On the variation of the Einstein-Hilbert action in pseudohermitian geometry
(with an appendix by Xiaodong Wang)

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Abstract

In this paper we compute the first and second variation of the normalized Einstein-Hilbert functional on CR manifolds. We characterize critical points as pseudo-Einstein structures. We then turn to the second variation on standard spheres. While the situation is quite similar to the Riemannian case in dimension greater or equal to five, in three dimension we observe a crucial difference, which mainly depends on the embeddable character of the perturbed CR structure.

1 Introduction

The *Einstein-Hilbert action* is the functional on the space of Riemannian metrics on a closed manifold M of dimension $n \geq 3$ defined by

$$\mathscr{R}(g) = \int_M R_g dv_g, \tag{1}$$

where R_g is the scalar curvature of the metric g. This functional is of great importance in differential geometry, and its variation is given by

$$d\mathscr{R}(g)[h] = \int_M E^{ij} h_{ij} dv_g$$

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where E_{ij} are the components of the Einstein tensor

$$E_{ij} = R_{ij} - \frac{1}{2} R_g g_{ij},$$

with R_{ij} the Ricci tensor of g. This implies that the critical points of $\mathscr{R}$ are Ricci-flat metrics.

Its variation on asymptotically flat metrics led to the notion of *ADM mass*, see e.g. [LP], a well-studied concept in geometric relativity which has been important, among other questions, in the resolution of some cases of Yamabe's conjecture, see [S1].

It is also relevant to extremize the functional $\mathscr{R}$ constrained to the class of metrics with volume normalized to 1, or equivalently to consider the scaling-invariant version

$$\tilde{\mathscr{R}}(g) = Vol_g(M)^{\frac{2-n}{n}} \mathscr{R}(g). \tag{2}$$

The first variation of $\tilde{\mathscr{R}}$ is given by

$$d\tilde{\mathscr{R}}(g)[h] = -V(g)^{\frac{2-n}{n}} \left(\frac{n-2}{2n} V(g)^{-1} \mathscr{R}(g) \int_M \mathrm{tr}_g h \, dv_g + \int_M E^{ij} h_{ij} dv_g \right).$$

The restriction of the Einstein-Hilbert functional to a given conformal class of metrics, usually called the *Yamabe functional*, plays a central role in conformal geometry (see [Tr], [A1], [S1], [A2], [LP]). In fact, along a conformal variation of metric that infinitesimally preserves the volume, i.e. taking $h_{ij} = \eta \, g_{ij}$ with $\int_M \eta \, dv_g = 0$, by the above formula one has

$$d\tilde{\mathscr{R}}(g)[h] = \left(\frac{n}{2} - 1 \right) V(g)^{\frac{2-n}{n}} \int_M R_g \eta \, dv_g.$$

Criticality subject to the above constraint for η then implies that R_g is constant. With this condition the first variation of $\tilde{\mathscr{R}}$ becomes

$$d\tilde{\mathscr{R}}(g)[h] = V(g)^{\frac{2-n}{n}} \int_M \left(\frac{R_g}{n} g^{ij} - R^{ij} \right) h_{ij} dv_g = 0,$$

whose vanishing for all h implies that g is Einstein.

The round metric of S^n is a saddle point for $\tilde{\mathscr{R}}(\cdot)$, being a minimum restricted to its conformal class, and a local maximum in the orthogonal directions (see [S2], [K], [FM], [V]). To see this, one can split the family of variations h of g_{S^n} as $S_0 \oplus S_1 \oplus S_2$, where S_0 stands for the Lie derivatives of g_{S^n}, S_1 the pure trace tensors and S_2 the family of TT-variations, i.e. with zero trace and divergence. The former give zero contribution since they just induce a family of diffeomorphisms.

Concerning the component S_1, one has that for $h = \eta g_{S^n}$ with η of zero average, the volume of the metric is preserved at first order. The second derivative in this direction is then given by

$$d^2\tilde{\mathscr{R}}(g_{S^n})[h,h] = \frac{(n-1)(n-2)}{2}\int_{S^n}\left(|\nabla_{g_{S^n}}\eta|^2 - n\eta^2\right)dv_{g_{S^n}}\,, \qquad \int_{S^n}\eta\, dv_{g_{S^n}} = 0.$$

On the round sphere S^n the first non-zero eigenvalue of the Laplacian is equal to n, with eigenspace generated by the affine functions of $\mathbf{R}^{n+1}$ restricted to S^n, and therefore the quadratic form in the latter formula is non-negative definite.

Since the second variation of $\mathscr{R}$ diagonalizes with repsect to S_0, S_1 and S_2, we can therefore restrict ourselves to $h \in S_2$, and for this choice one finds (see e.g. Section 5.1 in [V])

$$d^2\tilde{\mathscr{R}}(g_{S^n})[h,h] = V(g)^{\frac{2-n}{n}}\int_{S^n}\langle h, \Delta h - h\rangle dv_g, \tag{3}$$

where

$$(\Delta h)_{ij} = \nabla^k\nabla_k h_{ij}.$$

One sees in this way that the second variation of $\tilde{\mathscr{R}}$ is strictly negative-definite on the subspace of variations S_2.

More in general, on Einstein manifolds different from the standard sphere the second variation in the conformal directions is strictly positive-definite due to the eigenvalue estimate in [L], while on manifolds of constant curvature $K > 0$ the second variation in the TT directions is strictly negative-definite, since in (3) the term $-h$ is replaced by $-K\,h$.

The purpose of this paper is to study the situation on pseudohermitian manifolds, about which we recall the definition and properties in Section 2. Here we limit ourselves to mention that they are $(2n+1)$-dimensional manifolds with an n-dimensional complex sub-bundle $\mathscr{H}$ of the complexified tangent bundle $T^{\mathbf{C}}M$ of M, such that $\mathscr{H} \cap \overline{\mathscr{H}} = \{0\}$, $[\mathscr{H}, \mathscr{H}] \subseteq \mathscr{H}$, and on which a complex rotation J acts. Letting $H(M) = \mathfrak{Re}(\mathscr{H} \oplus \overline{\mathscr{H}})$, there exists a one-form θ such that $H(M) = \ker\theta$. Classical examples are hypersurfaces of $\mathbf{C}^n$ or the Heisenberg group. If the *Levi form* $L_\theta(W, \overline{Z}) := -id\theta(W, \overline{Z})$ is non degenerate, pseudohermitian manifolds carry a natural connection, the *Tanaka-Webster connection* (see [DT], [L2], [We]), and from it curvature operators can be built in analogy with the Riemannian case. The pseudohermitian counterpart of the scalar curvature is called *Webster curvature*, and $\theta \wedge (d\theta)^n$ acts as a natural volume form.

In the study of properties of CR geometry one observes a difference between low and high dimension related to the embeddable character of the underlying structures. By a classical theorem by Boutet de Monvel, see [Bo], if a closed CR manifold M of dimension $n \geq 5$ is strictly pseudoconvex (that is, the Levi form is positive-definite), then it can be CR-embedded in $\mathbf{C}^N$ for some natural integer N. This is not always true in dimension three (see [CS]) and, as we will see, this fact has repercussions on the analogies between CR and Riemannian geometry.

Some conditions in three dimensions that characterize embeddability are as follows. In [CCY], Chanillo, Chiu and Yang found sufficient conditions for embeddability related to the spectral properties of the *CR Paneitz operator* (see [Tk] for a partial converse). In [CMY1] a positive mass theorem for embeddable three-dimensional CR manifolds was proved, while, as a counterexample by the same authors in [CMY2] shows, in the non embeddable case the pseudohermitian mass can be negative. For pseudohermitian structures which are perturbations of the standard one on S^3, Bland characterized in [Bl] embeddability in terms of the Fourier expansion of the deformation tensor, as it will be recalled below.

In analogy with (1) and (2), denoting by $W_{J,\theta}$ the Webster curvature on a CR manifold M, we set

$$\mathscr{W}(J,\theta) = \int_M W_{J,\theta}\,\theta \wedge (d\theta)^n;$$

$$\tilde{\mathscr{W}}(J,\theta) = \left(\int_M \theta \wedge (d\theta)^n \right)^{-\frac{Q-2}{Q}} \mathscr{W}(J,\theta),$$

where $Q = 2n+2$ stands for the *homogeneous dimension* of the manifold M.

We have first the following result.

Theorem 1.1. *Suppose (J,θ) is critical for $\tilde{\mathscr{W}}$. Then the Webster curvature of (M,J,θ) is constant and the torsion vanishes identically. Moreover, if $c_1(\mathcal{H}) = 0$ then (M,J,θ) is pseudo-Einstein.*

Remark 1.2. *Recall that a pseudohermitian structure is called* pseudo-Einstein *when the Ricci tensor is a constant multiple of the Levi form, see [L2] and Section 2, which is trivial when $n=1$. We notice that by Proposition D in [L2], the requirement that $c_1(\mathcal{H}) = 0$ is necessary for the existence of a pseudo-Einstein structure.*

The constancy of the Webster curvature and the vanishing of the torsion follow from the first variation formula for $\tilde{\mathscr{W}}$. By Theorem E in [L2], there exists a conformal choice of contact form which is pseudo-Einstein. The fact that this occurs for the critical structure itself is a consequence of a divergence formula by Xiaodong Wang, displayed in the appendix of the paper (see also [Wa] for a related result).

We next specialize to the case of the spheres S^{2n+1} endowed with the standard CR structure inherited from $\mathbf{C}^{n+1}$, i.e. $\mathcal{H}(S^{2n+1}) = T^{1,0}\mathbf{C}^{n+1} \cap T_p^{\mathbf{C}} S^{2n+1}$ and the complex rotation J_0 is the restriction of the ambient complex one to the holomorphic tangent space. A standard choice of contact form is given by

$$\theta_0 = \frac{i}{2} \sum_{k=1}^{2} \left(z^k dz^{\overline{k}} - z^{\overline{k}} dz^k \right),$$

in which case the volume element induced by θ_0 compares to the Euclidean one as

$$\theta_0 \wedge (d\theta_0)^n = 2^n n!\, dv_{\mathrm{Eucl.}}.$$

This can be easily seen for example by evaluating the two volume forms at the point $(1, 0, \ldots, 0) \in \mathbf{R}^{2n+2}$ and by using the homogeneity of the spherical structure.

For $u \in C^\infty(S^{2n+1})$, we also define the *sub-gradient* of u as

$$\nabla_b u = \Pi \nabla_{g_{S^{2n+1}}} u$$

where Π denotes the orthogonal projection onto $H(S^{2n+1})$.

In this paper we are going to exhibit another relationship between embeddability and geometric properties of CR structures. Starting from the three-dimensional case, we define next a variation of J as

$$\dot{J} = 2E = 2E_1{}^{\overline{1}}\theta^1 \otimes Z_{\overline{1}} + \text{conj.}, \tag{4}$$

where, on S^3, we consider the standard generator of $\mathcal{H}$ and its dual form

$$Z_1 = z^{\overline{2}} \frac{\partial}{\partial z^1} - z^{\overline{1}} \frac{\partial}{\partial z^2}, \qquad \theta^1 = z^2 dz^1 - z^1 dz^2. \tag{5}$$

For z_1, z_2 the complex coordinates of $\mathbf{C}^2 \supseteq S^3$, we define then the subspace

$$\Gamma_m := \left\{ f \in C^\infty(S^3; \mathbf{C}) \mid f(e^{i\theta} z_1, e^{i\theta} z_2) = e^{im\theta} f(z_1, z_2) \right\}. \tag{6}$$

For products of powers in the z- and $\overline{z}$-coordinates, the index m counts the number of holomorphic factors minus the anti-holomorphic ones.

