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Enhanced dissipation and Lyapunov exponents for stochastic transport-diffusion equations with small molecular diffusivity and small noise intensity

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Abstract

The paper deals with linear passive scalar transport-diffusion equation subject to a velocity field which is white noise in time and is mainly active at small scales in space. The purpose is investigating the enhancement of dissipation and decay given by the small-scale transport. We modify and improve estimates from the previous work by Flandoli et al. (J. Differ. Equations **394**, 237–277, 2024), in order to investigate a different regime, namely small molecular diffusion and small noise intensity – corresponding to small turbulent kinetic energy. The noise specification is carefully tuned to Kolmogorov theory of turbulent fluids.

Keywords Passive scalar · Turbulence · Dissipation enhancement · Dissipation time · Lyapunov exponent

1 Introduction

Consider the stochastic scalar transport-diffusion equation on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ ($d \geq 2$) with molecular diffusivity $\kappa > 0$:

$$\begin{aligned} d\theta + \circ dW \cdot \nabla\theta &= \kappa \Delta\theta \, dt, \\ \theta|_{t=0} &= \theta_0 \in L^2\left(\mathbb{T}^d\right), \end{aligned} \tag{1.1}$$

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where the $\mathbb{R}^d$-valued divergence free random field $W = W(x, t)$ has the form

$$W(x, t) = \sqrt{2\nu} \sum_{k \in \mathbb{Z}_0^d} \sum_{i=1}^{d-1} \Gamma_k \sigma_{k,i}(x) W_t^{k,i} \tag{1.2}$$

for a parameter $\nu > 0$ having the meaning of *eddy* viscosity[1], real coefficients $(\Gamma_k)_{k \in \mathbb{Z}_0^d}$ having the role to normalize the sum over Fourier modes, with $\mathbb{Z}_0^d = \mathbb{Z}^d \setminus \{0\}$ being the set of nonzero lattice points, divergence free smooth vector fields $\{\sigma_{k,i}\}_{k \in \mathbb{Z}_0^d,\, i \in \{1,\ldots,d-1\}}$ related to a classical orthonormal basis of solenoidal zero average $L^2(\mathbb{T}^d; \mathbb{R}^d)$-fields, and independent real Brownian motions $(W^{k,i})_{k \in \mathbb{Z}_0^d,\, i \in \{1,\ldots,d-1\}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with expectation $\mathbb{E}$. See the next section for more details on the structure of $W(x, t)$.

The equation (1.1) has a unique adapted solution with paths of class

$$C\left([0, T]; L^2\left(\mathbb{T}^d\right)\right) \cap L^2\left(0, T; W^{1,2}\left(\mathbb{T}^d\right)\right)$$

for every $T > 0$ and, by elementary arguments,

$$\mathbb{E}\left[\|\theta_t\|_{L^2}^2\right] \le e^{-8\pi^2\kappa(t-s)}\, \mathbb{E}\left[\|\theta_s\|_{L^2}^2\right] \tag{1.3}$$

for every $t \ge s \ge 0$. This result is independent of the noise. Introducing the dissipation time (cf. [10, 15])

$$\tau_{\theta_0} = \min\left\{t \ge 0 : \mathbb{E}\left[\|\theta_{t+s}\|_{L^2}^2\right] \le \frac{1}{2^2}\, \mathbb{E}\left[\|\theta_s\|_{L^2}^2\right] \text{ for all } s \ge 0\right\},$$

we have at least the inequality

$$\tau_{\theta_0} \le \tau_{\text{free}} := \frac{\log 2}{4\pi^2\kappa} \tag{1.4}$$

for every $\theta_0 \in L^2\left(\mathbb{T}^d\right)$. Introducing the upper average Lyapunov exponent

$$\Lambda_{\theta_0} = \limsup_{t \to \infty} \frac{1}{2t} \log \mathbb{E}\left[\|\theta_t\|_{L^2}^2\right]$$

we have

$$\Lambda_{\theta_0} \le \Lambda_{\text{free}} := -4\pi^2\kappa \tag{1.5}$$

[1] To facilitate the interpretation, let us anticipate that we put ourselves in a framework where the eddy viscosity is a multiple of the kinematic viscosity, that we take equal to the molecular diffusivity κ. Hence ν will be a multiple of κ.

for every $\theta_0 \in L^2\left(\mathbb{T}^d\right)$. Notice also that, due to the monotonicity of $t \mapsto \mathbb{E}\left[\|\theta_t\|_{L^2}^2\right]$, for every $T > 0$ we also have

$$\Lambda_{\theta_0} = \limsup_{n\to\infty} \frac{1}{2nT} \log \mathbb{E}\left[\|\theta_{nT}\|_{L^2}^2\right].$$

Therefore, from $\mathbb{E}\left[\|\theta_{n\tau_{\theta_0}}\|_{L^2}^2\right] \leq \frac{1}{2^{2n}}\|\theta_0\|_{L^2}^2$ one can deduce the general relation

$$\Lambda_{\theta_0} \leq -\frac{\log 2}{\tau_{\theta_0}}. \tag{1.6}$$

In particular this implies the inequality (1.5) from the inequality (1.4). In the following we concentrate on estimates on τ_{θ_0}, deducing estimates on Λ_{θ_0} from (1.6). Let us also remark that, by suitable Borel-Cantelli arguments (see for instance [1, 2, 9, 12, 14]), one can deduce pathwise Lyapunov exponents from estimates of the previous form, but we do not stress this topic here.

The purpose of a large recent body of research (see for instance the references just quoted and also [10, 15]) is proving that the bounds (1.5) and (1.4) can be improved, thanks to suitable random/deterministic velocity fields, here $\partial_t W(x, t)$. This is also the topic of the present note. More precisely, our purpose is to complement the result of [12], based on high intensity noise as in [11, 13], by a result for small intensity noise and small molecular diffusivity, which is a more natural regime for applications. Let us explain the issue in more detail.

