)}(-m)\right| \le C_k|z|^k\log(1+|z|^{-1})\sup_{D_\infty}|\varphi|. \tag{6.13}$$

We will give the proof of this proposition in much detail as it differs significantly from the case $k=1$.

Proof. As the proofs of (6.12) and (6.13) are similar, we will first prove (6.13) and later give the differences for the proof of (6.12). For the following, we set $\widetilde{\varphi} = \sum_{\mathcal{M},k} \varphi$. Then, Proposition 6.6 and Proposition 6.1-(i) imply

$$\widetilde{\varphi} = \varphi + T_k \widetilde{\varphi} = \varphi + \sum_{m=1}^{\infty} L_{k,g(m)} \widetilde{\varphi}.$$

By the above equation and (5.9), the expression on the left-hand side of (6.13) can thus be summarized as

$$\begin{aligned} &\left| \sum_{\mathcal{M},k} \varphi(z) - \varphi(z) - \sum_{m=1}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \left(\sum_{\mathcal{M}} \varphi \right)^{(n)} (-m) \right| \\ &= \left| T_k \widetilde{\varphi}(z) - \sum_{m=1}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \widetilde{\varphi}^{(n)}(-m) \right| = \left| \sum_{m=1}^{\infty} z^k \left(\widetilde{\varphi} \left(\frac{1}{z} - m \right) - \widetilde{\varphi}(-m) \right) \right|. \end{aligned} \tag{6.14}$$

We will split the sum in the following way:

$$\left| \sum_{\mathcal{M},k} \varphi(z) - \varphi(z) \sum_{m=1}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \left(\sum_{\mathcal{M},k} \varphi \right)^{(n)} (-m) \right|$$

$$\leq \left| \sum_{m=1}^{\lfloor 3|z|^{-1} \rfloor + 1} z^k \left(\widetilde{\varphi} \left(\frac{1}{z} - m \right) - \widetilde{\varphi}(-m) \right) \right| \tag{6.15}$$

$$+ \left| \sum_{m=\lfloor 3|z|^{-1} \rfloor + 2}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \widetilde{\varphi}^{(n)}(-m) \right| \tag{6.16}$$

$$+ \left| \sum_{m=\lfloor 3|z|^{-1} \rfloor + 2}^{\infty} z^k \left(\widetilde{\varphi} \left(\frac{1}{z} - m \right) - \widetilde{\varphi}(-m) \right) - \sum_{m=\lfloor 3|z|^{-1} \rfloor + 2}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \widetilde{\varphi}^{(n)}(-m) \right|. \tag{6.17}$$

We first estimate the first summand (6.15). We note that for $m \in \mathbb{N}_{\leq \lfloor 3|z|^{-1} \rfloor + 1}$ and $z \in D_0 \cup H_0 \cup H_0'$ we have that $-m \in D_\infty$ and $1/z - m \in D_\infty$.

Applying then (5.1) with $\mathbb{C} \backslash \overline{V} = D_\infty$ yields

$$\begin{aligned} &\left| \sum_{m=1}^{\lfloor 3|z|^{-1} \rfloor + 1} z^k \left(\widetilde{\varphi} \left(\frac{1}{z} - m \right) - \widetilde{\varphi}(-m) \right) \right| \\ &\leq \sum_{m=1}^{\lfloor 3|z|^{-1} \rfloor + 1} c_{1,k} \, |z|^k \cdot \left(\frac{1}{m^k} + \left| -m + \frac{1}{z} \right|^{-k} \right) \cdot \sup_{D_\infty} |\widetilde{\varphi}| \end{aligned}$$

$$\leq c_{2,k}\,|z|^k \cdot \left(\sum_{m=1}^{\lfloor 3|z|^{-1}\rfloor+1} \left(\frac{1}{m} + \left| -m + \frac{1}{z} \right|^{-1} \right) \right) \cdot \sup_{D_\infty} |\widetilde{\varphi}|$$

with $c_{2,k} = c_{1,k} \sup_{z \in D_0 \cup H_0 \cup H_0', m \in \mathbb{N}} \left| -m + \frac{1}{z} \right|^{-k+1} = c_{1,k}\, 2^{k-1}$. Noting that for all $z \in D_0 \cup H_0 \cup H_0'$ we have

$$\sum_{m=1}^{\lfloor 3|z|^{-1}\rfloor+1} \frac{1}{m} \leq 1 + \int_1^{\lfloor 3|z|^{-1}\rfloor+1} \frac{1}{m}\,\mathrm{d}m \leq 1 + \log(\lfloor 3|z|^{-1}\rfloor + 1),$$

and since $\left| \frac{1}{z} - \left(\left\lfloor \mathrm{Re}\left(\frac{1}{z}\right) \right\rfloor - n \right) \right| \geq \left| \frac{1}{z} - \left(\left\lfloor \mathrm{Re}\left(\frac{1}{z}\right) \right\rfloor + n \right) \right|$ for $1 \leq n \leq \lfloor \mathrm{Re}(1/z) \rfloor$ and $\mathrm{Re}(z) \geq 0$, we have

$$\begin{aligned}
\sum_{m=1}^{\lfloor 3|z|^{-1}\rfloor+1} \left| m - \frac{1}{z} \right|^{-1} &= \sum_{\substack{\{m\colon 1\leq m \leq \lfloor 3|z|^{-1}\rfloor+1,\\ |m-\lfloor \mathrm{Re}(1/z)\rfloor| \leq 2\}}} \left| m - \frac{1}{z} \right|^{-1} \\
&\quad + \sum_{\substack{\{m\colon 1\leq m \leq \lfloor 3|z|^{-1}\rfloor+1,\\ |m-\lfloor \mathrm{Re}(1/z)\rfloor| > 2\}}} \left| m - \frac{1}{z} \right|^{-1} \\
&\leq 10 + 2 \sum_{m=\lfloor \mathrm{Re}(1/z)\rfloor+3}^{\lfloor 3|z|^{-1}\rfloor+1} \frac{1}{m - \lfloor \mathrm{Re}(1/z)\rfloor - 1} \\
&\leq 10 + 2 \int_{m=1}^{\lfloor 3|z|^{-1}\rfloor - \lfloor \mathrm{Re}(1/z)\rfloor} \frac{1}{m}\,\mathrm{d}m \\
&\leq 10 + 2\log(\lfloor 3|z|^{-1}\rfloor - \lfloor \mathrm{Re}(1/z)\rfloor).
\end{aligned}$$

The last two inequalities even hold true if $\mathrm{Re}(1/z) < 0$. The fact that $\lfloor 3|z|^{-1}\rfloor + 1 \leq (|z|^{-1}+1)^3$ implies that there exists another constant $c_{3,k} > 0$ such that, for all $\widetilde{\varphi} \in \mathcal{O}^k(\overline{\mathbb{C}} \setminus [0,1])$ and $z \in D_0 \cup H_0 \cup H_0'$, we have

$$\left| \sum_{m=1}^{\lfloor 3|z|^{-1}\rfloor+1} z^k \left(\widetilde{\varphi}\left(\frac{1}{z} - m \right) - \widetilde{\varphi}(-m) \right) \right| \leq c_{3,k}\,|z|^k \cdot \log\left(1 + |z|^{-1}\right) \cdot \sup_{D_\infty} |\widetilde{\varphi}|\,. \tag{6.18}$$

For the second summand (6.16), from (6.3), for all $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}} \setminus [0,1])$ and $z \in D_0 \cup H_0 \cup H_0'$ we have

$$\left| \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \widetilde{\varphi}^{(n)}(-m) \right|$$

$$\leq \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} \sum_{n=1}^{k} \frac{|z|^{k-n}}{n!} \cdot c'_{k,n} n! \cdot \frac{1}{m^{k+n}} \cdot \sup_{D_\infty} |\widetilde{\varphi}|$$

$$\leq \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} |z|^{k-1} \cdot 2^{k-1} c'_{k,n} \cdot \frac{1}{m^{k+1}} \cdot \sup_{D_\infty} |\widetilde{\varphi}| \, .$$

Noting that

$$\sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} m^{-k-1} < \int_{\lfloor 3|z|^{-1}\rfloor+1}^{\infty} m^{-k-1} \mathrm{d}m = k^{-1} \left(\lfloor 3|z|^{-1}\rfloor + 1\right)^{-k} < \left(\frac{|z|}{3}\right)^{k} \tag{6.19}$$

yields that for all $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}} \setminus [0,1])$ and $z \in D_0 \cup H_0 \cup H_0'$ we have

$$\left| \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \, \widetilde{\varphi}^{(n)}(-m) \right| \leq |z|^{2k-1} \cdot c'_{k,n} \cdot \sup_{D_\infty} |\widetilde{\varphi}| \, . \tag{6.20}$$

Finally, we estimate the last summand (6.17) using (5.25)

$$\left| \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} z^k \left(\widetilde{\varphi}\left(\frac{1}{z} - m\right) - \widetilde{\varphi}(-m) \right) - \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \widetilde{\varphi}^{(n)}(-m) \right|$$
$$\leq \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} \left| \frac{1}{z\, k!} \int_0^1 \widetilde{\varphi}^{(k+1)} \left(\frac{t}{z} - m\right) (1-t)^k \mathrm{d}t \right| . \tag{6.21}$$

Using Cauchy's integral formula yields for R sufficiently small (possibly depending on m and z) to be defined later that

$$\left| \int_0^1 \widetilde{\varphi}^{(k+1)} \left(\frac{t}{z} - m\right) (1-t)^k \mathrm{d}t \right|$$
$$\leq \left| \int_0^1 \frac{(k+1)!}{2\pi i} \int_{|\omega - (\frac{t}{z} - m)| = R} \frac{\widetilde{\varphi}(\omega)}{\left(\omega - \left(\frac{t}{z} - m\right)\right)^{k+2}} \mathrm{d}\omega \cdot (1-t)^k \mathrm{d}t \right|$$
$$\leq \sup_{|\omega - (\frac{t}{z} - m)| = R} \left| \widetilde{\varphi}(\omega) \right| \cdot \frac{(k+1)!}{R^{k+2}} \int_0^1 (1-t)^k \mathrm{d}t = \sup_{|\omega - (\frac{t}{z} - m)| = R} \left| \widetilde{\varphi}(\omega) \right| \cdot \frac{2\pi k!}{R^{k+1}} .$$

Now, we specify the radius R. It is sufficient that $-m + t/|z| + R < \sqrt{3}/2 - 1$ is fulfilled, i.e. it is sufficient that $R < \sqrt{3}/2 - 1 + m - 1/|z|$ is fulfilled and thus we can choose $R = m/3$. Hence, using the above estimate and inserting it into (6.21) yields for all $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}} \setminus [0,1])$

$$\left| \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} z^k \left(\widetilde{\varphi}\left(\frac{1}{z} - m\right) - \widetilde{\varphi}(-m) \right) - \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} \sum_{n=1}^{k} \frac{z^{k-n}}{n!} \widetilde{\varphi}^{(n)}(-m) \right|$$

$$\leq \sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} 2\pi\cdot 3^{k+1}\cdot |z|^{-1}\cdot m^{-k-1}\cdot \sup_{\left|\omega-\left(\frac{t}{z}-m\right)\right|=m/3} |\widetilde{\varphi}|\,. \tag{6.22}$$

Next, we further estimate $\sup_{\left|\omega-\left(\frac{t}{z}-m\right)\right|=m/3}|\widetilde{\varphi}|$. Using (5.1) and noting that we have $D_{z,m,t} := \{\omega : \left|\omega-\left(\frac{t}{z}-m\right)\right| = m/3\} \subset D_\infty$ and thus $\inf_{\omega\in D_{z,m,t}}|\omega| \geq \left|\frac{t}{z}-m\right| - \frac{m}{3} \geq m-|z|^{-1}-\frac{m}{3} \geq \frac{m}{3}$ we obtain that there exists $c_{4,k}$ such that

$$\sup_{D_{z,m,t}} |\widetilde{\varphi}| \leq c_{4,k} \sup_{D_\infty}|\widetilde{\varphi}|\cdot \sup_{D_{z,m,t}}|\omega|^{-k} \leq c_{4,k}\sup_{D_\infty}|\widetilde{\varphi}|\cdot\left(\frac{m}{3}\right)^{-k} \tag{6.23}$$

and for $m \geq \lfloor 3|z|^{-1}\rfloor + 2$, we have

$$\sup_{D_{z,m,t}} |\widetilde{\varphi}| \leq c_{4,k}\sup_{D_\infty}|\widetilde{\varphi}|\cdot|z|^k\,.$$

Combining this with (6.22) and (6.19) yields

$$\left|\sum_{m=\lfloor 3|z|^{-1}\rfloor+2}^{\infty} z^k\left(\widetilde{\varphi}\left(\frac{1}{z}-m\right)-\widetilde{\varphi}(-m)\right) - \sum_{m=1}^{\infty}\sum_{n=1}^{k}\frac{z^{k-n}}{n!}\widetilde{\varphi}^{(n)}(-m)\right|$$
$$\leq 2\pi c_{4,k}\cdot 3^{k+1}\cdot|z|^{-1}\cdot\left(\frac{|z|}{3}\right)^{2k}\cdot\sup_{D_\infty}|\widetilde{\varphi}| \leq 2\pi c_{4,k}\cdot|z|^{2k-1}\cdot\sup_{D_\infty}|\widetilde{\varphi}|\,. \tag{6.24}$$

Combining the sum of (6.15), (6.16), and (6.17) estimated in (6.18), (6.20), and (6.24), we obtain that there exists a constant $C_k > 0$ such that for all $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}}\setminus[0,1])$ and $z \in D_0 \cup H_0 \cup H_0'$ we have the statement of (6.13).

The left-hand side of (6.14) equals the left-hand side of (6.12) if we replace φ by $\widetilde{\varphi}$. All the remaining calculations stay the same with the difference that we have to take the supremum over U instead of D_∞. However, there are only two instances where this is important. The first is the choice of R which has to fulfill $R < \gamma_0 + m - 1/|z|$ instead of $R < \sqrt{3}/2 - 1 + m - 1/|z|$ and for both $R \leq m/3$ is sufficient. The second is the occurrence of $\inf_{x\in D_\infty}|x|^{-k}$ in (6.23) which we estimated by $c_{1,k}$ and which has to be changed to an estimate by a constant depending on I and k. □

Our next concern will be the behaviour of $\sum_{\mathcal{M},k}\varphi$ for $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}}\setminus[0,1])$ near 1. The following proposition is a generalisation of [26, Prop. 4.2].