Theorem 1.3. *Consider the standard pseudohermitian structure (S^3, J_0, θ_0).*

(i) Let $\eta \in C^\infty(S^3)$ be such that $\int_{S^3} \eta\, \theta_0 \wedge d\theta_0 = 0$. Then

$$d^2_\theta \tilde{\mathscr{W}}(J_0, \theta_0)[\eta\theta_0, \eta\theta_0] = c_3 \int_{S^3} (|\nabla_b \eta|^2 - 3\eta^2)\theta_0 \wedge d\theta_0 \geq 0$$

for some positive constant c_3, with equality holding if and only is η is the restriction to S^3 of some linear function on $\mathbf{C}^2$.

(ii) The second derivative of $\tilde{\mathscr{W}}(J_0, \theta_0)$ diagonalizes with respect to the splitting in Γ_m: precisely, setting $E_1^{\overline{1}} = \sum_{m \in \mathbf{Z}} E^{(m)}$, one has

$$d^2_J \tilde{\mathscr{W}}(J_0, \theta_0)[E, E] = \sum_m (m+4) \int_{S^3} |E^{(m)}|^2 \theta_0 \wedge d\theta_0 \tag{7}$$

Some comments on this result are in order. First, the behaviour of the functional in the conformal directions is completely analogous to the Riemannian case. However, a difference appears in the sign of second variation in the complementary directions, which we show to be tightly related to the embeddability properties of the (infinitesimally) perturbed CR structures. In fact, it was proved in [Bl] via a normal form that the perturbed structures that are embeddable are precisely characterized by having vanishing Fourier components $E^{(m)}$ for $m \leq -4$. Notice the difference in notation by a conjugation of the coefficient from (13.7) in [Bl] and (4), so the present condition on m changes by a sign compared to Theorem 15.1 in [Bl]. We then observe a situation similar to the Riemannian one as long as the CR structure stays embeddable, with a reversed sign if we stay infinitesimally orthogonal to these and to the conformal deformations. This fact is coherent with the results obtained in [CMY1] and [CMY2], where estimates on the Sobolev quotient were derived for a class of embeddable three-manifolds and for the (non-embeddable) Rossi spheres.

We consider next the higher-dimensional case, which turns out to be always in analogy with the Riemannian one: recall from the previous discussion that in dimensions greater or equal to five the CR structures are always embeddable.

Theorem 1.4. *Consider the standard pseudohermitian structure (J_0, θ_0) on S^{2n+1} with $n > 1$.*

(i) Let $\eta \in C^\infty(S^{2n+1})$ be such that $\int_{S^{2n+1}} \eta\, \theta_0 \wedge d\theta_0 = 0$. Then

$$d^2_\theta \tilde{\mathscr{W}}(J_0, \theta_0)[\eta\theta_0, \eta\theta_0] = c_n \int_{S^{2n+1}} (|\nabla_b \eta|^2 - n\eta^2)\theta_0 \wedge d\theta_0 \geq 0$$

for some positive constant c_n, with equality holding if and only is η is the restriction to S^{2n+1} of some linear function on $\mathbf{C}^{n+1}$.

(ii) For the variation in J we have instead

$$d^2_J \tilde{\mathscr{W}}(J_0, \theta_0)[E, E] > 0 \qquad \text{for any } E \not\equiv 0.$$

Even for $n > 1$ a formula similar to (7) holds true, but since we would need to introduce extra notation, we chose to postpone it to the final section of the paper and state in the above theorem only its consequence.

While in three dimensions it is a single complex function to determine a variation E of the CR structure J, in higher dimensions we need to work on a vector bundle over S^{2n+1}. To carry over the calculation of first- and second-order variations we use a frame approach as in e.g. [L2], leading to formula (35) for the second variation of the integral of the Webster curvature. However, to understand its sign it will be more practical for us to use the formalism in [G], that employs a basis of vector fields constructed from the ambient coordinates. Although this family does not form a linearly independent system, it has the advantage of leading to constant-coefficient quantities, that are more suitable to be analyzed via Fourier modes. We then rely crucially on the results in [BD1] and [BD2], where a parametrization of deformations of the CR structure of the sphere is performed via a suitable Banach manifold and via Fourier analysis.

The plan on the paper is as follows. In Section 2 we review some preliminary material that includes basic notions in pseudohermitian geometry: we derive in particular some first properties of deformations of the pseudohermitian structure. In Section 3 we then derive useful formulas for the variation of interesting geometric quantities, and we derive in particular an expression for the second variation of the Webster curvature. In Section 4 we prove our main theorems, with the completion of Theorem 1.1 performed in the Appendix.

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2 Some preliminary facts

In this section we recall some useful basic material on pseudohermitian manifolds, as well as some calculation concerning the variation of the contact form or of the CR structure.

About the forthcoming review material, we refer the reader to [DT] or [L2]. We recall that a *CR manifold* is a real smooth manifold M endowed with a complex sub-bundle $\mathscr{H} = T_{1,0}M$ of the complexified tangent bundle of M, $T^{\mathbf{C}}M$, such that $\mathscr{H} \cap \overline{\mathscr{H}} = \{0\}$ and $[\mathscr{H}, \mathscr{H}] \subseteq \mathscr{H}$. We will assume M to be of hypersurface type, that is $\dim M = 2n+1$ and $\dim_{\mathbf{C}} \mathscr{H} = n$. Let $H(M)$ denote the space $\mathfrak{Re}(\mathscr{H} \oplus \overline{\mathscr{H}})$. Then there exists a natural complex structure on $H(M)$ given by

$$J(Z + \overline{Z}) = i(Z - \overline{Z}).$$

The CR structure is uniquely determined by $H(M)$ and J. For $H(M)$ and J to generate a CR structure it is necessary that $\mathscr{H}$ is closed under Lie bracket operation. In three dimensions $\mathscr{H}$ is one-dimensional, so the condition $[\mathscr{H}, \mathscr{H}] \subseteq \mathscr{H}$ is automatically satisfied.

There exists a non-zero real differential form θ whose kernel at every point coincides with $H(M)$; it is unique up to scalar multiplication by a non-zero function. A triple (M, J, θ) as above is called a *pseudohermitian structure*. On a pseudohermitian manifold, the Levi form is defined as

$$L_\theta(V, \overline{W}) = -i d\theta(V, \overline{W}) = i\theta([V, \overline{W}]).$$

A CR manifold is said to be strictly *pseudoconvex* (respectively, *non-degenerate*) if it admits a positive definite (respectively, non-degenerate) Levi form. Non-degeneracy is equivalent to the fact that θ is a contact form (see Proposition 1.9 and formula (1.66) in [DT]). In this case, there exists a unique vector field T such that $i_T d\theta = 0$ and $\theta(T) = 1$. For example, if $z_1, \ldots, z_{n+1}$ are standard complex coordinates on $\mathbf{C}^{n+1}$, then on the unit sphere S^{2n+1} standard structures are given by

$$T = T_0 = \frac{i}{2} \sum_{\alpha=1}^{n+1} \left(z_\alpha \frac{\partial}{\partial z_\alpha} - \overline{z}_\alpha \frac{\partial}{\partial \overline{z}_\alpha} \right);$$

$$\theta = \theta_0 = \frac{i}{2} \sum_{\alpha=1}^{n+1} \left(z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha \right).$$

As mentioned in the introduction, classical examples of pseudohermitian manifolds are the Heisenberg group or boundaries of pseudoconvex domains in complex spaces.

On a nondegenerate pseudohermitian manifold one can introduce a connection, called the *Tanaka-Webster connection*. To define it, we recall some useful facts, mostly from [L2]. If $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$ is a frame dual to $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$, we express the Levi form as

$$L_\theta(f^\alpha Z_\alpha, g^{\overline{\beta}} Z_{\overline{\beta}}) = h_{\alpha\overline{\beta}} f^\alpha g^{\overline{\beta}}.$$

The matrix $h_{\alpha\overline{\beta}} = \delta_\alpha^\beta$ will be used in a standard way to raise and lower indices. The Webster connection forms ω_α^β and the torsion forms $\tau_\beta = A_{\beta\alpha}\theta^\alpha$ are defined by the equations

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta; \qquad \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha} = dh_{\alpha\overline{\beta}}; \qquad A_{\alpha\beta} = A_{\beta\alpha}. \tag{8}$$

Also, the *curvature forms*

$$\Pi_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$$

satisfy the *structure equations*

$$\Pi_\alpha^\beta = R_{\alpha\rho\overline{\sigma}}^\beta \theta^\rho \wedge \theta^{\overline{\sigma}} + W_{\alpha\gamma}^\beta \theta^\gamma \wedge \theta - W_{\alpha\overline{\gamma}}^\beta \theta^{\overline{\gamma}} \wedge \theta + i\theta_\alpha \wedge \tau^\beta - i\tau_\alpha \wedge \theta^\beta.$$

The *Ricci tensor* and the pseudohermitian scalar (or *Webster*) curvature are defined by the contractions

$$R_{\rho\overline{\sigma}} = R_{\alpha\rho\overline{\sigma}}^\alpha; \qquad W = R_\alpha^\alpha.$$

Recall that (M, J, θ) is *pseudo-Einstein* when the Ricci tensor is a scalar multiple of the Levi metric. A consequence of the Bianchi identities from [L2] is the constancy of the Webster curvature for pseudo-Einstein structures with vanishing torsion.

The covariant differentiation is characterized by the formulas

$$\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta; \qquad \nabla Z_{\overline{\alpha}} = \omega_{\overline{\alpha}}^{\overline{\beta}} \otimes Z_{\overline{\beta}}; \qquad \nabla T = 0. \tag{9}$$

For a tensor S with components $S_{\cdot}^{\cdot}$ we will use the notation

$$S_{\cdot,\alpha}^{\cdot} := (\nabla_{Z_\alpha} S)_{\cdot}^{\cdot}; \qquad S_{\cdot,\overline{\alpha}}^{\cdot} := (\nabla_{Z_{\overline{\alpha}}} S)_{\cdot}^{\cdot}.$$

We also have the following commutation relations for second-order covariant derivatives of functions u and $(1,0)$-forms $\sigma = \sigma_\alpha \theta^\alpha$, see e.g. Lemma 2.3 in [L2]:

$$u_{,\alpha\overline{\beta}} - u_{,\overline{\beta}\alpha} = i h_{\alpha\overline{\beta}} u_{,0}; \qquad u_{,\alpha\beta} = u_{,\beta\alpha}; \qquad u_{,0\alpha} - u_{,\alpha 0} = A_{\alpha\beta} u^{,\beta};$$

$$\sigma_{\alpha,\beta\gamma} - \sigma_{\alpha,\gamma\beta} = i A_{\alpha\gamma}\sigma_\beta - i A_{\alpha\beta}\sigma_\gamma;$$

$$\sigma_{\alpha,\overline{\beta}\overline{\gamma}} - \sigma_{\alpha,\overline{\gamma}\overline{\beta}} = i h_{\alpha\overline{\beta}} A_{\overline{\gamma}\overline{\rho}} \sigma^{\overline{\rho}} - i h_{\alpha\overline{\gamma}} A_{\overline{\beta}\overline{\rho}} \sigma^{\overline{\rho}};$$

$$\sigma_{\alpha,\beta\overline{\gamma}} - \sigma_{\alpha,\overline{\gamma}\beta} = i h_{\beta\overline{\gamma}} \sigma_{\alpha,0} + R^{\rho}_{\alpha\beta\overline{\gamma}} \sigma_\rho;$$

$$\sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} = \sigma^{\gamma}_{\alpha,} A_{\gamma\beta} - \sigma_\gamma A^{\gamma}_{\alpha\beta,}; \qquad \sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} = \sigma_{\alpha,\gamma} A^{\gamma}_{\beta} + \sigma_\gamma A^{\gamma}_{\beta,\alpha}.$$

As a consequence of Bianchi's identities, see e.g. Lemma 2.2 in [L2], we have in particular that

$$A_{\alpha\beta,\gamma} = A_{\alpha\gamma,\beta} \quad \text{for all indices } \alpha, \beta, \gamma. \tag{10}$$

The sub-Laplacian of a scalar function $u \in C^\infty(M)$ is then defined as (see e.g. formula (4.10) in [L1])

$$\Delta_b u = u^{\alpha}_{,\alpha} + u^{\overline{\alpha}}_{,\overline{\alpha}}.$$

The following result will be useful to derive tensorial identities because of the vanishing of some terms in their expressions.