In [12, Theorem 1.9], it was proved that, given any $\kappa > 0$, the dissipation time τ_{θ_0} can be made arbitrarily small, and $-\Lambda_{\theta_0}$ arbitrarily large, by choosing suitable coefficients $(\Gamma_k)_{k\in\mathbb{Z}_0^d}$ and (especially) suitably large ν. The first result of [14], up to a number of differences in the model and techniques, is in the same direction: large noise may provide arbitrarily strong dissipation (see Theorem 1.1 therein). The results of [1, 2] are different, either because the process representing the velocity field is not white noise in time (e.g. solutions to stochastic Navier-Stokes equations), or because $\kappa = 0$ from the beginning.

Here we want to investigate an opposite regime. In many (maybe most) examples, the molecular diffusivity is positive but very small. Moreover, the noise is also of small intensity, representing small turbulent scales of a fluid, which typically have relatively low kinetic energy. We modify in a few essential points the proof of [12, Theorem 1.9], to reach this target. The proof is essentially based on the semigroup approach and the mild formula, methods developed in [5–7] for a myriad of uses and applications and specifically initiated by Giuseppe Da Prato in [3, 4] for stochastic transport-diffusion equations.

To be more precise, not only do we have to change the proof but we also have to focus more clearly on a model representing turbulent scales. This requires some work, developed in Section 2 below. Assuming that the energy dissipation rate ϵ in the Kolmogorov inertial range is constant, we get a noise model of the form (1.2) where both ν and $(\Gamma_k)_{k\in\mathbb{Z}_0^d}$ depend only on two parameters (the precise expressions are given in equation (2.3) below):

1. the molecular diffusivity κ itself,
2. the length $\log \lambda$ (in logarithmic scale) of the turbulent inertial range.

Therefore the final model depends only on the pair of parameters (κ, λ). Call $\tau_{\theta_0}(\kappa, \lambda)$ and $\Lambda_{\theta_0}(\kappa, \lambda)$ the corresponding dissipation time and upper average Lyapunov exponent. We prove the following result in Section 3.

Theorem 1 *For every $\lambda > 1$, there exists $\kappa_0 = \kappa_0(\lambda, \epsilon) > 0$ such that for all $\kappa \in (0, \kappa_0)$ one has*

$$\tau_{\theta_0}(\kappa, \lambda) \leq \frac{4}{C_d \lambda^{4/3}} \tau_{\text{free}},$$
$$-\Lambda_{\theta_0}(\kappa, \lambda) \geq -\frac{C_d \lambda^{4/3}}{4} \Lambda_{\text{free}}$$

for all $\theta_0 \in L^2(\mathbb{T}^d)$. Here, $C_d > 0$ is a dimensional constant.

Therefore, if the turbulent inertial range is not too small (in practical examples λ can be of the order of 10^2–10^3), there is a great improvement in the dissipation time and in the Lyapunov exponent, for infinitesimal molecular diffusivity κ.

Remark 2 The expression (2.3) of ν and $(\Gamma_k)_{k \in \mathbb{Z}_0^d}$ in terms of κ and λ comes out very naturally from turbulence arguments but may appear exotic or very peculiar from an abstract viewpoint. Let us then give a general interpretation, free of turbulence details. The expression of ν is simply

$$\nu = C_d \lambda^{\frac{4}{3}} \kappa$$

hence a large multiple of κ. The square-root coefficient $\sqrt{C_d \lambda^{\frac{4}{3}} \kappa}$ in front of the noise, compared to the coefficient of the Laplacian, is one of the classical scalings of additive noise when (small viscosity)-(small noise) asymptotics are investigated. Here we do the same for a transport noise, with a large multiplicative constant between the two coefficients. Clarified the dependence of ν on κ and λ, the (apparently complicated and special) form of $(\Gamma_k)_{k \in \mathbb{Z}_0^d}$ in terms of κ and λ is just dictated by two requirements: normalization to "one" of the average size of sum over all Fourier modes, plus the fact that energy is concentrated at high modes.

The latter requirement, namely that *the velocity field is mainly active at large modes*, is the key for dissipation enhancement and improvement of the Lyapunov exponent. The notations of the paper and the conditions of the theorem could be rewritten in such a more general form, which however would hide the fact that the conditions imposed are just those emerging by themselves from a natural turbulence modeling.

2 Noise specification

Let us recall the scaling properties of Kolmogorov inertial range. Let ϵ be the dissipation rate of turbulence kinetic energy which will be fixed as a constant in the sequel.

We shall identify the kinematic viscosity of the fluid as the molecular diffusivity κ, cf. [16, Chap. XV, Sect. 118]. We have a range of length-scales ℓ from the smallest one, the Kolmogorov scale

$$\eta \sim \left(\frac{\kappa^3}{\epsilon}\right)^{\frac{1}{4}} \tag{2.1}$$

to the largest one, which we denote by

$$\ell_{\max} = \lambda\eta$$

for some $\lambda \geq 1$. We call inertial range the range of these scales.

At each scale ℓ in the inertial range the typical velocity (or velocity increment) is, in norm, of the order

$$u_\ell \sim \epsilon^{\frac{1}{3}}\ell^{\frac{1}{3}}.$$

The revolution (or relaxation) time of a structure at such scale is

$$\tau_\ell \sim \epsilon^{-\frac{1}{3}}\ell^{\frac{2}{3}}$$

(from $\tau_\ell = \ell/u_\ell$).

