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Dynamics of strongly-coupled chiral gauge theories

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Abstract.

We study the dynamics of $SU(N)$ chiral gauge theories with massless fermions belonging to various combinations of the symmetric, antisymmetric or fundamental representations. We limit ourselves to the gauge-anomaly-free and asymptotically free systems. These theories have a global symmetry group with the associated 't Hooft anomaly-matching conditions severely limiting the possible RG flows. Recent developments on the applications of the generalized symmetries and the stronger requirement of the matching of the mixed anomalies also give further indications on the possible IR dynamics. In vectorlike theories such as the quantum chromodynamics (QCD), gauge-invariant "quark-antiquark" condensates form and characterize the IR dynamics, and the anomaly matching involves the Nambu-Goldstone (NG) bosons. In some other special cases, such as the Bars-Yankielowicz (BY) or Georgi-Glashow (GG) models, a hypothetical solution was proposed in the literature, with no global symmetry breaking and with some simple set of composite massless fermions saturating all the anomalies. For the BY and GG systems, actually, a more plausible candidate for their IR physics is the dynamical Higgs phase, with a few simple bi-fermion color-flavor locked condensates, breaking the color and flavor symmetries, partially or totally. Remarkably, the 't Hooft anomaly-matching (and generalized anomaly-matching) conditions are automatically satisfied in this phase. Another interesting possibility, occurring in some chiral gauge theories, is dynamical Abelianization, familiar from $\mathcal{N} = 2$ supersymmetric gauge theories. We explore here even more general types of possible IR phases than the ones mentioned above, for wider classes of models. With the help of large-N arguments we look for IR free theories, whereas the MAC (maximal attractive channel) criterion might suggest some simple bi-fermion condensates characterizing the IR dynamics of the systems. In many cases the low-energy effective theories are found to be described by quiver-like gauge theories, some of the (nonAbelian) gauge groups are infrared-free while some others might be asymptotically free.

1. Introduction

It is the purpose of this review work to discuss a wide class of strongly-coupled $SU(N)$ chiral gauge theories and their possible IR physics, based on the work [1–8] done in the last several years. In spite of many years of efforts [9–20], and in spite of important possible applications in the context of realistic model building beyond the standard model of the fundamental interactions, our understanding of the dynamics of *chiral* gauge theories is still quite unsatisfactory today. Such a consideration motivated us to study wide classes of $SU(N)$ gauge theories with massless matter fermions in various possible representations, systematically. The main tool of the analysis is the constraints following from the requirement that the global symmetries of the systems be correctly reflected as the system RG-flows towards IR, and as the interactions become strong. Of particular importance in this context is the consideration of the 't Hooft anomalies [21] (the obstructions in gauging the global symmetries by introducing external, arbitrarily weakly coupled gauge fields), and their consequences in the RG flow. These anomalies must be faithfully reproduced by the degrees of freedom in the low-energy effective theory, either by massless composite fermions for unbroken symmetries (the 't Hooft anomaly matching constraints) or by interactions involving the Nambu-Goldstone (NG) bosons if some of the global symmetry is spontaneously broken, or in some cases by both.

As is well known, the choice of the fermion representations is restricted by the gauge anomaly-free conditions. Another condition we impose is that the system is asymptotically free. In our studies below, we further restrict possible reducible representations for the fermions to those involving the fundamental representation, the rank-two symmetric or antisymmetric representations, and their conjugates, only. As we will see they cover already very rich variety of systems of physical interest.

An aspect of these constraints that various symmetries impose on the RG flow (physics in the IR), which is perhaps less familiar, is the low-energy manifestations of the classical $U(1)$ symmetry, made anomalous by the strong gauge interactions ("the strong anomaly"). The so-called $U(1)_A$ problem and its solution in quantum chromodynamics (QCD) - which is the consequence of such a strong anomaly - is well-known [22–27], but for some reason this particular aspect of the symmetry consideration has been applied in the context of chiral gauge theories only recently [6] (see however [28]).

What makes these problems highly nontrivial and interesting is the fact that the strong $SU(N)$ gauge interactions themselves can manifest in different phases at low energies, depending on the matter fermions present, such as confinement, Higgs, Coulomb, or still other phases. To understand the interplay between the types of massless matter fermions and these phases of the system under study, is one of the central themes of this work.

Recently the ideas of generalized, higher-form symmetries and their gauging have been applied to gain deeper insights into these problems [29–36]. One of the key tools, which leads to many interesting consequences, is the $\mathbb{Z}_N$ center symmetry of $SU(N)$ theories. Although the idea of the center symmetry itself is a familiar one,[1] it becomes more powerful when combined with the idea of "gauging" such a discrete center symmetry [29–36]. In some cases, this leads to mixed ([0-form]-[1-form]) 't Hooft anomalies; the matching of these new types of anomalies imposes stronger

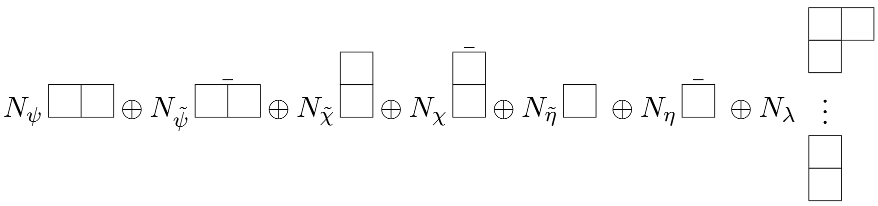
[1] A precursor of the ideas is indeed the center symmetry $\mathbb{Z}_N$ in Euclidean $SU(N)$ Yang-Mills theory at finite temperature, which acts on the Polyakov loop. The unbroken (or broken) center symmetry by the VEV of the Polyakov loop, is a criterion of confinement (or de-confinement) phase [37].

constraints on the possible infrared dynamics of the system, as compared with the conventional 't Hooft anomaly matching conditions.

This work is basically a review of our previous results [1–8], but a special emphasis will be put on an exploration of more general, new types of phases and low-energy effective actions, than those already discussed in literature or by ourselves. Keeping such an aim in mind, the discussions on the generalized symmetries and applications of the mixed anomalies will be left to another review work [7]. The present work is organized as follows. In Sec. 2 we discuss the classes of models to be analyzed below. In the following sections, Sec. 3, Sec. 4, and in Sec. 5, we review the three possible different types of phases which may be realized, depending on the matter content of our theories, i.e., a hypothetical confining flavor-symmetric vacuum, dynamical Higgs phase, and dynamical Abelianization, respectively. In Sec. 6 we explore other possible phases, which are generalizations of the dynamical Higgs/Abelianization cases discussed in Sec. 4, and in Sec. 5. The content of Sec. 6, which is the tentative study of more general IR dynamics, is mostly new. Sec. 7 reviews the implications of the strong anomaly to possible dynamical scenarios in the IR in some simple chiral gauge theories. Conclusive remarks are in Sec. 8.

2. Models

In this paper we shall focus on asymptotically-free $SU(N)$ gauge theories with chiral fermions. We restrict further our playground to the class of theories which possess a large N limit. The last requirement imposes that the matter content is restricted to a combination of a few irreducible representations (irreps) of $SU(N)$,[2]



ψ $\tilde\psi$ $\tilde\chi$ χ $\tilde\eta$ η λ , (2.1)

inclusion of larger irreps will render the theory IR-free at large N.

Not all the choices of N_ψ, $N_{\tilde\psi}$, $N_{\tilde\chi}$, N_χ, $N_{\tilde\eta}$, N_η, N_λ are permitted: gauge anomaly cancellation imposes that

$$(N_\psi - N_{\tilde\psi})(N+4) + (N_{\tilde\chi} - N_\chi)(N-4) + (N_{\tilde\eta} - N_\eta)N = 0 \,, \tag{2.2}$$

which reduces our parameter spaces from seven to six integers. Asymptotic freedom[3]

$$b_0 = 11N - (N_\psi + N_{\tilde\psi})(N+2) - (N_{\tilde\chi} + N_\chi)(N-2) - (N_{\tilde\eta} + N_\eta) - 2NN_\lambda > 0 \,. \tag{2.3}$$

This gives an inequality on the parameter space, thus it doesn't reduce the "dimensionality" of

[2] All fermions are taken to be left-handed.
[3] In general these theories become strongly coupled in IR, and develop a dynamically generated energy scale, Λ. Some of them might however flow to an interacting CFT, as can be suggested by the analysis which takes into account the higher coefficients of the beta function.

the parameter spaces, but renders the number of theories we can consider, for fixed N, finite.

Before giving a convenient parametrization of this six-dimensional set of theories, let us see that it contains many interesting theories.

Clearly, it contains all the vector-like theories that have a ('t Hooft) large-N limit. In particular, for $N_{\tilde{\eta}} = N_\eta$ and setting all rest Ns to zero one obtains ordinary QCD. Similarly, by allowing only $N_\psi = N_{\tilde{\psi}}$ ($N_{\tilde{\chi}} = N_\chi$) to be non-zero one obtains (S)QCD ((A)QCD), in which the "quarks" are in the second-rank tensor representations of $SU(N)$. Lastly, by allowing only $N_{adj.} \neq 0$, one obtains adjoint-QCD.

This family contains also many examples of chiral gauge theories. For example, taking $N_S = 1$, $N_\eta = (N+4)$ and setting all the others Ns to zero one obtains an $SU(N)$ gauge theory with left-handed fermions in the reducible, complex representation,

$$\square\!\square \oplus (N+4)\,\overline{\square}\ . \tag{2.4}$$

This is the simplest of the so-called Bars-Yankielowicz (BY) models [12]. We call it "$\psi\eta$" model, for short: its IR phase has been studied extensively, see [3–7].

Another simple example is $N_{\tilde{\chi}} = 1$, $N_\eta = (N-4)$, $N \geq 5$, i.e.

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus (N-4)\,\overline{\square}. \tag{2.5}$$

This is the simplest Georgi-Glashow (GG) model. We will refer to it as "$\tilde{\chi}\eta$" model.

A still another simple model can be constructed by taking $N_\psi = 1,\ N_\chi = 1,\ N_\eta = 8$, i.e.,

$$\square\!\square \oplus \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus 8\,\overline{\square}\ . \tag{2.6}$$

This theory that we called $\psi\chi\eta$ model, has been studied earlier in [11, 13, 15] and more recently, in [1, 2, 8, 38].

As introduced in [2] we consider the (N_ψ, N_χ) models. They are minimal models of the type (2.1) that do not have any sub-vectorial sector [4]. There are three categories of models. We denote them as type I, II and III. For the first one:

$$\begin{aligned}
&\text{type (I)}\,, \quad N_\psi \geq N_\chi \geq 0\ , \\
&N_\psi\, \square\!\square \oplus N_\chi\, \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus \big(N_\psi(N+4) - N_\chi(N-4)\big)\, \overline{\square}\ , \\
&b_0 = (11 - 2N_\psi)N - 6N_\psi - 2N_\chi\ .
\end{aligned} \tag{2.7}$$

For asymptotic large N, N_ψ can go up to 5 while retaining asymptotic freedom (AF). For the

[4] N_ψ, when positive, stands for the number of $\square\!\square$ and when negative stands for the number of $\overline{\square\!\square}$. N_χ, when positive, stands for the number of $\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and when negative stands for the number of $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. We choose the values (N_ψ, N_χ) and then we add the minimal number of fundamental, or anti-fundamental to cancel the gauge anomaly.

second class:

$$\begin{aligned} &\text{type (II)}\,, \quad N_\chi > N_\psi \geq 0 \ , \\ &N_\psi \, \yng(2) \oplus N_\chi \, \bar{\yng(1,1)} \oplus \big(N_\chi(N-4) - N_\psi(N+4)\big) \, \yng(1) \ , \\ &b_0 = (11 - 2N_\chi)N + 2N_\psi + 6N_\chi \ . \end{aligned} \tag{2.8}$$

For large N, $N_\chi \leq 5$ to have AF. For the third class:

$$\begin{aligned} &\text{type (III)}\,, \quad N_\psi \geq 0 \geq N_\chi \ , \\ &N_\psi \, \yng(2) \oplus (-N_\chi) \, \yng(1,1) \oplus \big(N_\psi(N+4) - N_\chi(N-4)\big) \, \bar{\yng(1)} \ , \\ &b_0 = (11 - 2(N_\psi - N_\chi))N - 6N_\psi - 6N_\chi \ . \end{aligned} \tag{2.9}$$

For large N, $N_\psi - N_\chi \leq 5$ to have AF.

