

Research



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Eddy heat exchange at the boundary under white noise turbulence

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We prove the existence of an eddy heat diffusion coefficient coming from an idealized model of turbulent fluid. A difficulty lies in the presence of a boundary, with also turbulent mixing and the eddy diffusion coefficient going to zero at the boundary. Nevertheless, enhanced diffusion takes place.

This article is part of the theme issue 'Scaling the turbulence edifice (part 2)'.

1. Introduction

Understanding the effect of turbulent transport on the diffusion of passive scalars is a classical subject, investigated in the literature of both theoretical physics and pure mathematics; see for instance [1–9]. A large part of this literature is based on a white noise velocity field which, although being an idealization, has the power to lead to special quantitative results. It has been widely used, mainly with the purpose of identifying anomalous scaling exponents of interest for turbulence. Other models, mainly related to homogenization theory (see [7]), have been investigated with a purpose more similar to ours, namely to show that the behaviour under a turbulent velocity field is similar to the behaviour under a diffusion operator with enhanced diffusion, the so-called eddy diffusion coefficient, observed in experiments and real situations.

This purpose (proving eddy diffusion starting from turbulent transport) has not been investigated for models based on white noise, to our knowledge. That the average of the solution of such a stochastic transport–diffusion equation satisfies a deterministic diffusion equation with enhanced diffusion operator is an obvious fact well recognized in the literature; see for instance [7, p. 419]. But the behaviour of a random field and its average can be completely different; a precise investigation of conditions under which they are close, and thus also the random field itself behaves (in the weak sense) as being subject to a modified diffusion operator (eddy dissipation), seems to be new. We have initiated this research in [10,11] and, in the present work, we approach the more difficult case of a bounded domain with zero boundary conditions (opposite to the periodic boundary conditions of previous works), with no-slip boundary condition for the velocity field, which leads to the fact that the enhanced diffusion part emerging from the transport term is degenerate at the boundary; this leads to difficulties treated here for the first time.

The results just discussed are strictly scalar: when a vector field (like a magnetic field) is passively advected by a white noise velocity field, the average of the solution still satisfies a deterministic equation with enhanced diffusion but the question whether the random solution is close to its average is very difficult and still open.

We consider the scalar temperature field $T = T(t, \mathbf{x})$ subject to the equation

$$\text{and} \quad \left. \begin{aligned} \partial_t T &= \kappa \Delta T + \mathbf{u}_Q \circ \nabla T && \text{in } [0, T] \times D \\ T|_{\partial D} &= 0, \quad T|_{t=0} = T_0 && \text{in } D, \end{aligned} \right\} \quad (1.1)$$

in an open connected bounded domain $D \subset \mathbb{R}^d$ with piecewise regular boundary; $\kappa > 0$ is the diffusion constant, that we should think to be small; the velocity field is a given random, divergence free, vector field $\mathbf{u}_Q(t, \mathbf{x})$, Gaussian, white noise in time, with a prescribed covariance matrix function $Q(\mathbf{x}, \mathbf{y})$ in space, simulating in a simplified fashion an incompressible turbulent fluid.

Without noise and fluid motion, the temperature would decay to zero due to the Dirichlet boundary conditions (the cold boundary absorbs heat) but the rate of decay would be given by $\kappa \lambda_D$, where $-\lambda_D$ is the first eigenvalue of the Laplacian operator Δ with zero boundary condition. But when the fluid is turbulent, we expect a faster decay thanks to the appearance of eddy diffusion.

There are specific technical difficulties due to the boundary that we have to overcome to prove the result. One problem is that the fluid is at rest on ∂D , due to no-slip condition, namely $Q(\mathbf{x}, \mathbf{x}) = 0$ for $\mathbf{x} \in \partial D$. Hence the strength of the mixing mechanism is depleted near the boundary, exactly where the fluid comes in interaction with the cold boundary which is responsible for cooling. We therefore have to understand the balance between these phenomena.

One of the main ideas used in this work goes back to [10,11], see also [12,13], but several other aspects are new: first of all the way to overcome the difficulties due to the boundary, but also the more quantitative presentation of the results, which required new proofs.

Let us state the main result of this work. We denote by $H^k(D)$ the classical Sobolev spaces of square integrable k -times weakly differentiable functions. Let J be a finite or countable index set and $(\mathbf{u}_j(\mathbf{x}))_{j \in J}$ be divergence free vector fields $\mathbf{u}_j : \bar{D} \rightarrow \mathbb{R}^d$

$$\mathbf{u}_j|_{\partial D} = 0, \quad \operatorname{div} \mathbf{u}_j = 0,$$

with smoothness $\sum_{j \in J} \|\mathbf{u}_j\|_{H^1(D) \cap C(\bar{D})}^2 < \infty$, which in particular allows us to define the covariance matrix-valued function $Q : \bar{D} \times \bar{D} \rightarrow \mathbb{R}^{d \times d}$

$$Q(\mathbf{x}, \mathbf{y}) = \sum_{j \in J} \mathbf{u}_j(\mathbf{x}) \otimes \mathbf{u}_j(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \bar{D}.$$

Associated with it define the bounded linear operator

$$\mathbb{Q} : L^2(D; \mathbb{R}^d) \rightarrow L^2(D; \mathbb{R}^d), \quad (\mathbb{Q}\mathbf{v})(\mathbf{x}) = \int_D Q(\mathbf{x}, \mathbf{y}) \mathbf{v}(\mathbf{y}) \, d\mathbf{y},$$

and introduce two important quantities

$$\left. \begin{aligned} q(\mathbf{x}) &:= \min_{\xi \neq 0} \frac{\xi^T Q(\mathbf{x}, \mathbf{x}) \xi}{\xi^T \xi}, \\ \epsilon_Q &:= \|\mathbb{Q}^{1/2}\|_{L^2 \rightarrow L^2}^2 = \sup_{\mathbf{v} \neq 0} \frac{\int_D \int_D \mathbf{v}(\mathbf{x})^T Q(\mathbf{x}, \mathbf{y}) \mathbf{v}(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}}{\int_D \mathbf{v}(\mathbf{x})^T \mathbf{v}(\mathbf{x}) \, d\mathbf{x}}. \end{aligned} \right\} \quad (1.2)$$

Denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a filtered probability space with expectation $\mathbb{E}$, let $(W_t^j)_{j \in J}$ be a family of independent Brownian motions; the generalized process

$$\mathbf{u}_Q(t, \mathbf{x}) = \sum_{j \in J} \mathbf{u}_j(\mathbf{x}) \frac{dW_t^j}{dt}$$

is a white noise in time, divergence free, with space-covariance $Q(\mathbf{x}, \mathbf{y})$. As it is always done in investigations of passive scalars subject to white noise velocity fields [1,2,4,6,7,9], which is motivated by the Wong–Zakai principle [14], proved also for transport–diffusion equations by many authors (see for instance [15]), we interpret the equation (1.1) as a stochastic equation with Stratonovich noise (the precise interpretation is in weak form with smooth test functions)

$$dT = \kappa \Delta T \, dt + \sum_{j \in J} \mathbf{u}_j \cdot \nabla T \circ dW_t^j. \quad (1.3)$$