The turbulent kinetic energy at scale ℓ is (up to the factor $\frac{1}{2}$ that we omit as other constants)

$$k_{T,\ell} \sim u_\ell^2 \sim \epsilon^{\frac{2}{3}}\ell^{\frac{2}{3}}.$$

In particular, the turbulent kinetic energy at the largest scale $\ell_{\max}$ is

$$k_{T,\ell_{\max}} \sim \epsilon^{\frac{2}{3}}\lambda^{\frac{2}{3}}\eta^{\frac{2}{3}}.$$

The global turbulent kinetic energy of the inertial range is (removing constant factors as above)

$$k_T \sim \sum_{\ell=\eta,\dots,\lambda\eta} \epsilon^{\frac{2}{3}}\ell^{\frac{2}{3}} \sim \epsilon^{\frac{2}{3}}\lambda^{\frac{2}{3}}\eta^{\frac{2}{3}}$$

as the one of the largest scale, since the tail of a geometric sum goes like the largest value.

Let us build a Gaussian model, on the unitary torus $\mathbb{T}^d$, of the previous inertial range velocity field $u_S(x,t)$, where the subscript S stands for small-scale components. First, let $\mathbb{Z}_0^d = \mathbb{Z}_+^d \cup \mathbb{Z}_-^d$ be a partition of $\mathbb{Z}_0^d$ with $\mathbb{Z}_+^d = -\mathbb{Z}_-^d$. For each $k \in \mathbb{Z}_+^d$, let $\{a_{k,i}\}_{i=1,\dots,d-1}$ be an orthonormal basis of $k^\perp := \{y \in \mathbb{R}^d : k \cdot y = 0\}$; set $a_{k,i} = a_{-k,i}$ for all $k \in \mathbb{Z}_-^d$. Let

$$\mathrm{e}_k(x) = \sqrt{2}\begin{cases} \cos(2\pi k \cdot x), & k \in \mathbb{Z}_+^d, \\ \sin(2\pi k \cdot x), & k \in \mathbb{Z}_-^d. \end{cases}$$

Then the divergence free vector fields are given by $\sigma_{k,i}(x) = a_{k,i}\mathrm{e}_k(x)$ for $k \in \mathbb{Z}_0^d$ and $i \in \{1,\dots,d-1\}$. Next, taking a family of independent two-sided standard Brownian motions $\{W_t^{k,i}\}_{k\in\mathbb{Z}_0^d,\, i=\{1,\dots,d-1\}}$, we consider the stationary solutions $\left(Z_t^{k,i}\right)_{t\geq 0}$ of

$$dZ_t^{k,i} = -\frac{1}{\tau_{|k|^{-1}}} Z_t^{k,i}\, dt + \sqrt{\frac{2}{\tau_{|k|^{-1}}}}\, dW_t^{k,i}.$$

We have $\mathbb{E}\big[Z_t^{k,i} Z_s^{k',i'}\big] = \delta_{kk'}\delta_{ii'} \exp\Big(-\frac{|t-s|}{\tau_{|k|^{-1}}}\Big)$.

We can now define a turbulent velocity field having the required features

$$u_S(x,t) = \sum_{(\lambda\eta)^{-1}\leq |k|\leq \eta^{-1}} \sum_{i=1}^{d-1} \frac{u_{|k|^{-1}}}{|k|^{\frac{d}{2}}}\, \sigma_{k,i}(x) Z_t^{k,i}$$

(it is not yet the final form, since we have to perform an approximation in order to start with model (1.2)). Let us explain all elements of this choice. The field $u_S(x,t)$ is the sum of contributions

$$u_S^{\ell}(x,t) = u_\ell \sum_{\ell^{-1}\leq |k|\leq 2\ell^{-1}} |k|^{-\frac{d}{2}} \sum_{i=1}^{d-1} \sigma_{k,i}(x) Z_t^{k,i}$$

corresponding to every space-scale $\ell \in [\eta, \lambda\eta]$. Each one of these contributions $u_S^{\ell}(x,t)$ has a typical (average) velocity modulus u_ℓ, a typical time-scale $\tau_{|k|^{-1}}$ (the relaxation time of each $Z_t^{k,i}$), and it is decomposed in all possible modes with space-scale near ℓ. Notice the exponential structure of the decomposition (it is additive in logarithmic scale of ℓ), related to the concept of Richardson (or Kolmogorov) cascade, where eddies of size 2ℓ generate by instability eddies of size ℓ, cf. [17]. The factor $|k|^{-\frac{d}{2}}$ is introduced precisely to compensate this latter decomposition into several modes, i.e., it is a normalizing factor. Indeed, we have

$$\mathbb{E}\Bigg[\Bigg| \sum_{\ell^{-1}\leq |k|\leq 2\ell^{-1}} |k|^{-\frac{d}{2}} \sum_{i=1}^{d-1} \sigma_{k,i}(x) Z_t^{k,i}\Bigg|^2\Bigg] = \sum_{\ell^{-1}\leq |k|\leq 2\ell^{-1}} \sum_{i=1}^{d-1} |k|^{-d} \left|\sigma_{k,i}(x)\right|^2,$$

considering the form of $\sigma_{k,i}$ and the property $\sin^2 r + \cos^2 r = 1$, up to constants the order of magnitude of this sum is (see more details below)

$$\int_{\ell^{-1}\leq r\leq 2\ell^{-1}} r^{-d} r^{d-1} dr = \log 2.$$

Therefore, thanks to the factor $|k|^{-\frac{d}{2}}$, u_ℓ is (up to a constant) the typical velocity modulus at scale ℓ.