Proposition 6.8. *There exists $C_k > 0$ such that for all $\varphi \in \mathcal{O}^k(\overline{\mathbb{C}}\setminus[0,1])$ and all $z \in D_1$ we have*

$$\left|T_k\varphi(z) + z^k\varphi\left(\frac{1}{z}-1\right) - \sum_{m=1}^{\infty}\sum_{n=1}^{k}\frac{z^{k-n}}{n!}\varphi^{(n)}(-m)\right| \leq C_k|z-1|\sup_{D_\infty}|\varphi|. \tag{6.25}$$

Proof. We first notice that (5.9) implies

$$\begin{aligned}&\left|T_k\varphi(z)+z^k\varphi\left(\frac{1}{z}-1\right)-\sum_{m=1}^{\infty}\sum_{n=1}^{k}\frac{z^{k-n}}{n!}\varphi^{(n)}(-m)\right|\\&=\left|\sum_{m=2}^{\infty}z^k\varphi\left(\frac{1}{z}-m\right)-\sum_{m=1}^{\infty}z^k\varphi(-m)\right|\\&\leq|z|^k\cdot\left|\sum_{m=2}^{\infty}\left(\varphi\left(\frac{1}{z}-m\right)-\varphi(1-m)\right)\right|.\end{aligned}\tag{6.26}$$

For $z\in D_1$, we have $1/z-1\in D_0$. Equation (6.3) implies that there exists a constant $c_{1,k}>0$ such that $|\varphi'(w)|\leq c_{1,k}m^{-k-1}\sup_{D_\infty}|\varphi|$. This implies

$$\begin{aligned}&\left|T_k\varphi(z)+z^k\varphi\left(\frac{1}{z}-1\right)-\sum_{m=1}^{\infty}\sum_{n=1}^{k}\frac{z^{k-n}}{n!}\varphi^{(n)}(-m)\right|\\&\leq|z|^k\cdot\sum_{m=2}^{\infty}c_{1,k}m^{-k-1}\sup_{D_\infty}|\varphi|\cdot\left|\frac{1}{z}-1\right|=|z|^{k-1}\cdot|1-z|\sup_{D_\infty}|\varphi|\sum_{m=2}^{\infty}c_{1,k}m^{-k-1}\\&=C_k\cdot|1-z|\sup_{D_\infty}|\varphi|,\end{aligned}$$

where $C_k:=\sup_{z\in D_1}|z|^{k-1}c_{1,k}\sum_{m=2}^{\infty}m^{-k-1}$. $\square$

We can finally use the previous results and the complex continued fraction algorithm to study the behaviour of $T_k^n\varphi(z)$ when z is close to the boundary. We recall that $D(m_1,\cdots,m_n)$ is the set of $z_0\in D$ whose first n complex continued fraction entries equal $\{m_i\}_{i=1}^n$. Next, we will prove a proposition which is an analog of [26, Prop. 4.4].

Proposition 6.9. *For all $\varphi\in\mathcal{O}^k(\overline{\mathbb{C}}\setminus[0,1])$, $n\in\mathbb{N}$, and $z_0\in D(m_1,\cdots,m_n)$ we have*

$$\begin{aligned}&T_k^n\varphi(z_0)=(-1)^{n(k+1)}(p_{n-1}-q_{n-1}z_0)^k(\varphi(z_n)+\varphi(z_n-1)+\varepsilon_n\varphi(z_n+1))\\&-(-1)^{(n-1)(k+1)}(p_{n-2}-q_{n-2}z_0)^k(1+z_{n-1})^k\varepsilon_{n-1}\varphi\left(-\frac{z_{n-1}}{1+z_{n-1}}\right)+R_k^{[n]}\varphi(z_0).\end{aligned}\tag{6.27}$$

The remainder term $R_k^{[n]}$ is holomorphic in the interior of $D(m_1,\cdots,m_n)$ and continuous in $D(m_1,\cdots,m_n)$ and there exists $C_k>0$ such that for all $\varphi\in\mathcal{O}^k(\overline{\mathbb{C}}\setminus[0,1])$, $n\in\mathbb{N}$, and $z_0\in D(m_1,\cdots,m_n)$ we have

$$\left|R_k^{[n]}\varphi(z_0)\right|\leq C_k\,n\left(\frac{\sqrt{5}-1}{2}\right)^{nk}\sup_{D_\infty}|\varphi|.\tag{6.28}$$

Proof. We start by an estimate for $n = 1$. For $z_0 \in D(m_1)$, by definition, we have

$$\begin{aligned} T_k\varphi(z_0) &= -\sum_{m=1}^{\infty} z_0^k \left(\varphi\left(\frac{1}{z_0} - m\right) - \varphi(-m)\right) + \sum_{m=1}^{\infty}\sum_{j=1}^{k} \frac{z_0^{k-j}}{j!}\varphi^{(j)}(-m) \\ &= -z_0^k(\varphi(z_1) + \varphi(z_1 - 1) + \varepsilon_1\varphi(z_1 + 1)) + R_k^{(m_1)}(\varphi)(z_0), \end{aligned}$$

with

$$\begin{aligned} R_k^{(m_1)}(\varphi)(z_0) = &\sum_{\substack{m\geq 1 \\ |m-m_1|\leq 1}} z_0^k\varphi(-m) + \sum_{m=1}^{\infty}\sum_{j=1}^{k} \frac{z_0^{k-j}}{j!}\varphi^{(j)}(-m) \\ &- \sum_{\substack{m\geq 1 \\ |m-m_1|>1}} z_0^k(\varphi(m_1 + z_1 - m) - \varphi(-m)) \\ =:&\, I_0 + I_1 - I_2. \end{aligned}$$

Then, $R_k^{(m_1)}(\varphi)$ is holomorphic in a neighbourhood of $D(m_1)$.

We will first show that there is a constant $c_{1,k}$ such that

$$|R_k^{(m_1)}(\varphi)(z_0)| \leq c_{1,k} \sup_{D_\infty} |\varphi| \quad \text{for all } z_0 \in D(m_1). \tag{6.29}$$

Note that this however is not an estimate of (6.27) for $n = 1$ as the terms in the second line are all contained in the error term $R_k^{(m_1)}(\varphi)(z_0)$. We start by estimating I_0. Since $|z_0| \leq 1$ and $I_0 = z_0^k\big(\varphi(-m_1 - 1) + \varphi(-m_1) + \varepsilon_1\varphi(-m_1 + 1)\big)$, we have

$$|I_0| \leq 3 \sup_{D_\infty} |\varphi|. \tag{6.30}$$

From (6.3) with $\{\omega : |\omega + m| = m - 0.8\} \subset D_\infty$, we have

$$|\varphi^{(j)}(-m)| \leq c'_{k,j} \frac{j! \sup_{D_\infty} |\varphi|}{(m - 0.8)^{k+j}}.$$

Let $c_{2,k} = \sum_{m=1}^{\infty}\sum_{j=1}^{k} \frac{c'_{k,j}}{(m-0.8)^{k+j}}$. Since $\sum_{m=1}^{\infty} \frac{1}{(m-0.8)^{k+j}} < \infty$ for $k, j \geq 1$,

$$c_{2,k} = \sum_{m=1}^{\infty}\sum_{j=1}^{k} \frac{c'_{k,j}}{(m - 0.8)^{k+j}} = \sum_{j=1}^{k}\sum_{m=1}^{\infty} \frac{c'_{k,j}}{(m - 0.8)^{k+j}} < \infty.$$

Then we have

$$|I_1| \leq c_{2,k} \sup_{D_\infty} |\varphi|. \tag{6.31}$$

Since $z_0 \in D(m_1)$, we have $1/z_0 = z_1 + m_1 \in \Delta + m_1$ (see (4.7) for the definition of Δ). Furthermore, $\frac{1}{m_1+1} \leq |z_0| \leq \frac{1}{m_1-1}$ if $m_1 \neq 1$ and $1/2 \leq |z_0| \leq$

1 if $m_1 = 1$. Let us consider

$$I_2' := \sum_{m \geq m_1+2} z_0^k(\varphi(1/z_0 - m) - \varphi(-m)),$$

and $I_2'' := I_2 - I_2'$. Then, $I_2'' = 0$ if $m_1 \leq 2$, and

$$I_2'' = \sum_{1 \leq m \leq m_1-2} z_0^k(\varphi(1/z_0 - m) - \varphi(-m)) \quad \text{if } m_1 \geq 3.$$

If $m \geq m_1+2$, then $\mathrm{Re}(1/z_0-m) \leq -1$. For z in the segment $[-m, 1/z_0-m] \subset D_\infty$, the circle $\{\omega : |\omega - z| = m - 0.8\}$ is contained in D_∞. Thus, by (6.3), we have $|\varphi'(z)| \leq c'_{k,1} \frac{\sup_{D_\infty}|\varphi|}{(m-0.8)^{k+1}}$ and thus

$$|I_2'| \leq \sum_{m \geq m_1+2} \frac{c'_{k,1}}{(m-0.8)^{k+1}} \sup_{D_\infty} |\varphi|.$$

If $m_1 \geq 3$, then $1/z_0 - m \in \Delta + (m_1 - m) \subset D_\infty$ for $1 \leq m \leq m_1 - 2$. Thus,

$$|I_2''| \leq \frac{2(m_1-2)\sup_{D_\infty}|\varphi|}{(m_1-1)^k} \leq 2 \sup_{D_\infty} |\varphi|.$$

Then, $|I_2| \leq c_{3,k} \sup_{D_\infty} |\varphi|$, where $c_{3,k} := \sum_{m=3}^\infty \frac{c'_{k,1}}{(m-0.8)^{k+1}} + 2$. Hence, by combining the last estimate with (6.30) and (6.31) and setting $c_{1,k} := c_{2,k} + c_{3,k} + 3$, we obtain (6.29).

In the following we will estimate $T_k^n\varphi(z_0)$ itself which we split into a number of summands to be estimated separately. Recall z_i as in (4.9) and (4.11). Iterating T_k n times, we get

$$\begin{aligned} T_k^n\varphi(z_0) &= (\varphi(z_n) + \varphi(z_n - 1) + \varepsilon_n\varphi(z_n+1)) \prod_{i=0}^{n-1}(-z_i^k) \\ &\quad + \sum_{j=1}^{n-1} T_k^{n-j}\varphi(z_j - 1) \prod_{i=0}^{j-1}(-z_i^k) \\ &\quad + \sum_{j=1}^{n-1} \varepsilon_j T_k^{n-j}\varphi(z_j+1) \prod_{i=0}^{j-1}(-z_i^k) \\ &\quad + \sum_{j=1}^{n} R_k^{(m_j)}(T_k^{n-j}\varphi)(z_{j-1}) \prod_{i=0}^{j-2}(-z_i^k) \\ &=: J_1 + J_2 + J_3 + J_4, \end{aligned}$$

where $z_{-1} = 1$. We recall (4.12) and (4.14), i.e., $\prod_{i=0}^{n-1}(-z_i^k) = (-1)^{n(k+1)}(p_{n-1} - q_{n-1}z_0)^k$ and $|p_{n-1} - q_{n-1}z_0| < q_n^{-1}$. Thus J_1 is the first term of (6.27). In the following steps (1)-(3) we estimate the summands J_2, J_3, and J_4.

(1) We continue by estimating J_4. By using (4.12), (4.14) and (6.29) and by applying Proposition 6.3-(ii) to each of the summands and noting that $q_j \geq \left(\frac{\sqrt{5}-1}{2}\right)^{-j}$, we obtain

$$\left|\sum_{j=1}^{n} R_k^{(m_j)}(T_k^{n-j}\varphi)(z_{j-1})\prod_{i=0}^{j-2}(-z_i^k)\right| \leq \sum_{j=1}^{n}\frac{c_{1,k}\sup_{D_\infty}|T_k^{n-j}\varphi|}{q_{j-1}^k}$$
$$\leq c_{1,k}\sum_{j=1}^{n}\overline{C}_{0,k}\left(\frac{\sqrt{5}-1}{2}\right)^{(n-1)k}\sup_{D_\infty}|\varphi|$$
$$\leq c_{4,k}\cdot n\left(\frac{\sqrt{5}-1}{2}\right)^{nk}\sup_{D_\infty}|\varphi| \tag{6.32}$$

with $c_{4,k} = c_{1,k}\overline{C}_{0,k}\left(\frac{\sqrt{5}+1}{2}\right)^k$.

(2) In the next steps, we will show that $|J_2| \leq c_{5,k} n (\frac{\sqrt{5}-1}{2})^{nk}\sup_{D_\infty}|\varphi|$ for some $c_{5,k}$. If $z_j - 1 \notin D_0$ for $1 \leq j < n$, then we have

$$|T_k^{n-j}\varphi(z_j-1)| \leq C_{K,k}\sup_{D_\infty}|(T_k^{n-j-1}\varphi)| \tag{6.33}$$

by Proposition 6.1-(i) with $I=[0,1]$ and $K = \overline{(D-1)\setminus D_0}$. On the other hand, let us consider the case $z_j - 1 \in D_0$. From (6.12), we deduce that

$$|T_k^{n-j}\varphi(z_j-1)| \leq C_{I,k}|z_j-1|^k\log(1+|z_j-1|^{-1})\sup_{D_\infty}|(T_k^{n-j-1}\varphi)|$$
$$+\left|\sum_{m=1}^{\infty}\sum_{i=1}^{k}\frac{(z_j-1)^{k-i}}{i!}(T_k^{n-j-1}\varphi)^{(i)}(-m)\right|.$$

From (6.31) and $|z_j-1|^k\log(1+|z_j-1|^{-1}) \leq \log 2$, we see that

$$|T_k^{n-j}\varphi(z_j-1)| \leq (C_{I,k}\log 2 + c_{2,k})\sup_{D_\infty}|(T_k^{n-j-1}\varphi)|. \tag{6.34}$$

Let $c_{6,k} = \max\{C_{K,k}, C_{I,k}\log 2 + c_{2,k}\}$. Then, by combining the calculations leading to (6.32), (6.33) and (6.35), we get

$$\left|\sum_{j=1}^{n-1} T_k^{n-j}\varphi(z_j-1)\prod_{i=0}^{j-1}(-z_i^k)\right| \leq c_{5,k}\, n\left(\frac{\sqrt{5}-1}{2}\right)^{nk}\sup_{D_\infty}|\varphi|,$$

where $c_{5,k} = \overline{C}_{0,k}\, c_{6,k}\left(\frac{\sqrt{5}+1}{2}\right)^k$.