Call $D(A)$ the space $H^2(D) \cap H_0^1(D)$ where $H_0^1(D)$ is the set of $H^1(D)$ -functions equal to zero at the boundary. Define two linear operators $A, A_Q : D(A) \rightarrow L^2(D)$ by setting

$$Af = \kappa \Delta f \quad \text{and} \quad A_Q f = (\kappa \Delta + \mathcal{L}_Q)f,$$

where

$$(\mathcal{L}_Q f)(\mathbf{x}) = \frac{1}{2} \sum_{\alpha, \beta=1}^d \partial_\beta (Q_{\alpha\beta}(\mathbf{x}, \mathbf{x}) \partial_\alpha f(\mathbf{x})),$$

is a second-order differential operator representing the enhanced dissipation due to eddy diffusion. Both operators A, A_Q generate strongly continuous semigroups, which we denote by e^{tA}, e^{tA_Q} (recall that $(e^{tA})_{t \geq 0}$ is a family of bounded linear maps in $L^2(D)$ such that $e^{(t+s)A} = e^{tA} e^{sA}$, e^{0A} is the identity operator and $(d/dt)e^{tA}h = Ae^{tA}h$ for every $h \in D(A)$; similarly for e^{tA_Q}); these semigroups are moreover analytic (see [16, §2.5]), but strictly speaking we do not need such fact. The generation of these semigroups is proved in [16, §7.2]. We use the semigroups to rewrite the stochastic equation in the so-called mild form, essential for the proof of our main theorem. The function $T_Q(t, \mathbf{x}) := (e^{tA_Q} T_0)(\mathbf{x})$ solves the modified heat equation

$$\partial_t T_Q(t, \mathbf{x}) = \operatorname{div}[(\kappa I + \tfrac{1}{2} Q(\mathbf{x}, \mathbf{x})) \nabla T_Q(t, \mathbf{x})], \quad (1.4)$$

and thus, in view of the following result, we may call $Q(\mathbf{x}, \mathbf{x})$ the **eddy diffusion coefficient**.

We have denoted above by $\kappa \lambda_D$ the first eigenvalue of $-A$; denote by $\lambda_{D, \kappa, Q}$ the first eigenvalue of $-A_Q$; a priori we only know that $\lambda_{D, \kappa, Q} \geq \kappa \lambda_D$.

Remark 1.1. $\mathcal{L}_Q$ is a degenerate elliptic operator: since $\mathbf{u}_j|_{\partial D} = 0$ we have also $Q|_{\partial D} = 0$. Therefore, it is not clear a priori that A_Q is more ‘elliptic’ than A . However, we shall prove that $\lambda_{D, \kappa, Q}$ can be much larger than $\kappa \lambda_D$.

Denote by $L_{\mathcal{F}_0}^2(\Omega; L^2(D))$ the space of square integrable random variables with values in $L^2(D)$, adapted to $\mathcal{F}_0$.

Theorem 1.2. Assume $T_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(D))$. Then, for every $\phi \in L^\infty(D)$,

$$\mathbb{E} \left[\left(\int_D \phi(\mathbf{x}) T(t, \mathbf{x}) \, d\mathbf{x} - \int_D \phi(\mathbf{x}) T_Q(t, \mathbf{x}) \, d\mathbf{x} \right)^2 \right] \leq \frac{\epsilon_Q}{2\kappa} \mathbb{E}[\|T_0\|_{L^2}^2] \|\phi\|_\infty^2,$$

where ϵ_Q is defined in (1.2). In particular, if $T_0 \geq 0$,

$$\mathbb{E} \left[\left(\int_D |T(t, \mathbf{x})| \, d\mathbf{x} \right)^2 \right] \leq \left(\frac{\epsilon_Q}{\kappa} + 2|D| \exp(-2\lambda_{D,\kappa,Q}t) \right) \mathbb{E}[\|T_0\|_{L^2}^2]. \quad (1.5)$$

Here, $|D|$ is the Lebesgue measure of D . This theorem shows (in a quantitative way) that decay is improved on *finite time intervals* $[0, \tau]$ if

- (i) ϵ_Q is very small,
- (ii) $\lambda_{D,\kappa,Q} \gg \kappa \lambda_D$.

The parameter τ can be computed by equating the coefficients on the right-hand side of (1.5)

$$\tau = \frac{1}{2\lambda_{D,\kappa,Q}} \log \frac{2\kappa|D|}{\epsilon_Q}.$$

For $t \in [0, \tau]$, it holds that $(\epsilon_Q/\kappa) \leq 2|D| \exp(-2\lambda_{D,\kappa,Q}t) \ll 2|D| \exp(-2\kappa\lambda_D t)$ and thus we have improved decay.

Denote by D_δ the set

$$D_\delta = \{\mathbf{x} \in D : \text{dist}(\mathbf{x}, \partial D) > \delta\},$$

and define

$$\sigma^2 = \inf_{\mathbf{x} \in D_\delta} q(\mathbf{x}). \quad (1.6)$$

For very general domains D , we have

Theorem 1.3. Let D be an open, bounded, Lipschitz domain in $\mathbb{R}^d$. Then, for any fixed $\kappa > 0$, it holds

$$\lim_{(\sigma, \delta) \rightarrow (+\infty, 0)} \lambda_{D,\kappa,Q} = +\infty.$$

Under more restrictive assumptions on the domain D , we may also provide the following quantitative lower bound on $\lambda_{D,\kappa,Q}$:

Theorem 1.4. There exists a constant $C_{D,d} > 0$ such that

$$\lambda_{D,\kappa,Q} \geq C_{D,d} \min \left(\sigma^2, \frac{\kappa}{\delta} \right),$$

for every Q such that

$$q(\mathbf{x}) \geq \sigma^2 \quad \text{in } D_\delta.$$

When D is the unit ball, asymptotically as $\delta \rightarrow 0$ one can take $C_{D,d} = d/2$ and one also has $\lambda_{D,\kappa,Q} \geq (\kappa d/(\kappa + \delta\sigma^2))\sigma^2$.

Recall that, when D is the unit ball, the first Dirichlet eigenvalue $\lambda_D = j_{d/2-1,1}^2$, where $j_{d/2-1,1}$ is the first zero of the Bessel function $J_{d/2-1}$ (see [17, p. 45, theorem 4]); in particular, if $d = 2$, then $j_{0,1} \approx 2.4048$.

We prove all the theorems above in §3. The consequence of the last two theorems is that $\lambda_{D,\kappa,Q}$ is large if the noise has a large intensity function $q(\mathbf{x})$, up to a small layer around the boundary ∂D . Summarizing, the information given by theorems 1.2–1.4 is that decay is improved on *finite time intervals* $[0, \tau]$ if

- (i) ϵ_Q is very small,
- (ii) $q(\mathbf{x})$ is large, except for a small layer around ∂D .

The question then is: Can we find a noise (namely a covariance function $Q(\mathbf{x}, \mathbf{y})$) with both properties, and possibly a similarity with the statistics observed in turbulent fluids?

Remark 1.5. Note that ϵ_Q , by definition, is given by the operator norm $\|\mathbb{Q}^{1/2}\|_{L^2 \rightarrow L^2}^2$ and thus, loosely speaking, it is related to the operator norm of $\mathbb{Q}$; and $q(\mathbf{x})$ is, loosely speaking, related to the trace of the operator $\mathbb{Q}$:

$$\text{Tr}(\mathbb{Q}) = \int_D \text{Tr} Q(\mathbf{x}, \mathbf{x}) \, d\mathbf{x}.$$

The requirement that ϵ_Q is small and $q(\mathbf{x})$ is large, heuristically translated into the requirements that the operator norm of $\mathbb{Q}$ is small and the trace is large is not strange: many operators have finite norm and infinite trace.