For simplicity of notation, let us set

$$\Lambda_\eta := \left\{k \in \mathbb{Z}_0^d : (\lambda\eta)^{-1} \leq |k| \leq \eta^{-1}\right\}. \tag{2.2}$$

Still in the preliminary phase of heuristic modeling, we remark that the process u_S can be approximated as

$$\begin{aligned} u_S(x,t) &= \sum_{k\in\Lambda_\eta}\sum_{i=1}^{d-1}\frac{u_{|k|^{-1}}}{|k|^{\frac{d}{2}}}\sigma_{k,i}(x)\int_{-\infty}^{t}\sqrt{\frac{2}{\tau_{|k|^{-1}}}}\,e^{-\frac{t-s}{\tau_{|k|^{-1}}}}\,dW_s^{k,i} \\ &= \sum_{k\in\Lambda_\eta}\sum_{i=1}^{d-1}\sqrt{2\tau_{|k|^{-1}}}\frac{u_{|k|^{-1}}}{|k|^{\frac{d}{2}}}\sigma_{k,i}(x)\int_{-\infty}^{t}\frac{1}{\tau_{|k|^{-1}}}e^{-\frac{t-s}{\tau_{|k|^{-1}}}}\frac{dW_s^{k,i}}{ds}\,ds \\ &\sim \sum_{k\in\Lambda_\eta}\sum_{i=1}^{d-1}\frac{\sqrt{2\tau_{|k|^{-1}}k_{T,|k|^{-1}}}}{|k|^{\frac{d}{2}}}\sigma_{k,i}(x)\frac{dW_t^{k,i}}{dt}, \end{aligned}$$

where the distributional derivative $\frac{dW_t^{k,i}}{dt}$ is the white noise. Recall that $\tau_{|k|^{-1}} = \epsilon^{-\frac{1}{3}}|k|^{-\frac{2}{3}}$ and $k_{T,|k|^{-1}} = \epsilon^{\frac{2}{3}}|k|^{-\frac{2}{3}}$, we rewrite $u_S(x,t)$ as

$$u_S(x,t) \sim \sqrt{2\epsilon^{\frac{1}{3}}}\sum_{k\in\Lambda_\eta}\sum_{i=1}^{d-1}\frac{1}{|k|^{\frac{d}{2}+\frac{2}{3}}}\sigma_{k,i}(x)\frac{dW_t^{k,i}}{dt}.$$

We remark that the above passage from the Ornstein-Uhlenbeck model u_S to the white-in-time noise on the right-hand side is only heuristic; a rigorous justification of this fact is a nontrivial research topic in itself, see e.g. [8].

Based on the above heuristic arguments, we define the random field

$$W(x,t) = \sqrt{2\epsilon^{\frac{1}{3}}}\sum_{k\in\Lambda_\eta}\sum_{i=1}^{d-1}\frac{1}{|k|^{\frac{d}{2}+\frac{2}{3}}}\sigma_{k,i}(x)W_t^{k,i}$$

and we use $\frac{\partial W(x,t)}{\partial t}$ (in Stratonovich form, for obvious heuristic reasons related to Wong-Zakai approximation, and also to preserve the natural conservation properties of the system) as a model of the turbulent velocity field, in equation (1.1).

Notice that

$$\epsilon^{\frac{1}{3}}\sum_{k\in\Lambda_\eta}\frac{1}{|k|^{d+\frac{4}{3}}} \sim \epsilon^{\frac{1}{3}}\int_{(\lambda\eta)^{-1}}^{\eta^{-1}}\frac{c_d}{r^{\frac{7}{3}}}\,dr = c_d\,\epsilon^{\frac{1}{3}}(\lambda^{\frac{4}{3}}-1)\,\eta^{\frac{4}{3}},$$

where c_d is the surface area of $d-1$ dimensional sphere. From the expression of Kolmogorov scale η in (2.1), we obtain

$$\epsilon^{\frac{1}{3}}\sum_{k\in\Lambda_\eta}\frac{1}{|k|^{d+\frac{4}{3}}} \sim c_d\,(\lambda^{\frac{4}{3}}-1)\,\kappa \overset{\text{large }\lambda}{\sim} C_d\lambda^{\frac{4}{3}}\kappa.$$

Therefore the variance of the random field $W(x,1)$ is asymptotically equal to $C_d \lambda^{\frac{4}{3}} \kappa$ as $\kappa \to 0$, for a suitable constant C_d, see the end of this section for more detailed computations. For this reason we rewrite the random field $W(x,t)$ in the form

$$\begin{aligned} W(x,t) &= \sqrt{2C_d \lambda^{\frac{4}{3}} \kappa} \sum_{k\in\Lambda_\eta} \sum_{i=1}^{d-1} \sqrt{\frac{\epsilon^{\frac{1}{3}}}{C_d \lambda^{\frac{4}{3}} \kappa} \frac{1}{|k|^{\frac{d}{2}+\frac{2}{3}}}} \sigma_{k,i}(x) W_t^{k,i} \\ &= \sqrt{2\nu} \sum_{k\in\mathbb{Z}_0^d} \sum_{i=1}^{d-1} \Gamma_k \sigma_{k,i}(x)\, W_t^{k,i} \end{aligned}$$

with

$$\nu = C_d \lambda^{\frac{4}{3}} \kappa, \quad \Gamma_k = \mathbf{1}_{\{(\lambda\eta)^{-1} \le |k| \le \eta^{-1}\}} \sqrt{\frac{\epsilon^{\frac{1}{3}}}{C_d \lambda^{\frac{4}{3}} \kappa} \frac{1}{|k|^{\frac{d}{2}+\frac{2}{3}}}}. \tag{2.3}$$

This is the form (1.2) assumed in the Introduction. The role of the coefficients Γ_k is to normalize the sum over the Fourier modes. By (2.1), we see that the noise is determined by the two parameters λ and κ.

