On the Boussinesq hypothesis for a stochastic Proudman-Taylor model

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Abstract

We introduce a stochastic version of Proudman-Taylor model, a 2D-3C fluid approximation of the 3D Navier-Stokes equations, with the small-scale turbulence modeled by a transport-stretching noise. For this model we may rigorously take a scaling limit leading to a deterministic model with additional viscosity on large scales. In certain choice of noises without mirror symmetry, we identify an anisotropic kinetic alpha (AKA) effect. This is the first example with a 3D structure and a stretching noise term.

Keywords: Boussinesq hypothesis, Proudman-Taylor model, turbulence, scaling limit, AKA effect

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1 Introduction

1.1 General comments

The purpose of this paper is proving that a certain model of a 3D fluid, incorporating small-scale turbulence by means of a suitable stochastic modification, manifests the property of dissipation on large scales predicted by Joseph Boussinesq in 1877 (cf. [4]). See Section 1.3 below for an illustration of the property in the framework of stochastic 3D Navier-Stokes equations in vorticity form; the latter will be heuristically derived in Section 1.2 as a starting point of investigation of the Boussinesq property. This property is at the foundation of Large Eddy Simulation (LES) models but its validity is critical [16, 24]. And its proof under suitable circumstances is a question which may count a number of contributions but not a definite answer, see for instance [3, 16, 18, 27].

From the numerical or experimental side, it is perhaps true that this property is better verified in certain 3D cases. In 2D, inverse cascade seems to deteriorate the structure of small-scale turbulence and scale separation needed to have dissipation at large scales, so the result is true but for a limited time [5], and after a while perturbations appear which even lead to a phenomenon often quoted as negative viscosity. In 3D, small-scale turbulence is more persistent and thus better oriented to the validity of the property investigated here; however, other elements in 3D may work against (for instance the direct cascade) and the picture is perhaps more difficult than in 2D, see e.g. [27].

Concerning simplified models in 2D with stochastic inputs as those discussed in [7, 9, 13, 19], it has been proved that an additional positive turbulent viscosity emerges (the validity of such

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result compared to direct numerical simulations [5] is limited to an initial time range). Similar unpublished computations, however, seem to indicate that the property fails for some 3D fluid models, due to a lack of control on the stochastic stretching term, as discussed in Section 1.3 (see also [12, 20] which involve only the transport noise). It emerges, among the possible ideas, the one that only a suitable geometry of the involved vector fields could remedy the excess of fluctuations in the stretching term. In particular, if the large-scale vorticity field and the small-scale turbulent velocity field would be orthogonal, this would cancel the above mentioned stretching contribution. But such geometry is too extreme and essentially boils down to the 2D problem.

Here we investigate a model which is in between the 2D and the 3D ones, the so-called 2D-3C model (called also 2.5D model [25]), where the dependence on space variables is 2D but the vector fields have three components. The physics behind this simplification has been identified by Proudman [22] and Taylor [26] and corresponds to situations with large Coriolis or rotation forces [1, 2]. In the transient, corrections to the 2D-3C model should be included, but they will be the object of future investigation. We think that the phenomena appearing in the simple 2D-3C model are interesting enough to justify a separate study.

The model is formulated below in Section 1.4. The small scales are modeled by a noise and in a suitable scaling limit we prove the existence of the additional dissipation at large scales predicted by Boussinesq [4].

The noise, as opposed to most previous works [7, 9, 15] where it was defined by Fourier decomposition, is modeled on idealized but still meaningful 3D vortex structures, following [11]. A Fourier description is possible and illustrated in Section 2.2, but the assumptions on the covariance function are suggested by the vortex structure interpretation. In Section 1.2 we explain the heuristic behind this choice. This new noise has been essential for us to identify the presence of the so-called AKA (anisotropic kinetic alpha) effect [27]. It is the presence of a term with first order derivatives, in addition to the dissipative second order term predicted by Boussinesq. As shown below and in analogy with previous studies based on different models and techniques like [27], the AKA effect is absent when mirror-symmetry (namely parity invariance) is imposed on the small-scale turbulence. Removing mirror symmetry, still we may have or not have the AKA effect (as in [27]); we provide examples of both cases.

In the rest of the introduction, we first describe, via the idea of separation of scales, the origin of noise in the vorticity formulation of 3D Navier-Stokes equations, then we discuss the Boussinesq hypothesis for such stochastic fluid equations, and finally we specialize the system in the 2D-3C setting to get the stochastic model studied in the paper.

1.2 Origin of noise in stochastic 3D Navier-Stokes equations

Consider a 3D Newtonian viscous fluid described, in vorticity form, by the equations

$$\begin{aligned}\partial_t \omega + v \cdot \nabla \omega - \omega \cdot \nabla v &= \nu \Delta \omega, \\ \omega &= \operatorname{curl} v, \\ \omega|_{t=t_0} &= \omega(t_0),\end{aligned}$$

where $\nu > 0$ is the fluid viscosity, ω is the vorticity field and v the velocity field. In the rigorous analysis below we shall assume that the space variable is in a torus, in order to avoid the difficult question of vorticity at a solid boundary and inessential troubles with the lack of compactness of full space.

We have taken as initial condition a time t_0 because the heuristic idea described here is that, if at a certain time t_0 the vorticity $\omega(t_0)$ is the sum of a large-scale component $\omega_L(t_0)$

plus a small-scale component $\omega_S(t_0)$ made of several small vortex structures $\omega_S^i(t_0)$:

$$\omega(t_0) = \omega_L(t_0) + \sum_{i=1}^{N} \omega_S^i(t_0),$$

then, at least on a short time interval $[t_0, t_0 + \Delta]$, the system

$$\begin{aligned} \partial_t \omega_L + v \cdot \nabla \omega_L - \omega_L \cdot \nabla v &= \nu \Delta \omega_L, \\ \partial_t \omega_S^i + v \cdot \nabla \omega_S^i - \omega_S^i \cdot \nabla v &= \nu \Delta \omega_S^i, \\ \omega_L|_{t=t_0} = \omega_L(t_0), \quad \omega_S^i|_{t=t_0} &= \omega_S^i(t_0) \end{aligned}$$

represents quite well the evolution of the different vortex structures. This kind of approach has been used for instance to investigate the small vortex-blob limit to point vortices [21] in 2D. The idea is that for a short while the small vortex structures ω_S^i maintain their integrity, as individual objects, before complicated merging and instability mechanisms take place. The system above is equivalent to the original one, in the sense that if $(\omega_L, \omega_S^1, \ldots, \omega_S^N)$ is a solution of the system, then $\omega = \omega_L + \sum_{i=1}^{N} \omega_S^i$ is a solution of the original equation.

The next step is considering only the equation for the large scales, isolating the term which is not closed, namely depends on the small scales:

$$\partial_t \omega_L + v_L \cdot \nabla \omega_L - \omega_L \cdot \nabla v_L - \nu \Delta \omega_L = -\sum_{i=1}^{N} \left(v_S^i \cdot \nabla \omega_L - \omega_L \cdot \nabla v_S^i \right).$$

Here v_L, with $\operatorname{div} v_L = 0$, has the property $\operatorname{curl} v_L = \omega_L$; namely v_L is reconstructed from ω_L by the Biot-Savart law: $v_L = K * \omega_L$ where K is the Biot-Savart kernel; and v_S^i corresponds to ω_S^i in the same way.

Until now we have not modified the true Navier-Stokes equations, just rewritten in a suitable way. The reformulation may not have the desired properties when the vortex structures ω_S^i start to merge or the large-scale field ω_L starts to develop small-scale instabilities, but it is reasonable to expect that in a short time the structure is roughly maintained.

Now we make a relevant approximation, whose validity is a major open problem. We idealize the small-scale structures v_S^i by a stochastic process, a priori given, delta correlated in time. The idealization of delta correlation in time is very strong, vaguely motivated by the very short time-scale of ω_S^i compared to ω_L. The idealization of Gaussianity and independence on ω_L are strong limitations which should be better investigated in the future. A large body of rigorous activity exists, see for instance [6, 14], aimed to justify this approximation starting from the original model, suitably perturbed or simplified.

Essentially, the modeling assumption made below replaces the small-scale velocity field $\sum_{i=1}^{N} v_S^i(t, x)$ by

$$\sum_k \sigma_k(x) \frac{dW_t^k}{dt},$$

where $\{\sigma_k\}_k$ are suitable divergence free fields and $\{W_t^k\}_k$ are independent scalar Brownian motions. In the replacement, Stratonovich integrals are used, in accordance to the Wong-Zakai principle (see rigorous results in [6, 14]).

1.3 Boussinesq hypothesis and its difficulty in 3D fluids

The presentation of this subsection is heuristic, the purpose being only to describe what we mean by Boussinesq hypothesis for the class of stochastic 3D Navier-Stokes equations derived above, and to explain which is the difficulty to solve the problem in general (motivating the detailed analysis of the particular case treated in this work). From now on we omit the subscript L for the large scales, since many other indices will appear. But the reader should remember, as a matter of interpretation, that v and ω are large-scale fields, $\{\sigma_k\}_k$ describe the small-scale turbulent components. We also conventionally set $t_0 = 0$. To simplify notation, for two vector fields X and Y, we write $\mathcal{L}_X Y$ for the Lie derivative $\mathcal{L}_X Y = X \cdot \nabla Y - Y \cdot \nabla X$.

Motivated by the heuristic discussions in the previous subsection, we consider the following stochastic 3D Navier-Stokes equations

$$d\omega_\epsilon + (\mathcal{L}_{v_\epsilon}\omega_\epsilon - \nu\Delta\omega_\epsilon)\,dt = -\sum_k \mathcal{L}_{\sigma_k^\epsilon}\omega_\epsilon \circ dW_t^k,$$
$$\omega_\epsilon|_{t=0} = \omega^0$$

where $\{\sigma_k^\epsilon\}_k$ are given vector fields, now parametrized by $\epsilon > 0$, hence also the solution ω_ϵ depends on ϵ. The stochastic multiplication is understood in the Stratonovich sense, which is formally equivalent to the following Itô equation with a correction term:

$$d\omega_\epsilon + (\mathcal{L}_{v_\epsilon}\omega_\epsilon - \nu\Delta\omega_\epsilon)\,dt = -\sum_k \mathcal{L}_{\sigma_k^\epsilon}\omega_\epsilon\,dW_t^k + \frac{1}{2}\sum_k \mathcal{L}_{\sigma_k^\epsilon}\mathcal{L}_{\sigma_k^\epsilon}\omega_\epsilon\,dt.$$

We say that Boussinesq hypothesis holds for a family of coefficients $\{\sigma_k^\epsilon\}_k$ if ω_ϵ converges as $\epsilon \to 0$, in an appropriate sense, to a solution $\overline{\omega}$ of the deterministic equation

$$\partial_t\overline{\omega} + \mathcal{L}_{\overline{v}}\overline{\omega} = (\nu\Delta + \mathcal{D})\,\overline{\omega},$$
$$\overline{\omega}|_{t=0} = \omega^0$$

where $\overline{v} = K * \overline{\omega}$, $\mathcal{D}$ is a suitable second order differential operator, limit of $\frac{1}{2}\sum_k \mathcal{L}_{\sigma_k^\epsilon}\mathcal{L}_{\sigma_k^\epsilon}$ as $\epsilon \to 0$. This operator represents the enhancement of viscosity produced by the noise in the scaling limit, and possibly incorporating other effects than viscosity, like the AKA effect. Loosely speaking, we may consider the Boussinesq hypothesis as a mean field result.

Finding families $\{\sigma_k^\epsilon\}_k$ such that the iterated Lie derivative operator $\frac{1}{2}\sum_k \mathcal{L}_{\sigma_k^\epsilon}\mathcal{L}_{\sigma_k^\epsilon}$ has a nontrivial and interesting limit $\mathcal{D}$ is not difficult. The difficulty is proving that the Itô terms $\int_0^T \mathcal{L}_{\sigma_k^\epsilon}\omega_\epsilon(t)\,dW_t^k$ converge to zero in a certain sense (in the heuristic mean field vision, they are the fluctuations). And, to be more precise, the difficulty in proving this result for the Itô terms lies in *the control of* ω_ϵ. Imposing assumptions on the noise covariance in order to have convergence to zero of expressions like

$$\sum_k \int_0^T \left\langle f(t), \mathcal{L}_{\sigma_k^\epsilon}\phi \right\rangle dW_t^k$$

for a given process f and smooth test function ϕ, assumptions compatible with those leading to a non-trivial limit $\mathcal{D}$ of $\frac{1}{2}\sum_k \mathcal{L}_{\sigma_k^\epsilon}\mathcal{L}_{\sigma_k^\epsilon}$, is possible. But a control on ω_ϵ is needed and this is highly non-trivial for 3D models.

The difficulty is only marginally due to the nonlinear term $\mathcal{L}_{v_\epsilon}\omega_\epsilon$. The same difficulty arises in the investigation of the Boussinesq hypothesis for a passive magnetic field M_ϵ:

$$dM_\epsilon - \eta\Delta M_\epsilon\,dt = -\sum_k \mathcal{L}_{\sigma_k^\epsilon}M_\epsilon \circ dW_t^k,$$
$$M_\epsilon|_{t=0} = M^0$$

($\eta > 0$). The mean field equation

$$\partial_t \overline{M} = (\eta\Delta + \mathcal{D})\,\overline{M},$$
$$\overline{M}|_{t=0} = M^0$$

is, in this linear case, rigorously obtained as the limit of the average of M_ϵ (see for instance [17]). However, the difficulty in the control of M_ϵ remains and it is due to the fact that the random stretching terms $M_\epsilon \cdot \nabla\sigma_k^\epsilon$ have a stronger and stronger effect as $\epsilon \to 0$.

1.4 The stochastic 2D-3C model and main result

As in the last subsection, we start from the following stochastic version of the 3D Navier-Stokes equations

$$\begin{aligned} d\omega + (v\cdot\nabla\omega - \omega\cdot\nabla v - \nu\Delta\omega)\,dt &= -\sum_k (\sigma_k\cdot\nabla\omega - \omega\cdot\nabla\sigma_k)\circ dW_t^k, \\ \omega|_{t=0} &= \omega_0. \end{aligned} \tag{1.1}$$

We still consider this equation at a formal level (but only to shorten the next preliminary steps, since this could be made rigorous).

Assume that, with the notation $x = (x_1, x_2, x_3)$, the initial condition and the noise are independent of x_3:

$$\omega_0 = \omega_0(x_1, x_2), \qquad \sigma_k = \sigma_k(x_1, x_2).$$

However, all vector fields still take values in $\mathbb{R}^3$ and we denote the canonical basis of $\mathbb{R}^3$ by $\{e_1, e_2, e_3\}$. Under the previous assumption, a solution of the stochastic equations above is given by

$$v(t, x_1, x_2) = v_H(t, x_1, x_2) + v_3(t, x_1, x_2)\,e_3,$$
$$\omega(t, x_1, x_2) = \omega_H(t, x_1, x_2) + \omega_3(t, x_1, x_2)\,e_3,$$

where v_H, ω_H live in the horizontal plane, namely take values in the span of $\{e_1, e_2\}$. Notice that the relation $\omega = \operatorname{curl} v$ implies

$$\omega_3 = -\nabla_H^\perp \cdot v_H, \quad \omega_H = \nabla_H^\perp v_3,$$

so the horizontal and vertical components are interlaced. Here, $\nabla_H^\perp = (\partial_2, -\partial_1)$ with $\partial_i = \frac{\partial}{\partial x_i}$. The model incorporates some degree of helicity (vorticity and velocity are not orthogonal). Decompose also the noise:

$$W = \sum_{k\in\mathcal{I}} \sigma_k W^k = \sum_{k\in\mathcal{I}} \sigma_k^H W^k + \sum_{k\in\mathcal{I}} \sigma_k^3 W^k e_3,$$

where $\mathcal{I}$ is some index set. Then we get the model ($\nabla_H = (\partial_1, \partial_2)$ and $\Delta_H = \partial_1^2 + \partial_2^2$)

$$d\omega_3 + (v_H\cdot\nabla_H\omega_3 - \omega_H\cdot\nabla_H v_3 - \nu\Delta_H\omega_3)\,dt = -\sum_k \left(\sigma_k^H\cdot\nabla_H\omega_3 - \omega_H\cdot\nabla_H\sigma_k^3\right)\circ dW_t^k,$$
$$dv_3 + (v_H\cdot\nabla_H v_3 - \nu\Delta_H v_3)\,dt = -\sum_k \sigma_k^H\cdot\nabla_H v_3 \circ dW_t^k,$$
$$\omega_3|_{t=0} = \omega_3^0, \quad v_3|_{t=0} = v_3^0,$$

with obvious meaning of ω_3^0 and v_3^0, given ω_0. Indeed, taking curl of the second equation, and using the simple identity

$$\nabla_H^\perp(u\cdot\nabla_H v_3) = u\cdot\nabla_H\omega_H - \omega_H\cdot\nabla_H u$$

which holds for any 2D divergence free vector field u, we obtain the first two components of the initial vorticity equations (1.1). We remark that the stretching term $\omega_H\cdot\nabla_H v_3$ in the equation for ω_3 vanishes because $\omega_H = \nabla_H^\perp v_3$ is orthogonal to $\nabla_H v_3$. Before moving forward, let us mention that v_3 satisfies a stochastic advection-diffusion equation with pure transport noise, and thus it enjoys nice pathwise energy estimates, see (5.2) below. Combining this fact with the properties of noise given in Hypothesis 3.3, we can derive uniform bound on ω_3 in spite of the presence of stretching part of noise; such uniform estimates play important role in the scaling limit arguments in Section 5.1. Similar results are not available in the true 3D problem illustrated in Section 1.3, see for instance [12, Section 6].

In the next sections we make a rigorous analysis of the above stochastic 2D-3C model and its scaling limit. Roughly speaking, under a suitable scaling limit of the noise introduced in Section 2.1, we will show that the above stochastic model converges weakly to the following deterministic system

$$\begin{cases} \partial_t\omega_3 + v_H\cdot\nabla_H\omega_3 = \big(\nu\Delta_H + \mathcal{L}_{\bar Q}\big)\omega_3 + \nabla_H\cdot(A\,\omega_H), \\ \partial_t v_3 + v_H\cdot\nabla_H v_3 = \big(\nu\Delta_H + \mathcal{L}_{\bar Q}\big)v_3, \end{cases} \tag{1.2}$$

where $\bar Q$ and A are 2×2 constant matrices, $\bar Q$ being nonnegative definite and $\mathcal{L}_{\bar Q} f = \frac{1}{2}\nabla_H\cdot\big[\bar Q\,\nabla_H f\big]$, see Theorem 3.4 below for more details. The system (1.2) can be rewritten in the more familiar vorticity formulation as follows: taking curl of the second equation leads to

$$\partial_t\omega_H + v_H\cdot\nabla_H\omega_H - \omega_H\cdot\nabla_H v_H = \big(\nu\Delta_H + \mathcal{L}_{\bar Q}\big)\,\omega_H;$$

recalling that $\omega_H\cdot\nabla_H v_3 = 0$, we combine the first equation in (1.2) and the above one to obtain

$$\partial_t\omega + v_H\cdot\nabla_H\omega - \omega_H\cdot\nabla_H v = \big(\nu\Delta_H + \mathcal{L}_{\bar Q}\big)\,\omega + \nabla_H\cdot(A\,\omega_H)\,e_3.$$

Since $\partial_3\omega = \partial_3 v = 0$, the nonlinear part can be further written as $v\cdot\nabla\omega - \omega\cdot\nabla v$ with $\nabla = (\partial_1,\partial_2,\partial_3)$. We see that there is an extra dissipation term $\mathcal{L}_{\bar Q}\,\omega$ predicted by Boussinesq's hypothesis, and a first order term $\nabla_H\cdot(A\,\omega_H)e_3$ responsible for the possible AKA effect; the latter is a scalar object multiplied by e_3, but we may also write it in the general 3D form, which reduces to the above one due to special cancellations of our set up.