First, we would like to explain a heuristic idea which, however, we think of relevance. It deals with Kraichnan noise. This noise is usually considered in full space and only in that case we are able to verify that, for certain values of the parameters, it satisfies our conditions. Such noise however has been generalized to the case of no-slip boundary conditions in [5]; to fit with our assumption one has to modify the proposal of [5] to be divergence free and prove that it satisfies our assumptions. Although this is plausible, it is not trivial and postponed to future research. Consider the homogeneous covariance ($Q(\mathbf{x}, \mathbf{y}) = Q(\mathbf{x} - \mathbf{y})$) of Kraichnan type

$$Q(\mathbf{z}) = \sigma^2 k_0^\zeta \int_{k_0 \leq |\mathbf{k}| \leq k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} e^{i\mathbf{k} \cdot \mathbf{z}} \left(I - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2} \right) d\mathbf{k}.$$

There are two cases where conditions (i) and (ii) above are satisfied:

- if $\zeta > 0$, $k_1 = +\infty$, σ^2 large, and k_0 is so large that $\sigma^2 k_0^{-d}$ is small, then $q(\mathbf{x})$ is large and ϵ_Q is small; recall [3] that K41 is $\zeta = \frac{4}{3}$;
- if $-d \leq \zeta \leq 0$, $k_0 = 1$, σ^2 small, and k_1 is so large that $\sigma^2 \int_{1 \leq k \leq k_1} (1/k^{\zeta+1}) dk$ is large, then $q(\mathbf{x})$ is large and ϵ_Q is small; notice that $\zeta = -d$ is the case of white in space; and $\zeta = 0$ is, in dimension 2, the so-called enstrophy measure.

In §2a below, we prove these claims. The previous arguments require an excellent quantitative spectral knowledge which is not so obvious in bounded domains; one could work with the eigenfunctions and eigenvalues of Stokes operator, mimicking the previous claims, but it is difficult to have explicit information to control the quantities. We have preliminary results corresponding to the white noise case ($\zeta = -d$), not reported here. Below, §2b, we present a different class of noise which, we believe, is new, suitable for bounded domains and of interest in itself.

2. Vortex patch noise

The purpose of this section is the construction of a noise, in two dimensions, based on the idea of vortex patches. The reader will recognize that a similar construction can be done also in dimension 3 but the resulting objects look artificial, since coherent vortex structures in three dimensions are closer to curves and surfaces (see [18] for a construction of a mixing flow in a domain in $\mathbb{R}^3$ with no-slip boundaries). But before, in order to identify a key step, we show why Kraichnan noise works.

(a) Preliminaries on Kraichnan noise

Above we have claimed that Kraichnan noise produces large $q(\mathbf{x})$ and small ϵ_Q under certain conditions. Let us prove that claim because it requires a nontrivial argument in one step (see (2.1) and the discussions at the end of this part); missing that detail would spoil the understanding of

the vortex patch noise in §2b. The control, for Kraichnan noise, on $q(\mathbf{x})$ is given by

$$\begin{aligned}\xi^T Q(\mathbf{x}, \mathbf{x}) \xi &= \xi^T Q(0) \xi = \sigma^2 k_0^\zeta \int_{k_0 \leq |\mathbf{k}| \leq k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} \left(|\xi|^2 - \frac{(\mathbf{k} \cdot \xi)^2}{|\mathbf{k}|^2} \right) d\mathbf{k} \\ &\geq \frac{3}{4} |\xi|^2 \sigma^2 k_0^\zeta \int_{\substack{k_0 \leq |\mathbf{k}| \leq k_1 \\ |\mathbf{k} \cdot \xi| \leq |\mathbf{k}| |\xi|/2}} \frac{1}{|\mathbf{k}|^{d+\zeta}} d\mathbf{k} \\ &= \frac{3}{4} |\xi|^2 \sigma^2 k_0^\zeta C \int_{k_0}^{k_1} \frac{1}{r^{d+\zeta}} r^{d-1} dr = \frac{3}{4} |\xi|^2 \sigma^2 C' \left(1 - \left(\frac{k_0}{k_1} \right)^\zeta \right),\end{aligned}$$

for suitable constants $C, C' > 0$. The control on ϵ_Q , is given by

$$\begin{aligned}\iint \mathbf{v}(\mathbf{x})^T Q(\mathbf{x}, \mathbf{y}) \mathbf{v}(\mathbf{y}) d\mathbf{x} d\mathbf{y} &= \sigma^2 k_0^\zeta \int_{k_0 \leq |\mathbf{k}| \leq k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} \left(|\widehat{\mathbf{v}}(\mathbf{k})|^2 - \frac{|\mathbf{k} \cdot \widehat{\mathbf{v}}(\mathbf{k})|^2}{|\mathbf{k}|^2} \right) d\mathbf{k} \\ &\leq \sigma^2 k_0^{-d} \int_{k_0 \leq |\mathbf{k}| \leq k_1} |\widehat{\mathbf{v}}(\mathbf{k})|^2 d\mathbf{k} \leq \sigma^2 k_0^{-d} \|\mathbf{v}\|_{L^2}^2.\end{aligned}\quad (2.1)$$

It is here that one step must be performed in the right way. If we just estimate from above as

$$\iint \mathbf{v}(\mathbf{x})^T Q(\mathbf{x}, \mathbf{y}) \mathbf{v}(\mathbf{y}) d\mathbf{x} d\mathbf{y} \leq \iint \sigma^2 k_0^\zeta \int_{k_0 \leq |\mathbf{k}| \leq k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} |\mathbf{v}(\mathbf{x})| |\mathbf{v}(\mathbf{y})| d\mathbf{k} d\mathbf{x} d\mathbf{y},$$

then, first, we are in trouble since the L^1 norm of $\mathbf{v}$ is difficult to estimate. Second, even if the space domain is a Torus (in this case, the integral over wavenumbers is a series) we would end-up with an estimate of the form

$$\leq \sigma^2 k_0^\zeta \|\mathbf{v}\|_{L^2}^2 \sum_{k_0 \leq |\mathbf{k}| \leq k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} \leq C \sigma^2 \|\mathbf{v}\|_{L^2}^2 \left(1 - \left(\frac{k_0}{k_1} \right)^\zeta \right),$$

which is not sufficient. The result would be that there is no difference in estimating the norm or the trace. The key is using the presence of an orthonormal family of functions (here $e^{i\mathbf{k} \cdot \mathbf{z}}$).

(b) The vortex noise in two dimensions

Thus consider $d=2$ and assume that D is a smooth bounded connected open domain. We are going to describe a noise of the form $\sum_{j \in J} \mathbf{u}_j(\mathbf{x}) dW_t^j$ with

$$\mathbf{u}_j(\mathbf{x}) = \mathbf{w}_r(\mathbf{x} - \mathbf{x}_j), \quad \mathbf{w}_r(\mathbf{x}) = r^{-1} \mathbf{w} \left(\frac{\mathbf{x}}{r} \right),$$

for suitable r and $\mathbf{w}$. The ingredients are therefore the points $\mathbf{x}_j$, called the ‘centres’ of the vortex blobs below, and a vector field $\mathbf{w}$.