(3) To complete the proof, it is enough to show that there is a constant $c_{7,k}$ such that

$$\Big|J_3 + \prod_{i=0}^{n-2}(-z_i^k)(1+z_{n-1})^k\varepsilon_{n-1}\varphi\Big(-\frac{z_{n-1}}{1+z_{n-1}}\Big)\Big| \le c_{7,k}\, n \left(\frac{\sqrt{5}-1}{2}\right)^{nk} \sup_{D_\infty}|\varphi|. \tag{6.35}$$

We have

$$\begin{aligned}
&J_3 + \prod_{i=0}^{n-2}(-z_i^k)(1+z_{n-1})^k\varepsilon_{n-1}\varphi\Big(-\frac{z_{n-1}}{1+z_{n-1}}\Big)\\
&= \sum_{j=1}^{n-2}\varepsilon_j T_k^{n-j}\varphi(z_j+1)\prod_{i=0}^{j-1}(-z_i^k)\\
&\qquad + \left(T_k\varphi(z_{n-1}+1) + (1+z_{n-1})^k\varphi\left(-\frac{z_{n-1}}{1+z_{n-1}}\right)\right)\prod_{i=0}^{n-2}(-z_i^k)\\
&=: I_4 + I_5.
\end{aligned}$$

(3a) We will estimate I_4. If m_{j+1} is small enough so that $z_j+1 \notin D_1$, then by Proposition 6.1-(i) applied to $I=[0,1]$ and $K' = \overline{(D+1)\setminus D_1}$ we have

$$|T_k^{n-j}\varphi(z_j+1)| \le C_{K',k}\sup_{D_\infty}|(T_k^{n-j-1}\varphi)|.$$

When m_{j+1} is sufficiently large, then $z_j+1\in D_1$. Then, by Proposition 6.8 and (6.31), we deduce that

$$\begin{aligned}
&|T_k^{n-j}\varphi(z_j+1)|\\
&\le |1+z_j|^k\left|(T_k^{n-j-1}\varphi)\left(-\frac{z_j}{1+z_j}\right)\right| + C_k|z_j|\sup_{D_\infty}|T_k^{n-j-1}\varphi|\\
&\qquad + \left|\sum_{m=1}^{\infty}\sum_{i=1}^{k}\frac{(z_j+1)^{k-i}}{i!}(T_k^{n-j-1}\varphi)^{(i)}(-m)\right|\\
&\le |1+z_j|^k\left|(T_k^{n-j-1}\varphi)\left(-\frac{z_j}{1+z_j}\right)\right| + (C_k + c_{2,k}(2\sqrt{3}/3)^k)\sup_{D\infty}|T_k^{n-j-1}\varphi|.
\end{aligned}$$

Since $-\frac{z_j}{1+z_j}\in D_0$ and $j\le n-2$, by Proposition 6.7, we have

$$\begin{aligned}
&|1+z_j|^k\left|(T_k^{n-j-1}\varphi)\left(-\frac{z_j}{1+z_j}\right)\right|\\
&\le C_{I,k}|1+z_j|^k\left|-\frac{z_j}{1+z_j}\right|^k\log\left(1+\left|-\frac{z_j}{1+z_j}\right|^{-1}\right)\sup_{D_\infty}|T_k^{n-j-2}\varphi|
\end{aligned}$$

$$+ |1+z_j|^k \left| \sum_{m=1}^{\infty} \sum_{i=1}^{k} \frac{(-z_j/(1+z_j))^{k-i}}{i!} (T_k^{n-j-2}\varphi)^{(i)}(-m) \right|.$$

Note that

$$|1+z_j|^k \left| -\frac{z_j}{1+z_j} \right|^k \log \left(1 + \left| -\frac{z_j}{1+z_j} \right|^{-1} \right)$$
$$= |z_j|^k \log \left(1 + \left| 1 + \frac{1}{z_j} \right| \right) \le |z_j|^k \log \left(2 + \frac{1}{|z_j|} \right).$$

Since $x^k \log(2+1/x) \to 0$ as $x \to 0$, it is increasing and $|z_j| < 1$, it follows that $|z_j|^k \log(2+1/|z_j|) \le \log 3$. Thus, by a similar way to the proof of (6.35), it follows that

$$|1+z_j|^k \left| (T_k^{n-j-1}\varphi) \left(-\frac{z_j}{1+z_j} \right) \right| \le \left(C_{I,k} \log 3 + c_{2,k} \left(2\sqrt{3}/3 \right)^k \right)$$
$$\sup_{D_\infty} |(T_k^{n-j-2}\varphi)|.$$

By letting $c_{8,k} = \max \left\{ C_{K',k},\ C_k + c_{2,k} \left(2\sqrt{3}/3 \right)^k, C_{I,k} \log 3 + c_{2,k} (2\sqrt{3}/3)^k \right\}$, we have

$$|I_4| \le c_{8,k} \sum_{j=1}^{n-2} \frac{\sup_{D_\infty} |T_k^{n-j-1}\varphi| + \sup_{D_\infty} |T_k^{n-j-2}\varphi|}{q_j^k}. \tag{6.36}$$

(3b) We will estimate I_5. If $z_{n-1}+1 \notin D_1$, then $\mathrm{Re} \left(\frac{1}{(z_{n-1}+1)} - 1 \right) < \frac{\sqrt{3}}{2} - 1$. Thus, $\frac{1}{(z_{n-1}+1)} - 1 \in D_\infty$. By Proposition 6.1-(i), we have

$$\left| T_k\varphi(z_{n-1}+1) + (1+z_{n-1})^k \varphi \left(-\frac{z_{n-1}}{1+z_{n-1}} \right) \right| \le (C_{K',k} + 2^k) \sup_{D_\infty} |\varphi|.$$

By Proposition 6.8 in the same manner as showing (6.31), we obtain that $z_{n-1}+1 \in D_1$ implies

$$\left| T_k\varphi(z_{n-1}+1) + (1+z_{n-1})^k \varphi \left(-\frac{z_{n-1}}{1+z_{n-1}} \right) \right|$$
$$\le C_k |z_{n-1}| \sup_{D_\infty} |\varphi| + \left| \sum_{m=1}^{\infty} \sum_{i=1}^{k} \frac{(z_{n-1}+1)^{k-i}}{i!} \varphi^{(i)}(-m) \right|$$
$$\le \left(C_k + c_{2,k} (2\sqrt{3}/3)^k \right) \sup_{D_\infty} |\varphi|.$$

Let $c_{9,k} = \max\{C_{K',k} + 2^k,\ C_k + c_{2,k}(2\sqrt{3}/3)^k\}$. Then, we have

$$|I_5| \le \frac{c_{9,k} \sup_{D_\infty} |\varphi|}{q_{n-1}^k}. \tag{6.37}$$

Letting $c_{10,k} = \max\{c_{8,k},\ c_{9,k}\}$, from (6.36), (6.37) and (6.32), we obtain

$$|I_4| + |I_5| \leq 2c_{10,k} \sum_{j=0}^{n-1} \frac{\sup_{D_\infty} |T_k^{n-j-1}\varphi|}{q_{j-1}^k} \leq c_{7,k} n \left(\frac{\sqrt{5}-1}{2}\right)^{nk} \sup_{D_\infty} |\varphi|,$$

where $c_{7,k} = 2c_{10,k}\overline{C}_{0,k}(\frac{\sqrt{5}+1}{2})^{2k}$. Note that $c_{7,k}$ depends only on k.

□

The next proposition is an analog of [[26], Prop. 4.1]. We can show the following proposition in the same manner as in the proof in [26] by using (5.28) and Proposition 6.3-(2) for $k = 1$ and the fact that $S = -T$.

Proposition 6.10. *The following two statements hold true:*

(i) Let $I = [\gamma_0, \gamma_1]$, $\gamma_0 > -1$. There exists $C_I > 0$ such that for all $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus I)$ and for all $z \in D_0 \cup H_0 \cup H_0'$, one has

$$|S\varphi(z) + \sum_{m\geq 1} \varphi'(-m)| \leq C_I |z| \log(1 + |z|^{-1}) \sup_U |\varphi|, \tag{6.38}$$

where U is as in (6.11).

(ii) There exists $C > 0$ such that for all $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$ and for all $z \in D_0 \cup H_0 \cup H_0'$, one has

$$|\overline{\sum_{\mathcal{M}}}\varphi(z) - \varphi(z) + \sum_{m\geq 1} (\overline{\sum_{\mathcal{M}}}\varphi)'(-m)| \leq C|z|(1 + \log|z|^{-1}) \sup_{D_\infty} |\varphi|. \tag{6.39}$$

Concerning the boundary behaviour of the sum leading to the complex Wilton function, we simply observe that since $S = -T$ Propositions 4.1 and 4.4 of [26] hold. The same is true for Proposition 4.11 of [26], concerning the action of $T = -S$ on the space of analytic functions on $\overline{\mathbb{C}} \setminus [0,1]$ with bounded real part. However, the Wilton function will not have a bounded real part we thus omit its discussion.

7. Complex k-Brjuno and complex Wilton functions

In this section, we finally define the complex k-Brjuno and the complex Wilton functions. Let

$$\varphi_0(z) := -\frac{1}{\pi}\mathrm{Li}_2\left(\frac{1}{z}\right) \quad \text{with} \quad \mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \tag{7.1}$$

Note that $\varphi_0 \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$, $\mathrm{Im}\varphi_0(x) = 0$ if $x \in \mathbb{R} \setminus [0,1]$ and

$$\mathrm{Im}\varphi_0(x \pm i0) = \pm \log\left(\frac{1}{x}\right) \quad \text{for } x \in (0,1]. \tag{7.2}$$

We define $\tau := \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$,

$$\varphi_{1,k} := \left(L_{k,g(1)} + L_{k,\tau}\right)\varphi_0 \quad \text{and} \quad \overline{\varphi}_1 := \left(\overline{L}_{g(1)} + \overline{L}_{\tau}\right)\varphi_0.$$

Note that $\overline{\varphi}_1 = (-L_{1,g(1)} + L_{1,\tau})\varphi_0$. Further, we note that $\varphi_0 \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$ implies $\varphi_{1,k},\ \overline{\varphi}_1 \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [1/2,2])$.

Similarly as in [26, Section 5] we note that from (6.8) we have that $\sum_{\mathcal{M},k}\varphi_{1,k} = \varphi_{1,k} + \sum_{\mathcal{M},k} T_k\varphi_{1,k}$. Since $Z\mathcal{M} = Z\mathcal{M}g(1) \sqcup Z\mathcal{M}\tau$, we have

$$\sum_{Z\mathcal{M},k}\varphi_0(z) = \sum_{Z,k}[\varphi_{1,k}(z) + \sum_{\mathcal{M},k}(T_k\varphi_{1,k}(z))] = \sum_{Z\mathcal{M},k}\varphi_{1,k}(z). \tag{7.3}$$

Similarly, we have $\overline{\sum_{Z\mathcal{M}}}\varphi_0(z) = \overline{\sum_{Z\mathcal{M}}}\overline{\varphi}_1(z)$. We then deduce by simply applying the definitions of $L_{k,g}$ and $\overline{L}_g$

$$\varphi_{1,k}(z) = -\,z^k\varphi_0\left(\frac{1}{z}-1\right) + \varphi_0(z-1) + \sum_{n=0}^{k}\frac{z^{k-n}}{n!}\varphi_0^{(n)}(-1)$$

and

$$\overline{\varphi}_1(z) = z\left(\varphi_0\left(\frac{1}{z}-1\right) - \varphi_0(-1)\right) + \varphi_0(z-1) - \varphi_0'(-1). \tag{7.4}$$

We deduce that

$$\varphi_{1,k}(z) = \frac{1}{\pi}\left[z^k\mathrm{Li}_2\left(\frac{z}{1-z}\right) - \mathrm{Li}_2\left(\frac{1}{z-1}\right)\right] + \sum_{n=0}^{k}\frac{z^{k-n}}{n!}\varphi_0^{(n)}(-1), \tag{7.5}$$

and since we additionally have $\varphi_0(-1) = \frac{\pi}{12}$ and $\varphi_0'(-1) = \frac{\log 2}{\pi}$, we also have

$$\overline{\varphi}_1(z) = -\frac{1}{\pi}\left[z\mathrm{Li}_2\left(\frac{z}{1-z}\right) + \mathrm{Li}_2\left(\frac{1}{z-1}\right)\right] - z\,\frac{\pi}{12} - \frac{\log 2}{\pi}. \tag{7.6}$$

By (7.2), we have

$$\mathrm{Im}\varphi_{1,k}(x \pm i0) = \begin{cases} \pm x^k\log\frac{x}{1-x}, & \text{if } 1/2 \le x < 1,\\ \pm\log\frac{1}{x-1}, & \text{if } 1 < x \le 2 \end{cases} \tag{7.7}$$

and

$$\mathrm{Im}\overline{\varphi}_1(x \pm i0) = \begin{cases} \mp x\log\frac{x}{1-x} & \text{if } 1/2 \le x < 1,\\ \pm\log\frac{1}{x-1} & \text{if } 1 < x \le 2. \end{cases} \tag{7.8}$$

Since T_k is defined exactly in such a way that $T_k\varphi(x) = x^k\,\varphi(1/x)$ as an operator on the space of real functions and the k-Brjuno function fulfills the recursion $B_k(x) = -\log x + x^k\,B_k(1/x)$, we have $(1-T_k)\,B_k)(x) = -\log x$ and therefore

$$B_k(x) = ((1-T_k)^{-1}f)(x) \quad \text{with} \quad f(x) = \sum_{n\in\mathbb{Z}}\mathrm{Im}\varphi_0(x+i0-n). \tag{7.9}$$

We note here that by the above considerations $f(x) = \sum_{n\in\mathbb{Z}} \mathrm{Im}\varphi_0(x+i0-n) = \log\{x\}$.

Then, the complex analytic extension of the real k-Brjuno function is $(1-T_k)^{-1}\left(\sum_Z \varphi_0\right)$ and we have $(1-T_k)^{-1}\left(\sum_Z \varphi_0\right) = \sum_{Z\mathcal{M},k} \varphi_0$ by applying (5.13), here T_k is the extended operator as in (5.9). Hence, it is natural to define *the complex k-Brjuno functions* by

$$\mathcal{B}_k := \sum_{Z\mathcal{M},k} \varphi_0.$$

By applying the above consideration to W instead of B_k and S instead of T_k, we may define the complex analytic extension of W by

$$\mathcal{W} := \overline{\sum_{Z\mathcal{M}}} \varphi_0, \tag{7.10}$$

which we call *the complex Wilton function*.