Let us compute more accurately the covariance function

$$\begin{aligned} Q(x,y) &:= \mathbb{E}[W(x,1) \otimes W(y,1)] \\ &= 2\epsilon^{\frac{1}{3}} \sum_{k\in\Lambda_\eta} \sum_{i=1}^{d-1} \frac{1}{|k|^{d+\frac{4}{3}}} a_{k,i} \otimes a_{k,i}\, \mathrm{e}_k(x)\, \mathrm{e}_k(y) \\ &= 2\epsilon^{\frac{1}{3}} \sum_{k\in\Lambda_\eta} \frac{1}{|k|^{d+\frac{4}{3}}} \left(I_d - \frac{k\otimes k}{|k|^2} \right) \mathrm{e}_k(x)\, \mathrm{e}_k(y), \end{aligned}$$

where I_d is the $d \times d$ unit matrix. By the definitions of $\{\mathrm{e}_k\}_k$, one easily get

$$Q(x,y) = 4\epsilon^{\frac{1}{3}} \sum_{k\in\Lambda_\eta\cap\mathbb{Z}_+^d} \frac{1}{|k|^{d+\frac{4}{3}}} \left(I_d - \frac{k\otimes k}{|k|^2} \right) \cos(2\pi k \cdot (x-y)).$$

So we can denote it as $Q(x-y)$ for some matrix valued function $Q : \mathbb{T}^d \to \mathbb{R}^{d\times d}$. In particular, taking $x = y$ yields

$$\begin{aligned} Q(0) &= 4\epsilon^{\frac{1}{3}} \sum_{k\in\Lambda_\eta\cap\mathbb{Z}_+^d} \frac{1}{|k|^{d+\frac{4}{3}}} \left(I_d - \frac{k\otimes k}{|k|^2} \right) \\ &= 2\epsilon^{\frac{1}{3}} \sum_{k\in\Lambda_\eta} \frac{1}{|k|^{d+\frac{4}{3}}} \left(I_d - \frac{k\otimes k}{|k|^2} \right). \end{aligned}$$

It is easy to see that the off-diagonal entries of $Q(0)$ are identically 0, while the diagonal ones are all equal to

$$Q_{11}(0) = \frac{2}{d}\epsilon^{\frac{1}{3}} \sum_{k\in\Lambda_\eta} \frac{1}{|k|^{d+\frac{4}{3}}} \sum_{i=1}^{d}\left(1-\frac{k_i^2}{|k|^2}\right) = \frac{2(d-1)}{d}\epsilon^{\frac{1}{3}} \sum_{k\in\Lambda_\eta} \frac{1}{|k|^{d+\frac{4}{3}}}.$$

By the definition of Λ_η in (2.2) and approximating the sum by integral, we get (as already sketched above)

$$\begin{aligned} Q_{11}(0) &\sim \frac{2(d-1)}{d}\epsilon^{\frac{1}{3}} \int_{(\lambda\eta)^{-1}\le|x|\le\eta^{-1}} \frac{1}{|x|^{d+\frac{4}{3}}}\,dx \\ &= \frac{2(d-1)}{d}\epsilon^{\frac{1}{3}} \int_{(\lambda\eta)^{-1}}^{\eta^{-1}} \frac{c_d}{r^{\frac{7}{3}}}\,dr \\ &= \frac{2(d-1)}{d} c_d\, \epsilon^{\frac{1}{3}} (\lambda^{\frac{4}{3}}-1)\, \eta^{\frac{4}{3}} \\ &= \frac{2(d-1)}{d} c_d (\lambda^{\frac{4}{3}}-1)\kappa \overset{\lambda\gg1}{\sim} \frac{2(d-1)}{d} c_d \lambda^{\frac{4}{3}}\kappa. \end{aligned}$$

The above computations imply that, for suitably chosen normalizing constant $C_d > 0$, one has

$$\sum_{k\in\mathbb{Z}_0^d}\sum_{i=1}^{d-1} \Gamma_k^2\, \sigma_{k,i}(x)\otimes\sigma_{k,i}(x) \equiv I_d \quad \text{for all } x\in\mathbb{T}^d. \tag{2.4}$$

3 Proof of Theorem 1

First, we have the following energy equality: $\mathbb{P}$-a.s.,

$$\|\theta_t\|_{L^2}^2 + 2\kappa\int_s^t \|\nabla\theta_r\|_{L^2}^2\,dr = \|\theta_s\|_{L^2}^2, \quad 0\le s<t. \tag{3.1}$$

Next, similarly to the computations at the end of the last section, the equation (1.1) reads in Itô formulation as

$$d\theta + dW\cdot\nabla\theta = (\kappa+\nu)\Delta\theta\,dt,$$

which can be further written in mild form ($P_t = e^{t(\kappa+\nu)\Delta}$):

$$\theta_t = P_{t-s}\theta_s - \int_s^t P_{t-r}(dW_r\cdot\nabla\theta_r), \quad 0\le s<t.$$

We follow the main idea in [12, Section 5.2] to prove the following estimate, but with some key modifications. Compared to [12, Theorem 1.9], as both parameters κ

and ν are assumed here to be very small, we fix some (big) $T > 0$ and study the relation between θ_{s+T} and θ_s (in [12] we simply take $T = 1$). Moreover, we employ a clever trick by computing the time integral on the interval $[s + T/2, s + T]$ which has a positive distance to s; this is crucial for obtaining a small exponential factor in case of small $\kappa + \nu$, see the proof below for details.

Lemma 3 *Take* $\alpha, \beta > 0$ *such that*

$$0 < \alpha < 1 \le \frac{d}{2} < \beta < \frac{d}{2} + 2.$$

There exists $\gamma = \gamma(\kappa, \lambda, T, \alpha, \beta) > 0$ *such that*

$$\mathbb{E}\|\theta_{s+T}\|_{L^2}^2 \le \gamma\, \mathbb{E}\|\theta_s\|_{L^2}^2 \quad \textit{for all } s \ge 0.$$

Proof We fix any $s \ge 0$. Noting that $t \mapsto \|\theta_t\|_{L^2}^2$ is decreasing and using the above mild formulation, it holds

$$\begin{aligned} \|\theta_{s+T}\|_{L^2}^2 &\le \frac{2}{T}\int_{s+T/2}^{s+T} \|\theta_t\|_{L^2}^2\, dt \\ &\le \frac{4}{T}\int_{s+T/2}^{s+T} \|P_{t-s}\theta_s\|_{L^2}^2\, dt \\ &\quad + \frac{4}{T}\int_{s+T/2}^{s+T} \Big\| \int_s^t P_{t-r}(dW_r\cdot\nabla\theta_r)\Big\|_{L^2}^2 dt. \end{aligned} \tag{3.2}$$