We conclude the introduction with the organization of the paper. We introduce in Section 2 the vortex noise which will be used to perturb the model. Section 3 contains the statements of our main results, including the well-posedness of stochastic 2D-3C models and a scaling limit result to the deterministic model mentioned above. We prove these results in Sections 4 and 5, respectively.

2 The vortex noise

In this section let us describe in detail the vortex noise W that we will use to perturb the 2D-3C model introduced above.

2.1 The covariance function

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus; we may identify it as $[-\frac{1}{2}, \frac{1}{2}]^2$ with periodic boundary condition. Recall that for $i \in \{1,2\}$, $\partial_i = \frac{\partial}{\partial x_i}$, $\nabla_H = (\partial_1, \partial_2)$ and $\nabla_H^\perp = (\partial_2, -\partial_1)$.

Let us assume that the velocity field of a vortex takes the form

$$\sigma(x) = \sigma_H(x) + \sigma_3(x)\, e_3, \quad x \in \mathbb{T}^2,$$

where $\sigma_H(x) = (\sigma_1(x), \sigma_2(x))$ is a divergence free vector field on $\mathbb{T}^2$; $\{\sigma_i\}_{i=1}^3$ are smooth periodic functions. For simplicity, we shall assume the vortex to be symmetric, namely,

$$\sigma_H(-x) = -\sigma_H(x), \quad \sigma_3(-x) = \sigma_3(x), \quad x \in \mathbb{T}^2. \tag{2.1}$$

As a typical example, we may take

$$\sigma_H = \Gamma K * \theta, \quad \sigma_3 = \gamma G * \chi, \tag{2.2}$$

where $\Gamma, \gamma > 0$ are vortex intensities, K is the Biot-Savart kernel and G the Green function on $\mathbb{T}^2$, and $\theta, \chi \in C_c^\infty\big((-\frac{1}{2}, \frac{1}{2})^2\big)$ are symmetric functions.

The associated vorticity field

$$\omega(x) = \omega_H(x) + \omega_3(x)\, e_3$$

is determined by the following relations:

$$\omega_H = \nabla_H^\perp \sigma_3, \quad \omega_3 = -\nabla_H^\perp \cdot \sigma_H.$$

Let X be a uniform random variable with values in $\mathbb{T}^2$, we can now define the velocity field of the random 3D vortex as follows:

$$\Sigma(x) = \sigma(x - X) = \sigma_H(x - X) + \sigma_3(x - X)\, e_3, \quad x \in \mathbb{T}^2.$$

Remark 2.1. *We describe here the "intuitive picture" of the vortex noise: choose a random location X in the horizontal torus $\mathbb{T}^2$, create a vertical 3D vortex object, similar to a sort of "vertical cylinder", with a certain velocity component in the vertical direction. Repeating this choice at high frequency, choosing at random the location X and the orientation (namely, multiply $\omega(x)$ by ± 1), we get a "random walk" of vortex structures that converges to a noise in a suitable scaling limit, see [11].*

Consider the covariance matrix function of the random field Σ:

$$Q(x,y) = \mathbb{E}[\Sigma(x) \otimes \Sigma(y)] = \mathbb{E}[\sigma(x - X) \otimes \sigma(y - X)] = \int_{\mathbb{T}^2} \sigma(x - z) \otimes \sigma(y - z)\, dz.$$

It is space-homogeneous, that is, equal to $Q(x - y)$ with

$$Q(a) = \int_{\mathbb{T}^2} \sigma(a - w) \otimes \sigma(-w)\, dw, \tag{2.3}$$

where we use the change of variables $y - z = -w$ in the integral above. As a matrix valued function, Q is smooth on $\mathbb{T}^2$.

We decompose $Q(a)$ in the form

$$Q(a) = \begin{pmatrix} Q_H(a) & Q_{H,3}(a) \\ Q_{3,H}(a) & Q_3(a) \end{pmatrix},$$

where, using (2.1),

$$\begin{aligned} Q_H(a) &= -\int_{\mathbb{T}^2} \sigma_H(a-x) \otimes \sigma_H(x)\,dx, \\ Q_3(a) &= \int_{\mathbb{T}^2} \sigma_3(a-x) \otimes \sigma_3(x)\,dx, \\ Q_{H,3}(a) &= \int_{\mathbb{T}^2} \sigma_H(a-x) \otimes \sigma_3(x)\,dx, \\ Q_{3,H}(a) &= -\int_{\mathbb{T}^2} \sigma_3(a-x) \otimes \sigma_H(x)\,dx. \end{aligned} \tag{2.4}$$

Remark 2.2. *Recall that, by the definition of covariance,*

$$Q(-a) = Q(a)^*$$

and that mirror symmetry means

$$Q(-a) = Q(a),$$

which also means that $Q(a)$ is a symmetric matrix. Notice that the covariance matrix of a random vector is always positive definite but, in spite of the name, $Q(a)$ is not the covariance matrix of a vector but the mutual covariance between two vectors, which may have any sign.

We have, by (2.1),

$$Q_H(-a) = -\int_{\mathbb{T}^2} \sigma_H(-a-x) \otimes \sigma_H(x)\,dx = \int_{\mathbb{T}^2} \sigma_H(a+x) \otimes \sigma_H(x)\,dx;$$

changing variable $z = -x$ yields

$$Q_H(-a) = \int_{\mathbb{T}^2} \sigma_H(a-z) \otimes \sigma_H(-z)\,dz = Q_H(a).$$

Similarly,

$$\begin{aligned} Q_3(-a) &= Q_3(a), \\ Q_{3,H}(-a) &= -Q_{3,H}(a), \\ Q_{H,3}(-a) &= -Q_{H,3}(a), \end{aligned}$$

hence *mirror symmetry* does not hold.

Remark 2.3 (Diagonality of $Q_H(0)$)**.** *Let the horizontal velocity field σ_H be given as in (2.2); by the definition of horizontal covariance function Q_H, we have*

$$\begin{aligned} Q_H(0) &= \int_{\mathbb{T}^2} \sigma_H(x) \otimes \sigma_H(x)\,dx = \Gamma^2 \int_{\mathbb{T}^2} (K * \theta)(x) \otimes (K * \theta)(x)\,dx \\ &= \Gamma^2 \int_{\mathbb{T}^2} (\nabla_H^\perp G * \theta)(x) \otimes (\nabla_H^\perp G * \theta)(x)\,dx. \end{aligned}$$

If we assume that θ is radially symmetric, *then $Q_H(0) = 2\kappa I_2$ for some $\kappa > 0$, here I_2 is the 2×2 unit matrix. Indeed, we have*

$$Q_H^{1,1}(0) = \Gamma^2 \int_{\mathbb{T}^2} \big[(\partial_2 G * \theta)(x)\big]^2\,dx, \quad Q_H^{2,2}(0) = \Gamma^2 \int_{\mathbb{T}^2} \big[(\partial_1 G * \theta)(x)\big]^2\,dx.$$

Using the radial symmetry of θ, one can easily show that

$$(\partial_2 G * \theta)(x_2, x_1) = (\partial_1 G * \theta)(x_1, x_2),$$

therefore, changing variable $(x_1, x_2) \to (x_2, x_1)$, we have

$$Q_H^{1,1}(0) = \Gamma^2 \int_{\mathbb{T}^2} \big[(\partial_2 G * \theta)(x_2, x_1)\big]^2 \, dx_2 dx_1 = \Gamma^2 \int_{\mathbb{T}^2} \big[(\partial_1 G * \theta)(x_1, x_2)\big]^2 \, dx_1 dx_2 = Q_H^{2,2}(0).$$

Next, for the off-diagonal entries,

$$Q_H^{1,2}(0) = -\Gamma^2 \int_{\mathbb{T}^2} (\partial_2 G * \theta)(x) \, (\partial_1 G * \theta)(x) \, dx = Q_H^{2,1}(0).$$

Again by the radial symmetry of θ, one can show that

$$(\partial_1 G * \theta)(-x_1, x_2) = -(\partial_1 G * \theta)(x_1, x_2), \quad (\partial_1 G * \theta)(x_1, -x_2) = (\partial_1 G * \theta)(x_1, x_2),$$
$$(\partial_2 G * \theta)(-x_1, x_2) = (\partial_2 G * \theta)(x_1, x_2), \quad (\partial_2 G * \theta)(x_1, -x_2) = -(\partial_2 G * \theta)(x_1, x_2).$$

These properties immediately imply that $Q_H^{1,2}(0) = Q_H^{2,1}(0) = 0$. Thus the assertion holds with $\kappa = \frac{1}{2} Q_H^{i,i}(0) = \frac{1}{4} \mathrm{Tr}(Q_H(0))$.

Remark 2.4. *Later on we will make a scaling of the vortex noise: for $\ell \in (0,1)$, let $\theta_\ell(x) = \ell^{-2}\theta(\ell^{-1}x)$ and define $\sigma_H^\ell = \Gamma K * \theta_\ell$. Then, as $\ell \to 0$, σ_H approaches a point vortex. In this case, if θ is a probability density with compact support in $(-\frac{1}{2}, \frac{1}{2})^2$, it is a classical result that its energy is of order $\log \ell^{-1}$; more precisely,*

$$\mathrm{Tr}(Q_H(0)) \sim \frac{\Gamma^2}{4\pi} \log \ell^{-1} \quad \text{as } \ell \to 0, \tag{2.5}$$

see Proposition 3.6 and its proof in Section 5.2, for instance, Lemma 5.5.

Let $\mathbb{Q}_H$ (resp. $\mathbb{Q}_3$) be the covariance operator associated to the covariance matrix Q_H (resp. covariance function Q_3) defined above. We conclude this part with estimates on the operator norms of $\mathbb{Q}_H$ and $\mathbb{Q}_3$ when the vortex velocity field is given by (2.2). Recall that the covariance operators are defined as follows:

$$(\mathbb{Q}_H V)(x) = \int_{\mathbb{T}^2} Q_H(x-y) V(y) \, dy = (Q_H * V)(x), \quad x \in \mathbb{T}^2, \, V \in L^2(\mathbb{T}^2, \mathbb{R}^2);$$
$$(\mathbb{Q}_3 f)(x) = \int_{\mathbb{T}^2} Q_3(x-y) f(y) \, dy = (Q_3 * f)(x), \quad x \in \mathbb{T}^2, \, f \in L^2(\mathbb{T}^2, \mathbb{R}).$$

Lemma 2.5. *Assume that the vortex velocity field is defined as in (2.2), then one has*

$$\|\mathbb{Q}_H\|_{L^2 \to L^2} \le \Gamma^2 \|K\|_{L^1}^2 \|\theta\|_{L^1}^2 \quad \text{and} \quad \|\mathbb{Q}_3\|_{L^2 \to L^2} \le \gamma^2 \|G\|_{L^1}^2 \|\chi\|_{L^1}^2.$$

Proof. We only prove the first estimate since the other proof is similar. By the classical inequality for convolution, one has

$$\|\mathbb{Q}_H V\|_{L^2} = \|Q_H * V\|_{L^2} \le \|Q_H\|_{L^1} \|V\|_{L^2}$$

which implies $\|\mathbb{Q}_H\|_{L^2 \to L^2} \le \|Q_H\|_{L^1}$.

Recalling the definition of Q_H, we have

$$\begin{aligned} \|Q_H\|_{L^1} &= \int_{\mathbb{T}^2} \bigg| \int_{\mathbb{T}^2} \sigma_H(x-y) \otimes \sigma_H(y) \, dy \bigg| \, dx \\ &\le \int_{\mathbb{T}^2} |\sigma_H(y)| \, dy \int_{\mathbb{T}^2} |\sigma_H(x-y)| \, dx \\ &= \|\sigma_H\|_{L^1}^2. \end{aligned}$$

Note that $\sigma_H = \Gamma K * \theta$, we use again the convolutional inequality and get

$$\|Q_H\|_{L^1} \le \Gamma^2 \|K * \theta\|_{L^1}^2 \le \Gamma^2 \|K\|_{L^1}^2 \|\theta\|_{L^1}^2.$$

Combining the above estimates we finish the proof. □

2.2 Series expansion of noise

The smoothness of Q implies that the corresponding covariance operator $\mathbb{Q}$ is of trace class. Let $\{f_k\}_k$ be a CONS of $\mathcal{H} = L^2_{\rm sol}(\mathbb{T}^3, \mathbb{R}^3)$ (sol means solenoidal fields) consisting of eigenvectors of $\mathbb{Q}$:

$$\mathbb{Q} f_k = \lambda_k f_k, \quad \lambda_k \geq 0, \quad k \geq 1.$$

For simplicity, we assume $\lambda_k > 0$ for all $k \geq 1$. From the above identity we deduce that the field f_k depends only on the horizontal variable $x = (x_1, x_2)$. Indeed, for any $\tilde{x} = (x, x_3) \in \mathbb{T}^3$, by the definition of $\mathbb{Q}$,

$$\begin{aligned} \lambda_k f_k(\tilde{x}) &= \int_{\mathbb{T}^3} Q(\tilde{x} - \tilde{y}) f_k(\tilde{y})\, d\tilde{y} = \int_{\mathbb{T}^3} Q(x - y) f_k(\tilde{y})\, d\tilde{y} \\ &= \int_{\mathbb{T}^2} Q(x-y) \bigg[\int_{\mathbb{T}} f_k(y, y_3)\, dy_3 \bigg] dy, \end{aligned}$$

where y is the horizontal part of $\tilde{y} \in \mathbb{T}^3$. The right-hand side depends only on x, thus $f_k(\tilde{x}) = f_k(x, x_3)$ is independent of x_3; this implies $\int_{\mathbb{T}} f_k(y, y_3)\, dy_3 = f_k(y)$. The above equality can be rewritten as

$$\lambda_k f_k(x) = \int_{\mathbb{T}^2} Q(x-y) f_k(y)\, dy.$$

The property of convolution implies that f_k is also smooth on $\mathbb{T}^2$.

Now let $W = \{W(t)\}_{t \geq 0}$ be a $\mathbb{Q}$-Brownian motion on $\mathcal{H}$; then there exists a sequence of independent real Brownian motions $\{W^k\}_{k \geq 1}$ such that

$$W(t) = \sum_k \sqrt{\lambda_k}\, W^k_t f_k = \sum_k \sigma_k W^k_t,$$

where we have denoted for simplicity that $\sigma_k = \sqrt{\lambda_k} f_k$, $k \geq 1$. We remark that

$$Q(x-y) = \sum_k \sigma_k(x) \otimes \sigma_k(y) \quad \text{for all } x, y \in \mathbb{T}^2, \tag{2.6}$$

and, since Q is smooth, the series converges in $C^m(\mathbb{T}^2 \times \mathbb{T}^2)$ for any $m \geq 1$. Similarly, we write the field σ_k as

$$\sigma_k(x) = \sigma^H_k(x) + \sigma^3_k(x)\, e_3,$$

where the horizontal part $\sigma^H_k = (\sigma^1_k, \sigma^2_k)$ is a divergence free field on $\mathbb{T}^2$. Accordingly, we can write the noise $W(t)$ as

$$\begin{aligned} W(t,x) &= W_H(t,x) + W_3(t,x)\, e_3, \\ W_H(t,x) &= \sum_k \sigma^H_k(x) W^k_t, \quad W_3(t,x) = \sum_k \sigma^3_k(x) W^k_t. \end{aligned}$$

3 Main results

In this part we first show in Section 3.1 the well-posedness of the stochastic 2D-3C models perturbed by the vortex noise introduced above, then in Section 3.2 we shall rescale the noise and show that the stochastic models converge to deterministic 2D-3C model which might exhibit AKA (short for anisotropic kinetic alpha) effect.

In the sequel, since the space variables x are always two dimensional, we omit for simplicity the subscript H in Δ_H, ∇_H, $\nabla_H^\perp$ etc, but we still write v_H, ω_H to distinguish them from v_3, ω_3.

We denote the usual Sobolev spaces on $\mathbb{T}^2$ as $H^s = H^s(\mathbb{T}^2)$, which will also be used for vector valued functions. When we want to specify the target space, we write more explicitly as $H^s(\mathbb{T}^2, \mathbb{R}^d)$. The norm in H^s is written as $\|\cdot\|_{H^s}$ and in case $s = 0$, we write H^0 as L^2 with the norm $\|\cdot\|_{L^2}$. We assume the spaces H^s consist of functions with zero spatial mean on $\mathbb{T}^2$. The notation $\langle \cdot, \cdot \rangle$ will be used for the scalar product in L^2 or the duality between H^s and H^{-s}. Sometimes we denote the norm in $L^p(0, T; H^s)$ simply as $\|\cdot\|_{L^p H^s}$, $p \geq 1, s \in \mathbb{R}$. In the sequel, we write $a \lesssim b$ if there exists some unimportant constant $C > 0$ such that $a \leq Cb$; the notation $\lesssim_{\nu,T}$ means that the constant C is dependent on ν, T.

3.1 Well-posedness of stochastic 2D-3C models

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space and $\{W^k\}_k$ a family of independent standard Brownian motions. Recall the vector fields $\{\sigma_k^H\}_k$ and functions $\{\sigma_k^3\}_k$ introduced in Section 2.2. Motivated by the discussions in Section 1.4, we consider the following stochastic 2D-3C model:

$$\begin{cases} d\omega_3 + (v_H \cdot \nabla \omega_3 - \nu \Delta \omega_3)\, dt = -\sum_k \left(\sigma_k^H \cdot \nabla \omega_3 - \omega_H \cdot \nabla \sigma_k^3\right) \circ dW_t^k, \\ dv_3 + (v_H \cdot \nabla v_3 - \nu \Delta v_3)\, dt = -\sum_k \sigma_k^H \cdot \nabla v_3 \circ dW_t^k. \end{cases} \tag{3.1}$$

where $\circ d$ means Stratonovich stochastic differential. We recall that

$$\omega_3 = -\nabla^\perp \cdot v_H, \quad \omega_H = \nabla^\perp v_3;$$

the latter identity explains why the stretching term $\omega_H \cdot \nabla v_3$ vanishes in the first equation; however, the random stretching terms $\sum_k \omega_H \cdot \nabla \sigma_k^3 \circ dW_t^k$ remain and, depending on certain condition, they might be the origin of the AKA term. Moreover, by the Biot-Savart law, $v_H = K * \omega_3$ where K is the Biot-Savart kernel on $\mathbb{T}^2$. In Itô formulation, the system becomes (see Section 4.1 for the derivations)

$$\begin{cases} d\omega_3 + (v_H \cdot \nabla \omega_3 - \nu \Delta \omega_3)\, dt = -\sum_k \left(\sigma_k^H \cdot \nabla \omega_3 - \omega_H \cdot \nabla \sigma_k^3\right) dW_t^k \\ \qquad\qquad\qquad\qquad + \left(\mathcal{L}\omega_3 + \nabla \cdot [\nabla Q_{H,3}(0)\, \omega_H]\right) dt, \\ dv_3 + (v_H \cdot \nabla v_3 - \nu \Delta v_3)\, dt = -\sum_k \sigma_k^H \cdot \nabla v_3\, dW_t^k + \mathcal{L} v_3\, dt, \end{cases} \tag{3.2}$$

where $\nabla Q_{H,3}(0)$ is a constant matrix given by

$$\nabla Q_{H,3}(0) = \begin{pmatrix} \partial_1 Q_{1,3}(0) & \partial_2 Q_{1,3}(0) \\ \partial_1 Q_{2,3}(0) & \partial_2 Q_{2,3}(0) \end{pmatrix} \tag{3.3}$$

and $\mathcal{L}$ is a second order differential operator given by

$$\mathcal{L} f = \frac{1}{2} \sum_k \sigma_k^H \cdot \nabla(\sigma_k^H \cdot \nabla f), \quad f \in C^2(\mathbb{T}^2).$$

As the vector fields $\{\sigma_k^H\}_k$ are divergence free, we have

$$\mathcal{L} f = \frac{1}{2} \sum_k \nabla \cdot \left[(\sigma_k^H \cdot \nabla f) \sigma_k^H\right] = \frac{1}{2} \sum_k \nabla \cdot \left[(\sigma_k^H \otimes \sigma_k^H) \nabla f\right] = \frac{1}{2} \nabla \cdot \left[Q_H(0) \nabla f\right], \tag{3.4}$$

where the last step is due to horizontal part of (2.6).