(i) The centres of the vortex blobs

Given a positive integer N such that $(1/N) \leq \delta$, consider the set Λ_N of all points of D_δ having coordinates of the form $(k/N, h/N)$ with $k, h \in \mathbb{Z}$. For the purpose of the example developed here, the centres $\mathbf{x}_j$ of the blobs will be taken equal to the points of Λ_N ; with some effort one can generalize to more flexible distributions of points, also random.

The index set J will be Λ_N itself and points of Λ_N will be denoted by $\mathbf{z}$. Notations below in this section will adapt to this choice; for instance, we write the noise in the form

$$\Gamma \sum_{\mathbf{z} \in \Lambda_N} \mathbf{w}_r(\mathbf{x} - \mathbf{z}) dW_t^{\mathbf{z}}.$$

We have

$$\min_{\mathbf{z}_1 \neq \mathbf{z}_2 \in \Lambda_N} |\mathbf{z}_1 - \mathbf{z}_2| = \frac{1}{N}, \quad \min_{\mathbf{z} \in \Lambda_N} d(\mathbf{z}, \partial D) \geq \delta.$$

Given a positive integer M (in the sequel M will be finite, while $N \rightarrow \infty$), the set Λ_N is decomposed as the disjoint union of the sets

$$\Lambda_N = \bigcup_{(k_0, h_0) \in \{0, 1, \dots, M-1\}^2} \Lambda_N^{(M, k_0, h_0)},$$

defined as follows: the points $(k/N, h/N)$ of $\Lambda_N^{(M, k_0, h_0)}$ have the property that $k = Mn + k_0$, $h = Mm + h_0$, with $n, m \in \mathbb{Z}$. Therefore,

$$\min_{\mathbf{z}_1 \neq \mathbf{z}_2 \in \Lambda_N^{(M, k_0, h_0)}} |\mathbf{z}_1 - \mathbf{z}_2| = \frac{M}{N},$$

for each $(k_0, h_0) \in \{0, 1, \dots, M-1\}^2$.

(ii) The vector field $\mathbf{w}$

The construction of vector field $\mathbf{w}$ requires some care. First, in order to have that $\sum_{\mathbf{z} \in \Lambda_N} \mathbf{w}_r(\mathbf{x} - \mathbf{z}) dW_t^{\mathbf{z}}$ is an admissible noise for our investigation, we need that each $\mathbf{u}_{\mathbf{z}}(\mathbf{x}) := \mathbf{w}_r(\mathbf{x} - \mathbf{z})$ is divergence free, smooth enough and zero at ∂D . Therefore, we need $\operatorname{div} \mathbf{w} = 0$, $\mathbf{w}$ smooth enough; and we look for a vector field with compact support, say in the closed ball $\overline{B(0, 1)}$, so that for $r \in (0, \delta)$ and $\mathbf{z} \in \Lambda_N \subset D_\delta$ the rescaled and shifted vector field $\mathbf{w}_r(\mathbf{x} - \mathbf{z})$ is zero on ∂D . Moreover, we need other two properties.

One is that $\mathbf{w}(\mathbf{x})$ is close to $(1/2\pi)(\mathbf{x}^\perp/|\mathbf{x}|^2)$ near $\mathbf{x} = 0$; this is central to the proof that the function $q(\mathbf{x})$ is large. The other is that the vector fields $\mathbf{w}_r(\mathbf{x} - \mathbf{z})$ are (up to the constant $\int |\mathbf{w}(\mathbf{x})|^2 d\mathbf{x}$, which is not zero since $\mathbf{w}$ is close to $(1/2\pi)(\mathbf{x}^\perp/|\mathbf{x}|^2)$ near $\mathbf{x} = 0$) ‘almost’ orthonormal in L^2 , which is guaranteed by the fact that the supports are ‘almost’ disjoint. To be precise, if we take truly disjoint supports, then the action of $\mathbf{w}_r(\mathbf{x} - \mathbf{z})$ does not cover the full set $D_{2\delta}$: there are intermediate zones between the supports, where $\mathbf{w}_r(\mathbf{x} - \mathbf{z})$ does not move space points and this is in contrast with the requirement that $q(\mathbf{x})$ should be large everywhere in $D_{2\delta}$. This is why we have introduced M and the sets $\Lambda_N^{(M, k_0, h_0)}$ above: inside each one of these classes the supports will be disjoint and this is sufficient for our estimates; in order to have the supports disjoint for elements of $\Lambda_N^{(M, k_0, h_0)}$ we ask $r \leq (M/2N)$.

Therefore, summarizing, we look for a vector field $\mathbf{w}$, defined on $\mathbb{R}^2$, smooth, with compact support in $\overline{B(0, 1)}$, $\operatorname{div} \mathbf{w} = 0$, close to $(1/2\pi)(\mathbf{x}^\perp/|\mathbf{x}|^2)$ near $\mathbf{x} = 0$. We construct it as

$$\mathbf{w} = \nabla^\perp \psi,$$

so that it is divergence free. Thus we look for a smooth function ψ on $\mathbb{R}^2$, compactly supported in $\overline{B(0, 1)}$, close to $(1/2\pi) \log |\mathbf{x}|$ near $\mathbf{x} = 0$. Such function exists and can be constructed as follows.

Let $\psi_0 \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ be a radial function such that

$$\psi_0(\mathbf{x}) = \frac{1}{2\pi} \log |\mathbf{x}| \text{ for } |\mathbf{x}| \leq \frac{1}{3} \quad \text{and} \quad \psi_0(\mathbf{x}) = 0 \text{ for } |\mathbf{x}| > \frac{2}{3}.$$

Let $f \in C^\infty(\mathbb{R}^2)$ be a probability density function with support in $B(0, 1)$. Given $\epsilon > 0$ small (at least $\epsilon < \frac{1}{6}$), define

$$f_\epsilon(\mathbf{x}) = \epsilon^{-2} f\left(\frac{\mathbf{x}}{\epsilon}\right) \quad \text{and} \quad \psi(\mathbf{x}) = \int_{\mathbb{R}^2} \psi_0(\mathbf{x} - \mathbf{y}) f_\epsilon(\mathbf{y}) d\mathbf{y}.$$

This function satisfies our requirements: its support is in $\overline{B(0, 1)}$, it is smooth everywhere and, if we take ϵ small, it is close to ψ_0 which is equal to $(1/2\pi) \log |\mathbf{x}|$ near $\mathbf{x} = 0$. The corresponding vector field $\mathbf{w} = \nabla^\perp \psi$ has the required properties.