For the first term, we have

$$\|P_{t-s}\theta_s\|_{L^2} \le e^{-4\pi^2(t-s)(\kappa+\nu)}\|\theta_s\|_{L^2} \le e^{-2\pi^2 T(\kappa+\nu)}\|\theta_s\|_{L^2}$$

for all $t \in [s + T/2, s + T]$, thus

$$\begin{aligned} \frac{4}{T}\int_{s+T/2}^{s+T} \|P_{t-s}\theta_s\|_{L^2}^2\, dt &\le \frac{4}{T}\int_{s+T/2}^{s+T} e^{-4\pi^2 T(\kappa+\nu)}\|\theta_s\|_{L^2}^2\, dt \\ &= 2e^{-4\pi^2 T(\kappa+\nu)}\|\theta_s\|_{L^2}^2. \end{aligned} \tag{3.3}$$

Next we treat the last term in (3.2). Denote the stochastic convolution by

$$Z_{s,t} = \int_s^t P_{t-r}(dW_r\cdot\nabla\theta_r) = \sqrt{2\nu}\sum_{k,i}\Gamma_k\int_s^t P_{t-r}(\sigma_{k,i}\cdot\nabla\theta_r)\, dW_r^{k,i},$$

where $\sum_{k,i}$ stands for $\sum_{k\in\mathbb{Z}_0^d}\sum_{i=1}^{d-1}$. Recalling the parameters $\alpha, \beta > 0$, by interpolation, we have

$$\|Z_{s,t}\|_{L^2} \le \|Z_{s,t}\|_{H^\alpha}^{\beta/(\alpha+\beta)}\|Z_{s,t}\|_{H^{-\beta}}^{\alpha/(\alpha+\beta)},$$

therefore,

$$
\begin{aligned}
&\int_{s+T/2}^{s+T} \mathbb{E}\|Z_{s,t}\|_{L^2}^2\,dt \\
&\le \int_{s+T/2}^{s+T} \mathbb{E}\Big(\|Z_{s,t}\|_{H^\alpha}^{2\beta/(\alpha+\beta)}\|Z_{s,t}\|_{H^{-\beta}}^{2\alpha/(\alpha+\beta)}\Big)\,dt \\
&\le \bigg[\int_{s+T/2}^{s+T} \mathbb{E}\|Z_{s,t}\|_{H^\alpha}^2\,dt\bigg]^{\frac{\beta}{\alpha+\beta}}\bigg[\int_{s+T/2}^{s+T} \mathbb{E}\|Z_{s,t}\|_{H^{-\beta}}^2\,dt\bigg]^{\frac{\alpha}{\alpha+\beta}}.
\end{aligned}
$$

It remains to estimate the two terms.

First, by the definition of $Z_{s,t}$ and heat semigroup property,

$$
\begin{aligned}
\mathbb{E}\|Z_{s,t}\|_{H^\alpha}^2 &= 2\nu\,\mathbb{E}\bigg[\sum_{k,i}\Gamma_k^2\int_s^t \|P_{t-r}(\sigma_{k,i}\cdot\nabla\theta_r)\|_{H^\alpha}^2\,dr\bigg] \\
&\lesssim 2\nu\,\mathbb{E}\bigg[\sum_{k,i}\Gamma_k^2\int_s^t \frac{\|\sigma_{k,i}\cdot\nabla\theta_r\|_{L^2}^2}{(\kappa+\nu)^\alpha(t-r)^\alpha}\,dr\bigg] \\
&= \frac{2\nu}{(\kappa+\nu)^\alpha}\,\mathbb{E}\int_s^t \frac{\|\nabla\theta_r\|_{L^2}^2}{(t-r)^\alpha}\,dr,
\end{aligned}
$$

where in the last step we have used the identity (2.4). As a result,

$$
\begin{aligned}
\int_{s+T/2}^{s+T}\mathbb{E}\|Z_{s,t}\|_{H^\alpha}^2\,dt &\le 2\nu^{1-\alpha}\int_s^{s+T}\mathbb{E}\int_s^t \frac{\|\nabla\theta_r\|_{L^2}^2}{(t-r)^\alpha}\,drdt \\
&= 2\nu^{1-\alpha}\,\mathbb{E}\int_s^{s+T}\|\nabla\theta_r\|_{L^2}^2\int_r^{s+T}\frac{dt}{(t-r)^\alpha}\,dr \\
&\le \frac{2\nu^{1-\alpha}}{1-\alpha}T^{1-\alpha}\,\mathbb{E}\int_s^{s+T}\|\nabla\theta_r\|_{L^2}^2\,dr \\
&\le \frac{\nu^{1-\alpha}}{1-\alpha}T^{1-\alpha}\kappa^{-1}\,\mathbb{E}\|\theta_s\|_{L^2}^2,
\end{aligned}
\tag{3.4}
$$

where the last step follows from the energy identity (3.1).

Next, taking $\delta = (2\beta - d)/4 \in (0,1)$, we have

$$
\begin{aligned}
\mathbb{E}\|Z_{s,t}\|_{H^{-\beta}}^2 &= 2\nu\,\mathbb{E}\bigg[\sum_{k,i}\Gamma_k^2\int_s^t \|P_{t-r}(\sigma_{k,i}\cdot\nabla\theta_r)\|_{H^{-d/2-2\delta}}^2\,dr\bigg] \\
&\lesssim 2\nu\,\mathbb{E}\bigg[\sum_{k,i}\Gamma_k^2\int_s^t \frac{\|\sigma_{k,i}\cdot\nabla\theta_r\|_{H^{-1-d/2-\delta}}^2}{(\kappa+\nu)^{1-\delta}(t-r)^{1-\delta}}\,dr\bigg] \\
&\lesssim \frac{2\nu}{(\kappa+\nu)^{1-\delta}}\,\mathbb{E}\bigg[\sum_{k,i}\Gamma_k^2\int_s^t \frac{\|\sigma_{k,i}\,\theta_r\|_{H^{-d/2-\delta}}^2}{(t-r)^{1-\delta}}\,dr\bigg],
\end{aligned}
\tag{3.5}
$$

where in the last step we have used the elementary estimate

$$\|\sigma_{k,i}\cdot\nabla\theta_r\|_{H^{-1-d/2-\delta}}=\|\nabla\cdot(\sigma_{k,i}\,\theta_r)\|_{H^{-1-d/2-\delta}}\lesssim\|\sigma_{k,i}\,\theta_r\|_{H^{-d/2-\delta}}.$$