We first give the meaning of solutions to the system (3.2).

Definition 3.1. *Let $(\omega_3^0, v_3^0) \in (L^2)^2$. A pair of stochastic processes (ω_3, v_3) defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $(\mathcal{F}_t)$-Brownian motions $\{W^k\}_k$, which are $(\mathcal{F}_t)$-predictable and have trajectories in*

$$L^\infty(0,T;L^2) \cap L^2(0,T;H^1),$$

is called a weak solution to (3.2) if for any $\phi \in H^1(\mathbb{T}^2)$, $\mathbb{P}$-a.s. for all $t \in [0,T]$, one has

$$\begin{aligned} \langle \omega_3(t), \phi\rangle &= \langle \omega_3^0, \phi\rangle + \int_0^t \langle \omega_3(s), v_H(s)\cdot\nabla\phi\rangle\, ds - \int_0^t \big\langle \omega_H(s), (\nabla Q_{H,3}(0))^*\nabla\phi\big\rangle\, ds \\ &\quad - \int_0^t \Big\langle \nabla\omega_3(s), \Big(\nu I_2 + \frac{1}{2}Q_H(0)\Big)\nabla\phi\Big\rangle\, ds \\ &\quad + \sum_k \int_0^t \big[\big\langle \omega_3(s), \sigma_k^H\cdot\nabla\phi\big\rangle - \big\langle \sigma_k^3, \omega_H(s)\cdot\nabla\phi\big\rangle\big]\, dW_s^k, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \langle v_3(t), \phi\rangle &= \langle v_3^0, \phi\rangle + \int_0^t \langle v_3(s), v_H(s)\cdot\nabla\phi\rangle\, ds + \sum_k \int_0^t \big\langle v_3(s), \sigma_k^H\cdot\nabla\phi\big\rangle\, dW_s^k \\ &\quad - \int_0^t \Big\langle \nabla v_3(s), \Big(\nu I_2 + \frac{1}{2}Q_H(0)\Big)\nabla\phi\Big\rangle\, ds. \end{aligned} \tag{3.6}$$

We remark that if $\omega_3, v_3 \in L^2\big(\Omega, L^\infty(0,T;L^2)\cap L^2(0,T;H^1)\big)$, then all the terms in (3.5) and (3.6) are well defined. According to the above definition, any solution (ω_3, v_3) can be decomposed as

$$\omega_3(t) = \omega_3^0 + V_\omega(t) + M_\omega(t), \quad v_3(t) = v_3^0 + V_v(t) + M_v(t),$$

where V_ω, V_v take values in $W^{1,2}(0,T;H^{-1})$, while M_ω, M_v are L^2-valued continuous local martingales. For instance, one has

$$\begin{aligned} V_\omega(t) &= \int_0^t \big(-v_H\cdot\nabla\omega_3 + \nu\Delta\omega_3 + \mathcal{L}\omega_3 + \nabla\cdot[\nabla Q_{H,3}(0)\,\omega_H]\big)(s)\, ds, \\ M_\omega(t) &= -\sum_k \int_0^t \big(\sigma_k^H\cdot\nabla\omega_3 - \omega_H\cdot\nabla\sigma_k^3\big)(s)\, dW_s^k; \end{aligned}$$

similar (and simpler) expressions hold for $V_v(t)$ and $M_v(t)$. Using the regularity properties of ω_3 and $v_H = K*\omega_3$, it is easy to see that

$$\begin{aligned} \|v_H(s)\cdot\nabla\omega_3(s)\|_{H^{-1}} &= \|\nabla\cdot(v_H(s)\omega_3(s))\|_{H^{-1}} \lesssim \|v_H(s)\omega_3(s)\|_{L^2} \\ &\lesssim \|\omega_3(s)\|_{L^2}\|v_H(s)\|_{H^2} \lesssim \|\omega_3\|_{L^\infty L^2}\|\omega_3(s)\|_{H^1}, \end{aligned}$$

and thus $\mathbb{P}$-a.s. $v_H\cdot\nabla\omega_3 \in L^2(0,T;H^{-1})$; the other terms in the expression of V_ω clearly enjoy the same property, therefore $\partial_t V_\omega \in L^2(0,T;H^{-1})$. The regularity on the martingale part M_ω can also be verified by classical estimates, cf. [8, Lemma 3.2] for detailed computations. From these discussions we see that the conditions in [23, Theorem 2.13] are verified, thus the Itô formula therein is applicable which implies that $t \mapsto \|\omega_3(t)\|_{L^2}$ is continuous. We conclude from this and the continuity of $\langle\omega_3(t), \phi\rangle$ in t that ω_3 has continuous trajectories in L^2. The same result holds for v_3.

The purpose of this part is to show the well-posedness of the system (3.2). Let $\|\cdot\|_M$ be a norm on the space of 2×2 matrices.

Theorem 3.2. *Let Q be the smooth covariance matrix function as defined in Section 2.1. For any $(\omega_3^0, v_3^0) \in (L^2)^2$, the stochastic 2D-3C model* (3.2) *admits a probabilistically weak solution (ω_3, v_3) in the sense of Definition 3.1, satisfying the bounds*

$$\mathbb{P}\text{-a.s. for all } t \in [0,T], \quad \|v_3(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla v_3(s)\|_{L^2}^2\, ds \le \|v_3^0\|_{L^2}^2, \tag{3.7}$$

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|\omega_3(t)\|_{L^2}^2 + \nu \int_0^T \|\nabla \omega_3(s)\|_{L^2}^2\, ds \bigg] \le C_{\nu,Q}\big(\|v_3^0\|_{L^2}^2 + \|\omega_3^0\|_{L^2}^2\big), \tag{3.8}$$

where $C_{\nu,Q}$ is a constant depending on $\nu, \|\nabla Q_{H,3}(0)\|_M, \|\nabla^2 Q_3(0)\|_M$.

Moreover, if $\|\nabla^2 Q_3(0)\|_M \le 2\nu$, then pathwise uniqueness holds for (3.2) *and thus the system has a unique probabilistically strong solution.*

This result will be proved in Section 4. The existence part follows from the classical Galerkin approximations and compactness arguments, therefore, we only provide some a priori estimates needed for showing tightness of laws of approximating solutions. Concerning the proof of pathwise uniqueness, the discussions below Definition 3.1 allow us to apply the Itô formula in [23, Theorem 2.13], then the uniqueness follows from some relatively standard arguments.

3.2 Scaling limit for stochastic 2D-3C models

In this part, following the recent works [7, 15, 19], we want to take a suitable scaling limit of the noise W in (3.1), and show that the solutions converge to some limit which solves a deterministic 2D-3C model with an extra dissipation term, and possibly also a first order term responsible for the AKA effect. The latter has been discussed in physics literature for a long time (see e.g. [27]), but escaped from rigorous mathematical treatment so far.

To this purpose, we shall take a family of vortex noise, defined as in Section 2.1, where the velocity field now depends on some parameter $\ell \in (0,1)$:

$$\sigma^\ell(x) = \sigma_H^\ell(x) + \sigma_3^\ell(x)\, e_3, \quad x \in \mathbb{T}^2,$$

satisfying the symmetry property (2.1); accordingly, we have the corresponding covariance matrix function

$$Q^\ell(x) = \int_{\mathbb{T}^2} \sigma^\ell(x-y) \otimes \sigma^\ell(-y)\, dy = \begin{pmatrix} Q_H^\ell(x) & Q_{H,3}^\ell(x) \\ Q_{3,H}^\ell(x) & Q_3^\ell(x) \end{pmatrix}, \quad x \in \mathbb{T}^2,$$

where Q_H^ℓ, $Q_{H,3}^\ell$ and Q_3^ℓ admit similar expressions as in (2.4). We can define the covariance operator $\mathbb{Q}^\ell$ associated to Q^ℓ, acting on functions in $L^2(\mathbb{T}^2, \mathbb{R}^3)$ and with operator norm $\|\mathbb{Q}^\ell\|_{L^2\to L^2}$; in the same way, we have the operators $\mathbb{Q}_H^\ell$ and $\mathbb{Q}_3^\ell$ with associated norms. Moreover, we denote

$$\nabla Q_{H,3}^\ell(0) = \begin{pmatrix} \partial_1 Q_{1,3}^\ell(0) & \partial_2 Q_{1,3}^\ell(0) \\ \partial_1 Q_{2,3}^\ell(0) & \partial_2 Q_{2,3}^\ell(0) \end{pmatrix}.$$

Our basic assumptions on the covariance functions are as follows:

Hypothesis 3.3. *(a) the limit $\bar{Q} := \lim_{\ell\to 0} Q_H^\ell(0)$ exists and is a nonnegative definite 2×2 matrix;*

(b) it holds that $\lim_{\ell\to 0} \|\mathbb{Q}_H^\ell\|_{L^2\to L^2} \vee \|\mathbb{Q}_3^\ell\|_{L^2\to L^2} = 0$;

(c) the limit $A := \lim_{\ell\to 0} \nabla Q_{H,3}^\ell(0)$ exists (might be a zero matrix);

(d) it holds that $\sup_{\ell\in(0,1)} \|\nabla^2 Q_3^\ell(0)\|_M < +\infty$.

Let $W^\ell = \{W^\ell(t)\}_{t\geq 0}$ be a $\mathbb{Q}^\ell$-Brownian motion in $\mathcal{H}$; similarly to Section 2.2, we decompose W^ℓ as follows:

$$W^\ell(t,x) = W_H^\ell(t,x) + W_3^\ell(t,x)\, e_3,$$
$$W_H^\ell(t,x) = \sum_k \sigma_k^{\ell,H}(x) W_t^k, \quad W_3^\ell(t,x) = \sum_k \sigma_k^{\ell,3}(x) W_t^k,$$

where $\sigma_k^{\ell,H} = \big(\sigma_k^{\ell,1}, \sigma_k^{\ell,2}\big)$. Now we consider the stochastic 2D-3C model driven by W^ℓ:

$$\begin{cases} d\omega_3^\ell + \big(v_H^\ell\cdot\nabla\omega_3^\ell - \nu\Delta\omega_3^\ell\big)\,dt = -\sum_k \big(\sigma_k^{\ell,H}\cdot\nabla\omega_3^\ell - \omega_H^\ell\cdot\nabla\sigma_k^{\ell,3}\big)\circ dW_t^k, \\ dv_3^\ell + \big(v_H^\ell\cdot\nabla v_3^\ell - \nu\Delta v_3^\ell\big)\,dt = -\sum_k \sigma_k^{\ell,H}\cdot\nabla v_3^\ell\circ dW_t^k. \end{cases}$$

where $\omega_H^\ell = \nabla^\perp v_3^\ell$ and $\omega_3^\ell = -\nabla^\perp\cdot v_H^\ell$. The associated Itô formulation is

$$\begin{cases} d\omega_3^\ell + v_H^\ell\cdot\nabla\omega_3^\ell\,dt = -\sum_k \big(\sigma_k^{\ell,H}\cdot\nabla\omega_3^\ell - \omega_H^\ell\cdot\nabla\sigma_k^{\ell,3}\big)\,dW_t^k \\ \qquad\qquad + \big(\nu\Delta + \mathcal{L}^\ell\big)\omega_3^\ell\,dt + \nabla\cdot\big[\nabla Q_{H,3}^\ell(0)\,\omega_H^\ell\big]\,dt, \\ dv_3^\ell + v_H^\ell\cdot\nabla v_3^\ell\,dt = -\sum_k \sigma_k^{\ell,H}\cdot\nabla v_3^\ell\,dW_t^k + \big(\nu\Delta+\mathcal{L}^\ell\big)\omega_3^\ell\,dt, \end{cases} \tag{3.9}$$

where the operator $\mathcal{L}^\ell f = \frac{1}{2}\nabla\cdot\big[Q_H^\ell(0)\nabla f\big]$. Given an ℓ-independent initial data $(\omega_3(0), v_3(0))$, under Hypothesis 3.3, Theorem 3.2 asserts that the system has probabilistically weak solutions $\big(\omega_3^\ell, v_3^\ell\big)$, satisfying bounds as (3.7) and (3.8) which are uniform in $\ell\in(0,1)$. We remark that if we strengthen condition (d) in Hypothesis 3.3 to be $\sup_{\ell\in(0,1)}\|\nabla^2 Q_3^\ell(0)\|_M \leq 2\nu$, then the solutions $\big(\omega_3^\ell, v_3^\ell\big)$ are also strong in the probabilistic sense.

Our main result reads as follows:

Theorem 3.4. *Assume Hypothesis 3.3; for any $\ell\in(0,1)$, let $\big(\omega_3^\ell, v_3^\ell\big)$ be a probabilistic weak solution to* (3.9) *with the same initial data $(\omega_3(0), v_3(0))$. Then the family $\{\eta^\ell\}_{\ell\in(0,1)}$ of laws of $\big\{\big(\omega_3^\ell, v_3^\ell\big)\big\}_{\ell\in(0,1)}$ is tight on $L^2(0,T;L^2)$, and any weakly convergent subsequence of $\{\eta^\ell\}_{\ell\in(0,1)}$ converges to a limit probability measure which is supported on the weak solution (ω_3, v_3) of the deterministic 2D-3C model:*

$$\begin{cases} \partial_t\omega_3 + v_H\cdot\nabla\omega_3 = \big(\nu\Delta + \mathcal{L}_{\bar Q}\big)\omega_3 + \nabla\cdot(A\,\omega_H), \\ \partial_t v_3 + v_H\cdot\nabla v_3 = \big(\nu\Delta+\mathcal{L}_{\bar Q}\big)v_3, \end{cases} \tag{3.10}$$

where $\mathcal{L}_{\bar Q} f = \frac{1}{2}\nabla\cdot\big[\bar Q\,\nabla f\big]$ and, as usual, $\omega_3 = -\nabla^\perp\cdot v_H$ and $\omega_H = \nabla^\perp v_3$.

Moreover, if the matrices $\bar Q$ and A satisfy

$$x\cdot\bar Q x + y\cdot\bar Q y + 2x\cdot A y^\perp \geq 0 \quad \textit{for any } x,y\in\mathbb{R}^2, \tag{3.11}$$

then the system (3.10) *admits a unique weak solution (ω_3, v_3) in the class $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$; in this case, the whole family $\big\{\big(\omega_3^\ell, v_3^\ell\big)\big\}_{\ell\in(0,1)}$ converges in law, in the topology of $L^2(0,T;L^2)$, to (ω_3, v_3).*

This result will be proved in Section 5.1, where the key ingredient is to prove that the martingale parts vanish in a weak sense. The proof of uniqueness of (3.10) is standard and follows by applying the Lions-Magenes lemma. As $\bar{Q}$ is nonnegative definite, it is clear that (3.11) holds in the case $A = 0$; in the example given below Remark 3.5, we have $\bar{Q} = 2\kappa I_2$ and thus (3.11) also holds if q_0 defined in (3.14) belongs to the interval $[-1, 1]$.

Remark 3.5. *Let us interpret the above result. First, Boussinesq hypothesis is verified, but, as expected from the geometric structure of the problem, the additional dissipation acts in the horizontal direction only. Second, the presence of a random stretching term (absent in previous works) may give rise to a first order term, $\nabla \cdot (A\,\omega_H)$, which could increase exponentially the size of the vorticity, like in the dynamo problem of magnetic fields subject to turbulence [25].*

Next, we provide an example of vortex noise verifying the above conditions. We begin with heuristic discussions on how to choose the right scaling of noise. Recall (2.2) for typical choices of velocity fields of the vortex; we assume that the function $\theta \in C_c^\infty\big((-1/2, 1/2)^2\big)$ is a radially symmetric probability density function, then by Remark 2.3, the horizontal covariance matrix $Q_H(0)$ is diagonal. In this case, the second order differential operator $\mathcal{L}$ in Itô equations (3.2) takes the form

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}(Q_H(0))\Delta.$$

As in Remark 2.4, we define rescaled vortex: $\theta_\ell(x) = \ell^{-2}\theta(\ell^{-1}x)$, $x \in \mathbb{T}^2$, where $\ell \in (0,1)$ stands for the "size" of the vortex; then we define

$$\sigma_H^\ell = \Gamma_\ell\, K * \theta_\ell.$$

We have to determine the dependence of the (horizontal) vortex intensity Γ_ℓ on ℓ. By (2.5), if Γ_ℓ were a constant independent of ℓ, then $\mathrm{Tr}(Q_H(0)) \sim \|K * \theta_\ell\|_{L^2}^2$ explodes at the rate of $\log \ell^{-1}$. In order that the vortex has constant energy as $\ell \to 0$, we choose Γ_ℓ to be a function of ℓ in such a way that $Q_H^\ell(0)$ is a constant matrix. Thus, we fix some $\kappa > 0$ and define

$$\Gamma_\ell = 2\sqrt{\kappa}\, \|K * \theta_\ell\|_{L^2}^{-1} \sim (\log \ell^{-1})^{-1/2}. \tag{3.12}$$

It remains to introduce the rescaled vertical velocity σ_3^ℓ of the vortex:

$$\sigma_3^\ell = \gamma_\ell\, G * \chi_\ell, \tag{3.13}$$

where γ_ℓ vanishes as $\ell \to 0$, G is the Green function on $\mathbb{T}^2$ and $\chi \in C_c^\infty\big((-1/2, 1/2)^2\big)$ is radially symmetric; we remark that γ_ℓ may not be the same as Γ_ℓ.

Having σ_H^ℓ and σ_3^ℓ in mind, we can finally define as above the covariance matrices Q^ℓ, Q_H^ℓ etc., and the associated operators $\mathbb{Q}^\ell$, $\mathbb{Q}_H^\ell$. Then by Remark 2.3 and (3.12), $Q_H^\ell(0) = 2\kappa I_2$ is a constant matrix and thus (a) of Hypothesis 3.3 is verified; in particular,

$$\mathrm{Tr}(Q_H^\ell(0)) = 4\kappa.$$

Moreover, by Lemma 2.5, one has $\|\mathbb{Q}_H^\ell\|_{L^2\to L^2} \le \Gamma_\ell^2 \|K\|_{L^1}^2$ because θ is a probability density; (3.12) implies that $\|\mathbb{Q}_H^\ell\|_{L^2\to L^2}$ vanishes as $\ell \to 0$. In the same way, $\|\mathbb{Q}_3^\ell\|_{L^2\to L^2} \le \gamma_\ell^2 \|G\|_{L^1}^2 \|\chi\|_{L^1}^2$ vanishes as well in the limit $\ell \to 0$. Therefore, condition (b) in Hypothesis 3.3 holds too.

In order to verify conditions (c) and (d) in Hypothesis 3.3, we need some other assumptions.