Among the (N_ψ, N_χ) models $(1, 0)$ correspond to the $\psi\eta$ model, $(1, 1)$ corresponds to the $\psi\chi\eta$ model and $(0, 1)$ corresponds to the $\chi\tilde{\eta}$ model. These three types, plus their complex conjugate $(\bar{\text{I}})$, $(\bar{\text{II}})$ $(\bar{\text{III}})$ with $N_\psi \leq 0$ are shown in Figure 1 in the (N_ψ, N_χ) plane . The boundaries are set by the asymptotic-freedom requirement $b_0 > 0$ evaluated at large N. We introduce a quiver notation which may be useful to visualize the theories and their differences. We use the following notation for the quiver diagram: circles with a number inside n represent a gauge group $SU(n)$, squares with a number m represent a global symmetry $SU(m)$, fermions are lines connecting the groups, arrows on the line indicate if is fundamental (ingoing) or antifundamental (outgoing), little "o" or "x" within a line indicates if the ends are symmetric or anti-symmetric. See figure 2 for the diagrams of the (N_ψ, N_χ) models of types (I), (II) and (III).

The complete six-parameter set of theories of (2.1), (2.2) can be then parametrize by a (N_ψ, N_χ) model plus various vectorial copies of QCD, (S)QCD, (A)QCD or adjoint QCD.

As an example, starting from the $\psi\eta$ model one can obtain all the Bars-Yankielowicz (BY) models

$$\yng(2) \oplus (N + 4 + p) \, \bar{\yng(1)} \oplus p \, \yng(1) \ , \tag{2.10}$$

where the number of the extra fundamental pairs p is limited by $\frac{9}{2}N - 3$ before asymptotic freedom is lost.

Also the Georgi-Glashow models

$$\yng(1,1) \oplus (N - 4 + p) \, \bar{\yng(1)} \oplus p \, \yng(1) \ . \tag{2.11}$$

can be constructed starting from the $(N_\psi = 0, N_\chi = -1)$ model (the $\tilde{\chi}\eta$ model), by adding p Dirac fundamental fermions. Here p will be assumed to be less than $\frac{9}{2}N + 3$ so as to maintain AF.

These theories can have various types of gauge-invariant operators. Whenever there is a

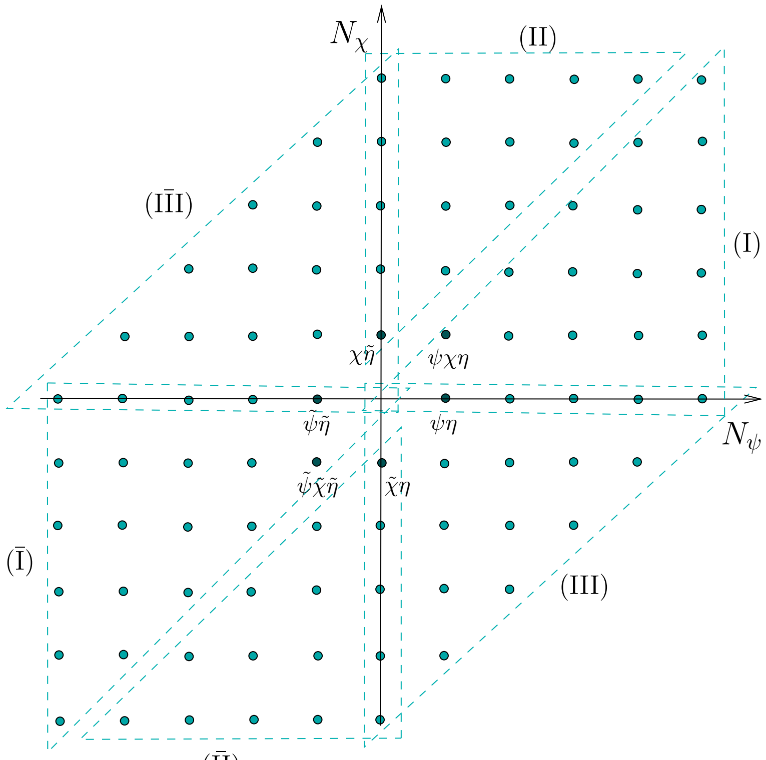


Figure 1: All the possible (N_ψ, N_χ) models that are AF at large N.

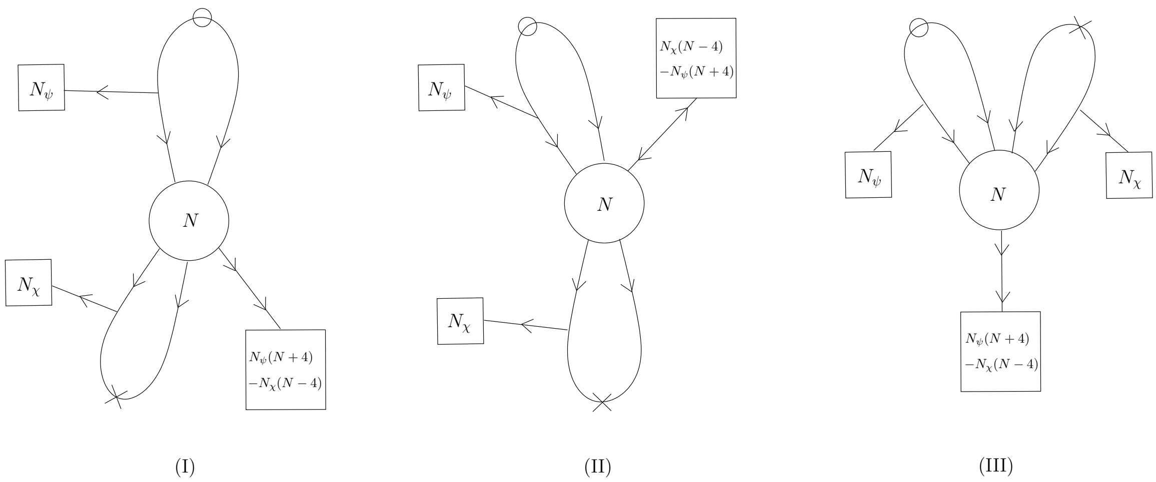


Figure 2: Quiver diagram to (N_ψ, N_χ) models models of type (I), (II) and (III). There are other $U(1)$ global charges which are not shown in the diagram.

vectorial part like QCD, (S)QCD or (A)QCD, there are gauge invariant bi-fermions $\eta\tilde{\eta}$, $\tilde{\psi}\psi$ or $\chi\tilde{\chi}$. These operators can be scalars, and we expect them to condense as they are usually in the strongest possible attractive channel, and also because they do in normal QCD. In the (N_ψ, N_χ) models it is not possible to have bi-fermion gauge-invariant operators. For type (I) and (II) theories we have an adjoint bi-fermion operator $\psi\chi$ which can be made gauge invariant with the addition of some gluon operator and then traced appropriately. These operators can also be scalars and in principle they could condense, although we do not consider this possibility. There are in general three-fermion gauge invariant objects, for example $\psi\eta\eta$, $\bar{\chi}\eta\eta$ in type (I), $\chi\tilde{\eta}\tilde{\eta}$, $\bar{\psi}\tilde{\eta}\tilde{\eta}$ in type (II) and $\psi\eta\eta$, $\tilde{\chi}\eta\eta$ in type (II). These can have a role in the IR as massless composite fermions. There are four-fermion gauge-invariant operators $\bar{\eta}\bar{\psi}\psi\eta$ $\bar{\eta}\chi\bar{\chi}\eta$ or $\psi\chi\psi\chi$ in type (I) and similar for the others. These states might play a role as mesonic states. These operators can also be scalar and, in principle, they could condense, although we do not explore this possibility here.

The Large N limit of these theories contains a mixture of features some are typical of 't Hooft, Veneziano or (S,A)QCD-type, large N limits. "Open-strings" type states, like mesons $\eta\tilde{\eta}$ in QCD have a three-vertex interaction that scales like $g_s \sim \frac{1}{\sqrt{N}}$ at large-N. The same happens for all open-strings type states like $\eta\psi\eta$, $\eta\tilde{\chi}\eta$, $\bar{\eta}\bar{\psi}\psi\eta$, ... "Closed-strings" type states, like mesons $\tilde{\psi}\psi$ or $\chi\tilde{\chi}$ in (S,A)QCD have a three-vertex interaction that scales like $g_s \sim \frac{1}{N}$ at large-N. The same happens for all states made of multiple ψ's and χ's traced. Large N argument suggests that the IR theory is weakly coupled for $N \to \infty$. We will look for IR free theories, but we should keep in mind that there is always the possibility of IR interacting fixed points, especially near the border of the diagram of Fig.1.

Inside this class there are models without fundamentals [3, 39]. (S)QCD, (A)QCD and adjoint QCD are however vectorial. Chiral theories can be built with fermions

$$\left(N_\psi = \frac{N-4}{k}\right) \square\!\square \oplus \left(N_\chi = \frac{N+4}{k}\right) \overline{\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}}\,, \tag{2.12}$$

where k is a common divisor of $N-4$ and $N+4$. These examples are the first theories of type (II) (2.8) where the number of fundamental is zero, $N_\psi(N+4) - N_\chi(N-4) = 0$, they are represented in Figure 3. This type of theories cannot be scaled to large N without fundamentals. On top of (2.12) we can add multiple copies of N_ψ (S)QCD, $N_{\tilde{\chi}}$ (A)QCD and N_λ adjoint QCD. An interesting feature of these models is that they have exact $\mathbb{Z}_2$ center symmetry for N even. Another feature is that they do not have fermionic gauge-invariant operator when $N = 4k$, in a sense they are 'purely bosonic' theories.

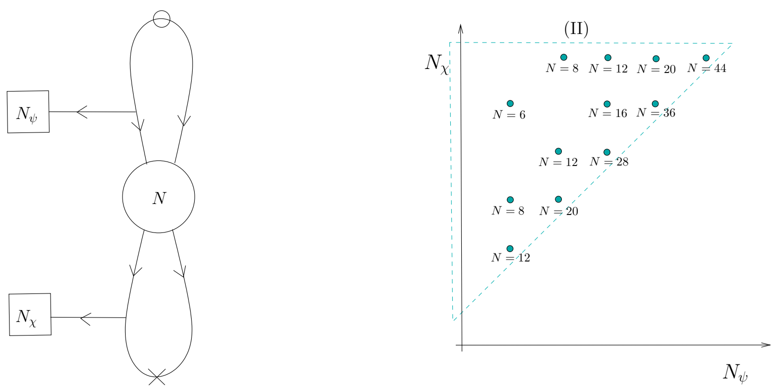


Figure 3: (N_ψ, N_χ) models without fundamentals.

2.1. Constraints for AF and infrared fixed-point conformal theory (CFT)

The requirement of AF used in Table 1 to restrict the classes of theories, is actually subtler than that implied by the first coefficient of the beta function, b_0, in (2.7)-(2.9). A theory with $b_0 > 0$ may actually flow into a CFT, as shown by Banks and Zaks [40] for QCD at large N and N_F near $N_F = 11N/2$, and by Seiberg [41] for supersymmetric QCD in the range $3N/2 < N_F < 3N$.