Lemma 2.1. *([L2]) For any point $p \in M$, there exists a neighborhood U_p and an admissible co-frame $(\theta^\alpha)_\alpha$ such that $\omega_\alpha^\beta = 0$ at p.*

We recall next the transformation law of the Webster curvature under conformal changes of contact form, see e.g. [JL2]. If one writes $Q = 2n+2$ for the *homogeneous dimension* of (M, J, θ), and $\tilde{\theta} = u^{\frac{4}{Q-2}}\theta$, then the Webster curvature $W = W_\theta$ transforms as

$$-\left(2 + \frac{2}{n}\right)\Delta_b u + W_\theta u = W_{\tilde{\theta}} u^{\frac{Q+2}{Q-2}}. \tag{11}$$

It will then be useful to recall some spectral properties of the sub-Laplace operator on spheres, see e.g. [F] or [G].

Proposition 2.2. *Let $H_{p,q,n}$ denote the restriction to S^{2n+1} of the homogeneous complex harmonic polynomials of degree $p+q$, where p is the holomorphic homogeneity and q the antiholomorphic one. Then one has*

$$-\Delta_b \phi = \lambda_{p,q,n}\phi \quad \textit{for all } \phi \in H_{p,q,n}; \qquad \lambda_{p,q,n} = pq + \frac{n}{2}(p+q).$$

Moreover, let

$$\Gamma_m = \left\{ u \in C^\infty(S^{2n+1}) \mid u(e^{i\theta}z_1, \dots e^{i\theta}z_n) = e^{im\theta}u(z_1, \dots, z_n) \right\}.$$

Then one has

$$T\phi = i\frac{m}{2}\phi \quad \textit{for all} \quad \phi \in \Gamma_m.$$

We next consider a deformation of the CR structure, keeping the contact form fixed: we introduce some notation for this purpose and derive some preliminary properties.

Given the contact bundle ξ, consider a smooth family $t \mapsto J_{(t)}$ of CR structures on ξ. Then, for all values of t, $J_{(t)} : \xi \to \xi$ satisfies $J_{(t)}^2 = -Id$. Take a basis of eigenvectors $(Z_{\alpha(t)})_\alpha$ such that $J_{(t)}Z_{\alpha(t)} = iZ_{\alpha(t)}$, which implies for the conjugate vector fields $J_{(t)}\overline{Z}_{\alpha(t)} = -i\overline{Z}_{\alpha(t)}$. In this way $J_{(t)}$ writes as

$$J_{(t)} = i\theta^\alpha_{(t)} \otimes Z_{\alpha(t)} - i\theta^{\overline{\alpha}}_{(t)} \otimes Z_{\overline{\alpha}(t)}.$$

We begin with the following simple result.

Lemma 2.3. *Setting $\dot{J} := \dot{J}_{(t)} = \frac{d}{dt}J_{(t)}$, one has the simplified relation*

$$\dot{J} = 2E = 2E_\alpha{}^{\overline{\beta}}\theta^\alpha \otimes Z_{\overline{\beta}} + \text{conj}.$$

Proof. By the above formula we get $\dot{J}(Z_\alpha) = 2E_\alpha{}^{\overline{\beta}}Z_{\overline{\beta}} + 2E_\alpha{}^\beta Z_\beta$. Differentiate the relation $J_{(t)}^2 = -Id$ with respect to t (at $t = 0$) to obtain:

$$\dot{J} \circ J + J \circ \dot{J} = 0.$$

Expressing this relation with respect to the basis $(Z_\alpha)_\alpha$ and $(Z_{\overline{\alpha}})_\alpha$, we obtain

$$\begin{pmatrix} E_\alpha{}^\beta & E_\alpha{}^{\overline{\beta}} \\ E_{\overline{\alpha}}{}^\beta & E_{\overline{\alpha}}{}^{\overline{\beta}} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} E_\alpha{}^\beta & E_\alpha{}^{\overline{\beta}} \\ E_{\overline{\alpha}}{}^\beta & E_{\overline{\alpha}}{}^{\overline{\beta}} \end{pmatrix} = 0,$$

which implies $E_\alpha{}^\beta = E_{\overline{\alpha}}{}^{\overline{\beta}} = 0$, as claimed. $\square$

We consider next the variation of the basis vector fields with respect to t.

Lemma 2.4. *Let us write the derivative of the unitary frame $(Z_\alpha)_\alpha$ and its dual forms $(\theta^\alpha)_\alpha$ as*

$$\dot{Z}_\alpha = F_\alpha{}^\beta Z_\beta + G_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}}; \qquad \dot{\theta}^{\overline{\gamma}} = iE_l^{\overline{\gamma}}\theta^l - F_{\overline{l}}^{\overline{\gamma}}\theta^{\overline{l}},$$

Then we can assume that ${F_\alpha}^\beta \in \mathbf{R}$*, and that there holds*

$$
{G_\alpha}^{\overline{\beta}} = -i{E_\alpha}^{\overline{\beta}}; \qquad {F_\alpha}^\beta + {F_\beta}^\alpha = 0. \tag{12}
$$

Moreover, at $t = 0$ *we can take* ${F_\beta}^\alpha = 0$*.*

Proof. For notational simplicity we will often write F^α_β instead of ${F_\beta}^\alpha$, $E^{\overline{\beta}}_\alpha$ instead of ${E_\alpha}^{\overline{\beta}}$, etc..

We have $d\theta = i \sum \theta^\alpha_{(t)} \wedge \theta^{\overline{\alpha}}_{(t)}$ (from $h_{\alpha\overline{\beta}} = \delta^\beta_\alpha$), compare also to Lemma 2.1 in [CL], so

$$
-2id\theta(Z_{\alpha(t)} \wedge Z_{\overline{\beta}(t)}) = \delta^\beta_\alpha.
$$

Differentiating this relation in t, we get

$$
\begin{aligned}
0 &= -2id\theta(\dot{Z}_\alpha \wedge Z_{\overline{\beta}}) - 2id\theta(Z_\alpha \wedge \dot{Z}_{\overline{\beta}}) \\
&= -2id\theta(({F_\alpha}^\gamma Z_\gamma + {G_\alpha}^{\overline{\gamma}} Z_{\overline{\gamma}}) \wedge Z_{\overline{\beta}}) - 2id\theta(Z_\alpha \wedge (\overline{{F_\beta}^\gamma} Z_{\overline{\gamma}} + \overline{{G_\beta}^{\overline{\gamma}}} Z_\gamma)) \\
&= -2i{F_\alpha}^\beta - 2i\overline{{F_\beta}^\alpha}.
\end{aligned} \tag{13}
$$

On the other hand, we can always compose the frame $(Z_\alpha)_\alpha$ with an element of $SU(n)$, which infinitesimally means adding to F^β_α a matrix B^β_α such that $B^\beta_\alpha = -\overline{B}^\alpha_\beta$. We can choose for example to add $\overline{F}^\beta_\alpha$, which satisfies this property by the above relation (13): this means that we can take F to be real, and implies the second relation in (12).

To get the first one, differentiate $J_{(t)} Z_{\alpha(t)} = iZ_{\alpha(t)}$ with respect to t, to find

$$
\begin{aligned}
0 &= \dot{J} Z_\alpha + J\dot{Z}_\alpha - i\dot{Z}_\alpha \\
&= 2{E_\alpha}^{\overline{\beta}} Z_{\overline{\beta}} + {F_\alpha}^\beta iZ_\beta + {G_\alpha}^{\overline{\beta}}(-i)Z_{\overline{\beta}} - i({F_\alpha}^\beta Z_\beta + {G_\alpha}^{\overline{\beta}} Z_{\overline{\beta}}) \\
&= 2{E_\alpha}^{\overline{\beta}} Z_{\overline{\beta}} - 2i{G_\alpha}^{\overline{\beta}} Z_{\overline{\beta}},
\end{aligned}
$$

so ${G_\alpha}^{\overline{\beta}} = -i{E_\alpha}^{\overline{\beta}}$, as desired.

To prove that we can take $F = 0$ at $t = 0$, let $\mathfrak{F} = F^\beta_\alpha$ at $t = 0$ and consider the new frame

$$
\tilde{Z}_{\alpha(t)} = (e^{-t\mathfrak{F}})^\beta_\alpha Z_\beta.
$$

Then, by cancellation

$$
\tilde{Z}_{\alpha(0)} = Z_{\alpha(0)} \quad \text{and} \quad \frac{d}{dt}_{t=0} \tilde{Z}_{\alpha(t)} = -iE^{\overline{\beta}}_{\alpha(0)} \tilde{Z}_{\overline{\beta}(0)},
$$

concluding the proof. ∎

We next derive some consequences of the integrability conditions

$$
\theta([Z_\alpha, Z_\beta]) = 0; \qquad \theta^{\overline{\gamma}}([Z_\alpha, Z_\beta]) = 0, \tag{14}
$$

which hold along all the deformation, i.e. for all t. We have the following result.

Lemma 2.5. *For all indices α, β, γ we have that*

$$E_{\alpha\beta} = E_{\beta\alpha}; \qquad E^{\overline{\gamma}}_{\alpha,\beta} = E^{\overline{\gamma}}_{\beta,\alpha}.$$

Proof. We differentiate in t the first relation in (14), obtaining

$$\begin{aligned}\frac{d}{dt}[Z_\alpha, Z_\beta] &= [\dot{Z}_\alpha, Z_\beta] + [Z_\alpha, \dot{Z}_\beta] \\ &= [F^\gamma_\alpha Z_\gamma - iE^{\overline{\gamma}}_\alpha Z_{\overline{\gamma}}, Z_\beta] + [Z_\alpha, F^\gamma_\beta Z_\gamma - iE^{\overline{\gamma}}_\beta Z_{\overline{\gamma}}],\end{aligned}$$

which by integrability yields

$$0 = \theta\left(\frac{d}{dt}[Z_\alpha, Z_\beta]\right) = -iE^{\overline{\gamma}}_\alpha \theta([Z_{\overline{\gamma}}, Z_\beta]) - iE^{\overline{\gamma}}_\beta \theta([Z_\alpha, Z_{\overline{\gamma}}]).$$

Notice that

$$[Z_{\overline{\gamma}}, Z_\beta] = ih_{\beta\overline{\gamma}}T + \omega^l_\beta(Z_{\overline{\gamma}})Z_l - \omega^{\bar{l}}_{\overline{\gamma}}(Z_\beta)Z_{\bar{l}},$$

which in turn implies

$$0 = -iE^{\overline{\gamma}}_\alpha ih_{\beta\overline{\gamma}} + iE^{\overline{\gamma}}_\beta ih_{\alpha\overline{\gamma}}.$$

Since $h_{\beta\overline{\gamma}} = \delta^\gamma_\beta$, we deduce the first assertion of the lemma.