Proposition 3.6. *Assume that $\theta, \chi \in C_c^\infty\big((-\frac{1}{2}, \frac{1}{2})^2\big)$ are radially symmetric probability density functions.*

(1) If $\gamma_\ell = o(\Gamma_\ell)$ as $\ell \to 0$, then the limit $A := \lim_{\ell\to 0} \nabla Q^\ell_{H,3}(0)$ is null, i.e. $A = 0$.

(2) Assume that

$$q_0 := \lim_{\ell\to 0} \frac{\gamma_\ell}{\Gamma_\ell} \neq 0, \tag{3.14}$$

then the matrix

$$A = 2\,\kappa\, q_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(3) If $\gamma_\ell = O(\Gamma_\ell)$ as $\ell \to 0$, then $\sup_{\ell\in(0,1)} \|\nabla^2 Q^\ell_3(0)\|_M < +\infty$.

Remark 3.7. *Assume condition (3.14). From the proof of Lemma 5.4 in Section 5.2, it is clear that we can relax the nonnegativity condition on χ, but assuming $a_0 := \int_{\mathbb{T}^2} \chi\, dx \neq 0$; of course, the constant in A would be different. In this case, one can show as in the proof of Lemma 5.4 that*

$$(\partial_i G_{\mathbb{R}^2}) * \chi(x) \sim a_0\, \partial_i G_{\mathbb{R}^2}(x) \quad \textit{for } |x| \gg 1,$$

where $G_{\mathbb{R}^2}$ is the Green function on $\mathbb{R}^2$. However, if $a_0 = 0$, then $(\partial_i G_{\mathbb{R}^2}) * \chi(x)$ decays too fast as $|x| \to \infty$; this would result in that A is the trivial null matrix.

4 A priori estimates and proof of Theorem 3.2

This section consists of three parts: in Section 4.1 we first provide the computations which lead the system (3.1) to its Itô formulation (3.2), then we derive in Section 4.2 the a priori estimates necessary for proving the existence of solutions, finally we give a sketched proof of Theorem 3.2 in Section 4.3, focusing on the pathwise uniqueness of (3.2).

4.1 Itô-Stratonovich corrector

Recall the stochastic 2D-3C model in Stratonovich form:

$$\begin{cases} d\omega_3 = (-v_H \cdot \nabla \omega_3 + \nu\Delta\omega_3)\, dt - \sum_k \big(\sigma_k^H \cdot \nabla\omega_3 - \omega_H \cdot \nabla \sigma_k^3\big) \circ dW_t^k, \\ dv_3 = (-v_H \cdot \nabla v_3 + \nu \Delta v_3)\, dt - \sum_k \sigma_k^H \cdot \nabla v_3 \circ dW_t^k. \end{cases} \tag{4.1}$$

As the equation for v_3 is driven by pure transport noise, it is well known that the corresponding Itô equation is

$$dv_3 = (-v_H \cdot \nabla v_3 + \nu\Delta v_3 + \mathcal{L} v_3)\, dt - \sum_k \sigma_k^H \cdot \nabla v_3\, dW_t^k,$$

where the second order differential operator $\mathcal{L}$ is defined in (3.4).

We turn to deriving the Itô equation for ω_3:

$$\begin{aligned} d\omega_3 + (v_H \cdot \nabla\omega_3 - \nu\Delta\omega_3)\, dt = &- \sum_k \big(\sigma_k^H \cdot \nabla \omega_3 - \omega_H \cdot \nabla\sigma_k^3\big)\, dW_t^k \\ &- \frac{1}{2} \sum_k d\big[\sigma_k^H \cdot \nabla\omega_3 - \omega_H \cdot \nabla \sigma_k^3, W^k\big]_t. \end{aligned} \tag{4.2}$$

Noting that $\omega_H = \nabla^\perp v_3$, it holds

$$
\begin{aligned}
d\big(\sigma_k^H \cdot \nabla\omega_3 - \omega_H \cdot \nabla\sigma_k^3\big) &= \sigma_k^H \cdot \nabla(d\omega_3) - \nabla^\perp(dv_3)\cdot\nabla\sigma_k^3 \\
&= V\,dt - \sum_l \sigma_k^H \cdot \nabla\big(\sigma_l^H \cdot \nabla\omega_3 - \omega_H \cdot \nabla\sigma_l^3\big) \circ dW_t^l \\
&\quad + \sum_l \big(\nabla^\perp(\sigma_l^H \cdot \nabla v_3)\cdot\nabla\sigma_k^3\big) \circ dW_t^l,
\end{aligned}
$$

where $V\,dt$ is the finite variation part with

$$
V = \sigma_k^H \cdot \nabla(-v_H \cdot \nabla\omega_3 + \nu\Delta\omega_3) - \nabla^\perp(-v_H\cdot\nabla v_3 + \nu\Delta v_3)\cdot\nabla\sigma_k^3,
$$

thus

$$
\begin{aligned}
&d\big[\sigma_k^H \cdot \nabla\omega_3 - \omega_H \cdot \nabla\sigma_k^3, W^k\big]_t \\
&= -\sigma_k^H \cdot \nabla\big(\sigma_k^H \cdot \nabla\omega_3 - \omega_H \cdot \nabla\sigma_k^3\big)\,dt + \big(\nabla^\perp(\sigma_k^H \cdot \nabla v_3)\cdot\nabla\sigma_k^3\big)\,dt.
\end{aligned}
$$

Therefore, the last term in (4.2) (modulo dt) is

$$
\begin{aligned}
&\frac{1}{2}\sum_k \sigma_k^H \cdot \nabla\big(\sigma_k^H \cdot \nabla\omega_3 - \omega_H \cdot \nabla\sigma_k^3\big) - \frac{1}{2}\sum_k \nabla^\perp(\sigma_k^H\cdot\nabla v_3)\cdot\nabla\sigma_k^3 \\
&= \frac{1}{2}\sum_k \sigma_k^H \cdot \nabla(\sigma_k^H\cdot\nabla\omega_3) - \frac{1}{2}\sum_k \sigma_k^H\cdot\nabla\big(\omega_H\cdot\nabla\sigma_k^3\big) \\
&\quad - \frac{1}{2}\sum_k \nabla^\perp(\sigma_k^H\cdot\nabla v_3)\cdot\nabla\sigma_k^3 \\
&=: \mathcal{L}\omega_3 - \frac{1}{2}(I+J).
\end{aligned}
\tag{4.3}
$$

It is sufficient to compute the last two sums, denoted by I and J respectively.

Lemma 4.1. *One has*

$$
I = J = \sum_k (\sigma_k^H \cdot \nabla\omega_H)\cdot\nabla\sigma_k^3 - \omega_H \cdot \begin{pmatrix} \sum_k (\partial_1\sigma_k^H)\cdot\nabla\sigma_k^3 \\ \sum_k (\partial_2\sigma_k^H)\cdot\nabla\sigma_k^3 \end{pmatrix}.
$$

Proof. We have

$$
\begin{aligned}
I &= \sum_k \sigma_k^H \cdot \nabla\big(\omega_H\cdot\nabla\sigma_k^3\big) \\
&= \sum_k (\sigma_k^H\cdot\nabla\omega_H)\cdot\nabla\sigma_k^3 + \omega_H \cdot \bigg(\sum_k \sigma_k^H\cdot\nabla(\nabla\sigma_k^3)\bigg).
\end{aligned}
$$

The second term can be slightly simplified: for $i \in \{1,2\}$,

$$
\begin{aligned}
\sum_k \sigma_k^H \cdot \nabla(\partial_i\sigma_k^3) &= \sum_k \sigma_k^H \cdot \partial_i(\nabla\sigma_k^3) \\
&= \sum_k \big[\partial_i(\sigma_k^H\cdot\nabla\sigma_k^3) - (\partial_i\sigma_k^H)\cdot\nabla\sigma_k^3\big] \\
&= \partial_i\bigg[\sum_k \sigma_k^H\cdot\nabla\sigma_k^3\bigg] - \sum_k(\partial_i\sigma_k^H)\cdot\nabla\sigma_k^3 \\
&= -\sum_k (\partial_i\sigma_k^H)\cdot\nabla\sigma_k^3,
\end{aligned}
$$

where we have used the fact that

$$\sum_k \sigma_k^H \cdot \nabla \sigma_k^3 = \sum_{j=1}^2 \sum_k \sigma_k^j\, \partial_j \sigma_k^3 = -\sum_{j=1}^2 \partial_j Q_{j,3}(0)$$

is a constant matrix, due to the space homogeneity of covariance function. From this we immediately get the expression for I.

It remains to compute

$$\begin{aligned} J &= \sum_k \nabla^\perp (\sigma_k^H \cdot \nabla v_3) \cdot \nabla \sigma_k^3 \\ &= \sum_k \nabla^\perp (\sigma_k^1 \partial_1 v_3 + \sigma_k^2 \partial_2 v_3) \cdot \nabla \sigma_k^3 \\ &= \sum_k \big[(\partial_1 v_3) \nabla^\perp \sigma_k^1 + (\partial_2 v_3) \nabla^\perp \sigma_k^2 \big] \cdot \nabla \sigma_k^3 \\ &\quad + \sum_k \big[\sigma_k^1 \nabla^\perp (\partial_1 v_3) + \sigma_k^2 \nabla^\perp (\partial_2 v_3) \big] \cdot \nabla \sigma_k^3 \,; \end{aligned}$$

note that $\nabla^\perp(\partial_i v_3) = \partial_i \nabla^\perp v_3 = \partial_i \omega_H$, thus,

$$\begin{aligned} J &= (\partial_1 v_3) \sum_k \nabla^\perp \sigma_k^1 \cdot \nabla \sigma_k^3 + (\partial_2 v_3) \sum_k \nabla^\perp \sigma_k^2 \cdot \nabla \sigma_k^3 \\ &\quad + \sum_k \big[\sigma_k^1 \partial_1 \omega_H + \sigma_k^2 \partial_2 \omega_H \big] \cdot \nabla \sigma_k^3 \\ &= \omega_H \cdot \begin{pmatrix} \sum_k \nabla^\perp \sigma_k^2 \cdot \nabla \sigma_k^3 \\ -\sum_k \nabla^\perp \sigma_k^1 \cdot \nabla \sigma_k^3 \end{pmatrix} + \sum_k (\sigma_k^H \cdot \nabla \omega_H) \cdot \nabla \sigma_k^3. \end{aligned}$$

We see that I and J contain a common term; in order to show that they coincide, it suffices to check that the remaining terms are the same. Indeed,

$$\partial_1 \sigma_k^H = \begin{pmatrix} \partial_1 \sigma_k^1 \\ \partial_1 \sigma_k^2 \end{pmatrix} = \begin{pmatrix} -\partial_2 \sigma_k^2 \\ \partial_1 \sigma_k^2 \end{pmatrix} = -\nabla^\perp \sigma_k^2.$$

In the same way, $\partial_2 \sigma_k^H = \nabla^\perp \sigma_k^1$; as a consequence, $I = J$. □

Now we show that the second addend in I and J vanishes.

Lemma 4.2. *It holds that*

$$I = J = \sum_k (\sigma_k^H \cdot \nabla \omega_H) \cdot \nabla \sigma_k^3. \tag{4.4}$$

Proof. First of all, we prove a simple fact: let $Q = (Q_{i,j})_{1 \le i,j \le 3}$ be the covariance matrix, then for any $i, j \in \{1,2,3\}$, $m, n \in \{1,2\}$, it holds

$$-\partial_m \partial_n Q_{i,j}(0) = \sum_k \partial_m \sigma_k^i(x)\, \partial_n \sigma_k^j(x) = \sum_k \partial_n \sigma_k^i(x)\, \partial_m \sigma_k^j(x). \tag{4.5}$$

Indeed, recalling that for $i, j \in \{1,2,3\}$,

$$Q_{i,j}(x-y) = \sum_k \sigma_k^i(x)\, \sigma_k^j(y), \quad x, y \in \mathbb{T}^2,$$

thus we have

$$\partial_{x_m}\partial_{y_n}[Q_{i,j}(x-y)] = \sum_k \partial_{x_m}\sigma_k^i(x)\,\partial_{y_n}\sigma_k^j(y);$$

that is,

$$-\partial_m\partial_n Q_{i,j}(x-y) = \sum_k \partial_m\sigma_k^i(x)\,\partial_n\sigma_k^j(y).$$

Letting $x=y$ gives the first identity. For the second one, we have

$$\partial_{x_n}\partial_{y_m}[Q_{i,j}(x-y)] = \sum_k \partial_{x_n}\sigma_k^i(x)\,\partial_{y_m}\sigma_k^j(y);$$

from here we get the equality in the same way as above.

Now we start proving equality (4.4); recall that

$$I = \sum_k (\sigma_k^H\cdot\nabla\omega_H)\cdot\nabla\sigma_k^3 + \omega_H\cdot\begin{pmatrix} -\sum_k(\partial_1\sigma_k^H)\cdot\nabla\sigma_k^3 \\ -\sum_k(\partial_2\sigma_k^H)\cdot\nabla\sigma_k^3\end{pmatrix},$$

we have

$$\begin{aligned}\sum_k(\partial_1\sigma_k^H)\cdot\nabla\sigma_k^3 &= \sum_k\big(\partial_1\sigma_k^1\,\partial_1\sigma_k^3 + \partial_1\sigma_k^2\,\partial_2\sigma_k^3\big)\\ &= \sum_k\big(-\partial_2\sigma_k^2\,\partial_1\sigma_k^3 + \partial_1\sigma_k^2\,\partial_2\sigma_k^3\big),\end{aligned}$$

where we have used $\partial_1\sigma_k^1+\partial_2\sigma_k^2=0$. By the above identity (4.5),

$$\sum_k(\partial_1\sigma_k^H)\cdot\nabla\sigma_k^3 = \partial_1\partial_2 Q_{2,3}(0) - \partial_1\partial_2 Q_{2,3}(0) = 0.$$

In the same way,

$$\begin{aligned}\sum_k(\partial_2\sigma_k^H)\cdot\nabla\sigma_k^3 &= \sum_k\big(\partial_2\sigma_k^1\,\partial_1\sigma_k^3 + \partial_2\sigma_k^2\,\partial_2\sigma_k^3\big)\\ &= \sum_k\big(\partial_2\sigma_k^1\,\partial_1\sigma_k^3 - \partial_1\sigma_k^1\,\partial_2\sigma_k^3\big)\\ &= -\partial_1\partial_2 Q_{1,3}(0) + \partial_1\partial_2 Q_{1,3}(0) = 0.\end{aligned}$$

This implies (4.4) and completes the proof of Lemma 4.2. □

Recall the constant matrix $\nabla Q_{H,3}(0)$ defined in (3.3); we can further simplify $I=J$ as follows.

Proposition 4.3. *Let* $Q_{H,3}(x) = \big(Q_{1,3}(x), Q_{2,3}(x)\big)^*$ $(x\in\mathbb{T}^2)$ *be a vector valued function. Then,*

$$I = J = -\nabla\cdot[\nabla Q_{H,3}(0)\,\omega_H].$$

Proof. Recalling equality (4.4), we have

$$\begin{aligned} I &= \sum_k\big[(\partial_1\sigma_k^3)\,\sigma_k^H\cdot\nabla\omega_1 + (\partial_2\sigma_k^3)\,\sigma_k^H\cdot\nabla\omega_2\big]\\ &= \sum_k\begin{pmatrix}\sigma_k^1\partial_1\sigma_k^3\\ \sigma_k^2\partial_1\sigma_k^3\end{pmatrix}\cdot\nabla\omega_1 + \sum_k\begin{pmatrix}\sigma_k^1\partial_2\sigma_k^3\\ \sigma_k^2\partial_2\sigma_k^3\end{pmatrix}\cdot\nabla\omega_2\\ &= -\begin{pmatrix}\partial_1 Q_{1,3}(0)\\ \partial_1 Q_{2,3}(0)\end{pmatrix}\cdot\nabla\omega_1 - \begin{pmatrix}\partial_2 Q_{1,3}(0)\\ \partial_2 Q_{2,3}(0)\end{pmatrix}\cdot\nabla\omega_2\\ &= -\partial_1 Q_{H,3}(0)\cdot\nabla\omega_1 - \partial_2 Q_{H,3}(0)\cdot\nabla\omega_2.\end{aligned}$$

Then,

$$I = -\nabla \cdot \big[\omega_1\, \partial_1 Q_{H,3}(0)\big] - \nabla \cdot \big[\omega_2\, \partial_2 Q_{H,3}(0)\big] = -\nabla \cdot [\nabla Q_{H,3}(0)\, \omega_H],$$

which finishes the proof. $\square$

Combining (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned} d\omega_3 + (v_H \cdot \nabla\omega_3 - \nu\Delta\omega_3)\, dt = &- \sum_k \big(\sigma_k^H \cdot \nabla\omega_3 - \omega_H \cdot \nabla\sigma_k^3\big)\, dW_t^k \\ &+ \big(\mathcal{L}\omega_3 + \nabla \cdot [\nabla Q_{H,3}(0)\, \omega_H]\big)\, dt, \end{aligned} \tag{4.6}$$

where $\nabla Q_{H,3}(0)$ and the operator $\mathcal{L}$ are defined, respectively, in (3.3) and (3.4).

4.2 A priori estimates

This section is devoted to proving a priori estimates for the stochastic 2D-3C model (4.1) (i.e. (3.1)). First, as v_3 satisfies a stochastic advection-diffusion equation with divergence free fields v_H and σ_k^H, one has $\mathbb{P}$-a.s.,

$$\|v_3(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla v_3(s)\|_{L^2}^2\, ds \le \|v_3^0\|_{L^2}^2 \quad \text{for all } t \ge 0. \tag{4.7}$$

Next, we turn to dealing with the estimate on ω_3; recall that $\|\cdot\|_M$ is a matrix norm.

Lemma 4.4. *We have*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|\omega_3(t)\|_{L^2}^2 + \nu \int_0^T \|\nabla\omega_3(s)\|_{L^2}^2\, ds\bigg] \le C_{\nu,Q}\big(\|v_3^0\|_{L^2}^2 + \|\omega_3^0\|_{L^2}^2\big), \tag{4.8}$$

where $C_{\nu,Q}$ is a constant depending on $\nu, \|\nabla Q_{H,3}(0)\|_M, \|\nabla^2 Q_3(0)\|_M$.

