Dissipation properties of transport noise in the two–layer quasi–geostrophic model

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Abstract

A stochastic version of the two–layer quasi–geostrophic model (2LQG) with multiplicative transport noise is analysed. This popular intermediate complexity model describes large scale atmosphere and ocean dynamics at the mid–latitudes. The transport noise, which acts on both layers, accounts for the unresolved small scales. After establishing the well–posedness of the perturbed equations, we show that, under a suitable scaling of the noise, the solutions converge to the deterministic 2LQG model with enhanced dissipation. Moreover, these solutions converge to the deterministic stationary ones on the long time horizon.

Keywords: SPDEs; Stochastic geophysical flow models; Dissipation enhancement; Transport noise.

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1 Introduction

The quasi–geostrophic model, first mathematically described by Charney [6] in the first half of the XX century, is an approximation of the three dimensional Navier–Stokes equation in vorticity formulation which captures the large–scale phenomena of atmosphere and ocean dynamics. Quasi–geostrophic models with several layers, like the one considered in this article, are particularly suitable to describe baroclinic instabilities, a crucial mechanism behind most common weather patterns at the mid–latitudes. In fact the quasi–geostrophic equations is a popular model in theoretical meteorology given the richness of phenomena it can describe (see e.g. [29, 26]). From a mathematical perspective, it can be seen as a model of intermediate complexity between two and three dimensional Navier–Stokes equation. The model has been shown to be well–posed and to exhibit a global attractor in its deterministic version (see [1]) and in a stochastic version with additive noise [7, 4]. In this paper a stochastically perturbed version of the two–layer quasi–geostrophic (2LQG) model with multiplicative noise of transport type on both layers is studied and shown to be well–posed.

The nature of this perturbation is crucial both from a physical interpretation of the results and for the novelty of mathematical tools it requires. Recent developments on the physical justification for transport noise in fluid dynamics models include [20]. There the transport noise is systematically introduced in models relevant for geophysical applications, including the quasi–geostrophic model, in such a way to retain conservation laws crucial for the description of fluids, in particular circulation. Recently [17] proposed a rigorous interpretation and justification of transport noise as additive noise on smaller scales. For more on the interpretation of transport noise in stochastic partial differential equation refer to [17] and references therein.

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From a mathematical perspective the literature on transport noise for fluid dynamics models is vast, especially for Euler and Navier–Stokes equation. Most relevant for this work are [13, 12]. In these works the authors make use a suitable scaling, first introduced in [18], to show that weak solutions of a stochastic Euler model converge to weak solutions of the deterministic Navier–Stokes equation, and provide a quantitative estimate of the rate of convergence. Similarly, for the 2LQG model we will see that the transport noise provides an enhancement of eddy dissipation effects. These results highlight the regularization action of the transport noise, which, despite being energy-preserving in general, acts as a dissipative force. On the relevance of transport noise for dissipation and mixing properties we refer as well to [14, 11].

Thanks to dissipation properties of the transport noise we will also show that the solution of the 2LQG model approaches the deterministic stationary solution for large times. The well–posedness of the stationary system associated to the 2LQG equations would require in general a large model viscosity, hard to justify from a physical perspective. However, we will show that an appropriate choice of the transport noise ensures stability on the long run of the solutions. This results extends those obtained for the heat equations in [15] to a nonlinear system like the quasi–geostrophic model here under analysis. We expect this approach to generalise to the two–dimensional Navier–Stokes equations and other models with similar structure.

1.1 The Model and the Main Results

The two-layer quasi–geostrophic model (2LQG) is one of the most used model to describe the motion of atmosphere as well as of the ocean at the mid–latitudes. In particular it captures the large scale dynamics of two layers of fluid of fixed height h_1, h_2 respectively and with density ρ_1 and ρ_2 with $\rho_1 < \rho_2$. We consider the β-plane approximation to the Coriolis effect for which the Coriolis parameter f_c can be expressed as $f_c(y) = f_0 + \beta y$, with f_0 and β assigned positive constants. In the model we include the effect of eddy viscosity on both layers, of the bottom friction on the second layer, to account for the interaction with the Eckmann layer, and a deterministic additive forcing on the first layer, for example to account for the wind forcing on the upper ocean. In order to give a mathematical formulation of the model let us introduce a spatial domain $\mathcal{D}$, squared domain $\mathcal{D} = [0, L] \times [0, L] \subset \mathbb{R}^2$ (where L will be e.g. 10^5 m for the ocean).

Consider the following equations for the variables $\boldsymbol{\psi}(t, \mathbf{x}) = (\psi_1(t, \mathbf{x}), \psi_2(t, \mathbf{x}))^t$, streamfunction of the fluid, and $\mathbf{q}(t, \mathbf{x}) = (q_1(t, \mathbf{x}), q_2(t, \mathbf{x}))^t$, the so-called quasi–geostrophic (QG) potential vorticity

$$\begin{aligned} dq_1 + \left(\nabla^\perp\psi_1 \cdot \nabla q_1\right) dt &= \left(\nu\Delta^2\psi_1 - \beta\partial_x\psi_1 + F(t)\right) dt \\ dq_2 + \left(\nabla^\perp\psi_2 \cdot \nabla q_2\right) dt &= \left(\nu\Delta^2\psi_2 - \beta\partial_x\psi_2 - r\Delta\psi_2\right) dt. \end{aligned} \tag{1}$$

Here $\nu > 0$ is the eddy viscosity parameter, $r > 0$ accounts for the bottom friction and $F(t)$ is a deterministic forcing with zero spatial averages, namely

$$\int_{\mathcal{D}} F(t, \mathbf{x})\, d\mathbf{x} = 0 \quad \text{for all } \ t \geq 0. \tag{2}$$

The QG potential vorticities $\mathbf{q}$ and the streamfunction $\boldsymbol{\psi}$ are linked by the equations

$$\begin{aligned} q_1 &= \Delta\psi_1 + S_1(\psi_2 - \psi_1) \\ q_2 &= \Delta\psi_2 + S_2(\psi_1 - \psi_2), \end{aligned} \tag{3}$$

where S_1, S_2 are positive constants such that

$$h_1 S_1 = h_2 S_2 =: S. \tag{4}$$

As stated in the introduction, the goal of the present work is to study the eddy dissipation properties of transport noise for this model. There are several motivations to consider transport noise, as the effect of small scales on large scales in fluid dynamical problems, see [16], [20] for several discussion on this topic. Loosely speaking, small scale transport noise produces in the limit an extra dissipative term, which can be called eddy dissipation. There is an extended literature devoted to these kind of topics both in the endogenous and the exogenous case. See for example [12], [13], [15], [11].

Let us now introduce the stochastic perturbation we will consider. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. Let K be a finite set of indexes, $\{W_t^{j,k}\}_{k\in K},\ j \in \{1,\ 2\}$ be two sequences of real independent Brownian motions adapted to $\mathcal{F}_t$ and consider two corresponding sequences of divergence free vector fields $\{\boldsymbol{\sigma}_{j,k}\}_{k=1}^{K} \subseteq C^\infty(\mathcal{D};\mathbb{R}^2),\ j \in \{1,\ 2\}$. In general, less can be required on the regularity of the coefficients $\boldsymbol{\sigma}_{j,k}$ and the cardinality of K, but it is not our goal to stretch the boundaries of regularity of the noise as it is not critical to obtain the desired final result. Now we can consider the following stochastically perturbed two-layer quasi–geostrophic model

$$\begin{aligned} dq_1 + \left(\nabla^\perp\psi_1 \cdot \nabla q_1\right) dt &= \left(\nu\Delta^2\psi_1 - \beta\partial_x\psi_1 + F\right) dt + \sum_{k\in K} \left(\boldsymbol{\sigma}_{1,k} \circ dW^{1,k}\right) \cdot \nabla q_1 \\ dq_2 + \left(\nabla^\perp\psi_2 \cdot \nabla q_2\right) dt &= \left(\nu\Delta^2\psi_2 - \beta\partial_x\psi_2 - r\Delta\psi_2\right) dt + \sum_{k\in K} \left(\boldsymbol{\sigma}_{2,k} \circ dW^{2,k}\right) \cdot \nabla q_2 \end{aligned} \tag{5}$$

which in Itô formulation reads

$$\begin{aligned} dq_1 + \left(\nabla^\perp\psi_1 \cdot \nabla q_1\right) dt &= \left(\nu\Delta^2\psi_1 - \beta\partial_x\psi_1 + F\right)\ dt + \sum_{k\in K} \boldsymbol{\sigma}_{1,k} \cdot \nabla q_1\, dW^{1,k} + \tfrac{1}{2}\sum_{k\in K} \boldsymbol{\sigma}_{1,k} \cdot \nabla(\boldsymbol{\sigma}_{1,k} \cdot \nabla q_1)\, dt \\ dq_2 + \left(\nabla^\perp\psi_2 \cdot \nabla q_2\right) dt &= \left(\nu\Delta^2\psi_2 - \beta\partial_x\psi_2 - r\Delta\psi_2\right)\ dt + \sum_{k\in K} \boldsymbol{\sigma}_{2,k} \cdot \nabla q_2\, dW^{2} + \tfrac{1}{2}\sum_{k\in K} \boldsymbol{\sigma}_{2,k} \cdot \nabla(\boldsymbol{\sigma}_{2,k} \cdot \nabla q_2)\, dt. \end{aligned} \tag{6}$$

We assume periodic boundary conditions for the streamfunction $\boldsymbol{\psi}$ in both directions and that

$$\int_{\mathcal{D}} \boldsymbol{\psi}(t, \mathbf{x})\, d\mathbf{x} = 0.$$

Let us set some notation before stating the main contributions of this work. Let $(H^k(\mathcal{D}), \|\cdot\|_{H^k}), k \in \mathbb{R}$ be the standard Sobolev spaces of L-periodic functions satisfying condition (2). We will denote by $\langle\cdot,\cdot\rangle_{H^k}$ the corresponding scalar products. With a slight abuse of notation, for $k > 0$ we denote the dual pairing with

$$\langle T, \varphi\rangle_{H^{-k},H^k} = T(\varphi) \quad \text{for all } T \in H^{-k},\ \varphi \in H^k.$$

In case of $k = 0$ we will write $L^2(\mathcal{D})$ instead of $H^0(\mathcal{D})$ and we will neglect the subscript in the notation for the norm. Similarly, we introduce the Sobolev spaces of zero mean vector fields

$$\mathbf{H}^k = \{(u_1, u_2)^t :\ u_1, u_2 \in H^k(\mathcal{D})\},\ \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{H}^k} = \langle u_1, v_1\rangle_{H^k} + \langle u_2, v_2\rangle_{H^k}, \text{ for } k \in \mathbb{R}.$$

Again, in case of $k = 0$ we will write $\mathbf{L}^2$ instead of $\mathbf{H}^0$ and we will neglect the subscript in the notation for the norm and the scalar product. Sometimes, on the space $\mathbf{H}^k$ we will consider the norms

$$|||\cdot|||^2_{\mathbf{H}^k} = h_1\|\cdot_1\|^2_{H^k} + h_2\|\cdot_2\|^2_{H^k} \tag{7}$$

which are straightforwardly equivalent to the standard ones that we denote by $\|\cdot\|_{\mathbf{H}^k}$. Furthermore we define the following functional spaces

$$\mathcal{H} = \mathbf{L}^2,\ \mathcal{V} = \mathbf{H}^1,\ D\left(\boldsymbol{\Delta}\right) = \mathbf{H}^2$$

where $\boldsymbol{\Delta} : D\left(\boldsymbol{\Delta}\right) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\boldsymbol{\Delta}(q_1, q_2)^t = (\Delta q_1, \Delta q_2)^t.$$

It is well known that $\boldsymbol{\Delta}$ is the infinitesimal generator of analytic semigroup of negative type and moreover $\mathcal{V}$ can be identified with $D((-\boldsymbol{\Delta})^{1/2})$, see e.g. [25].

A first issue related to the analysis of the dissipation properties of the transport noise is the well–posedness of such system. In fact the existence of strong probabilistic solution is outside the framework treated in earlier works, see for example [4, 8, 9, 20]. Thus, first, we need to define our notion of solution for system (6) and state our well–posedness result.

Let Z be a separable Hilbert space, with associated norm $\|\cdot\|_Z$. For $p \geq 1$ denote by $L^p(\mathcal{F}_{t_0}, Z)$ the space of p-integrable random variables with values in Z, measurable with respect to $\mathcal{F}_{t_0}$. Moreover, denote by $C_{\mathcal{F}}([0,T];Z)$ the space of continuous adapted processes $(X_t)_{t\in[0,T]}$ with values in Z such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_Z^2\right] < \infty$$

and by $L^p_{\mathcal{F}}(0,T;Z)$ the space of progressively measurable processes $(X_t)_{t\in[0,T]}$ with values in Z such that

$$\mathbb{E}\left[\int_0^T \|X_t\|_Z^p \, dt\right] < \infty.$$

Define for $j \in \{1,2\}$ the operators

$$F_j(q) := \frac{1}{2}\sum_{k\in K} \boldsymbol{\sigma}_{j,k} \cdot \nabla(\boldsymbol{\sigma}_{j,k} \cdot \nabla q), \tag{8}$$

$$G_j^k(q) := \boldsymbol{\sigma}_{j,k} \cdot \nabla q, \text{ for } k \in \{1,\dots,K\} \tag{9}$$

which can be easily shown to be bounded linear operators

$$F_j \in \mathcal{L}(H^2(\mathcal{D}), L^2(\mathcal{D})), \quad G_j^k \in \mathcal{L}(H^1(\mathcal{D}); L^2(\mathcal{D})).$$

We then consider the following concept of weak solution (see e.g. [10, Chapter 7]) for (6):

Definition 1.1. A stochastic process

$$\mathbf{q} \in C_{\mathcal{F}}([0,T];\mathcal{H}) \cap L^2_{\mathcal{F}}(0,T;\mathcal{V})$$

is a weak solution of equation (6) if, for every $\boldsymbol{\phi} = (\phi_1,\phi_2)^t \in D(\boldsymbol{\Delta})$, we have

$$\begin{aligned}
\langle q_1(t),\phi_1\rangle &= \langle q_{1,0},\phi_1\rangle + \int_0^t \langle \Delta\psi_1(s), \nu\Delta\phi_1\rangle\, ds + \int_0^t \langle q_1(s), F_1\phi_1\rangle\, ds + \int_0^t \langle q_1(s), \nabla^\perp\psi_1(s)\cdot\nabla\phi_1\rangle ds \\
&\quad + \int_0^t \langle \beta\partial_x\psi_1(s), \phi_1\rangle\, ds + \int_0^t \langle F(s),\phi_1\rangle\, ds - \sum_{k\in K}\int_0^t \left\langle q_1(s), G_1^k\phi_1\right\rangle dW_s^{1,k} \\
\langle q_2(t),\phi_2\rangle &= \langle q_{2,0},\phi_2\rangle + \int_0^t \langle \Delta\psi_2(s), \nu\Delta\phi_2\rangle\, ds + \int_0^t \langle q_2(s), F_2\phi_2\rangle\, ds + \int_0^t \langle q_2(s), \nabla^\perp\psi_2(s)\cdot\nabla\phi_2\rangle ds \\
&\quad + \int_0^t \langle \beta\partial_x\psi_2(s), \phi_2\rangle\, ds - r\int_0^t \langle \Delta\psi_2(s),\phi_2\rangle\, ds - \sum_{k\in K}\int_0^t \left\langle q_2(s), G_2^k\phi_2\right\rangle dW_s^{2,k}
\end{aligned}$$

for every $t \in [0,T]$, $\mathbb{P}$-a.s., where $\mathbf{q}$ and $\boldsymbol{\psi}$ are linked by relation (3).

The well posedness of this system is guaranteed by the following result which we will prove in Section 3.

Theorem 1.2. *For every $\mathbf{q}_0 \in L^4_{\mathcal{F}_0}(\Omega,\mathcal{H})$ and $F \in L^4_{\mathcal{F}}\left(0,T;L^2(\mathcal{D})\right)$, there exists one and only one weak solution of equation* (6).

Remark 1.3. We stated Theorem 1.2 in full generality in order to provide a complete framework for the well posedness of such stochastic system, that, to the best of these authors knowledge, is unavailable in the literature. This result is redundant for the scope of this work though. In fact, in order to exploit the eddy dissipation properties of the transport noise we will consider deterministic initial conditions and a deterministic and time–independent forcing.

Remark 1.4. It is well known that, in absence of external forcing or dissipation, the quasi–geostrophic model has an infinite number of preserved quantities. As a consequence, in that setting and the one–layer case, [19] showed existence of Gaussian invariant measure induced by the quadratic first integrals. The noise introduced in (5) will preserve the L^2-norm of the quasi–geostrophic potential vorticity $\mathbf{q}$, namely the potential enstrophy of the system. For more on the role of enstrophy in the two-layer quasi–geostrophic dynamics see e.g. [22], [29, Section 5.6.3, Section 9.2.2].

After showing the well posedness of system (6), our goal is to provide sufficient conditions in order to model the dissipation properties of the transport noise. Let us explain briefly the adopted strategy before going into the details. First, following the approach introduced in [12], we will show that under an appropriate scaling of the noise, first introduced by Galeati in [18], the solution of the stochastic system (5) converges, in a suitable sense which we will clarify later, to the solution of the associated deterministic system with an extra diffusion.

Second, for time–independent forcings F and same properties of the noise, solutions of the stochastic system (5) will converge to those of the associated stationary deterministic 2LQG model.

In order to precisely formulate these statements, let us introduce some notation and the precise formulation of the noise. Let a, b be two positive numbers, then we write $a \lesssim b$ if there exists a positive constant C such that $a \le Cb$ and $a \lesssim_\alpha b$ when we want to highlight the dependence of the constant C on a parameter α. Let $e_k(\mathbf{x}) = L^{-1}\exp\{\frac{2\pi i}{L} k\cdot \mathbf{x}\}$, $k \in \mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus \{(0,0)\}$, orthonormal basis of $L^2(\mathcal{D})$ made by eigenfunctions of $-\Delta$.

