A generating function for the cubic interactions of higher spin fields

Ruben Manvelyan [a,b,∗], Karapet Mkrtchyan [a,b], Werner Rühl [a]

[a] *Department of Physics, Erwin Schrödinger Straße, Technical University of Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany*
[b] *Yerevan Physics Institute, Alikhanian Br. Str. 2, 0036 Yerevan, Armenia*

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ABSTRACT

We present an off-shell generating function for all cubic interactions of higher spin gauge fields constructed in Manvelyan et al. (2010) [1]. It is a generalization of the on-shell generating function proposed by Sagnotti and Taronna [2], is written in a very compact way, and turns out to have a remarkable structure.

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1. Introduction and notations

Higher spin gauge field theory is one of the most important and puzzling problems in modern quantum field theory. It is a subject of many articles and always stays in the center of attention during last thirty years. Despite the fact that consistent equations of motion for higher spin gauge fields are known over twenty years [3], the question of existence of Lagrangians for interacting higher spin gauge fields is still open. The subject of special interest is a minimal selfinteraction of even spin gauge fields, where one can naively expect the existence of an Einstein–Hilbert type nonlinear action for any single even spin gauge field. Although there are known restrictions on higher spin theories in flat space–time, the recent development [1] has shown that *there is a local higher derivative cubic Lagrangian for gauge fields with any higher spins*. This shifts the no-go theorems to the quartic power of fields in interaction Lagrangians, where one can expect the final battle for the existence of local (or nonlocal) Lagrangians for interacting HS gauge field theory in flat space.

The free Lagrangian for higher spin gauge fields both in flat space and in constantly curved backgrounds (dS and AdS) are known over thirty years [4]. In contrast to free theory, attempts to construct Lagrangians for interacting theories haven't been successful yet beyond the cubic vertices. In this Letter we are going to discuss only trilinear interactions of higher spin gauge fields.

Our recent results [1,5–8] on higher spin gauge field cubic interactions in flat space, which certainly reproduce the flat limit of the Fradkin–Vasiliev vertex for higher spin coupling to gravity [9], show that all interactions of higher spin gauge fields with any spins s_1, s_2, s_3 both in flat space and in dS or AdS are unique.[1] This was already proven for some low spin cases of both the Fradkin–Vasiliev vertex for $2, s, s$ and the nonabelian vertex for $1, s, s$ in [10].

The first important step towards cubic interactions in higher spin gauge field theory was done in 1984 by Berends, Burgers and van Dam [11]. They constructed a cubic selfinteraction Lagrangian for spin three gauge fields and proved impossibility of extension to higher orders. Their arguments are based on gauge algebra, which does not close for a single spin three nonabelian field. The authors give an optimistic hope that it will be possible to extend this Lagrangian to higher orders if one takes into account corrections from interactions with gauge fields with spins higher than three. A recent discussion on this subject appeared from Bekaert, Boulanger and Leclerq [12]. They show the impossibility to close this (spin 3) algebra taking into account corrections from interactions of other fields with spins higher (or lower) than three. It is not yet known whether there is or there is not any nonlinear self-interaction for a spin three gauge field in the background space with nonzero cosmological constant (pure spin three theory).

Another important step was done by Fradkin and Vasiliev in [9] where coupling of higher spin gauge fields to linearized gravity was constructed in the constantly curved background. The interesting property of this Lagrangian is it's non-analyticity in the cos-

∗ Corresponding author at: Department of Physics, Erwin Schrödinger Straße, Technical University of Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.

E-mail addresses: manvel@physik.uni-kl.de (R. Manvelyan), karapet@yerphi.am (K. Mkrtchyan), ruehl@physik.uni-kl.de (W. Rühl).