Proof. In the sequel we will write $\nabla^\perp v_3$ instead of ω_H, in order to make use of the equation for v_3. By the first equation in (4.1), we have

$$d\, \|\omega_3\|_{L^2}^2 + 2\nu\, \|\nabla\omega_3\|_{L^2}^2\, dt = -2 \sum_k \langle \nabla\omega_3 \cdot \nabla^\perp v_3, \sigma_k^3\rangle \circ dW_t^k.$$

Transforming in Itô differential yields

$$\begin{aligned} d\, \|\omega_3\|_{L^2}^2 + 2\nu\, \|\nabla\omega_3\|_{L^2}^2\, dt = &-2 \sum_k \langle \nabla\omega_3 \cdot \nabla^\perp v_3, \sigma_k^3\rangle\, dW_t^k \\ &- \sum_k d\big[\langle \nabla\omega_3 \cdot \nabla^\perp v_3, \sigma_k^3\rangle, W^k\big]_t. \end{aligned} \tag{4.9}$$

We have

$$d\langle \nabla\omega_3 \cdot \nabla^\perp v_3, \sigma_k^3\rangle = \big\langle \nabla(d\omega_3) \cdot \nabla^\perp v_3, \sigma_k^3\big\rangle + \big\langle \nabla\omega_3 \cdot \nabla^\perp (dv_3), \sigma_k^3\big\rangle,$$

and, using the equations in (4.1) for ω_3 and v_3,

$$\begin{aligned} d\langle \nabla\omega_3 \cdot \nabla^\perp v_3, \sigma_k^3\rangle = \tilde{V}\, dt &- \sum_l \big\langle \nabla(\sigma_l^H \cdot \nabla\omega_3) \cdot \nabla^\perp v_3, \sigma_k^3\big\rangle \circ dW_t^l \\ &+ \sum_l \big\langle \nabla(\nabla^\perp v_3 \cdot \nabla\sigma_l^3) \cdot \nabla^\perp v_3, \sigma_k^3\big\rangle \circ dW_t^l \\ &- \sum_l \big\langle \nabla\omega_3 \cdot \nabla^\perp(\sigma_l^H \cdot \nabla v_3), \sigma_k^3\big\rangle \circ dW_t^l, \end{aligned}$$

where $\tilde V\,dt$ is the finite variation part; as a result,

$$\begin{aligned}-d\big[\langle\nabla\omega_3\cdot\nabla^\perp v_3,\sigma_k^3\rangle,W^k\big]_t&=\big\langle\nabla(\sigma_k^H\cdot\nabla\omega_3)\cdot\nabla^\perp v_3,\sigma_k^3\big\rangle\,dt\\&\quad-\big\langle\nabla(\nabla^\perp v_3\cdot\nabla\sigma_k^3)\cdot\nabla^\perp v_3,\sigma_k^3\big\rangle\,dt\\&\quad+\big\langle\nabla\omega_3\cdot\nabla^\perp(\sigma_k^H\cdot\nabla v_3),\sigma_k^3\big\rangle\,dt.\end{aligned}$$

Integration by parts:

$$\begin{aligned}-d\big[\langle\nabla\omega_3\cdot\nabla^\perp v_3,\sigma_k^3\rangle,W^k\big]_t&=-\big\langle(\sigma_k^H\cdot\nabla\omega_3)\nabla^\perp v_3,\nabla\sigma_k^3\big\rangle\,dt\\&\quad+\big\langle(\nabla^\perp v_3\cdot\nabla\sigma_k^3)\nabla^\perp v_3,\nabla\sigma_k^3\big\rangle\,dt\\&\quad-\big\langle(\nabla\omega_3)(\sigma_k^H\cdot\nabla v_3),\nabla^\perp\sigma_k^3\big\rangle\,dt.\end{aligned}\tag{4.10}$$

Let us consider the first term on the right-hand side:

$$\begin{aligned}\big\langle(\sigma_k^H\cdot\nabla\omega_3)\nabla^\perp v_3,\nabla\sigma_k^3\big\rangle&=\int(\sigma_k^H\cdot\nabla\omega_3)(\nabla^\perp v_3\cdot\nabla\sigma_k^3)\,dx\\&=\int(\nabla\omega_3)^*\sigma_k^H(\nabla\sigma_k^3)^*\nabla^\perp v_3\,dx.\end{aligned}$$

We have

$$\sum_k\sigma_k^H(\nabla\sigma_k^3)^*=\sum_k\begin{pmatrix}\sigma_k^1\partial_1\sigma_k^3&\sigma_k^1\partial_2\sigma_k^3\\\sigma_k^2\partial_1\sigma_k^3&\sigma_k^2\partial_2\sigma_k^3\end{pmatrix}=-\begin{pmatrix}\partial_1Q_{1,3}(0)&\partial_2Q_{1,3}(0)\\\partial_1Q_{2,3}(0)&\partial_2Q_{2,3}(0)\end{pmatrix}=-\nabla Q_{H,3}(0);\tag{4.11}$$

as a result,

$$\begin{aligned}-\sum_k\big\langle(\sigma_k^H\cdot\nabla\omega_3)\nabla_H^\perp v_3,\nabla\sigma_k^3\big\rangle&=\int(\nabla\omega_3)^*\nabla Q_{H,3}(0)\nabla^\perp v_3\,dx\\&\le\|\nabla Q_{H,3}(0)\|_M\|\nabla\omega_3\|_{L^2}\|\nabla v_3\|_{L^2}\\&\le\frac{\nu}{2}\|\nabla\omega_3\|_{L^2}^2+\frac{\|\nabla Q_{H,3}(0)\|_M^2}{2\nu}\|\nabla v_3\|_{L^2}^2.\end{aligned}$$

The third term in (4.10) can be estimated in the same way, thus

$$\begin{aligned}&-\sum_k\big\langle(\sigma_k^H\cdot\nabla\omega_3)\nabla^\perp v_3,\nabla\sigma_k^3\big\rangle-\sum_k\big\langle(\nabla\omega_3)(\sigma_k^H\cdot\nabla v_3),\nabla^\perp\sigma_k^3\big\rangle\\&\le\nu\|\nabla\omega_3\|_{L^2}^2+\nu^{-1}\|\nabla Q_{H,3}(0)\|_M^2\|\nabla v_3\|_{L^2}^2.\end{aligned}\tag{4.12}$$

Next, for the second term in (4.10), we have

$$\sum_k\big\langle(\nabla^\perp v_3\cdot\nabla\sigma_k^3)\nabla^\perp v_3,\nabla\sigma_k^3\big\rangle=\sum_k\big\|\nabla^\perp v_3\cdot\nabla\sigma_k^3\big\|_{L^2}^2.\tag{4.13}$$

Recall that

$$Q_3(x-y)=Q_{3,3}(x-y)=\sum_k\sigma_k^3(x)\,\sigma_k^3(y),\quad x,y\in\mathbb{T}^2;$$

by the computations at the beginning of the proof of Lemma 4.2, it holds

$$-\partial_j\partial_iQ_3(0)=\sum_k\partial_i\sigma_k^3(x)\,\partial_j\sigma_k^3(x).$$

In matrix form, it reads as

$$-\nabla^2 Q_3(0) = \sum_k \nabla\sigma_k^3(x)\,(\nabla\sigma_k^3(x))^*. \tag{4.14}$$

Therefore,

$$\begin{aligned}\sum_k \|\nabla^\perp v_3\cdot\nabla\sigma_k^3\|_{L^2}^2 &= \sum_k \int \big(\nabla^\perp v_3\cdot\nabla\sigma_k^3\big)^2\,dx\\ &= \sum_k \int (\nabla^\perp v_3)^*\,(\nabla\sigma_k^3)\,(\nabla\sigma_k^3)^*(\nabla^\perp v_3)\,dx\\ &= -\int (\nabla^\perp v_3)^*\,\nabla^2 Q_3(0)\,(\nabla^\perp v_3)\,dx\\ &= -\big\langle \nabla^\perp v_3, \nabla^2 Q_3(0)\,(\nabla^\perp v_3)\big\rangle.\end{aligned}$$

As a result, by (4.13),

$$\begin{aligned}\sum_k \big\langle(\nabla^\perp v_3\cdot\nabla\sigma_k^3)\nabla^\perp v_3, \nabla\sigma_k^3\big\rangle &= -\big\langle \nabla^\perp v_3, \nabla^2 Q_3(0)\,(\nabla^\perp v_3)\big\rangle\\ &\le \|\nabla^2 Q_3(0)\|_M \|\nabla v_3\|_{L^2}^2.\end{aligned} \tag{4.15}$$

Combining the above estimate with (4.10) and (4.12), we arrive at

$$-\sum_k d\big[\langle\nabla\omega_3\cdot\nabla^\perp v_3, \sigma_k^3\rangle, W^k\big]_t \le \nu\|\nabla\omega_3\|_{L^2}^2\,dt + \tilde C_{\nu,Q}\|\nabla v_3\|_{L^2}^2\,dt,$$

where $\tilde C_{\nu,Q}$ is a constant defined as

$$\tilde C_{\nu,Q} = \nu^{-1}\|\nabla Q_{H,3}(0)\|_M^2 + \|\nabla^2 Q_3(0)\|_M.$$

Substituting this estimate into (4.9) and noticing that $\langle\nabla\omega_3\cdot\nabla^\perp v_3, \sigma_k^3\rangle = -\langle\omega_3, \nabla^\perp v_3\cdot\nabla\sigma_k^3\rangle$, we arrive at

$$d\|\omega_3\|_{L^2}^2 + \nu\|\nabla\omega_3\|_{L^2}^2\,dt \le 2\sum_k\langle\omega_3, \nabla^\perp v_3\cdot\nabla\sigma_k^3\rangle\,dW_t^k + \tilde C_{\nu,Q}\|\nabla v_3\|_{L^2}^2\,dt. \tag{4.16}$$

Let $M(t)$ be the martingale part and define the stopping times $\tau_n = \inf\{t\ge 0 : \|\omega_3(t)\|_{L^2}\ge n\}$, $n\ge 1$; then

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\|\omega_3(t\wedge\tau_n)\|_{L^2}^2\Big] &\le \|\omega_3^0\|_{L^2}^2 + \mathbb{E}\Big[\sup_{t\in[0,T]}|M(t\wedge\tau_n)|\Big] + \tilde C_{\nu,Q}\int_0^T \|\nabla v_3(t)\|_{L^2}^2\,dt\\ &\le \|\omega_3^0\|_{L^2}^2 + \mathbb{E}\Big[\sup_{t\in[0,T]}|M(t\wedge\tau_n)|\Big] + \tilde C_{\nu,Q}\,\nu^{-1}\|v_3^0\|_{L^2}^2,\end{aligned}$$

where the second step is due to (4.7). We have

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|M(t\wedge\tau_n)|\Big] &\lesssim \mathbb{E}\bigg[\Big(\sum_k\int_0^{T\wedge\tau_n}\big\langle\omega_3(t), \nabla^\perp v_3(t)\cdot\nabla\sigma_k^3\big\rangle^2\,dt\Big)^{1/2}\bigg]\\ &\le \mathbb{E}\bigg[\Big(\int_0^{T\wedge\tau_n}\|\omega_3(t)\|_{L^2}^2\sum_k\big\|\nabla^\perp v_3(t)\cdot\nabla\sigma_k^3\big\|_{L^2}^2\,dt\Big)^{1/2}\bigg]\\ &\le C_Q\,\mathbb{E}\bigg[\Big(\int_0^{T\wedge\tau_n}\|\omega_3(t)\|_{L^2}^2\|\nabla v_3(t)\|_{L^2}^2\,dt\Big)^{1/2}\bigg],\end{aligned}$$

where the last step follows from (4.15) and $C_Q = \|\nabla^2 Q_3(0)\|_M^{1/2}$. Therefore, by (4.7) and Cauchy's inequality,

$$\begin{aligned}\mathbb{E}\bigg[\sup_{t\in[0,T]} |M(t\wedge\tau_n)|\bigg] &\le C_Q\,\mathbb{E}\bigg[\sup_{t\in[0,T]} \|\omega_3(t\wedge\tau_n)\|_{L^2}\Big(\int_0^T \|\nabla v_3(t)\|_{L^2}^2\,dt\Big)^{1/2}\bigg] \\ &\le C_Q\,\mathbb{E}\bigg[\nu^{-1/2}\|v_3^0\|_{L^2}\sup_{t\in[0,T]} \|\omega_3(t\wedge\tau_n)\|_{L^2}\bigg] \\ &\le \frac12\,\mathbb{E}\bigg[\sup_{t\in[0,T]} \|\omega_3(t\wedge\tau_n)\|_{L^2}^2\bigg] + \frac{C_Q^2}{2\nu}\|v_3^0\|_{L^2}^2.\end{aligned}$$

Combining the above estimates, we obtain

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|\omega_3(t\wedge\tau_n)\|_{L^2}^2\bigg] \le \hat C_{\nu,Q}\big(\|v_3^0\|_{L^2}^2 + \|\omega_3^0\|_{L^2}^2\big).$$

By Fatou's lemma, letting $n\to\infty$ yields the first estimate. The estimate on $\mathbb{E}\int_0^T \|\nabla\omega_3(t)\|_{L^2}^2 dt$ follows easily from (4.16). □

In order to apply the compactness method for proving existence of weak solutions, we need also to establish the following estimates.

Lemma 4.5. *Fix some $\alpha\in(0,1/2)$. It holds that*

$$\mathbb{E}\bigg[\int_0^T\!\!\int_0^T \frac{\|v_3(t)-v_3(s)\|_{H^{-2}}^2}{|t-s|^{1+2\alpha}}\,dtds + \int_0^T\!\!\int_0^T \frac{\|\omega_3(t)-\omega_3(s)\|_{H^{-2}}^2}{|t-s|^{1+2\alpha}}\,dtds\bigg] \le \tilde C < +\infty.$$

Proof. Using the equation for v_3, it is not difficult to show that

$$\mathbb{E}\big(\|v_3(t)-v_3(s)\|_{H^{-2}}^2\big) \le C_{\nu,Q,T}(t-s)\big(1+\|\omega_3^0\|_{L^2}^4 + \|v_3^0\|_{L^2}^4\big). \tag{4.17}$$

We omit the proof here since they are easier than that of the estimate on ω_3 given below.

By the equation (4.6) for ω_3, we have

$$\begin{aligned}\omega_3(t)-\omega_3(s) = &-\int_s^t (v_H\cdot\nabla\omega_3)(r)\,dr + \int_s^t \big[\nu\Delta\omega_3 + \mathcal{L}\omega_3 + \nabla\cdot(\nabla Q_{H,3}(0)\,\omega_H)\big](r)\,dr \\ &- \sum_k \int_s^t \big(\sigma_k^H\cdot\nabla\omega_3 - \omega_H\cdot\nabla\sigma_k^3\big)(r)\,dW_r^k;\end{aligned}$$

therefore,

$$\|\omega_3(t)-\omega_3(s)\|_{H^{-2}}^2 \lesssim I_1+I_2+I_3,$$

where

$$\begin{aligned} I_1 &\le (t-s)\int_s^t \|(v_H\cdot\nabla\omega_3)(r)\|_{H^{-2}}^2\,dr, \\ I_2 &\le (t-s)\int_s^t \big\|\big[\nu\Delta\omega_3 + \mathcal{L}\omega_3 + \nabla\cdot(\nabla Q_{H,3}(0)\,\omega_H)\big](r)\big\|_{H^{-2}}^2\,dr, \\ I_3 &\le \bigg\|\sum_k \int_s^t \big(\sigma_k^H\cdot\nabla\omega_3 - \omega_H\cdot\nabla\sigma_k^3\big)(r)\,dW_r^k\bigg\|_{H^{-2}}^2.\end{aligned}$$

First, note that v_H is a divergence free vector field on $\mathbb{T}^2$, it holds

$$\|(v_H\cdot\nabla\omega_3)(r)\|_{H^{-2}} \lesssim \|v_H(r)\,\omega_3(r)\|_{H^{-1}} \lesssim \|v_H(r)\,\omega_3(r)\|_{L^{3/2}} \le \|v_H(r)\|_{L^6}\|\omega_3(r)\|_{L^2},$$

where in the last two steps we have used the embedding $L^{3/2} \hookrightarrow H^{-1}$ and Hölder's inequality. Using the Sobolev embedding $H^1 \hookrightarrow L^6$, we arrive at

$$\|(v_H \cdot \nabla \omega_3)(r)\|_{H^{-2}} \lesssim \|v_H(r)\|_{H^1} \|\omega_3(r)\|_{L^2} \lesssim \|\omega_3(r)\|_{L^2}^2$$

since $v_H(r) = K * \omega_3(r)$ where K is the Biot-Savart kernel on $\mathbb{T}^2$. Recalling the definition of I_1, we arrive at

$$\mathbb{E} I_1 \lesssim (t-s) \int_s^t \mathbb{E}\big(\|\omega_3(r)\|_{L^2}^4\big)\, dr \lesssim_{\nu,Q,T} (t-s)^2 \big(\|\omega_3^0\|_{L^2}^4 + \|v_3^0\|_{L^2}^4\big),$$

where in the second step we have used Lemma 4.6 at the end of this section.

Next, for I_2, recalling that $\mathcal{L}\omega_3 = \frac{1}{2}\nabla \cdot \big[Q_H(0)\nabla\omega_3\big]$ (see (3.4)) and $\omega_H = \nabla^\perp v_3$, we have

$$\begin{aligned} I_2 &\lesssim (t-s) \int_s^t \Big[\nu^2 \|\omega_3(r)\|_{L^2}^2 + \|Q_H(0)\nabla\omega_3(r)\|_{H^{-1}}^2 + \big\|\nabla Q_{H,3}(0)\nabla^\perp v_3(r)\big\|_{H^{-1}}^2\Big]\, dr \\ &\lesssim_{\nu,Q} (t-s) \int_s^t \big[\|\omega_3(r)\|_{L^2}^2 + \|v_3(r)\|_{L^2}^2\big]\, dr; \end{aligned}$$

therefore, by Lemma 4.4 and (4.7), it holds

$$\mathbb{E} I_2 \lesssim_{\nu,Q,T} (t-s)^2 \big(\|\omega_3^0\|_{L^2}^2 + \|v_3^0\|_{L^2}^2\big).$$

Finally, we have

$$\begin{aligned} \mathbb{E} I_3 &\lesssim \mathbb{E}\Big[\sum_k \int_s^t \big\|\sigma_k^H \cdot \nabla\omega_3(r) - \nabla^\perp v_3(r) \cdot \nabla\sigma_k^3\big\|_{H^{-2}}^2\, dr\Big] \\ &\lesssim \mathbb{E}\Big[\sum_k \int_s^t \Big(\big\|\sigma_k^H \cdot \nabla\omega_3(r)\big\|_{H^{-2}}^2 + \big\|\nabla^\perp v_3(r) \cdot \nabla\sigma_k^3\big\|_{H^{-2}}^2\Big)\, dr\Big]. \end{aligned}$$

As $\{\sigma_k^H\}_k$ are divergence free, it holds

$$\sum_k \big\|\sigma_k^H \cdot \nabla\omega_3(r)\big\|_{H^{-2}}^2 \lesssim \sum_k \big\|\sigma_k^H \omega_3(r)\big\|_{H^{-1}}^2 \lesssim \sum_k \big\|\sigma_k^H \omega_3(r)\big\|_{L^2}^2 \lesssim \mathrm{Tr}(Q_H(0)) \|\omega_3(r)\|_{L^2}^2;$$

moreover,

$$\begin{aligned} \sum_k \big\|\nabla^\perp v_3(r) \cdot \nabla\sigma_k^3\big\|_{H^{-2}}^2 = \sum_k \big\|\nabla^\perp \cdot \big(v_3(r)\nabla\sigma_k^3\big)\big\|_{H^{-2}}^2 &\lesssim \sum_k \big\|v_3(r)\nabla\sigma_k^3\big\|_{L^2}^2 \\ &\lesssim \|\nabla^2 Q_3(0)\|_M \|v_3(r)\|_{L^2}^2, \end{aligned}$$

where in the last step we have used similar derivations of (4.15). Substituting these estimates into the inequality above, we obtain

$$\mathbb{E} I_3 \lesssim_Q \mathbb{E} \int_s^t \big(\|\omega_3(r)\|_{L^2}^2 + \|v_3(r)\|_{L^2}^2\big)\, dr \lesssim_{\nu,Q,T} (t-s)\big(\|\omega_3^0\|_{L^2}^2 + \|v_3^0\|_{L^2}^2\big).$$

Summarizing the above estimates on I_1, I_2 and I_3, we arrive at

$$\mathbb{E}\big(\|\omega_3(t) - \omega_3(s)\|_{H^{-2}}^2\big) \le C_{\nu,Q,T}(t-s)\big(1 + \|\omega_3^0\|_{L^2}^4 + \|v_3^0\|_{L^2}^4\big).$$

Combining this estimate with (4.17), we immediately obtain the desired result. □

In the estimate of I_1 in the above proof, we have used the following result.