We have examined whether or not some of the theories listed in Table 1 (all with $b_0 > 0$) can flow into a CFT, by taking into account the second coefficient of the beta function, at large N [39, 42]. The result is shown in the Table below for type (I) and (II)

$$\alpha_{N_\psi,N_\chi} = \begin{pmatrix} -\frac{22\pi}{17N} & -\frac{24\pi}{13N} & -\frac{28\pi}{5N} & \frac{40\pi}{19N} & \frac{\pi}{2N} & \frac{8\pi}{77N} \\ -\frac{24\pi}{13N} & -\frac{2\pi}{N} & -\frac{8\pi}{N} & \frac{20\pi}{11N} & \frac{8\pi}{17N} & \frac{\pi}{10N} \\ -\frac{28\pi}{5N} & -\frac{8\pi}{N} & -\frac{14\pi}{N} & \frac{8\pi}{5N} & \frac{4\pi}{9N} & \frac{8\pi}{83N} \\ \frac{40\pi}{19N} & \frac{20\pi}{11N} & \frac{8\pi}{5N} & \frac{10\pi}{7N} & \frac{8\pi}{19N} & \frac{4\pi}{43N} \\ \frac{\pi}{2N} & \frac{8\pi}{17N} & \frac{4\pi}{9N} & \frac{8\pi}{19N} & \frac{2\pi}{5N} & \frac{8\pi}{89N} \\ \frac{8\pi}{77N} & \frac{\pi}{10N} & \frac{8\pi}{83N} & \frac{4\pi}{43N} & \frac{8\pi}{89N} & \frac{2\pi}{23N} \end{pmatrix} + O(1/N^2)\,, \tag{2.13}$$

where the rows and columns refer to the values of $N_\psi, N_\chi = 0, 1, \ldots, 5$. The positive value of the possible fixed-point coupling α_{N_ψ,N_χ} might indicate the presence of an IR CFT. This table might suggest that the e models towards the boundaries in Fig. 1 are in the IR interacting CFT. The result in the Table below are for type (III)

$$\alpha_{N_\psi,N_{\tilde\chi}} = \begin{pmatrix} -\frac{22\pi}{17N} & -\frac{24\pi}{13N} & -\frac{28\pi}{5N} & \frac{40\pi}{19N} & \frac{\pi}{2N} & \frac{8\pi}{77N} \\ -\frac{24\pi}{13N} & -\frac{28\pi}{5N} & \frac{40\pi}{19N} & \frac{\pi}{2N} & \frac{8\pi}{77N} & \\ -\frac{28\pi}{5N} & \frac{40\pi}{19N} & \frac{\pi}{2N} & \frac{8\pi}{77N} & & \\ \frac{40\pi}{19N} & \frac{\pi}{2N} & \frac{8\pi}{77N} & & & \\ \frac{\pi}{2N} & \frac{8\pi}{77N} & & & & \\ \frac{8\pi}{77N} & & & & & \end{pmatrix} + O(1/N^2)\,. \tag{2.14}$$

The first row (column) corresponds to $N_\psi = 0$ ($N_{\tilde\chi} = 0$), so coincides with the first row (column) of the previous table.

Unfortunately, all these putative fix-points (if they really exist) are non-perturbative, even if $\alpha \ll 1$. The reason is that the coefficients b_i of the beta function scales as $O(N^{i+1})$ so for $\alpha \sim O(1/N)$ all the terms of

$$\beta(g) = -\frac{g}{12\pi}\left(b_0\frac{\alpha}{4\pi} + b_1\left(\frac{\alpha}{4\pi}\right)^2 + \ldots\right) \tag{2.15}$$

are of the same order. Another, equivalent, way to state this state of matter is that if [5] $b_0 \sim O(N)$, the correct perturbative expansion for large N is the 't Hooft coupling $\lambda = Ng^2$ which is $O(1)$ in these models. We shall not pursue further this problem in this work.

[5] By adding vector-like matter, e.g. by cranking up p in (2.10) or in (2.11), one can reach a point where $b_0 \sim O(1)$, while b_1 is still $O(N^2)$. In this case, one obtains $\alpha \sim O(1/N^2)$, and the fix-point is perturbative. The scenario should be similar to the Banks-Zaks fix-point in QCD. We postpone any further discussion on this point to future work.

3. Hypothetical confining, symmetric phase

Most of the theories studied here contain a large, non-Abelian flavor symmetry group. Confinement without symmetry breaking, with no condensate formation, would require that the spectrum of massless gauge-invariant composite fermions be such that it matches all the 't Hooft anomaly triangles of the UV theory. This represents quite a nontrivial constraint on the IR theory, and it is somewhat surprising that some models in the family, namely the BY [12] and the generalized GG models, apparently allow for solutions to these constraints, with a simple set of massless baryons [11, 12].

We review first these solutions, and then explain why one cannot expect any solution of this sort in other more general models, at least for large N.

Let us start with the $\psi\eta$ model. The matter content of the theory

$$\psi^{\{ij\}} \,, \quad \eta_i^A \,, \qquad i,j = 1,2,\ldots,N \,, \quad A = 1,2,\ldots,N+4 \,, \tag{3.1}$$

$(i,j$ are $SU(N)$ color indices, A is an $SU(8)$ flavor index) transforms as in (2.4) under [6]

$$G_F = SU(N+4) \times U(1)_{\psi\eta} \,, \tag{3.2}$$

where $U(1)_{\psi\eta}$ is the anomaly-free combination of $U(1)_\psi$ and $U(1)_\eta$,

$$U(1)_{\psi\eta} : \ \psi \to \mathrm{e}^{\mathrm{i}(N+4)\alpha}\psi \,, \qquad \eta \to \mathrm{e}^{-\mathrm{i}(N+2)\alpha}\eta \,. \qquad \alpha \in \mathbb{R} \,. \tag{3.3}$$

The surprising feature of this theory is that all the $SU(N+4) \times U(1)_{\psi\eta}$ anomaly triangles are saturated by a massless "baryon"

$$\mathcal{B}^{[AB]} = \psi^{ij}\eta_i^A\eta_j^B \,, \qquad A,B = 1,2,\ldots,N+4 \,, \tag{3.4}$$

antisymmetric in $A \leftrightarrow B$, the $SU(N+4)$ indices. The fact that the matching holds, for now, has no simple explanation: the best we can do is check it by inspection of Table 1.

	fields	$SU(N)_{\rm c}$	$SU(N+4)$	$U(1)_{\psi\eta}$
UV	ψ	$\square\square$	$\frac{N(N+1)}{2}\cdot(\cdot)$	$N+4$
	η^A	$(N+4)\cdot\bar{\square}$	$N\cdot\square$	$-(N+2)$
IR	$\mathcal{B}^{[AB]}$	$\frac{(N+4)(N+3)}{2}\cdot(\cdot)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$-N$

Table 1: Chirally symmetric phase of the $(1,0)$ model. The multiplicity, charges and the representation are shown for each set of fermions. $(\cdot)$ stands for a singlet representation.

It is possible to generalize this solution for all the BY models ($N_\psi = 1, \ldots$), defined in (2.10).

[6] The group (3.2) is actually not the true symmetry group of our system, but its covering group. Its global structures however contain some redundancies, which must be modded out appropriately in order to eliminate the double counting. They depend on whether N is odd or even. Such an observation was fundamental in the study of the mixed anomalies and generalized anomaly-matching requirement studied in [4].

The fermions

$$\psi^{[ij]}\,, \quad \eta_i^A \quad \xi^{i,b}\,, \qquad i,j=1,2,\ldots,N\,, \quad A=1,2,\ldots,N+p+4 \quad b=1,2,\ldots,p\,, \tag{3.5}$$

transform under an enlarged global symmetry

$$G_F = \begin{cases} SU(N+5)_\eta \times U(1)_{\psi\eta} \times U(1)_{\psi\xi}\,, & \text{for } p=1\,, \\ SU(N+4+p)_\eta \times SU(p)_\xi \times U(1)_{\psi\eta} \times U(1)_{\psi\xi}\,, & \text{for } p>1\,, \end{cases} \tag{3.6}$$

Here, again,

$$U(1)_{\psi\eta}: \qquad \psi \to \mathrm{e}^{\mathrm{i}(N+4+p)\alpha}\psi\,, \quad \eta \to \mathrm{e}^{-\mathrm{i}(N+2)\alpha}\eta\,, \tag{3.7}$$

with $\alpha \in \mathbb{R}$, and

$$U(1)_{\psi\xi}: \qquad \psi \to \mathrm{e}^{\mathrm{i}p\beta}\psi\,, \quad \xi \to \mathrm{e}^{-\mathrm{i}(N+2)\beta}\xi\,, \tag{3.8}$$

with $\beta \in \mathbb{R}$, is a particular choice of the two, Abelian, anomaly free symmetries.

In this case the solution 3.4 generalizes to

$$(\mathcal{B}_1)^{[AB]} = \psi^{ij}\eta_i^A\eta_j^B\,, \qquad (\mathcal{B}_2)_A^a = \bar{\psi}_{ij}\bar{\eta}_A^i\xi^{j,a}\,, \qquad (\mathcal{B}_3)_{\{ab\}} = \psi^{ij}\bar{\xi}_{i,a}\bar{\xi}_{j,b}\,, \tag{3.9}$$

where the first baryon is anti-symmetric in $A \leftrightarrow B$ and the third is symmetric in $a \leftrightarrow b$. Their charges are given in Table 2, and again, one must check all the 't Hooft anomaly matching condition by inspection.

	$SU(N)_\mathrm{c}$	$SU(N+4+p)$	$SU(p)$	$U(1)_{\psi\eta}$	$U(1)_{\psi\xi}$
$\mathcal{B}_1$	$\frac{(N+4+p)(N+3+p)}{2}\cdot(\cdot)$	$\begin{matrix}\square\\\square\end{matrix}$	$\frac{(N+4+p)(N+3+p)}{2}\cdot(\cdot)$	$-N+p$	p
$\mathcal{B}_2$	$(N+4+p)p\cdot(\cdot)$	$p\cdot\bar{\square}$	$(N+4+p)\cdot\square$	$-(p+2)$	$-(N+p+2)$
$\mathcal{B}_3$	$\frac{p(p+1)}{2}\cdot(\cdot)$	$\frac{p(p+1)}{2}\cdot(\cdot)$	$\overline{\square\square}$	$N+4+p$	$2N+4+p$

Table 2: Chirally symmetric phase of the BY model.

All anomaly triangles are indeed saturated by these candidate massless composite fermions, as seen in Table 3, taken from [4].

There is a similar solution also for all the generalized Georgi-Glashow models 2.11.

The problem is that we do not have any physical understanding (apart from that of the general principle) of why and how these matching equations are satisfied. This in in stark contrast to the case of the dynamical Higgs phase discussed below, Sec. 4.

Moreover, it seems that these solutions *cannot* be extended to other models. One can give a simple argument for this, for large N.