We next differentiate in t the second relation in (14) to get

$$0 = \dot{\theta}^{\overline{\gamma}}([Z_\alpha, Z_\beta]) + \theta^{\overline{\gamma}}\left(\frac{d}{dt}[Z_\alpha, Z_\beta]\right).$$

Since by Lemma 2.4

$$\dot{\theta}^{\overline{\gamma}} = iE^{\overline{\gamma}}_l \theta^l - F^{\overline{\gamma}}_{\bar{l}} \theta^{\bar{l}},$$

and

$$[Z_\alpha, Z_\beta] = \omega^l_\beta(Z_\alpha)Z_l - \omega^l_\alpha(Z_\beta)Z_l,$$

we obtain

$$\begin{aligned}0 &= iE^{\overline{\gamma}}_l \theta^l(\omega^l_\beta(Z_\alpha) - \omega^l_\alpha(Z_\beta)) \\ &\quad + \theta^{\overline{\gamma}}\Big([Z_\alpha, F^\gamma_\beta Z_\gamma - iE^{\overline{\gamma}}_\beta Z_{\overline{\gamma}}] + [Z_\alpha, F^\gamma_\beta Z_\gamma - iE^{\overline{\gamma}}_\beta Z_{\overline{\gamma}}]\Big) \\ &= -i\omega^l_\alpha(Z_\beta)E^{\overline{\gamma}}_l + i\omega^l_\beta(Z_\alpha)E^{\overline{\gamma}}_l + iE^{\bar{l}}_\alpha \omega^{\overline{\gamma}}_{\bar{l}}(Z_\beta) \\ &\quad + iZ_\beta(E^{\overline{\gamma}})_\alpha - iE^{\bar{l}}_\beta \omega^{\overline{\gamma}}_{\bar{l}}(Z_\alpha) - iZ_\alpha(E^{\bar{l}}_\beta).\end{aligned}$$

This implies

$$iE^{\overline{\gamma}}_{\alpha,\beta} - iE^{\overline{\gamma}}_{\beta,\alpha} = 0,$$

which is the second assertion. $\square$

3 Variation of geometric quantities

In this section we compute the variation of several relevant geometric quantities along the deformation of J. Our final goal is to derive a second variation formula for the normalized integral of the Webster curvature at its critical points.

We begin with the variation of connection and torsion.

Lemma 3.1. *For all t, the variation of the torsion is given by*

$$\dot{A}^{\alpha}_{\overline{\gamma}} = -iE^{\alpha}_{\overline{\gamma},0} + A^{\alpha}_{\overline{l}}F^{\overline{l}}_{\overline{\gamma}} - F^{\alpha}_{l}A^{l}_{\overline{\gamma}}, \tag{15}$$

while for the variation of the connection we have

$$\begin{aligned}\dot{\omega}^{\alpha}_{\beta} = &\left[i(A^{\alpha}_{\overline{\gamma}}E^{\overline{\gamma}}_{\beta} + E^{\alpha}_{\overline{\gamma}}A^{\overline{\gamma}}_{\beta}) + F^{\alpha}_{\beta,0}\right]\theta \\ &+ (-iE^{\overline{\beta}}_{\gamma,\overline{\alpha}} - F^{\overline{\beta}}_{\overline{\alpha},\gamma})\theta^{\gamma} + (-iE^{\alpha}_{\overline{\gamma},\beta} + F^{\alpha}_{\beta,\overline{\gamma}})\theta^{\overline{\gamma}}.\end{aligned} \tag{16}$$

Proof. We start by differentiating in t the relations

$$\theta^{\alpha}_{(t)}(Z_{\beta(t)}) = \delta^{\alpha}_{\beta}; \qquad \theta^{\alpha}_{(t)}(Z_{\overline{\beta}(t)}) = \theta^{\alpha}_{(t)}(T) = 0,$$

to get

$$\dot{\theta}^{\alpha}(Z_{\beta}) + \theta^{\alpha}(\dot{Z}_{\beta}) = 0; \qquad \dot{\theta}^{\alpha}(Z_{\overline{\beta}}) + \theta^{\alpha}(\dot{Z}_{\overline{\beta}}) = 0; \qquad \dot{\theta}^{\alpha}(T) = 0.$$

Recall that by Lemma 2.4 one has

$$\dot{\theta}^{\alpha} = -iE^{\alpha}_{\overline{\beta}}\theta^{\overline{\beta}} - F^{\alpha}_{\beta}\theta^{\beta}. \tag{17}$$

Differentiate now in t the structure equation

$$d\theta^{\alpha} = \theta^{\beta}_{(t)} \wedge \omega^{\alpha}_{\beta(t)} + A^{\alpha}_{\overline{\gamma}(t)}\theta \wedge \theta^{\overline{\gamma}}_{(t)}$$

to get

$$\begin{aligned}d\dot{\theta}^{\alpha} &= \dot{\theta}^{\beta} \wedge \omega^{\alpha}_{\beta} + \theta^{\beta} \wedge \dot{\omega}^{\alpha}_{\beta} + \dot{A}^{\alpha}_{\overline{\gamma}}\theta \wedge \theta^{\overline{\gamma}} + A^{\alpha}_{\overline{\gamma}}\theta \wedge \dot{\theta}^{\overline{\gamma}} \\ &= -idE^{\alpha}_{\overline{\beta}} \wedge \theta^{\overline{\beta}} - iE^{\alpha}_{\overline{\beta}}d\theta^{\overline{\beta}} - dF^{\alpha}_{\beta} \wedge \theta^{\beta} - F^{\alpha}_{\beta}d\theta^{\beta}.\end{aligned} \tag{18}$$

At a given point p we may assume that $\omega^{\alpha}_{\beta} = 0$, by Lemma 2.1. Therefore we obtain at p

$$-idE^{\alpha}_{\overline{\beta}} \wedge \theta^{\overline{\beta}} = -iE^{\alpha}_{\overline{\beta},0}\theta \wedge \theta^{\overline{\beta}} - iE^{\alpha}_{\overline{\beta},\gamma}\theta^{\gamma} \wedge \theta^{\overline{\beta}} - iE^{\alpha}_{\overline{\beta},\overline{\gamma}}\theta^{\overline{\gamma}} \wedge \theta^{\overline{\beta}};$$

$$d\theta^{\overline{\beta}} = \theta^{\overline{\gamma}} \wedge \omega^{\overline{\beta}}_{\overline{\gamma}} + A^{\overline{\beta}}_{\gamma}\theta \wedge \theta^{\gamma} = A^{\overline{\beta}}_{\gamma}\theta \wedge \theta^{\gamma};$$

$$-dF^{\alpha}_{\beta} \wedge \theta^{\beta} - F^{\alpha}_{\beta}d\theta^{\beta} = -F^{\alpha}_{\beta,0}\theta \wedge \theta^{\beta} - F^{\alpha}_{\beta,\gamma}\theta^{\gamma} \wedge \theta^{\beta} - F^{\alpha}_{\beta,\overline{\gamma}}\theta^{\overline{\gamma}} \wedge \theta^{\beta} - F^{\alpha}_{\beta}A^{\beta}_{\overline{\gamma}}\theta \wedge \theta^{\overline{\gamma}}.$$

Comparing the coefficients of $\theta \wedge \theta^{\overline{\gamma}}$ in (18), we deduce the first assertion.

Write next

$$\dot{\omega}_\beta^\alpha = x_\beta^\alpha \theta + y_{\beta\gamma}^\alpha \theta^\gamma + y_{\beta\overline{\gamma}}^\alpha \theta^{\overline{\gamma}}. \tag{19}$$

From the relation $\omega_\alpha^\beta + \omega_{\overline{\beta}}^{\overline{\alpha}} = 0$, which implies $\dot{\omega}_\alpha^\beta + \dot{\omega}_{\overline{\beta}}^{\overline{\alpha}} = 0$, we get the system

$$\begin{cases} x_\beta^\alpha + \overline{x_{\overline{\alpha}}^{\overline{\beta}}} = 0; \\ y_{\beta\gamma}^\alpha = -\overline{y_{\alpha\overline{\gamma}}^\beta} := -y_{\overline{\alpha}\gamma}^{\overline{\beta}}. \end{cases}$$

Substituting (17), (15) and (19) into (18) we obtain

$$\begin{cases} x_\beta^\alpha = iE_{\overline{\gamma}}^\alpha A_\beta^{\overline{\gamma}} + iA_{\overline{\gamma}}^\alpha E_\beta^{\overline{\gamma}} + F_{\beta,0}^\alpha; \\ y_{\beta\overline{\gamma}}^\alpha = -iE_{\overline{\gamma},\beta}^\alpha + F_{\beta,\overline{\gamma}}^\alpha, \end{cases} \tag{20}$$

which implies in particular $y_{\beta\gamma}^\alpha = -iE_{\gamma,\overline{\alpha}}^{\overline{\beta}} - F_{\overline{\alpha},\gamma}^{\overline{\beta}}$. We also get the relations

$$(y_{\beta\gamma}^\alpha - F_{\beta,\gamma}^\alpha)\theta^\beta \wedge \theta^\gamma = 0; \qquad -iE_{\overline{\beta},\overline{\gamma}}^\alpha \theta^{\overline{\gamma}} \wedge \theta^{\overline{\beta}} = 0,$$

giving the following constraints on the deformation tensors

$$\begin{cases} y_{\beta\gamma}^\alpha - F_{\beta,\gamma}^\alpha = y_{\gamma\beta}^\alpha - F_{\gamma,\beta}^\alpha; \\ E_{\overline{\beta},\overline{\gamma}}^\alpha = E_{\overline{\gamma},\overline{\beta}}^\alpha. \end{cases}$$

In this way, we obtain the second assertion as well. □

We can now compute the derivative of the curvature tensor with respect to t, together with its contractions.

Proposition 3.2. *For the curvature tensor, the Ricci tensor and the Webster curvature we have the following variation formulas*

$$\begin{aligned} \dot{R}_{\beta\rho\overline{\sigma}}^\alpha &= y_{\beta\overline{\sigma}\rho}^\alpha - y_{\beta\rho\overline{\sigma}}^\alpha + ix_\beta^\alpha \delta_{\rho\overline{\sigma}} + R_{\beta l\overline{\sigma}}^\alpha F_\rho^l + R_{\beta\rho\overline{l}}^\alpha F_{\overline{\sigma}}^{\overline{l}} \\ &\quad + A_{\overline{\alpha}\overline{\sigma}} E_\rho^{\overline{\beta}} + A_{\beta\rho} E_{\overline{\sigma}}^\alpha - A_{\overline{\alpha}\overline{\gamma}} E_\rho^{\overline{\gamma}} \delta_{\beta\overline{\sigma}} - A_{\beta\gamma} E_{\overline{\sigma}}^\gamma \delta_{\alpha\rho}; \end{aligned} \tag{21}$$

$$\dot{R}_{\rho\overline{\sigma}(t)} = iE_{\rho,\overline{\gamma}\overline{\sigma}}^{\overline{\gamma}} - iE_{\overline{\sigma},\gamma\rho}^\gamma - (A_l^{\overline{\gamma}} E_{\overline{\gamma}}^l + A_{\overline{\gamma}}^l E_l^{\overline{\gamma}})\delta_{\rho\overline{\sigma}} + R_{l\overline{\sigma}} F_\rho^l + R_{\rho\overline{l}} E_{\overline{\sigma}}^{\overline{l}}, \tag{22}$$

and

$$\dot{W} = \dot{R}_{\alpha\overline{\alpha}} = iE_{l,\overline{\gamma}\overline{l}}^{\overline{\gamma}} - iE_{\overline{l},\gamma l}^\gamma - (A_l^{\overline{\gamma}} E_{\overline{\gamma}}^l + A_{\overline{\gamma}}^l E_l^{\overline{\gamma}})n + R_{l\overline{\gamma}} F_\gamma^l + R_{r\overline{\gamma}} F_{\overline{r}}^{\overline{\gamma}}. \tag{23}$$