There is a two-to-one correspondence between $Z\mathcal{M}$ and $\mathbb{Q}$. From (5.11) and (5.12), we obtain analogous formulations for $\mathcal{B}_k$ and $\mathcal{W}$ respectively such that

$$\begin{aligned}\mathcal{B}_k(z) = -\sum_{p/q\in\mathbb{Q}} &\det\begin{pmatrix} p' & p \\ q' & q\end{pmatrix}^{k+1} \\ &\left\{\frac{1}{\pi}\left[(p'-q'z)^k \mathrm{Li}_2\left(\frac{p'-q'z}{qz-p}\right) - (q''z-p'')^k \mathrm{Li}_2\left(\frac{p''-q''z}{qz-p}\right)\right]\right. \\ &\quad + \sum_{n=0}^{k} \frac{1}{n!}\det\begin{pmatrix} p' & p \\ q' & q\end{pmatrix}^{n} \\ &\qquad \left.\left[\frac{(p'-q'z)^{k-n}}{(q')^n}\varphi_0^{(n)}\left(-\frac{q}{q'}\right) - \frac{(q''z-p'')^{k-n}}{(q'')^n}\varphi_0^{(n)}\left(-\frac{q}{q''}\right)\right]\right\},\end{aligned} \tag{7.11}$$

where $[\frac{p'}{q'}, \frac{p''}{q''}]$ is the Farey interval such that $\frac{p}{q} = \frac{p'+p''}{q'+q''}$ (with the convention $p' = p-1$, $q' = 1$, $p'' = 1$, $q'' = 0$ if $q = 1$). To obtain a representation as in (1.10) we remember the definition of φ_0 from (7.1) and note that

$$\varphi_0'(z) = -\frac{1}{\pi}\cdot\frac{1}{z}\log\left(1-\frac{1}{z}\right).$$

If we let $f(z) = \frac{1}{z}$ and $g(z) = \log\left(1-\frac{1}{z}\right)$, then $\varphi_0'(z) = -\frac{1}{\pi}f(z)g(z)$ and

$$f^{(j)}(z) = \frac{(-1)^j j!}{z^{j+1}} \text{ for } j \geq 0,$$

and

$$g^{(j)}(z) = \frac{(-1)^{j-1}(j-1)!}{z^j}\left(\left(\frac{z}{z-1}\right)^j - 1\right) \text{ for } j \geq 1$$

holds. Since

$$(fg)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)},$$

we have, for $n \geq 1$,

$$\begin{aligned}
\varphi_0^{(n)}(z) &= -\frac{1}{\pi} \sum_{i=0}^{n-1} \binom{n-1}{i} f^{(n-1-i)} g^{(i)} \\
&= -\frac{1}{\pi} \Bigg[\frac{(-1)^{n-1}(n-1)!}{z^n} \log\left(1 - \frac{1}{z}\right) \\
&\qquad + \sum_{i=1}^{n-1} \frac{(n-1)!}{(n-i-1)!i!} \frac{(-1)^{n-1-i}(n-1-i)!}{z^{n-i}} \frac{(-1)^{i-1}(i-1)!}{z^i} \\
&\qquad\qquad \cdot \left(\left(\frac{z}{z-1} \right)^i - 1 \right) \Bigg] \\
&= -\frac{1}{\pi} \Bigg[\frac{(-1)^{n-1}(n-1)!}{z^n} \log\left(1 - \frac{1}{z}\right) \\
&\quad + \frac{(-1)^n (n-1)!}{z^n} \sum_{i=1}^{n-1} \frac{1}{i} \left(\left(\frac{z}{z-1} \right)^i - 1 \right) \Bigg] \\
&= \frac{(-1)^n (n-1)!}{\pi z^n} \left[\log\left(1 - \frac{1}{z}\right) - \sum_{i=1}^{n-1} \frac{1}{i} \left(\left(\frac{z}{z-1} \right)^i - 1 \right) \right]
\end{aligned}$$

and by (7.11) we obtain (1.10). For the complex Wilton function, we obtain by the same considerations as above the representation as in (1.11).

7.1. Behaviour of $\mathcal{B}_k$ and $\mathcal{W}$ at rational points

7.1.1. Behaviour of $\mathcal{B}_k$ at rational points.

Let

$$R_k(z) := \sum_{n=0}^{k} \frac{z^{k-n}}{n!} \varphi_0^{(n)}(-1),$$

which coincides with the last term in (7.5). Note that $R_k(1) \in \mathbb{R}$ is well defined for all $k \in \mathbb{N}$, as $\varphi_0 \in C^\infty$.

Lemma 7.1. *The function $\varphi_{1,k}(z) + i\log(1-z)$ is continuous on $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ and its value at 1 is $R_k(1) + \frac{\pi}{2}$.*

Proof. Recall that for $t \in \mathbb{C} \setminus [0, +\infty)$ we have Euler's functional equation

$$\mathrm{Li}_2(t) + \mathrm{Li}_2\left(\frac{1}{t}\right) = -\frac{1}{2}(\log(-t))^2 - \frac{\pi^2}{6}. \tag{7.12}$$

If $z \notin [0,+\infty)$, then $\frac{z}{1-z}, \frac{1}{z-1} \notin [0,+\infty)$. Thus, applying (7.12) with $t = \frac{z}{1-z}$ and $t = \frac{1}{z-1}$, and substituting it into (7.5) gives

$$\begin{aligned}\varphi_{1,k}(z) =& R_k(z) - z^k\frac{\pi}{6} + \frac{\pi}{6} - \frac{1}{\pi}\left(z^k \mathrm{Li}_2\left(\frac{1-z}{z}\right) - \mathrm{Li}_2(z-1)\right)\\ &- \frac{1}{2\pi}\left(z^k \log^2\left(\frac{-z}{1-z}\right) - \log^2\left(\frac{1}{1-z}\right)\right).\end{aligned}$$

The function $z^k \mathrm{Li}_2\left(\frac{1-z}{z}\right) - \mathrm{Li}_2(z-1)$ is continuous on $\overline{\mathbb{H}}$ and it is in $O(|z-1|)$ in the neighbourhood of 1. We then observe that

$$\begin{aligned}& z^k \log^2\left(\frac{-z}{1-z}\right) - \log^2\left(\frac{1}{1-z}\right)\\ &= (z^k-1)\log^2\left(\frac{-z}{1-z}\right) + \log^2\left(\frac{-z}{1-z}\right) - \log^2\left(\frac{1}{1-z}\right),\end{aligned}$$

and $(z^k-1)\log^2\left(\frac{-z}{1-z}\right) = O\big(|1-z|\log\left(\frac{1}{|1-z|}\right)\big)$ in the neighbourhood of 1. Also, we have

$$\begin{aligned}\log^2\left(\frac{-z}{1-z}\right) - \log^2\left(\frac{1}{1-z}\right) &= \left(\log(-z) + \log\left(\frac{1}{1-z}\right)\right)^2 - \log^2\left(\frac{1}{1-z}\right)\\ &= \log^2(-z) + 2\log(-z)\log\left(\frac{1}{1-z}\right).\end{aligned}$$

In the neighbourhood of 1 in $\overline{\mathbb{H}}$, we have $\log(-z) + i\pi = O(|z-1|)$. Thus,

$$\log^2\left(\frac{-z}{1-z}\right) - \log^2\left(\frac{1}{1-z}\right) = -\pi^2 - 2i\pi\log\left(\frac{1}{1-z}\right) + O\left(|z-1|\,\log\left(|1-z|\right)\right).$$

Therefore, in the neighbourhood of 1, we have

$$\varphi_{1,k}(z) = R_k(z) + \frac{\pi}{2} + i\log\left(\frac{1}{1-z}\right) + O\left(|z-1|\log\left(|1-z|\right)\right) \quad (7.13)$$

and the result follows. □

Corollary 7.2. *The real part of $\varphi_{1,k}$ is bounded in $\overline{\mathbb{C}} \setminus [1/2, 2]$. It has an extension to a continuous function on $\overline{\mathbb{C}} \setminus \{1\}$ and*

$$\lim_{x\to 1^{\pm}} \mathrm{Re}(\varphi_{1,k}(x)) = R_k(1) \mp \frac{\pi}{2}.$$

Proof. By (7.13), we have

$$\begin{aligned}\lim_{x\to 1^{\pm}} \mathrm{Re}(\varphi_{1,k}(x)) &= R_k(1) + \frac{\pi}{2} + \lim_{x\to 1^{\pm}} \mathrm{Re}\left(i\log\left(\frac{1}{1-x}\right)\right)\\ &= R_k(1) + \frac{\pi}{2} - \lim_{x\to 1^{\pm}} \arg\left(\frac{1}{1-x}\right).\end{aligned}$$

The result follows from the fact that

$$\lim_{x\to 1^+} \arg\left(\frac{1}{1-x}\right) = \pi \quad \text{and} \quad \lim_{x\to 1^-} \arg\left(\frac{1}{1-x}\right) = 0.$$

□

Theorem 7.3. *The real part of the complex k-Brjuno function has a decreasing jump of $\frac{\pi}{q^k}$ at each rational number $\frac{p}{q}$.*

Proof. The proof follows similar arguments as that in [26, Section 5.2]. The space $\widehat{\mathbb{H}/\mathbb{Z}}$ is defined by

$$\widehat{\mathbb{H}/\mathbb{Z}} = \mathbb{H}/\mathbb{Z} \sqcup (\mathbb{R}\backslash\mathbb{Q})/\mathbb{Z} \sqcup \left(\overline{\mathbb{Q}/\mathbb{Z}} \times \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]\right),$$

where $\overline{\mathbb{Q}/\mathbb{Z}} = \mathbb{Q}/\mathbb{Z} \cup \{\infty\}$. The space corresponds to a compactification of $(\overline{\mathbb{H}}\backslash\mathbb{Q})/\mathbb{Z}$ by attaching a semicircle on each $q \in \mathbb{Q} \cup \{\infty\}$. Then the value of a function on $\overline{\mathbb{Q}/\mathbb{Z}} \times [-\pi/2, \pi/2]$ is defined by

$$\varphi(\alpha, \theta) := \lim_{\substack{z\to\alpha \\ z\in\ell}} \varphi(z),$$

where ℓ is a ray emitting from α and the angle from $i\mathbb{R}$ to ℓ in the clockwise direction is θ.

We note that the topology induced by $\widehat{\mathbb{H}/\mathbb{Z}}$ is the same as the topology induced by $\mathbb{H}/\mathbb{Z} \sqcup (\mathbb{R}\backslash\mathbb{Q})/\mathbb{Z}$. This implies the continuity of $\mathrm{Re}\sum_{Z\mathcal{M},k}\varphi_0$ on $\widehat{\mathbb{H}/\mathbb{Z}}$ and hence the real part $\mathrm{Re}\sum_{Z\mathcal{M},k}\varphi_0$ of the complex k-Brjuno function is continuous on $\mathbb{H}/\mathbb{Z} \sqcup (\mathbb{R}\backslash\mathbb{Q})/\mathbb{Z}$ in the usual sense. For $(\alpha_0, \pi/2) \in \overline{\mathbb{Q}/\mathbb{Z}} \times [-\pi/2, \pi/2]$, the value $\mathrm{Re}\sum_{Z\mathcal{M},k}\varphi_0(\alpha_0, \pi/2)$ (resp. $(\alpha_0, -\pi/2)$), with $\alpha_0 \in \mathbb{Q}/\mathbb{Z}$, is the right (resp. left) limit of $\mathrm{Re}\sum_{Z\mathcal{M},k}\varphi_0(\alpha)$, as $\alpha \in (\mathbb{R}\backslash\mathbb{Q})/\mathbb{Z}$ tends to α_0. Recalling Corollary 7.2, one has

$$\mathrm{Re}\varphi_{1,k}(1, \pi/2) - \mathrm{Re}\varphi_{1,k}(1, -\pi/2) = -\pi$$

and more precisely, by a similar argument as in the proof of Corollary 7.2, from (7.13), we have

$$\begin{aligned}
&\mathrm{Re}\varphi_{1,k}(1,\theta) - \mathrm{Re}\varphi_{1,k}(1,0) \\
&= -\lim_{t\to 0^+} \arg\left(\frac{1}{1-(1+t\mathrm{e}^{i(\pi/2-\theta)})}\right) + \lim_{t\to 0^+} \arg\left(\frac{1}{1-(1+t\mathrm{e}^{i(\pi/2)})}\right) \\
&= -\lim_{t\to 0^+} \arg\left(t^{-1}\mathrm{e}^{i(\pi/2+\theta)}\right) + \lim_{t\to 0^+} \arg\left(t^{-1}\mathrm{e}^{i(\pi/2)}\right) = -\theta.
\end{aligned}$$

If $\alpha_0 \in \mathbb{Q}$, $\alpha_0 \neq 1$, then $\mathrm{Re}\varphi_{1,k}(\alpha_0, \theta) = \mathrm{Re}\varphi_{1,k}(\alpha_0, 0)$ for all $\theta \in [-\pi/2, \pi/2]$. Thus, by (7.3) one obtains that for all $p/q \in \mathbb{Q}$, $(p \wedge q = 1)$

$$\mathrm{Re}\sum_{Z\mathcal{M},k} \varphi_0(p/q, \theta) = \mathrm{Re}\sum_{Z\mathcal{M},k} \varphi_0(p/q, 0) - \theta/q^k. \tag{7.14}$$

To show (7.14), we use the fact that

$$\mathrm{Re}(L_{k,g}\varphi(\alpha,\theta)) = \det(g)^{k+1}(a-c\alpha)^k\left[\mathrm{Re}\varphi\left(\frac{d\alpha-b}{a-c\alpha},\det(g)\theta\right)-\varphi\left(-\frac{d}{c}\right)\right] \\ -\sum_{n=1}^{k}\frac{\det(g)^{k-n+1}(a-c\alpha)^{k-n}}{c^n n!}\varphi^{(n)}\left(-\frac{d}{c}\right). \qquad (7.15)$$

To verify this formula we note that for any $m\in\mathbb{N}$ we have

$$\lim_{t\to 0+}\arg\left(\begin{pmatrix}0&1\\1&m\end{pmatrix}.\left(\alpha+t\cdot e^{i(\pi/2-\theta)}\right)-\begin{pmatrix}0&1\\1&m\end{pmatrix}.\alpha\right) \\ =\lim_{t\to 0+}\arg\left(-\frac{t\cdot e^{i(\pi/2-\theta)}}{(\alpha+m)(\alpha+m+t\cdot e^{i(\pi/2-\theta)})}\right)=-\theta-\frac{\pi}{2}.$$

(This is the ray we get after applying the Möbius transform.) Now, if φ is analytic on the upper half-plane and real on the real line, we can apply Schwarz' reflection principle and obtain

$$\mathrm{Re}\varphi(\alpha,\theta)=\lim_{t\to 0+}\mathrm{Re}\varphi(\alpha+t\cdot e^{i(\pi/2-\theta)})=\lim_{t\to 0+}\mathrm{Re}\overline{\varphi}(\alpha+t\cdot e^{-i(\pi/2-\theta)}) \\ =\lim_{t\to 0+}\mathrm{Re}\varphi(\alpha+t\cdot e^{-i(\pi/2-\theta)}).$$

Hence, if $\psi(z)=\varphi(g(m).z)$, then $\mathrm{Re}\psi(\alpha,\theta)=\mathrm{Re}\varphi(g(m).\alpha,-\theta)$ and generally if $\psi(z)=\varphi(g.z)$ with $g\in\mathcal{M}$, then $\mathrm{Re}\psi(\alpha,\theta)=\mathrm{Re}\varphi(g.\alpha,\det(g)\theta)$.