[1] The cubic interaction Lagrangian is unique up to partial integration and field redefinition.

mological constant, therefore excluding a flat space limit. However it was shown already in [10] that after rescaling of higher spin gauge fields one can observe a flat limit for the Fradkin–Vasiliev interactions. In our approach the spin s gauge field has scaling dimension [*length*]$^{s-2}$, therefore the Fradkin–Vasiliev vertex has a flat limit with $2s-2$ derivatives (minimal possible number) in the $2-s-s$ interaction which has the same scaling dimension as the Einstein–Hilbert Lagrangian terms. As it was shown by Metsaev in [13] using a light cone gauge approach, there are three different couplings to linearized gravity with different numbers of derivatives for any higher spin s field, and in general min$\{s_1, s_2, s_3\} + 1$ different possibilities with different numbers of derivatives for the $s_1 - s_2 - s_3$ interaction. All these interactions were derived in a covariant off-shell formulation in [1].

In the recent paper [2] by Sagnotti and Taronna the authors proposed an on-shell generating function for the general interaction presented in [1] from a string theory consideration. In this article we are going to present an off-shell extension of that generating function which can surprisingly be enhanced with a beautiful Grassmann structure, the string origin of which is not clear yet.

For some important results on higher spin cubic interactions see [14–18] and references therein. For recent reviews see [19]. We should note also that a connection between nonabelian tensor gauge theories and string amplitudes was also explored in [20].

In this Letter we consider a higher spin gauge field theory in Fronsdal's formulation. The spin s field is a rank s symmetric, double traceless tensor and we consider here only one copy for any spin, so these interactions are for even spin gauge field (self)interactions only.

To continue with this subject we introduce here briefly our standard notations coming from our previous papers about HSF (see for example [21]). As usual we utilize instead of symmetric tensors such as $h^{(s)}_{\mu_1\mu_2\cdots\mu_s}(z)$ the homogeneous polynomials in the vector a^μ of degree s at the base point z

$$h^{(s)}(z;a) = \sum_{\mu_i}\left(\prod_{i=1}^s a^{\mu_i}\right)h^{(s)}_{\mu_1\mu_2\cdots\mu_s}(z). \tag{1}$$

Then we can write the symmetrized gradient, trace and divergence[2]

$$\text{Grad}: h^{(s)}(z;a) \Rightarrow \text{Grad}\, h^{(s+1)}(z;a) = (a\nabla)h^{(s)}(z;a), \tag{2}$$

$$\text{Tr}: h^{(s)}(z;a) \Rightarrow \text{Tr}\, h^{(s-2)}(z;a) = \frac{1}{s(s-1)}\Box_a h^{(s)}(z;a), \tag{3}$$

$$\text{Div}: h^{(s)}(z;a) \Rightarrow \text{Div}\, h^{(s-1)}(z;a) = \frac{1}{s}(\nabla\partial_a)h^{(s)}(z;a). \tag{4}$$

Here we only present Fronsdal's Lagrangian in terms of these conventions[3]:

$$\mathcal{L}_0\big(h^{(s)}(a)\big) = -\frac{1}{2}h^{(s)}(a) *_a \mathcal{F}^{(s)}(a)$$
$$+ \frac{1}{8s(s-1)}\Box_a h^{(s)}(a) *_a \Box_a \mathcal{F}^{(s)}(a), \tag{5}$$

where $\mathcal{F}^{(s)}(z;a)$ is the Fronsdal tensor

$$\mathcal{F}^{(s)}(z;a) = \Box h^{(s)}(z;a) - s(a\nabla)D^{(s-1)}(z;a), \tag{6}$$

and $D^{(s-1)}(z;a)$ is the deDonder tensor or traceless divergence of the higher spin gauge field

$$D^{(s-1)}(z;a) = \text{Div}\, h^{(s-1)}(z;a) - \frac{s-1}{2}(a\nabla)\text{Tr}\, h^{(s-2)}(z;a), \tag{7}$$

$$\Box_a D^{(s-1)}(z;a) = 0. \tag{8}$$

The initial gauge variation of order zero in the spin s field is

$$\delta_{(0)}h^{(s)}(z;a) = s(a\nabla)\epsilon^{(s-1)}(z;a), \tag{9}$$

with a traceless gauge parameter for the double traceless gauge field

$$\Box_a \epsilon^{(s-1)}(z;a) = 0, \tag{10}$$

$$\Box_a^2 h^{(s)}(z;a) = 0. \tag{11}$$

2. Free Lagrangian for all higher spin gauge fields

We introduce a generating function for HS gauge fields by

$$\Phi(z;a) = \sum_{s=0}^{\infty}\frac{1}{s!}h^{(s)}(z;a) \tag{12}$$

where we assume that the spin s field has scaling dimension $s-2$, the a_i vectors have dimension -1, and therefore all terms in the generating function for higher spin gauge fields (12) have the same dimension -2. A zeroth order gauge transformation for this field reads as