Proof. Differentiate in t the structure equation (see Section 2)

$$\begin{aligned} d\omega_{\beta(t)}^\alpha - \omega_{\beta(t)}^\gamma \wedge \omega_{\gamma(t)}^\alpha &= R_{\beta\rho\overline{\sigma}(t)}^\alpha \theta_{(t)}^\rho \wedge \theta_{(t)}^{\overline{\sigma}} + W_{\beta\rho(t)}^\alpha \theta_{(t)}^\rho \wedge \theta - W_{\beta\overline{\rho}}^\alpha \theta_{(t)}^{\overline{\rho}} \wedge \theta \\ &\quad + i\theta_{\beta(t)} \wedge \tau_{(t)}^\alpha - i\tau_{\beta(t)} \wedge \theta_{(t)}^\alpha \end{aligned}$$

where, we recall,

$$\tau_\beta = A_{\beta\gamma}\theta^\gamma; \qquad \tau^\alpha = A_{\overline{\gamma}}^\alpha \theta^{\overline{\gamma}},$$

and $A^{\alpha}_{\overline{\gamma}} = A_{\overline{\alpha\gamma}}$ since $h_{\alpha\overline{\beta}} = \delta^{\beta}_{\alpha}$. We then deduce

$$\begin{aligned} d\dot{\omega}^{\alpha}_{\beta} - \dot{\omega}^{\gamma}_{\beta} \wedge \omega^{\alpha}_{\gamma} - \omega^{\gamma}_{\beta} \wedge \dot{\omega}^{\alpha}_{\gamma} = {} & \dot{R}^{\alpha}_{\beta\rho\overline{\sigma}}\theta^{\rho} \wedge \theta^{\overline{\sigma}} + R^{\alpha}_{\beta\rho\overline{\sigma}}(\dot{\theta}^{\rho} \wedge \theta^{\overline{\sigma}} + \theta^{\rho} \wedge \dot{\theta}^{\bar{\sigma}}) \\ & + i\dot{\theta}^{\overline{\beta}} \wedge A_{\overline{\alpha\gamma}}\theta^{\overline{\gamma}} + i\dot{A}_{\overline{\alpha\gamma}}\theta^{\overline{\beta}} \wedge \theta^{\overline{\gamma}} + iA_{\overline{\alpha\gamma}}\theta^{\overline{\beta}} \wedge \dot{\theta}^{\overline{\gamma}} \\ & - iA_{\beta\gamma}\dot{\theta}^{\gamma} \wedge \theta^{\alpha} - i\dot{A}_{\beta\gamma}\theta^{\gamma} \wedge \theta^{\alpha} - iA_{\beta\gamma}\theta^{\gamma} \wedge \dot{\theta}^{\alpha} \\ & \text{mod.} \quad \theta^{\alpha} \wedge \theta \quad \text{and} \quad \theta^{\overline{\alpha}} \wedge \theta. \end{aligned} \tag{24}$$

Writing

$$d\dot{\omega}^{\alpha}_{\beta} = x^{\alpha}_{\beta}d\theta + y^{\alpha}_{\beta\gamma\bar{l}}\theta^{\bar{l}} \wedge \theta^{\gamma} + y^{\alpha}_{\beta\gamma\bar{l}}\theta^{\bar{l}} \wedge \theta^{\gamma} + y^{\alpha}_{\beta\overline{\gamma}l}\theta^{l} \wedge \theta^{\overline{\gamma}},$$

keeping only terms of the type $\theta^{\rho} \wedge \theta^{\overline{\sigma}}$ and using $\omega^{\alpha}_{\gamma}(p) = 0$, we get

$$\begin{aligned} \dot{R}^{\alpha}_{\beta\rho\overline{\sigma}} = {} & y^{\alpha}_{\beta\overline{\sigma}\rho} - y^{\alpha}_{\beta\rho\overline{\sigma}} + ix^{\alpha}_{\beta}\delta_{\rho\overline{\sigma}} + R^{\alpha}_{\beta l\overline{\sigma}}F^{l}_{\rho} + R^{\alpha}_{\beta\rho\bar{l}}F^{\bar{l}}_{\overline{\sigma}} \\ & + A_{\overline{\alpha\sigma}}E^{\overline{\beta}}_{\rho} + A_{\beta\rho}E^{\alpha}_{\overline{\sigma}} - A_{\overline{\alpha\gamma}}E^{\overline{\gamma}}_{\rho}\delta_{\beta\overline{\sigma}} - A_{\beta\gamma}E^{\gamma}_{\overline{\sigma}}\delta_{\alpha\rho}. \end{aligned} \tag{25}$$

Recall that, from (20),

$$y^{\alpha}_{\beta\overline{\sigma}} = -iE^{\alpha}_{\overline{\sigma},\beta} + F^{\alpha}_{\beta,\overline{\sigma}} \quad \Rightarrow \quad y^{\alpha}_{\beta\overline{\sigma}\rho} = -iE^{\alpha}_{\overline{\sigma},\beta\rho} + F^{\alpha}_{\beta,\overline{\sigma}\rho};$$

$$y^{\alpha}_{\beta\gamma} = -iE^{\overline{\beta}}_{\rho,\overline{\alpha}} - F^{\overline{\beta}}_{\overline{\alpha},\rho} \quad \Rightarrow \quad y^{\alpha}_{\beta\rho\overline{\sigma}} = -iE^{\overline{\beta}}_{\rho,\overline{\alpha\sigma}} - F^{\overline{\beta}}_{\overline{\alpha},\rho\overline{\sigma}}$$

and that $x^{\alpha}_{\beta} = i(A^{\overline{\gamma}}_{\beta}E^{\alpha}_{\overline{\gamma}} + A^{\alpha}_{\overline{\gamma}} + A^{\alpha}_{\overline{\gamma}}E^{\overline{\gamma}}_{\beta}) + F^{\alpha}_{\beta,0}$. These last formulas, together with (25), yield (21).

Contracting then (21), for the Ricci tensor $R_{\rho\overline{\sigma}(t)} := R^{\gamma}_{\gamma\rho\overline{\sigma}(t)}$ we obtain, after some cancellation that uses $A_{\alpha\beta} = A_{\beta\alpha}$

$$\dot{R}_{\rho\overline{\sigma}(t)} = iE^{\overline{\gamma}}_{\rho,\overline{\gamma\sigma}} - iE^{\gamma}_{\overline{\sigma},\gamma\rho} - (A^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{l})\delta_{\rho\overline{\sigma}} + R_{l\overline{\sigma}}F^{l}_{\rho} + R_{\rho\bar{l}}E^{\bar{l}}_{\overline{\sigma}}. \tag{26}$$

We then obtain for the Webster curvature $W_{(t)} = R_{\alpha\overline{\alpha}}$ with a further contraction (recall that $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$)

$$\dot{W} = \dot{R}_{\alpha\overline{\alpha}} = iE^{\overline{\gamma}}_{l,\overline{\gamma}\bar{l}} - iE^{\gamma}_{\bar{l},\gamma l} - (A^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{l})n + R_{l\overline{\gamma}}F^{l}_{r} + R_{\gamma\bar{l}}F^{\bar{l}}_{\overline{\gamma}},$$

where we used $F^{\bar{l}}_{\overline{r}} = \overline{F^{l}_{\gamma}} = F^{l}_{\gamma}$. □

We can now pass to the calculation of the second derivative of the Webster curvature with respect to t.

Proposition 3.3. *For the second variation of $W = W_{(t)}$ along the deformation $J_{(t)}$, we have the following formula at $t = 0$*

$$\begin{aligned} \ddot{W} = {} & i\dot{E}^{\overline{\gamma}}_{l,\overline{\gamma}\bar{l}} - A^{\overline{\gamma}}_{l}\dot{E}^{l}_{\overline{\gamma}}n + R_{l\overline{\gamma}}\dot{F}^{l}_{\gamma} - n\dot{A}^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} \\ & - E^{l}_{\overline{\rho}}E^{\overline{\gamma}}_{\rho,\overline{\gamma}l} - E^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{\rho,l\overline{\rho}} - E^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma},\rho\overline{\rho}} - E^{\bar{l}}_{\rho}E^{l}_{\overline{\gamma},\gamma\overline{\rho}} - E^{l}_{\overline{\gamma},\overline{\rho}}E^{\overline{\gamma}}_{\rho,l} \\ & - E^{\overline{\gamma}}_{l,\overline{\rho}}E^{l}_{\overline{\gamma},\rho} - E^{\bar{l}}_{\rho,\overline{\rho}}E^{l}_{\overline{\gamma},\gamma} - E^{l}_{\overline{\rho},\rho}E^{\overline{\gamma}}_{l,\overline{\gamma}} + \text{conj..} \end{aligned} \tag{27}$$

Proof. From formula (23), we see that the contribution in the second variation from $\dot{E}$ and $\dot{F}$ is given by

$$-(A_l^{\overline{\gamma}}\dot{E}_{\overline{\gamma}}^l + A_{\overline{\gamma}}^l\dot{E}_l^{\overline{\gamma}})n + R_{l\overline{\gamma}}\dot{F}_\gamma^l + R_{r\overline{\gamma}}\dot{F}_{\overline{r}}^{\overline{\gamma}},$$

giving the second and third term in the right-hand side of (27), plus their conjugates. To compute the remaining terms, we can therefore assume that $\dot{E}=0$ and $\dot{F}=0$.

We will need first some preliminary calculation: recall that

$$E_{\alpha,\beta}^{\overline{\gamma}} = Z_\beta(E_\alpha^{\overline{\gamma}}) - \omega_\alpha^l(Z_\beta)E_l^{\overline{\gamma}} + \omega_{\overline{l}}^{\overline{\gamma}}(Z_\beta)E_\alpha^{\overline{l}}.$$

Taking the t-derivative and using that $\dot{E}=0$ at $t=0$ we get

$$0 = \dot{Z}_\beta(E_\alpha^{\overline{\gamma}}) - \left[\dot{\omega}_\alpha^l(Z_\beta) + \omega_\alpha^l(\dot{Z}_\beta)\right]E_l^{\overline{\gamma}} + \left[\dot{\omega}_{\overline{l}}^{\overline{\gamma}}(Z_\beta) + \omega_{\overline{l}}^{\overline{\gamma}}(\dot{Z}_\beta)\right]E_\alpha^{\overline{l}}.$$

Recalling that at $t=0$ we can take $F=0$, we have

$$\dot{\omega}_\alpha^l = i(A_{\overline{\gamma}}^l E_\alpha^{\overline{\gamma}} + E_{\overline{\gamma}}^l A_\alpha^{\overline{\gamma}})\theta - iE_{\overline{\gamma},\alpha}^l\theta^{\overline{\gamma}} - iE_{\gamma,\overline{l}}^{\overline{\alpha}}\theta^\gamma,$$

which after some calculation implies

$$(E_{\alpha,\beta}^{\overline{\gamma}})^{\cdot} = -iE_\alpha^{\overline{l}}E_{\beta,\overline{l}}^{\overline{\gamma}} + iE_{\alpha,\overline{l}}^{\overline{\beta}}E_l^{\overline{\gamma}} + iE_\beta^{\overline{l}}E_{\alpha,\overline{l}}^{\overline{\gamma}}.$$

In a similar manner, from the formula

$$E_{\alpha,\overline{\beta}}^{\overline{\gamma}} = Z_{\overline{\beta}}(E_\alpha^{\overline{\gamma}}) - \omega_\alpha^l(Z_{\overline{\beta}})E_l^{\overline{\gamma}} + \omega_{\overline{l}}^{\overline{\gamma}}(Z_{\overline{\beta}})E_\alpha^{\overline{l}}$$

one finds, for $t=0$

$$(E_{\alpha,\overline{\beta}}^{\overline{\gamma}})^{\cdot} = iE_{\overline{\beta}}^\rho E_{\alpha,\rho}^{\overline{\gamma}} + iE_l^{\overline{\gamma}}E_{\overline{\beta},\alpha}^l + iE_\alpha^{\overline{l}}E_{\overline{\beta},\gamma}^l.$$

We analyze the terms with second-order covariant derivatives. Notice that

$$E_{\alpha,\overline{\beta}\overline{\rho}}^{\overline{\gamma}} = Z_{\overline{\rho}}(E_{\alpha,\overline{\beta}}^{\overline{\gamma}}) - \omega_\alpha^l(Z_{\overline{\rho}})E_{l,\overline{\beta}}^{\overline{\gamma}} - \omega_{\overline{\beta}}^{\overline{l}}(Z_{\overline{\rho}})E_{\alpha,\overline{l}}^{\overline{\gamma}} + \omega_{\overline{l}}^{\overline{\gamma}}(Z_{\overline{\rho}})E_{\alpha,\overline{\beta}}^{\overline{l}}.$$