Since every element of $\mathcal{M}$ can be written as a product of matrices $g(m)$, (7.15) follows.

By (7.3), we have

$$\mathrm{Re}\sum_{\mathcal{ZM},k}\varphi_0(\alpha,\theta)-\mathrm{Re}\sum_{\mathcal{ZM},k}\varphi_0(\alpha,0) \\ =\sum_{n\in\mathbb{Z},g\in\mathcal{M}}\mathrm{Re}\left[L_{k,g}\varphi_{1,k}(\alpha-n,\theta)-L_{k,g}\varphi_{1,k}(\alpha-n,0)\right].$$

By Corollary 7.2, the real part $\mathrm{Re}\big[L_{k,g}\varphi_{1,k}(\alpha-n,\theta)-L_{k,g}\varphi_{1,k}(\alpha-n,0)\big]$ is 0 if and only if $\frac{d(\alpha-n)-b}{a-c(\alpha-n)}\neq 1$ and by (7.15),

$$\mathrm{Re}\big[L_{k,g}\varphi_{1,k}(\alpha-n,\theta)-L_{k,g}\varphi_{1,k}(\alpha-n,0)\big]=-\left(\det(g)(a-c(\alpha-n))\right)^k\theta,$$

if $\frac{d(\alpha-n)-b}{a-c(\alpha-n)}=1$. There exist unique $n\in\mathbb{N}$ and $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathcal{M}$ such that $\frac{d(\alpha-n)-b}{a-c(\alpha-n)}=1$, i.e. $\alpha=n+\frac{a+b}{c+d}$. If $\alpha=p/q$, then $c+d=q$. Thus,

$$\mathrm{Re}\sum_{\mathcal{ZM}}\varphi_0(p/q,\theta)-\mathrm{Re}\sum_{\mathcal{ZM}}\varphi_0(p/q,0)=-\frac{\det(g)^{2k}}{(c+d)^k}\theta=-\frac{\theta}{q^k}.$$

Thus, $\mathrm{Re}(\mathcal{B}_k)$ has at each rational $p/q\in\mathbb{Q}/\mathbb{Z}$ a decreasing jump of π/q^k. $\square$

7.1.2. A comment on the Wilton function. It would be interesting to know the behaviour of the Wilton function at rational points. However, the results can not easily be transferred to the Wilton function. By a similar argument as in the proof of Lemma 7.1, we will get from the next lemma that $\overline{\varphi}_1(z) + i\log(1-z) - \frac{1}{2\pi}(z+1)\log^2\left(1-\frac{1}{z}\right)$ is continuous on $\overline{\mathbb{H}}$ and in a neighbourhood of 1 it behaves like

$$\begin{aligned}\overline{\varphi}_1(z) &= \frac{3\pi}{4} - \frac{\log 2}{\pi} - i\log(1-z) + \frac{1}{2\pi}(z+1)\log^2\left(1-\frac{1}{z}\right) \\ &\quad + O\left(|1-z|\log\left(|1-z|\right)\right).\end{aligned}$$

Thus, we do not have an analog of Corollary 7.2 since $\mathrm{Re}\big(\log^2\left(1-\frac{1}{z}\right)\big)$ does not go to a finite value for $z \to 1$.

7.2. Behaviour of the imaginary part of $\mathcal{W}$

In this section, we will explain the behaviour of the imaginary part of the Wilton function near the real axis, distinguishing rational points, Wilton numbers and Diophantine numbers. Many arguments are slight adaptations of those given in [26], thus they will only be sketched and many proofs will be given in the Appendix.

The same behaviour which we observe for the imaginary part of the Wilton function will also characterize the approach to the real axis of the imaginary part of the k-Brjuno function, but we will limit ourselves to proving the results for the former, the arguments being very similar.

The following lemma is an analog to [26, Lem. 5.2] and Lemma 7.1. However, here we are considering the function $\overline{\varphi}_1(z) + i\log(1-z) - \frac{1}{2\pi}(z+1)\log^2\left(1-\frac{1}{z}\right)$ instead of $\varphi_1(z) + i\log(1-z)$ in [26]. We will later remark on how this further influences the proof of Theorem 7.5.

Lemma 7.4. *The function $\overline{\varphi}_1(z) + i\log(1-z) - \frac{1}{2\pi}(z+1)\log^2\left(1-\frac{1}{z}\right)$ is continuous on $\overline{\mathbb{H}}$ and its value at 1 is $\frac{3\pi}{4} - \frac{\log 2}{\pi}$.*

Proof. By combining (7.6) and Euler's functional equation (7.12) it follows that

$$\begin{aligned}\overline{\varphi}_1(z) =& \frac{1}{\pi}\left[z\mathrm{Li}_2\left(\frac{1-z}{z}\right) + \mathrm{Li}_2(z-1)\right] + \frac{1}{2\pi}\left[z\log^2\left(\frac{1-z}{-z}\right) + \log^2(-z+1)\right] \\ &+ \frac{\pi}{12}z + \frac{\pi}{6} - \frac{\log 2}{\pi}.\end{aligned} \tag{7.16}$$

The function $z\mathrm{Li}_2\left(\frac{1-z}{z}\right) + \mathrm{Li}_2(z-1)$ is regular and vanishing at $z=1$. We have

$$z\log^2\left(\frac{1-z}{-z}\right) + \log^2(-z+1) = (z+1)\log^2\left(\frac{1-z}{-z}\right)$$

$$+\log^2(-z+1)-\log^2\left(\frac{1-z}{-z}\right).$$

As in the proof of [26, Lem. 5.2], it follows that

$$\log^2(-z+1)-\log^2\left(\frac{1-z}{-z}\right)=\pi^2-2\pi i\log(1-z)+O\left(|z-1|\log\left(|z-1|\right)\right),$$

which completes the proof. $\square$

From now on, we write $f(z)\lesssim g(z)$ if there exists a constant $C>0$ such that $f(z)\le Cg(z)$ for all z. The following is an analog to [26, Thm. 5.10].

Theorem 7.5. *For $n\ge 0$, $m_1,\cdots,m_n\ge 1$ and $z_0\in H(m_1,\cdots,m_n)$, we have*

$$\begin{aligned}\mathrm{Im}\mathcal{W}(z_0)=&-W_{\text{finite}}\left(\frac{p_n}{q_n}\right)+(-1)^n(p_{n-1}-q_{n-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_n+1)\\&+r_n(z_0),\end{aligned}$$

with $|r_n(z_0)|\lesssim\frac{\log q_n}{q_n}|z_n|\log^2(1+|z_n|^{-1})$.

As the proof is very similar to the one in [26, Thm. 5.11], it is given in the Appendix. However, the summand and the remainder term differ slightly from those in [26] about which we will comment in the following.

Remark 7.6. Here we give a remark on the summand $(-1)^n(p_{n-1}-q_{n-1}\mathrm{Re}(z_0))$ $\mathrm{Im}\overline{\varphi}_1(z_n+1)$ of Theorem 7.5. We have

$$\begin{aligned}&(-1)^n(p_{n-1}-q_{n-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_n+1)\\&=\left(q_n^{-1}\log\frac{1}{|z_n|}\right)\left(1+\frac{2}{\pi}\mathrm{Arg}(1+1/z_n)+o(1)\right).\end{aligned}$$

It can be derived as follows.

By (4.13), we have

$$q_n(p_{n-1}-q_{n-1}\mathrm{Re}(z_0))=(-1)^n\mathrm{Re}\left(\frac{q_n}{q_{n-1}z_n+q_n}\right)\to(-1)^n\quad\text{as } z_n\to 0.$$

By $\mathrm{Re}(\log^2(w))=\log^2|w|-(\mathrm{Arg}(w))^2$ and $\mathrm{Im}(\log^2(x))=2\log|w|\mathrm{Arg}(w)$, we have

$$\begin{aligned}&\mathrm{Im}\left((z_n+2)\log^2\left(1+\frac{1}{z_n}\right)\right)=\\&2(\mathrm{Re}(z_n)+2)\mathrm{Arg}(1+1/z_n)(\log|z_n+1|+\log|z_n|^{-1})\\&+\mathrm{Im}(z_n)(\log^2|1+1/z_n|-(\mathrm{Arg}(1+1/z_n))^2).\end{aligned}$$

In the above equation, $2(\mathrm{Re}(z_n)+2)\mathrm{Arg}(1+1/z_n)\log|z_n+1|-\mathrm{Im}(z_n)(\mathrm{Arg}(1+1/z_n))^2$ is uniformly bounded. Since $|\mathrm{Im}(z_n)\log^2|1+1/z_n|\le|z_n|\log^2(1+$

$1/|z_n|) \to 0$ as $z_n \to 0$, from Lemma 7.4, we have

$$\frac{\mathrm{Im}\overline{\varphi}_1(z_n+1)}{\left(\log\frac{1}{|z_n|}\right)} = \frac{-\log|z_n| + \frac{1}{2\pi}2(\mathrm{Re}(z_n)+2)\mathrm{Arg}(1+1/z_n)\log|z_n|^{-1}}{\log|z_n|^{-1}} + o(1)$$

$$= 1 + \frac{2}{\pi}\mathrm{Arg}(1+1/z_n) + o(1) = 1 + O(1),$$

but not in $1+o(1)$.

Finally, we remark here on the estimate of the remainder term in Theorem 7.5 which differs from the remainder term in [26, Thm. 5.10] as we have an additional factor of $\log q_n \cdot \log(1+|z_n|^{-1})$. It is caused by the signs of the terms $z\varphi_0(1/z-1)$ and $\varphi_0(z-1)$ in the definition of $\overline{\varphi}_1$ in (7.4). The functions $\varphi_0(1/z-1)$ and $\varphi_0(z-1)$ are not continuous at 1 because they can be written as a sum of a function continuous near 1 and a constant multiple of $\log^2(1-z)$ (see the proof of Lemma 7.4). Unlike in the case of the Brjuno function, the non-continuous parts cancel out (see [26, Lem. 5.2]).

Finally, we will show an analog of [26, Thm. 5.19]. We will give a sufficient condition for $W(x)$ to be approximated by the imaginary part of $\mathcal{W}(z)$ if z approaches x. For $R>0$ and $0<r<1/2$, let

$$\begin{aligned} U_R &= \{u \in \mathbb{H}:\ \mathrm{Im}(u) \geq |\mathrm{Re}(u)|^R\} \quad \text{and} \\ \widetilde{U}_r &= \left\{u \in \mathbb{H}:\ \mathrm{Im}(u) \geq \exp\left(-\frac{1}{|\mathrm{Re}(u)|^r}\right)\right\}. \end{aligned}$$

Theorem 7.7. *We have the following statements:*

(i) For any Wilton number x and any $R>0$, we have

$$\lim_{u\to 0,\ u\in U_R} \mathrm{Im}\mathcal{W}(u+x) = -W(x).$$

(ii) Let x be an irrational Diophantine number and $0<r<1/2$ such that

$$\liminf_{q\to\infty} \|qx\|_{\mathbb{Z}}\ q^{1/r-1} = \infty,$$

where $\|\cdot\|_{\mathbb{Z}}$ denotes the distance from the nearest integer. Then,

$$\lim_{u\to 0,\ u\in \widetilde{U}_r} \mathrm{Im}\mathcal{W}(u+x) = -W(x).$$

Since the singular behaviour of the k-Brjuno function as the real line is approached is tamer than the singular behaviour of the Brjuno function, we expect that the same properties proven in [26] and stated in the theorem above for the Wilton function hold by a modest adaptation of the proofs given there.

Proof of Theorem 7.7. Let $x \in (0,1)$ be an irrational number whose continued fraction expansion is

$$x = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_n + \ddots}}}}.$$

Let $x_\ell = G^\ell(x)$. Let $\{p_n/q_n\}_{n\geq 0}$ be the sequence of the partial quotients of x. Let u be a point near 0 such that $|\mathrm{Im}(u)| \leq 1/2$ and $z = u + x \in \Delta$. Then there are $N_1 \cdots, N_L$ such that $z \in H(N_1, \cdots, N_L)$. Let p/q be a rational number such that

$$p/q = \cfrac{1}{N_1 + \cfrac{1}{N_2 + \cfrac{1}{\ddots + \cfrac{1}{N_L}}}}.$$

We distinguish two cases.

(i) Assume that $p/q = p_L/q_L$, i.e., $m_i = N_i$ for $i = 1, \cdots, L$. From Theorem 7.5, Remark 7.6 and Lemma 3.1, there exist uniform constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned}
&|\mathrm{Im}\mathcal{W}(u+x) + W(x)| \\
&= \Big| - W_{\text{finite}}\left(\frac{p_L}{q_L}\right) + \sum_{\ell=0}^{L-1} (-1)^\ell \beta_{\ell-1}(x) \log\frac{1}{x_\ell} + \sum_{\ell=L}^{\infty} (-1)^\ell \beta_{\ell-1}(x) \log\frac{1}{x_\ell} \\
&\qquad + (-1)^n (p_{L-1} - q_{L-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_L + 1) + r_n(z_0)\Big| \qquad (7.17)\\
&\leq c_1 x_L q_n^{-1} + c_2 q_L^{-1} \log|z_L|^{-1} + c_3 q_L^{-1} \log(q_L)|z_L| \log^2(1 + |z_L|^{-1}) \\
&\qquad + \Big|\sum_{\ell=L}^{\infty} (-1)^\ell \beta_{\ell-1}(x) \log\frac{1}{x_\ell}\Big|.
\end{aligned}$$

This case follows in an analogous way to the proof of [26, Thm. 5.19(I)] where we note that the considered terms also stay small if we consider a Wilton instead of a Brjuno number.