$$\delta_\Lambda^0 \Phi(z;a) = (a\nabla)\Lambda(z;a), \tag{13}$$

$$\delta_\Lambda^0 D_a \Phi(z;a) = \Box\Lambda(z;a), \tag{14}$$

$$\delta_\Lambda^0 \Box_a \Phi(z;a) = 2(\nabla\partial_a)\Lambda(z;a), \tag{15}$$

where

$$\Lambda(z;a) = \sum_{s=1}^{\infty}\frac{1}{(s-1)!}\epsilon^{(s-1)}(z;a), \tag{16}$$

is the generating function of the gauge parameters and is dimensionless.[4]

Fronsdal's constraint for the gauge parameter reads as

$$\Box_a \Lambda(z;a) = 0. \tag{17}$$

For a spin s field gauge variation we get as expected

$$\delta_\epsilon^0 h^{(s)}(z;a) = s(a\nabla)\epsilon^{(s-1)}(z;a). \tag{18}$$

The second Fronsdal constraint of the gauge field reads in these notations

$$\Box_a^2 \Phi(z;a) = 0. \tag{19}$$

We introduced the "deDonder" operator

$$D_{a_i} = (\partial_{a_i}\nabla_i) - \frac{1}{2}(a_i\nabla_i)\Box_{a_i}. \tag{20}$$

This operator is "linear" in ∂_{a_i}.

Here we write the quadratic Lagrangian for free higher spin gauge fields in general form using the generating function for HS fields (12). First we introduce Fronsdal's operator

$$\mathcal{F}_{a_i} = \Box_i - (a_i\nabla_i)(\nabla_i\partial_{a_i}) + \frac{1}{2}(a_i\nabla_i)^2\Box_{a_i}, \tag{21}$$

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or with the help of (20)

$$\mathcal{F}_{a_i} = \Box_i - (a_i \nabla_i) D_{a_i}. \tag{22}$$

The operator of the equation of motion can be written in the form

$$\mathcal{G}_{a_i} = \mathcal{F}_{a_i} - \frac{a_i^2}{4} \Box_{a_i} \mathcal{F}_{a_i}. \tag{23}$$

Now we can write the free Lagrangian for all gauge fields of any spin in a symmetric elegant form

$$\mathcal{L}^{free}(z) = \frac{\kappa}{2} \exp[\lambda^2 \partial_{a_1} \partial_{a_2}] \int_{z_1 z_2} \delta(z_1 - z)\delta(z_2 - z)$$

$$\times \left\{ (\nabla_1 \nabla_2) - \lambda^2 D_{a_1} D_{a_2} - \frac{\lambda^4}{4} (\nabla_1 \nabla_2) \Box_{a_1} \Box_{a_2} \right\}$$

$$\times \Phi(z_1; a_1)\Phi(z_2; a_2)|_{a_1 = a_2 = 0} \tag{24}$$

where λ has scaling dimension -1, therefore λ^2 compensates the dimension of the operator in the exponent. We will see that all relative coupling constants of HS interactions can be expressed as powers of λ. The parameter κ is a constant which makes the action dimensionless (analogous to Einstein's constant and simply connected with the latter). It has scaling dimension $6 - d$, where d is the space–time dimension. For Einstein's constant κ_E we get

$$\kappa_E^{-2} = \kappa \lambda^4. \tag{25}$$

It is now obvious that in the free Lagrangian there is no mixing between gauge fields of different spin. It can also be written in such forms

$$\mathcal{L}^{free}(z) = -\frac{1}{2} \exp[\lambda^2 \partial_{a_1} \partial_{a_2}]$$

$$\times \int_{z_1} \delta(z_1 - z)(\mathcal{G}_{a_1})\Phi(z_1; a_1)\Phi(z; a_2)|_{a_1 = a_2 = 0}$$

$$= -\frac{1}{2} \exp[\lambda^2 \partial_{a_1} \partial_{a_2}]$$

$$\times \int_{z_2} \delta(z_2 - z)(\mathcal{G}_{a_2})\Phi(z; a_1)\Phi(z; a_2)|_{a_1 = a_2 = 0}. \tag{26}$$

These expressions reproduce Fronsdal's Lagrangians for all gauge fields with any spin.