Since $\omega=0$ at $t=0$ and at a given point p, this implies

$$\begin{aligned}(E_{\alpha,\overline{\beta}\overline{\rho}}^{\overline{\gamma}})^{\cdot} &= \dot{Z}_{\overline{\rho}}(E_{\alpha,\overline{\beta}}^{\overline{\gamma}}) + Z_{\overline{\rho}}(\dot{E}_{\alpha,\overline{\beta}}^{\overline{\gamma}}) \\ &\quad - \dot{\omega}_\alpha^l(Z_{\overline{\rho}})E_{l,\overline{\beta}}^{\overline{\gamma}} - \dot{\omega}_{\overline{\beta}}^{\overline{l}}(Z_{\overline{\rho}})E_{\alpha,\overline{l}}^{\overline{\gamma}} + \dot{\omega}_{\overline{l}}^{\overline{\gamma}}(Z_{\overline{\rho}})E_{\alpha,\overline{\beta}}^{\overline{l}}.\end{aligned}$$

After some straightforward calculation, one then finds

$$\begin{aligned}&(E_{\alpha,\overline{\beta}\overline{\rho}}^{\overline{\gamma}})^{\cdot} = iE_{\overline{\rho}}^l E_{\alpha,\overline{\beta}l}^{\overline{\gamma}} + iE_{\overline{\rho},\alpha}^l E_{l,\overline{\beta}}^{\overline{\gamma}} + iE_{\overline{\rho},\gamma}^l E_{\alpha,\overline{\beta}}^{\overline{l}} - iE_{\overline{\rho},l}^\beta E_{\alpha,\overline{l}}^{\overline{\gamma}} \\ &+ i\left(E_{\overline{\beta},\overline{\rho}}^l E_{\alpha,l}^{\overline{\gamma}} + E_{\overline{\beta}}^l E_{\alpha,l\overline{\rho}}^{\overline{\gamma}} + E_{l,\overline{\rho}}^{\overline{\gamma}}E_{\overline{\beta},\alpha}^l + E_l^{\overline{\gamma}}E_{\overline{\beta},\alpha\overline{\rho}}^l + E_{\alpha,\overline{\rho}}^{\overline{l}}E_{\overline{\beta},\gamma}^l + E_\alpha^{\overline{l}}E_{\overline{\beta},\gamma\overline{\rho}}^l\right).\end{aligned}$$

In particular, taking the trace we obtain after some cancellation

$$
\begin{aligned}
(E^{\overline{\gamma}}_{\rho,\overline{\beta}\overline{\rho}})^{\cdot} &= i\left(E^{l}_{\overline{\rho}}E^{\overline{\gamma}}_{\rho,\overline{\gamma}l} + E^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{\rho,l\overline{\rho}} + E^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma},\rho\overline{\rho}} + E^{\overline{l}}_{\rho}E^{l}_{\overline{\gamma},\gamma\overline{\rho}}\right)\\
&\quad + i\left(E^{l}_{\overline{\gamma},\overline{\rho}}E^{\overline{\gamma}}_{\rho,l} + E^{\overline{\gamma}}_{l,\overline{\rho}}E^{l}_{\overline{\gamma},\rho} + E^{\overline{l}}_{\rho,\overline{\rho}}E^{l}_{\gamma,\overline{\gamma}} + E^{l}_{\overline{\rho},\rho}E^{\overline{\gamma}}_{l,\overline{\gamma}}\right).
\end{aligned}
$$

We have that

$$
\begin{aligned}
\ddot{W} &= i(E^{\overline{\gamma}}_{l,\overline{\gamma}\overline{l}})^{\cdot} - i(E^{\gamma}_{\overline{l},\gamma l})^{\cdot} - (\dot{A}^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} + \dot{A}^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{l})n\\
&\quad + (\dot{R}_{l\overline{\gamma}}F^{l}_{\gamma} + R_{l\overline{\gamma}}\dot{F}^{l}_{\gamma}) + (\dot{R}_{\gamma\overline{l}}F^{\overline{l}}_{\overline{\gamma}} + R_{\gamma\overline{l}}\dot{F}^{\overline{l}}_{\overline{\gamma}}).
\end{aligned}
$$

The second line indeed vanishes since $F^{l}_{\gamma} = 0$ at $t = 0$, and since $F^{l}_{l} = 0$ implies

$$
(R_{l\overline{\gamma}} + R_{\gamma\overline{l}})\dot{F}^{l}_{\gamma} = \frac{1}{n}(\delta_{l\gamma} + \delta_{\gamma l})\dot{F}^{l}_{\gamma} = \frac{2}{n}W\dot{F}^{l}_{l} = 0.
$$

This concludes the proof. □

4 Proof of the theorems

In this section we prove our main results, starting from conformal variations and then passing to variations of the CR structure.

Proof of Theorem 1.1. The constancy of Webster's curvature is classical and can be obtained as in [JL2]. In fact, from formula (11) and an integration by parts we have that

$$
\begin{aligned}
\tilde{\mathscr{W}}(J, u^{\frac{4}{Q-2}}\theta) &= \frac{\int_M u\left[-\left(2+\frac{2}{n}\right)\Delta_b u + W_{J,\theta}u\right]\theta\wedge(d\theta)^n}{\left(\int_M u^{\frac{2Q}{Q-2}}\theta\wedge(d\theta)^n\right)^{\frac{Q-2}{Q}}} \qquad (28)\\
&= \frac{\int_M \left[\left(2+\frac{2}{n}\right)|\nabla_b u|^2 + W_{J,\theta}u^2\right]\theta\wedge(d\theta)^n}{\left(\int_M u^{\frac{2Q}{Q-2}}\theta\wedge(d\theta)^n\right)^{\frac{Q-2}{Q}}}. \qquad (29)
\end{aligned}
$$

Given the scaling-invariant character of $\tilde{\mathscr{W}}$, when taking conformal variations of the type $(u+tv)^{\frac{2}{n}}\theta$ at $u \equiv 1$ we can assume that $\int_M v\,\theta\wedge(d\theta)^n = 0$, so we obtain

$$
\int_M W_{J,\theta}v\,\theta\wedge(d\theta)^n = 0 \qquad \text{for all } v \text{ such that } \int_M v\,\theta\wedge(d\theta)^n = 0,
$$

implying that $W_{J,\theta}$ is constant.

Let us now consider variations in J (leaving then θ fixed) of $\tilde{\mathscr{W}}$. We see that the first two terms in the right-hand side of (23) vanish after integration, and that the last two terms also vanish pointwise since we can take $F = 0$.

Therefore, choosing $E_{\alpha\beta} = A_{\alpha\beta}$, we deduce the vanishing of the torsion by integration of the formula over M. Notice that such variations are admissible since by the third equation in (13) and (10) we have the constraints given in Lemma 2.5.

Let us now check the pseudo-Einstein condition: vanishing of the torsion implies that the Reeb vector field T corresponding to θ generates an infinitesimal transverse symmetry, see e.g. (2.12) and Proposition 2.2 in [We]. By Theorem E in [L2], if we also assume that $c_1(T_{1,0}(M)) = 0$, then there exists $u \in C^\infty(M)$ such that $(M, J, e^{2u}\theta)$ is pseudo-Einstein. In the Appendix it is indeed shown that u can be taken identically zero, see Proposition 5.2. $\square$

Before proving the next theorems, we compute the second variation of $\tilde{\mathscr{W}}$ in the conformal directions. The first conformal variation is given by

$$\frac{d}{dt}|_{t=0}\tilde{\mathscr{W}}(J,(u+tv)^{\frac{4}{Q-2}}\theta) = 2\frac{\int_M (b_n \nabla_b u \cdot \nabla_b v + W_{J,\theta} uv)\,\theta \wedge (d\theta)^n}{\left(\int_M u^{\frac{2Q}{Q-2}}\theta \wedge (d\theta)^n\right)^{\frac{Q-2}{Q}}} \tag{30}$$

$$- 2\frac{\int_M \left(|\nabla_b u|^2 + W_{J,\theta} u^2\right)\theta \wedge (d\theta)^n}{\left(\int_M u^{\frac{2Q}{Q-2}}\theta \wedge (d\theta)^n\right)^{2\frac{Q-2}{Q}}} \int_M u^{\frac{Q+2}{Q-2}} v\,\theta \wedge (d\theta)^n. \tag{31}$$

In this way, if $W_{J,\theta}$ is constant, one sees that criticality occurs when

$$Vol_\theta(M)^{\frac{2}{Q}} u^{\frac{4}{Q-2}} = 1. \tag{32}$$

The second variation at a stationary point is the following

$$\frac{d^2}{dt^2}|_{t=0}\tilde{\mathscr{W}}(J,(1+tv)^{\frac{4}{Q-2}}\theta) = 2\frac{\int_M \left(b_n|\nabla_b v|^2 + W_{J,\theta} v^2\right)\theta \wedge (d\theta)^n}{Vol_{\theta,u}(M)^{\frac{Q-2}{Q}}}$$
$$- 2\frac{Q+2}{Q-2}\frac{W_{J,\theta} Vol_\theta(M)}{Vol_{\theta,u}(M)^{2\frac{Q-2}{Q}}}\int_M v^2\,\theta \wedge (d\theta)^n,$$

where $Vol_{\theta,u}(M) = \int_M u^{\frac{2Q}{Q-2}}\theta \wedge (d\theta)^n$. Inserting (32) into the latter formula we see that the second variation becomes

$$\frac{2}{Vol_{\theta,u}(M)^{\frac{Q-2}{Q}}}\int_M \left(b_n|\nabla_b v|^2 - \frac{4}{Q-2}W_{J,\theta} v^2\right)\theta \wedge (d\theta)^n. \tag{33}$$

For the standard spheres $(S^{2n+1}, J_0, \theta_0)$, recalling from [JL2] and [We] that

$$b_n = 2 + \frac{2}{n}; \qquad W_{J_0,\theta_0} = n(n+1), \tag{34}$$

we get the first statements in Theorem 1.3 and Theorem 1.4 (see also [MU]).

Remark 4.1. *More in general, if we are on a pseudo-Einstein manifold with zero torsion other than the standard sphere, using Theorem 1.1 in [Ch] and Theorem 3 in [LiWa] for $n = 1$ and $n > 1$ respectively, by formula* (33) *the second conformal variation is strictly positive-definite.*

We consider next the second variation of $\tilde{\mathscr{W}}$ on standard spheres with respect to the deformation of the CR structure.

Lemma 4.2. *For the standard structure $(S^{2n+1}, J_0, \theta_0)$ we have that*

$$\frac{d^2}{dt^2}|_{t=0}\tilde{\mathscr{W}}(J_{(t)}, \theta_0) = -i\, n \int_{S^{2n+1}} E_{\alpha}{}^{\overline{\gamma}}{}_{,0} E_{\overline{\gamma}}{}^{\alpha}\, \theta_0 \wedge (d\theta_0)^n + \text{conj.}, \tag{35}$$

where $E = 2\frac{d}{dt}|_{t=0} J_{(t)}$.

Proof. Since θ_0 remains fixed, we just need to integrate $\ddot{W}$ with respect to the volume form $\theta_0 \wedge (d\theta_0)^n$.

Recalling (27), we first notice that the terms involving $\dot{E}$ and $\dot{F}$ vanish since they correspond to the first variation of $\tilde{\mathscr{W}}$ in the direction $\dot{E}$, but we are at a stationary point.

Concerning the quadratic terms in E, we observe that after integrating and using Lemma 2.5, we obtain cancellation in (27) of the first with the seventh, of the second with the fifth, of the third with the sixth and of the fourth with the eighth. We are then left with

$$\ddot{\mathscr{W}} = -n \int_{S^{2n+1}} (\dot{A}_l^{\overline{\gamma}} E_{\overline{\gamma}}^l + \dot{A}_{\overline{\gamma}}^l E_l^{\overline{\gamma}})\, \theta_0 \wedge (d\theta_0)^n.$$

Recalling formula (15) and the fact that we can take $F_\beta^\alpha = 0$ at $t = 0$, we obtain the desired conclusion. □

To understand the second variation formula in (35) on the sphere S^{2n+1}, instead of employing the above moving frame approach, we use instead a basis of the complexified tangent space that is induced from the ambient space $\mathbf{C}^{n+1}$, introduced in [G]. This basis does not consist of linearly independent vectors, but it has the advantage of leading to computable quantities, with coefficients that are either constant or that are powers of the z- and $\overline{z}$-coordinates.