Lemma 4.6. *One has, for all $t \in [0,T]$,*

$$\mathbb{E}\big(\|\omega_3(t)\|_{L^2}^4\big) \le \hat{C}_{\nu,Q}\big(\|\omega_3^0\|_{L^2}^4 + \|v_3^0\|_{L^2}^4\big).$$

Proof. By (4.16) and Itô's formula, we have

$$\begin{aligned} d\|\omega_3\|_{L^2}^4 &= 2\|\omega_3\|_{L^2}^2\, d\|\omega_3\|_{L^2}^2 + d\big[\|\omega_3\|_{L^2}^2, \|\omega_3\|_{L^2}^2\big]_t \\ &\le 2\|\omega_3\|_{L^2}^2\big(-\nu\|\nabla\omega_3\|_{L^2}^2 + C_{\nu,Q}\|\nabla v_3\|_{L^2}^2\big)\,dt + 4\sum_k \langle \omega_3, \nabla^\perp v_3\cdot\nabla\sigma_k^3\rangle^2\,dt \\ &\quad + 4\|\omega_3\|_{L^2}^2\sum_k \langle \omega_3, \nabla^\perp v_3\cdot\nabla\sigma_k^3\rangle\,dW_t^k, \end{aligned}$$

where $C_{\nu,Q} = \nu^{-1}\|\nabla Q_{H,3}(0)\|_M^2 + \|\nabla^2 Q_3(0)\|_M$. By Cauchy's inequality,

$$\sum_k \langle \omega_3, \nabla^\perp v_3\cdot\nabla\sigma_k^3\rangle^2 \le \|\omega_3\|_{L^2}^2 \sum_k \|\nabla^\perp v_3\cdot\nabla\sigma_k^3\|_{L^2}^2$$

where the sum on the right-hand side is the same as (4.13), and thus by (4.15),

$$\sum_k \langle \omega_3, \nabla^\perp v_3\cdot\nabla\sigma_k^3\rangle^2 \le \|\omega_3\|_{L^2}^2\|\nabla^2 Q_3(0)\|_M\|\nabla v_3\|_{L^2}^2.$$

Substituting this estimate into the inequality above gives us

$$d\|\omega_3\|_{L^2}^4 \le \tilde{C}_{\nu,Q}\|\omega_3\|_{L^2}^2\|\nabla v_3\|_{L^2}^2\,dt + 4\|\omega_3\|_{L^2}^2\sum_k \langle \omega_3, \nabla^\perp v_3\cdot\nabla\sigma_k^3\rangle\,dW_t^k. \tag{4.18}$$

Introduce the stopping time $\tau_n = \inf\{t\ge 0 : \|\omega_3(t)\|_{L^2}\ge n\}$; we have

$$\begin{aligned} &\mathbb{E}\int_0^{t\wedge\tau_n} \|\omega_3(s)\|_{L^2}^4 \sum_k \langle \omega_3(s), \nabla^\perp v_3(s)\cdot\nabla\sigma_k^3\rangle^2\,ds \\ &\le n^6\|\nabla^2 Q_3(0)\|_M\,\mathbb{E}\int_0^{t\wedge\tau_n}\|\nabla v_3(s)\|_{L^2}^2\,ds \\ &\le n^6\|\nabla^2 Q_3(0)\|_M\,\nu^{-1}\|v_3^0\|_{L^2}^2, \end{aligned}$$

where the last step follows from (4.7). This implies the last term in (4.18) is a locally square integrable martingale, thus we have

$$\begin{aligned} \mathbb{E}\big(\|\omega_3(t\wedge\tau_n)\|_{L^2}^4\big) &\le \|\omega_3^0\|_{L^2}^4 + \tilde{C}_{\nu,Q}\,\mathbb{E}\int_0^{t\wedge\tau_n}\|\omega_3(s)\|_{L^2}^2\|\nabla v_3(s)\|_{L^2}^2\,ds \\ &\le \|\omega_3^0\|_{L^2}^4 + \tilde{C}_{\nu,Q}\,\mathbb{E}\bigg[\sup_{s\in[0,T]}\|\omega_3(s)\|_{L^2}^2\int_0^t\|\nabla v_3(s)\|_{L^2}^2\,ds\bigg] \\ &\le \|\omega_3^0\|_{L^2}^4 + \tilde{C}_{\nu,Q}\,\nu^{-1}\|v_3^0\|_{L^2}^2\,\mathbb{E}\bigg[\sup_{s\in[0,T]}\|\omega_3(s)\|_{L^2}^2\bigg], \end{aligned}$$

which, combined with Lemma 4.4, gives us

$$\mathbb{E}\big(\|\omega_3(t\wedge\tau_n)\|_{L^2}^4\big) \le \hat{C}_{\nu,Q}\big(\|\omega_3^0\|_{L^2}^4 + \|v_3^0\|_{L^2}^4\big).$$

By Fatou's lemma, letting $n\to\infty$ we finish the proof. □

4.3 Proof of Theorem 3.2

This subsection is devoted to the proof of Theorem 3.2. To show the existence of weak solutions with desired regularity, we may adopt the classical methods of Galerkin approximation and compactness arguments. Thanks to the a priori estimates in Section 4.2, the proof is quite standard and can be found in many references, see for instance [10, 7] and also [8, Section 3]. Therefore, we omit the details of the proof of existence part, and concentrate on the pathwise uniqueness.

Proof of Theorem 3.2: pathwise uniqueness. Let $(\tilde\omega_3, \tilde v_3)$ and $(\bar\omega_3, \bar v_3)$ be two solutions to (3.2) defined on the same probability space Ω corresponding to the same initial data and Brownian motions $\{W^k\}_k$, such that $\tilde\omega_3$ and $\bar\omega_3$ (resp. $\tilde v_3$ and $\bar v_3$) fulfill the estimate (3.8) (resp. (3.7)). According to the discussions below Definition 3.1, we have the decompositions

$$\omega_3 := \tilde\omega_3 - \bar\omega_3 = V_{\tilde\omega_3} - V_{\bar\omega_3} + M_{\tilde\omega_3} - M_{\bar\omega_3}, \quad v_3 := \tilde v_3 - \bar v_3 = V_{\tilde v_3} - V_{\bar v_3} + M_{\tilde v_3} - M_{\bar v_3},$$

where $V_{\tilde\omega_3}, V_{\bar\omega_3}$ and $V_{\tilde v_3}, V_{\bar v_3}$ take values in $W^{1,2}(0,T;H^{-1})$, while $M_{\tilde\omega_3}, M_{\bar\omega_3}$ and $M_{\tilde v_3}, M_{\bar v_3}$ are L^2-valued continuous local martingales. One can check that the conditions of [23, Theorem 2.13] are verified for ω_3 and v_3, and thus the Itô formula therein is applicable.

More precisely, letting (K is the Biot-Savart kernel on $\mathbb{T}^2$)

$$v_H = \tilde v_H - \bar v_H = K * (\tilde\omega_3 - \bar\omega_3) = K * \omega_3,$$
$$\omega_H = \tilde\omega_H - \bar\omega_H = \nabla^\perp(\tilde v_3 - \bar v_3) = \nabla^\perp v_3,$$

then we have

$$\begin{aligned} d\omega_3 &= \big(- v_H\cdot\nabla\tilde\omega_3 - \bar v_H\cdot\nabla\omega_3 + \nu\Delta\omega_3 + \mathcal{L}\omega_3 + \nabla\cdot[\nabla Q_{H,3}(0)\,\omega_H]\big)\,dt \\ &\quad - \sum_k \big(\sigma_k^H\cdot\nabla\omega_3 - \omega_H\cdot\nabla\sigma_k^3\big)\,dW_t^k, \\ dv_3 &= \big(- v_H\cdot\nabla\tilde v_3 - \bar v_H\cdot\nabla v_3 + \nu\Delta v_3 + \mathcal{L}v_3\big)\,dt - \sum_k \sigma_k^H\cdot\nabla v_3\,dW_t^k. \end{aligned}$$

By the Itô formula in [23, Theorem 2.13],

$$\begin{aligned} d\|\omega_3\|_{L^2}^2 &= -2\langle\omega_3, v_H\cdot\nabla\tilde\omega_3\rangle\,dt - 2\nu\|\nabla\omega_3\|_{L^2}^2\,dt - \langle\nabla\omega_3, Q_H(0)\nabla\omega_3\rangle\,dt \\ &\quad - 2\langle\nabla\omega_3, \nabla Q_{H,3}(0)\,\omega_H\rangle\,dt + 2\sum_k\langle\omega_3, \omega_H\cdot\nabla\sigma_k^3\rangle\,dW_t^k \\ &\quad + \sum_k \big\|\sigma_k^H\cdot\nabla\omega_3 - \omega_H\cdot\nabla\sigma_k^3\big\|_{L^2}^2\,dt, \end{aligned}$$

where we have used $\langle\omega_3, \bar v_H\cdot\nabla\omega_3\rangle = 0 = \big\langle\omega_3, \sigma_k^H\cdot\nabla\omega_3\big\rangle$ since $\bar v_H$ and σ_k^H are divergence free. It is easy to know that

$$\begin{aligned} I &:= \sum_k \big\|\sigma_k^H\cdot\nabla\omega_3 - \omega_H\cdot\nabla\sigma_k^3\big\|_{L^2}^2 \\ &= \sum_k \Big(\big\|\sigma_k^H\cdot\nabla\omega_3\big\|_{L^2}^2 + \big\|\omega_H\cdot\nabla\sigma_k^3\big\|_{L^2}^2 - 2\big\langle\sigma_k^H\cdot\nabla\omega_3, \omega_H\cdot\nabla\sigma_k^3\big\rangle\Big) \\ &= \langle\nabla\omega_3, Q_H(0)\nabla\omega_3\rangle - \langle\omega_H, \nabla^2 Q_3(0)\,\omega_H\rangle + 2\langle\nabla\omega_3, \nabla Q_{H,3}(0)\,\omega_H\rangle, \end{aligned}$$

where in the last step we have used (4.11) and (4.14); therefore, the identity above reduces to

$$\begin{aligned} d\|\omega_3\|_{L^2}^2 &= -2\langle\omega_3, v_H\cdot\nabla\tilde\omega_3\rangle\,dt - 2\nu\|\nabla\omega_3\|_{L^2}^2\,dt \\ &\quad + 2\sum_k\langle\omega_3, \omega_H\cdot\nabla\sigma_k^3\rangle\,dW_t^k - \langle\omega_H, \nabla^2 Q_3(0)\,\omega_H\rangle\,dt. \end{aligned} \tag{4.19}$$

In the same way,

$$d\|v_3\|_{L^2}^2 = -2\langle v_3, v_H \cdot \nabla \tilde v_3\rangle\, dt - 2\nu \|\nabla v_3\|_{L^2}^2\, dt. \tag{4.20}$$

The estimate of the first term on the right-hand side of (4.19) is quite standard: recalling that $v_H = K * \omega_3$, we have

$$\begin{aligned} |\langle \omega_3, v_H \cdot \nabla \tilde\omega_3\rangle| &\le \|\omega_3\|_{L^2} \|v_H\|_{L^\infty} \|\nabla \tilde\omega_3\|_{L^2} \le \|\omega_3\|_{L^2} \|K * \omega_3\|_{H^2} \|\nabla \tilde\omega_3\|_{L^2} \\ &\le \|\omega_3\|_{L^2} \|\omega_3\|_{H^1} \|\nabla \tilde\omega_3\|_{L^2} \le \frac{\nu}{2} \|\nabla \omega_3\|_{L^2}^2 + \frac{1}{2\nu} \|\omega_3\|_{L^2}^2 \|\nabla \tilde\omega_3\|_{L^2}^2, \end{aligned} \tag{4.21}$$

as a result,

$$\begin{aligned} d\|\omega_3\|_{L^2}^2 + \nu \|\nabla \omega_3\|_{L^2}^2\, dt &\le \nu^{-1} \|\omega_3\|_{L^2}^2 \|\nabla \tilde\omega_3\|_{L^2}^2\, dt - \langle \omega_H, \nabla^2 Q_3(0)\, \omega_H\rangle\, dt \\ &\quad + 2\sum_k \langle \omega_3, \omega_H \cdot \nabla \sigma_k^3\rangle\, dW_t^k. \end{aligned}$$

Similar calculations yield

$$d\|v_3\|_{L^2}^2 + 2\nu \|\nabla v_3\|_{L^2}^2\, dt \le \nu \|\nabla \omega_3\|_{L^2}^2\, dt + \nu^{-1} \|v_3\|_{L^2}^2 \|\nabla \tilde v_3\|_{L^2}^2\, dt.$$

Taking the sum of the two estimates above, we arrive at

$$\begin{aligned} d\big(\|\omega_3\|_{L^2}^2 + \|v_3\|_{L^2}^2\big) &\le -\big(2\nu \|\nabla v_3\|_{L^2}^2 + \langle \omega_H, \nabla^2 Q_3(0)\, \omega_H\rangle\big)\, dt + 2\sum_k \langle \omega_3, \omega_H \cdot \nabla \sigma_k^3\rangle\, dW_t^k \\ &\quad + \nu^{-1}\big(\|\nabla \tilde\omega_3\|_{L^2}^2 + \|\nabla \tilde v_3\|_{L^2}^2\big)\big(\|\omega_3\|_{L^2}^2 + \|v_3\|_{L^2}^2\big)\, dt. \end{aligned}$$

Recall that we assume $\|\nabla^2 Q_3(0)\|_M \le 2\nu$; hence,

$$|\langle \omega_H, \nabla^2 Q_3(0)\, \omega_H\rangle| \le \|\omega_H\|_{L^2} \|\nabla^2 Q_3(0)\, \omega_H\|_{L^2} \le 2\nu \|\omega_H\|_{L^2}^2 = 2\nu \|\nabla v_3\|_{L^2}^2,$$

thus, denoting by M_t the martingale part, we obtain

$$d\big(\|\omega_3\|_{L^2}^2 + \|v_3\|_{L^2}^2\big) \le dM_t + \nu^{-1}\big(\|\nabla \tilde\omega_3\|_{L^2}^2 + \|\nabla \tilde v_3\|_{L^2}^2\big)\big(\|\omega_3\|_{L^2}^2 + \|v_3\|_{L^2}^2\big)\, dt. \tag{4.22}$$

We have, by Itô's isometry and Cauchy's inequality,

$$\begin{aligned} \mathbb{E}[M]_t &= 4\,\mathbb{E}\bigg[\sum_k \int_0^t \langle \omega_3(s), \omega_H(s) \cdot \nabla \sigma_k^3\rangle^2\, ds\bigg] \\ &\le 4\,\mathbb{E}\bigg[\sum_k \int_0^t \|\omega_3(s)\|_{L^2}^2 \big\|\omega_H(s) \cdot \nabla \sigma_k^3\big\|_{L^2}^2\, ds\bigg] \\ &\le 8\nu\,\mathbb{E}\bigg[\int_0^t \|\omega_3(s)\|_{L^2}^2 \|\omega_H(s)\|_{L^2}^2\, ds\bigg], \end{aligned}$$

where we have used (4.15) and the bound $\|\nabla^2 Q_3(0)\|_M \le 2\nu$. Note that $\omega_H = \nabla^\perp v_3 = \nabla^\perp \tilde v_3 - \nabla^\perp \bar v_3$, by (3.7), we have

$$\begin{aligned} \mathbb{E}[M]_t &\le 8\nu\,\mathbb{E}\bigg[\bigg(\sup_{s\in[0,T]} \|\omega_3(s)\|_{L^2}^2\bigg) \int_0^T \|\nabla^\perp v_3(s)\|_{L^2}^2\, ds\bigg] \\ &\le C\|v_3^0\|_{L^2}^2\, \mathbb{E}\bigg(\sup_{s\in[0,T]} \|\omega_3(s)\|_{L^2}^2\bigg) \\ &\le C_{\nu,Q}\big(\|v_3^0\|_{L^2}^2 + \|\omega_3^0\|_{L^2}^2\big)^2, \end{aligned}$$

where in the third step we have used (3.8). This implies that M_t is a square integrable martingale. Define a positive and decreasing process

$$\rho_t := \exp\bigg[-\nu^{-1}\int_0^t \big(\|\nabla\tilde\omega_3(s)\|_{L^2}^2 + \|\nabla\tilde v_3(s)\|_{L^2}^2\big)\, ds\bigg], \quad t\geq 0;$$

then $d\rho_t = -\nu^{-1}\rho_t\big(\|\nabla\tilde\omega_3(t)\|_{L^2}^2 + \|\nabla\tilde v_3(t)\|_{L^2}^2\big)\, dt$. By (4.22) and Itô's formula,

$$d\big[\rho_t\big(\|\omega_3\|_{L^2}^2 + \|v_3\|_{L^2}^2\big)\big] \leq \rho_t\, dM_t.$$

Integrating from 0 to t and taking expectation lead to

$$\mathbb{E}\big[\rho_t\big(\|\omega_3(t)\|_{L^2}^2 + \|v_3(t)\|_{L^2}^2\big)\big] \leq \mathbb{E}\int_0^t \rho_s\, dM_s = 0,$$

where in the last step we used the fact that $\big\{\int_0^t \rho_s\, dM_s\big\}_t$ is a martingale since $0<\rho_t\leq 1$ and $\{M_t\}_t$ is a square integrable martingale. We conclude that $\|\omega_3(t)\|_{L^2} = \|v_3(t)\|_{L^2} \equiv 0$ and the proof is finished. □

5 Scaling limit and proof of Theorem 3.4

In this part we first show that, under Hypothesis 3.3 in Section 3.2, the stochastic 2D-3C models (3.9) with the noise W^ℓ converge weakly to deterministic model; then we prove the technical results in Proposition 3.6.

5.1 Proof of Theorem 3.4

We first recall the setting: we are given covariance functions $\{Q^\ell\}_{\ell\in(0,1)}$ satisfying Hypothesis 3.3, and $\{\mathbb{Q}^\ell\}_{\ell\in(0,1)}$ are the corresponding covariance operators; let W^ℓ be a $\mathbb{Q}^\ell$-Wiener process with the expression

$$W^\ell(t,x) = \sum_k \sigma_k^{\ell,H}(x) W_t^k + \sum_k \sigma_k^{\ell,3}(x) W_t^k e_3,$$

where $\sigma_k^{\ell,H} = \big(\sigma_k^{\ell,1}, \sigma_k^{\ell,2}\big)$ are divergence free vector fields on $\mathbb{T}^2$, and $\{W^k\}_k$ are independent standard Brownian motions. For any $\ell\in(0,1)$, by Theorem 3.2, the following system

$$\begin{cases} d\omega_3^\ell + v_H^\ell\cdot\nabla\omega_3^\ell\, dt = -\sum_k \big(\sigma_k^{\ell,H}\cdot\nabla\omega_3^\ell - \omega_H^\ell\cdot\nabla\sigma_k^{\ell,3}\big)\, dW_t^k \\ \qquad\qquad\qquad\qquad + \big(\nu\Delta + \mathcal{L}^\ell\big)\omega_3^\ell\, dt + \nabla\cdot\big[\nabla Q_{H,3}^\ell(0)\,\omega_H^\ell\big]\, dt, \\ dv_3^\ell + v_H^\ell\cdot\nabla v_3^\ell\, dt = -\sum_k \sigma_k^{\ell,H}\cdot\nabla v_3^\ell\, dW_t^k + \big(\nu\Delta + \mathcal{L}^\ell\big)\omega_3^\ell\, dt \end{cases} \tag{5.1}$$

has a probabilistic weak solution $\big(\omega_3^\ell, v_3^\ell\big)$ satisfying the estimates:

$$\mathbb{P}\text{-a.s. for all } t\in[0,T], \quad \big\|v_3^\ell(t)\big\|_{L^2}^2 + 2\nu\int_0^t \big\|\nabla v_3^\ell(s)\big\|_{L^2}^2\, ds \leq \|v_3(0)\|_{L^2}^2, \tag{5.2}$$

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \big\|\omega_3^\ell(t)\big\|_{L^2}^2 + \nu\int_0^T \big\|\nabla\omega_3^\ell(s)\big\|_{L^2}^2\, ds\bigg] \leq C_{\nu,Q}\big(\|v_3(0)\|_{L^2}^2 + \|\omega_3(0)\|_{L^2}^2\big), \tag{5.3}$$

where $C_{\nu,Q}$ is a constant depending on ν, $\sup_{\ell\in(0,1)}\|\nabla Q_{H,3}^\ell(0)\|_M$ and $\sup_{\ell\in(0,1)}\|\nabla^2 Q_3^\ell(0)\|_M$, which are finite by Hypothesis 3.3. We point out that, since we are dealing with weak solutions, the probability and expectation should be written more accurately as $\mathbb{P}^\ell$ and $\mathbb{E}^\ell$, but we omit such dependence for simplicity of notation. Finally, using the equations in (5.1), we can also obtain similar estimates as in Lemma 4.5.