It holds

$$\begin{aligned}\sum_{k,i}\Gamma_k^2\|\sigma_{k,i}\,\theta_r\|_{H^{-d/2-\delta}}^2&\le\|\Gamma\|_{\ell^\infty}^2\sum_{k,i}\sum_l\frac{|\langle\sigma_{k,i}\,\theta_r,\mathrm{e}_l\rangle|^2}{|l|^{d+2\delta}}\\&\le\|\Gamma\|_{\ell^\infty}^2\sum_l\frac{1}{|l|^{d+2\delta}}\sum_k|\langle\theta_r,\mathrm{e}_{l-k}\rangle|^2\\&\lesssim\|\Gamma\|_{\ell^\infty}^2\|\theta_r\|_{L^2}^2\,\delta^{-1},\end{aligned}\tag{3.6}$$

where we have estimated the series by integral as follows:

$$\sum_l\frac{1}{|l|^{d+2\delta}}\lesssim\int_{|x|>1/2}\frac{dx}{|x|^{d+2\delta}}=c_d\int_{1/2}^\infty\frac{dr}{r^{1+2\delta}}\lesssim\frac{1}{\delta}.$$

Substituting (3.6) into the inequality (3.5), we obtain

$$\begin{aligned}\mathbb{E}\|Z_{s,t}\|_{H^{-\beta}}^2&\lesssim\frac{2\nu}{(\kappa+\nu)^{1-\delta}}\|\Gamma\|_{\ell^\infty}^2\delta^{-1}\big(\mathbb{E}\|\theta_s\|_{L^2}^2\big)\int_s^t(t-r)^{\delta-1}\,dr\\&\lesssim 2\nu^\delta\|\Gamma\|_{\ell^\infty}^2\delta^{-2}\big(\mathbb{E}\|\theta_s\|_{L^2}^2\big)(t-s)^\delta.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{s+T/2}^{s+T}\mathbb{E}\|Z_{s,t}\|_{H^{-\beta}}^2\,dt&\lesssim 2\nu^\delta\|\Gamma\|_{\ell^\infty}^2\delta^{-2}\big(\mathbb{E}\|\theta_s\|_{L^2}^2\big)\frac{T^{1+\delta}}{1+\delta}\\&\lesssim\nu^\delta T^{1+\delta}\|\Gamma\|_{\ell^\infty}^2\delta^{-2}\,\mathbb{E}\|\theta_s\|_{L^2}^2.\end{aligned}\tag{3.7}$$

Combining the two estimates (3.4) and (3.7), we arrive at

$$\begin{aligned}&\int_{s+T/2}^{s+T}\mathbb{E}\|Z_{s,t}\|_{L^2}^2\,dt\\&\le\bigg[\frac{\nu^{1-\alpha}}{1-\alpha}T^{1-\alpha}\kappa^{-1}\,\mathbb{E}\|\theta_s\|_{L^2}^2\bigg]^{\frac{\beta}{\alpha+\beta}}\Big[\nu^\delta T^{1+\delta}\|\Gamma\|_{\ell^\infty}^2\delta^{-2}\,\mathbb{E}\|\theta_s\|_{L^2}^2\Big]^{\frac{\alpha}{\alpha+\beta}}\\&\lesssim(1-\alpha)^{-\frac{\beta}{\alpha+\beta}}\nu^{\frac{(1-\alpha)\beta+\alpha\delta}{\alpha+\beta}}T^{\frac{(1-\alpha)\beta+\alpha(1+\delta)}{\alpha+\beta}}\kappa^{-\frac{\beta}{\alpha+\beta}}\delta^{-\frac{2\alpha}{\alpha+\beta}}\|\Gamma\|_{\ell^\infty}^{\frac{2\alpha}{\alpha+\beta}}\,\mathbb{E}\|\theta_s\|_{L^2}^2.\end{aligned}$$

As a result, the last term in (3.2) admits the bound

$$\frac{4}{T}\int_{s+T/2}^{s+T}\mathbb{E}\|Z_{s,t}\|_{L^2}^2\,dt\le C_0\,\nu^{\frac{(1-\alpha)\beta+\alpha\delta}{\alpha+\beta}}T^{-\frac{\alpha(\beta-\delta)}{\alpha+\beta}}\kappa^{-\frac{\beta}{\alpha+\beta}}\|\Gamma\|_{\ell^\infty}^{\frac{2\alpha}{\alpha+\beta}}\,\mathbb{E}\|\theta_s\|_{L^2}^2$$

for some constant $C_0=C_0(\alpha,\beta,d)>0$.