Following the approach first introduced in [18] consider the following explicit representations of the coefficients $\boldsymbol{\sigma}_{j,k},\ k \in \mathbb{Z}^2_0,\ j \in \{1,2\}$

$$\boldsymbol{\sigma}_{j,k}(\mathbf{x}) = \sqrt{2\kappa}\boldsymbol{a}_{j,k} e_k(\mathbf{x}) = \begin{cases} \sqrt{2\kappa}\theta_{j,k} e_k(\mathbf{x})\frac{k^\perp}{|k|} & \textit{if } k \in \mathbb{Z}^2_+ \\ \sqrt{2\kappa}\theta_{j,k} e_k(\mathbf{x})\frac{-k^\perp}{|k|} & \textit{if } k \in \mathbb{Z}^2_-, \end{cases} \tag{10}$$

where $\mathbb{Z}^2_+,\ \mathbb{Z}^2_-$ is a partition of $\mathbb{Z}^2_0$ with $\mathbb{Z}^2_+ = -\mathbb{Z}^2_-$, and the parameters $\theta_{j,k}$ satisfy the following conditions:

1. $\sum_{k\in\mathbb{Z}^2_0} \theta^2_{j,k} = 1$;
2. $\theta_{j,k} = 0$ if $|k|$ is large enough. We will denote by K the finite set of k where $\theta_{j,k} \neq 0$;
3. $\theta_{j,k} = \theta_{j,l}$ if $|k| = |l|$.

Furthermore take an infinite sequence of complex standard Brownian motions such that $\overline{W^{j,k}} = W^{j,-k}$ and $W^{i,k}$ is independent from $W^{j,l}$ if $|k| \neq |l|$ or $i \neq j$. Thus the noise we consider is parameterized by the coefficients $\kappa,\ \theta_{j,k}$ and the set K. Under this setting, as described for example in [18, 11], equation (6) can be reformulated as

$$\begin{aligned} dq_1 + \left(\nabla^\perp\psi_1\cdot\nabla q_1\right)dt &= \left(\kappa\Delta q_1 + \nu\Delta^2\psi_1 - \beta\partial_x\psi_1 + F\right)dt + \sqrt{2\kappa}\sum_{k\in K}\boldsymbol{a}_{1,k}e_k\cdot\nabla q_1\, dW^{1,k} \\ dq_2 + \left(\nabla^\perp\psi_2\cdot\nabla q_2\right)dt &= \left(\kappa\Delta q_2 + \nu\Delta^2\psi_2 - \beta\partial_x\psi_2 - r\Delta\psi_2\right)dt + \sqrt{2\kappa}\sum_{k\in K}\boldsymbol{a}_{2,k}e_k\cdot\nabla q_2\, dW^{2,k}. \end{aligned} \tag{11}$$

The corresponding deterministic system is

$$\begin{aligned} d\bar{q}_1 + \left(\nabla^\perp\bar{\psi}_1\cdot\nabla\bar{q}_1\right)dt &= \left(\kappa\Delta\bar{q}_1 + \nu\Delta^2\bar{\psi}_1 - \beta\partial_x\bar{\psi}_1 + F\right)dt \\ d\bar{q}_2 + \left(\nabla^\perp\bar{\psi}_2\cdot\nabla\bar{q}_2\right)dt &= \left(\kappa\Delta\bar{q}_2 + \nu\Delta^2\bar{\psi}_2 - \beta\partial_x\bar{\psi}_2 - r\Delta\bar{\psi}_2\right)dt, \end{aligned} \tag{12}$$

where, as before, $\bar{\mathbf{q}}$ and $\bar{\boldsymbol{\psi}}$ are linked by relation (3).

Thanks to Theorem 1.2 and classical results on two dimensional deterministic quasi–geostrophic equations, see for example [1], under the assumptions $\mathbf{q}_0 \in \mathcal{H},\ F \in L^4(0,T;L^2(\mathcal{D}))$ there exists a unique weak solutions $\mathbf{q}$ (resp. $\bar{\mathbf{q}}$) of the problem (11) (resp. (12)). Now we can state one of the main results of our work which allows to quantify the difference between the behavior of the stochastic and the deterministic system.

Theorem 1.5. *Let $\mathbf{q}$ and $\bar{\mathbf{q}}$ be weak solutions to* (11) *and* (12) *respectively. Then for any $\alpha \in (0,1)$, there exists C depending from α and all the parameters of the model except for the noise such that for any $\epsilon \in (0,\alpha]$ one has*

(i)

$$\mathbb{E}\left[\|\mathbf{q}-\bar{\mathbf{q}}\|^2_{C([0,T];\mathbf{H}^{-\alpha})}\right] \lesssim_{\alpha,M,\epsilon,T} \kappa^\epsilon \|\theta\|^{2(\alpha-\epsilon)}_{\ell^\infty} R_T^2 \exp\left(T\frac{\nu^2+\beta^2+r^2}{\kappa+\nu}\right) \exp\left(\frac{CTR_T^2}{(\kappa+\nu)^2}(1+\kappa+\nu) + \frac{C}{(\kappa+\nu)^2}\int_0^T \|F(s)\|^2\, ds\right),$$

(ii)

$$\mathbb{E}\left[\|\mathbf{q}-\bar{\mathbf{q}}\|^2_{C([0,T];\mathbf{H}^{-\alpha})}\right] \lesssim_{\alpha,M,\epsilon,T} \kappa^{\epsilon}\|\theta\|^{2(\alpha-\epsilon)}_{\ell^\infty} R_T^2 \exp\left(T\frac{\nu^2+\beta^2+r^2}{\kappa+\nu}\right)$$
$$\exp\left(\frac{C}{\nu^2}\left(TR_T^2+\int_0^T \|F(s)\|^2\, ds\right)\right),$$

where R_T^2 is a constant independent of the noise defined in Section 4.1.

Next, given the model (5) now with a time–independent forcing F, consider the associated stationary system. On the one hand, similarly to the Navier–Stokes system, existence and uniqueness of the solution of the stationary system associated to quasi–geostrophic equations is guaranteed under, generally unfeasible, assumptions on the viscosity. On the other hand, as we will show in Section 4.2, in our framework these assumptions will be satisfied thanks to the dissipation properties of the transport noise. More precisely, consider $\tilde{\mathbf{q}}$ solution of the following system

$$\begin{aligned}\nabla^\perp\tilde{\psi}_1\cdot\nabla\tilde{q}_1 &= \kappa\Delta\tilde{q}_1+\nu\Delta^2\tilde{\psi}_1-\beta\partial_x\tilde{\psi}_1+F\\ \nabla^\perp\tilde{\psi}_2\cdot\nabla\tilde{q}_2 &= \kappa\Delta\tilde{q}_2+\nu\Delta^2\tilde{\psi}_2-\beta\partial_x\tilde{\psi}_2-r\Delta\tilde{\psi}_2,\end{aligned} \tag{13}$$

where $F\in L^2(\mathcal{D})$ and, as always, $\tilde{\mathbf{q}}$ and $\tilde{\boldsymbol{\psi}}$ are linked by relation (3). Then, thanks to a particular parametrization of the noise, we will show in Section 4.2 the following:

Theorem 1.6. *For κ large enough, for each $\delta>0$ and $\alpha\in(0,1)$, it exists $\overline{T}=\overline{T}(\delta)$ and a sequence $\{\theta_{j,k}\}_{k\in K, j\in\{1,2\}}$ depending from δ, $\overline{T}$, α such that for each $t\in[\overline{T},2\overline{T}]$*

$$\mathbb{E}\left[\sup_{t\in[\overline{T},2\overline{T}]}\|\mathbf{q}(t)-\tilde{\mathbf{q}}\|^2_{\mathbf{H}^{-\alpha}}\right]\le\delta.$$

2 Preliminaries

In this section we recall several technical tools crucial for the next sections, see for example [2],[5],[12],[25] for more details.

First, similarly to [5], we define the linear operator $\tilde{A}:\mathbf{H}^{k+2}\to\mathbf{H}^k,\ k\in\mathbb{R}$ connecting the streamfunction with the quasi–geostrophic potential vorticity

$$\tilde{A}\boldsymbol{\psi} := (-\boldsymbol{\Delta}-M)\,\boldsymbol{\psi},\ \textit{where } M=\begin{pmatrix}-S_1 & S_1\\ S_2 & -S_2\end{pmatrix}.$$

It is well known that it has a bounded inverse $(-\boldsymbol{\Delta}-M)^{-1}:\mathbf{H}^k\to\mathbf{H}^{k+2}$. Thanks to this fact, for each $\mathbf{q}\in\mathbf{H}^k$, there exists a unique $\boldsymbol{\psi}\in\mathbf{H}^{k+2}$ such that $\mathbf{q}=(\boldsymbol{\Delta}+M)\,\boldsymbol{\psi}$ and moreover for each $k\in\mathbb{R}$ there exists two constants $c_{1,k}\le c_{2,k}$ such that

$$c_{1,k}\|\boldsymbol{\psi}\|_{\mathbf{H}^{k+2}}\le\|\mathbf{q}\|_{\mathbf{H}^k}\le c_{2,k}\|\boldsymbol{\psi}\|_{\mathbf{H}^{k+2}}.$$

In this work we will also use extensively the relation

$$\psi_1-\psi_2=(-\Delta+S_1+S_2)^{-1}(q_2-q_1) \tag{14}$$

which follows directly from (3).

We recall three technical lemmata which can be proved by classical arguments and we refer to [12, Section 2.1].

Lemma 2.1 ([12, Lemma 2.1])**.** *Given a divergence free vector field $V\in L^2(\mathcal{D};\mathbb{R}^2)$ the following bounds hold true.*

1. *If $V\in L^\infty\left(\mathcal{D};\mathbb{R}^2\right)$, $f\in L^2(\mathcal{D})$, then we have*

$$\|V\cdot\nabla f\|_{H^{-1}}\lesssim\|V\|_{L^\infty}\|f\|.$$

2. *Let* $\alpha \in (1,2],\ \beta \in (0,\alpha-1),\ V \in H^\alpha(\mathcal{D};\mathbb{R}^2),\ f \in H^{-\beta}(\mathcal{D})$, *we have*

$$\|V\cdot\nabla f\|_{H^{-1-\beta}} \lesssim_{\alpha,\beta} \|V\|_{H^\alpha}\|f\|_{H^{-\beta}}.$$

3. *Let* $\beta \in (0,1)$, *then for any* $f \in H^\beta(\mathcal{D}),\ g \in H^{1-\beta}(\mathcal{D})$ *it holds*

$$\|fg\| \lesssim_\beta \|f\|_{H^\beta}\|g\|_{H^{1-\beta}}.$$

4. *Let* $\beta \in (0,1),\ V \in H^{1-\beta}(\mathcal{D};\mathbb{R}^2),\ f \in L^2(\mathcal{D})$, *then one has*

$$\|V\cdot\nabla f\|_{H^{-1-\beta}} \lesssim_\beta \|V\|_{H^{1-\beta}}\|f\|.$$

Remark 2.2. With some abuse of notation, if $\boldsymbol{\psi}$ and $\mathbf{q}$ are two vector fields, we will denote

$$\nabla^\perp\boldsymbol{\psi}\cdot\nabla\mathbf{q} := \left(\nabla^\perp\psi_1\cdot\nabla q_1, \nabla^\perp\psi_2\cdot\nabla q_2\right)^t$$

and the results of previous lemma continue to hold in this framework. Then this first lemma generalises fundamental estimates for the nonlinearity of the quasi–geostrophic model presented for example in [5, 7].

The second lemma provides classical estimates on the semigroup generated by $\boldsymbol{\Delta}$:

Lemma 2.3 ([12, Lemma 2.2])**.** *Let* $\mathbf{q} \in \mathbf{H}^\alpha,\ \alpha \in \mathbb{R}$. *Then:*

1. *for any* $\rho \ge 0$, *it holds* $\|e^{t\boldsymbol{\Delta}}\mathbf{q}\|_{\mathbf{H}^{\alpha+\rho}} \le C_\rho t^{-\rho/2}\|\mathbf{q}\|_{\mathbf{H}^\alpha}$ *for some constant increasing in* ρ;
2. *for any* $\rho \in [0,2]$, *it holds* $\|\left(I - e^{t\boldsymbol{\Delta}}\right)\mathbf{q}\|_{\mathbf{H}^{\alpha-\rho}} \lesssim_\rho t^{\rho/2}\|\mathbf{q}\|_{\mathbf{H}^\alpha}$.

The semigroup $e^{\delta(t-s)\boldsymbol{\Delta}}$ has also regularising effects as stated in the following:

Lemma 2.4 ([12, Lemma 2.3])**.** *For any* $\delta > 0,\ \alpha \in \mathbb{R},\ \mathbf{q} \in L^2(0,T;\mathbf{H}^\alpha)$, *it holds*

$$\left\|\int_0^t e^{\delta(t-s)\boldsymbol{\Delta}}\mathbf{q}(s)\ ds\right\|^2_{\mathbf{H}^{\alpha+1}} \lesssim \frac{1}{\delta}\int_0^t \|\mathbf{q}(s)\|^2_{\mathbf{H}^\alpha}\ ds \quad \forall t \in [0,T].$$

Finally consider the following classical result.

Lemma 2.5 ([2, Proposition B.3])**.** *If* $\{Q_N\}_{N\ge1} \subseteq L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathbb{R}))$ *are continuous stochastic processes,* $\{\sigma_M\}_{M\ge1}$ *are* $\mathcal{F}_t$*-stopping times such that*

$$\lim_{M\to+\infty}\mathbb{P}(\sigma_M < T) = 0, \tag{15}$$

$$\sup_{N\ge1}\mathbb{E}\left[|Q_N(T)|^2\right] < +\infty, \tag{16}$$

$$\lim_{N\to+\infty}\mathbb{E}\left[|Q_N(\sigma_M)|\right] = 0 \quad \forall M \ge 1, \tag{17}$$

then $\mathbb{E}\left[|Q_N(T)|\right] \to 0$.

3 Well–posedness

In this section we will show Theorem 1.2 holds following a classical approach by means of the Galerkin approximation.

3.1 Galerkin Approximation and Limit Equations

As described in Section 1.1, let $\{e_i\}_{i\in\mathbb{N}}$ be the orthonormal basis of $L^2(\mathcal{D})$ made by eigenvectors of $-\Delta$ and λ_i the corresponding eigenvalues, λ_i are positive and non decreasing. Given $\mathcal{H}^N = \text{span}\{e_1,\ldots,\ e_N\} \subseteq L^2(\mathcal{D})$ let $P^N : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ be the orthogonal projector of $L^2(\mathcal{D})$ on $\mathcal{H}^N$. As we are looking for a finite dimensional approximation of the solution of equation (6), define

$$q_j^N(t) := \sum_{i=1}^N c_{i,j,N}(t)e_i(x).$$

The coefficients $c_{i,j,N}$ are chosen in such a way to satisfy for all eigenfunctions $e_i,\ \ 1 \le i \le N$ and all $t\in[0,T]$ the system

$$\begin{aligned}
\langle q_1^N(t), e_i\rangle &= \langle q_{1,0}^N, e_i\rangle + \int_0^t \langle \Delta\psi_1^N(s), \nu\Delta e_i\rangle\, ds + \int_0^t \langle q_1^N(s), F_1^N e_i\rangle\, ds + \int_0^t \langle q_1^N(s), \nabla^\perp\psi_1^N(s)\cdot\nabla e_i\rangle ds \\
&\quad + \int_0^t \langle \beta\partial_x\psi_1^N(s), e_i\rangle\, ds + \int_0^t \langle F(s), e_i\rangle\, ds - \sum_{k\in K}\int_0^t \langle q_1^N(s), G_1^k e_i\rangle\, dW_s^{1,k} \\
\langle q_2^N(t), e_i\rangle &= \langle q_{2,0}^N, e_i\rangle + \int_0^t \langle \Delta\psi_2^N(s), \nu\Delta e_i\rangle\, ds + \int_0^t \langle q_2^N(s), F_2^N e_i\rangle\, ds + \int_0^t \langle q_2^N(s), \nabla^\perp\psi_2(s)\cdot\nabla e_i\rangle ds \\
&\quad + \int_0^t \langle \beta\partial_x\psi_2^N(s), e_i\rangle\, ds - r\int_0^t \langle \Delta\psi_2^N(s), e_i\rangle\, ds - \sum_{k\in K}\int_0^t \langle q_2^N(s), G_2^k e_i\rangle\, dW_s^{2,k}.
\end{aligned} \tag{18}$$

Here $\mathbf{q}_0^N = P^N\mathbf{q}_0$ and the variables $\boldsymbol{\psi}^N$ and $\mathbf{q}^N$ are linked by relation (3), and the operators F_j^N, $j\in\{1,2\}$, are defined similarly to (8), namely

$$F_j^N\phi = \frac{1}{2}\sum_{k\in K} P^N\left(\boldsymbol{\sigma}_{j,k}\cdot\nabla P^N(\boldsymbol{\sigma}_{j,k}\cdot\nabla\phi)\right),\ j\in\{1,2\} \quad \text{for all } \ \phi\in\mathcal{H}^N.$$

The local well–posedness of (18) follows from classical results about stochastic differential equations with locally Lipshitz coefficients, see for example [21],[28]. The global well–posedness follows from the following a priori estimates:

Lemma 3.1. *Given the system* (18) *the following energy estimate holds:*

$$\begin{aligned}
\frac{d\lvert\lvert\lvert\mathbf{q}^N\rvert\rvert\rvert^2}{2} + \nu\lvert\lvert\lvert\nabla\mathbf{q}^N\rvert\rvert\rvert^2 dt &= \left(-\beta h_1\langle\partial_x\psi_1^N, q_1^N\rangle - \beta h_2\langle\partial_x\psi_2^N, q_2^N\rangle + h_1\langle F, q_1^N\rangle - rh_2\|q_2^N\|^2 + Sr\langle\psi_1^N-\psi_2^N, q_2^N\rangle\right) dt \\
&\quad + \left(S\nu\|q_1^N - q_2^N\|^2 + S\nu(S_1+S_2)\langle\psi_1^N-\psi_2^N, q_1^N-q_2^N\rangle\right)\, dt.
\end{aligned} \tag{19}$$

Furthermore for any $\mathbf{q}_0\in L^4_{\mathcal{F}_0}(\Omega,\mathcal{H})$ *and* $F\in L^4_{\mathcal{F}}(0,T;L^2(\mathcal{D}))$ *the following a priori and integral bounds are satisfied uniformly in* $N\in\mathbb{N}$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\lvert\lvert\lvert\mathbf{q}^N(t)\rvert\rvert\rvert^2\right] \le \mathbb{E}\left[\lvert\lvert\lvert\mathbf{q}_0\rvert\rvert\rvert^2 + 2\int_0^T \|F(s)\|^2 ds\right]e^{CT}, \tag{20}$$

$$\nu\mathbb{E}\left[\int_0^T \lvert\lvert\lvert\nabla\mathbf{q}^N(s)\rvert\rvert\rvert^2\ ds\right] \le \mathbb{E}\left[\int_0^T \|F(s)\|^2\ ds\right] + CT\mathbb{E}\left[\lvert\lvert\lvert\mathbf{q}_0\rvert\rvert\rvert^2 + 2\int_0^T\|F\|^2 ds\right]e^{CT}, \tag{21}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\lvert\lvert\lvert\mathbf{q}^N(t)\rvert\rvert\rvert^4\right] + \mathbb{E}\left[\int_0^T \lvert\lvert\lvert\mathbf{q}^N(s)\rvert\rvert\rvert^2\lvert\lvert\lvert\nabla\mathbf{q}^N(s)\rvert\rvert\rvert^2\ ds\right] \le C(T), \tag{22}$$

$$\mathbb{E}\left[\left(\int_0^T \lvert\lvert\lvert\nabla\mathbf{q}^N(s)\rvert\rvert\rvert^2\ ds\right)^2\right] \le C(T), \tag{23}$$

where C *is a constant possibly changing its value line by line, but independent of* N.