3. Cubic interactions

We are going to present a very beautiful and compact form of all HS gauge field interactions derived in [1]. First we rewrite the leading term of a general trilinear interaction of higher spin gauge fields with any spins s_1, s_2, s_3[5]

$$\mathcal{L}^{leading}_{(1)}\left(h^{(s_1)}(z), h^{(s_2)}(z), h^{(s_3)}(z)\right)$$

$$= \sum_{\alpha + \beta + \gamma = n} \frac{1}{\alpha!\beta!\gamma!} \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3)$$

$$\times \left[(\nabla_1 \partial_c)^{s_3 - n + \gamma} (\nabla_2 \partial_a)^{s_1 - n + \alpha} (\nabla_3 \partial_b)^{s_2 - n + \beta} \right.$$

$$\times (\partial_a \partial_b)^\gamma (\partial_b \partial_c)^\alpha (\partial_c \partial_a)^\beta \right]$$

$$\times h^{(s_1)}(a; z_1) h^{(s_2)}(b; z_2) h^{(s_3)}(c; z_3), \tag{27}$$

where the number of derivatives is

5 $\nabla_2 \partial_a = \frac{\partial}{\partial a^\mu} \nabla_2^\mu$ and so on.

$$\Delta = s_1 + s_2 + s_3 - 2n, \tag{28}$$

$$0 \leqslant n \leqslant \min(s_1, s_2, s_3). \tag{29}$$

As we see, the minimal and maximal possible numbers of derivatives are

$$\Delta_{\min} = s_1 + s_2 + s_3 - 2\min(s_1, s_2, s_3), \tag{30}$$

$$\Delta_{\max} = s_1 + s_2 + s_3. \tag{31}$$

These interactions trivialize only if we have two equal spin values and the third value is odd. This we call the $\ell - s - s$ case, where ℓ is odd. In that case we should have a multiplet of spin s fields, with at least two charges to couple to the spin ℓ field. In the case of $\ell - \ell - \ell$ odd spin self interaction, the number of possible charges in the multiplet should be at least 3. The case of $\Delta_{\min}$ is important also because only in that case the interaction (27) has the same dimension as the lowest spin field free Lagrangian.

The same Lagrangian can be written in the following way (due to a constant normalization factor 2^Δ)

$$\mathcal{L}^{leading}_{(1)}\left(h^{(s_1)}(z), h^{(s_2)}(z), h^{(s_3)}(z)\right)$$

$$= \sum_{\alpha + \beta + \gamma = n} \frac{1}{\alpha!\beta!\gamma!} \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3)$$

$$\times \left[(\nabla_{12} \partial_c)^{s_3 - n + \gamma} (\nabla_{23} \partial_a)^{s_1 - n + \alpha} (\nabla_{31} \partial_b)^{s_2 - n + \beta} \right.$$

$$\times (\partial_a \partial_b)^\gamma (\partial_b \partial_c)^\alpha (\partial_c \partial_a)^\beta \right]$$

$$\times h^{(s_1)}(a; z_1) h^{(s_2)}(b; z_2) h^{(s_3)}(c; z_3), \tag{32}$$

where

$$\nabla_{12} = \nabla_1 - \nabla_2, \tag{33}$$

$$\nabla_{23} = \nabla_2 - \nabla_3, \tag{34}$$

$$\nabla_{31} = \nabla_3 - \nabla_1. \tag{35}$$

Now we can see that the following expression is a generating function for the leading term of all interactions of HS gauge fields.