Let us first discuss the possibility of an IR phase of this sort for the $\psi\chi\eta$ model, that is the $(N_\psi, N_\chi) = (1,1)$ model. Take for example the 't Hooft $[SU(8)_\eta]^3$ anomaly, where $SU(8)_\eta$ is the global symmetry acting on the anti-fundamentals η. This anomaly is N in UV, because they are anti-fundamentals fields under $SU(N)$. To saturate it in the IR it would require $\propto N$ distinct gauge invariant composite fermions charged under $SU(8)_\eta$. The possibility that the

	UV	IR
$SU(N+4+p)^3$	N	$N+p-p$
$SU(p)^3$	N	$N+4+p-(p+4)$
$SU(N+4+p)^2-U(1)_{\psi\eta}$	$-N(N+2)$	$-(N+2+p)(N-p)-p(p+2)$
$SU(N+4+p)^2-U(1)_{\psi\xi}$	0	$(N+2+p)p-p(N+p+2)$
$SU(p)^2-U(1)_{\psi\eta}$	0	$-(N+4+p)(p+2)+(p+2)(N+p+4)$
$SU(p)^2-U(1)_{\psi\xi}$	$-N(N+2)$	$-(N+4+p)(N+p+2)+(p+2)(2N+p+4)$
$U(1)^3_{\psi\eta}$	$\frac{N(N+1)}{2}(N+4+p)^3-N(N+4+p)(N+2)^3$	$-\frac{(N+4+p)(N+3+p)}{2}(N-p)^3-(N+4+p)p(p+2)^3+$ $+\frac{p(p+1)}{2}(N+4+p)^3$
$U(1)^3_{\psi\xi}$	$\frac{N(N+1)}{2}p^3-Np(N+2)^3$	$\frac{(N+4+p)(N+3+p)}{2}p^3-(N+4+p)p(N+p+2)^3+$ $+\frac{p(p+1)}{2}(2N+4+p)^3$
$\mathrm{Grav}^2-U(1)_{\psi\eta}$	$\frac{N(N+1)}{2}(N+4+p)-N(N+4+p)(N+2)$	$-\frac{(N+4+p)(N+3+p)}{2}(N-p)-(N+4+p)p(p+2)+$ $+\frac{p(p+1)}{2}(N+4+p)$
$\mathrm{Grav}^2-U(1)_{\psi\xi}$	$\frac{N(N+1)}{2}p-Np(N+2)$	$\frac{(N+4+p)(N+3+p)}{2}p-(N+4+p)p(N+p+2)+$ $+\frac{p(p+1)}{2}(2N+4+p)$
$SU(N+4+p)^2-(\mathbb{Z}_{N+2})_\psi$	0	$N+2+p-p=0 \bmod N+2$
$SU(p)^2-(\mathbb{Z}_{N+2})_\psi$	0	$-(N+4+p)+p+2=0 \bmod N+2$
$\mathrm{Grav}^2-(\mathbb{Z}_{N+2})_\psi$	1	$1-1+1$

Table 3: Anomaly matching checks for the IR chiral symmetric phase of the BY model.

system confines, with no global symmetry breaking and with some massless "baryons" saturating the 't Hooft anomalies, thus does not appear to be likely here, at least in the large N limit [2,13,15].

Next, consider the possibility of such IR phase for the $(N_\psi > 1, 0)$ model, that is N_ψ families of the $\psi\eta$ multiplet. We do have simple gauge invariant fermion, e.g. $\psi\eta\eta$ in $\square\square$ or $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of $SU(N_\psi(N+4))_\eta$ and in fundamental of of $SU(N_\psi)_\psi$, we also have composite of these ones, and more made with the epsilon tensor. Take for example the $[SU((N_\psi(N+4))_\eta]^3$ anomaly which is N in the UV. For this anomaly $\psi\eta\eta$ would give a contribution $(N_\psi((N_\psi(N+4)\mp 4)$ in the IR. Clearly $\psi\eta\eta$, no matter its representation, cannot saturate these anomalies (only in the special case s $N_\psi = 1$ of the $\psi\eta$ model the saturation works). Take for example the 't Hooft $[SU((N_\psi)_\psi]^3$ anomaly which is $N(N+1)/2$ n the UV, $\psi\eta\eta$ would give a contribution $(N_\psi(N+4)((N_\psi(N+4)\mp 1)/2$ in the IR. Cannot be excluded, but confinement without symmetry breaking for the $(N_\psi > 1, 0)$ model is not a plausible solution for large N. Similar considerations for the $(N_\chi > 1, 0)$ model, that is N_χ families of the $\chi\tilde{\eta}$ multiplet, or any model where we have more than one ψ or χ, essentially any model of the type (2.1), (2.2) apart from BY and GG for which we saw the easy solution before.

Any solution, even if any can be found, is rather unnatural, very contrived and highly dependent of N. See for example [43] with three copies of $\chi\eta$ model with $N = 5$, this is the Georgi-Glashow GUT model with three families, which has a clear interest for phenomenological purposes.

Most significantly, the recent series of studies based on the generalized 't Hooft (mixed-anomaly) matching conditions, see [4–7], cast serious doubt on the consistency of such a confining, fully symmetric phase, reviewed in this section.

4. Dynamical Higgs phase

A natural candidate for the IR dynamics of the BY and GG models is what might be called dynamical Higgs phase. Namely, as the interactions become strong towards the infrared, bi-fermion condensates are formed so as to break both color $SU(N)$ and (part of) the global symmetry of the system. A particularly interesting possibility is the color-flavor locked bi-fermion condensates are formed and characterize the IR system, as studied in [2, 4–7].

A remarkable property of this dynamical Higgs phase is that all the 't Hooft anomaly matching equations are automatically satisfied. The IR spectrum of the massless fermions in such a system is found by decomposing the UV fermion representations as direct sums of the irreps of the unbroken symmetry group (gauge and flavor). All Dirac-like pairs of fermions with respect to the unbroken group can be assumed to get mass and decouple. The spectrum of the remaining fermions is then seen to be *identical* in UV and IR (their charges and multiplicities) hence the anomaly matching (including the generalized, mixed anomalies) is automatic, and does not require checking any arithmetic equations such as in Table 3. In other words, we understand why the 't Hooft anomaly constraints are satisfied. The matching of the anomalies of the spontaneously broken part of the symmetries, instead, is taken care of by the coupling of the Nambu Goldstone bosons with the background gauge fields.

As the 't Hooft anomaly matching requirement does not lead to any new constraint here, it is difficult to know which particular condensate and which symmetry-breaking pattern is actually realized. To determine the correct one, one can appeal to several heuristic arguments:

- The Maximally Attractive Channel (MAC) criterion [9] suggests that the condensate that actually forms is the one that maximizes (in the absolute value) the quantity

$$C_2(\mathcal{R}_c) - C_2(\mathcal{R}_1) - C_2(\mathcal{R}_2) \tag{4.1}$$

  where $C_2(\mathcal{R})$ is the quadratic Casimir of the irrep $\mathcal{R}$. It represents the strength of the one-gluon exchange force (4.1) in various bifermion (made of $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$) composite-scalar $(\mathcal{R}_c)$ channels.
  Just to have some quantitative idea, let us compare the strength of the attraction (4.1) in various bifermion scalar channels, formed by two out of the three types of fermions, ψ, χ and η. Some of the most probable channels are

$$\begin{aligned}
A:&\quad \psi\left(\yng(2)\right)\,\psi\left(\yng(2)\right) \quad \text{forming} \quad \yng(2,2)\,;\\
B:&\quad \chi\left(\bar{\yng(1,1)}\right)\,\chi\left(\bar{\yng(1,1)}\right) \quad \text{forming} \quad \bar{\yng(1,1,1,1)}\,;\\
C:&\quad \eta\left(\bar{\yng(1)}\right)\,\eta\left(\bar{\yng(1)}\right) \quad \text{forming} \quad \bar{\yng(1,1)}\,;\\
D:&\quad \chi\left(\bar{\yng(1,1)}\right)\,\eta\left(\bar{\yng(1)}\right) \quad \text{forming} \quad \bar{\yng(1,1,1)}\,;\\
E:&\quad \psi\left(\yng(2)\right)\,\chi\left(\bar{\yng(1,1)}\right) \quad \text{forming an adjoint representation } (\tilde{\phi})\,;\\
F:&\quad \psi\left(\yng(2)\right)\,\eta\left(\bar{\yng(1)}\right) \quad \text{forming} \quad \yng(1) \quad (\phi);\\
G:&\quad \tilde{\eta}\left(\yng(1)\right)\,\eta\left(\bar{\yng(1)}\right) \quad \text{forming} \quad (\cdot) \quad (\text{singlet})\,.
\end{aligned} \tag{4.2}$$

The one-gluon exchange strength is, in the six cases above, proportional to

$$
\begin{aligned}
A:&\quad \frac{2(N^2-4)}{N}-\frac{(N+2)(N-1)}{N}-\frac{(N+2)(N-1)}{N}=-\frac{2(N+2)}{N}\;;\\
B:&\quad \frac{2(N+1)(N-4))}{N}-\frac{(N+1)(N-2)}{N}-\frac{(N+1)(N-2)}{N}=-\frac{4(N+1)}{N}\;;\\
C:&\quad \frac{(N+1)(N-2))}{N}-\frac{N^2-1}{2N}-\frac{N^2-1}{2N}=-\frac{N+1}{N}\;;\\
D:&\quad \frac{3(N+1)(N-3))}{2N}-\frac{N^2-1}{2N}-\frac{(N+1)(N-2)}{N}=-\frac{2N+2}{N}\;;\\
E:&\quad N-\frac{(N+2)(N-1)}{N}-\frac{(N+1)(N-2)}{N}=-\frac{N^2-4}{N}\;;\\
F:&\quad \frac{N^2-1}{2N}-\frac{N^2-1}{2N}-\frac{(N+2)(N-1)}{N}=-\frac{(N+2)(N-1)}{N}\;;\\
G:&\quad 0-\frac{N^2-1}{2N}-\frac{N^2-1}{2N}=-\frac{N^2-1}{N}\;, \qquad (4.3)
\end{aligned}
$$

respectively. We note that the $\tilde{\phi}$ $(\psi\chi)$ and ϕ $(\psi\eta)$ condensates considered by us (cases E and F, respectively), together with the "quark-antiquark condensate" (case G), correspond precisely to the three most attractive channels, at large N. Their attraction strength scales as $O(N)$ in contrast to the other four channels which scale as $O(1)$.
Note in particular that the bifermion $\psi\eta$ (or $\chi\eta$) condensate channels assumed in the $\psi\eta$ (and $\chi\eta$) models, see (4.5), (4.9) below, have the same attraction strength as in the familiar color-singlet $\langle\bar{q}q\rangle$ condensates in the standard QCD.

- A second prescription is that the right condensate to consider is the one that minimizes a_{IR}, a quantity defined at a fixed point (either IR or UV) from the conformal anomaly. In a weakly coupled theory, a can be computed from the massless spectrum:

$$a = (\#\text{ bosons}) + \frac{11}{2}(\#\text{Weyl fermions}) + 62(\#\text{vector bosons})\;. \qquad (4.4)$$

This prescription arises from the a-theorem, proposed a long time ago [44] and proven only recently [45]. Unfortunately, the theorem imposes only that $a_{UV} > a_{IR}$, but this does not mean that a_{IR} is the smallest possible between a set of candidates: it should simply be smaller than the a in the UV.

- A related idea is the ACS prescription [10].

The simplest example of color-flavor locking phases is the $\psi\eta$ model. It is natural to assume (see (4.2), (4.3)) that a bifermion condensate

$$\langle\psi^{\{ij\}}\eta_i^b\rangle = c\,\Lambda^3\delta^{jb} \neq 0\;, \qquad j,b=1,2,\ldots N\;, \qquad c\sim O(1) \qquad (4.5)$$

forms, breaking the gauge-flavor symmetries as

$$SU(N)_{\rm c}\times SU(N+4)\times U(1)_{\psi\eta} \to SU(N)_{\rm cf}\times SU(4)_{\rm f}\times U(1)'\,. \qquad (4.6)$$

Here $U(1)'$ symmetry is the diagonal combination of $U(1)_{\psi\eta}$ and the elements of $SU(N+4)$ generated by

$$\begin{pmatrix} -2\,\mathbf{1}_N & \\ & \frac{N}{2}\mathbf{1}_4 \end{pmatrix} \qquad (4.7)$$

left unbroken by the condensate.

In the IR, alongside the $8N+1$ Nambu-Goldstone (NG) bosons associated with the breaking 4.6, there are a few leftover massless fermions that enforce the 't Hooft anomaly matching of the unbroken symmetry. As explained already, the decomposition of the UV fermions as a direct sum of the irreps of the unbroken group, see the upper half of Table 4, is sufficient to show that all the matching requirements are met by construction.

	fields	$SU(N)_{\rm cf}$	$SU(4)_{\rm f}$	$U(1)'$
UV	ψ	$\square\!\square$	$\frac{N(N+1)}{2}\cdot(\cdot)$	$N+4$
	η^{A_1}	$\overline{\square\!\square} \oplus \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$N^2\ \cdot\ (\cdot)$	$-(N+4)$
	η^{A_2}	$4\cdot\bar{\square}$	$N\ \cdot\ \square$	$-\frac{N+4}{2}$
IR	$\mathcal{B}^{[A_1B_1]}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	$-(N+4)$
	$\mathcal{B}^{[A_1B_2]}$	$4\cdot\bar{\square}$	$N\ \cdot\ \square$	$-\frac{N+4}{2}$

Table 4: Color-flavor locked phase in the $(1,0)$ model. A_1 or B_1 stand for $1,2,\ldots,N$, A_2 or B_2 the rest of the flavor indices, $N+1,\ldots,N+4$.