Proof. Let us start by showing (19). We first apply the finite dimensional Itô's formula to the system (18) to get

$$\frac{d\|q_1^N\|^2}{2} = \left(\nu\langle\Delta\psi_1^N, \Delta q_1^N\rangle - \beta\langle\partial_x\psi_1^N, q_1^N\rangle + \langle F, q_1^N\rangle\right) dt$$
$$\frac{d\|q_2^N\|^2}{2} = \left(\nu\langle\Delta\psi_2^N, \Delta q_2^N\rangle - \beta\langle\partial_x\psi_2^N, q_2^N\rangle - r\langle\Delta\psi_2^N, q_2^N\rangle\right) dt,$$

then multiply each equation by h_1 and h_2 respectively and sum them up to obtain

$$\frac{d|||\mathbf{q^N}|||^2}{2} = \left(h_1\nu\langle\Delta\psi_1^N, \Delta q_1^N\rangle + h_2\nu\langle\Delta\psi_2^N, \Delta q_2^N\rangle\right) dt + \left(-\beta h_1\langle\partial_x\psi_1^N, q_1^N\rangle - \beta h_2\langle\partial_x\psi_2^N, q_2^N\rangle\right) dt$$
$$\left(+h_1\langle F, q_1^N\rangle - rh_2\langle\Delta\psi_2^N, q_2^N\rangle\right) dt. \quad (24)$$

Now, thanks to relations (3), (4) and (14) we have

$$\begin{aligned} h_1\langle\Delta\psi_1^N, \Delta q_1^N\rangle + h_2\nu\langle\Delta\psi_2^N, \Delta q_2^N\rangle &= h_1\langle q_1^N + S_1(\psi_1 - \psi_2), \Delta q_1^N\rangle + h_2\langle q_2^N + S_2(\psi_2 - \psi_1), \Delta q_2^N\rangle \\ &= -|||\nabla\mathbf{q}^N|||^2 + S\langle\Delta\psi_1^N - \Delta\psi_2^N, q_1^N - q_2^N\rangle \\ &= -|||\nabla\mathbf{q}^N|||^2 + S\|q_1^N - q_2^N\|^2 + S(S_1 + S_2)\langle\psi_1^N - \psi_2^N, q_1^N - q_2^N\rangle. \end{aligned}$$

Using (3) to treat similarly the term $rh_2\langle\Delta\psi_2^N, q_2^N\rangle$, from (24) we get the desired result

$$\frac{d|||\mathbf{q}^N|||^2}{2} + \nu|||\nabla\mathbf{q}^N|||^2 dt = \left(-\beta h_1\langle\partial_x\psi_1^N, q_1^N\rangle - \beta h_2\langle\partial_x\psi_2^N, q_2^N\rangle + h_1\langle F, q_1^N\rangle - rh_2\|q_2^N\|^2\right) dt$$
$$+ \left(Sr\langle\psi_1^N - \psi_2^N, q_2^N\rangle + S\nu\|q_1^N - q_2^N\|^2 + \nu S(S_1 + S_2)\langle\psi_1^N - \psi_2^N, q_1^N - q_2^N\rangle\right) dt. \quad (25)$$

All the terms in the right hand side of (25) except for $h_1\langle F, q_1^N\rangle$ can be estimate by $C\|\boldsymbol{\psi}^N\|_{\mathbf{H}^2}|||\mathbf{q}^N|||$ via Cauchy–Schwarz inequality. Therefore, exploiting the continuity of $(-\boldsymbol{\Delta} - M)^{-1} : \mathcal{H} \to \mathbf{H}^2$ we have

$$\|\boldsymbol{\psi}^N\|_{\mathbf{H}^2}|||\mathbf{q}^N||| \leq C|||\mathbf{q}^N|||^2. \quad (26)$$

For what concerns $h_1\langle F, q_1^N\rangle$, it can be bounded easily by Young's inequality in the following way

$$h_1\langle F, q_1^N\rangle \leq \|F\|^2 + C|||\mathbf{q}^N|||^2. \quad (27)$$

Combining equations (26) and (27) it follows that there exists a constant C independent of N such that

$$\frac{d|||\mathbf{q}^N|||^2}{2} + \nu|||\nabla\mathbf{q}^N|||^2 dt \leq \left(C|||\mathbf{q}^N|||^2 + \|F\|^2\right) dt. \quad (28)$$

Thus, via Grönwall's inequality and exploiting the fact that P^N are projections on $L^2(\mathcal{D})$, we have that

$$|||\mathbf{q}^N(t)|||^2 \leq \left(|||\mathbf{q}_0|||^2 + 2\int_0^t \|F(s)\|^2 ds\right) e^{Ct},$$

hence (20) holds taking the supremum over $[0, T]$ and the expectation on both sides. Next, integrating in time (28) and using the estimate just obtained we have

$$\nu\int_0^T |||\nabla\mathbf{q}^N(s)|||^2 \, ds \leq \int_0^T \|F(s)\|^2 \, ds + CT\left(|||\mathbf{q}_0|||^2 + 2\int_0^T \|F(s)\|^2 ds\right) e^{CT}.$$

from which follows (21).

Moving on to (22), since $d|||\mathbf{q}^N|||^4 = 2|||\mathbf{q}^N|||^2 d|||\mathbf{q}^N|||^2$, exploiting the energy estimate (19) we obtain

$$d|||\mathbf{q}^N|||^4 + 4\nu|||\mathbf{q}^N|||^2|||\nabla\mathbf{q}^N|||^2 dt = 4|||\mathbf{q}^N|||^2\left(-\beta h_1\langle\partial_x\psi_1^N, q_1^N\rangle - \beta h_2\langle\partial_x\psi_2^N, q_2^N\rangle + h_1\langle F, q_1^N\rangle\right) dt+$$
$$4|||\mathbf{q}^N|||^2\left(-rh_2\|q_2^N\|^2 + Sr\langle\psi_1^N - \psi_2^N, q_2^N\rangle\right) dt + 4|||\mathbf{q}^N|||^2\left(S\nu\|q_1^N - q_2^N\|^2 + S\nu(S_1 + S_2)\langle\psi_1^N - \psi_2^N, q_1^N - q_2^N\rangle\right) dt. \quad (29)$$

As argued for (20), by appropriate use of Cauchy-Schwartz inequality and of the continuity of the operator $(-\Delta - M)^{-1}$, we can estimate the right hand side of (29) as follows

$$
\begin{aligned}
d\left|\left|\left|\mathbf{q}^N\right|\right|\right|^4 + 4\nu\left|\left|\left|\mathbf{q}^N\right|\right|\right|^2\left|\left|\left|\nabla\mathbf{q}^N\right|\right|\right|^2 dt &\leq \left(C\left|\left|\left|\mathbf{q}^N\right|\right|\right|^4 + C\|F\|\left|\left|\left|\mathbf{q}^N\right|\right|\right|^3\right)\, dt \\
&\leq \left(C\left|\left|\left|\mathbf{q}^N\right|\right|\right|^4 + \|F\|^4\right)\, dt.
\end{aligned} \tag{30}
$$

Thus, via Grönwall's inequality and the properties of the projection P^N, we get

$$
\left|\left|\left|\mathbf{q}^N(t)\right|\right|\right|^4 \leq \left(|||\mathbf{q}_0|||^4 + \int_0^t \|F(s)\|^4\, ds\right) e^{Ct}.
$$

From this relation it follows immediately that

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\left|\left|\mathbf{q}^N(t)\right|\right|\right|^4\right] + \mathbb{E}\left[\int_0^T |||\mathbf{q}^N(s)|||^2\left|\left|\left|\nabla\mathbf{q}^N(s)\right|\right|\right|^2\, ds\right] \leq C
$$

where $C = C(T)$ is a constant dependent on the time T but crucially independent of N.

Finally, to prove (23) we integrate (28) over $[0,T]$ to get

$$
\nu\int_0^T \left|\left|\left|\nabla\mathbf{q}^N(s)\right|\right|\right|^2 ds \leq \tfrac{1}{2}|||\mathbf{q}_0|||^2 + C\int_0^T \left|\left|\left|\mathbf{q}^N(s)\right|\right|\right|^2 ds + \int_0^T \|F(s)\|^2\, ds.
$$

Squaring and simply bounding the right hand side we have

$$
\nu^2\left(\int_0^T \left|\left|\left|\nabla\mathbf{q}^N(s)\right|\right|\right|^2\, ds\right)^2 \leq \tfrac{3}{2}|||\mathbf{q}_0|||^4 + 3\left(CT\sup_{t\in[0,T]}\left|\left|\left|\mathbf{q}^N(s)\right|\right|\right|^2\right)^2 + 3\left(\int_0^T \|F(s)\|^2\, ds\right)^2.
$$

By (20) we have

$$
\nu^2\left(\int_0^T \left|\left|\left|\nabla\mathbf{q}^N(s)\right|\right|\right|^2\, ds\right)^2 \leq \tfrac{3}{2}|||\mathbf{q}_0|||^4 + 3\left[CTe^{CT}\left(|||\mathbf{q}_0|||^2 + 2\int_0^T \|F(s)\|^2\, ds\right)\right]^2 + 3\left(\int_0^T \|F(s)\|^2\, ds\right)^2.
$$

Then taking the expectation on both side, (23) holds with constant C depending on T, $\mathbb{E}\left[|||\mathbf{q}_0|||^4\right]$ and $\mathbb{E}\left[\|F\|^4_{L^2(0,T;\mathcal{H})}\right]$.
□

From the energy estimates for $\mathbf{q}^N$ shown in Lemma 3.1, it follows that there exists a subsequence, which we will denote again for simplicity by $\mathbf{q}^N$, converging to a stochastic process $\mathbf{q}$ $*$-weakly in $L^4(\Omega; L^\infty(0,T;\mathcal{H}))$, and weakly in $L^4(\Omega; L^2(0,T;\mathcal{V}))$. Furthermore there exist two unknown processes, B_1^*, B_2^* such that

$$
\nabla^\perp\psi_j^N \cdot \nabla q_j^N \rightharpoonup B_j^* \quad \text{in } L^2(\Omega; L^2(0,T; H^{-1}(\mathcal{D})), \quad j\in\{1,2\}. \tag{31}
$$

Moreover, thanks to the converging properties of the projector P^N for $N\to+\infty$, the processes $\mathbf{q}$ and B_j^*, $j\in\{1,2\}$ satisfies $\mathbb{P}$-a.s. for each $i\in\mathbb{N}$ and $t\in[0,T]$

$$
\begin{aligned}
\langle q_1(t), e_i\rangle + \int_0^t \langle B_1^*(s), e_i\rangle_{H^{-1},H^1} ds &= \langle q_{1,0}, e_i\rangle + \int_0^t \langle \Delta\psi_1(s), \nu\Delta e_i\rangle\, ds + \int_0^t \langle q_1(s), F_1 e_i\rangle\, ds \\
&\quad + \int_0^t \langle \beta\partial_x\psi_1(s), e_i\rangle\, ds + \int_0^t \langle F(s), e_i\rangle\, ds - \sum_{k\in K}\int_0^t \left\langle q_1(s), G_1^k e_i\right\rangle dW_s^{1,k} \\
\langle q_2(t), e_i\rangle + \int_0^t \langle B_2^*(s), e_i\rangle_{H^{-1},H^1} ds &= \langle q_{2,0}, e_i\rangle + \int_0^t \langle \Delta\psi_2(s), \nu\Delta e_i\rangle\, ds + \int_0^t \langle q_2(s), F_2 e_i\rangle\, ds \\
&\quad + \int_0^t \langle \beta\partial_x\psi_2(s), e_i\rangle\, ds - r\int_0^t \langle \Delta\psi_2(s), e_i\rangle\, ds - \sum_{k\in K}\int_0^t \left\langle q_2(s), G_2^k e_i\right\rangle dW_s^{2,k}.
\end{aligned} \tag{32}
$$

Let us expand on the convergence of the term

$$\int_0^t \langle q_j^N(s), F_j^N e_i\rangle\, ds.$$

We know that for each $\alpha \geq 0$ and $x \in D((-\Delta)^\alpha)$, $\|P^N x - x\|_{D((-\Delta)^\alpha)}$ vanishes on the limit $N \to \infty$. Thus, since for each $k \in K$, $\beta \geq 1/2$, the operator $\boldsymbol{\sigma}_{j,k} \cdot (\nabla \cdot)$ is in $L(D((-\Delta)^\beta), D((-\Delta)^{\beta-1/2}))$, then for $\phi \in D((-\Delta)^\beta)$, $\|P^N(\boldsymbol{\sigma}_{j,k} \cdot \phi) - \boldsymbol{\sigma}_{j,k} \cdot \phi\|_{D((-\Delta)^{\beta-1/2})}$ converges to zero for increasing N. Starting from these observations it is easy to show that for each i, $\|F_j^N e_i - F_j e_i\| \to 0$. Then, thanks to the weak convergence of q_j^N to q_j, we have the required convergence of $\int_0^t \langle q_j^N(s), F_j^N e_i\rangle ds$.

For what concerns the continuity in $\mathcal{H}$ of the process $\mathbf{q}$ we can argue in the following way via Itô's formula and Kolmogorov continuity theorem. From the weak formulation above we get the weak continuity in $\mathcal{H}$ of $\mathbf{q}$ applying the Kolmogorov continuity theorem for (32). Then, applying the Itô's formula to $|||\mathbf{q}(t)|||^2$ we get, arguing as in the proof of Lemma 3.1,

$$\begin{aligned}\frac{d|||\mathbf{q}|||^2}{2} &= -\nu|||\nabla\mathbf{q}|||^2 dt \left(-\beta h_1\langle\partial_x\psi_1, q_1\rangle - \beta h_2\langle\partial_x\psi_2, q_2\rangle + h_1\langle F, q_1\rangle - rh_2\|q_2\|^2 + Sr\langle\psi_1 - \psi_2, q_2\rangle\right) dt \\ &\quad + \left(S\nu\|q_1 - q_2\|^2 + S\nu(S_1 + S_2)\langle\psi_1 - \psi_2, q_1 - q_2\rangle\right) dt - \left(h_1\langle B_1^*, q_1\rangle + h_2\langle B_2^*, q_2\rangle\right) dt.\end{aligned}$$

From this, we get the continuity of $\|\mathbf{q}(t)\|^2$ thanks to the integrability properties of $\mathbf{q}$. Weak continuity and continuity of the norm implies strong continuity, thus we have the strong continuity of $\mathbf{q}$ as a process taking values in $\mathcal{H}$. Alternatively the strong continuity in $\mathcal{H}$ of $\mathbf{q}$ follows from the results in [24].

3.2 Existence, Uniqueness and Further Results

First, it is easy to show that the solution of (6), when it exists, it is unique.