$$\mathcal{A}^{00} = \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3) \exp W$$

$$\times \Phi_1\left(z_1; a_1 + \frac{1}{2}\nabla_{23}\right) \Phi_2\left(z_2; a_2 + \frac{1}{2}\nabla_{31}\right)$$

$$\times \Phi_3\left(z_3; a_3 + \frac{1}{2}\nabla_{12}\right)\bigg|_{a_1 = a_2 = a_3 = 0} \tag{36}$$

with

$$W = \frac{\lambda^2}{2}\left[(\partial_{a_1}\partial_{a_2})(\partial_{a_3}\nabla_{12}) + (\partial_{a_2}\partial_{a_3})(\partial_{a_1}\nabla_{23}) \right.$$

$$\left. + (\partial_{a_3}\partial_{a_1})(\partial_{a_2}\nabla_{31}) \right]. \tag{37}$$

This can be written in another form

$$\mathcal{A}^{00}(\Phi(z)) = \int_{z_1, z_2, z_3} \delta(z - z_{1,2,3}) \exp \hat{W}$$

$$\times \Phi(z_1; a_1)\Phi(z_2; a_2)\Phi(z_3; a_3)|_{a_1 = a_2 = a_3 = 0} \tag{38}$$

where

$$\hat{W} = \frac{\lambda^2}{2}\left[(\partial_{a_1}\partial_{a_2})(\partial_{a_3}\nabla_{12}) + (\partial_{a_2}\partial_{a_3})(\partial_{a_1}\nabla_{23}) + (\partial_{a_3}\partial_{a_1})(\partial_{a_2}\nabla_{31}) \right]$$

$$+ \frac{1}{2}\left[(\partial_{a_3}\nabla_{12}) + (\partial_{a_1}\nabla_{23}) + (\partial_{a_2}\nabla_{31}) \right], \tag{39}$$

$$\int_{z_1, z_2, z_3} \delta(z - z_{1,2,3}) = \int_{z_1, z_2, z_3} \delta(z - z_1)\delta(z - z_2)\delta(z - z_3) \tag{40}$$

for brevity. Furthermore we will always assume this integration with delta functions, without writing it explicitly. The operator in the second row of (39) is a dimensionless operator, therefore it does not need any dimensional constant multiplier.

Now we can derive all other terms in the Lagrangian using the following important relation

$$[\exp \hat{W}, A] = \exp \hat{W}[\hat{W}, A] + \exp \hat{W}[\hat{W}, [\hat{W}, A]]$$
$$+ \exp \hat{W}[\hat{W}, [\hat{W}, [\hat{W}, A]]] + \cdots \quad (41)$$

for any operator A. And therefore

$$[\exp \hat{W}, (a_1 \nabla_1)] = \exp \hat{W}[\hat{W}, (a_1 \nabla_1)], \quad (42)$$

$$[\exp \hat{W}, (a_2 \nabla_2)] = \exp \hat{W}[\hat{W}, (a_2 \nabla_2)], \quad (43)$$

$$[\exp \hat{W}, (a_3 \nabla_3)] = \exp \hat{W}[\hat{W}, (a_3 \nabla_3)]. \quad (44)$$

The following commutators will be used many times while deriving trace and divergence terms

$$[\hat{W}, (a_1 \nabla_1)] = -\frac{\lambda^2}{4} [(\partial_{a_2} \nabla_2)(\partial_{a_3} \nabla_{12}) + (\partial_{a_3} \nabla_3)(\partial_{a_2} \nabla_{31})]$$
$$+ \frac{1}{2} [\lambda^2 (\partial_{a_2} \partial_{a_3}) + 1] \nabla_1 \nabla_{23}, \quad (45)$$

$$[\hat{W}, (a_2 \nabla_2)] = -\frac{\lambda^2}{4} [(\partial_{a_3} \nabla_3)(\partial_{a_1} \nabla_{23}) + (\partial_{a_1} \nabla_1)(\partial_{a_3} \nabla_{12})]$$
$$+ \frac{1}{2} [\lambda^2 (\partial_{a_3} \partial_{a_1}) + 1] \nabla_2 \nabla_{31}, \quad (46)$$

$$[\hat{W}, (a_3 \nabla_3)] = -\frac{\lambda^2}{4} [(\partial_{a_1} \nabla_1)(\partial_{a_2} \nabla_{31}) + (\partial_{a_2} \nabla_2)(\partial_{a_1} \nabla_{23})]$$
$$+ \frac{1}{2} [\lambda^2 (\partial_{a_1} \partial_{a_2}) + 1] \nabla_3 \nabla_{12}. \quad (47)$$