(ii) Assume that p/q is not one of the partial quotients of x. We denote by $\{p'_\ell/q'_\ell\}_{0\leq\ell\leq L}$ the partial quotient of p/q and by n the largest integer such that $p'_n/q'_n = p_n/q_n$. Clearly, one has $n < L$ and $p'_L/q'_L = p/q$. Note that $q_n^{-1}\log(q_n) \geq (q'_L)^{-1}\log(q'_L)$ if $q_n \geq 3$. As in the proof of [26, Thm. 5.19], from the fact $|x - p/q| \geq (2q^2)^{-1}$, if $u \in U_R$, then there exists a uniform constant $c > 0$ such that

$$q^{-1}\log|z_L|^{-1} \leq q^{-1}(c + (2R-2)\log q) \leq q_n^{-1}(c + (2R-2)\log q_n)$$

and if $u \in \widetilde{U_r}$, then we have

$$q^{-1}\log|z_L|^{-1} \le q^{-1}cq^{2r} \le q_n^{-1}cq_n^{2r}.$$

From Theorem 7.5, Remark 7.6 and Lemma 3.1, there exist uniform constants $c_1, c_2, C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}
&|\mathrm{Im}\mathcal{W}(u+x) + W(x)| \\
&= \Big| -W_{\text{finite}}\left(\frac{p_L'}{q_L'}\right) + \sum_{\ell=0}^{n-1}(-1)^\ell \beta_{\ell-1}(x)\log\frac{1}{x_\ell} + \sum_{\ell=n}^{\infty}(-1)^\ell\beta_{\ell-1}(x)\log\frac{1}{x_\ell} \\
&\quad + (-1)^L(p_{L-1}' - q_{L-1}'\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_L+1) + r_L(z_0)\Big| \\
&\le \Big| -W_{\text{finite}}\left(\frac{p_L'}{q_L'}\right) + W_{\text{finite}}\left(\frac{p_n}{q_n}\right)\Big| + \Big| -W_{\text{finite}}\left(\frac{p_n}{q_n}\right) + \sum_{\ell=0}^{n-1}(-1)^\ell\beta_{\ell-1}(x)\log\frac{1}{x_\ell}\Big| \\
&\quad + \Big|\sum_{\ell=n}^{\infty}(-1)^\ell\beta_{\ell-1}(x)\log\frac{1}{x_\ell}\Big| + c_1\frac{\log|z_L|^{-1}}{q_L'} + c_2\frac{\log q_L'}{q_L'}|z_L|\log^2(1+|z_L|^{-1}) \\
&\le \Big| -W_{\text{finite}}\left(\frac{p_L'}{q_L'}\right) + W_{\text{finite}}\left(\frac{p_n}{q_n}\right)\Big| + \Big|\sum_{\ell=n}^{\infty}(-1)^\ell\beta_{\ell-1}(x)\log\frac{1}{x_\ell}\Big| \\
&\quad + C_3\frac{x_n}{q_n} + C_1\frac{1}{q_n^\gamma} + C_2\frac{\log q_n}{q_n}
\end{aligned}$$

since $|z_L|\log^2(1+|z_L|^{-1})$ are bounded, where $\gamma = 1$ for the case of $u \in U_R$ and $\gamma = 1-2r$ for the case of $u \in \widetilde{U}_r$. Thus, it is enough to show that $W_{\text{finite}}(p/q)$ is close to $W_{\text{finite}}(p_n/q_n)$. Let

$$\rho = \max_{n\le\ell<L}(q_\ell')^{-1}\log\frac{q_{\ell+1}'}{q_\ell'}(\ell-n+1)^2.$$

By Lemma 3.1, we have

$$\begin{aligned}
&|W_{\text{finite}}(p_n/q_n) - W_{\text{finite}}(p/q)| \\
&\le \Big| W_{\text{finite}}(p_n/q_n) - \sum_{\ell=0}^{n-1}(-1)^\ell\beta_{\ell-1}(x)\log\frac{1}{x_\ell} \\
&\quad + \sum_{\ell=0}^{L-1}(-1)^\ell\beta_{\ell-1}(x)\log\frac{1}{x_\ell} - W_{\text{finite}}(p/q)\Big| \\
&\le Cx_nq_n^{-1} + Cx_L(q_L')^{-1} + \left|\sum_{\ell=n}^{L-1}(-1)^\ell\beta_{\ell-1}(x)\log\frac{1}{x_\ell}\right| \\
&\le C'q_n^{-1} + \left|\sum_{\ell=n}^{L-1}(-1)^\ell\beta_{\ell-1}(x)\log\frac{1}{x_\ell}\right|.
\end{aligned}$$

We have

$$\sum_{\ell=n}^{L-1} \beta_{\ell-1}(x) \log \frac{1}{x_\ell} \lesssim \sum_{\ell=n}^{L-1} \frac{1}{q'_\ell} \log \frac{q'_{\ell+1}}{q'_\ell} \leq \rho \sum_{\ell=r}^{L-1} \frac{1}{(\ell-n+1)^2} \lesssim \rho.$$

With the same argument as in the proof of [26, Thm. 5.19] we have the conclusion.

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Appendix A. Convergence condition of F_s for $1/2 < s < 1$

In this appendix, we show that the convergence of $\mathsf{F}_s(x) = \sum_{n=1}^{\infty} \frac{e^{i\pi n^2 x}}{n^s}$ for $s \in (1/2, 1)$ in [32] is equivalent to the convergence of a Brjuno-type function considered in [27]. Rivoal and Seuret proved the following:

Theorem 7.8. [32, Theorem 1-(i)] *Let $x \in (0,1)\backslash\mathbb{Q}$ and $s \in (1/2, 1)$. If*

$$\sum_{n=0}^{\infty} \frac{(q_{k+1}(x))^{(1-s)/2}}{(q_k(x))^{s/2}} < \infty, \tag{A.1}$$

then $\mathsf{F}_s(x)$ *is absolutely convergent.*

Let us consider the function $B_{\sigma,\nu}$ for $\sigma > 0$ and $\nu > 0$ which satisfies

$$\begin{cases} B_{\sigma,\nu}(x+1) = B_{\sigma,\nu}(x) & \text{for } x \in \mathbb{R}, \\ B_{\sigma,\nu}(x) = x^{-1/\sigma} + x^{\nu} B_{\sigma,\nu}(1/x) & \text{for } x \in (0,1). \end{cases} \tag{A.2}$$

When $\nu = 1$, it can be obtained by replacing $-\log(x)$ by $x^{-1/\sigma}$ in (1.1) and it is recalled from [27, Eq. (3.6)] but here we consider a more general exponent $\nu > 0$. Then, we have

$$B_{\sigma,\nu}(x) = \sum_{n=0}^{\infty} (\beta_{n-1}(x))^{\nu} x_n^{-1/\sigma}.$$

In the same way an operator T with $Tf(x) = x^{\nu} f(1/x)$ could be defined such that $(1-T)B_{\sigma,\nu}(x) = x^{-1/\sigma}$. Furthermore, note that $x_n^{-1/\sigma} = \left(\frac{|p_n - q_n x|}{|p_{n-1} - q_{n-1} x|}\right)^{-1/\sigma}$ $\sim (q_n/q_{n+1})^{-1/\sigma}$ and $\beta_{n-1}(x) \sim 1/q_n$, here $f(x) \sim g(x)$ means that there is a constant $c > 1$ such that $1/c < f(x)/g(x) < c$. Thus, $B_{\sigma,\nu}(x) < \infty$ is equivalent to

$$\sum_{n=0}^{\infty} \frac{(q_{n+1}(x))^{1/\sigma}}{(q_n(x))^{\nu(1+1/\sigma)}} < \infty.$$

If we take $\sigma = \frac{2}{1-s}$ and $\nu = \frac{s}{3-s}$, then we have the condition as in (A.1).

Appendix B. Proof of Proposition 6.3

Proof. From the fact that $\mathcal{M}$ is the free monoid generated by the matrices $g(m)$, we have that $z \in D_\infty$ and $g \in \mathcal{M}$ imply $g^{-1}.z \in D_\infty$. Also, by this fact combined with [26, Lem. 3.2], we then have that

$$z \in V_\rho(D_\infty) \text{ implies } g^{-1}.z \in V_\rho(D_\infty). \tag{B.1}$$

Let $\mathcal{M}^{(r)} = \{g \in \mathcal{M} \mid g = g(m_1) \cdots g(m_r),\ m_i \in \mathbb{N}\}$. Then we have for all $\psi \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus I)$ that

$$(T_k^{(k+2)})^r \psi = \sum_{g \in \mathcal{M}^{(r)}} L_g^{(k+2)} \psi = \sum_{g' \in \mathcal{M}^{(r-1)}} L_{g'}^{(k+2)} (T_k^{(k+2)} \psi). \tag{B.2}$$

Let $g' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, $z \in V_\rho(D_\infty)$ and $z' = (g')^{-1}.z$. Then for all $m \in \mathbb{N}$, we have

$$g' g(m) = \begin{pmatrix} b' & a' + mb' \\ d' & c' + md' \end{pmatrix}.$$

Thus, by using the definition of $T_k^{(k+2)}$ given in Proposition 6.1 together with (5.26) and (5.27) we obtain

$$\begin{aligned} L_{g'}^{(k+2)}\left(T_k^{(k+2)}\psi\right) &= L_{g'}^{(k+2)}\left(\sum_{m=1}^{\infty} L_{g(m)}^{(k+2)}\psi\right) = L_{g'}^{(k+2)}\left(\sum_{m=1}^{\infty} -z^{-(k+2)}\psi\left(\frac{1}{z}-m\right)\right) \\ &= \det(g)^{k+1}\left(a'-c'z\right)^{-(k+2)}\left(-\sum_{m=1}^{\infty}\left(\frac{a'-c'z}{d'z-b'}\right)^{k+2}\psi\left(\frac{1}{z'}-m\right)\right) \\ &= -\det(g)^{k+1}(d'z-b')^{-(k+2)}\sum_{m=1}^{\infty}\psi\left(\frac{1}{z'}-m\right). \end{aligned}$$

We remark that if $r=1$, then we consider $g'\in\mathcal{M}^{(0)}=\{\mathrm{id}\}$ and the following arguments work well.

It follows from (5.1) and (B.1) that

$$\sum_{m=1}^{\infty}\left|\psi\left(\frac{1}{z'}-m\right)\right| \le \sum_{m=1}^{\infty} c_{1,\rho,k}\left|\frac{1}{z'}-m\right|^{-(k+2)}\sup_{V_\rho(D_\infty)}|\psi|$$

for some constant $c_{1,\rho,k}$. Since $\sup_{z\in D_\infty}\mathrm{Re}(1/z)=1$ (see Figure 7b), there exists $c_2>0$ such that $|1/z-m|^{-1}\le c_2 m^{-1}$ for all $z\in D_\infty$ and $m\ge 2$. Thus, a similar argument as the one showing (6.2) in the proof of Proposition 6.1 yields that for all $z'\in V_\rho(D_\infty)$

$$\sum_{m=1}^{\infty}\left|\psi\left(\frac{1}{z'}-m\right)\right| \le c_{3,\rho,k}\sup_{V_\rho(D_\infty)}|\psi|$$

for some constant $c_{3,\rho,k}$.

Also, there exists $c_{4,\rho}>0$ such that $|z-\frac{b'}{d'}|^{-1}\le c_{4,\rho}$ for all $z\in V_\rho(D_\infty)$ and $\frac{b'}{d'}\in[0,1]$. Here, d' can be considered as the denominator of the rth principal convergent of some number. Finally,

$$\min_{g'\in\mathcal{M}^{(r-1)}} d' \ge \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^{r-1} \quad\text{and}\quad \sum_{g'\in\mathcal{M}^{(r-1)}} d'^{-2}\le c_5, \tag{B.3}$$

for some $c_5>0$, see for example [26, Prop. A1.1 in Appendix 1]. Furthermore, since $g'\in\mathcal{M}^{(r-1)}$, we have that $\left|\det(g')^{k+1}\right|=1$. Combining these results we obtain the statement of (i).

In the following, we will show (ii) for $r\ge 2$, and we will discuss the case $r=1$ separately. Starting in an analogous way as in the proof of (i), using (5.23), we obtain for $\varphi\in\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$ that

$$\begin{aligned} L_{k,g'}\left(T_k\varphi\right) &= L_{k,g'}\left(\sum_{m=1}^{\infty} L_{k,g(m)}\varphi\right) \\ &= \frac{(c')^{-(k+1)}\det(g')^{k+1}}{k!}\left(a'-c'z\right)^{-1} \end{aligned}$$

$$\cdot \int_0^1 (1-t)^k \sum_{m=1}^{\infty} L_{k,g(m)}^{(k+2)} \varphi^{(k+1)} \left(-\frac{d'}{c'} + \frac{\det(g')t}{c'(a'-c'z)} \right) \mathrm{d}t.$$

We have

$$\begin{aligned}\xi_{z,t} : &= -\frac{d'}{c'} + \frac{\det(g')t}{c'(a'-c'z)} = -\frac{d'}{c'}t + \frac{\det(g')t}{c'(a'-c'z)} + \frac{d'}{c'}t - \frac{d'}{c'} \\ &= tg'^{-1}.z - (1-t)\frac{d'}{c'}.\end{aligned}$$

If we denote $\iota(z) := 1/z$, see Fig. 7b, then we can inductively show that $g'^{-1}.z$ is contained in a ρ-neighbourhood of $\bigcup_{m\geq 1}(\iota(D_\infty)-m) \subset D_\infty$ with respect to the Poincaré metric. Since $d'/c' \in (-\infty,-1]$, the line segment between $-d'/c'$ and $g^{-1}.z$ is contained in $V_\rho(D_\infty \cap \bigcup_{m\geq 1}(\iota(D_\infty)-m))$.

Thus, there is a constant $c_{6,\rho} > 0$ such that $|\xi_{z,t}| \geq c_{6,\rho}$. The constant $c_{6,\rho}$ increases when $\rho \to 0$. Since $\xi_{z,t}$ is in a ρ-neighbourhood of i, $-i$, or $\mathrm{Re}(\xi_{z,t}) \leq 0$, we have $|1/\xi_{z,t} - m| \geq m$.