Theorem 3.2. *There is at most one weak solution of problem* (6) *in the sense of Definition 1.1.*

Proof. Let $\mathbf{q}$, $\tilde{\mathbf{q}}$ be two solutions and let $\mathbf{v}$ be their difference. Let $\boldsymbol{\psi}$, $\tilde{\boldsymbol{\psi}}$ be the corresponding streamfunctions and $\boldsymbol{\chi}$ be their difference. Let us consider the difference between two weak solutions and applying Itô's formula, which can be justified arguing as in the proof of Proposition 3.6 below, we get

$$\begin{aligned}\frac{d\|v_1\|^2}{2} &= \Big(\nu\langle\Delta\chi_1, \Delta v_1\rangle - \langle\nabla^\perp\psi_1 \cdot \nabla q_1, v_1\rangle + \langle\nabla^\perp\tilde{\psi}_1 \cdot \nabla\tilde{q}_1, v_1\rangle_{H^{-1},H^1} - \beta\langle\partial_x\chi_1, v_1\rangle\Big) dt \\ \frac{d\|v_2\|^2}{2} &= \Big(\nu\langle\Delta\chi_2, \Delta v_2\rangle - \langle\nabla^\perp\psi_2 \cdot \nabla q_2, v_2\rangle_{H^{-1},H^1} + \langle\nabla^\perp\tilde{\psi}_2 \cdot \nabla\tilde{q}_2, v_2\rangle - \beta\langle\partial_x\chi_2, v_2\rangle - r\langle\Delta\chi_2, v_2\rangle\Big) dt.\end{aligned}$$

Let us rewrite better $-\langle\nabla^\perp\psi_1 \cdot \nabla q_1, v_1\rangle_{H^{-1},H^1} + \langle\nabla^\perp\tilde{\psi}_1 \cdot \nabla\tilde{q}_1, v_1\rangle_{H^{-1},H^1}$, the other one is analogous.

$$\begin{aligned}&-\langle\nabla^\perp\psi_1 \cdot \nabla q_1, v_1\rangle_{H^{-1},H^1} + \langle\nabla^\perp\tilde{\psi}_1 \cdot \nabla\tilde{q}_1, v_1\rangle_{H^{-1},H^1} \pm \langle\nabla^\perp\tilde{\psi}_1 \cdot \nabla q_1, v_1\rangle_{H^{-1},H^1} \\ &= -\langle\nabla^\perp\chi_1 \cdot q_1, v_1\rangle_{H^{-1},H^1} - \langle\nabla^\perp\tilde{\psi}_1 \cdot \nabla v_1, v_1\rangle_{H^{-1},H^1} = \langle\nabla^\perp\chi_1 \cdot \nabla v_1, q_1\rangle_{H^{-1},H^1}.\end{aligned}$$

Multiplying the equations by h_1 and h_2 respectively and summing up, if we call $\tilde{\beta}_j = h_j\beta$, for $j \in \{1, 2\}$, and $\tilde{r} = rh_2$ we get

$$\begin{aligned}\frac{d|||\mathbf{v}|||^2}{2} &= \Big(\nu h_1\langle\Delta\chi_1, \Delta v_1\rangle + h_1\langle\nabla^\perp\chi_1 \cdot v_1, q_1\rangle_{H^{-1},H^1} - \tilde{\beta}_1\langle\partial_x\chi_1, v_1\rangle\Big) dt \\ &\quad + \Big(\nu h_2\langle\Delta\chi_2, \Delta v_2\rangle + h_2\langle\nabla^\perp\chi_2 \cdot v_2, q_2\rangle_{H^{-1},H^1} - \tilde{\beta}_2\langle\partial_x\chi_2, v_2\rangle - \tilde{r}\langle\Delta\chi_2, v_2\rangle\Big) dt =: RHSdt.\end{aligned}$$

Let $R(t) = \exp\Big(-\int_0^T \eta|||\nabla\mathbf{q}(s)|||^2\, ds\Big)$, where η is a positive constant fixed below. Therefore we have

$$\frac{dR(t)|||\mathbf{v}|||^2}{2} = R(t)RHS\, dt - \frac{\eta}{2}|||\mathbf{v}|||^2|||\nabla\mathbf{q}|||^2 R(t)\, dt.$$

Exploiting relation (3) we obtain

$$\begin{aligned}\frac{dR(t)\|\!|\mathbf{v}|\!\|^2}{2}+\nu R(t)\|\!|\nabla\mathbf{v}|\!\|^2dt=&-\frac{\eta}{2}\|\!|\mathbf{v}|\!\|^2\|\!|\nabla\mathbf{q}|\!\|^2R(t)dt-R(t)\nu S\langle\Delta(\chi_2-\chi_1),v_1-v_2\rangle\ dt\\&\left(-\tilde{\beta}_1R(t)\langle\partial_x\chi_1,v_1\rangle-\tilde{\beta}_2R(t)\langle\partial_2\chi_2,v_2\rangle-\tilde{r}R(t)\|v_2\|^2\right)\ dt\\&+\big(\tilde{r}R(t)\langle\chi_1-\chi_2,v_2\rangle+h_1R(t)\langle\nabla^\perp\chi_1\cdot v_1,q_1\rangle_{H^{-1},H^1}\\&+h_2R(t)\langle\nabla^\perp\chi_2\cdot v_2,q_2\rangle_{H^{-1},H^1}\big)\ dt.\end{aligned}$$

The terms which comes from the linear part of the equations can be estimated easily, up to some constant C, by $R(t)\|\!|\mathbf{v}(t)|\!\|^2$, thus we need only to analyze the nonlinear ones. Again we treat only one of the two in order to avoid repetitions.

$$\begin{aligned}\langle\nabla^\perp\chi_1\cdot v_1,q_1\rangle_{H^{-1},H^1}&\le\|\nabla q_1\|\|v_1\|_{L^4}\|\nabla^\perp\chi_1\|_{L^4}\le C\|\nabla q_1\|\|\!|\mathbf{v}|\!\|\|\!|\nabla\mathbf{v}|\!\|\\&\le\frac{\nu}{4h_1}\|\!|\nabla\mathbf{v}|\!\|^2+C\|\nabla q_1\|^2\|\!|\mathbf{v}|\!\|^2.\end{aligned}$$

Therefore, taking η large enough we get

$$\frac{dR\|\!|\mathbf{v}|\!\|^2}{2}+\frac{\nu}{2}R\|\!|\nabla\mathbf{v}|\!\|^2dt\le CR\|\!|\mathbf{v}|\!\|^2dt$$

and the thesis follows immediately by Grönwall's Lemma. □

Harder task is to ensure the existence of the solutions of equations (6). To show it we need the following crucial lemma which, for an appropriate stopping time τ_M, ensures convergence of the associated stopped Galerkin process $\mathbf{q}^N$ to the stopped process $\mathbf{q}$. This procedure, introduced in [3], is standard in stochastic analysis, see for example [27], [23].

Lemma 3.3. *Let* $\tau_M=\inf\{t\in[0,T]:\|\!|\mathbf{q}(t)|\!\|^2\ge M\}\wedge\inf\{t\in[0,T]:\int_0^t\|\!|\mathbf{q}(s)|\!\|_{\mathcal{V}}^2\ ds\ge M\}\wedge T$, *then*

$$\mathbb{1}_{[0,\tau_M]}(\mathbf{q}^N-\mathbf{q})\to0\quad in\ L^2(\Omega,L^2(0,T;\mathcal{H})).$$

Proof. We have to show that

$$\mathbb{E}\int_0^T\mathbb{1}_{[0,\tau_M]}(s)\left|\!\left|\!\left|\mathbf{q}^N(s)-\mathbf{q}(s)\right|\!\right|\!\right|^2ds\tag{33}$$

converges to zero in N. Let us call $\tilde{\psi}_j^N=P^N\psi_j\ \tilde{q}_j^N=P^Nq_j,\ j\in\{1,2\}$ and $\tilde{\boldsymbol{\psi}}^N,\ \tilde{\mathbf{q}}^N$ satisfy relation (3). Then, by the triangular inequality

$$(33)\le2\mathbb{E}\int_0^T\mathbb{1}_{[0,\tau_M]}(s)\left|\!\left|\!\left|\tilde{\mathbf{q}}^N(s)-\mathbf{q}(s)\right|\!\right|\!\right|^2ds+2\mathbb{E}\int_0^T\mathbb{1}_{[0,\tau_M]}(s)\left|\!\left|\!\left|\tilde{\mathbf{q}}^N(s)-\mathbf{q}^N(s)\right|\!\right|\!\right|^2ds.$$

Thanks to the properties of the projector P^N and dominated convergence theorem, it can be shown that $\tilde{\mathbf{q}}^N\to\mathbf{q}$ in $L^2(\Omega,L^2(0,T;\mathcal{V}))\cap L^2(\Omega,C(0,T;\mathcal{H}))$, and also in weaker topologies. Therefore, we are left to show the convergence of

$$\mathbb{E}\int_0^{\tau_M}\left|\!\left|\!\left|\tilde{\mathbf{q}}^N(s)-\mathbf{q}^N(s)\right|\!\right|\!\right|^2ds.\tag{34}$$

Let us start by taking the difference of (32) and (18), pair the two components respectively with $h_1\left(\tilde{q}_1^N-q_1^N\right)$ and $h_2\left(\tilde{q}_2^N-q_2^N\right)$, and add them together to get an equation for $\left|\!\left|\!\left|\tilde{\mathbf{q}}^N-\mathbf{q}^N\right|\!\right|\!\right|$. With Itô formula and some elementary manipulation using (14), we have

$$\begin{aligned}\frac{1}{2}d\left|\!\left|\!\left|\tilde{\mathbf{q}}^N-\mathbf{q}^N\right|\!\right|\!\right|^2+\nu\left|\!\left|\!\left|\nabla(\tilde{\mathbf{q}}^N-\mathbf{q}^N)\right|\!\right|\!\right|^2\,dt=&-h_1\langle B_1^*-B_1^N,\tilde{q}_1^N-q_1^N\rangle_{H^{-1},H^1}\,dt-h_2\langle B_2^*-B_2^N,\tilde{q}_2^N-q_2^N\rangle_{H^{-1},H^1}\,dt\\&-\nu S(S_1+S_2)\|(-\Delta+S_1+S_2)^{-1/2}\left((\tilde{q}_1^N-\tilde{q}_2^N)-(q_1^N-q_2^N)\right)\|^2\\&-\tilde{\beta}_1\langle\partial_x(\psi_1-\psi_1^N),\tilde{q}_1^N-q_1^N\rangle\,dt-\tilde{\beta}_2\langle\partial_x(\psi_2-\psi_2^N),\tilde{q}_2^N-q_2^N\rangle\,dt\\&+\nu S\|(\tilde{q}_1^N-\tilde{q}_2^N)-(q_1^N-q_2^N)\|^2-S_1\sum_{k\in K}\langle q_1-q_1^N,G_1^k(\tilde{q}_1^N-q_1^N)\rangle dW_t^{1,k}\\&-S_2\sum_{k\in K}\langle q_2-q_2^N,G_2^k(\tilde{q}_2^N-q_2^N)\rangle dW_t^{2,k}\\&+S_1\langle q_1,F_1(\tilde{q}_1^N-q_1^N)\rangle\,dt-S_1\langle q_1^N,F_1^N(\tilde{q}_1^N-q_1^N)\rangle\,dt\\&+S_2\langle q_2,F_2(\tilde{q}_2^N-q_2^N)\rangle\,dt-S_2\langle q_2^N,F_2^N(\tilde{q}_2^N-q_2^N)\rangle\,dt\\&-\tilde{r}\langle\Delta(\tilde{\psi}_2^N-\psi_2^N),\tilde{q}_2^N-q_2^N\rangle\,dt\\&+\frac{S_1}{2}\sum_{k\in K}\sum_{i=1}^N\langle q_1-q_1^N,G_1^ke_i\rangle^2\,dt+\frac{S_2}{2}\sum_{k\in K}\sum_{i=1}^N\langle q_2-q_2^N,G_2^ke_i\rangle^2\,dt,\end{aligned}\tag{35}$$

where we denoted $\tilde{r}:=h_2r$, $\tilde{\beta}_j:=h_j\beta$ and $B_j^N=\nabla^\perp\psi_j^N\cdot\nabla q_j^N$ for $j\in\{1,2\}$. Now we can estimate the RHS in (35). In the following C will denote a generic constant independent of N.

$$\|\langle\Delta(\tilde{\psi}_2^N-\psi_2^N),\tilde{q}_2^N-q_2^N\rangle\|\le\|\Delta(\tilde{\psi}_2^N-\psi_2^N)\|\|\tilde{q}_2^N-q_2^N\|\|\le C\left|\!\left|\!\left|\tilde{\mathbf{q}}^N-\mathbf{q}^N\right|\!\right|\!\right|^2;\tag{36}$$

$$\|(\tilde{q}_1^N-\tilde{q}_2^N)-(q_1^N-q_2^N)\|^2\le C\left|\!\left|\!\left|\tilde{\mathbf{q}}^N-\mathbf{q}^N\right|\!\right|\!\right|^2;\tag{37}$$

$$\begin{aligned}\|\tilde{\beta}_1\langle\partial_x(\psi_1-\psi_1^N),\tilde{q}_1^N-q_1^N\rangle+\tilde{\beta}_2\langle\partial_x(\psi_2-\psi_2^N),\tilde{q}_2^N-q_2^N\rangle\|&\le C\left|\!\left|\!\left|\mathbf{q}-\mathbf{q}^N\right|\!\right|\!\right|\left|\!\left|\!\left|\tilde{\mathbf{q}}^N-\mathbf{q}^N\right|\!\right|\!\right|\\&\le C\left|\!\left|\!\left|\tilde{\mathbf{q}}^N-\mathbf{q}^N\right|\!\right|\!\right|^2+C\left|\!\left|\!\left|\mathbf{q}-\tilde{\mathbf{q}}^N\right|\!\right|\!\right|\left|\!\left|\!\left|\tilde{\mathbf{q}}^N-\mathbf{q}^N\right|\!\right|\!\right|\\&\le C\left|\!\left|\!\left|\tilde{\mathbf{q}}^N-\mathbf{q}^N\right|\!\right|\!\right|^2+C\left|\!\left|\!\left|\mathbf{q}-\tilde{\mathbf{q}}^N\right|\!\right|\!\right|^2.\end{aligned}\tag{38}$$

Next, to better understand the behavior of the terms

$$2\langle q_1,F_1(\tilde{q}_1^N-q_1^N)\rangle-2\langle q_1^N,F_1^N(\tilde{q}_1^N-q_1^N)\rangle+\sum_{k\in K}\sum_{i=1}^N\langle q_1-q_1^N,G_1^ke_i\rangle^2\tag{39}$$

we will first write them in an equivalent form. By definition of F_1 (8) and Green's theorem we have

$$\begin{aligned}2\langle q_1,F_1(\tilde{q}_1^N-q_1^N)\rangle-2\langle q_1^N,F_1^N(\tilde{q}_1^N-q_1^N)\rangle&=-\sum_{k\in K}\langle\boldsymbol{\sigma}_{1,k}\cdot\nabla(\tilde{q}_1^N-q_1^N),\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1\rangle\\&\quad+\sum_{k\in K}\langle P^N\left(\boldsymbol{\sigma}_{1,k}\cdot\nabla(\tilde{q}_1^N-q_1^N)\right),\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N\rangle\\&=-\sum_{k\in K}\langle\boldsymbol{\sigma}_{1,k}\cdot\nabla(\tilde{q}_1^N-q_1^N),\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1-P^N\left(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N\right)\rangle.\end{aligned}$$

Similarly, recalling the definition (9) of G_1^k, $k=1,\ldots,K$, we have

$$\begin{aligned}\sum_{k\in K}\sum_{i=1}^N\langle q_1-q_1^N,G_1^ke_i\rangle^2&=\sum_{k\in K}\sum_{i=1}^N\langle\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1-\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N,e_i\rangle^2\\&=\sum_{k\in K}\langle\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1-P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N),P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1)-P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N)\rangle.\end{aligned}$$

Therefore, by some additional manipulations, it can be shown that

$$(39) = -\sum_{k\in K}\langle \boldsymbol{\sigma}_{1,k}\cdot\nabla q_1 - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N), (I-P^N)(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1)\rangle + \sum_{k\in K}\langle \boldsymbol{\sigma}_{1,k}\cdot\nabla q_1 - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N), \boldsymbol{\sigma}_{1,k}\cdot\nabla(q_1-\tilde{q}_1^N)\rangle$$
$$+ \sum_{k\in K}\langle \boldsymbol{\sigma}_{1,k}\cdot\nabla q_1 - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N), (I-P^N)(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N)\rangle.$$

We can now estimate all terms by Cauchy-Schwartz inequality to get

$$(39) \le \sum_{k\in K}\|\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1 - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N)\|\|(I-P^N)(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1)\|$$
$$+ \sum_{k\in K} C\|\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1 - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N)\|\|\nabla(q_1-\tilde{q}_1^N)\| + \sum_{k\in K} C\|(I-P^N)(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1)\|\|\nabla q_1^N\|. \quad (40)$$

The same estimate holds for the terms in (35) relative to the second component.