Let

$$Z_{jk} = \overline{z}_j \frac{\partial}{\partial z_k} - \overline{z}_k \frac{\partial}{\partial z_j}, \quad \theta_{jk} = z_j dz_k - z_k dz_j, \qquad j \neq k. \tag{36}$$

We have that

$$\begin{aligned}\theta_{\ell m}(Z_{jk}) &= (z_\ell dz_m - z_m dz_\ell)\left(\overline{z}_j \frac{\partial}{\partial z_k} - \overline{z}_k \frac{\partial}{\partial z_j}\right) \\ &= z_\ell \overline{z}_j \delta_{km} - z_\ell \overline{z}_k \delta_{jm} - z_m \overline{z}_j \delta_{k\ell} + z_m \overline{z}_k \delta_{j\ell}.\end{aligned} \tag{37}$$

As proven in [G], every form of type $(0,1)$ can be written as

$$\eta = \sum_{0\le j<k\le n} \eta(\overline{Z}_{jk})\overline{\theta}_{jk}.$$

We warn that the coefficients $\eta(\overline{Z}_{jk})$ are some functions which may not coincide with η applied to $\overline{Z}_{jk}$.

Similarly, any form of type $(1,0)$ can be written as

$$\eta = \sum_{0\le j<k\le n} \eta(Z_{jk})\theta_{jk},$$

and any vector field of type $(0,1)$ as

$$X = \sum_{0\le j<k\le n} \overline{\theta}_{jk}(X)\overline{Z}_{jk}.$$

Starting with tensor products of objects of the above form, by linearity a tensor S of type $((0,1);(1,0))$ can be written as

$$S = \sum_{j<k,\ell<m} S(\overline{\theta}_{jk}, Z_{\ell m})\overline{Z}_{jk}\otimes\theta_{\ell m}. \tag{38}$$

Define the musical flat operator $\sharp^{-1} : \overline{\mathcal{H}} \to \Omega^{1,0}(M)$ by

$$\sharp^{-1}(\overline{Z}_{jk}) = id\theta(\cdot, \overline{Z}_{jk}).$$

Lemma 4.3. *We have the following relations*

$$\nabla_T Z_{jk} = -iZ_{jk}, \qquad \nabla_{Z_{jk}} Z_{pq} = 0 \quad \textit{for all } j,k,p,q;$$

$$\nabla_{Z_{jk}}\overline{Z}_{lm} = (\delta_{kl}\overline{z}_j - \delta_{jl}\overline{z}_k)(\overline{\partial}_m - z_m\sigma^\sharp) + (\delta_{jm}\overline{z}_k - \delta_{lm}\overline{z}_j)(\overline{\partial}_l - z_l\sigma^\sharp),$$

where $\sigma^\sharp = \sum_{\nu=1}^{n+1}\overline{z}_\nu\overline{\partial}_\nu$. *Moreover, if* $\sharp^{-1}$ *is as above, one has that*

$$\sharp^{-1}(\overline{Z}_{jk}) = \theta_{jk} \qquad \textit{for all } j,k.$$

Proof. Since the pseudohermitian torsion is zero and T is parallel

$$\begin{aligned}\nabla_T Z_{jk} = [T, Z_{jk}] &= \frac{i}{2}\sum\left[z_\alpha\frac{\partial}{\partial z_\alpha} - \overline{z}_\alpha\frac{\partial}{\partial\overline{z}_\alpha}, \overline{z}_j\frac{\partial}{\partial z_k} - \overline{z}_k\frac{\partial}{\partial z_j}\right]\\ &= \frac{i}{2}\left(-\overline{z}_j\frac{\partial}{\partial z_k} + \overline{z}_k\frac{\partial}{\partial z_j} - \overline{z}_j\frac{\partial}{\partial z_k} + \overline{z}_k\frac{\partial}{\partial z_j}\right)\\ &= i\left(-\overline{z}_j\frac{\partial}{\partial z_k} + \overline{z}_k\frac{\partial}{\partial z_j}\right) = -iZ_{jk},\end{aligned}$$

giving the first assertion. The third one is proved in (2.4) of [C].

Let us now turn to the second assertion. From Lemma 3.2 (2) in [Tn] (in our notation we have an extra factor 1/2 in front of the contact form) one has that

$$d\theta(\nabla_X Y, \overline{Z}) = X d\theta(Y, \overline{Z}) - d\theta(Y, [X, \overline{Z}]_{\overline{\mathcal{H}}}); \qquad X, Y \in \mathcal{H}, Z \in \overline{\mathcal{H}},$$

where $[X, \overline{Z}]_{\overline{\mathcal{H}}}$ stands for the anti-holomorphic projection. From (the conjugation of) the formula in Lemma 3.2 (i) of [Tn] and Lemma 4.3 one has that

$$\begin{aligned}&[Z_{jk}, \overline{Z}_{lm}]_{\overline{\mathcal{H}}} = \nabla_{Z_{jk}} \overline{Z}_{lm} \\ &= (\delta_{kl}\overline{z}_j - \delta_{jl}\overline{z}_k)(\overline{\partial}_m - z_m\sigma^\sharp) + (\delta_{jm}\overline{z}_k - \delta_{lm}\overline{z}_j)(\overline{\partial}_l - z_l\sigma^\sharp).\end{aligned}$$

A direct computation shows, after some cancellation

$$\begin{aligned}d\theta(Z_{pq}, [Z_{jk}, \overline{Z}_{lm}]_{\overline{\mathcal{H}}}) &= 2i\overline{z}_p\left[\delta_{kl}\overline{z}_k\delta_{qm} + \delta_{jm}\overline{z}_k\delta_{ql}\right] \\ &\quad - 2i\overline{z}_q\left[\delta_{kl}\overline{z}_j\delta_{pm} + \delta_{jm}\overline{z}_k\delta_{pl}\right].\end{aligned}$$

On the other hand, we have that

$$d\theta(Z_{pq}, \overline{Z}_{lm}) = 2i\overline{z}_p(z_l\delta_{qm} - z_m\delta_{ql}) - 2i\overline{z}_q(z_l\delta_{pm} - z_m\delta_{pl}),$$

which implies

$$\begin{aligned}&Z_{jk}(d\theta(Z_{pq}, \overline{Z}_{lm})) = \overline{z}_j\left[2i\overline{z}_p(\delta_{kl}\delta_{qm} - \delta_{km}\delta_{ql}) - 2i\overline{z}_q(\delta_{kl}\delta_{pm} - \delta_{km}\delta_{pl})\right] \\ &- \overline{z}_k\left[2i\overline{z}_p(\delta_{jl}\delta_{qm} - \delta_{jm}\delta_{ql}) - 2i\overline{z}_q(\delta_{jl}\delta_{pm} - \delta_{jm}\delta_{pl})\right].\end{aligned}$$

This means that $Z_{jk}(d\theta(Z_{pq}, \overline{Z}_{lm})) - d\theta(Z_{pq}, [Z_{jk}, \overline{Z}_{lm}]_{\overline{\mathcal{H}}}) = 0$, and therefore $d\theta(\nabla_{Z_{jk}} Z_{pq}, \overline{Z}_{lm}) = 0$ for all l, m: hence we obtain the second assertion too.

To prove the last property, we notice that

$$id\theta = \sum_\alpha (dz_\alpha \wedge d\overline{z}_\alpha),$$

and therefore

$$id\theta(Z_{lm}, \overline{Z}_{jk}) = \overline{z}_l z_j \delta_{km} - \overline{z}_l z_k \delta_{jm} - \overline{z}_m z_j \delta_{lk} + \overline{z}_m z_k \delta_{lj}.$$

From (37) we then get

$$id\theta(Z_{lm}, \overline{Z}_{jk}) = \theta_{jk}(Z_{lm}) \qquad \text{for all indices } j, k, l, m,$$

proving also the last assertion. □

We can now prove our second and third main results.

Proof of Theorem 1.3. The first statement follows from formula (33), recalling (34), so it remains to prove formula (7).

Notice that, by Leibnitz's rule

$$E_{,0} := \nabla_T(E_1^{\overline{1}}\theta^1 \otimes Z_{\overline{1}}) = T(E_1^{\overline{1}})\theta^1 \otimes Z_{\overline{1}} + E_1^{\overline{1}}((\nabla_T\theta^1) \otimes Z^{\overline{1}} + \theta^1 \otimes (\nabla_T Z_{\overline{1}})). \quad (39)$$

In our notation, comparing (5) and (36), and recalling Lemma 4.3 we have that

$$\nabla_T Z_{\overline{1}} = iZ_{\overline{1}}; \qquad \nabla_T\theta^1 = i\theta^1.$$

This implies

$$E_{,0} = i\left(\frac{m}{2} + 2\right) \qquad \text{for } E_1^{\overline{1}} \in \Gamma_m,$$

and in turn

$$\ddot{\mathscr{W}} = \sum_{m\in\mathbf{Z}} (m+4)\int_{S^3} \left|E^{(m)}\right|^2 \theta_0 \wedge d\theta_0,$$

by Lemma 4.2. This gives the desired conclusion. □

Proof of Theorem 1.4. In the notation of [Bl] and [BD1], the eigenspace of a deformed structure corresponding to the eigenvalue i is written as

$$\hat{\mathcal{H}} = \left\{X - \bar{\phi}(X) \ : \ X \in \mathcal{H}_0\right\}; \qquad \phi : \mathcal{H} \to \mathcal{H}.$$

If ϕ is infinitesimal, then we have the following relation to the tensor E from Lemma 2.3:

$$\phi_\alpha^{\overline{\gamma}} = iE_\alpha^{\overline{\gamma}}.$$

By Theorem 4.1 in [BD1], ϕ is of the form

$$\phi = \overline{\partial}_b(\overline{\partial}_b f)^\sharp + h_\sigma(h), \quad (40)$$

where f, h are a complex-valued function and a two-form of type $(0,2)$ whose negative Fourier components are zero (with h determined by f). Here $\overline{\partial}_b$ denotes the holomorphic differential of f and $\sharp$ the musical isomorphism from $\Omega^{1,0}(S^{2n+1})$ to $T^{0,1}(S^{2n+1})$. In analogy with (6), the *m-th Fourier eigenspace* Γ_m for a tensor on S^{2n+1} is defined by the action of the flow generated by the vector field T.

Both the operators $\overline{\partial}_b$ and $(\overline{\partial}_b\cdot)^\sharp$ commute with the Lie derivative by T, and it is noticed on page 102 of [BD2] that h_σ preserves the Fourier decomposition. Therefore the tensor ϕ, and hence E as well, only consist of non-negative Fourier modes.

As for (38), let us write

$$E = \sum_{j<k,\ell<s} E(\overline{\theta}_{jk}, Z_{\ell s})\overline{Z}_{jk} \otimes \theta_{\ell s} : \quad (41)$$

since both $\overline{Z}_{jk}$ and $\theta_{\ell s}$ are invariant under the action of T, we must have that $E(\overline{\theta}_{jk}, Z_{\ell s})$ also has only non-negative Fourier modes.

Arguing then as for (39) and using the latter formula, we still obtain that $E_{,0} = i\left(\frac{m}{2}+2\right)$ for $E \in \Gamma_m$, which by Lemma 4.2 gives

$$\ddot{\mathscr{W}} = n \sum_{m \in Z} (m+4) \int_{S^3} \left| E^{(m)} \right|^2 \theta_0 \wedge d\theta_0.$$

Recalling that $E^{(m)} = 0$ for $m < 0$, we obtain the conclusion.