Since $\varphi^{(k+1)} \in \mathcal{O}^{k+2}(\overline{\mathbb{C}} \setminus [0,1])$, by using (5.26) and (6.3), we have

$$\begin{aligned}\sup_{z\in V_\rho(D_\infty)} |L_{k,g'}(T_k\varphi)| &\leq \frac{(c')^{-(k+2)}}{k! \inf_{z\in V_\rho(D_\infty)} |z-\frac{a'}{c'}|} \sum_{m\geq 1} \int_0^1 \frac{|\varphi^{(k+1)}(1/\xi_{z,t}-m)|}{|\xi_{z,t}|^{k+2}} dt \\ &\leq \frac{(c')^{-(k+2)}}{k! \inf_{z\in V_\rho(D_\infty)} |z-\frac{a'}{c'}|} \sum_{m\geq 1} \frac{c'_{1,k+1}(k+1)!}{c_{6,\rho}^{k+2}(m-0.2)^{k+2}} \sup_{z\in V_\rho(D_\infty)} |\varphi(z)| \\ &\leq \frac{c_{7,k}c'_{1,k+1}(k+1)}{c_{6,\rho}^{k+2}} \cdot \frac{(c')^{-(k+2)}}{\inf_{z\in V_\rho(D_\infty)} |z-\frac{a'}{c'}|} \cdot \sup_{z\in V_\rho(D_\infty)} |\varphi(z)|,\end{aligned}$$

where $c_{7,k} = \sum_{m\geq 1}(m-0.2)^{-(k+2)}$.

Since $a'/c' \in [0,1]$, we have $|z-\frac{a'}{c'}|^{-1} \leq c_{4,\rho}$ for $z \in V_\rho(D_\infty)$. Using (B.3) and (6.3), we obtain

$$\begin{aligned}\sup_{z\in V_\rho(D_\infty)} |(T_k^r\varphi)(z)| &\leq \frac{2^k c_{7,k}c'_{1,k+1}(k+1)}{c_{6,\rho}^{k+2}} \left(\frac{\sqrt{5}-1}{2}\right)^{k(r-2)} \\ &\qquad \sum_{g'\in\mathcal{M}^{(r-1)}} (c')^{-2} \cdot \sup_{z\in V_\rho(D_\infty)} |\varphi(z)| \\ &\leq \overline{C}_{\rho,k,\geq 2} \left(\frac{\sqrt{5}-1}{2}\right)^{rk} \sup_{z\in V_\rho(D_\infty)} |\varphi(z)|, \qquad \text{(B.4)}\end{aligned}$$

where $\overline{C}_{\rho,k,\geq 2} = 2^k(k+1)(\frac{\sqrt{5}+1}{2})^{2k} c'_{1,k+1} c_5 c_{4,\rho} c_{6,\rho}^{-(k+2)} c_{7,k}$, implying (ii) for $r \geq 2$.

For the case $r = 1$, we will estimate

$$|T_k\varphi(z)| \leq \sum_{m\geq 1} |L_{k,g(m)}\varphi(z)|$$

by using (5.24) and (5.25). We consider ℓ denoting the right arc of ∂D_∞ (which is the arc of the circle $\left|z-\frac{\sqrt{3}}{3}\right|=\frac{\sqrt{3}}{3}$ between $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ and $\frac{\sqrt{3}}{2}-\frac{1}{2}i$ containing $\frac{2\sqrt{3}}{3}$), see Fig. 7a. Since $1/z\in V_\rho(\iota(D_\infty))$, from the shape of $\iota(D_\infty)$, we can conclude that $t/z\in V_\rho(\iota(D_\infty))$ for $0\le t\le 1$. Thus, $t/z-m\in V_\rho(\iota(D_\infty)-m)$. If $m\ge 2$ or z is not in the neighbourhood of ℓ, then there is $c>0$ such that $|t/z-m|\ge cm$, thus by Cauchy's estimate as in (6.3), we have

$$\left|\varphi^{(k+1)}\left(\frac{t}{z}-m\right)\right|\le c'_{1,k+1}\frac{(k+1)!\sup_{V_\rho(D_\infty)}|\varphi|}{cm^{k+2}}.$$

Thus, if z is not in the neighbourhood of ℓ, then by using (5.25), we have

$$\begin{aligned}|T_k\varphi(z)|&\le\frac{1}{c_{6,\rho}k!}\sum_{m\ge 1}c'_{1,k+1}\frac{(k+1)!\sup_{V_\rho(D_\infty)}|\varphi|}{cm^{k+2}}\\&\le\overline{C}'_{\rho,k}\left(\frac{\sqrt{5}-1}{2}\right)^k\sup_{V_\rho(D_\infty)}|\varphi|,\end{aligned}\tag{B.5}$$

where $\overline{C}'_{\rho,k}=\left(\frac{\sqrt{5}-1}{2}\right)^{-k}(k+1)c'_{1,k+1}c_{6,\rho}^{-1}c^{-1}\sum_{m\ge 1}\frac{1}{m^{k+2}}$. We are left to show the case that z is in a neighbourhood of ℓ. For $z\in V_\rho(D_\infty)$ in a neighbourhood of ℓ, we can assume that $|z|\le 2$. Thus, by (5.24) and (6.3), we have

$$\begin{aligned}|L_{k,g(1)}\varphi(z)|&\le|z|^k\left|\varphi\left(\frac{1}{z}-1\right)\right|+\sum_{n=0}^{k}\frac{|z|^{k-n}}{n!}|\varphi^{(n)}(-1)|\\&\le|z|^k\sup_{V_\rho(D_\infty)}|\varphi|+\sum_{n=0}^{k}\frac{|z|^{k-n}}{n!}c'_{1,n}\frac{n!\sup_{V_\rho(D_\infty)}|\varphi|}{0.8^{n+1}}\\&\le\left(|z|^k+\sum_{n=1}^{k}\frac{c'_{1,n}|z|^{k-n}}{0.8^{n+1}}\right)\sup_{V_\rho(D_\infty)}|\varphi|\le C_k\left(\frac{\sqrt{5}-1}{2}\right)^k\sup_{V_\rho(D_\infty)}|\varphi|,\end{aligned}\tag{B.6}$$

where $C_k=\left(\frac{\sqrt{5}-1}{2}\right)^{-k}2^k\left(1+\sum_{n=1}^{k}\frac{c'_{1,n}}{2^n 0.8^{n+1}}\right)$. By replacing the summand of $m=1$ in (B.5) with (B.6) and combining this result with the estimate for $r\ge 2$ in (B.4), we have the conclusion when we take $\overline{C}_{\rho,k}=\max\{C_k+\overline{C}'_{\rho,k},\overline{C}_{\rho,k,\ge 2}\}$.
$\square$

Appendix C. Proof of Theorem 7.5.

Analogously as in [26] the strategy of the proof of the theorem is to start from the formula

$$\mathrm{Im}\mathcal{W}(z) = \sum_{n\in\mathbb{Z}}\sum_{\ell\geq 0} \mathrm{Im}S^{\ell}\overline{\varphi}_1(z+n). \tag{C.1}$$

We will only give the parts of the proof which are different.

We have the following lemmas and propositions which are analogs to [26, Lem. 5.11, 5.14 and Prop. 5.15, 5.16] in the same way by using $S = -T$ and substituting $\overline{\varphi}_1$ for φ_1. For the following we denote for a bounded closed interval $I \subset \mathbb{R}$, the space of functions $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}}\backslash I)$ whose real part is bounded, we denote it by $E(I)$, endowed with the norm $\|\varphi\|_{E(I)} = \sup_{\mathbb{C}\backslash I} |\mathrm{Re}\varphi|$. If it is clear which interval I is meant, we also write $\|\varphi\|$ instead of $\|\varphi\|_{E(I)}$. Note that $\overline{\varphi}_1 \in E([1/2,2])$. If $\psi \in E([1/2,2])$ is real on the real axis outside $[1/2,2]$, then ψ fulfils the arguments about φ_1 in the proof of [26, Lem. 5.11, 5.14 and Prop. 5.15, 5.16].

Lemma 7.9. *For $z \in \Delta \setminus [0,1]$, if we write*

$$\begin{aligned}\mathrm{Im}\mathcal{W}(z) =&\mathrm{Im}\overline{\varphi}_1(z) + \mathrm{Im}\overline{\varphi}_1(z+1) + \mathrm{Im}\overline{\varphi}_1(z+2) + \mathrm{Im}S\overline{\varphi}_1(z-1) + \mathrm{Im}S\overline{\varphi}_1(z)\\ &+ \sum_{\ell>1}\left[\mathrm{Im}S^{\ell}\overline{\varphi}_1(z-1) + \mathrm{Im}S^{\ell}\overline{\varphi}_1(z) + \mathrm{Im}S^{\ell}\overline{\varphi}_1(z+1)\right] + r^0(z),\end{aligned}$$

then we have

$$|r^0(z)| \leq C|\mathrm{Im}(z)|\|\overline{\varphi}_1\|_{E([1/2,2])}.$$

Lemma 7.10. *For $z \in \Delta \setminus [0,1]$, we have*

$$\begin{aligned}\mathrm{Im}\mathcal{W}(z) =&(1-\varepsilon_1)\mathrm{Im}\overline{\varphi}_1(z) + \mathrm{Im}\overline{\varphi}_1(z+1) + [1-\varepsilon_2(1-\varepsilon_1)]\mathrm{Im}S\overline{\varphi}_1(z)\\ &+ \sum_{\ell>1}\mathrm{Im}S^{\ell}\overline{\varphi}_1(z) + r^0(z) + r^1(z),\end{aligned} \tag{C.2}$$

and

$$|r^1(z)| \leq C|\mathrm{Im}(z)|\log(1+|\mathrm{Im}(z)|^{-1})\|\overline{\varphi}_1\|_{E([1/2,2])}.$$

Proposition 7.11. *For $n \geq 1$, $m_1,\dots,m_n \geq 1$, $z_0 \in D(m_1,\dots,m_n)$, $\ell > 0$ we have*

$$\begin{aligned}&\left|(-1)^n\mathrm{Im}\left(S^{n+\ell}\overline{\varphi}_1(z_0)\right) - \left[\mathrm{Im}\left(S^{\ell}\overline{\varphi}_1(z_n)\right)\right](p_{n-1}-q_{n-1}\mathrm{Re}(z_0))\right|\\ &\leq Cq_n^{-1}|\mathrm{Im}(z_n)|\log(1+|\mathrm{Im}(z_n)|^{-1}) \times \begin{cases}\|S^{\ell-2}\overline{\varphi}_1\| & \text{if } \ell>1,\\ \|\overline{\varphi}_1\| & \text{if } \ell=1.\end{cases}\end{aligned}$$

Proposition 7.12. *For $n \geq 1$, $m_1,\dots,m_n \geq 1$, $z_0 \in D(m_1,\dots,m_n)$ we have*

$$|(-1)^n\mathrm{Im}(S^n\overline{\varphi}_1(z_0)) - (p_{n-1}-q_{n-1}\mathrm{Re}(z_0))[(1-\varepsilon_{n+1})\mathrm{Im}(\overline{\varphi}_1(z_n))$$

$$+ \varepsilon_n \mathrm{Im}(\overline{\varphi}_1(z_n+1))]| \leq C q_n^{-1} |\mathrm{Im}(z_n)| \log(1+|\mathrm{Im}(z_n)|^{-1}) \|\overline{\varphi}_1\| \,.$$

The following is an analog of [26, Prop. 5.17]. We recall the definition of $H(m_1, \cdots, m_n)$ given in (4.15).

Proposition 7.13. *For $n \geq 0$, $m_1, \ldots, m_n \geq 1$, $z_0 \in H(m_1, \ldots, m_n)$ we have*

$$\begin{aligned} \mathrm{Im}\mathcal{W}(z_0) = &\sum_{\ell=0}^{n} \varepsilon_\ell (-1)^\ell (p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0)) \mathrm{Im}\overline{\varphi}_1(z_\ell+1) \\ &+ \sum_{\ell=0}^{n-1} (1-\varepsilon_{\ell+1})(-1)^\ell (p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0)) \mathrm{Im}\overline{\varphi}_1(z_\ell) + r_n(z_0), \end{aligned}$$

with $\varepsilon_0 = 1$, and $|r_n(z_0)| \leq C q_n^{-1} |z_n| \log(1+|z_n|^{-1})$.

Proof. Since $\overline{\varphi}_1$ is real on the real axis outside $[0,1]$, by Proposition 6.10, when $z \in H$,

$$|\mathrm{Im} S^n \overline{\varphi}_1(z)| \lesssim |z| \log(1+|z|^{-1}) \sup_{D_\infty} |S^{n-1}\overline{\varphi}_1|. \tag{C.3}$$

We obtain $\sum_{n=1}^{\infty} \sup_{D_\infty} |S^n \overline{\varphi}_1| < \infty$ by using Proposition 6.3-(ii). Thus, from (C.3) and Lemma 7.10, we get

$$\mathrm{Im}\mathcal{W}(z) = (1-\varepsilon_1)\mathrm{Im}\overline{\varphi}_1(z) + \mathrm{Im}\overline{\varphi}_1(z+1) + r(z), \tag{C.4}$$

with $|r(z)| \lesssim |z| \log(1+|z|^{-1})$.