We now move on to treating the nonlinear term

$$-h_1\langle B_1^* - B_1^N, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1}\,dt - h_2\langle B_2^* - B_2^N, \tilde{q}_2^N - q_2^N\rangle_{H^{-1},H^1}\,dt$$

in (35) and again we only treat $\langle B_1^* - B_1^N, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1}$ as for the second component the argument is identical. By definition $B_1^N = \nabla^\perp\psi_1^N\cdot\nabla q_1^N$, hence

$$-h_1\langle B_1^* - B_1^N, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1} = h_1\langle\nabla^\perp(\tilde{\psi}_1^N - \psi_1^N)\cdot\nabla(\tilde{q}_1^N - q_1^N), \tilde{q}_1^N\rangle_{H^{-1},H^1}$$
$$+ h_1\langle\nabla^\perp\tilde{\psi}_1^N\cdot\nabla\tilde{q}_1^N - B_1^*,\, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1}. \quad (41)$$

We can estimate the first term on the right hand side of the last display as follows

$$|\langle\nabla^\perp(\tilde{\psi}_1^N - \psi_1^N)\cdot\nabla(\tilde{q}_1^N - q_1^N), \tilde{q}_1^N\rangle_{H^{-1},H^1}| \le \|\nabla\tilde{q}_1^N\|\|\tilde{q}_1^N - q_1^N\|_{L^4}\|\|\nabla^\perp(\tilde{\psi}_1^N - \psi_1^N)\|_{L^4}$$
$$\le C\|\nabla q_1\|\|\tilde{q}_1^N - q_1^N\|_{L^4}^2$$

and, by Ladyzhenskaya's and Young's inequalities we have

$$\le C\|\nabla q_1\|\|\tilde{q}_1^N - q_1^N\|\|\nabla(\tilde{q}_1^N - q_1^N)\| \le C_\nu\|\nabla q_1\|^2\|\tilde{q}_1^N - q_1^N\|^2 + \frac{\nu}{2}\|\nabla(\tilde{q}_1^N - q_1^N)\|^2. \quad (42)$$

The second term in (41) can be rewritten as

$$\langle\nabla^\perp\tilde{\psi}_1^N\cdot\nabla\tilde{q}_1^N - B_1^*,\, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1} = \langle\nabla^\perp\psi_1\cdot\nabla q_1 - B_1^*,\, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1}$$
$$- \langle\nabla^\perp\psi_1\cdot\nabla q_1 - \nabla^\perp\tilde{\psi}_1^N\cdot\nabla\tilde{q}_1^N, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1}.$$

Since $\tilde{\mathbf{q}}^N - \mathbf{q}^N \rightharpoonup 0$ in $L^2(\Omega; L^2(0,T;\mathcal{V}))$, the expected value of first term will go to 0 easily. For what concerns the second one, adding and subtracting $\langle\nabla^\perp\psi_1\cdot\nabla\tilde{q}_1^N, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1}$ we have

$$\langle\nabla^\perp\psi_1\cdot\nabla q_1 - \nabla^\perp\tilde{\psi}_1^N\cdot\nabla\tilde{q}_1^N, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1} = \langle\nabla^\perp\psi_1\cdot(\nabla q_1 - \nabla\tilde{q}_1^N), \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1}$$
$$+ \langle(\nabla^\perp\psi_1 - \nabla^\perp\tilde{\psi}_1^N)\cdot\nabla\tilde{q}_1^N, \tilde{q}_1^N - q_1^N\rangle_{H^{-1},H^1}$$

so that, again by Ladyzhenskaya's inequality,

$$\le \|\nabla^\perp\psi_1\|_{L^4}\|q_1 - \tilde{q}_1^N\|^{1/2}\|\nabla(q_1 - \tilde{q}_1^N)\|^{1/2}\|\nabla(\tilde{q}_1^N - q_1^N)\| + \|\nabla^\perp\psi_1 - \nabla^\perp\tilde{\psi}_1^N\|_{L^4}\|\tilde{q}_1^N\|^{1/2}\|\nabla\tilde{q}_1^N\|^{1/2}\|\nabla(\tilde{q}_1^N - q_1^N)\|.$$

In summary, we showed

$$\langle\nabla^\perp\tilde{\psi}_1^N\cdot\nabla\tilde{q}_1^N - B_1^*,\, \tilde{q}_1^N - q_1^N\rangle = \langle\nabla^\perp\psi_1\cdot\nabla q_1 - B_1^*,\, \tilde{q}_1^N - q_1^N\rangle$$
$$+ \|\nabla^\perp\psi_1\|_{L^4}\|q_1 - \tilde{q}_1^N\|^{1/2}\|\nabla(q_1 - \tilde{q}_1^N)\|^{1/2}\|\nabla(\tilde{q}_1^N - q_1^N)\| \quad (43)$$
$$+ \|\nabla^\perp\psi_1 - \nabla^\perp\tilde{\psi}_1^N\|_{L^4}\|\tilde{q}_1^N\|^{1/2}\|\nabla\tilde{q}_1^N\|^{1/2}\|\nabla(\tilde{q}_1^N - q_1^N)\|.$$

Consider now the auxiliary function

$$R(t) := \tfrac{1}{2}\exp\biggl(-\eta_1 t - \eta_2 \int_0^t \|\nabla \mathbf{q}(s)\|^2\, ds\biggr), \tag{44}$$

with η_1 and η_2 two positive constants to be defined later, and let us then compute via the Itô's formula $R(t)\left\|\tilde{\mathbf{q}}^N(t) - \mathbf{q}^N(t)\right\|^2$. We exploit previous estimates (36),(37),(38),(40) and (42) and take the expected value for $t = \tau_M$ obtaining

$$\begin{aligned}
&\mathbb{E}\left[\tfrac{1}{2}R(\tau_M)\left\|\tilde{\mathbf{q}}^N(\tau_M) - \mathbf{q}^N(\tau_M)\right\|^2\right] + \nu\mathbb{E}\left[\int_0^{\tau_M} R(s)\left\|\nabla(\tilde{\mathbf{q}}^N(s) - \mathbf{q}^N(s))\right\|^2 ds\right] \le \\
&- \frac{\eta_1}{2}\mathbb{E}\left[\int_0^{\tau_M} R(s)\left\|\tilde{\mathbf{q}}^N(s) - \mathbf{q}^N(s)\right\|^2 ds\right] - \frac{\eta_2}{2}\mathbb{E}\left[\int_0^{\tau_M} R(s)\|\nabla \mathbf{q}(s)\|^2\left\|\tilde{\mathbf{q}}^N(s) - \mathbf{q}^N(s)\right\|^2 ds\right] \\
&+ h_1\left|\mathbb{E}\left[\int_0^{\tau_M} R(s)\langle B_1^*(s) - \nabla^\perp\tilde{\psi}_1^N(s)\cdot\nabla\tilde{q}_1^N(s), \tilde{q}_1^N(s) - q_1^N(s)\rangle\ \ ds\right]\right| \\
&+ h_2\left|\mathbb{E}\left[\int_0^{\tau_M} R(s)\langle B_2^*(s) - \nabla^\perp\tilde{\psi}_2^N(s)\cdot\nabla\tilde{q}_2^N(s), \tilde{q}_2^N(s) - q_2^N(s)\rangle\ \ ds\right]\right| \\
&+ C\mathbb{E}\left[\int_0^{\tau_M} R(s)\|\nabla \mathbf{q}(s)\|^2\left\|\tilde{\mathbf{q}}^N(s) - \mathbf{q}^N(s)\right\|^2 ds\right] + \frac{\nu}{2}\mathbb{E}\left[\int_0^{\tau_M} R(s)\left\|\nabla\tilde{\mathbf{q}}(s)^N - \nabla\mathbf{q}^N(s)\right\|^2 ds\right] \\
&+ C\mathbb{E}\left[\int_0^{\tau_M} R(s)\left\|\tilde{\mathbf{q}}^N(s) - \mathbf{q}^N(s)\right\|^2 ds\right] + C\mathbb{E}\left[\int_0^{\tau_M} R(s)\left\|\tilde{\mathbf{q}}^N(s) - \mathbf{q}(s)\right\|^2 ds\right] \\
&+ \sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1(s) - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N(s))\|\|(I-P^N)(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1(s))\|\ ds\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1(s) - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N(s))\|\|\nabla(q_1(s) - \tilde{q}_1^N(s))\|\ ds\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|(I-P^N)(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1(s))\|\|\nabla q_1^N(s)\|\ ds\right] \\
&+ \sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|\boldsymbol{\sigma}_{2,k}\cdot\nabla q_2(s) - P^N(\boldsymbol{\sigma}_{2,k}\cdot\nabla q_2^N(s))\|\|(I-P^N)(\boldsymbol{\sigma}_{2,k}\cdot\nabla q_2(s))\|\ ds\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|\boldsymbol{\sigma}_{2,k}\cdot\nabla q_2(s) - P^N(\boldsymbol{\sigma}_{2,k}\cdot\nabla q_2^N(s))\|\|\nabla(q_2(s) - \tilde{q}_2^N(s))\|\ ds\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|(I-P^N)(\boldsymbol{\sigma}_{2,k}\cdot\nabla q_2(s))\|\|\nabla q_2^N(s)\|\ ds\right].
\end{aligned} \tag{45}$$

Taking η_1, η_2 large enough and exploiting the convergence of $\tilde{\mathbf{q}}^N$ to $\mathbf{q}$ we can neglect several terms in the right hand side. Let us consider the remaining terms. Recalling that from the weak convergence of $\mathbf{q}^N$ it follows that $\mathbb{E}\left[\int_0^T \left\|\nabla\mathbf{q}^N(s)\right\|^2 ds\right] \le C$, applying Cauchy–Schwarz inequality where it is needed, we get

$$\begin{aligned}
&\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1(s) - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N(s))\|\|(I-P^N)(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1(s))\|\ ds\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1(s) - P^N(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1^N(s))\|\|\nabla(q_1(s) - \tilde{q}_1^N(s))\|\ ds\right] \\
&+ C\sum_{k\in K}\mathbb{E}\left[\int_0^{\tau_M}\|(I-P^N)(\boldsymbol{\sigma}_{1,k}\cdot\nabla q_1(s))\|\|\nabla q_1^N(s)\|\ ds\right] \\
&\le C\sum_{k\in K}\mathbb{E}\left[\int_0^T\|(I-P^N)(\sigma_{1,k}\cdot\nabla q_1(s))\|^2\ ds\right]^{1/2} + C\mathbb{E}\left[\int_0^T\|\nabla(q_1(s) - \tilde{q}_1^N(s))\|^2\ ds\right]^{1/2} \to 0
\end{aligned}$$

and similarly for the second component. Lastly, let us consider

$$\left|\mathbb{E}\left[\int_0^{\tau_M} R(s)\langle B_1^*(s) - \nabla^\perp \tilde{\psi}_1^N(s)\cdot\nabla\tilde{q}_1^N(s), \tilde{q}_1^N(s) - q_1^N(s)\rangle \ \ ds\right]\right|.$$

By (43) we have that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau_M} R(s)\langle\nabla^\perp\tilde{\psi}_1^N(s)\cdot\nabla\tilde{q}_1^N(s) - B_1^*(s),\, \tilde{q}_1^N(s) - q_1^N(s)\rangle\, ds\right]\\
&\leq \mathbb{E}\left[\int_0^{\tau_M} R(s)\langle\nabla^\perp\psi_1(s)\cdot\nabla q_1(s) - B_1^*(s),\, \tilde{q}_1^N(s) - q_1^N(s)\rangle\, ds\right]\\
&+\mathbb{E}\left[\int_0^{\tau_M} R(s)\|\nabla^\perp\psi_1(s)\|_{L^4}\|q_1(s)-\tilde{q}_1^N(s)\|^{1/2}\|\nabla(q_1(s)-\tilde{q}_1^N(s))\|^{1/2}\|\nabla(\tilde{q}_1^N(s)-q_1^N(s))\|\, ds\right]\\
&+\mathbb{E}\left[\int_0^{\tau_M} R(s)\|\nabla^\perp\psi_1(s)-\nabla^\perp\tilde{\psi}_1^N(s)\|_{L^4}\|\tilde{q}_1^N(s)\|^{1/2}\|\nabla\tilde{q}_1^N(s)\|^{1/2}\|\nabla(\tilde{q}_1^N(s)-q_1^N(s))\|\, ds\right].
\end{aligned} \tag{46}$$

As already observed the first term converges to zero as $\tilde{q}^N - q^N \rightharpoonup 0$ in $L^2(\Omega; L^2(0,T;\mathcal{V}))$. For what concerns the two remaining terms, thanks to Hölder inequality, we have

$$\begin{aligned}
&\leq C\mathbb{E}\left[\int_0^T \|q_1(s)-\tilde{q}_1^N(s)\|^2\ ds\right]^{1/4}\mathbb{E}\left[\int_0^T \|\nabla(q_1(s)-\tilde{q}_1^N(s))\|^2\ ds\right]^{1/4}\mathbb{E}\left[\int_0^T \|\nabla(q_1^N(s)-\tilde{q}_1^N(s))\|^2\ ds\right]^{1/2}\\
&+C\mathbb{E}\left[\int_0^T \|q_1(s)-\tilde{q}_1^N(s)\|^2\ ds\right]^{1/4}\mathbb{E}\left[\int_0^{\tau_M} \|\nabla\tilde{q}_1^N(s)\|^2\ ds\right]^{1/4}\mathbb{E}\left[\int_0^T \|\nabla(q_1^N(s)-\tilde{q}_1^N(s))\|^2\ ds\right]^{1/2}.
\end{aligned}$$

In the last inequality we exploit the fact that $\left|\!\left|\!\left|q(t)\right|\!\right|\!\right|^2 \leq M$, for all $t\leq\tau_M$, and $\tilde{q}^N - q^N \rightharpoonup 0$ in $L^2(\Omega; L^2(0,T;\mathcal{V}))$.

In conclusion, in (45) all the terms on the right hand side converge to zero as $N\to\infty$, namely we have the following relation

$$\mathbb{E}\left[\frac{1}{2}R(\tau_M)\left|\!\left|\!\left|\tilde{\mathbf{q}}^N(\tau_M)-\mathbf{q}^N(\tau_M)\right|\!\right|\!\right|^2\right]+\frac{\nu}{2}\mathbb{E}\left[\int_0^{\tau_M} R(s)\left|\!\left|\!\left|\nabla(\tilde{\mathbf{q}}^N(s)-\mathbf{q}^N(s))\right|\!\right|\!\right|^2 ds\right]\to 0. \tag{47}$$

From relation (47), $R(t)\geq C_M > 0$, for all $t\leq\tau_M$, and the properties of P^N, via triangular inequality the thesis follows. □

The lemma just shown allows to treat the nonlinearity $\nabla^\perp\psi_j^N\cdot\nabla q_j^N$, more precisely, to show that for both $j=1,2$

$$\nabla^\perp\psi_j^N\cdot\nabla q_j^N \rightharpoonup \nabla^\perp\psi_j\cdot\nabla q_j.$$

Lemma 3.4. *Let B_1^*, B_2^* be the limit processes as in* (31). *Then $B_1^* = \nabla^\perp\psi_1\cdot\nabla q_1$ and $B_2^* = \nabla^\perp\psi_2\cdot\nabla q_2$ in $L^2(\Omega, L^2(0,T;H^{-1}(\mathcal{D})))$.*

Proof. Thanks to estimate (22) and (23) we know that $\nabla^\perp\psi_{1,2}\cdot\nabla q_{1,2}^N$ and $\nabla^\perp\psi_{1,2}^N\cdot\nabla q_{1,2}$ converge to $\nabla^\perp\psi_{1,2}\cdot\nabla q_{1,2}$ weakly in $L^2(\Omega; L^2(0,T;H^{-1}(\mathcal{D})))$. We do the explicit computations just for one of them, the others are analogous.

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \|\nabla^\perp\psi_{1,2}(s)\cdot\nabla q_{1,2}^N(s)\|_{H^{-1}}^2\ ds\right] &\leq C\mathbb{E}\left[\int_0^T \|\Delta\psi(s)\|^2\|\nabla q^N(s)\|^2\ ds\right]\\
&\leq C\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{q}(t)\|^2\int_0^T\|\nabla\mathbf{q}^N(s)\|^2\ ds\right]\\
&\leq C\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{q}(t)\|^4\right]+C\mathbb{E}\left[\left(\int_0^T\|\nabla\mathbf{q}^N(s)\|^2\ ds\right)^2\right]\leq C.
\end{aligned}$$

Let now $\phi \in L^\infty(\Omega; L^\infty(0,T; H^1(\mathcal{D})))$, then $\nabla\phi \cdot \nabla^\perp\psi_{1,2} \in L^2(\Omega; L^2(0,T; L^2(\mathcal{D})))$. Thus, from the convergence properties of $\mathbf{q}^N$, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \langle \nabla^\perp\psi_{1,2}(s)\cdot\nabla q_{1,2}^N(s), \phi\rangle_{H^{-1},H^1} ds\right] &= -\mathbb{E}\left[\int_0^T \langle \nabla^\perp\psi_{1,2}(s)\cdot\nabla\phi, q_{1,2}^N(s)\rangle ds\right] \\
&\to -\mathbb{E}\left[\int_0^T \langle \nabla^\perp\psi_{1,2}(s)\cdot\nabla\phi, q_{1,2}(s)\rangle ds\right] \\
&= \mathbb{E}\left[\int_0^T \nabla^\perp\psi_{1,2}(s)\cdot\nabla q_{1,2}(s), \phi\rangle_{H^{-1},H^1} ds\right].
\end{aligned}$$

From the density of $L^\infty(\Omega; L^\infty(0,T; H^1(\mathcal{D})))$ in $L^2(\Omega; L^2(0,T; H^1(\mathcal{D})))$ and the uniform boundedness of $\nabla^\perp\psi_{1,2}\cdot\nabla q_{1,2}^N$ in $L^2(\Omega; L^2(0,T; H^{-1}(\mathcal{D})))$ we have the required claim. For what concerns the convergence of the nonlinear term, first note that, arguing as above, the sequence $\{\nabla^\perp\psi_{1,2}^N \cdot \nabla q_{1,2}^N\}$ is uniformly bounded in $L^2(\Omega; L^2(0,T; H^{-1}(\mathcal{D})))$. Moreover we have

$$\begin{aligned}
\nabla^\perp\psi_{1,2}\cdot\nabla q_{1,2} - \nabla^\perp\psi_{1,2}^N\cdot\nabla q_{1,2}^N &= \nabla^\perp\psi_{1,2}\cdot\nabla(q_{1,2}-q_{1,2}^N) + \nabla^\perp\psi_{1,2}\cdot\nabla q_{1,2}^N \\
&+ \nabla^\perp\psi_{1,2}^N\cdot\nabla(q_{1,2}-q_{1,2}^N) - \nabla^\perp\psi_{1,2}\cdot\nabla q_{1,2}^N =: I_1+I_2+I_3+I_4.
\end{aligned}$$

Thanks to the previous observations $I_1+I_2+I_4$ converges weakly to 0 in $L^2(\Omega; L^2(0,T; H^{-1}(\mathcal{D})))$. For what concerns I_3, let us take $\phi \in L^\infty(\Omega; L^\infty(0,T; H^2(\mathcal{D})))$ and τ_M defined as in Lemma 3.3, then we have

$$\mathbb{E}\left[\int_0^{\tau_M} \langle \nabla^\perp\psi_{1,2}^N(s)\cdot\nabla(q_{1,2}(s)-q_{1,2}^N(s)), \phi\rangle\ ds\right] \le C\mathbb{E}\left[\int_0^{\tau_M} \|\mathbf{q}^N(s)\|\|\mathbf{q}(s)-\mathbf{q}^N(s)\|\ ds\right] \to 0$$

thanks to Hölder's inequality and Lemma 3.3. Since it holds $\tau_M \nearrow T$ *a.s.*, then the thesis follows, thanks to Lemma 3.1. Then the thesis follows by the density of $L^\infty(\Omega; L^\infty(0,T; H^2(\mathcal{D})))$ in $L^2(\Omega; L^2(0,T; H^1(\mathcal{D})))$ and the uniform boundedness of $\nabla^\perp\psi_{1,2}^N\cdot\nabla q_{1,2}^N$ in $^2(\Omega; L^2(0,T; H^{-1}(\mathcal{D})))$. □

This series of lemmas identify the nonlinear term and conclude the proof of Theorem 1.2. Actually, thanks to some abstract results on stochastic processes it can be shown something more, namely that the full sequence $\mathbf{q}^N$ converges to $\mathbf{q}$ in $L^2(\Omega; L^2(0,T; \mathcal{V}))$ and, for each $t \in [0,T]$, $\mathbf{q}^N(t)$ converges to $\mathbf{q}(t)$ in $L^2(\Omega; \mathcal{H})$. This is an easy corollary of previous result and Lemma 2.5.