Generically, the symmetric part of η^{A_1} and ψ form a massive Dirac pair, leaving in IR the symmetric part of η^{A_1} and η^{A_2}. At this point one can always dress the IR particles with the condensate to make them invariant under the UV gauge group:

$$\mathcal{B}^{[A_1,B_1]}=\psi^{ij}\eta_j^{[A_1}\eta_i^{B_1]} \quad \text{and} \quad \mathcal{B}^{[A_1,B_1]}=\psi^{ij}\eta_j^{[A_1}\eta_i^{B_2]} \tag{4.8}$$

These fermions can be thought of as what remains of the massless baryon 3.4, if one enforces the symmetry breaking 4.6. However we should stress that these two phases are different: the symmetries and the massless spectrum distinguish them.

One can consider a similar solution for the $\tilde{\chi}\eta$ model. Again, one has a bi-fermion condensate

$$\langle\tilde{\chi}^{[ij]}\eta_i^b\rangle = c\,\Lambda^3\delta^{jb}\neq 0\ , \qquad j,b=1,2,\ldots N\ , \qquad c\sim O(1) \tag{4.9}$$

that breaks

$$SU(N)_c\times SU(N-4)_\eta\times U(1)_{\chi\eta}\to SU(4)_c\times SU(N-4)_{cf}\times\tilde{U}(1) \tag{4.10}$$

where $\tilde{U}(1)$ is a particular combination of $U(1)_{\chi\eta}$ and a $SU(N)_c$ rotation generated by

$$\tilde{Q}=Q_{\chi\eta}+\frac{1}{2}\begin{pmatrix}4\ \cdot\mathbf{1}_{(N-4)\times(N-4)} & \\ & -(N-4)\ \cdot\mathbf{1}_{4\times 4}\end{pmatrix}\ . \tag{4.11}$$

Notice that the structure of the global symmetry group is the same as we have in UV: there are no NGBs and the charges of any gauge invariant operator under $SU(N-4)_{cf}\times\tilde{U}(1)$ and under $SU(N-4) \times U(1)_{\chi\eta}$ are identical[7].

To figure out the spectrum it is useful to decompose the fundamental fermions under the leftover symmetry group, as in Table 5. One can see that χ_1 and η_2, χ_2, and η_3 form complex conjugate pairs, so they can be gapped together. Similarly, η_3 is self-conjugate, so, as $SU(4)$ confines, also χ_3 becomes massive, and disappears from the spectrum. In the IR only η_1 remains.

	fields	$SU(4)_c$	$SU(N-4)_{\rm cf}$	$\tilde{U}(1)$
UV	χ_1	$(\cdot)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	N
	χ_2	$\square$	$\square$	$N/2$
	χ_3	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$(\cdot)$	0
	η_1	$(\cdot)$	$\overline{\square\square}$	$-N$
	η_2	$(\cdot)$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$-N$
	η_3	$\bar{\square}$	$\bar{\square}$	$-N/2$

Table 5: Color-flavor locked phase in the $(0,1)$ model. A_1 or B_1 stand for $1, 2, \ldots, N$, A_2 or B_2 the rest of the flavor indices, $N+1, \ldots, N+4$.

Dressing η_1 with the condensate

$$\left(\chi^{[ij]}\eta_j^{\{A}\right)\eta_i^{B\}} = \mathcal{B}^{\{AB\}} \tag{4.12}$$

would make it gauge invariant. Such "baryons" are identical to the massless composite fermions which would saturate the full 't Hooft anomaly triangles in the hypothetical, confining flavor-symmetric vacuum, as those discussed in Sec. 3, for the $\tilde{\chi}\eta$ model (see [4, 6] for more details). This might lead some to suspect that in the $\tilde{\chi}\eta$ model the so-called complementarity [46] is at work, i.e., that dynamical Higgs and confinement without symmetry breaking might be actually the same phase.

As discussed carefully in Section 4 of [6], however, there are reasons to believe that these two phases are actually physically distinct. There is no complementarity in the $\tilde{\chi}\eta$ model. Only one of them can be the correct IR phase of the theory. As discussed in [4–6] the dynamical Higgs phase appears to describe correctly the IR physics of the $\tilde{\chi}\eta$ model.

5. Dynamical Abelianization

Another remarkable infrared phase which might describe the physics of some of the models in the infrared, is dynamical Abelianization, a phenomenon familiar from the exactly solved $\mathcal{N}=2$ supersymmetric gauge theories [47, 48]. What is perhaps not quite familiar is the fact that dynamical Abelianization can occur in some class of nonsupersymmetric, chiral gauge theories we are interested in.

[7] Up to a complex conjugation. This is because we defined the generator of $SU(N-4)_{cf}$ as $T^a_{cf} = T^a_c - T^a_f$.

The first carefully studied case concerns the $\psi\chi\eta$ model, (2.6). We assume that a bifermion condensate in the adjoint representation forms:

$$\langle \psi^{ik}\chi_{kj}\rangle = \Lambda^3 \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_N \end{pmatrix}^i_j , \qquad \langle \psi^{ij}\eta_j^A\rangle = 0 , \tag{5.1}$$

$$c_n \in \mathbb{C} , \qquad \sum_n c_n = 0 , \qquad c_m - c_n \neq 0 , \quad m \neq n , \tag{5.2}$$

(with no other particular relations among c_j's), inducing dynamical Abelianization of the system. The unbroken gauge group $U(1)^{N-1}$ is generated by the Cartan subalgebra. More precisely [8], we require that the condensate (5.1), (5.2) induce the symmetry breaking

$$SU(N) \to U(1)^{N-1} . \tag{5.3}$$

As the effective composite scalar fields $\phi \sim \psi\chi$ are in the adjoint representation of $SU(N)$, it can be parametrized as a linear combination,

$$\phi \sim \psi\chi = \phi^A T^A = \phi^{(\alpha)} E_\alpha + \phi^{(-\alpha)} E_{-\alpha} + \phi^{(i)} H^i , \tag{5.4}$$

where ϕ^A are complex fields and T^A are the Hermitian generators of $SU(N)$ in the fundamental representation ($A = 1, 2, \ldots, N^2 - 1$). $E_{\pm\alpha}$ are the raising and lowering generators in the Cartan basis, associated with the various root vectors α. The condition for the dynamical Abelianization (5.1) is that the fields that condense are in the Cartan subalgebra,

$$\phi \sim \psi\chi = \phi^{(i)} H^i , \qquad \langle \phi^i \rangle \neq 0 , \qquad \forall i , \tag{5.5}$$

whereas

$$\langle \phi^{(\alpha)} \rangle = \langle \phi^{(-\alpha)} \rangle = 0 , \qquad \forall \alpha . \tag{5.6}$$

See [8] for more about the associated (would-be) NG bosons.

The fields η_i^A which do not participate in the condensate remain massless and weakly coupled to the gauge bosons from the Cartan subalgebra which we will refer to as the photons. Also, some of the fermions ψ^{ij} do not participate in the condensates. Due to the fact that $\psi^{\{ij\}}$ are symmetric whereas $\chi_{[ij]}$ are antisymmetric, actually only the nondiagonal elements of $\psi^{\{ij\}}$ condense and get mass. The diagonal fields $\psi^{\{ii\}}$, $i = 1, 2, \ldots, N$, together with all of η_i^a remain massless. Also there is one physical NG boson. All of the anomaly triangles, $[SU(8)]^3$, $\tilde{U}(1) - [SU(8)]^2$, $[\tilde{U}(1)]^3$, $\tilde{U}(1) - [\text{gravity}]^2$, are easily seen to match, on inspection of Table 6. Perhaps the only non-trivial ones are the ones that do not involve $SU(8)$. The unbroken gauge group $\prod_{\ell=1}^{N-1} U(1)_\ell$ is generated by the subalgebra,

$$t^1 = \frac{1}{2}\text{diag}\,(1, -1, 0, \ldots, 0) \;, \quad t^2 = \frac{1}{2}\text{diag}\,(0, 1, -1, 0, \ldots, 0) \;, \\ \cdots , \quad t^{N-1} = \frac{1}{2}\text{diag}\,(0, \ldots, 0, 1, -1) \;. \tag{5.7}$$

In this basis the IR theory can be represented as the $U(1)$ quiver diagram of figure 4.

	fields	$SU(8)$	$\tilde{U}(1)$
UV	ψ	$\frac{N(N+1)}{2} \cdot (\cdot)$	$\frac{N(N+1)}{2} \cdot (2)$
	χ	$\frac{N(N-1)}{2} \cdot (\cdot)$	$\frac{N(N-1)}{2} \cdot (-2)$
	η^A	$N \cdot \square$	$8N \cdot (-1)$
IR	ψ^{ii}	$N \cdot (\cdot)$	$N \cdot (2)$
	η^A	$N \cdot \square$	$8N \cdot (-1)$

Table 6: Full dynamical Abelianization in the $\psi\chi\eta$ model.

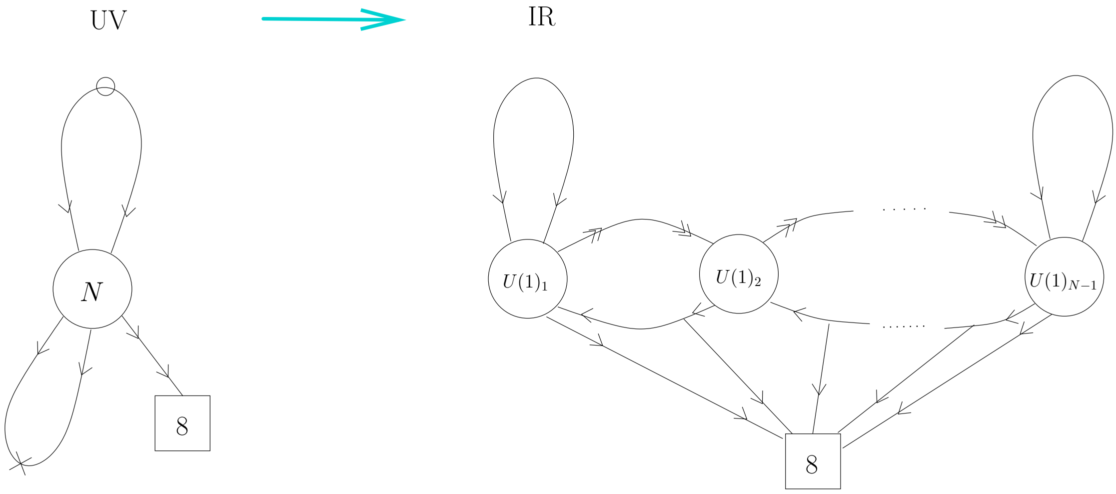


Figure 4: $U(1)$'s quiver model representing dynamical Abelianization for the $\psi\chi\eta$ model

The dynamical Abelianization proposal (5.1), has been shown to be fully consistent with a stronger requirement of the matching of the mixed anomalies, in [8]. Also, as discussed in [2,3,39] dynamical Abelianizatio may well describe the IR phase of many other models.

6. Non-Abelian gauge groups in the IR

Let us now discuss generalizations of the dynamical Abelianization in (N_ψ, N_χ) models of type I and II. All these models have a $\psi\chi$ composite scalar in the adjoint representation. The condensation of such a composte adjoint scalar field can lead to dynamical Abelianization, as discussed in the previous section. It is, however, possible that the condensation of the adjoint scalar $\langle\psi\chi\rangle$ leads to more general types of gauge symmetry breaking. Let us now discuss the possibility that the RG flow actually brings the system towards more general low-energy effective gauge groups, containing various nonAbelian factors.