Note that

$$\nabla_1 \nabla_{23} = \Box_3 - \Box_2, \quad (48)$$

$$\nabla_2 \nabla_{31} = \Box_1 - \Box_3, \quad (49)$$

$$\nabla_3 \nabla_{12} = \Box_2 - \Box_1, \quad (50)$$

which is obvious because[6]

$$\nabla_1 + \nabla_2 + \nabla_3 = 0. \quad (51)$$

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[6] We always understand partial integrations to be performed, working with a Lagrangian as with an action.

We are working with the same type of diagram as in [1].

$\Box_{a_i}$ \ D_{a_i}	0	1	2	3
0	$\mathcal{A}^{00}$	$\mathcal{A}^{10}$	$\mathcal{A}^{20}$	$\mathcal{A}^{30}$
1	$\mathcal{A}^{01}$	$\mathcal{A}^{11}$	$\mathcal{A}^{21}$	
2	$\mathcal{A}^{02}$	$\mathcal{A}^{12}$		
3	$\mathcal{A}^{03}$			

$$(52)$$

Now we take a gauge variation of $\mathcal{A}^{00}$, and find generating functions for all other terms in the cubic Lagrangian. A simple but elegant structure is exhibited by the first row of the diagram

$$\mathcal{A}^{10}(\Phi(z)) = \mathcal{A}^{30}(\Phi(z)) = 0, \quad (53)$$
$$\mathcal{A}^{20}(\Phi(z))$$
$$= \frac{1}{4} \exp \hat{W} \{ + [\lambda^2 (\partial_{a_1} \partial_{a_2}) + 1][\lambda^2 (\partial_{a_2} \partial_{a_3}) + 1] D_{a_3} D_{a_1}$$
$$+ [\lambda^2 (\partial_{a_2} \partial_{a_3}) + 1][\lambda^2 (\partial_{a_3} \partial_{a_1}) + 1] D_{a_1} D_{a_2}$$
$$+ [\lambda^2 (\partial_{a_3} \partial_{a_1}) + 1][\lambda^2 (\partial_{a_1} \partial_{a_2}) + 1] D_{a_2} D_{a_3} \}$$
$$\times \Phi(z_1; a_1) \Phi(z_2; a_2) \Phi(z_3; a_3)|_{a_1 = a_2 = a_3 = 0}. \quad (54)$$

Other terms are

$$\mathcal{A}^{01}(\Phi(z)) = 0, \quad (55)$$
$$\mathcal{A}^{11}(\Phi(z))$$
$$= \frac{\lambda^2}{16} \exp \hat{W} \{ + [\lambda^2 (\partial_{a_1} \partial_{a_2}) + 1](\partial_{a_1} \nabla_{23}) \Box_{a_3} D_{a_2}$$
$$- [\lambda^2 (\partial_{a_1} \partial_{a_2}) + 1](\partial_{a_2} \nabla_{31}) \Box_{a_3} D_{a_1}$$
$$+ [\lambda^2 (\partial_{a_2} \partial_{a_3}) + 1](\partial_{a_2} \nabla_{31}) \Box_{a_1} D_{a_3}$$
$$- [\lambda^2 (\partial_{a_2} \partial_{a_3}) + 1](\partial_{a_3} \nabla_{12}) \Box_{a_1} D_{a_2}$$
$$+ [\lambda^2 (\partial_{a_3} \partial_{a_1}) + 1](\partial_{a_3} \nabla_{12}) \Box_{a_2} D_{a_1}$$
$$- [\lambda^2 (\partial_{a_3} \partial_{a_1}) + 1](\partial_{a_1} \nabla_{23}) \Box_{a_2} D_{a_3} \}$$
$$\times \Phi(z_1; a_1) \Phi(z_2; a_2) \Phi(z_3; a_3)|_{a_1 = a_2 = a_3 = 0} \quad (56)$$

and so on.