Therefore, if $|\mathbf{x}| \leq \frac{1}{6}$ and $\epsilon < \frac{1}{6}$ (so that the support of f_ϵ is in $B(0, \frac{1}{6})$) we have

$$\mathbf{w}(\mathbf{x}) = \int_{\mathbb{R}^2} \nabla^\perp \psi_0(\mathbf{x} - \mathbf{y}) f_\epsilon(\mathbf{y}) \, d\mathbf{y} = \int_{\mathbb{R}^2} \frac{1}{2\pi} \frac{(\mathbf{x} - \mathbf{y})^\perp}{|\mathbf{x} - \mathbf{y}|^2} f_\epsilon(\mathbf{y}) \, d\mathbf{y}$$

and

$$\mathbf{w}_r(\mathbf{x}) = \frac{1}{2\pi r} \int_{\mathbb{R}^2} \frac{(\mathbf{x}/r - \mathbf{y})^\perp}{|\mathbf{x}/r - \mathbf{y}|^2} f_\epsilon(\mathbf{y}) \, d\mathbf{y} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\mathbf{x} - \mathbf{y})^\perp}{|\mathbf{x} - \mathbf{y}|^2} (\epsilon r)^{-2} f\left(\frac{\mathbf{y}}{\epsilon r}\right) \, d\mathbf{y}.$$

(iii) Estimates on $q(\mathbf{x})$ and ϵ_Q

We now check that, with proper choices of the parameters M , ϵ , Γ and also r , the noise $\Gamma \sum_{\mathbf{z} \in \Lambda_N} \mathbf{w}_r(\mathbf{x} - \mathbf{z}) \, dW_t^{\mathbf{z}}$ with $\mathbf{w}_r(\mathbf{x}) = r^{-1} \mathbf{w}(\mathbf{x}/r)$ has large $q(\mathbf{x})$ and small ϵ_Q .

We choose r with more than one constraint. We have already assumed above

$$r \leq \frac{M}{2N}, \quad r \leq \delta.$$

The first inequality implies that the supports of $\mathbf{w}_r(\mathbf{x} - \mathbf{z})$ are disjoint for $\mathbf{z}$ in the same subset $\Lambda_N^{(M, k_0, j_0)}$. The second inequality implies that they are zero at the boundary of D .

The covariance of this noise is

$$Q(\mathbf{x}, \mathbf{y}) = \Gamma^2 \sum_{\mathbf{z} \in \Lambda_N} \mathbf{w}_r(\mathbf{x} - \mathbf{z}) \otimes \mathbf{w}_r(\mathbf{y} - \mathbf{z}).$$

We therefore have, for the estimate of ϵ_Q ,

$$\begin{aligned} \iint \mathbf{v}(\mathbf{x})^T Q(\mathbf{x}, \mathbf{y}) \mathbf{v}(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} &= \Gamma^2 \sum_{\mathbf{z} \in \Lambda_N} \left(\int \mathbf{w}_r(\mathbf{x} - \mathbf{z}) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \right)^2 \\ &= \|\mathbf{w}\|_{L^2}^2 \Gamma^2 \sum_{(k_0, j_0) \in \{0, 1, \dots, M-1\}^2} \sum_{\mathbf{z} \in \Lambda_N^{(M, k_0, j_0)}} \left(\int \frac{\mathbf{w}_r(\mathbf{x} - \mathbf{z})}{\|\mathbf{w}\|_{L^2}} \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \right)^2 \\ &\leq M^2 \|\mathbf{w}\|_{L^2}^2 \Gamma^2 \|\mathbf{v}\|_{L^2}^2. \end{aligned}$$

We have used a basic property, similarly to the most important step in the verification done above for Kraichnan noise: the family $\{(\mathbf{w}_r(\mathbf{x} - \mathbf{z})/\|\mathbf{w}\|_{L^2})\}_{\mathbf{z} \in \Lambda_N^{(M, k_0, j_0)}}$ is orthonormal (not complete), because of the disjoint supports and the property $\int |\mathbf{w}_r(\mathbf{x})|^2 \, d\mathbf{x} = \|\mathbf{w}\|_{L^2}^2$. One can easily check that

$$\|\mathbf{w}\|_{L^2}^2 \leq C \log \frac{1}{\epsilon},$$

and, therefore, taking $\epsilon = 1/N$ leads to

$$\epsilon_Q \leq M^2 \Gamma^2 C \log N,$$

which is small if, given N , Γ is small enough.

Concerning $q(\mathbf{x})$, we have, for every $\mathbf{x} \in D$ and every unitary vector $\mathbf{v} \in \mathbb{R}^2$,

$$\mathbf{v}^T Q(\mathbf{x}, \mathbf{x}) \mathbf{v} = \Gamma^2 \sum_{\mathbf{z} \in \Lambda_N} (\mathbf{w}_r(\mathbf{x} - \mathbf{z}) \cdot \mathbf{v})^2.$$

Now, consider a point $\mathbf{x} \in D_{2\delta}$. If N is large enough with respect to the curvature of ∂D near $\mathbf{x}$, we may find $\mathbf{z} \in \Lambda_N$ close to $\mathbf{x}$, precisely with $1/2N \leq |\mathbf{x} - \mathbf{z}| < 2/N$, such that

$$\left| \mathbf{v} \cdot \frac{(\mathbf{x} - \mathbf{z})^\perp}{|\mathbf{x} - \mathbf{z}|} \right| \geq \frac{1}{4}.$$

Then, if $|\mathbf{x} - \mathbf{z}|/r \leq \frac{1}{6}$, which is true if $(2/rN) \leq \frac{1}{6}$, namely $r \geq 12/N$,

$$|\mathbf{w}_r(\mathbf{x} - \mathbf{z}) \cdot \mathbf{v}| = \left| \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathbf{v} \cdot \frac{(\mathbf{x} - \mathbf{z} - \mathbf{y})^\perp}{|\mathbf{x} - \mathbf{z} - \mathbf{y}|^2} (\epsilon r)^{-2} f\left(\frac{\mathbf{y}}{\epsilon r}\right) \, d\mathbf{y} \right|.$$

The constraints $r \leq M/2N$, $r \leq \delta$, $r \geq 12/N$ are all satisfiable if we take $M > 24$ and N large enough; of course, we may reduce quantitatively the constraint $M > 24$ by different choices of some parameters above. Recalling that $\epsilon = 1/N$; in the above integral, we have $|\mathbf{y}| \leq \epsilon r \sim 1/N^2$ which means that $\mathbf{y}$ is an infinitesimal perturbation of $\mathbf{x} - \mathbf{z}$ due to $|\mathbf{x} - \mathbf{z}| \sim 1/N$. Thus, for N big enough the last integral is bounded below by

$$\frac{1}{2\pi} \frac{1}{8} N = \frac{N}{16\pi}.$$

It follows

$$q(\mathbf{x}) \geq \frac{\Gamma^2 N}{16\pi}.$$

Therefore, we may choose N and Γ so that ϵ_Q is small as we want and $q(\mathbf{x})$, on $D_{2\delta}$, is large as we want.

3. Proofs

For reasons of space, we omit some secondary details in the following proofs; for instance we do not write explicitly the definition of solution, the proof that energy and maximum principle estimates are satisfied, the proof that we may pass from the weak to the mild formulation. We only mention that the energy estimate follows by applying the Stratonovich calculus (the same as ordinary calculus) to (1.3), while the maximum principle is a consequence of the fact that $Q(\mathbf{x}, \mathbf{x})$ is non-negative definite for all $\mathbf{x} \in D$.