A type (I) theory has fields

$$\begin{array}{ccc} \psi^{\{ij\},a} \;, & \chi^b_{[ij]} \;, & \eta^c_i \;, \\ a = 1, \dots N_\psi \;, & b = 1, \dots N_\chi \;, & c = 1, \dots, N_\psi(N+4) - N_\chi(N-4) \;. \end{array} \tag{6.1}$$

We assume that only one of the ψ's pairs with one of the χ's condense. The $\psi\chi$ condensate breaks the global symmetry as

$$\begin{aligned}&SU(N_\psi) \times SU(N_\chi) \times SU(N_\psi(N+4) - N_\chi(N-4)) \times U(1)^2 \\ &\longrightarrow SU(N_\psi - 1) \times SU(N_\chi - 1) \times SU(N_\psi(N+4) - N_\chi(N-4)) \times U(1)^3 \ .\end{aligned} \tag{6.2}$$

We assume that the condensate can be brought in a diagonal form

$$\langle \psi^{ik,1} \chi^1_{kj} \rangle = \Lambda^3 \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_N \end{pmatrix}^i_{\ j} \neq 0 \ , \tag{6.3}$$

$$c_n \in \mathbb{C} \ , \qquad \sum_n c_n = 0 \ , \tag{6.4}$$

as in the dynamical-Abelianization case, but this time we allow the possibility for some degeneracy among c's. Let us assume for instance that there a block of n coefficients which are degenerate:

$$c_1 = c_2 = \cdots = c_n \ . \tag{6.5}$$

This leaves a non-Abelian unbroken gauge group

$$SU(N) \longrightarrow SU(n) \times \dots \ . \tag{6.6}$$

The first coefficient of the beta function for $SU(n)$ is

$$\begin{aligned} b_0[SU(n)] &= (11 - 2N_\psi)n - 6N_\psi - 2N_\chi - (N_\psi + N_\chi - 2)(N - n) \\ &= (9 - N_\psi + N_\chi)n - (6 + N)N_\psi - (2 + N)N_\chi + 2N \end{aligned} \tag{6.7}$$

when $n = N$ this is exactly the b_0 of the original $SU(N)$ theory that we choose positive (**??**). For the $\psi\chi\eta$ model where $(N_\psi, N_\chi) = (1, 1)$ this is always the same $\psi\chi\eta$ model reduced to $SU(n)$, and $b_0 > 0$. Where any of N_ψ or N_χ is greater than 1 we can have a sign flip for certain values of n. For a certain value n^* we have a zero of (6.7). The change of sign for (6.7) happens at

$$n^* = \frac{(N_\psi + N_\chi - 2)N + 6N_\psi + 2N_\chi}{9 - N_\psi + N_\chi} \ . \tag{6.8}$$

We take $[n^*]$ the biggest integer smaller than n^*. If we than n in (6.5),(6.6) to be $[n^*]$ we have the biggest possible non-Abelian IR free sub-group. We can see that when N_ψ or N_χ is greater than 1 we have $[n^*] \simeq \frac{(N_\psi + N_\chi - 2)}{9 - N_\psi + N_\chi} N$ which is greater than one and a fraction of N.

Example of type (I) is $(N_\psi, N_\chi) = (2, 1)$:

$$2\,\square\!\square \oplus \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \oplus (N+12)\,\bar{\square} \ . \tag{6.9}$$

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This is the $\psi\eta$ model combined with the $\psi\chi\eta$ model. We have

$$b_0[SU(N)] = 7N - 14 \tag{6.10}$$

$N \geq 3$, and the first coefficient of the beta function for $SU(n)$

$$b_0[SU(n)] = 7n - 14 - (N - n) = 8n - N - 14\ . \tag{6.11}$$

The change of sign happens at

$$n^* = \frac{N + 14}{8}\ . \tag{6.12}$$

(6.13)

N	$[n^*]$	$SU(N) \to \cdots$
3	2	$SU(3) \to SU(2) \times U(1)$
4	2	$SU(4) \to SU(2) \times SU(2) \times U(1)$
5	2	$SU(5) \to SU(2) \times SU(2) \times U(1)^2$
6	2	$SU(6) \to SU(2) \times SU(2) \times SU(2) \times U(1)^2$
$\vdots$	$\vdots$	$\vdots$
$N \to \infty$	$[n^*]$	$SU(N) \to \prod_1^7 SU([n^*]) \times SU(N - 7[n^*]) \times U(1)^7$

In Figure 5 the quiver diagrams showing the RG flow from UV to IR are shown.

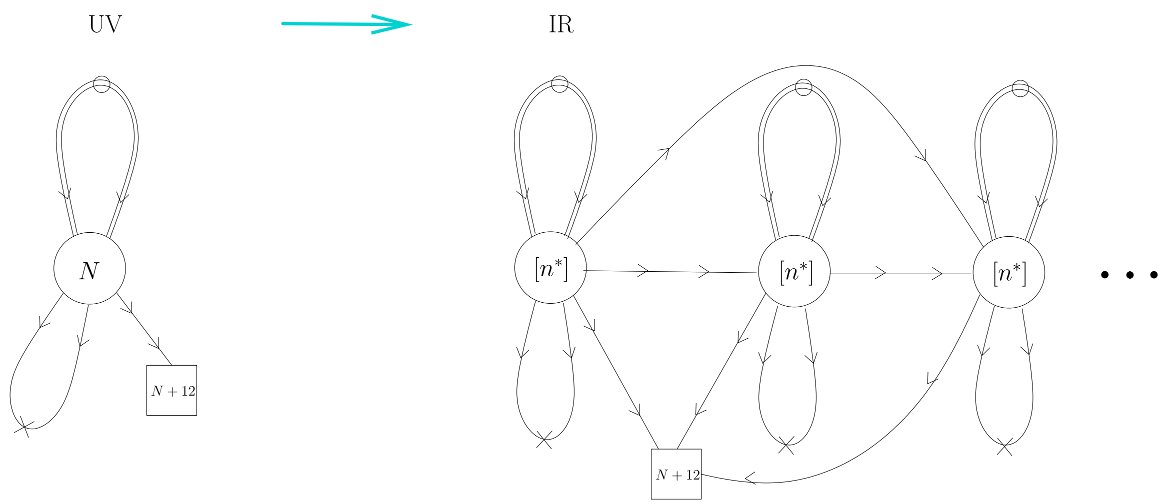

Figure 5: Example of type (I) is $(N_\psi, N_\chi) = (2, 1)$.

Another example of type (I) is $(N_\psi, N_\chi) = (2, 2)$:

$$2\,\yng(2) \oplus 2\,\bar{\yng(1,1)} \oplus 16\,\bar{\yng(1)}\ , \tag{6.14}$$

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This is two-family version of the $\psi\chi\eta$ model.

$$b_0[SU(N)] = 7N - 16 \tag{6.15}$$

$N \geq 3$. For $(2,2)$ the beta function of $SU(n)$ is

$$b_0[SU(N)] = 7n - 16 - 2(N-n) = 9n - 2N - 16 \tag{6.16}$$

The change of sign happens at

$$n^* = \frac{2N+16}{9} \ . \tag{6.17}$$

N	$[n^*]$	$SU(N) \to \cdots$
3	2	$SU(3) \to SU(2) \times U(1)$
4	2	$SU(4) \to SU(2) \times SU(2) \times U(1)$
5	2	$SU(5) \to SU(2) \times SU(2) \times U(1)^2$
6	3	$SU(6) \to SU(3) \times SU(3) \times U(1)$
$\vdots$	$\vdots$	$\vdots$
$N \to \infty$	$[n^*]$	$SU(N) \to \prod_1^4 SU([n^*]) \times SU(N-4[n^*]) \times U(1)^4$

(6.18)

In Figure 6 the quiver diagrams for the RG flow from UV to IR are illustrated.

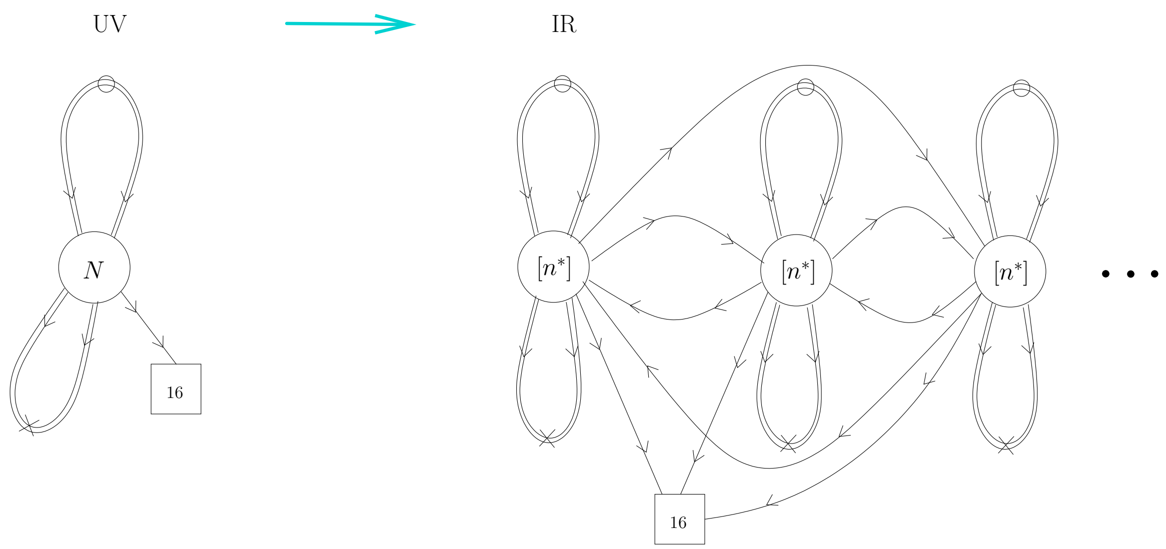

Figure 6: Example of type (I) is $(N_\psi, N_\chi) = (2,2)$.

A type (II) theory has fields

$$\psi^{\{ij\},a} \ , \qquad \chi_{[ij]}^{b} \ , \qquad \tilde{\eta}_i^c \ ,$$
$$a = 1, \ldots N_\psi \ , \qquad b = 1, \ldots N_\chi \ , \qquad c = 1, \ldots, N_\chi(N-4) - N_\psi(N+4) \ . \tag{6.19}$$

We assume that only one the ψ's pairs condenses with one of the χ's. The $\psi\chi$ condensate breaks the global symmetry as

$$\begin{aligned} & SU(N_\psi) \times SU(N_\chi) \times SU(N_\chi(N-4) - N_\psi(N+4)) \times U(1)^2 \\ & \longrightarrow SU(N_\psi - 1) \times SU(N_\chi - 1) \times SU(N_\chi(N-4) - N_\psi(N+4)) \times U(1)^3 \, . \end{aligned} \tag{6.20}$$

We assume that the condensate can be brought to a diagonal form like before (6.3) and that there a a block of n coefficients which are degenerate like (6.5) This leaves a non-Abelian unbroken gauge group as (6.6). The first coefficient of the beta function for $SU(n)$ is

$$b_0[SU(n)] = (11 - 2N_\chi)n + 2N_\psi + 6N_\chi - (N_\psi + N_\chi - 2)(N - n) \tag{6.21}$$
$$= (9 + N_\psi - N_\chi)n + (2 - N)N_\psi + (6 - N)N_\chi + 2N \tag{6.22}$$

when $n = N$ this is exactly the b_0 of the original $SU(N)$ theory (2.3) that we choose positive. For the $\psi\chi\eta$ model where $(N_\psi, N_\chi) = (1, 1)$ this is always the same $\psi\chi\eta$ model reduced to $SU(n)$, and $b_0 > 0$. Where any of N_ψ or N_χ is greater than 1 we can have a sign flip for certain values of n. For a certain value n^* we have a zero of (6.7). The change of sign for (6.7) happens at

$$n^* = \frac{(N_\psi + N_\chi - 2)N - 2N_\psi - 6N_\chi}{9 + N_\psi - N_\chi} \tag{6.23}$$

An example of type (II) is $(N_\psi, N_\chi) = (1, 2)$:

$$\square\!\square \oplus 2\,\overline{\begin{matrix}\square\\[-4pt]\square\end{matrix}} \oplus (N-12)\,\square \, , \tag{6.24}$$

$$b_0 = 11 \cdot N - (N+2) - 2(N-2) - (N-12) = 7N + 14 \tag{6.25}$$

$N \geq 12$. This for $N = 12$ is the $\psi\chi$ model with $k = 8$. All $\psi\chi$ models are particular cases of this class. For 12 the beta function of $SU(n)$ is

$$b_0[SU(n)] = 7n + 14 - (N - n) = 8n - N + 14 \tag{6.26}$$

The change of sign happens at

$$n^* = \frac{N - 14}{8} \, . \tag{6.27}$$

7. Strong anomaly

Up to now we discussed the IR dynamics of the theories focusing on the realization of their global symmetry group. However these theories, at the classical level, possess another independent $U(1)$ symmetry, broken by the ABJ anomaly, due to the nontrivial topological effects of the strong gauge interactions. The effect is sometimes simply called as the "strong anomaly". As is well-known in QCD (the $U_A(1)$ problem and its solution) [22–27] and in chiral theories [28] [6], it is actually useful to keep track of these symmetries, i.e., trying to reproduce their breaking in the

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IR theory appropriately. Let us briefly review how the story goes in QCD, before moving to the chiral theories of our interest.