One comment on the above proof is due, since the coefficients in the expansion (41) for E are not uniquely determined. Near each point of S^{2n+1} one could choose linearly independent bases of vector fields and forms, which would give the asserted property on the Fourier modes of $E(\overline{\theta}_{jk}, Z_{\ell s})$, proving that in any case $E_{,0} = i\left(\frac{m}{2}+2\right)$ for $E \in \Gamma_m$. $\square$

5 Appendix by Xiaodong Wang[1]

It is a well known fact in Kahler geometry that a Kahler metric ω on a closed complex manifold M with constant scalar curvature must be Kahler-Einstein if the Kahler class $[\omega]$ is proportional to the first Chern class $c_1(M)$. In this appendix we discuss a CR analogue of this result.

We still follow [L2] as our standard reference on CR geometry. Let M be a CR manifold of dimension $2n+1$. The first Chern class of the complex vector bundle $T^{1,0}M$ will be simply denoted by $c_1(M)$. Given a pseudohermitian structure θ, we always work with the Tanaka-Webster connection ∇, and $2\pi c_1(M)$ is then represented by the closed 2-form

$$\rho_\theta = i\left[R_{\mu\overline{\nu}}\theta^\mu \wedge \theta^{\overline{\nu}} + A_{\alpha\gamma,\overline{\alpha}}\theta^\gamma \wedge \theta - A_{\overline{\gamma}\overline{\alpha},\alpha}\theta^{\overline{\gamma}} \wedge \theta\right], \tag{42}$$

where $R_{\mu\overline{\nu}}$ is the Ricci curvature and A is the torsion of the Tanaka-Webster connection. Throughout this section, we always work with a local unitary frame $\{Z_\alpha : \alpha = 1, \cdots, n\}$.

Lemma 5.1. *Suppose $\phi = f_{\mu\overline{\nu}}\theta^\mu \wedge \theta^{\overline{\nu}}$ is a $(1,1)$-form. Let $\Lambda(\phi) = \sum_{\mu=1}^n f_{\mu\overline{\mu}}$ be its trace. Then*

$$-d^*\phi = f_{\alpha\overline{\nu},\overline{\alpha}}\theta^{\overline{\nu}} + f_{\mu\overline{\alpha},\alpha}\theta^\mu + i\Lambda(\phi)\theta,$$

where d^* the dual of d with respect to the adapted Riemannian metric.

Proof. By a standard formula, in terms of the Levi-Civita connection $\widetilde{\nabla}$

$$-d^*\phi = T \lrcorner \widetilde{\nabla}_T \phi + Z_\alpha \lrcorner \widetilde{\nabla}_{\overline{Z}_\alpha}\phi + \overline{Z}_\alpha \lrcorner \widetilde{\nabla}_{Z_\alpha}\phi.$$

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We recall the relationship between the Levi-Civita connection $\widetilde{\nabla}$ and the Tanka-Webster connection, that can be found in [DT]: for X, Y horizontal

$$\begin{aligned}
&\widetilde{\nabla}_T T = 0, \\
&\widetilde{\nabla}_X T = AX + \frac{1}{2} JX, \\
&\widetilde{\nabla}_X Y = \nabla_X Y - \left[\langle AX, Y \rangle + \frac{1}{2} d\theta \left(X, Y \right) \right] T.
\end{aligned}$$

We compute, using the above identities

$$\begin{aligned}
T \lrcorner \widetilde{\nabla}_T \phi &= f_{\mu\overline{\nu}} \left(\widetilde{\nabla}_T \theta^{\mu} \left(T \right) \theta^{\overline{\nu}} - \widetilde{\nabla}_T \theta^{\overline{\nu}} \left(T \right) \theta^{\mu} \right) \\
&= f_{\mu\overline{\nu}} \left(-\theta^{\mu} \left(\widetilde{\nabla}_T T \right) \theta^{\overline{\nu}} + \theta^{\overline{\nu}} \left(\widetilde{\nabla}_T T \right) \theta^{\mu} \right) = 0,
\end{aligned}$$

$$\begin{aligned}
&Z_\alpha \lrcorner \widetilde{\nabla}_{\overline{Z}_\alpha} \phi \\
&= \overline{Z}_\alpha f_{\alpha\overline{\nu}} \theta^{\overline{\nu}} + f_{\mu\overline{\nu}} \widetilde{\nabla}_{\overline{Z}_\alpha} \theta^{\mu} \left(Z_\alpha \right) \theta^{\overline{\nu}} - f_{\mu\overline{\nu}} \widetilde{\nabla}_{\overline{Z}_\alpha} \theta^{\overline{\nu}} \left(Z_\alpha \right) \theta^{\mu} + f_{\alpha\overline{\nu}} \widetilde{\nabla}_{\overline{Z}_\alpha} \theta^{\overline{\nu}} \\
&= \overline{Z}_\alpha f_{\alpha\overline{\nu}} \theta^{\overline{\nu}} - f_{\mu\overline{\nu}} \theta^{\mu} \left(\widetilde{\nabla}_{\overline{X}_\alpha} Z_\alpha \right) \theta^{\overline{\nu}} + f_{\mu\overline{\nu}} \theta^{\overline{\nu}} \left(\widetilde{\nabla}_{\overline{Z}_\alpha} Z_\alpha \right) \theta^{\mu} \\
&- f_{\alpha\overline{\nu}} \left(\theta^{\overline{\nu}} \left(\widetilde{\nabla}_{\overline{Z}_\alpha} Z_\beta \right) \theta^{\beta} + \theta^{\overline{\nu}} \left(\widetilde{\nabla}_{\overline{Z}_\alpha} \overline{Z}_\beta \right) \theta^{\overline{\beta}} + \theta^{\overline{\nu}} \left(\widetilde{\nabla}_{\overline{Z}_\alpha} T \right) \theta \right) \\
&= \overline{Z}_\alpha f_{\alpha\overline{\nu}} \theta^{\overline{\nu}} - f_{\mu\overline{\nu}} \theta^{\mu} \left(\nabla_{\overline{Z}_\alpha} Z_\alpha \right) \theta^{\overline{\nu}} + f_{\mu\overline{\nu}} \theta^{\overline{\nu}} \left(\nabla_{\overline{Z}_\alpha} Z_\alpha \right) \theta^{\mu} \\
&- f_{\alpha\overline{\nu}} \left(\theta^{\overline{\nu}} \left(\nabla_{\overline{Z}_\alpha} Z_\beta \right) \theta^{\beta} + \theta^{\overline{\nu}} \left(\nabla_{\overline{Z}_\alpha} Z_{\overline{\beta}} \right) \theta^{\overline{\beta}} + -\frac{i}{2} \delta^{\nu}_{\alpha} \theta \right) \\
&= f_{\alpha\overline{\nu},\overline{\alpha}} \theta^{\overline{\nu}} + \frac{i}{2} \Lambda \left(\phi \right) \theta,
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\overline{Z}_\alpha \lrcorner \widetilde{\nabla}_{Z_\alpha} \phi \\
&= Z_\alpha f_{\mu\overline{\alpha}} \theta^{\mu} - f_{\mu\overline{\nu}} \widetilde{\nabla}_{Z_\alpha} \theta^{\overline{\nu}} \left(\overline{X}_\alpha \right) \theta^{\mu} + f_{\mu\overline{\nu}} \widetilde{\nabla}_{Z_\alpha} \theta^{\mu} \left(\overline{Z}_\alpha \right) \theta^{\overline{\nu}} - f_{\mu\overline{\alpha}} \widetilde{\nabla}_{Z_\alpha} \theta^{\mu} \\
&= Z_\alpha f_{\mu\overline{\alpha}} \theta^{\mu} + f_{\mu\overline{\nu}} \theta^{\overline{\nu}} \left(\widetilde{\nabla}_{Z_\alpha} \overline{X}_\alpha \right) \theta^{\mu} - f_{\mu\overline{\nu}} \theta^{\mu} \left(\widetilde{\nabla}_{Z_\alpha} \overline{Z}_\alpha \right) \theta^{\overline{\nu}} \\
&+ f_{\mu\overline{\alpha}} \left(\theta^{\mu} \left(\widetilde{\nabla}_{Z_\alpha} Z_\beta \right) \theta^{\beta} + \theta^{\mu} \left(\widetilde{\nabla}_{Z_\alpha} \overline{Z}_\beta \right) \overline{\theta}^{\beta} + \theta^{\mu} \left(\widetilde{\nabla}_{Z_\alpha} T \right) \theta \right) \\
&= Z_\alpha f_{\mu\overline{\alpha}} \theta^{\mu} + f_{\mu\overline{\nu}} \theta^{\overline{\nu}} \left(\nabla_{Z_\alpha} \overline{Z}_\alpha \right) \theta^{\mu} - f_{\mu\overline{\nu}} \theta^{\mu} \left(\nabla_{Z_\alpha} \overline{Z}_\alpha \right) \theta^{\overline{\nu}} \\
&+ f_{\mu\overline{\alpha}} \left(\theta^{\mu} \left(\nabla_{Z_\alpha} Z_\beta \right) \theta^{\beta} + \theta^{\mu} \left(\nabla_{Z_\alpha} \overline{Z}_\beta \right) \overline{\theta}^{\beta} + \frac{i}{2} \delta^{\mu}_{\alpha} \theta \right) \\
&= f_{\mu\overline{\alpha},\alpha} \theta^{\mu} + \frac{i}{2} \Lambda \left(\phi \right) \theta.
\end{aligned}$$

We remark that in the calculations the torsion A does not appear because it anti-commutes with J and therefore maps a $(1,0)$-vector to a $(0,1)$-vector and vice versa. Combining these results yields

$$-d^* \phi = f_{\alpha\overline{\nu},\overline{\alpha}} \theta^{\overline{\nu}} + f_{\mu\overline{\alpha},\alpha} \theta^{\mu} + i \Lambda \left(\phi \right) \theta,$$

concluding the proof. ☐

Recall that a pseudohermitian structure θ is called pseudo-Einstein if $R_{\mu\overline{\nu}} = \frac{W}{n}\delta_{\mu\overline{\nu}}$, where W is the scalar curvature. When $n = 1$, this is always true. From now on we assume $n \geq 2$. Lee showed in [L2] that a necessary condition for the existence of a pseudo-Einstein structure is $c_1(M) = 0$.

Proposition 5.2. *Suppose M is a closed CR manifold of dimension $2n+1 \geq 5$ with $c_1(M) = 0$. If θ is a pseudohermitian structure with zero torsion and constant scalar curvature, then it is pseudo-Einstein.*

Proof. Let $\phi = \rho_\theta - \frac{W}{n}d\theta$. Since $A = 0$, we have

$$\phi = iR_{\mu\overline{\nu}}\theta^\mu \wedge \theta^{\overline{\nu}} - \frac{W}{n}d\theta = i\left(R_{\mu\overline{\nu}} - \frac{W}{n}\delta_{\mu\overline{\nu}}\right)\theta^\mu \wedge \theta^{\overline{\nu}},$$

It is a real $(1,1)$-form with $\Lambda(\phi) = 0$. As $A = 0$, we have $R_{\alpha\overline{\nu},\alpha} = R_{\mu\overline{\alpha},\alpha} = 0$ by the Bianchi identities. Together with W being constant, we see that $d^*\phi = 0$ by Lemma 5.1. As $c_1\left(T^{1,0}M\right) = 0$, ρ_θ is exact, i.e. there is a real 1-form χ s.t. $\rho_\theta = d\chi$. Then $\phi = d\widetilde{\chi}$, where $\widetilde{\chi} = \chi - \frac{W}{n}\theta$. It follows

$$\|\phi\|^2 = \langle\phi, d\widetilde{\chi}\rangle = \langle d^*\phi, \widetilde{\chi}\rangle = 0.$$

Therefore $\phi = 0$, i.e. θ is pseudo-Einstein. ☐

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