Theorem 3.5. *The entire Galerkin's sequence* $\mathbf{q}^N$ *satisfies*

$$\lim_{N\to+\infty} \mathbb{E}\left[\|\mathbf{q}^N(t)-\mathbf{q}(t)\|^2\right] = 0;$$
$$\lim_{N\to+\infty} \int_0^T \mathbb{E}\left[\|\nabla(\mathbf{q}^N(t)-\mathbf{q}(t))\|^2\right]\ dt = 0.$$

Proof. From the uniqueness of the solution of problem (6), we have that each subsequence $\mathbf{q}^{n_k}$ has a converging sub-subsequence $\mathbf{q}^{n_{k,k}}$ which satisfies Lemma 3.1, Lemma 3.3 and Lemma 3.4. In order to apply Lemma 2.5, we consider the stochastic processes

$$Q_N := \int_0^t \|\mathbf{q}(s)-\mathbf{q}^{n_{N,N}}(s)\|_{\mathcal{V}}^2\ ds \quad \text{or} \quad Q_N := \|(\mathbf{q}-\mathbf{q}^{n_{N,N}})(t)\|^2,$$

and we take σ_M to be τ_M, the stopping time introduced in Lemma 3.3. This choice for σ_M satisfies the conditions of Lemma 2.5 by the following argument. Condition (16) follows directly from Lemma 3.1. Condition (17) is satisfied thanks to relation (47) by exploiting the properties of P^N and the fact that $R(t) \ge C_M > 0$ for all $t \le \tau_M$. It remains to show that

$$\lim_{M\to+\infty} \mathbb{P}(\tau_M < T) = 0.$$

By the definition of τ_M, Markov's inequality, and Lemma 3.1, it follows:

$$\begin{aligned}\mathbb{P}(\tau_M < T) &\leq \mathbb{P}\left(\sup_{t\in[0,T]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbf{q}(t) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert^2 > M\right) + \mathbb{P}\left(\int_0^T \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbf{q}(s) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\mathcal{V}}^2 ds > M\right) \\ &\leq \frac{1}{M}\mathbb{E}\left[\sup_{t\in[0,T]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbf{q}(t) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert^2\right] + \frac{1}{M}\mathbb{E}\left[\int_0^T \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbf{q}(s) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\mathcal{V}}^2 ds\right] \\ &\leq \frac{1}{M}\limsup_{N\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbf{q}^{n_{N,N}}(t) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert^2\right] + \frac{1}{M}\limsup_{N\to+\infty}\mathbb{E}\left[\int_0^T \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbf{q}^{n_{N,N}}(s) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\mathcal{V}}^2 ds\right] \\ &\leq \frac{C}{M} \to 0 \quad \text{for } M\to\infty\end{aligned}$$

as desired. Then Lemma 2.5 ensures the result of the theorem. □

Lastly, we show that the energy estimate (19) in Lemma 3.1 continues to holds for $\mathbf{q}$, solution of problem (6). For what concerns the a priori and integral estimates in Lemma 3.1, they straightforwardly continue to hold by the weak convergence of $\mathbf{q}^N$ to $\mathbf{q}$, but they can proved independently starting from the energy estimate and repeating the same steps of Lemma 3.1.

Proposition 3.6. *Given* $\mathbf{q}$ *solution of* (6)*, then the following energy estimate holds*

$$\begin{aligned}\frac{d\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbf{q} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert^2}{2} + \nu\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \nabla\mathbf{q} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert^2 dt &= \left(-\beta h_1\langle\partial_x\psi_1, q_1\rangle - \beta h_2\langle\partial_x\psi_2, q_2\rangle + h_1\langle F, q_1\rangle - rh_2\|q_2\|^2 + Sr\langle\psi_1-\psi_2, q_2\rangle\right)dt \\ &\quad + \left(S\nu\|q_1-q_2\|^2 + S\nu(S_1+S_2)\langle\psi_1-\psi_2, q_1-q_2\rangle\right)dt.\end{aligned}$$

Proof. Let $\tilde{\mathbf{q}}^N$ be defined as in Lemma 3.3. We already know by the properties of the projector P^N that $\tilde{\mathbf{q}}^N \to \mathbf{q}$ in $L^2(0,T;\mathcal{V})\cap C(0,T;\mathcal{H})$ $\mathbb{P}$-a.s. As $\mathbf{q}$ satisfies the weak formulation of (6) with test functions e_i we have

$$\begin{aligned}\langle q_1(t), e_i\rangle &= \langle q_{1,0}, e_i\rangle + \int_0^t \langle\Delta\psi_1(s), \nu\Delta e_i\rangle\, ds + \int_0^t \langle q_1(s), F_1 e_i\rangle\, ds + \int_0^t \langle q_1(s), \nabla^\perp\psi_1(s)\cdot\nabla e_i\rangle ds \\ &\quad + \int_0^t \langle\beta\partial_x\psi_1(s), e_i\rangle\, ds + \int_0^t \langle F(s), e_i\rangle\, ds - \sum_{k\in K}\int_0^t \left\langle q_1(s), G_1^k e_i\right\rangle dW_s^{1,k} \\ \langle q_2(t), e_i\rangle &= \langle q_{2,0}, e_i\rangle + \int_0^t \langle\Delta\psi_2(s), \nu\Delta e_i\rangle\, ds + \int_0^t \langle q_2(s), F_2 e_i\rangle\, ds + \int_0^t \langle q_2(s), \nabla^\perp\psi_2(s)\cdot\nabla e_i\rangle ds \\ &\quad + \int_0^t \langle\beta\partial_x\psi_2(s), e_i\rangle\, ds - r\int_0^t \langle\Delta\psi_2(s), e_i\rangle\, ds - \sum_{k\in K}\int_0^t \left\langle q_2(s), G_2^k e_i\right\rangle dW_s^{2,k}.\end{aligned}$$

Multiplying each equation by e_i and summing up, we get

$$\begin{aligned}d\tilde{q_1}^N &= \nu\Delta^2\tilde{\psi}_1^N dt + \sum_{i=1}^N \left\langle\nabla^\perp\psi_1\cdot\nabla e_i, q_1\right\rangle e_i dt + \sum_{i=1}^N \langle F - \beta\partial_x\psi_1, e_i\rangle e_i dt \\ &\quad - \sum_{k\in K}\sum_{i=1}^N \langle G_1^k e_i, q_1\rangle\, dW^{1,k} + \sum_{i=1}^N \langle q_1, F_1 e_i\rangle e_i\, dt \\ d\tilde{q_2}^N &= \nu\Delta^2\tilde{\psi}_2^N dt - r\Delta\tilde{\psi}_2^N dt + \sum_{i=1}^N \left\langle\nabla^\perp\psi_2\cdot\nabla e_i, q_2\right\rangle e_i dt - \beta\sum_{i=1}^N \langle\partial_x\psi_2, e_i\rangle e_i dt \\ &\quad - \sum_{k\in K}\sum_{i=1}^N \langle G_2^k e_i, q_2\rangle\, dW^{2,k} + \sum_{i=1}^N \langle q_2, F_2 e_i\rangle e_i\, dt.\end{aligned}$$

Now we can apply the Itô's formula to the process $\frac{|||\tilde{\mathbf{q}}^N|||^2}{2} = \frac{h_1\|\tilde{q}_1^N\|^2 + h_2\|\tilde{q}_2^N\|^2}{2}$ obtaining, thanks to the relations (3) and (14),

$$\begin{aligned}\frac{d|||\tilde{\mathbf{q}}^N|||^2}{2} + \nu|||\nabla\tilde{\mathbf{q}}^N|||^2 dt &= \Big(-\beta h_1\langle\partial_x\psi_1^N, \tilde{q}_1^N\rangle - \beta h_2\langle\partial_x\psi_2^N, \tilde{q}_2^N\rangle + h_1\langle F, \tilde{q}_1^N\rangle - rh_2\|\tilde{q}_2^N\|^2 + Sr\langle\tilde{\psi}_1^N - \tilde{\psi}_2^N, \tilde{q}_2^N\rangle\Big)\, dt \\ &+ \Big(S\nu\|\tilde{q}_1^N - \tilde{q}_2^N\|^2 + \nu S(S_1+S_2)\langle\tilde{\psi}_1^N - \tilde{\psi}_2^N, \tilde{q}_1^N - \tilde{q}_2^N\rangle\Big)\, dt \\ &+ h_1\langle q_1, F_1\tilde{q}_1^N\rangle dt + h_2\langle q_2, F_2\tilde{q}_2^N\rangle dt + \frac{h_1}{2}\sum_{k\in K}\sum_{i=1}^N\langle G_1^k e_i, q_1\rangle^2 dt + \frac{h_2}{2}\sum_{k\in K}\sum_{i=1}^N\langle G_2^k e_i, q_2\rangle^2 dt \\ &- h_1\sum_{k\in K}\langle q_1, G_1^K\tilde{q}_1^N\rangle dW^{1,k} - h_2\sum_{k\in K}\langle q_2, G_1^K\tilde{q}_2^N\rangle dW^{2,k}.\end{aligned}$$

Then, thanks to the properties of the projector P^N we get easily the desired formula. The only thing we need to prove is that

$$h_1\langle q_1, F_1\tilde{q}_1^N\rangle + h_2\langle q_2, F_2\tilde{q}_2^N\rangle + \frac{h_1}{2}\sum_{k\in K}\sum_{i=1}^N\langle G_1^k e_i, q_1\rangle^2 + \frac{h_2}{2}\sum_{k\in K}\sum_{i=1}^N\langle G_2^k e_i, q_2\rangle^2 \to 0.$$

The last relation is true, in fact for each $k\in\{1,\ldots,K\},\ j\in\{1,2\}$ we have

$$\begin{aligned}&\sum_{i=1}^N\langle q_j(s), \boldsymbol{\sigma}_{j,k}\cdot\nabla e_i\rangle^2 + \langle q_j(s), \boldsymbol{\sigma}_{j,k}\cdot\nabla\left(\boldsymbol{\sigma}_{j,k}\cdot\nabla\tilde{q}_j^N\right)\rangle \\ &= -\langle q_j(s), \boldsymbol{\sigma}_{j,k}\cdot\nabla\left(P^N(\boldsymbol{\sigma}_{j,k}\cdot\nabla q_j(s))\right)\rangle + \langle q_j(s), \boldsymbol{\sigma}_{j,k}\cdot\nabla\left(\boldsymbol{\sigma}_{j,k}\cdot\nabla\tilde{q}_j^N\right)\rangle \\ &= \langle\boldsymbol{\sigma}_{j,k}\cdot\nabla q_j(s), P^N\left(\boldsymbol{\sigma}_{j,k}\cdot\nabla q_j(s)\right)\rangle - \langle\boldsymbol{\sigma}_{j,k}\cdot\nabla q_j(s), \boldsymbol{\sigma}_{j,k}\cdot\nabla\tilde{q}_j^N\rangle \to 0.\end{aligned}$$

□

4 Enhanced dissipation by transport

In this section we will focus first on the proof of Theorem 1.5 and then, in Section 4.2, we will study the long time behaviour of the solutions of (11) showing that Theorem 1.6 holds.

4.1 Convergence to the Deterministic Evolution Model

The first step in order to prove Theorem 1.5, is to rewrite equations (11)-(12) in an alternative way, in order to introduce explicitly the term $\boldsymbol{\Delta}\mathbf{q}$. For this reason, using once more the relation (3), we express $\boldsymbol{\Delta}^2\boldsymbol{\psi}$ as

$$\boldsymbol{\Delta}^2\boldsymbol{\psi} = \boldsymbol{\Delta}\left(\mathbf{q} - M\boldsymbol{\psi}\right) = \boldsymbol{\Delta}\mathbf{q} + \boldsymbol{\Delta}M(-\boldsymbol{\Delta} - M)^{-1}\mathbf{q}.$$

Thus if we call $\mathbf{F} = \begin{bmatrix} F \\ -r\Delta\psi_2\end{bmatrix}$, equation (11) can be rewritten as

$$\begin{aligned}d\mathbf{q} &= \left((\kappa+\nu)\boldsymbol{\Delta}\mathbf{q} + \nu\boldsymbol{\Delta}M(-\boldsymbol{\Delta}-M)^{-1}\mathbf{q} - \nabla^\perp\boldsymbol{\psi}\cdot\nabla\mathbf{q} - \beta\partial_x\boldsymbol{\psi} + \mathbf{F}\right)dt + \sqrt{2\kappa}\sum_{k\in K}\mathbf{a}_k e_k\cdot\nabla\mathbf{q}\,dW^k \\ \boldsymbol{\psi} &= -(-\boldsymbol{\Delta} - M)^{-1}\mathbf{q}\end{aligned} \tag{48}$$

with initial condition $\mathbf{q}(0) = \mathbf{q}_0$, where we denoted by

$$\mathbf{a}_k e_k\cdot\nabla\mathbf{q}\,dW^k := \begin{bmatrix}\mathbf{a}_{1,k}e_k\cdot\nabla q_1 dW^{1,k} \\ \mathbf{a}_{2,k}e_k\cdot\nabla q_2 dW^{2,k}\end{bmatrix}$$

Similarly, if we call $\bar{\mathbf{F}} = \begin{bmatrix} F \\ -r\Delta\bar{\psi}_2 \end{bmatrix}$, the deterministic equation (12) can be rewritten as

$$\begin{aligned} d\bar{\mathbf{q}} &= \left((\kappa+\nu)\boldsymbol{\Delta}\bar{\mathbf{q}} + \nu\boldsymbol{\Delta}M(-\boldsymbol{\Delta}-M)^{-1}\bar{\mathbf{q}} - \nabla^{\perp}\bar{\boldsymbol{\psi}}\cdot\nabla\bar{\mathbf{q}} - \beta\partial_x\bar{\boldsymbol{\psi}} + \bar{\mathbf{F}}\right)dt \\ \bar{\boldsymbol{\psi}} &= -(-\boldsymbol{\Delta}-M)^{-1}\bar{\mathbf{q}} \end{aligned} \tag{49}$$

with initial condition $\bar{\mathbf{q}}(0) = \mathbf{q}_0$.

First, we want to show that the weak solution $\mathbf{q}$ of (11) satisfies a mild formulation. Denote the stochastic integral and stochastic convolution as

$$\mathbf{M}(t) := \sqrt{2\kappa}\sum_{k\in K}\int_0^t \mathbf{a}_k e_k\cdot\nabla\mathbf{q}(s)dW_s^k \tag{50}$$

$$\mathbf{Z}(t) := \sqrt{2\kappa}\sum_{k\in K}\int_0^t e^{(\kappa+\nu)(t-s)\boldsymbol{\Delta}}\left(\mathbf{a}_k e_k\cdot\nabla\mathbf{q}(s)\right)dW_s^k. \tag{51}$$

Thanks to the results of Section 3 for the stochastic system the following relations can be shown to hold

$$\sup_{t\in[0,T]}\|\mathbf{q}(t)\|^2 \le C\left(\|\mathbf{q}_0\|^2 + \int_0^T \|F(s)\|^2 ds\right)e^{CT} =: R_T^2, \qquad \mathbb{P}\text{–a.s.} \tag{52}$$

$$\sup_{t\in[0,T]}\|\bar{\mathbf{q}}(t)\|^2 \le R_T^2 \tag{53}$$

$$\int_0^T \|\nabla\mathbf{q}(s)\|^2\, ds \le \frac{C}{\nu}\left(TR_T^2 + \int_0^T \|F(s)\|^2\, ds\right) \qquad \mathbb{P}\text{–a.s.} \tag{54}$$

$$\int_0^T \|\nabla\bar{\mathbf{q}}(s)\|^2\, ds \le \frac{C}{\kappa+\nu}\left(TR_T^2 + \int_0^T \|F(s)\|^2\, ds\right). \tag{55}$$

The constants C, R_T here above depend from all the parameters of the model (i.e. $T, \nu, r, F, \mathbf{q}_0, M, \beta$), except for the parameters of the noise. Thanks to these estimates, Assumption 2.4 of [12] holds. Thus, thanks to Corollary 2.6 in [12], the stochastic integral (50) and the stochastic convolution (51) are well defined and have the regularity prescribed by the following Lemma.