Let us consider the case $z_0 \in H$ which corresponds to the case $n = 0$. Since we set $\varepsilon_1 = 1$ for $z_0 \in \Delta \backslash D$, from (C.4), we obtain

$$|\mathrm{Im}\mathcal{W}(z_0) - \mathrm{Im}\overline{\varphi}_1(z_0+1)| \lesssim |z_0| \log(1+|z_0|^{-1}).$$

Next, we consider $z_0 \in H(m_1, \cdots, m_n)$ with $n \geq 1$ and $m_1, \cdots, m_n \geq 1$. By Lemma 7.10, we need to show that the modulus of

$$\begin{aligned} &\Big(\sum_{\ell=n+1}^{\infty} \mathrm{Im} S^\ell \overline{\varphi}_1(z_0) \Big) + (-1)^n (p_{n-1} - q_{n-1}\mathrm{Re}(z_0))(1-\varepsilon_{n+1}) \mathrm{Im}\overline{\varphi}_1(z_n) + r^0(z_0) + r^1(z_0) \\ &+ \Big[(1-\varepsilon_2(1-\varepsilon_1)) \mathrm{Im} S \overline{\varphi}_1(z_0) + (p_0 - q_0 \mathrm{Re}(z_0))(\varepsilon_1 \mathrm{Im}\overline{\varphi}_1(z_1+1) + (1-\varepsilon_2)\mathrm{Im}\overline{\varphi}_1(z_1)) \Big] \\ &+ \sum_{\ell=2}^{n} \Big[\mathrm{Im} S^\ell \overline{\varphi}_1(z_0) - (-1)^\ell (p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0)) \big(\varepsilon_\ell \mathrm{Im}\overline{\varphi}_1(z_\ell+1) + (1-\varepsilon_{\ell+1}) \mathrm{Im}\overline{\varphi}_1(z_\ell) \big) \Big] \end{aligned} \tag{C.5}$$

is in $O(q_n^{-1}|z_n| \log(1+|z_n|^{-1}))$. From [26, Prop. 4.11] using $S = -T$, we have $\sum_{j=0}^{\infty} \|S^j \overline{\varphi}_1\| < \infty$. Then, from Proposition 7.11, we have

$$\begin{aligned} &\Big| \Big(\sum_{\ell=n+1}^{\infty} \mathrm{Im} S^\ell \overline{\varphi}_1(z_0) \Big) - (p_{n-1} - q_{n-1}\mathrm{Re}(z_0)) \sum_{\ell=1}^{\infty} \mathrm{Im} S^\ell \overline{\varphi}_1(z_n) \Big| \\ &\lesssim q_n^{-1} |\mathrm{Im}(z_n)| \log(1+|\mathrm{Im}(z_n)|^{-1}). \end{aligned}$$

Since $z_n \in H$, by using (C.3), the first term of (C.5) is bounded by

$$\sum_{\ell=n+1}^{\infty} \left|\mathrm{Im} S^{\ell}\overline{\varphi}_1(z_0)\right| \lesssim q_n^{-1}|z_n|\log(1+|z_n|^{-1}).$$

For $z \in H$, we have

$$|\mathrm{Im}\overline{\varphi}_1(z)| \lesssim |\mathrm{Im}(z)|. \tag{C.6}$$

For the second term of (C.5), by using (C.6) and (4.14), we have

$$|(p_{n-1} - q_n - 1\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_n)(1-\varepsilon_{n+1})| \lesssim q_n^{-1}|\mathrm{Im}(z_n)|.$$

We have $|r^0(z_0) + r^1(z_0)| \lesssim |\mathrm{Im}(z_0)|\log(1+|\mathrm{Im}(z_0)|^{-1})\|\overline{\varphi}_1\|_{E([1/2,2])}$ in Lemma 7.9 and 7.10. Since

$$[1-\varepsilon_2(1-\varepsilon_1)](1-\varepsilon_2) = 1-\varepsilon_2 \quad \text{and} \quad [1-\varepsilon_2(1-\varepsilon_1)]\varepsilon_1 = \varepsilon_1,$$

the third term of (C.5) is 0 or

$$[\mathrm{Im} S\overline{\varphi}_1(z_0) + (p_0 - q_0\mathrm{Re}(z_0))(\varepsilon_1\mathrm{Im}\overline{\varphi}_1(z_1+1) + (1-\varepsilon_2)\mathrm{Im}\overline{\varphi}_1(z_1))].$$

By using Proposition 7.12, the last terms of (C.5) are in

$$O\left(\sum_{\ell=0}^{n} q_\ell^{-1}|\mathrm{Im}(z_\ell)|\log(1+|\mathrm{Im}(z_\ell)|^{-1})\|\overline{\varphi}_1\|_{E([1/2,2])}\right)$$

and we have

$$\sum_{\ell=0}^{n} q_\ell^{-1}|\mathrm{Im}(z_\ell)|\log(1+|\mathrm{Im}(z_\ell)|^{-1}) \le q_n^{-1}|\mathrm{Im}(z_n)|\log(1+|\mathrm{Im}(z_n)|^{-1})$$

as in the last part of the proof of [[26], Prop. 5.17]. Since $|\mathrm{Re}(z_0)| \le 1$, we have the conclusion. □

Proof of Theorem 7.5. Let $n \ge 0$ and $m_1, \cdots, m_n \ge 1$. The domain $H(m_1, \cdots, m_n)$ meets $\mathbb{R}$ in a unique point p_n/q_n. Let us denote $x_0 = p_n/q_n$ and consider its continued fraction

$$x_i^{-1} = m_{i+1} + x_{i+1}, \quad 0 \le i < n \quad \text{and} \quad x_n = 0,$$

here, x_i is the point of intersection of $H(m_{i+1}, \cdots, m_n)$ with $\mathbb{R}$.

Let $z_0 \in H(m_1, \cdots, m_n)$ and $\mathrm{Im}(z_0) > 0$. We will then have $(-1)^\ell\mathrm{Im}(z_\ell) > 0$ for $0 \le \ell \le n$. From (7.8), we deduce that

$$\mathrm{Im}\overline{\varphi}_1(x_\ell + 1 + (-1)^\ell i0) = (-1)^\ell \log\frac{1}{x_\ell}, \quad \text{for } 0 \le \ell < n. \tag{C.7}$$

Since $\frac{x_\ell}{1-x_\ell} = \frac{1}{x_{\ell+1}}$ if $\varepsilon_{\ell+1} = 0$, we obtain from (7.8)

$$(1-\varepsilon_{\ell+1})\mathrm{Im}\overline{\varphi}_1(x_\ell + (-1)^\ell i0) = (1-\varepsilon_{\ell+1})(-1)^{\ell+1}x_\ell\log\frac{1}{x_{\ell+1}},$$
$$\text{for } 0 \le \ell < n-1. \tag{C.8}$$

The above equations imply for W_{finite} extended to the complex plane that

$$\begin{aligned} W_{\text{finite}}(p_n/q_n) = &\sum_{\ell=0}^{n-1} \varepsilon_\ell (q_{\ell-1}x_0 - p_{\ell-1})(-1)^\ell \mathrm{Im}\overline{\varphi}_1(x_\ell + 1 + (-1)^\ell i0) \\ &+ \sum_{\ell=0}^{n-2} (1-\varepsilon_{\ell+1})(q_{\ell-1}x_0 - p_{\ell-1})(-1)^\ell \mathrm{Im}\overline{\varphi}_1(x_\ell + (-1)^\ell i0). \end{aligned} \tag{C.9}$$

Combining Proposition 7.13 with (C.9), we get

$$\begin{aligned} &\mathrm{Im}\mathcal{W}(z_0) + W_{\text{finite}}(p_n/q_n) - (-1)^n (p_{n-1} - q_{n-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_n+1) - r_n(z_0) \\ &= \sum_{\ell=0}^{n-1} \varepsilon_\ell (-1)^\ell \Big[(p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_\ell+1) \\ &\quad - (p_{\ell-1} - q_{\ell-1}x_0)\mathrm{Im}\overline{\varphi}_1(x_\ell + 1 + (-1)^\ell i0) \Big] \\ &+ \sum_{\ell=0}^{n-2} (1-\varepsilon_{\ell+1})(-1)^\ell \Big[(p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_\ell) \\ &\quad - (p_{\ell-1} - q_{\ell-1}x_0)\mathrm{Im}\overline{\varphi}_1(x_\ell + (-1)^\ell i0) \Big] \\ &- (1-\varepsilon_n)(-1)^n \Big[(p_{n-1} - q_{n-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_n+1) \\ &\quad + (p_{n-2} - q_{n-2}\mathrm{Re}(z_0))\mathrm{Im}(\overline{\varphi}_1(z_{n-1})) \Big]. \end{aligned} \tag{C.10}$$

Thus, we have to deal with the following expressions. For $0 \le \ell \le n-1$ such that $\varepsilon_\ell = 1$,

$$A_\ell := (p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_\ell+1) - (p_{\ell-1} - q_{\ell-1}x_0)\mathrm{Im}\overline{\varphi}_1(x_\ell + 1 + (-1)^\ell i0).$$

For $0 \le \ell \le n-2$ such that $\varepsilon_{\ell+1} = 0$,

$$B_\ell := (p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_\ell) - (p_{\ell-1} - q_{\ell-1}x_0)\mathrm{Im}\overline{\varphi}_1(x_\ell + (-1)^\ell i0).$$

When $\varepsilon_n = 0$,

$$C_n := (p_{n-1} - q_{n-1}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_n+1) + (p_{n-2} - q_{n-2}\mathrm{Re}(z_0))\mathrm{Im}\overline{\varphi}_1(z_{n-1}).$$

The first n digits of the continued fraction of $\mathrm{Re}(z_0)$ are also $m_1, \cdots, m_n$. This gives

$$|\mathrm{Re}(z_0) - x_0| \le |z_0 - p_n/q_n| = q_n^{-1}|q_{n-1}z_0 - p_{n-1}||z_n| \le q_n^{-2}|z_n|.$$

Since (7.6) implies that

$$\overline{\varphi}_1'(z) = \frac{1}{\pi}\left(-\mathrm{Li}_2\left(\frac{z}{1-z}\right) + \frac{1}{1-z}\log\left(\frac{1-2z}{1-z}\right) + \frac{1}{1-z}\log\left(\frac{z-2}{z-1}\right)\right) - \frac{\pi}{12},$$

we have

$$|\overline{\varphi}_1'(z)| \le C|z-1|^{-1}\log\frac{1}{|z-1|} \quad \text{near } 1. \tag{C.11}$$

For $0 \leq \ell \leq n$, the distance of x_ℓ and z_ℓ from 0 are comparable. Thus, for $0 \leq \ell \leq n$,

$$|\mathrm{Im}\overline{\varphi}_1(z_\ell + 1) - \mathrm{Im}\overline{\varphi}_1(x_\ell + 1 + (-1)^\ell i0)| \leq C x_\ell^{-1} \log \frac{1}{x_\ell} |z_\ell - x_\ell|,$$

and, for $0 \leq \ell \leq n-1$,

$$|\mathrm{Im}\overline{\varphi}_1(z_\ell) - \mathrm{Im}\overline{\varphi}_1(x_\ell + (-1)^\ell i0)| \leq C|x_\ell - 1|^{-1} \log \frac{1}{1 - x_\ell} |z_\ell - x_\ell|.$$

We have $|z_\ell - x_\ell| \lesssim |z_n| q_\ell^2 q_n^{-2}$, $x_\ell^{-1} \lesssim m_{\ell+1}$ and $|x_\ell - 1|^{-1} \lesssim x_{\ell+1}^{-1} \lesssim m_{\ell+2}$. Thus, according to (C.7) and (C.8), for $0 \leq \ell < n$, we have

$$\begin{aligned}
|A_\ell| &= \Big|(p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0))\Big(\mathrm{Im}\overline{\varphi}_1(z_\ell + 1) - \mathrm{Im}\overline{\varphi}_1(x_\ell + 1 + (-1)^\ell i0)\Big) \\
&\qquad + q_{\ell-1}(\mathrm{Re}(z_0) - x_0)\mathrm{Im}\overline{\varphi}_1(x_\ell + 1 + (-1)^\ell i0)\Big| \\
&\lesssim x_\ell^{-1} \log \frac{1}{x_\ell} |z_\ell - x_\ell| q_\ell^{-1} + q_{\ell-1} q_n^{-2} |z_n| \log \frac{1}{x_\ell} \\
&\lesssim |z_n| q_n^{-2} (q_\ell m_{\ell+1} + q_{\ell-1}) \log(m_{\ell+1}) \lesssim |z_n| q_n^{-2} q_{\ell+1} \log(m_{\ell+1}),
\end{aligned}$$

and, for $0 \leq \ell < n-1$ such that $\varepsilon_{\ell+1} = 0$,

$$\begin{aligned}
|B_\ell| &= \Big|(p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0))\Big(\mathrm{Im}\overline{\varphi}_1(z_\ell) - \mathrm{Im}\overline{\varphi}_1(x_\ell + (-1)^\ell i0)\Big) \\
&\qquad + q_{\ell-1}(\mathrm{Re}(z_0) - x_0)\mathrm{Im}\overline{\varphi}_1(x_\ell + (-1)^\ell i0)\Big| \\
&\lesssim |p_{\ell-1} - q_{\ell-1}\mathrm{Re}(z_0)||x_\ell - 1|^{-1} \log \frac{1}{|x_\ell - 1|} |z_\ell - x_\ell| + q_{\ell-1} q_n^{-2} |z_n| \log \frac{1}{x_{\ell+1}} \\
&\lesssim |x_\ell - 1|^{-1} \log \frac{1}{|x_\ell - 1|} |z_\ell - x_\ell| q_\ell^{-1} + q_{\ell-1} q_n^{-2} |z_n| \log \frac{1}{x_{\ell+1}} \\
&\lesssim |z_n| q_n^{-2} (q_\ell m_{\ell+2} + q_{\ell-1}) \log(m_{\ell+2}) \lesssim |z_n| q_n^{-2} q_{\ell+2} \log(m_{\ell+2}).
\end{aligned}$$

From $q_n \geq 2^{\lfloor (n-\ell)/2 \rfloor} q_\ell$, it follows that

$$\sum_{\ell=0}^{n-1} |A_\ell| + \sum_{\ell=0}^{n-2} |B_\ell| \lesssim |z_n| \frac{\log q_n}{q_n}.$$

Furthermore, when $\varepsilon_n = 0$, we have

$$\begin{aligned}
\overline{\varphi}_1(z_{n-1}) &= z_{n-1} \left(\varphi_0(z_n) - \varphi_0(-1)\right) - \varphi_0'(-1) + \varphi_0(z_{n-1} - 1). \\
\overline{\varphi}_1(1 + z_n) &= z_{n-1}^{-1} \left(\varphi_0(z_{n-1} - 1) - \varphi_0(-1)\right) - \varphi_0'(-1) + \varphi_0(z_n).
\end{aligned}$$

Since

$$\overline{\varphi}_1(z_{n-1}) = (1 - z_{n-1}) \left(\frac{\pi}{12} - \frac{1}{\pi} \log 2\right) + z_{n-1} \overline{\varphi}_1(1 + z_n),$$

we have

$$C_n = \mathrm{Im}\overline{\varphi}_1(z_n + 1)[(p_{n-1} - q_{n-1}\mathrm{Re}(z_0)) + (p_{n-2} - q_{n-2}\mathrm{Re}(z_0))\mathrm{Re}(z_{n-1})]$$

$$+ (p_{n-2} - q_{n-2}\mathrm{Re}(z_0))\mathrm{Im}(z_{n-1}) \left[-\left(\frac{\pi}{12} - \frac{1}{\pi}\log 2 \right) + \mathrm{Re}\overline{\varphi}_1(1+z_n)) \right].$$

By (7.16), we have $|\overline{\varphi}_1(1+z_n))| \lesssim \log^2(1+|z_n|^{-1})$. As in the proof of [26, Theorem 5.10],

$$|(p_{n-2} - q_{n-2}\mathrm{Re}(z_0))\mathrm{Im}(z_{n-1})| \lesssim q_n^{-1}|\mathrm{Im}(z_n)|,$$

and

$$[(p_{n-1} - q_{n-1}\mathrm{Re}(z_0)) + (p_{n-2} - q_{n-2}\mathrm{Re}(z_0))\mathrm{Re}(z_{n-1})] \lesssim q_n^{-1}|\mathrm{Im}(z_n)|.$$

Thus, we have $|C_n| \lesssim q_n^{-1}|\mathrm{Im}(z_n)| \log^2(1+|z_n|^{-1})$.

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