In QCD the axial $U(1)_A$ symmetry is broken both by the ABJ anomaly and by the chiral condensate

$$\langle \bar{\psi}_L \psi_R \rangle \sim -\Lambda^3 \neq 0 \,. \tag{7.1}$$

If one imagines the anomaly as a small explicit breaking on top of the spontaneous breaking due to the condensate, then one should have, alongside to the pions, another NGB, with a small mass due to the anomalous breaking. This scenario is quantitatively correct in the large N limit.

Supposing for the moment that the anomalous breaking is small, one can describe it directly in the chiral Lagrangian,

$$L_0 = \frac{F_\pi^2}{2} \, \mathrm{Tr} \, \partial_\mu U \partial^\mu U^\dagger + \mathrm{Tr} \, M \, U + \mathrm{h.c.} + \dots \,, \qquad U \equiv \bar{\psi}_R \psi_L \tag{7.2}$$

by adding a strong-anomaly effective Lagrangian [22–27]

$$\hat{L} = \frac{i}{2} q(x) \, \log \det U/U^\dagger + \frac{N}{a_0 F_\pi^2} q^2(x) - \theta \, q(x) \,, \tag{7.3}$$

$q(x)$ is the topological density

$$q(x) = \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu} \,, \tag{7.4}$$

a_0 is a constant of the order of unity, F_π the pion decay constant, and θ is the QCD vacuum parameter. The $U(1)_A$ anomaly under

$$\Delta S = 2 N_{\mathrm{f}} \alpha \int d^4x \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu} \,, \qquad \psi_L \to e^{i\alpha} \psi_L \,, \quad \psi_R \to e^{-i\alpha} \psi_R \,, \tag{7.5}$$

is reproduced by the $\log \det U/U^\dagger$ term of the effective action. Treating $q(x)$ as an auxiliary field, and integrating, one gets another form of the anomaly term:

$$\hat{L} = -\frac{F_\pi^2 \, a_0}{4N} (\theta - \frac{i}{2} \log \det U/U^\dagger)^2 \,. \tag{7.6}$$

The multivalued function $\hat{L}$ is actually well defined because

$$\langle U \rangle \propto \mathbf{1} \neq 0 \,. \tag{7.7}$$

Moreover, as promised, this breaking is small at large N, so the idea of treating the anomalous breaking as a perturbation to the chiral Lagrangian is a consistent one.

Expanding (7.6) around this VEV,

$$U \propto e^{i \frac{\pi^a t^a}{F_\pi} + i \frac{\eta \, t^0}{F_\pi^{(0)}}} = \mathbf{1} + i \frac{\pi^a t^a}{F_\pi} + i \frac{\eta \, t^0}{F_\pi^{(0)}} + \dots \,, \tag{7.8}$$

one finds the mass term for the would-be NG boson, η. In real-world QCD the mass of this

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pseudo-NGB is large, but one can still identify it with the η meson (in the two-flavored QCD) or with the η' meson (in the three-flavored QCD).

This discussion on the solution of the so-called $U(1)_A$ problem is well known. An observation which might not be as familiar, is that the logic of the argument may be reversed: one can actually argue that the presence of an effective action (7.6) needed for reproducing the strong anomaly *implies* a nonvanishing condensate, $\langle U \rangle = \langle \bar{\psi}_R \psi_L \rangle \neq 0$, and hence, indirectly, also the spontaneous breaking of *nonanomalous* chiral symmetry itself,

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R \to U(1)_V \ , \tag{7.9}$$

affecting the low-energy physics.

In the following we discuss how the anomalous and nonanomalous $U(1)$ (and other) symmetries can be correctly reproduced in terms of a low-energy effective action, in the $\psi\eta$, $\tilde{\chi}\eta$, and in $\psi\chi\eta$ models.

7.1. $\psi\eta$ model

Let us apply these ideas to the chiral gauge theories we are interested in here. The simplest chiral gauge theory we studied from this viewpoint [6] is the $\psi\eta$ model.

Let us review briefly the symmetries of the model. In UV we have as global symmetry group $SU(N+4) \times U(1)_{\psi\eta}$: any combinations of $U(1)_\psi \times U(1)_\eta$ different form $U(1)_{\psi\eta}$ is broken by a strong anomaly. We wish to describe this anomalous breaking in the low-energy effective Lagrangian. It is particularly convenient to chose the independent anomalous $U(1)$ symmetry as

$$U(1)_{an}: \begin{cases} \psi \to e^{i\alpha}\psi \ , \\ \eta \to e^{-i\alpha}\eta \ . \end{cases} \tag{7.10}$$

We note that the anomalous divergence

$$\partial_\mu J^\mu_{an} = \{(N+2) - (N+4)\} \frac{g^2}{32\pi^2} tr[F_{\mu\nu}F^{\mu\nu}] = -2\frac{g^2}{32\pi^2} tr[F_{\mu\nu}F^{\mu\nu}] \ , \tag{7.11}$$

scales as $\mathcal{O}(1)$ for $N \to \infty$, suggesting that, for large N, a situation similar to the one in QCD (with large N but fixed N_f): the anomalous breaking is a small effect.

Let us assume the color-flavor locked phase, discussed in Sec. 4, characterized by the bifermion condensate $\langle \psi\eta \rangle \sim \Lambda^3$ in (4.5). The system breaks the gauge and part of the global symmetries dynamically,

$$SU(N)_c \times SU(N+4)_\eta \times U(1)_{\psi\eta} \to SU(N)_{cf} \times SU(4)_f \times \tilde{U}(1) \ . \tag{7.12}$$

In order to understand the low-energy effective action, one must first identify correctly all the NG bosons. From the symmetry breaking one expects to find $8N$ nonAbelian NG bosons relative to $\frac{SU(N+4)}{SU(N)\times SU(4)\times U(1)_D}$, interpolated in a gauge invariant fashion by

$$\phi^A = (\psi^{ij}\eta_j^a)^*(T^A)_b^a(\psi^{ik}\eta_k^b) \ . \tag{7.13}$$

Here T_A are the $8N$ broken generators that connect the N dimensional subspace (where $SU(N)$ acts) and the 4 dimensional one (where $SU(4)$ acts).

The Abelian symmetries in this model are slightly subtle. The non-anomalous but spontaneously broken $U(1)_{\psi\eta}$ symmetry (which implies a massless NG boson) and anomalous (but unlike $U(1)_A$ in QCD, not-spontaneously-broken) $U(1)_{an}$ symmetry, must both correctly be reproduced in the low-energy effective action, analogous to the strong-anomaly effective action of QCD, (7.6). As almost any combination of $U(1)_{\psi\eta}$ and $U(1)_{an}$ is broken both by the condensate and by the anomaly one expects to have a massive (would be NG-) boson, alongside an exactly massless physical NG boson. The problem is that it is not straightforward to identify what kind of interpolating fields, constructed from ψ and η fields describe correctly the low-energy effective action, satisfying such requirements. Note that in the case of QCD, the flavor-singlet combination in $U = \bar{\psi}_R\psi_L$ describes indeed the massive would-be NG boson, η (or η'), see (7.6) and (7.8).

We start from the UV theory,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}^{\rm fermions} \,, \tag{7.14}$$

$$\mathcal{L}^{\rm fermions} = -i\overline{\psi}\bar{\sigma}^{\mu}\left(\partial + \mathcal{R}_{\rm S}(a)\right)_{\mu}\psi \; - i\overline{\eta}\bar{\sigma}^{\mu}\left(\partial + \mathcal{R}_{\rm F^*}(a)\right))_{\mu}\eta \,, \tag{7.15}$$

where a is the $SU(N)$ gauge field, and the matrix representations appropriate for ψ and η fields are indicated with $\mathcal{R}_{\rm S}$ and $\mathcal{R}_{\rm F^*}$. To match with the IR effective Lagrangian it is useful to perform a change of variable

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}^{\rm fermions} + {\rm Tr}[(\psi\eta)^*U] + {\rm h.c.} + B\,(\psi\eta\eta)^* + {\rm h.c.} \,, \tag{7.16}$$

where U is the composite scalars of $N \times (N+4)$ color-flavor mixed matrix form,

$$ {\rm Tr}[(\psi\eta)^*U] \equiv (\psi^{ij}\eta_j^m)^*U^{im} \,, \tag{7.17}$$

and B are the baryons $B \sim \psi\eta\eta$,

$$B^{mn} = \psi^{ij}\eta_i^m\eta_j^n \,, \qquad m,n = 1,2,\ldots,N+4 \,, \tag{7.18}$$

antisymmetric in $m \leftrightarrow n$. In writing down the Lagrangian (7.16) we have anticipated the fact that these baryon-like composite fields, present in the Higgs phase together with the composite scalars $\psi\eta$, are also needed to write down the strong-anomaly effective action.

Integrating ψ and η out, one gets

$$\mathcal{L}^{\rm eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + {\rm Tr}(\mathcal{D}U)^{\dagger}\mathcal{D}U - i\overline{B}\,\bar{\sigma}^{\mu}\partial_{\mu}B - V \,. \tag{7.19}$$

The potential V is assumed to be such that its minimum is of the form:

$$\langle U^{im}\rangle = c_{\psi\eta}\,\Lambda^3\delta^{im} \,, \qquad i,m = 1,2,\ldots N \,, \tag{7.20}$$

and contains the strong anomaly term,

$$V = V^{(0)} + \hat{L}_{\rm an} \ . \tag{7.21}$$

$\hat{L}_{\rm an}$ of the form,

$$\hat{L}_{\rm an} = {\rm const} \, \Big[\log \left(\epsilon \, BB \, \det U \right) - \log \left(\epsilon \, BB \, \det U \right)^{\dagger} \Big]^2 \tag{7.22}$$

which is analogue of (7.6) in QCD. The argument of the logarithm

$$\epsilon \, BB \, \det U \equiv \epsilon^{m_1, m_2, \ldots, m_{N+4}} \epsilon^{i_1, i_2, \ldots, i_N} B_{m_{N+1}, m_{N+2}} B_{m_{N+3}, m_{N+4}} U_{i_1 m_1} U_{i_2 m_2} \ldots U_{i_N m_N} \ , \tag{7.23}$$

is invariant under the full (nonanomalous) symmetry,

$$SU(N)_{\rm c} \times SU(N+4) \times U(1)_{\psi\eta} \tag{7.24}$$

as it should be. Moreover it contains $N+2$ ψ's and $N+4$ η's, the correct numbers of the fermion zeromodes in the instanton background: it corresponds to a 't Hooft's instanton n-point function, e.g.,

$$\langle \psi\eta\eta(x_1) \psi\eta\eta(x_2) \psi\eta(x_3) \ldots \psi\eta(x_{N+2}) \rangle \ . \tag{7.25}$$

This effective Lagrangian is well defined only if the argument of the logarithm takes a VEV. In particular it is natural to assume

$$\langle \epsilon^{(4)} BB \rangle \neq 0 \ , \qquad \langle \det U \rangle \neq 0 \ , \tag{7.26}$$

where

$$\epsilon^{(4)} BB = \epsilon_{\ell_1 \ell_2 \ell_3 \ell_4} B^{\ell_1 \ell_2} B^{\ell_1 \ell_2} \ , \qquad \ell_i = N+1, \ldots, N+4 \ . \tag{7.27}$$

As

$$\langle \det U \rangle \propto \mathbf{1}_{N \times N} \tag{7.28}$$

takes up all flavors up to N (the flavor $SU(N+4)$ symmetry can be used to orient the symmetry breaking this way), BB must be made of the four remaining flavors, as in (7.27). These baryons were not among those considered in the earlier studies [5, 11], but assumed to be massless here, and indicated as $B^{[A_2 B_2]}$ in Table 4. This is possible because these fermions do not have any perturbative anomaly with respect to the unbroken symmetry group, $SU(N) \times SU(4) \times U(1)$: 't Hooft anomaly matching considerations cannot tell if they are massive or not, either of the two options is possible.