Lemma 4.1. *Given the processes* (50) *and* (51) *the following statements hold true:*

(i) $\mathbf{M}(t)$ *is a continuous martingale with values in* $\mathbf{H}^{-1}$. *Moreover it holds*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{M}(t)\|^2_{\mathbf{H}^{-1}}\right] \lesssim \kappa R_T^2 T.$$

(ii) For each $\epsilon\in(0,1/2)$, $p\ge 1$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{Z}(t)\|^p_{\mathbf{H}^{-\epsilon}}\right]^{1/p} \lesssim_{\epsilon,p,T} \sqrt{\kappa(\nu+\kappa)^{\epsilon-1}}R_T, \tag{56}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{Z}(t)\|^p_{\mathbf{H}^{-1-\epsilon}}\right]^{1/p} \lesssim_{\epsilon,p,T} \sqrt{\kappa(\nu+\kappa)^{\epsilon-1}}\|\theta\|_{\ell^\infty}R_T. \tag{57}$$

(iii) For $\beta\in(0,1]$ *and* $\epsilon\in(0,\beta]$, $p\ge 1$ *it holds*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{Z}(t)\|^p_{\mathbf{H}^{-\beta}}\right]^{1/p} \lesssim_{\epsilon,p,T} \sqrt{\kappa(\nu+\kappa)^{\epsilon-1}}\|\theta\|^{\beta-\epsilon}_{\ell^\infty}R_T. \tag{58}$$

It is then easy to show that the weak solution $\mathbf{q}$ of the problem (12) satisfies also a mild formulation. In fact the following lemma holds.

Lemma 4.2. *If we denote by* $\mathbf{G}(t) := \nu \mathbf{\Delta} M(-\mathbf{\Delta} - M)^{-1}\mathbf{q}(t) - \nabla^{\perp}\boldsymbol{\psi}(t) \cdot \nabla \mathbf{q}(t) - \beta\partial_x \boldsymbol{\psi}(t) + \boldsymbol{F}(t)$*, then*

$$\mathbf{q}(t) = e^{(k+\nu)t\mathbf{\Delta}}\mathbf{q}_0 + \int_0^t e^{(k+\nu)(t-s)\mathbf{\Delta}}\mathbf{G}(s)ds + \mathbf{Z}(t), \quad \mathbb{P}-a.s. \ \ \forall t \in [0,T].$$

Proof. By definition, if $\mathbf{q}$ is a weak solution of problem (11), then it satisfies

$$\langle \mathbf{q}(t), \boldsymbol{\phi}\rangle - \langle \mathbf{q}_0, \boldsymbol{\phi}\rangle = \int_0^t \langle (\kappa+\nu)\mathbf{\Delta}\mathbf{q}(s) + \mathbf{G}(s), \boldsymbol{\phi}\ \rangle_{\mathbf{H}^{-2},\mathbf{H}^2} ds - \sqrt{2\kappa}\sum_{k\in K}\int_0^t \langle \mathbf{a}_k e_k \cdot \nabla \boldsymbol{\phi}, \mathbf{q}(s)\rangle dW^k_s$$

$\mathbb{P}$-a.s. for each $t \in [0,T]$ and $\boldsymbol{\phi} \in D(-\mathbf{\Delta})$. If take $\boldsymbol{e}_i := (e_i, e_i)^t$ as a test function, we have

$$\langle \mathbf{q}(t), \boldsymbol{e}_i\rangle - \langle \mathbf{q}_0, \boldsymbol{e}_i\rangle = \int_0^t \langle \lambda_i(\kappa+\nu)\mathbf{q}(s) + \mathbf{G}(s), \boldsymbol{e}_i\rangle_{\mathbf{H}^{-2},\mathbf{H}^2} ds - \sqrt{2\kappa}\sum_{k\in K}\int_0^t \langle \mathbf{a}_k e_k \cdot \nabla \boldsymbol{e}_i, \mathbf{q}(s)\rangle dW^k_s.$$

If we apply Itô formula to the process $e^{-t(\kappa+\nu)\lambda_i}\langle \mathbf{q}(t), \boldsymbol{e}_i\rangle$ and integrate in time we have

$$\begin{aligned}\langle \mathbf{q}(t), \boldsymbol{e}_i\rangle &= e^{-t(\kappa+\nu)\lambda_i}\langle \mathbf{q}_0, \boldsymbol{e}_i\rangle + \int_0^t e^{-(t-s)(\kappa+\nu)\lambda_i}\langle \mathbf{G}(s), \boldsymbol{e}_i\rangle ds \\ &\quad - \sqrt{2\kappa}\sum_{k\in K}\int_0^t e^{-(t-s)(\kappa+\nu)\lambda_i}\langle \mathbf{a}_k e_k \cdot \nabla \boldsymbol{e}_i, \mathbf{q}(s)\rangle dW^k_s.\end{aligned}$$

We can then find $\Gamma \subseteq \Omega$ of full probability such that the above equality holds for all $t \in [0,T]$and all $i \in \mathbb{Z}^2_0$. But this is exactly the mild formulation written in Fourier modes. □

Remark 4.3. Similarly to Lemma 4.2, also the solution $\bar{\mathbf{q}}$ of the associated determinsitic equation (12) satisfies for all $t \in [0,T]$ the integral relation

$$\bar{\mathbf{q}}(t) = e^{(k+\nu)t\mathbf{\Delta}}\mathbf{q}_0 + \int_0^t e^{(k+\nu)(t-s)\mathbf{\Delta}}\left(\nu \mathbf{\Delta} M(-\mathbf{\Delta}-M)^{-1}\bar{\mathbf{q}}(s) - \nabla^{\perp}\bar{\boldsymbol{\psi}}(s)\cdot\nabla\bar{\mathbf{q}}(s) - \beta\partial_x\bar{\boldsymbol{\psi}}(s) + \bar{\mathbf{F}}(s)\right) ds.$$

Before proving Theorem 1.5, we need a preliminary result on the nonlinearity of our problem.

Lemma 4.4. *Let* $\mathbf{q} \in \mathbf{L}^2$*,* $\bar{\mathbf{q}} \in \mathbf{H}^1$*, then, given*

$$\boldsymbol{R}(\mathbf{q}) = -\nabla^{\perp}(-\mathbf{\Delta} - M)^{-1}\mathbf{q} \cdot \nabla \mathbf{q},$$

for each $\alpha \in (0,1)$ *the following relation holds true*

$$\|\boldsymbol{R}(\mathbf{q}) - \boldsymbol{R}(\bar{\mathbf{q}})\|_{\mathbf{H}^{-1-\alpha}} \lesssim_{\alpha,M} \|\mathbf{q} - \bar{\mathbf{q}}\|_{\mathbf{H}^{-\alpha}}\left(\|\mathbf{q}\| + \|\bar{\mathbf{q}}\|_{\mathbf{H}^1}\right).$$

Proof. With a simple manipulation and the triangular inequality we have

$$\begin{aligned}\|\boldsymbol{R}(\mathbf{q}) - \boldsymbol{R}(\bar{\mathbf{q}})\|_{\mathbf{H}^{-1-\alpha}} &\le \|\nabla^{\perp}(-\mathbf{\Delta}-M)^{-1}\bar{\mathbf{q}}\cdot\nabla(\mathbf{q}-\bar{\mathbf{q}})\|_{\mathbf{H}^{-1-\alpha}} + \|\nabla^{\perp}(-\mathbf{\Delta}-M)^{-1}(\mathbf{q}-\bar{\mathbf{q}})\cdot\nabla\mathbf{q}\|_{\mathbf{H}^{-1-\alpha}} \\ &=: I_1 + I_2.\end{aligned}$$

Then the thesis follows by Lemma 2.1. In fact by point 2, we have

$$I_1 \lesssim_{\alpha,M} \|\nabla^{\perp}(-\mathbf{\Delta}-M)^{-1}\bar{\mathbf{q}}\|_{\mathbf{H}^2}\|\bar{\mathbf{q}}-\mathbf{q}\|_{\mathbf{H}^{-\alpha}} \lesssim \|\bar{\mathbf{q}}\|_{\mathbf{H}^1}\|\bar{\mathbf{q}}-\mathbf{q}\|_{\mathbf{H}^{-\alpha}},$$

and by point 4 it follows

$$I_2 \lesssim_{\alpha,M} \|\mathbf{q}\|\|\nabla^{\perp}(-\mathbf{\Delta}-M)^{-1}(\mathbf{q}-\bar{\mathbf{q}})\|_{\mathbf{H}^{1-\alpha}} \lesssim_{\alpha,M} \|\mathbf{q}\|\|\mathbf{q}-\bar{\mathbf{q}}\|_{\mathbf{H}^{-\alpha}}.$$

□

We are now ready to show the main result of this section, Theorem 1.5:

Proof of Theorem 1.5. We have seen in Lemma 4.2 and Remark 4.3 that both $\mathbf{q}$ and $\bar{\mathbf{q}}$ satisfies a mild formulation. Thus calling $\boldsymbol{\xi} = \mathbf{q} - \bar{\mathbf{q}}$ and $\boldsymbol{\chi} = \boldsymbol{\psi} - \bar{\boldsymbol{\psi}}$ we have

$$\boldsymbol{\xi}(t) = \int_0^t e^{(\kappa+\nu)(t-s)\boldsymbol{\Delta}} \left(\nu\boldsymbol{\Delta} M(-\boldsymbol{\Delta} - M)^{-1}\boldsymbol{\xi}(s) - \beta\partial_x\boldsymbol{\chi}(s) + \mathbf{F}(s) - \bar{\mathbf{F}}(s)\right) ds \\ - \int_0^t e^{(\kappa+\nu)(t-s)\boldsymbol{\Delta}} \left(\boldsymbol{R}(\mathbf{q}(s)) - \boldsymbol{R}(\bar{\mathbf{q}}(s))\right) ds + \boldsymbol{Z}(t).$$

By Lemma 2.4 we have

$$\|\boldsymbol{\xi}(t)\|^2_{\mathbf{H}^{-\alpha}} \lesssim_{\alpha,M} \|\boldsymbol{Z}(t)\|^2_{\mathbf{H}^{-\alpha}} + \frac{1}{\kappa+\nu}\int_0^t \|\boldsymbol{R}(\mathbf{q}(s)) - \boldsymbol{R}(\bar{\mathbf{q}}(s))\|^2_{\mathbf{H}^{-\alpha-1}}\, ds \\ + \frac{1}{\kappa+\nu}\int_0^t \nu^2\|\boldsymbol{\Delta} M(-\boldsymbol{\Delta} - M)^{-1}\boldsymbol{\xi}(s)\|^2_{H^{-\alpha-1}} + \beta^2\|\partial_x\boldsymbol{\chi}(s)\|^2_{\mathbf{H}^{-\alpha-1}} + r^2\|\Delta\chi_2(s)\|^2_{H^{-\alpha-1}}\, ds$$

and by the relation (3) it follows

$$\|\boldsymbol{\xi}(t)\|^2_{\mathbf{H}^{-\alpha}} \lesssim_{\alpha,M} \|\boldsymbol{Z}(t)\|^2_{\mathbf{H}^{-\alpha}} + \frac{1}{\kappa+\nu}\int_0^t \|\boldsymbol{R}(\mathbf{q}(s)) - \boldsymbol{R}(\bar{\mathbf{q}}(s))\|^2_{\mathbf{H}^{-\alpha-1}}\, ds + \frac{\beta^2+\nu^2+r^2}{\kappa+\nu}\int_0^t \|\boldsymbol{\xi}(s)\|^2_{\mathbf{H}^{-\alpha-1}}\, ds.$$

Last, thanks to Lemma 4.4

$$\|\boldsymbol{\xi}(t)\|^2_{\mathbf{H}^{-\alpha}} \lesssim_{\alpha,M} \|\boldsymbol{Z}(t)\|^2_{\mathbf{H}^{-\alpha}} + \frac{1}{\kappa+\nu}\int_0^t \|\boldsymbol{\xi}(s)\|^2_{\mathbf{H}^{-\alpha}}\left(\|\mathbf{q}(s)\|^2 + \|\bar{\mathbf{q}}(s)\|^2_V + \beta^2 + \nu^2 + r^2\right)\, ds.$$

Therefore, by Grönwall's lemma, there exists $C = C(\alpha, M)$ such that

$$\|\boldsymbol{\xi}(t)\|^2_{\mathbf{H}^{-\alpha}} \lesssim_{\alpha,M} \left(\sup_{t\in[0,T]}\|\boldsymbol{Z}(t)\|^2_{\mathbf{H}^{-\alpha}}\right) \exp\left(\frac{C}{\nu+\kappa}\int_0^T \|\mathbf{q}(s)\|^2 + \|\bar{\mathbf{q}}(s)\|^2_V\, ds\right) \exp\left(T\frac{\nu^2+\beta^2+r^2}{\kappa+\nu}\right). \tag{59}$$

Now we take the expectation of (59) and use relation (58) in Lemma 4.1 to estimate the stochastic convolution to get

$$\mathbb{E}\,\|\boldsymbol{\xi}(t)\|^2_{\mathbf{H}^{-\alpha}} \lesssim_{\alpha,M,\epsilon,T} \frac{\kappa}{(\nu+\kappa)^{1-\epsilon}}\|\theta\|^{2(\alpha-\epsilon)}_{\ell^\infty} R_T^2 \exp\left(T\frac{\nu^2+\beta^2+r^2}{\kappa+\nu}\right)\mathbb{E}\exp\left(\frac{C}{\nu+\kappa}\int_0^T \|\mathbf{q}(s)\|^2 + \|\bar{\mathbf{q}}(s)\|^2_V\, ds\right). \tag{60}$$

Now to prove statement (i) we use (52) and (55) to derive

$$\mathbb{E}\left[\|\mathbf{q} - \bar{\mathbf{q}}\|^2_{C([0,T];\mathbf{H}^{-\alpha})}\right] \lesssim_{\alpha,M,\epsilon,T} \kappa^\epsilon\|\theta\|^{2(\alpha-\epsilon)}_{\ell^\infty} R_T^2 \exp\left(T\frac{\nu^2+\beta^2+r^2}{\kappa+\nu}\right) \\ \exp\left(\frac{CTR_T^2}{(\kappa+\nu)^2}(1+\kappa+\nu) + \frac{C}{(\kappa+\nu)^2}\int_0^T \|F(s)\|^2\, ds\right).$$

We move on to proving statement (ii). Given (60) let us now estimate $\int_0^T \|\mathbf{q}(s)\|^2$ using the estimate (54) instead of (52), namely

$$\int_0^T \|\mathbf{q}(s)\|^2 ds \lesssim \int_0^T \|\mathbf{q}(s)\|^2_V ds \lesssim \frac{C}{\nu}\left(TR_T^2 + \int_0^T \|F(s)\|^2\, ds\right)$$

where we have also used that $\frac{1}{\nu} + \frac{1}{\kappa+\nu} \leq \frac{2}{\nu}$. Then the desired bound follows from the same arguments used for the estimate (i). ☐

Remark 4.5. Similarly to [12], we can consider also the p moment of the random variables treated in the previous theorem. We neglect this fact, which is not needed in order to prove Theorem 1.6.

4.2 Long Time Behavior

Let us recall our framework, $\mathbf{q}$ is the weak solution of the system

$$dq_1 + \left(\nabla^{\perp}\psi_1 \cdot \nabla q_1\right) dt = \left(\kappa\Delta q_1 + \nu\Delta^2\psi_1 - \beta\partial_x\psi_1 + F\right) dt + \sqrt{2\kappa}\sum_{k\in K} \boldsymbol{a}_{1,k} e_k \cdot \nabla q_1 \, dW^{1,k}$$
$$dq_2 + \left(\nabla^{\perp}\psi_2 \cdot \nabla q_2\right) dt = \left(\kappa\Delta q_2 + \nu\Delta^2\psi_2 - \beta\partial_x\psi_2 - r\Delta\psi_2\right) dt + \sqrt{2\kappa}\sum_{k\in K} \boldsymbol{a}_{2,k} e_k \cdot \nabla q_2 \, dW^{2,k}$$

with $\boldsymbol{\psi} = -(-\boldsymbol{\Delta} - M)^{-1}\mathbf{q}$, and initial condition $\mathbf{q}(0) = q_0$, and $\tilde{\mathbf{q}}$ is a weak solution of the stationary deterministic system

$$\begin{aligned} \nabla^{\perp}\tilde{\psi}_1 \cdot \nabla\tilde{q}_1 &= \kappa\Delta\tilde{q}_1 + \nu\Delta^2\tilde{\psi}_1 - \beta\partial_x\tilde{\psi}_1 + F \\ \nabla^{\perp}\tilde{\psi}_2 \cdot \nabla\tilde{q}_2 &= \kappa\Delta\tilde{q}_2 + \nu\Delta^2\tilde{\psi}_2 - \beta\partial_x\tilde{\psi}_2 - r\Delta\tilde{\psi}_2 \end{aligned} \tag{61}$$

with $\tilde{\boldsymbol{\psi}} = -(-\boldsymbol{\Delta} - M)^{-1}\tilde{\mathbf{q}}$. Thanks to this relation between $\tilde{\mathbf{q}}$ and $\tilde{\boldsymbol{\psi}}$, the system (61) can be rewritten as

$$0 = (\kappa + \nu)\boldsymbol{\Delta}\tilde{\mathbf{q}} + \nu\boldsymbol{\Delta}M(-\boldsymbol{\Delta} - M)^{-1}\tilde{\mathbf{q}} - \nabla^{\perp}\tilde{\boldsymbol{\psi}} \cdot \nabla\tilde{\mathbf{q}} - \beta\partial_x\tilde{\boldsymbol{\psi}} + \tilde{\mathbf{F}} \tag{62}$$

where

$$\tilde{\mathbf{F}} = \begin{bmatrix} F \\ -r\Delta\tilde{\psi}_2 \end{bmatrix}.$$

Then we consider the following concept of solution for (62):