Let η^ℓ be the law of $\big(\omega_3^\ell, v_3^\ell\big)$, $\ell \in (0,1)$; then the above arguments show that $\{\eta^\ell\}_\ell$ is bounded in probability in $L^2(0,T;H^1)\cap W^{\alpha,2}(0,T;H^{-2})$, the latter being compactly embedded in $L^2(0,T;L^2)$. Therefore, the family of laws $\{\eta^\ell\}_\ell$ is tight on $L^2(0,T;L^2)$. Then, by the Prohorov theorem, we can find a subsequence (not relabelled for simplicity) of $\{\eta^\ell\}_\ell$ converging weakly, in the topology of $L^2(0,T;L^2)$, to some probability measure η. Moreover, by Skorohod's representation theorem, there exists a new probability space $\big(\hat\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}\big)$, a sequence of processes $\big(\hat\omega_3^\ell, \hat v_3^\ell\big)$ and a limit process $(\hat\omega_3, \hat v_3)$ defined on $\hat\Omega$, such that

- $(\hat\omega_3, \hat v_3) \overset{d}{\sim} \eta$ and $\big(\hat\omega_3^\ell, \hat v_3^\ell\big) \overset{d}{\sim} \eta^\ell$ for any $\ell \in (0,1)$;
- $\hat{\mathbb{P}}$-a.s., $\big(\hat\omega_3^\ell, \hat v_3^\ell\big)$ converges as $\ell \to 0$ to $(\hat\omega_3, \hat v_3)$ in the topology of $L^2(0,T;L^2)$.

We remark that the family $\big\{\big(\hat\omega_3^\ell, \hat v_3^\ell\big)\big\}_\ell$ fulfills the same estimates as (5.2) and (5.3) with respect to $\hat{\mathbb{P}}$ and $\hat{\mathbb{E}}$, respectively; thus, up to a further subsequence, $\big(\hat\omega_3^\ell, \hat v_3^\ell\big)$ converges weakly in $L^2\big(\Omega, L^\infty(0,T;L^2)\cap L^2(0,T;H^1)\big)$ to $(\hat\omega_3, \hat v_3)$; in particular, the limit processes $(\hat\omega_3, \hat v_3)$ enjoy the same bounds. Meanwhile, we can also obtain the existence of a family of Brownian motions $\big\{\big(\hat W^{\ell,k}\big)_{k\ge1}\big\}_{\ell\in(0,1)}$ on $\hat\Omega$, such that for any $\ell\in(0,1)$, $\big(\hat\omega_3^\ell, \hat v_3^\ell, \big(\hat W^{\ell,k}\big)_{k\ge1}\big)$ has the same law as $\big(\omega_3^\ell, v_3^\ell, (W^k)_{k\ge1}\big)$; we omit the details here.

For any $\phi \in C^\infty(\mathbb{T}^2)$, since $\big(\hat\omega_3^\ell, \hat v_3^\ell, \big(\hat W^{\ell,k}\big)_{k\ge1}\big)$ has the same law as $\big(\omega_3^\ell, v_3^\ell, (W^k)_{k\ge1}\big)$, we have

$$\begin{aligned}
\big\langle \hat\omega_3^\ell(t), \phi\big\rangle &= \langle \omega_3(0), \phi\rangle + \int_0^t \big\langle \hat\omega_3^\ell(s), \hat v_H^\ell(s)\cdot\nabla\phi\big\rangle\,ds - \int_0^t \big\langle \hat\omega_H^\ell(s), (\nabla Q_{H,3}^\ell(0))^*\nabla\phi\big\rangle\,ds \\
&\quad - \int_0^t \Big\langle \nabla\hat\omega_3^\ell(s), \Big(\nu I_2 + \frac12 Q_H^\ell(0)\Big)\nabla\phi\Big\rangle\,ds \\
&\quad + \sum_k \int_0^t \big[\big\langle \hat\omega_3^\ell(s), \sigma_k^{\ell,H}\cdot\nabla\phi\big\rangle - \big\langle \sigma_k^{\ell,3}, \hat\omega_H^\ell(s)\cdot\nabla\phi\big\rangle\big]\,d\hat W_s^{\ell,k},
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
\big\langle \hat v_3^\ell(t), \phi\big\rangle &= \langle v_3(0), \phi\rangle + \int_0^t \big\langle \hat v_3^\ell(s), \hat v_H^\ell(s)\cdot\nabla\phi\big\rangle\,ds + \sum_k \int_0^t \big\langle \hat v_3^\ell(s), \sigma_k^{\ell,H}\cdot\nabla\phi\big\rangle\,d\hat W_s^{\ell,k} \\
&\quad - \int_0^t \Big\langle \nabla\hat v_3^\ell(s), \Big(\nu I_2 + \frac12 Q_H^\ell(0)\Big)\nabla\phi\Big\rangle\,ds.
\end{aligned} \tag{5.5}$$

It remains to take the limit $\ell \to 0$ in the above equations (5.4) and (5.5); thanks to the discussions in the previous paragraphs, it is standard to show the convergence of all the terms, including the nonlinear ones, except those involving stochastic integrals, which are the key ingredients for proving that the pair of limit processes $(\hat\omega_3, \hat v_3)$ is a weak solution to the deterministic system (cf. (3.10))

$$\begin{cases}
\partial_t\hat\omega_3 + \hat v_H\cdot\nabla\hat\omega_3 = \big(\nu\Delta + \mathcal{L}_{\bar Q}\big)\hat\omega_3 + \nabla\cdot(A\,\hat\omega_H), \\
\partial_t\hat v_3 + \hat v_H\cdot\nabla\hat v_3 = \big(\nu\Delta + \mathcal{L}_{\bar Q}\big)\hat v_3,
\end{cases} \tag{5.6}$$

where $\hat\omega_3 = -\nabla^\perp\cdot\hat v_H$, $\hat\omega_H = \nabla^\perp\hat v_3$. Therefore, we concentrate on the martingale parts in (5.4) and (5.5), trying to show that they vanish in a certain sense.

Lemma 5.1. *Assume Hypothesis 3.3-(b); then as $\ell \to 0$, the martingale part*

$$M_1^\ell(t) := \sum_k \int_0^t \big\langle \hat v_3^\ell(s), \sigma_k^{\ell,H} \cdot \nabla\phi \big\rangle \, d\hat W_s^{\ell,k}$$

tends to 0 in the mean square sense.

Proof. By Itô's isometry,

$$\begin{aligned}
\mathbb{E}\big(M_1^\ell(t)^2\big) &= \mathbb{E}\bigg(\sum_k \int_0^t \big\langle \hat v_3^\ell(s), \sigma_k^{\ell,H} \cdot \nabla\phi \big\rangle^2 \, ds \bigg) \\
&= \mathbb{E}\bigg(\sum_k \int_0^t \int_{(\mathbb{T}^2)^2} \big[\big(\hat v_3^\ell(s)\nabla\phi\big)(x)\big]^* \sigma_k^{\ell,H}(x) \big[\sigma_k^{\ell,H}(y)\big]^* \big(\hat v_3^\ell(s)\nabla\phi\big)(y) \, dxdyds \bigg) \\
&= \mathbb{E}\bigg(\int_0^t \int_{(\mathbb{T}^2)^2} \big[\big(\hat v_3^\ell(s)\nabla\phi\big)(x)\big]^* Q_H^\ell(x-y) \big(\hat v_3^\ell(s)\nabla\phi\big)(y) \, dxdyds \bigg);
\end{aligned}$$

recalling the definition of the covariance operator $\mathbb{Q}_H^\ell$, we have

$$\begin{aligned}
\mathbb{E}\big(M_1^\ell(t)^2\big) &= \mathbb{E}\bigg(\int_0^t \big\langle \hat v_3^\ell(s)\nabla\phi, \mathbb{Q}_H^\ell\big(\hat v_3^\ell(s)\nabla\phi\big) \big\rangle \, ds \bigg) \\
&\le \mathbb{E}\bigg(\int_0^t \big\| \hat v_3^\ell(s)\nabla\phi \big\|_{L^2}^2 \big\| \mathbb{Q}_H^\ell \big\|_{L^2\to L^2} \, ds \bigg) \\
&\le \big\| \mathbb{Q}_H^\ell \big\|_{L^2\to L^2} \|\nabla\phi\|_{L^\infty}^2 T \|v_3(0)\|_{L^2}^2,
\end{aligned}$$

where in the last step we have used estimate (5.2). Thanks to condition (b) in Hypothesis 3.3, we conclude that the last quantity vanishes as $\ell \to 0$. □

Using the uniform bound (5.3), we can show in the same way that

$$M_2^\ell(t) := \sum_k \int_0^t \big\langle \hat\omega_3^\ell(s), \sigma_k^{\ell,H} \cdot \nabla\phi \big\rangle \, d\hat W_s^{\ell,k}$$

vanishes in the sense of mean square; indeed, the last step of Lemma 5.1 becomes

$$\begin{aligned}
\mathbb{E}\big(M_2^\ell(t)^2\big) &\le \big\| \mathbb{Q}_H^\ell \big\|_{L^2\to L^2} \|\nabla\phi\|_{L^\infty}^2 \, \mathbb{E} \int_0^t \big\| \hat\omega_3^\ell(s) \big\|_{L^2}^2 \, ds \\
&\le \big\| \mathbb{Q}_H^\ell \big\|_{L^2\to L^2} \|\nabla\phi\|_{L^\infty}^2 T C_{\nu,Q} \big(\|v_3(0)\|_{L^2}^2 + \|\omega_3(0)\|_{L^2}^2 \big)
\end{aligned}$$

which tends to 0 as $\ell \to 0$. Finally, we consider the other martingale in (5.4).

Lemma 5.2. *Assume condition (b) in Hypothesis 3.3. The martingale part*

$$M_3^\ell(t) := \sum_k \int_0^t \big\langle \sigma_k^{\ell,3}, \hat\omega_H^\ell(s) \cdot \nabla\phi \big\rangle \, d\hat W_s^{\ell,k}$$

vanishes as $\ell \to 0$.

Proof. Again by Itô's isometry, we have

$$\mathbb{E}\big(M_3^\ell(t)^2\big) = \mathbb{E} \int_0^t \sum_k \big\langle \sigma_k^{\ell,3}, \hat\omega_H^\ell(s) \cdot \nabla\phi \big\rangle^2 \, ds.$$

It holds that

$$\sum_k \big\langle \sigma_k^{\ell,3}, \hat\omega_H^\ell(s)\cdot\nabla\phi\big\rangle^2 = \int_{(\mathbb{T}^2)^2} Q_3^\ell(x-y)(\hat\omega_H^\ell(s)\cdot\nabla\phi)(x)(\hat\omega_H^\ell(s)\cdot\nabla\phi)(y)\,dxdy$$
$$= \big\langle \hat\omega_H^\ell(s)\cdot\nabla\phi, \mathbb{Q}_3^\ell\big(\hat\omega_H^\ell(s)\cdot\nabla\phi\big)\big\rangle$$
$$\le \big\|\mathbb{Q}_3^\ell\big\|_{L^2\to L^2}\big\|\hat\omega_H^\ell(s)\cdot\nabla\phi\big\|_{L^2}^2$$
$$\le \big\|\mathbb{Q}_3^\ell\big\|_{L^2\to L^2}\|\nabla\phi\|_{L^\infty}^2\|\nabla\hat v_3^\ell(s)\|_{L^2}^2,$$

where we have used $\hat\omega_H^\ell = \nabla^\perp \hat v_3^\ell$. Thanks to the estimate (4.7), we arrive at

$$\mathbb{E}\big(M_3^\ell(t)^2\big) \le \big\|\mathbb{Q}_3^\ell\big\|_{L^2\to L^2}\|\nabla\phi\|_{L^\infty}^2\,\mathbb{E}\int_0^t \|\nabla\hat v_3^\ell(s)\|_{L^2}^2\,ds \lesssim_\nu \big\|\mathbb{Q}_3^\ell\big\|_{L^2\to L^2}\|\nabla\phi\|_{L^\infty}^2\|v_3(0)\|_{L^2}^2,$$

the last quantity vanishes due to condition (b) in Hypothesis 3.3. □

Finally we present

Proof of Theorem 3.4. Summarizing the arguments starting from Lemma 5.1, we see that the martingale parts in (5.4) and (5.5) vanish in the sense of mean square. Moreover, the discussions above Lemma 5.1 give us the convergence of other terms in the equations (5.4) and (5.5). Therefore, we have proved the first assertion in Theorem 3.4.

We turn to showing the uniqueness of weak solutions to the system (3.10) in the class $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$; the proof is similar to the uniqueness part of Theorem 3.2, using now the Lions-Magenes lemma. Let $(\tilde\omega_3,\tilde v_3)$ and $(\bar\omega_3,\bar v_3)$ be two weak solutions to (3.10) in $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$, with the same initial data, define $\omega_3=\tilde\omega_3-\bar\omega_3$, $v_3=\tilde v_3-\bar v_3$; then we have

$$\begin{cases} \partial_t\omega_3 + v_H\cdot\nabla\tilde\omega_3 + \bar v_H\cdot\nabla\omega_3 = \big(\nu\Delta+\mathcal{L}_{\bar Q}\big)\omega_3 + \nabla\cdot(A\,\omega_H),\\ \partial_t v_3 + v_H\cdot\nabla\tilde v_3 + \bar v_H\cdot\nabla v_3 = \big(\nu\Delta+\mathcal{L}_{\bar Q}\big)v_3,\end{cases}$$

where $v_H=\tilde v_H-\bar v_H$ and $\omega_H=\tilde\omega_H-\bar\omega_H$. Similarly to the discussions below Definition 3.1, we have $\omega_3\in L^2(0,T;H^1)$ and $\partial_t\omega_3\in L^2(0,T;H^{-1})$; thus by the Lions-Magenes lemma,

$$\frac{d}{dt}\|\omega_3\|_{L^2}^2 = -2\langle\omega_3, v_H\cdot\nabla\tilde\omega_3\rangle - 2\nu\|\nabla\omega_3\|_{L^2}^2 - \langle\nabla\omega_3,\bar Q\nabla\omega_3\rangle - 2\langle\nabla\omega_3, A\,\omega_H\rangle$$
$$\le -\nu\|\nabla\omega_3\|_{L^2}^2 + \nu^{-1}\|\omega_3\|_{L^2}^2\|\nabla\tilde\omega_3\|_{L^2}^2 - \langle\nabla\omega_3,\bar Q\nabla\omega_3\rangle - 2\langle\nabla\omega_3, A\,\omega_H\rangle,$$

where in the second step we have used the estimate (4.21). Similarly,

$$\frac{d}{dt}\|v_3\|_{L^2}^2 \le \nu\|\nabla\omega_3\|_{L^2}^2 + \nu^{-1}\|v_3\|_{L^2}^2\|\nabla\tilde v_3\|_{L^2}^2 - 2\nu\|\nabla v_3\|_{L^2}^2 - \langle\nabla v_3,\bar Q\nabla v_3\rangle.$$

Noting that $\omega_H=\nabla^\perp v_3$; summing up the above two inequalities leads to

$$\frac{d}{dt}\big(\|\omega_3\|_{L^2}^2+\|v_3\|_{L^2}^2\big) \le \nu^{-1}\big(\|\nabla\tilde\omega_3\|_{L^2}^2+\|\nabla\tilde v_3\|_{L^2}^2\big)\big(\|\omega_3\|_{L^2}^2+\|v_3\|_{L^2}^2\big)$$
$$- \langle\nabla\omega_3,\bar Q\nabla\omega_3\rangle - \langle\nabla v_3,\bar Q\nabla v_3\rangle - 2\langle\nabla\omega_3, A\nabla^\perp v_3\rangle$$
$$\le \nu^{-1}\big(\|\nabla\tilde\omega_3\|_{L^2}^2+\|\nabla\tilde v_3\|_{L^2}^2\big)\big(\|\omega_3\|_{L^2}^2+\|v_3\|_{L^2}^2\big)$$

where the second step follows from condition (3.11). Since $\tilde\omega_3$ and $\tilde v_3$ belong to $L^2(0,T;H^1)$, we finish the proof by applying Gronwall's inequality. □

5.2 Proof of Proposition 3.6

We first make some preparations. Recall that G is the Green function on $\mathbb{T}^2$; the Green function on $\mathbb{R}^2$ admits the expression

$$G_{\mathbb{R}^2}(x) = -\frac{1}{2\pi}\log|x|;$$

there exists a smooth function $\zeta : \mathbb{T}^2 \to \mathbb{R}$ such that

$$G(x) = G_{\mathbb{R}^2}(x) + \zeta(x), \quad x \in \mathbb{T}^2. \tag{5.7}$$

Given a function $f \in C(\mathbb{T}^2)$ with compact support in $\big(-\frac{1}{2}, \frac{1}{2}\big)^2$, we will regard it also as a function on $\mathbb{R}^2$, still supported in $\big(-\frac{1}{2}, \frac{1}{2}\big)^2$; for any $\ell \in (0,1)$, we denote $f_\ell(x) = \ell^{-2} f(\ell^{-1}x)$, $x \in \mathbb{T}^2$. Let $\mathbb{T}^2_{\ell^{-1}} = \big[-\frac{1}{2}\ell^{-1}, \frac{1}{2}\ell^{-1}\big]^2$ be the torus of size ℓ^{-1}. We first prove the following formula.