To sum up, combining the above estimate with (3.2) and (3.3), we arrive at

$$\mathbb{E}\|\theta_{s+T}\|_{L^2}^2 \le \left(\mathbb{E}\|\theta_s\|_{L^2}^2\right)\Big(2e^{-4\pi^2 T(\kappa+\nu)} + C_0\,\nu^{\frac{(1-\alpha)\beta+\alpha\delta}{\alpha+\beta}} T^{-\frac{\alpha(\beta-\delta)}{\alpha+\beta}} \kappa^{-\frac{\beta}{\alpha+\beta}} \|\Gamma\|_{\ell^\infty}^{\frac{2\alpha}{\alpha+\beta}}\Big). \tag{3.8}$$

Note that $\nu = C_d \lambda^{\frac{4}{3}} \kappa$ and

$$\|\Gamma\|_{\ell^\infty} \lesssim \epsilon^{\frac16}\lambda^{-\frac23}\kappa^{-\frac12}(\lambda\eta)^{\frac23+\frac{d}{2}} \sim \epsilon^{\frac16}\lambda^{\frac{d}{2}}\kappa^{-\frac12}\left[\left(\frac{\kappa^3}{\epsilon}\right)^{\frac14}\right]^{\frac23+\frac{d}{2}} = \epsilon^{-\frac{d}{8}}\lambda^{\frac{d}{2}}\kappa^{\frac38 d},$$

therefore,

$$\begin{aligned}
&C_0\nu^{\frac{(1-\alpha)\beta+\alpha\delta}{\alpha+\beta}} T^{-\frac{\alpha(\beta-\delta)}{\alpha+\beta}} \kappa^{-\frac{\beta}{\alpha+\beta}} \|\Gamma\|_{\ell^\infty}^{\frac{2\alpha}{\alpha+\beta}} \\
&\lesssim C_0 (C_d\lambda^{\frac43}\kappa)^{\frac{(1-\alpha)\beta+\alpha\delta}{\alpha+\beta}} T^{-\frac{\alpha(\beta-\delta)}{\alpha+\beta}} \kappa^{-\frac{\beta}{\alpha+\beta}} \left(\epsilon^{-\frac{d}{8}}\lambda^{\frac{d}{2}}\kappa^{\frac38 d}\right)^{\frac{2\alpha}{\alpha+\beta}} \\
&= C_0 C_d^{\frac{(1-\alpha)\beta+\alpha\delta}{\alpha+\beta}} \epsilon^{-\frac{d\alpha}{4(\alpha+\beta)}} \lambda^{\frac{4(1-\alpha)\beta+4\alpha\delta+3d\alpha}{3(\alpha+\beta)}} \kappa^{\frac{\alpha(3d+4\delta-4\beta)}{4(\alpha+\beta)}} T^{-\frac{\alpha(\beta-\delta)}{\alpha+\beta}} \\
&=: C_{d,\alpha,\beta,\epsilon,\delta}\, \lambda^{\frac{4(1-\alpha)\beta+4\alpha\delta+3d\alpha}{3(\alpha+\beta)}} \kappa^{\frac{\alpha(3d+4\delta-4\beta)}{4(\alpha+\beta)}} T^{-\frac{\alpha(\beta-\delta)}{\alpha+\beta}}.
\end{aligned}$$

Combined with (3.8), we define

$$\gamma := 2e^{-4\pi^2 T(\kappa+\nu)} + C_{d,\alpha,\beta,\epsilon,\delta}\, \lambda^{\frac{4(1-\alpha)\beta+4\alpha\delta+3d\alpha}{3(\alpha+\beta)}} \kappa^{\frac{\alpha(3d+4\delta-4\beta)}{4(\alpha+\beta)}} T^{-\frac{\alpha(\beta-\delta)}{\alpha+\beta}}$$

and finish the proof of Lemma 3. □

We slightly simplify the expression of γ. Fix $\delta \in (0, 1/2)$ and set

$$\alpha = 1 - 2\delta, \quad \beta = \frac{d}{2} + 2\delta;$$

it is clear that α and β verify the condition of Lemma 3, and $\alpha + \beta = \frac{d}{2} + 1$. Then we have

$$\gamma \le 2e^{-4\pi^2 C_d T\kappa\lambda^{4/3}} + C_{d,\delta,\epsilon}\lambda^{\frac{2d}{d+2} - \frac{4\delta(d-2-4\delta)}{3(d+2)}} \kappa^{\frac{d}{2(d+2)} - \frac{\delta(d+2-4\delta)}{d+2}} T^{-\frac{(1-2\delta)(d+2\delta)}{d+2}}$$

for some constant $C_{d,\delta,\epsilon} > 0$.

Now we are ready to provide the

Proof of Theorem 1 Recall the above estimate of γ; we take T sufficiently small such that $2e^{-4\pi^2 C_d T\kappa\lambda^{4/3}} = \frac18$, that is

$$T = \frac{\log 16}{4\pi^2 C_d\,\kappa\lambda^{4/3}} = \frac{4}{C_d\lambda^{4/3}}\tau_{\text{free}}. \tag{3.9}$$

We also need

$$C_{d,\delta,\epsilon}\lambda^{\frac{2d}{d+2}-\frac{4\delta(d-2-4\delta)}{3(d+2)}}\kappa^{\frac{d}{2(d+2)}-\frac{\delta(d+2-4\delta)}{d+2}}T^{-\frac{(1-2\delta)(d+2\delta)}{d+2}} \le \frac{1}{8}$$

which, by the first equality in (3.9), is equivalent to

$$\tilde{C}_{d,\delta,\epsilon}\lambda^{\frac{10d-4\delta(3d-4)}{3(d+2)}}\kappa^{\frac{3d(1-2\delta)}{2(d+2)}} \le \frac{1}{8}.$$

As $\delta \in (0, \frac{1}{2})$, for every $\lambda > 1$ fixed, it is clear that there is a $\kappa_0 = \kappa_0(\lambda) > 0$ such that the above inequality holds for all $\kappa \in (0, \kappa_0)$.

To sum up, we have proved that for the parameters T and κ chosen as above, it holds

$$\mathbb{E}\|\theta_{s+T}\|_{L^2}^2 \le \frac{1}{2^2}\,\mathbb{E}\|\theta_s\|_{L^2}^2 \quad \text{for all } s \ge 0.$$

By the definition of $\tau_{\theta_0}(\kappa, \lambda)$ we obtain the first assertion, while the second one follows by combining it with (1.5) and (1.6). □

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