All these expressions can be written in a very elegant form. First we introduce Grassmann variables by

$$\eta_{a_1}, \quad \bar{\eta}_{a_1}, \quad \eta_{a_2}, \quad \bar{\eta}_{a_2}, \quad \eta_{a_3}, \quad \bar{\eta}_{a_3}. \quad (57)$$

Then we change expressions in the formula (38) in a following way

$$(\partial_{a_1} \partial_{a_2}) \rightarrow (\partial_{a_1} \partial_{a_2}) + \frac{1}{4} \eta_{a_1} \bar{\eta}_{a_2} \Box_{a_2} + \frac{1}{4} \eta_{a_2} \bar{\eta}_{a_1} \Box_{a_1}, \quad (58)$$

$$(\partial_{a_2} \partial_{a_3}) \rightarrow (\partial_{a_2} \partial_{a_3}) + \frac{1}{4} \eta_{a_2} \bar{\eta}_{a_3} \Box_{a_3} + \frac{1}{4} \eta_{a_3} \bar{\eta}_{a_2} \Box_{a_2}, \quad (59)$$

$$(\partial_{a_3}\partial_{a_1}) \rightarrow (\partial_{a_3}\partial_{a_1}) + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_1}\Box_{a_1} + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_3}\Box_{a_3}, \tag{60}$$

$$(\partial_{a_1}\nabla_{23}) \rightarrow (\partial_{a_1}\nabla_{23}) + \eta_{a_1}\bar{\eta}_{a_2}D_{a_2} - \eta_{a_1}\bar{\eta}_{a_3}D_{a_3}, \tag{61}$$

$$(\partial_{a_2}\nabla_{31}) \rightarrow (\partial_{a_2}\nabla_{31}) + \eta_{a_2}\bar{\eta}_{a_3}D_{a_3} - \eta_{a_2}\bar{\eta}_{a_1}D_{a_1}, \tag{62}$$

$$(\partial_{a_3}\nabla_{12}) \rightarrow (\partial_{a_3}\nabla_{12}) + \eta_{a_3}\bar{\eta}_{a_1}D_{a_1} - \eta_{a_3}\bar{\eta}_{a_2}D_{a_2} \tag{63}$$

and can write

$$\begin{aligned}
\mathcal{A}(\Phi(z)) = \int &d\eta_{a_1}\,d\bar{\eta}_{a_1}\,d\eta_{a_2}\,d\bar{\eta}_{a_2}\,d\eta_{a_3}\,d\bar{\eta}_{a_3} \\
&\times (1 + \eta_{a_1}\bar{\eta}_{a_1})(1 + \eta_{a_2}\bar{\eta}_{a_2})(1 + \eta_{a_3}\bar{\eta}_{a_3}) \\
&\times \exp\hat{W}\,\Phi(z_1;a_1)\Phi(z_2;a_2)\Phi(z_3;a_3)|_{a_1=a_2=a_3=0}
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
\hat{W} = &\frac{1}{2}\left[1 + \lambda^2\left(\partial_{a_1}\partial_{a_2} + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_2}\Box_{a_2} + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_1}\Box_{a_1}\right)\right] \\
&\times [\partial_{a_3}\nabla_{12} + \eta_{a_3}\bar{\eta}_{a_1}D_{a_1} - \eta_{a_3}\bar{\eta}_{a_2}D_{a_2}] \\
&+ \frac{1}{2}\left[1 + \lambda^2\left(\partial_{a_2}\partial_{a_3} + \frac{1}{4}\eta_{a_2}\bar{\eta}_{a_3}\Box_{a_3} + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_2}\Box_{a_2}\right)\right] \\
&\times [\partial_{a_1}\nabla_{23} + \eta_{a_1}\bar{\eta}_{a_2}D_{a_2} - \eta_{a_1}\bar{\eta}_{a_3}D_{a_3}] \\
&+ \frac{1}{2}\left[1 + \lambda^2\left(\partial_{a_3}\partial_{a_1} + \frac{1}{4}\eta_{a_3}\bar{\eta}_{a_1}\Box_{a_1} + \frac{1}{4}\eta_{a_1}\bar{\eta}_{a_3}\Box_{a_3}\right)\right] \\
&\times [\partial_{a_2}\nabla_{31} + \eta_{a_2}\bar{\eta}_{a_3}D_{a_3} - \eta_{a_2}\bar{\eta}_{a_1}D_{a_1}].
\end{aligned} \tag{65}$$