Lemma 5.3. *Let $\phi, \psi \in C_c\big((-\frac{1}{2}, \frac{1}{2})^2\big)$. Then we have*

$$\begin{aligned}
\langle \partial_i G * \phi_\ell, \partial_j G * \psi_\ell \rangle = &\int_{\mathbb{T}^2_{\ell^{-1}}} (\partial_i G_{\mathbb{R}^2} * \phi)(x)\, (\partial_j G_{\mathbb{R}^2} * \psi)(x)\, dx \\
&+ \ell \int_{\mathbb{T}^2_{\ell^{-1}}} (\partial_i G_{\mathbb{R}^2} * \phi)(x) \bigg[\int_{\mathbb{T}^2} \partial_j \zeta(\ell(x-z))\, \psi(z)\, dz\bigg] dx \\
&+ \ell \int_{\mathbb{T}^2_{\ell^{-1}}} \bigg[\int_{\mathbb{T}^2} \partial_i \zeta(\ell(x-z))\, \phi(z)\, dz\bigg] (\partial_j G_{\mathbb{R}^2} * \psi)(x)\, dx \\
&+ \ell^2 \int_{\mathbb{T}^2_{\ell^{-1}}} \bigg[\int_{\mathbb{T}^2} \partial_i \zeta(\ell(x-z))\, \phi(z)\, dz\bigg] \bigg[\int_{\mathbb{T}^2} \partial_j \zeta(\ell(x-z))\, \psi(z)\, dz\bigg] dx.
\end{aligned}$$

Proof. We have

$$(\partial_i G * \phi_\ell)(x) = \int_{\mathbb{T}^2} \partial_i G(x-y)\, \phi_\ell(y)\, dy = \int_{\mathbb{T}^2_{\ell^{-1}}} \partial_i G(x-\ell y')\, \phi(y')\, dy',$$

where in the second step we have changed the variable $y = \ell y'$. As ϕ has compact support in $(-\frac{1}{2}, \frac{1}{2})^2$, we arrive at

$$(\partial_i G * \phi_\ell)(x) = \int_{\mathbb{T}^2} \partial_i G(x-\ell y)\, \phi(y)\, dy;$$

in the same way,

$$(\partial_j G * \psi_\ell)(x) = \int_{\mathbb{T}^2} \partial_j G(x-\ell z)\, \psi(z)\, dz.$$

Therefore,

$$\begin{aligned}
\langle \partial_i G * \phi_\ell, \partial_j G * \psi_\ell \rangle &= \int_{\mathbb{T}^2} \bigg[\int_{\mathbb{T}^2} \partial_i G(x-\ell y)\, \phi(y)\, dy\bigg] \bigg[\int_{\mathbb{T}^2} \partial_j G(x-\ell z)\, \psi(z)\, dz\bigg] dx \\
&= \ell^2 \int_{\mathbb{T}^2_{\ell^{-1}}} \bigg[\int_{\mathbb{T}^2} \partial_i G(\ell(x'-y))\, \phi(y)\, dy\bigg] \bigg[\int_{\mathbb{T}^2} \partial_j G(\ell(x'-z))\, \psi(z)\, dz\bigg] dx',
\end{aligned}$$

where we have changed variable $x = \ell x'$. Note that $\partial_i G_{\mathbb{R}^2}(\ell x) = \ell^{-1} \partial_i G_{\mathbb{R}^2}(x)$ for any $x \in \mathbb{R}^2 \setminus \{0\}$, therefore,

$$\int_{\mathbb{T}^2} \partial_i G_{\mathbb{R}^2}(\ell(x'-y))\, \phi(y)\, dy = \ell^{-1} (\partial_i G_{\mathbb{R}^2} * \phi)(x').$$

Combining this fact with (5.7), we obtain the desired expression. ☐

The next simple result shows that the convolution of $\nabla G_{\mathbb{R}^2}$ and a probability density, with compact support, is close to $\nabla G_{\mathbb{R}^2}$ in the region far from the origin. Let $B_R \subset \mathbb{R}^2$ be the open ball centered at the origin with radius $R > 0$.

Lemma 5.4. *Let $\psi \in C_c(B_R)$ be a probability density function. Then there exists a constant $C_R > 0$ such that*

$$\big|(\nabla G_{\mathbb{R}^2} * \psi)(x) - \nabla G_{\mathbb{R}^2}(x)\big| \le \frac{C_R}{|x|^2}, \quad \textit{for all } |x| \ge (2R) \vee 1.$$

Proof. Note that

$$\begin{aligned}(\nabla G_{\mathbb{R}^2} * \psi)(x) - \nabla G_{\mathbb{R}^2}(x) &= -\frac{1}{2\pi}\int_{B_R} \frac{x-y}{|x-y|^2}\psi(y)\,dy + \frac{1}{2\pi}\frac{x}{|x|^2} \\ &= -\frac{1}{2\pi}\int_{B_R}\left(\frac{x-y}{|x-y|^2} - \frac{x}{|x|^2}\right)\psi(y)\,dy,\end{aligned}$$

therefore,

$$\begin{aligned}\big|(\nabla G_{\mathbb{R}^2} * \psi)(x) - \nabla G_{\mathbb{R}^2}(x)\big| &\le \frac{1}{2\pi}\int_{B_R}\left|\frac{x-y}{|x-y|^2} - \frac{x}{|x|^2}\right|\psi(y)\,dy \\ &= \frac{1}{2\pi}\int_{B_R}\frac{\big||x|^2(x-y) - |x-y|^2 x\big|}{|x-y|^2|x|^2}\psi(y)\,dy.\end{aligned}$$

Since $|x|^2(x-y) - |x-y|^2 x = -|x|^2 y - |y|^2 x + 2(x\cdot y)x$, we have

$$\begin{aligned}\big|(\nabla G_{\mathbb{R}^2} * \psi)(x) - \nabla G_{\mathbb{R}^2}(x)\big| &\le \frac{1}{2\pi}\int_{B_R}\frac{|x|^2|y| + |y|^2|x| + 2|x\cdot y|\,|x|}{|x-y|^2|x|^2}\psi(y)\,dy \\ &\le \frac{1}{2\pi}\int_{B_R}\frac{|x|^2 R + R^2|x| + 2R|x|^2}{|x-y|^2|x|^2}\psi(y)\,dy \\ &\le \frac{R^2+3R}{2\pi}\int_{B_R}\frac{1}{|x-y|^2}\psi(y)\,dy\end{aligned}$$

for all $|x| \ge 1$. Now if $|x| \ge (2R) \vee 1$, we have

$$\begin{aligned}\big|(\nabla G_{\mathbb{R}^2} * \psi)(x) - \nabla G_{\mathbb{R}^2}(x)\big| &\le \frac{R^2+3R}{2\pi|x|^2}\int_{B_R}\frac{|x|^2}{|x-y|^2}\psi(y)\,dy \\ &\le \frac{2(R^2+3R)}{\pi|x|^2}\end{aligned}$$

since $\frac{|x|^2}{|x-y|^2} \le 4$ for $|y| \le R$ and $|x| \ge 2R$, and ψ is a probability density function. □

The next lemma gives the limit behavior of integrals of $\partial_i G$ outside a fixed ball.

Lemma 5.5. *For any $R > 0$, $i = 1, 2$, we have*

$$\lim_{\ell\to 0}\frac{1}{\log \ell^{-1}}\int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R}(\partial_i G_{\mathbb{R}^2}(x))^2\,dx = \frac{1}{4\pi}.$$

Proof. Since $\partial_i G_{\mathbb{R}^2}(x) = -\frac{1}{2\pi}\frac{x_i}{|x|^2}$, we have

$$\int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} (\partial_1 G_{\mathbb{R}^2}(x))^2\,dx = \frac{1}{4\pi^2}\int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} \frac{x_1^2}{|x|^4}\,dx,$$

by changing of variable $(x_1, x_2) \to (x_2, x_1)$, it is equal to

$$\frac{1}{4\pi^2}\int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} \frac{x_2^2}{|x|^4}\,dx = \int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} (\partial_2 G_{\mathbb{R}^2}(x))^2\,dx.$$

Therefore,

$$\int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} (\partial_1 G_{\mathbb{R}^2}(x))^2\,dx = \frac{1}{8\pi^2}\int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} \frac{dx}{|x|^2}. \tag{5.8}$$

We have, for any $\ell < R^{-1}$,

$$\int_{R\le|x|\le(2\ell)^{-1}} \frac{dx}{|x|^2} \le \int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} \frac{dx}{|x|^2} \le \int_{R\le|x|\le\ell^{-1}} \frac{dx}{|x|^2},$$

thus,

$$2\pi\log\frac{\ell^{-1}}{2R} \le \int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} \frac{dx}{|x|^2} \le 2\pi\log\frac{\ell^{-1}}{R}.$$

This implies

$$\lim_{\ell\to 0}\frac{1}{\log\ell^{-1}}\int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R} \frac{dx}{|x|^2} = 2\pi.$$

Combining this limit with (5.8), we finish the proof. ◻

We are ready to prove the following key result.

Proposition 5.6. *Let $\phi, \psi \in C_c\big((-\frac{1}{2}, \frac{1}{2})^2\big)$ be radially symmetric probability density functions. Then for $i, j \in \{1, 2\}$,*

$$\lim_{\ell\to 0}\frac{\langle \partial_i G * \phi_\ell, \partial_j G * \psi_\ell\rangle}{\log\ell^{-1}} = \frac{1}{4\pi}\delta_{i,j},$$

where $\delta_{i,j}$ is Kronecker's delta, i.e. it equals 1 if $i = j$ and 0 if $i \ne j$.

Proof. Recall Lemma 5.3 for the expression of $\langle \partial_i G * \phi_\ell, \partial_j G * \psi_\ell\rangle$; we denote the four terms by I^ℓ_n, $n = 1, 2, 3, 4$.

Step 1. We first show that

$$\lim_{\ell\to 0}\frac{|I^\ell_n|}{\log\ell^{-1}} = 0, \quad n = 2, 3, 4. \tag{5.9}$$

Indeed, for I^ℓ_2, since $\zeta \in C^\infty(\mathbb{T}^2)$, we have

$$\bigg|\int_{\mathbb{T}^2} \partial_j\zeta(\ell(x-z))\,\psi(z)\,dz\bigg| \le \|\partial_j\zeta\|_{L^\infty}\int_{\mathbb{T}^2}\psi(z)\,dz = \|\partial_j\zeta\|_{L^\infty},$$

therefore,

$$\begin{aligned}|I^\ell_2| &\le \ell\|\partial_j\zeta\|_{L^\infty}\int_{\mathbb{T}^2_{\ell^{-1}}} \big|(\partial_i G_{\mathbb{R}^2} * \phi)(x)\big|\,dx \\ &= \ell\|\partial_j\zeta\|_{L^\infty}\bigg(\int_{B_R} + \int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R}\bigg)\big|(\partial_i G_{\mathbb{R}^2} * \phi)(x)\big|\,dx\end{aligned}$$

for some fixed $R \in [1, (2\ell)^{-1})$. The first integral is bounded by some constant C_R; by Lemma 5.4, one has $\big|(\partial_i G_{\mathbb{R}^2} * \phi)(x)\big| \le C'_R/|x|$ for all $|x| \ge R$, and thus

$$\int_{\mathbb{T}^2_{\ell^{-1}} \setminus B_R} \big|(\partial_i G_{\mathbb{R}^2} * \phi)(x)\big|\, dx \le \int_{R \le |x| \le \ell^{-1}} \frac{C'_R}{|x|}\, dx = 2\pi C'_R(\ell^{-1} - R).$$

Summarizing these estimates we arrive at $|I_2^\ell| \le C_{1,R}\ell + C_{2,R}$, which implies that (5.9) holds for $n = 2$. The proofs for the other two limits are similar.

Step 2. Now we consider the term I_1^ℓ, and denote it with more precise notations

$$J^\ell_{i,j} = \int_{\mathbb{T}^2_{\ell^{-1}}} (\partial_i G_{\mathbb{R}^2} * \phi)(x)\, (\partial_j G_{\mathbb{R}^2} * \psi)(x)\, dx, \quad 1 \le i, j \le 2.$$

We begin with showing that the off-diagonal terms

$$J^\ell_{1,2} = J^\ell_{2,1} = \int_{\mathbb{T}^2_{\ell^{-1}}} (\partial_1 G_{\mathbb{R}^2} * \phi)(x)(\partial_2 G_{\mathbb{R}^2} * \psi)(x)\, dx$$

vanish for any $\ell \in (0, 1)$. Indeed, by symmetry of ϕ and ψ, one can show that

$$(\partial_1 G_{\mathbb{R}^2} * \phi)(-x_1, x_2) = -(\partial_1 G_{\mathbb{R}^2} * \phi)(x_1, x_2), \quad (\partial_1 G_{\mathbb{R}^2} * \phi)(x_1, -x_2) = (\partial_1 G_{\mathbb{R}^2} * \phi)(x_1, x_2),$$
$$(\partial_2 G_{\mathbb{R}^2} * \psi)(-x_1, x_2) = (\partial_2 G_{\mathbb{R}^2} * \psi)(x_1, x_2), \quad (\partial_2 G_{\mathbb{R}^2} * \psi)(x_1, -x_2) = -(\partial_2 G_{\mathbb{R}^2} * \psi)(x_1, x_2).$$

Due to cancellation of integrals in the four quadrants, we easily conclude that

$$J^\ell_{1,2} = J^\ell_{2,1} = 0. \tag{5.10}$$

Next, we show that $J^\ell_{1,1}$ has nontrivial limit; similar proof works for $J^\ell_{2,2}$. We have

$$J^\ell_{1,1} = \bigg(\int_{B_R} + \int_{\mathbb{T}^2_{\ell^{-1}} \setminus B_R}\bigg)(\partial_1 G_{\mathbb{R}^2} * \phi)(x)(\partial_1 G_{\mathbb{R}^2} * \psi)(x)\, dx,$$

where B_R is the ball centered at the origin with radius $R \ge 1$. It is clear that

$$\lim_{\ell \to 0} \frac{1}{\log \ell^{-1}} \int_{B_R} (\partial_1 G_{\mathbb{R}^2} * \phi)(x)(\partial_1 G_{\mathbb{R}^2} * \psi)(x)\, dx = 0,$$

thus it holds that

$$\lim_{\ell \to 0} \frac{J^\ell_{1,1}}{\log \ell^{-1}} = \lim_{\ell \to 0} \frac{1}{\log \ell^{-1}} \int_{\mathbb{T}^2_{\ell^{-1}} \setminus B_R} (\partial_1 G_{\mathbb{R}^2} * \phi)(x)(\partial_1 G_{\mathbb{R}^2} * \psi)(x)\, dx. \tag{5.11}$$

We have

$$\begin{aligned}
&\big|(\partial_1 G_{\mathbb{R}^2} * \phi)(x)(\partial_1 G_{\mathbb{R}^2} * \psi)(x) - (\partial_1 G_{\mathbb{R}^2}(x))^2\big| \\
&\quad \le |(\partial_1 G_{\mathbb{R}^2} * \psi)(x)|\, |(\partial_1 G_{\mathbb{R}^2} * \phi)(x) - \partial_1 G_{\mathbb{R}^2}(x)| \\
&\qquad + |\partial_1 G_{\mathbb{R}^2}(x)|\, |(\partial_1 G_{\mathbb{R}^2} * \psi)(x) - \partial_1 G_{\mathbb{R}^2}(x)|;
\end{aligned}$$

thanks to Lemma 5.4, for any $|x| \ge R \ge 1$,

$$\big|(\partial_1 G_{\mathbb{R}^2} * \phi)(x)(\partial_1 G_{\mathbb{R}^2} * \psi)(x) - (\partial_1 G_{\mathbb{R}^2}(x))^2\big| \le \frac{C_R}{|x|^3}.$$

Since

$$\limsup_{\ell \to 0} \frac{1}{\log \ell^{-1}} \int_{\mathbb{T}^2_{\ell^{-1}} \setminus B_R} \frac{C_R}{|x|^3}\, dx \le \limsup_{\ell \to 0} \frac{1}{\log \ell^{-1}} \int_{\mathbb{R}^2 \setminus B_R} \frac{C_R}{|x|^3}\, dx = 0,$$

we conclude from (5.11) that

$$\lim_{\ell\to 0}\frac{J_{1,1}^{\ell}}{\log \ell^{-1}} = \lim_{\ell\to 0}\frac{1}{\log \ell^{-1}}\int_{\mathbb{T}^2_{\ell^{-1}}\setminus B_R}(\partial_1 G_{\mathbb{R}^2}(x))^2\,dx = \frac{1}{4\pi},$$

where the last identity follows from Lemma 5.5. □

Recall the definition of Γ_ℓ in (3.12):

$$\Gamma_\ell = 2\sqrt{\kappa}\,\|K*\theta_\ell\|_{L^2}^{-1}.$$

Then we have

$$4\kappa = \Gamma_\ell^2\|K*\theta_\ell\|_{L^2}^2 = 2\Gamma_\ell^2\|\partial_1 G*\theta_\ell\|_{L^2}^2,$$

by Proposition 5.6, we deduce that

$$\lim_{\ell\to 0}\Gamma_\ell^2\log\ell^{-1} = 8\pi\kappa. \tag{5.12}$$

Now we are ready to provide

Proof of Proposition 3.6. First we derive the expression for $\nabla Q_{H,3}^{\ell}(0)$. By the definition of $Q_{H,3}^{\ell}$, we have

$$\begin{aligned}Q_{H,3}^{\ell}(a) &= \int_{\mathbb{T}^2}\sigma_H^{\ell}(x)\,\sigma_3^{\ell}(a-x)\,dx\\ &= \Gamma_\ell\gamma_\ell\int_{\mathbb{T}^2}(K*\theta_\ell)(x)\,(G*\chi_\ell)(a-x)\,dx.\end{aligned}$$

Then,

$$\nabla Q_{H,3}^{\ell}(a) = \Gamma_\ell\gamma_\ell\int_{\mathbb{T}^2}(K*\theta_\ell)(x)\otimes(\nabla G*\chi_\ell)(a-x)\,dx,$$

and thus,

$$\begin{aligned}\nabla Q_{H,3}^{\ell}(0) &= -\Gamma_\ell\gamma_\ell\int_{\mathbb{T}^2}(K*\theta_\ell)(x)\otimes(\nabla G*\chi_\ell)(x)\,dx\\ &= -\Gamma_\ell\gamma_\ell\begin{pmatrix}\langle\partial_2 G*\theta_\ell,\partial_1 G*\chi_\ell\rangle & \langle\partial_2 G*\theta_\ell,\partial_2 G*\chi_\ell\rangle\\ -\langle\partial_1 G*\theta_\ell,\partial_1 G*\chi_\ell\rangle & -\langle\partial_1 G*\theta_\ell,\partial_2 G*\chi_\ell\rangle\end{pmatrix}\\ &= -\Gamma_\ell\gamma_\ell\begin{pmatrix}0 & \langle\partial_2 G*\theta_\ell,\partial_2 G*\chi_\ell\rangle\\ -\langle\partial_1 G*\theta_\ell,\partial_1 G*\chi_\ell\rangle & 0\end{pmatrix},\end{aligned}$$

where the last step is due to similar proofs of (5.10).

Combining this formula with Proposition 5.6 and the limit (5.12), we immediately deduce that if $\gamma_\ell = o(\Gamma_\ell)$, then all the entries of the matrix $\nabla Q_{H,3}^{\ell}(0)$ have zero limit. Thus we obtain the first assertion of Proposition 3.6.

We turn to proving assertion (2). If $\gamma_\ell/\Gamma_\ell\to q_0$ as $\ell\to 0$, using again Proposition 5.6 and (5.12), we arrive at

$$\lim_{\ell\to 0}\Gamma_\ell\gamma_\ell\langle\partial_2 G*\theta_\ell,\partial_2 G*\chi_\ell\rangle = \lim_{\ell\to 0}\frac{\gamma_\ell}{\Gamma_\ell}\,(\Gamma_\ell^2\log\ell^{-1})\,\frac{\langle\partial_2 G*\theta_\ell,\partial_2 G*\chi_\ell\rangle}{\log\ell^{-1}} = 2\kappa q_0$$

which yields the desired result.

Finally we prove assertion (3) of Proposition 3.6. We have, for $i,j\in\{1,2\}$,

$$\partial_i\partial_j Q_3^{\ell}(0) = -\gamma_\ell^2\int_{\mathbb{T}^2}(\partial_i G*\chi_\ell)(x)\,(\partial_j G*\chi_\ell)(x)\,dx = -\gamma_\ell^2\,\langle\partial_i G*\chi_\ell,\partial_j G*\chi_\ell\rangle.$$

Noting that $\gamma_\ell = O(\Gamma_\ell)$ as $\ell\to 0$, it is clear from the above computations that all the terms are uniformly bounded in $\ell\in(0,1)$. □

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Data availability statements

This article has no additional data.

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