(a) Proof of theorem 1.2

The first key ingredient is the reformulation of the Stratonovich equation in Itô form

$$d_t T = (\kappa \Delta T + \tilde{\mathcal{L}}_Q T) dt + \sum_{j \in J} \mathbf{u}_j \cdot \nabla T dW_t^j,$$

where

$$(\tilde{\mathcal{L}}_Q T)(\mathbf{x}) := \frac{1}{2} \sum_{j \in J} \mathbf{u}_j(\mathbf{x}) \cdot \nabla (\mathbf{u}_j(\mathbf{x}) \cdot \nabla T(\mathbf{x})).$$

One has

$$\tilde{\mathcal{L}}_Q = \mathcal{L}_Q.$$

This is a well-known fact, see for instance [19]; indeed

$$\tilde{\mathcal{L}}_Q T = \frac{1}{2} \sum_{j \in J} \sum_{\alpha, \beta=1}^d u_j^\alpha \partial_\alpha u_j^\beta \partial_\beta T + \frac{1}{2} \sum_{j \in J} \sum_{\alpha, \beta=1}^d u_j^\alpha u_j^\beta \partial_\alpha \partial_\beta T.$$

The second sum is equal to $\frac{1}{2} \sum_{\alpha, \beta=1}^d Q_{\alpha\beta}(\mathbf{x}, \mathbf{x}) \partial_\alpha \partial_\beta T$. The first one, due to the property $\operatorname{div} \mathbf{u}_j = 0$, is equal to

$$\frac{1}{2} \sum_{j \in J} \sum_{\alpha, \beta=1}^d \partial_\alpha (u_j^\alpha u_j^\beta) \partial_\beta T = \frac{1}{2} \sum_{\alpha, \beta=1}^d \partial_\alpha Q_{\alpha\beta}(\mathbf{x}, \mathbf{x}) \partial_\beta T,$$

where we have also used the assumptions of uniform convergence of the series of the derivatives.

Recall that $T_Q(t, \mathbf{x}) := (e^{tA_Q} T_0)(\mathbf{x})$, see (1.4); from the previous facts we have

$$d_t(T - T_Q) = (\kappa \Delta + \mathcal{L}_Q)(T - T_Q) dt + \sum_{j \in J} \mathbf{u}_j \cdot \nabla T dW_t^j.$$

The mild formulation of this identity, furthermore applied in a weak sense to a smooth test function ϕ with compact support in D , is

$$\langle \phi, T(t) - T_Q(t) \rangle = \sum_{j \in J} \int_0^t \left\langle e^{(t-s)A_Q} \phi, \mathbf{u}_j \cdot \nabla T(s) \right\rangle dW_s^j,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(D)$ and we have also used the fact that the semigroup $e^{(t-s)A_Q}$ is self-adjoint. By the isometry formula for Itô integrals,

$$\mathbb{E} \left[\langle \phi, T(t) - T_Q(t) \rangle^2 \right] = \sum_{j \in J} \int_0^t \mathbb{E} \left[\left\langle e^{(t-s)A_Q} \phi, \mathbf{u}_j \cdot \nabla T(s) \right\rangle^2 \right] ds.$$

We have (we write $T_s(\mathbf{x})$ for $T(s, \mathbf{x})$ and $\phi_{t,s}(\mathbf{x})$ for $(e^{(t-s)A_Q} \phi)(\mathbf{x})$ to shorten notations)

$$\sum_{j \in J} \left\langle e^{(t-s)A_Q} \phi, \mathbf{u}_j \cdot \nabla T_s \right\rangle^2 = \sum_{\alpha, \beta=1}^d \int_D \int_D \phi_{t,s}(\mathbf{x}) \phi_{t,s}(\mathbf{y}) Q_{\alpha\beta}(\mathbf{x}, \mathbf{y}) \partial_\alpha T_s(\mathbf{x}) \partial_\beta T_s(\mathbf{y}) d\mathbf{x} d\mathbf{y}.$$

The semigroup e^{tA_Q} satisfies the maximum principle, namely $\|e^{tA_Q} \phi\|_\infty \leq \|\phi\|_\infty$. Hence, recalling the definition of $\mathbb{Q}$ and ϵ_Q ,

$$\begin{aligned} \sum_{j \in J} \left\langle e^{(t-s)A_Q} \phi, \mathbf{u}_j \cdot \nabla T_s \right\rangle^2 &\leq \epsilon_Q \int_D |\phi_{t,s}(\mathbf{x}) \nabla T_s(\mathbf{x})|^2 d\mathbf{x} \\ &\leq \|\phi\|_\infty^2 \epsilon_Q \int_D |\nabla T(s, \mathbf{x})|^2 d\mathbf{x}. \end{aligned}$$

Moreover, applying Stratonovich calculus to the stochastic equation (1.3) and using the fact that $\mathbf{u}_j$'s are divergence free, we obtain the inequality

$$\int_0^\infty \int_D |\nabla T(t, \mathbf{x})|^2 d\mathbf{x} dt \leq \frac{1}{2\kappa} \int_D T_0^2(\mathbf{x}) d\mathbf{x}.$$

Together they imply

$$\mathbb{E} \left[\langle \phi, T(t) - T_Q(t) \rangle^2 \right] \leq \frac{\epsilon_Q}{2\kappa} \mathbb{E}[\|T_0\|_{L^2}^2] \|\phi\|_\infty^2.$$

If $T_0 \geq 0$, then both $T(t)$ and $T_Q(t)$ are non-negative. Choose a sequence ϕ_n converging to 1 in D . We deduce

$$\mathbb{E} \left[\left(\int_D T(t, \mathbf{x}) d\mathbf{x} - \langle 1, e^{tA_Q} T_0 \rangle \right)^2 \right] \leq \frac{\epsilon_Q}{2\kappa} \mathbb{E}[\|T_0\|_{L^2}^2].$$

It implies

$$\begin{aligned} \mathbb{E} \left[\left(\int_D |T(t, \mathbf{x})| d\mathbf{x} \right)^2 \right] &\leq 2 \frac{\epsilon_Q}{2\kappa} \mathbb{E}[\|T_0\|_{L^2}^2] + 2 \mathbb{E} \left[\langle 1, e^{tA_Q} T_0 \rangle_{L^2}^2 \right] \\ &\leq \frac{\epsilon_Q}{\kappa} \mathbb{E}[\|T_0\|_{L^2}^2] + 2|D| \mathbb{E}[\|e^{tA_Q} T_0\|_{L^2}^2] \\ &\leq \left(\frac{\epsilon_Q}{\kappa} + 2|D| \exp(-2\lambda_{D,\kappa} Q t) \right) \mathbb{E}[\|T_0\|_{L^2}^2]. \end{aligned}$$

(b) Proof of theorems 1.3 and 1.4

(i) Proof of theorem 1.3

Note that the operator A_Q is self-adjoint, we can use the variational characterization of $\lambda_{D,\kappa,Q}$ given by

$$\lambda_{D,\kappa,Q} = \inf_{T \in H_0^1(D): \int_D T^2 dx = 1} \int_D \sum_{\alpha,\beta=1}^d (\kappa \delta_{\alpha\beta} + Q_{\alpha\beta}(\mathbf{x}, \mathbf{x})) \partial_\alpha T(\mathbf{x}) \partial_\beta T(\mathbf{x}) dx.$$

Recalling the definition of σ^2 in (1.6), we have $\lambda_{D,\kappa,Q} \geq \lambda_{\kappa,\sigma,\delta}$ where

$$\lambda_{\kappa,\sigma,\delta} := \inf_{T \in H_0^1(D): \int_D T^2 dx = 1} \int_D (\kappa + \sigma^2 \cdot 1_{D_\delta}(\mathbf{x})) |\nabla T(\mathbf{x})|^2 dx.$$