This operator generates all terms in the cubic interaction of any three HS fields with any possible number of derivatives Δ in the range $\Delta_{\min} \leqslant \Delta \leqslant \Delta_{\max}$. Another possible form of the $\hat{W}$ operator is

$$\begin{aligned}
\hat{W} = &\left[1 + \lambda^2\left(\partial_{a_1}\partial_{a_2} + \frac{1}{2}\eta_{a_1}\bar{\eta}_{a_2}\Box_{a_2}\right)\right] \\
&\times \left[(\partial_{a_3}\nabla_1) + \frac{1}{2}\eta_{a_3}\bar{\eta}_{a_1}D_{a_1} - \frac{1}{2}\eta_{a_3}\bar{\eta}_{a_2}D_{a_2} + \frac{1}{2}\eta_{a_3}\bar{\eta}_{a_3}D_{a_3}\right] \\
&+ \left[1 + \lambda^2\left(\partial_{a_2}\partial_{a_3} + \frac{1}{2}\eta_{a_2}\bar{\eta}_{a_3}\Box_{a_3}\right)\right] \\
&\times \left[(\partial_{a_1}\nabla_2) + \frac{1}{2}\eta_{a_1}\bar{\eta}_{a_2}D_{a_2} - \frac{1}{2}\eta_{a_1}\bar{\eta}_{a_3}D_{a_3} + \frac{1}{2}\eta_{a_1}\bar{\eta}_{a_1}D_{a_1}\right] \\
&+ \left[1 + \lambda^2\left(\partial_{a_3}\partial_{a_1} + \frac{1}{2}\eta_{a_3}\bar{\eta}_{a_1}\Box_{a_1}\right)\right] \\
&\times \left[(\partial_{a_2}\nabla_3) + \frac{1}{2}\eta_{a_2}\bar{\eta}_{a_3}D_{a_3} - \frac{1}{2}\eta_{a_2}\bar{\eta}_{a_1}D_{a_1} + \frac{1}{2}\eta_{a_2}\bar{\eta}_{a_2}D_{a_2}\right].
\end{aligned} \tag{66}$$

This case generates the Lagrangian derived in [1]. The leading term of that Lagrangian is (27). These two operators (65) and (66) generate two Lagrangians that differ from each other just by partial integration and field redefinition. All interactions of HS gauge fields with any number of derivatives are unique and are generated by both operators (65) and (66).

In the case of (65) we have

$\begin{matrix}&D_{a_i}\\ \Box_{a_i}&\end{matrix}$	0	1	2	3
0	$\mathcal{A}^{00}$	0	$\mathcal{A}^{20}$	0
1	0	$\mathcal{A}^{11}$	$\mathcal{A}^{21}$	
2	$\mathcal{A}^{02}$	$\mathcal{A}^{12}$		
3	$\mathcal{A}^{03}$			

$$\tag{67}$$

In the case of (66) we have

$\begin{matrix}&D_{a_i}\\ \Box_{a_i}&\end{matrix}$	0	1	2	3
0	$\mathcal{A}^{00}$	$\mathcal{A}^{10}$	$\mathcal{A}^{20}$	$\mathcal{A}^{30}$
1	0	$\mathcal{A}^{11}$	$\mathcal{A}^{21}$	
2	0	$\mathcal{A}^{12}$		
3	$\mathcal{A}^{03}$			

$$\tag{68}$$

Both forms of the same cubic Lagrangian are very useful for further investigations.

Note added

When the present work was on its final stage for submission, the paper [22] appeared in the archive which includes an analysis of off-shell cubic interactions in HS field theories from String theory point of view, using BRST technique. The two results are closely related.

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