Definition 4.6. A weak solution of the problem (62) is a function $\tilde{\mathbf{q}} \in \mathcal{V}$ such that for all $\boldsymbol{\phi} = (\phi_1, \phi_2)^t \in \mathcal{V}$ the following relation holds true

$$(\kappa + \nu)\langle\nabla\tilde{\mathbf{q}}, \nabla\boldsymbol{\phi}\rangle = \nu\langle\boldsymbol{\Delta}M(-\boldsymbol{\Delta} - M)^{-1}\tilde{\mathbf{q}}, \boldsymbol{\phi}\rangle - \langle\nabla^{\perp}\tilde{\boldsymbol{\psi}} \cdot \nabla\tilde{\mathbf{q}}, \boldsymbol{\phi}\rangle - \beta\langle\partial_x\tilde{\boldsymbol{\psi}}, \boldsymbol{\phi}\rangle + \langle\tilde{\mathbf{F}}, \boldsymbol{\phi}\rangle.$$

Proposition 4.7. *For κ large enough there exists a unique $\tilde{\mathbf{q}}$ weak solution of problem (62), moreover $\|\tilde{\mathbf{q}}(t) - \bar{\mathbf{q}}(t)\| \to 0$ exponentially fast as $t \to +\infty$.*

Proof. As usual we start by establishing a priori estimates. Assuming there exist a solution $\tilde{\mathbf{q}}$. Taking $\tilde{\mathbf{q}}$ as a test function in the weak formulation, we obtain, thanks to the fact that $\langle\nabla^{\perp}\tilde{\boldsymbol{\psi}} \cdot \nabla\tilde{\mathbf{q}}, \tilde{\mathbf{q}}\rangle = 0$ and to Poincaré inequality,

$$(\kappa + \nu)\|\nabla\tilde{\mathbf{q}}\|^2 \leq C\nu\|\nabla\tilde{\mathbf{q}}\|^2 + C\|F\|\|\nabla\tilde{\mathbf{q}}\| + C\|\nabla\tilde{\mathbf{q}}\|^2$$

where the constant C depends from $\beta, S_1, \ S_2, r, \mathcal{D}$ but it is independent of κ and ν. Thus, if κ is large enough we have

$$\|\nabla\tilde{\mathbf{q}}\| \leq \frac{C\|F\|}{\kappa + \nu - C\nu - C} = M_\kappa. \tag{63}$$

For κ large enough such that the a priori estimate (63) holds, let us consider the complete metric space

$$B_\kappa = \{\mathbf{q} \in \mathcal{V} : \|\mathbf{q}\|_V \leq M_\kappa\}.$$

Let us show that the map T which takes $\mathbf{q} \in B_\kappa$ and associate to it $\tilde{\mathbf{q}}$ which is the unique weak solution of the linear problem

$$(\kappa + \nu)\langle\nabla\tilde{\mathbf{q}}, \nabla\boldsymbol{\phi}\rangle = \nu\langle\boldsymbol{\Delta}M(-\boldsymbol{\Delta} - M)^{-1}\mathbf{q}, \boldsymbol{\phi}\rangle - \langle\nabla^{\perp}\boldsymbol{\psi} \cdot \nabla\tilde{\mathbf{q}}, \boldsymbol{\phi}\rangle - \beta\langle\partial_x\boldsymbol{\psi}, \boldsymbol{\phi}\rangle + \langle\mathbf{F}, \boldsymbol{\phi}\rangle. \tag{64}$$

is a contraction in B_κ if we take κ large enough. Here, of course $\boldsymbol{\psi}$ is obtained by $\mathbf{q}$ via relation (3) and

$$\mathbf{F} = \begin{bmatrix} F \\ -r\Delta\psi_2 \end{bmatrix}.$$

Existence and uniqueness of the solution of (64) follows immediately by Lax-Milgram Lemma thanks to the fact that the bilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by

$$a(\mathbf{q}_1, \mathbf{q}_2) = (\kappa + \nu)\langle \nabla \mathbf{q}_1, \nabla \mathbf{q}_2 \rangle - \langle \nabla^\perp \boldsymbol{\psi} \cdot \nabla \mathbf{q}_1, \mathbf{q}_2 \rangle$$

is continuous and coercive. Moreover, arguing as in the a priori estimate, $T(\mathbf{q})$ satisfies

$$(\kappa + \nu)\|\nabla T(\mathbf{q})\| \leq C\nu M_\kappa + CM_\kappa + C\|F\|$$

where C is the same costant as above. From this it follows immediately that

$$\|\nabla T(\mathbf{q})\| \leq M_\kappa.$$

Thus T is a map between B_κ and itself for κ large enough such that $M_\kappa > 0$. Lastly we need to show that T is a contraction. Let $\mathbf{q}_1, \mathbf{q}_2 \in B_\kappa$ and $T(\mathbf{q}_1)$, $T(\mathbf{q}_2)$ the corresponding solutions. Then $\forall \boldsymbol{\phi} \in \mathcal{V}$ we have

$$\begin{aligned}(\kappa + \nu)\langle \nabla (T(\mathbf{q}_1) - T(\mathbf{q}_2)), \nabla \boldsymbol{\phi} \rangle &= \nu \langle \boldsymbol{\Delta} M(-\boldsymbol{\Delta} - M)^{-1}(\mathbf{q}_1 - \mathbf{q}_2), \boldsymbol{\phi} \rangle - \langle \nabla^\perp \boldsymbol{\psi}_1 \cdot \nabla T(\mathbf{q}_1), \boldsymbol{\phi} \rangle \\ &\quad + \langle \nabla^\perp \boldsymbol{\psi}_2 \cdot \nabla T(\mathbf{q}_2), \boldsymbol{\phi} \rangle - \beta \langle \partial_x(\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2), \boldsymbol{\phi} \rangle + \langle \boldsymbol{F}_1 - \boldsymbol{F}_2, \boldsymbol{\phi} \rangle.\end{aligned}$$

Here, of course $\boldsymbol{\psi}_j$ is obtained by $\mathbf{q}_j$, for $j \in \{1, 2\}$, via relation (3) and

$$\mathbf{F}_{1,2} = \begin{bmatrix} F \\ -r(\Delta \boldsymbol{\psi}_{1,2})_2 \end{bmatrix}.$$

Taking $\boldsymbol{\phi} = T(\mathbf{q}_1) - T(\mathbf{q}_2)$ we have

$$\begin{aligned}(\kappa + \nu)\|\nabla (T(\mathbf{q}_1) - T(\mathbf{q}_2))\|^2 &\leq C\nu \|\nabla (T(\mathbf{q}_1) - T(\mathbf{q}_2))\| \|\nabla (\mathbf{q}_1 - \mathbf{q}_2)\| + C\|\nabla (T(\mathbf{q}_1) - T(\mathbf{q}_2))\| \|\nabla (\mathbf{q}_1 - \mathbf{q}_2)\| \\ &\quad - \langle \nabla^\perp \boldsymbol{\psi}_1 \cdot \nabla T(\mathbf{q}_1), T(\mathbf{q}_1) - T(\mathbf{q}_2) \rangle + \langle \nabla^\perp \boldsymbol{\psi}_2 \cdot \nabla T(\mathbf{q}_2), T(\mathbf{q}_1) - T(\mathbf{q}_2) \rangle \\ &\quad \pm \langle \nabla^\perp \boldsymbol{\psi}_1 \cdot \nabla T(\mathbf{q}_2), T(\mathbf{q}_1) - T(\mathbf{q}_2) \rangle \\ &\leq C\nu \|\nabla (T(\mathbf{q}_1) - T(\mathbf{q}_2))\| \|\nabla (\mathbf{q}_1 - \mathbf{q}_2)\| + C\|\nabla (T(\mathbf{q}_1) - T(\mathbf{q}_2))\| \|\nabla (\mathbf{q}_1 - \mathbf{q}_2)\| \\ &\quad + CM_\kappa \|\nabla (T(\mathbf{q}_1) - T(\mathbf{q}_2))\| \|\nabla (\mathbf{q}_1 - \mathbf{q}_2)\|.\end{aligned}$$

If we take κ large enough, thus T is a contraction and the thesis follows.

We conclude by showing the desired exponential convergence. Let $\bar{\boldsymbol{\psi}}$ and $\tilde{\boldsymbol{\psi}}$ be the stream functions associated to $\bar{\mathbf{q}}$ and $\tilde{\mathbf{q}}$ respectively and define $\boldsymbol{w} = \bar{\mathbf{q}} - \tilde{\mathbf{q}}$ and $\boldsymbol{\chi} = \bar{\boldsymbol{\psi}} - \tilde{\boldsymbol{\psi}}$. The following differential relation holds

$$\frac{d\|\boldsymbol{w}\|^2}{2dt} + (\kappa + \nu)\|\nabla \boldsymbol{w}\|^2 = \langle \nu \Delta M(-\Delta - M)^{-1}\boldsymbol{w}, \boldsymbol{w} \rangle - \beta \langle \partial_x \boldsymbol{\chi}, \boldsymbol{w} \rangle + \langle \bar{\mathbf{F}} - \tilde{\mathbf{F}}, \boldsymbol{w} \rangle - \langle \nabla^\perp \boldsymbol{\chi} \cdot \nabla \tilde{\mathbf{q}}, \boldsymbol{w} \rangle.$$

Using the definition of $\bar{\boldsymbol{F}}$ and $\tilde{\boldsymbol{F}}$, Lemma 2.1 and Young's inequality we have

$$\begin{aligned}\frac{d\|\boldsymbol{w}\|^2}{2dt} + (\kappa + \nu)\|\nabla \boldsymbol{w}\|^2 &\leq \nu C\|\boldsymbol{w}\|^2 + C\|\boldsymbol{w}\|^2 + C\|\boldsymbol{w}\|\|\nabla \boldsymbol{w}\|\|\nabla \tilde{\mathbf{q}}\| \\ &\leq \nu C\|\boldsymbol{w}\|^2 + C\|\boldsymbol{w}\|^2 + \frac{\nu}{2}\|\nabla \boldsymbol{w}\|^2 + \frac{C}{2\nu}\|\boldsymbol{w}\|^2\|\nabla \tilde{\mathbf{q}}\|^2,\end{aligned}$$

where C is a constant possibly depending from $\mathcal{D}$, β, r, M but it is independent of κ, ν and T. By Poincaré inequality, we have $\|\nabla \boldsymbol{w}\|^2 \geq \frac{1}{C_p}\|\boldsymbol{w}\|^2$, therefore

$$\frac{d\|\boldsymbol{w}\|^2}{2dt} + \left(\frac{\kappa + \nu/2}{C_p} - (\nu C + C + C\|\nabla \tilde{\mathbf{q}}\|^2) \right) \|\boldsymbol{w}\|^2 \leq 0.$$

Calling $\alpha = \frac{\kappa + \nu/2}{C_p} - (\nu C + C + C\|\nabla \tilde{\mathbf{q}}\|^2)$. If κ is large enough, $\alpha > 0$ thus by Grönwall's Lemma we have the exponential rate of convergence. □

Remark 4.8. 1. By the proof of previous theorem, κ must be large enough that the following inequalities are satisfied, here C is a constant possibly depending from $\beta, S_1, S_2, r, \mathcal{D}$ but independent of κ and ν.

$$\kappa + \nu - C\nu - C > 0;$$
$$\frac{1}{\kappa+\nu}\left(C\nu + C + \frac{C\|F\|}{\kappa+\nu-C\nu-C}\right) < 1;$$
$$\frac{\kappa+\nu/2}{C_p} - \left(\nu C + C + \frac{C\|F\|^2}{(\kappa+\nu-C\nu-C)^2}\right) > 0.$$

2. By elliptic regularity, it follows immediately that, actually, $\tilde{\mathbf{q}} \in D(\boldsymbol{\Delta})$.

Now we are able to prove our final result.

Proof of Theorem 1.6. Let $\bar{\mathbf{q}}$ be the weak solution of the deterministic problem (12). First we fix $\delta > 0,\ \alpha \in (0,1)$. If κ is large enough, by Proposition 4.7, we can find $\overline{T} = \overline{T}(\delta)$ such that

$$\|\bar{\mathbf{q}}(t) - \tilde{\mathbf{q}}\|^2 \leq \delta/4, \quad \text{for all } t \geq \overline{T}.$$

Now we use the results of Theorem 1.5 for $\epsilon = \alpha/2$, thus we have

$$\mathbb{E}\left[\|\mathbf{q} - \bar{\mathbf{q}}\|^2_{C([0,2\overline{T}];\mathbf{H}^{-\alpha})}\right] \lesssim_{\alpha,M,\epsilon,\overline{T}} \kappa^\epsilon \|\theta\|^{2(\alpha-\epsilon)}_{\ell^\infty} R^2_{2\overline{T}} \exp\left(2\overline{T}\frac{\nu^2+\beta^2+r^2}{\kappa+\nu}\right)$$
$$\exp\left(\frac{C\overline{T}R^2_{2\overline{T}}}{(\kappa+\nu)^2}(1+\kappa+\nu) + \frac{C}{(\kappa+\nu)^2}\int_0^{2\overline{T}} \|F(s)\|^2\, ds\right).$$

Since the constants appearing in previous equation are independent of the parameters of the noise, if we take θ to be such that the right hand side of the previous inequality can be bounded by $\delta/4$ then the thesis follows immediately. For example some possible choices of θ can be found in [12, Example 1.3]. ☐

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References

[1] C. Bernier. Existence of attractor for the quasi–geostrophic approximation of the Navier–Stokes equations and estimate of its dimension. *Adv. Math. Sci. Appl.*, 4(2):465–489, 1994.

[2] H. Breckner. Approximation and optimal control of the stochastic Navier–Stokes equation. *Mathematisch Naturwissenschaftlich Technischen Fakultät der Martin Luther Universität Halle Wittenberg, Diss*, 1999.

[3] H. Breckner. Galerkin approximation and the strong solution of the Navier–Stokes equation. *Journal of Applied Mathematics and Stochastic Analysis*, 13(3):239–259, 2000.

[4] G. Carigi. *Ergodic properties and response theory for a stochastic two-layer model of geophysical fluid dynamics.* PhD thesis, University of Reading, 2021. https://doi.org/10.48683/1926.00102181.

[5] G. Carigi, J. Bröcker, and T. Kuna. Exponential ergodicity for a stochastic two-layer quasi–geostrophic model. *Stochastics and Dynamics*, 2022. https://doi.org/10.1142/S0219493723500119.

[6] J. G. Charney. On the scale of atmospheric motions. *Geofysiske Publikasjoner*, 17(2):1–17, 1948.

[7] I. Chueshov, J. Duan, and B. Schmalfuss. Probabilistic dynamics of two-layer geophysical flows. *Stochastics and Dynamics*, 01(04):451–475, 2001.

[8] C. Cotter, D. Crisan, D. Holm, W. Pan, and I. Shevchenko. Data assimilation for a quasi–geostrophic model with circulation-preserving stochastic transport noise. *Journal of Statistical Physics*, pages 1–36, 2020.

[9] C. Cotter, D. Crisan, D. Holm, W. Pan, and I. Shevchenko. Modelling uncertainty using stochastic transport noise in a 2-layer quasi–geostrophic model. *Foundations of Data Science*, 2(2):173, 2020.

[10] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions.* Cambridge university press, 2014.

[11] Z. Dong, D. Luo, and B. Tang. Dissipation enhancement by transport noise for stochastic p-laplace equations. *arXiv preprint arXiv:2206.01376*, 2022.

[12] F. Flandoli, L. Galeati, and D. Luo. Quantitative convergence rates for scaling limit of spdes with transport noise. *arXiv preprint arXiv:2104.01740*, 2021.

[13] F. Flandoli, L. Galeati, and D. Luo. Eddy heat exchange at the boundary under white noise turbulence. *Philosophical Transactions of the Royal Society A*, 380(2219):20210096, 2022.

[14] F. Flandoli and D. Luo. High mode transport noise improves vorticity blow-up control in 3d navier–stokes equations. *Probability Theory and Related Fields*, 180(1-2):309–363, mar 2021.

[15] F. Flandoli and E. Luongo. Heat diffusion in a channel under white noise modeling of turbulence. *Mathematics in Engineering*, 4(4):1–21, 2022.

[16] F. Flandoli and E. Luongo. *Stochastic Partial Differential Equations in Fluid Mechanics.* Vol. 2328. Springer Nature, 2023.

[17] F. Flandoli and U. Pappalettera. From additive to transport noise in 2d fluid dynamics. *Stochastics and Partial Differential Equations: Analysis and Computations*, 2022.

[18] L. Galeati. On the convergence of stochastic transport equations to a deterministic parabolic one. *Stochastics and Partial Differential Equations: Analysis and Computations*, 8(4):833–868, 2020.

[19] F. Grotto and U. Pappalettera. Equilibrium statistical mechanics of barotropic quasi-geostrophic equations. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*: 24(1): 2150007, 2021.

[20] D. D. Holm. Variational principles for stochastic fluid dynamics. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2176):20140963, 2015.

[21] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, 2012.

[22] V. Lucarini, R. Blender, C. Herbert, F. Ragone, S. Pascale, and J. Wouters. Mathematical and physical ideas for climate science. *Reviews of Geophysics*, 52(4):809–859, 2014.

[23] E. Luongo. Inviscid limit for stochastic second-grade fluid equations. *arXiv preprint arXiv:2207.03174*, 2022.

[24] E. Pardoux. *Equations aux dérivées partielles stochastiques monotones.* PhD thesis, Université Paris-Sud, 1975.

[25] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[26] J. Pedlosky. *Geophysical fluid dynamics.* Springer Science & Business Media, 2013.

[27] P. A. Razafimandimby and M. Sango. Strong solution for a stochastic model of two-dimensional second grade fluids: existence, uniqueness and asymptotic behavior. *Nonlinear Analysis: Theory, Methods & Applications*, 75(11):4251–4270, 2012.

[28] A. V. Skorokhod. *Studies in the theory of random processes*, volume 7021. Courier Dover Publications, 1982.

[29] G. K. Vallis. *Atmospheric and Oceanic Fluid Dynamics.* Cambridge University Press, Cambridge, U.K., 2006.