We want to prove that

$$\lim_{(\sigma,\delta) \rightarrow (+\infty,0)} \lambda_{\kappa,\sigma,\delta} = +\infty.$$

Suppose this is not true, then we can find $C > 0$ and a sequence $(\sigma_n, \delta_n) \rightarrow (+\infty, 0)$ such that $\lambda_n := \lambda_{\kappa,\sigma_n,\delta_n} \leq C$; classical facts imply that there exists a minimizer $T_n \in H_0^1(D)$ such that $\|T_n\|_{L^2} = 1$ and

$$\int_D (\kappa + \sigma_n^2 \cdot 1_{D_{\delta_n}}(\mathbf{x})) |\nabla T_n(\mathbf{x})|^2 dx = \lambda_n \leq C \quad \forall n \in \mathbb{N}.$$

We deduce as a consequence that $\int_D |\nabla T_n|^2 dx \leq \kappa^{-1} C$ and the sequence $\{T_n\}_n$ is bounded in $H_0^1(D)$; by Rellich–Kondrakhov compactness theorem for $H_0^1(D)$, we can extract a (not relabelled) subsequence such that $T_n \rightarrow T$ strongly in $L^2(D)$ and $\nabla T_n \rightarrow \nabla T$ weakly in $L^2(D)$ for a suitable $T \in H_0^1(D)$. On the other hand,

$$\int_{D_{\delta_n}} |\nabla T_n(\mathbf{x})|^2 dx \leq \frac{C}{\sigma_n^2} \rightarrow 0,$$

which together with $D_\varepsilon \subset D_{\delta_n}$ for n large enough implies that $\nabla T(\mathbf{x}) = 0$ for a.e. $x \in D_\varepsilon$ and for any $\varepsilon > 0$. Overall, this implies that $\|T\|_{L^2} = 1$ and $\nabla T = 0$, thus T is a constant function which is 0 at the boundary ∂D , giving a contradiction.

(ii) Preparation to the proof of theorem 1.4

We give the proof of theorem 1.4 only in the case of the ball $D = B(0, 1)$. The case of a star-shaped domain with smooth boundary can be reduced to the ball by relatively easy arguments. We think that the result is true for much more general domains but the details are outside the scope of this work.

Therefore, now we have (with the notations of the previous section)

$$\lambda_{\kappa,\sigma,\delta} = \inf_{T \in H_0^1(B(0,1)): \int_{B(0,1)} T^2 dx = 1} \int_{B(0,1)} (\kappa + \sigma^2 \cdot 1_{B(0,1-\delta)}(\mathbf{x})) |\nabla T(\mathbf{x})|^2 dx.$$

Again by classical results, there is a unique minimizer for the variational problem which defines $\lambda_{\kappa,\sigma,\delta}$, and it is non-negative. Denote it by $T_{\kappa,\sigma,\delta}$. Since the functional is invariant by rotation, uniqueness implies that also the minimizer is invariant by rotation. Then

$$T_{\kappa,\sigma,\delta}(\mathbf{x}) = f_{\kappa,\sigma,\delta}(|\mathbf{x}|),$$

for some function $f_{\kappa,\sigma,\delta} \in H^1(0, 1)$. Called ω_d the surface of the unit sphere in $\mathbb{R}^d$, we have $\lambda_{\kappa,\sigma,\delta} = \omega_d \inf J(f)$,

$$J(f) = \kappa \int_0^1 f'(r)^2 r^{d-1} dr + \sigma^2 \int_0^{1-\delta} f'(r)^2 r^{d-1} dr, \quad (3.1)$$

the infimum being taken over all $f \in H^1(0, 1)$ such that $f(1) = 0$ and $\int_0^1 f(r)^2 r^{d-1} dr = 1/\omega_d$. The function $f_{\kappa,\sigma,\delta}$, non-negative, is non-increasing; let us prove this by contradiction. Indeed, if there are $r_1 < r_2$ with $f_{\kappa,\sigma,\delta}(r_1) < f_{\kappa,\sigma,\delta}(r_2)$, by continuity of $f_{\kappa,\sigma,\delta}$ (it is of class $H^1(0, 1)$) there exists a point

$r_{\min} < r_2$ of minimum in $[0, r_2]$, with $f_{\kappa, \sigma, \delta}(r_{\min}) < f_{\kappa, \sigma, \delta}(r_2)$. Given $l \in (f_{\kappa, \sigma, \delta}(r_{\min}), f_{\kappa, \sigma, \delta}(r_2))$, let r_l^+ be the minimum of all points $r > r_{\min}$ such that $f_{\kappa, \sigma, \delta}(r) = l$; it exists again by continuity of $f_{\kappa, \sigma, \delta}$. If $r_{\min} = 0$, we complete the contradiction as follows: for any such l we introduce the function $\tilde{f}_l$ equal to $f_{\kappa, \sigma, \delta}$ on $[r_l^+, 1]$ and constantly equal to $f_{\kappa, \sigma, \delta}(r_l^+) = l$ in $[0, r_l^+]$. It is of class $H^1(0, 1)$, $\tilde{f}_l(1) = 0$, $J(\tilde{f}_l) < J(f_{\kappa, \sigma, \delta})$. In itself this is not a contradiction yet because $a_l^2 := \int_0^1 \tilde{f}_l(r)^2 r^{d-1} dr$ is not equal to $1/\omega_d$; but $a_l^2 > 1/\omega_d$, because $\tilde{f}_l(r) = f_{\kappa, \sigma, \delta}(r_l^+) > f_{\kappa, \sigma, \delta}(r)$ in $[0, r_l^+]$; hence the function $\tilde{f}_l = \tilde{f}_l/(|a_l| \sqrt{\omega_d})$ satisfies all the constraints and has the property $J(\tilde{f}_l) < J(f_{\kappa, \sigma, \delta})$; hence $J(\tilde{f}_l) < J(f_{\kappa, \sigma, \delta})$. If $r_{\min} > 0$, it is sufficient to introduce the maximum r_l^- of all points $r < r_{\min}$ such that $f_{\kappa, \sigma, \delta}(r) = l$ and repeat the previous argument on $[r_l^-, r_l^+]$ instead of $[0, r_l^+]$. In this way, we conclude that the function $f_{\kappa, \sigma, \delta}$ is non-increasing.

(iii) Proof of theorem 1.4

Therefore, $\lambda_{\kappa, \sigma, \delta} = \omega_d J(f_{\kappa, \sigma, \delta})$, where we know that $f := f_{\kappa, \sigma, \delta}$ is of class $H^1(0, 1)$, non-negative and non-increasing, $f(1) = 0$, $\int_0^1 f(r)^2 r^{d-1} dr = 1/\omega_d$. We prove now several inequalities, some of them inspired by the Poincaré inequality. First,

$$\begin{aligned} \delta \int_0^1 f'(r)^2 r^{d-1} dr &\geq \delta \int_{1-\delta}^1 f'(r)^2 r^{d-1} dr \geq (1-\delta)^{d-1} \left(\int_{1-\delta}^1 |f'(r)| dr \right)^2 \\ &= (1-\delta)^{d-1} \left(- \int_{1-\delta}^1 f'(r) dr \right)^2 = (1-\delta)^{d-1} f^2(1-\delta). \end{aligned} \quad (3.2)$$

Second, since $f(r) = f(1-\delta) - \int_r^{1-\delta} f'(s) ds$, we have

$$f(r)^2 \leq (1+\gamma) f^2(1-\delta) + (1+\gamma^{-1}) \int_r^{1-\delta} f'(s)^2 ds,$$

for all $\gamma > 0$, hence we get, for $g(\delta) := \int_{1-\delta}^1 f(r)^2 r^{d-1} dr$,

$$\begin{aligned} \frac{1}{\omega_d} - g(\delta) &= \int_0^{1-\delta} f(r)^2 r^{d-1} dr \\ &\leq \frac{1+\gamma}{d} f^2(1-\delta) + (1+\gamma^{-1}) \int_0^{1-\delta} \int_r^{1-\delta} f'(s)^2 r^{d-1} ds dr. \end{aligned}$$

The double integral can be manipulated and shown to be equal to $(1/d) \int_0^{1-\delta} s^d f'(s)^2 ds$, and a factor s in this integral can be bounded above by 1. Note also that, by monotonicity of f , $g(\delta) \leq \delta f^2(1-\delta)$. We deduce

$$\int_0^{1-\delta} s^{d-1} f'(s)^2 ds \geq \frac{d}{(1+\gamma^{-1})\omega_d} - \frac{1+\gamma}{1+\gamma^{-1}} f^2(1-\delta) - \frac{\delta d}{1+\gamma^{-1}} f^2(1-\delta).$$

Therefore, combining this inequality with (3.2) and (3.1) yields

$$\lambda_{\kappa, \sigma, \delta} \geq \omega_d \left(\frac{\kappa}{\delta} (1-\delta)^{d-1} - \sigma^2 \frac{1+\gamma+\delta d}{1+\gamma^{-1}} \right) f^2(1-\delta) + \frac{d}{1+\gamma^{-1}} \sigma^2.$$

We now choose γ such that

$$\frac{1+\gamma+\delta d}{1+\gamma^{-1}} = \frac{\kappa}{\delta \sigma^2} (1-\delta)^{d-1},$$

which is easily seen to be always possible. With this choice we have $\lambda_{\kappa, \sigma, \delta} \geq (d/(1+\gamma^{-1}))\sigma^2$. The algebraic computations to complete the proof of the theorem are now elementary but cumbersome, so let us give them only asymptotically as $\delta \rightarrow 0$. We thus have $(1+\gamma)/(1+\gamma^{-1}) = \kappa/\delta\sigma^2$ which gives $\gamma = \kappa/\delta\sigma^2$; hence $\lambda_{\kappa, \sigma, \delta} \geq \kappa d/(\kappa + \delta\sigma^2)\sigma^2$, as stated in the theorem. It can also be rewritten as

$$\lambda_{\kappa, \sigma, \delta} \geq \frac{(\kappa/\delta)d}{(\kappa/\delta) + \sigma^2} \sigma^2,$$

which easily proves it is larger than $(d/2) \min(\sigma^2, \kappa/\delta)$ (if $\sigma^2 \leq \kappa/\delta$, then $(\kappa/\delta)d/((\kappa/\delta) + \sigma^2) \geq (\kappa/\delta)d/2(\kappa/\delta) = d/2$; similarly in the opposite case).

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References

1. Kraichnan RH. 1994 Anomalous scaling of a randomly advected passive scalar. *Phys. Rev. Lett.* **72**, 1016–1019. (doi:10.1103/PhysRevLett.72.1016)
2. Gawedzki K, Kupiainen A. 1995 Anomalous scaling of the passive scalar. *Phys. Rev. Lett.* **75**, 3834–3837. (doi:10.1103/PhysRevLett.75.3834)
3. Frisch U. 1995 *Turbulence. The legacy of A. N. Kolmogorov*. Cambridge, UK: Cambridge University Press.
4. Chertkov M, Falkovich G. 1996 Anomalous scaling exponents of a white-advected passive scalar. *Phys. Rev. Lett.* **76**, 2706–2709. (doi:10.1103/PhysRevLett.76.2706)
5. Eyink GL, Xin J. 1996 Existence and uniqueness of L^2 -solutions at zero-diffusivity in the Kraichnan model of a passive scalar. Preprint (<https://arxiv.org/abs/chao-dyn/9605008>).
6. Frisch U, Mazzino A, Vergassola M. 1998 Intermittency in passive scalar advection. *Phys. Rev. Lett.* **80**, 5532–5535. (doi:10.1103/PhysRevLett.80.5532)
7. Majda AJ, Kramer PR. 1999 Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena. *Phys. Rep.* **314**, 237–574. (doi:10.1016/S0370-1573(98)00083-0)
8. Le Jan Y, Raimond O. 2002 Integration of Brownian vector fields. *Ann. Probab.* **30**, 826–873. (doi:10.1214/aop/1023481009)
9. Sreenivasan KR. 2019 Turbulent mixing: a perspective. *Proc. Natl Acad. Sci. USA* **116**, 18 175–18 183. (doi:10.1073/pnas.1800463115)
10. Galeati L. 2020 On the convergence of stochastic transport equations to a deterministic parabolic one. *Stoch. Partial Differ. Equ. Anal. Comput.* **8**, 833–868. (doi:10.1007/s40072-019-00162-6)
11. Flandoli F, Galeati L, Luo D. 2021 Scaling limit of stochastic 2D Euler equations with transport noises to the deterministic Navier-Stokes equations. *J. Evol. Equ.* **21**, 567–600. (doi:10.1007/s00028-020-00592-z)
12. Flandoli F, Luo D. 2020 Convergence of transport noise to Ornstein-Uhlenbeck for 2D Euler equations under the enstrophy measure. *Ann. Probab.* **48**, 264–295. (doi:10.1214/19-AOP1360)
13. Flandoli F, Galeati L, Luo D. 2021 Delayed blow-up by transport noise. *Commun. Partial Differ. Equ.* **46**, 1757–1788. (doi:10.1080/03605302.2021.1893748)
14. Wong E, Zakai M. 1965 On the relation between ordinary and stochastic differential equations. *Int. J. Eng. Sci.* **3**, 213–219. (doi:10.1016/0020-7225(65)90045-5)
15. Brzezniak Z, Flandoli F. 1995 Almost sure approximation of Wong-Zakai type for stochastic partial differential equations. *Stoch. Process. Their Appl.* **55**, 329–358. (doi:10.1016/0304-4149(94)00037-T)
16. Pazy A 1983 *Semigroups of linear operators and applications to partial differential equations*, vol. **44**. Applied Mathematical Sciences. New York, NY: Springer.
17. Chavel I. 1984 *Eigenvalues in Riemannian geometry*. New York, NY: Academic Press, Inc.
18. MacKay RS. 2008 A steady mixing flow with no-slip boundaries. In *Chaos, complexity and transport theory and applications*, pp. 55–68. Singapore: World Scientific.
19. Coghi M, Flandoli F. 2016 Propagation of chaos for interacting particles subject to environmental noise. *Ann. Appl. Probab.* **26**, 1407–1442. (doi:10.1214/15-AAP1120)