Now we see how the apparent difficulty about the NG bosons hinted at above is solved. We can define the interpolating fields of the two NG bosons by expanding the condensates,

$$\begin{aligned} \det U &= \langle \det U \rangle + \ldots \propto \mathbf{1} + \frac{i}{F_{\pi}^{(0)}} \, \phi_0 + \ldots \ ; \\ \epsilon^{(4)} BB &= \langle \epsilon^{(4)} BB \rangle + \ldots \propto \mathbf{1} + \frac{i}{F_{\pi}^{(1)}} \, \phi_1 + \ldots \ , \end{aligned} \tag{7.29}$$

(here $F_{\pi}^{(0)}$ and $F_{\pi}^{(1)}$ are some constants with dimension of mass). Clearly in general the physical

NG boson and the anomalous would-be NG boson will be interpolated by *two* linear combinations of ϕ_0 and ϕ_1. The effective Lagraingian determines these linear combinations.

Indeed, the effective Lagrangian (7.22) is invariant under the nonanomalous symmetry group, and in particular $U(1)_{\psi\eta}$. The boson which appears in the strong-anomaly effective action as the fluctuation of $\epsilon BB\ \det U$,

$$\tilde{\phi} \equiv N_\pi \left[\frac{1}{F_\pi^{(0)}}\,\phi_0 + \frac{1}{F_\pi^{(1)}}\,\phi_1 \right] , \qquad N_\pi = \frac{F_\pi^{(0)} F_\pi^{(1)}}{\sqrt{\left(F_\pi^{(0)}\right)^2 + \left(F_\pi^{(1)}\right)^2}} , \tag{7.30}$$

cannot be the massless physical one: it is the would-be NG boson relative to the anomalous symmetry. Indeed the effective action provides a mass term for this would-be NG boson.

The orthogonal combination

$$\phi \equiv N_\pi \left[\frac{1}{F_\pi^{(1)}}\phi_0 - \frac{1}{F_\pi^{(0)}}\phi_1 \right] , \tag{7.31}$$

is the interpolating field of the physical NG boson, remaining massless.

Before we have included in the low-energy description some massless baryons which are neither required nor excluded by the 't Hooft anomaly matching. Now one can see their ultimate fate using the strong-anomaly effective Lagrangian. In particular (7.22) contains a 4-fermion coupling between these baryons, which, plugging the VEVs (7.26), provides a mass term for them. This realizes the general expectation that the system chooses the smallest amount of massless matter needed to satisfy 't Hooft anomaly matching, in line with a-theorem and the ACS condition.

In the above we have assumed that the system is in dynamical Higgs phase (as discussed in Sec. 4) and we have found out how the nonanomalous and anomalous symmetries of the model is correctly described in the low-energy effective action. As emphasized in [6], it does not seem to be possible to write down a strong-anomaly effective action, if one assumed that the system in the hypothetical, confining, symmetric vacuum, reviewed in Sec. 3. The reason is that by using only the presumed massless degrees of freedom of the low-energy theory (the baryons (3.4)) it is not possible to saturate all the fermion zeromodes needed to write down a local Lagrangian transforming appropriately under the anomalous $U(1)$ transformation.

7.2. $\chi\eta$ model

It is an interesting exercise to apply the same reasoning about the strong anomaly to the $\chi\eta$ model. We will find that there are good analogies with the $\psi\eta$ case studied above, but also quite significant differences. In this model any combination of $U(1)_\chi$ and $U(1)_\eta$, except $U(1)_{\chi\eta}$, is anomalous, therefore some term similar to (7.6) (in QCD) should appear.

In the dynamical Higgs scenario for the $\chi\eta$ model, there are two bi-fermion condensates,

$$\langle \chi^{ij} \eta_j^m \rangle = c_{\chi\eta}\, \delta^{im}\, \Lambda^3 , \qquad i, m = 1, 2, \ldots , N-4 , \tag{7.32}$$

and

$$\langle \chi\chi \rangle \neq 0 . \tag{7.33}$$

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This implements a two-step breaking,

$$\begin{aligned} SU(N) \times SU(N-4) \times U(1)_{\chi\eta} &\xrightarrow{\langle\chi\eta\rangle} SU(N-4)_{\rm cf} \times SU(4)_{\rm c} \times U(1)' \\ &\xrightarrow{\langle\chi\chi\rangle} SU(N-4)_{\rm cf} \times U(1)' \ . \end{aligned} \tag{7.34}$$

As before, in order to construct a fully consistent effective action, one should keep the full invariance of the original theory, either spontaneously broken or not. To do so, it is convenient to re-express the condensates (7.32) in a gauge invariant way. The answer is

$$\begin{aligned} U &= \epsilon_{i_1 i_2 \ldots i_N} \epsilon_{m_1 m_2 \ldots m_{N-4}} (\chi\eta)^{i_1 m_1} (\chi\eta)^{i_2 m_2} \ldots (\chi\eta)^{i_{N-4} m_{N-4}} \chi^{i_{N-3} i_{N-2}} \chi^{i_{N-1} i_N} \\ &\sim \epsilon\, (\chi\eta)^{N-4} (\chi\chi) \ , \end{aligned} \tag{7.35}$$

which encodes both types of the two possible composite scalar fields.

This choice suggests that the correct strong-anomaly effective action for the $\chi\eta$ model is

$$\frac{i}{2} q(x) \log \epsilon (\chi\eta)^{N-4} (\chi\chi) + \text{h.c.} \ , \tag{7.36}$$

where, again, $q(x)$ is the topological density defined in Eq.(7.4). Clearly it is by construction invariant under the whole (nonanomalous) symmetry group

$$SU(N)_{\rm c} \times SU(N-4) \times U(1)_{\chi\eta} \ . \tag{7.37}$$

This anomaly effective action (7.36) agrees with the one proposed by Veneziano [28] for the case of $SU(5)$, and generalizes it to all $SU(N)$ $\chi\eta$ models. A key observation we share with [28] and generalizes to models with any N, is that this strong anomaly effective action, which should be there in the low-energy theory to reproduce correctly the (anomalous and nonanomalous) symmetries of the UV theory, *implies* nonvanishing condensates,

$$\langle\chi\eta\rangle \neq 0 \ , \qquad \langle\chi\chi\rangle \neq 0 \ , \tag{7.38}$$

i.e., that the system is in dynamical Higgs phase.

Up to now the story has been very similar to the one about the $\psi\eta$ model. However there are some differences. Differently from the $\psi\eta$ model, where the baryon condensate must enter in the strong-anomaly effective action, here the structure of the effective action is simpler, and no baryonlike composites are needed. Moreover, contrary to the $\psi\eta$ model, the $\chi\eta$ system has no physical $U(1)$ NG boson: it is eaten by a color $SU(N)$ gauge boson. However the counting of the broken and unbroken $U(1)$ symmetries is basically similar in the two models. Of the two nonanomalous symmetries ($U(1)_{\rm c}$ and $U(1)_{\chi\eta}$), a combination remains a manifest physical symmetry, and the other becomes the longitudinal part of a color gauge boson. Still another, anomalous, $U(1)$ symmetry exists, which is any combination of $U(1)_\chi$ and $U(1)_\eta$ other than $U(1)_{\chi\eta}$. This symmetry is also spontaneously broken, hence it must be associated with a NG boson, even though it will get mass by the strong anomaly.

As in the $\psi\eta$ model, one can describe this situation explicitly, by expanding the composite $\chi\eta$

and $\chi\chi$ fields around their VEV's,

$$(\det U)' = \langle (\det U)' \rangle + \ldots \propto \mathbf{1} + i\frac{1}{F_\pi^{(0)}}\,\phi_0' + \ldots\ ;$$
$$\chi\chi = \langle \chi\chi \rangle + \ldots \propto \mathbf{1} + i\frac{1}{F_\pi^{(1)}}\,\phi_1' + \ldots\ , \tag{7.39}$$

where $(\det U)'$ is defined in the $N-4$ dimensional color-flavor mixed space, and

$$\chi\chi \equiv \epsilon_{i_1,i_2,i_3,i_4}\chi^{i_1 i_2}\chi^{i_3 i_4}\ , \qquad N-3 \le i_j \le N\ . \tag{7.40}$$

Now one can see that the strong-anomaly effective action (7.36) gives mass to

$$\tilde{\phi}' \equiv N_\pi \left[\frac{1}{F_\pi^{(0)}}\phi_0' + \frac{1}{F_\pi^{(1)}}\phi_1' \right]\ , \qquad N_\pi = \frac{F_\pi^{(0)} F_\pi^{(1)}}{\sqrt{(F_\pi^{(0)})^2 + (F_\pi^{(1)})^2}}\ , \tag{7.41}$$

whereas an orthogonal combination

$$\phi' \equiv N_\pi \left[\frac{1}{F_\pi^{(1)}}\phi_0' - \frac{1}{F_\pi^{(0)}}\phi_1' \right] \tag{7.42}$$

remains massless. The latter corresponds to the potential NG boson which is absorbed by the color $T_{\rm c}$ gauge boson.

7.3. $\psi\chi\eta$ model

It turns out to be quite an instructive exercise to study the IR effective action of the $\psi\chi\eta$ model. On the one hand, the strong anomaly manifests differently in the dynamically Abelianized $\psi\chi\eta$ system than in the $\psi\eta$ or $\tilde{\chi}\eta$ models studied above. Also, the way the nonanomalous but spontaneously broken $U(1)_{\psi\chi}$ symmetry is realized in the IR reveals some unusual, interesting features, on the other.

As explained in Sec. 5, the condensate $<\psi\chi>$ breaks the global and the gauge symmetry group

$$SU(N)_{\rm c} \times SU(8)_{\rm f} \times \tilde{U}(1) \times U(1)_{\psi\chi} \longrightarrow \prod_{\ell=1}^{N-1} U(1)_\ell \times SU(8)_{\rm f} \times \tilde{U}(1)\ . \tag{7.43}$$

There are three $U(1)$ symmetries in the model [8], two non-anomalous ones $\tilde{U}(1)$ and $U(1)_{\psi\chi}$ and an anomalous one $U(1)_{\rm an}$. It is convenient, to decouple the different effects we want to exhibit, to take the anomalous one to be

$$U(1)_{an} : \begin{cases} \psi \to e^{i\gamma}\psi\ , \\ \chi \to e^{-i\gamma}\chi\ , \\ \eta \to \eta\ . \end{cases} \tag{7.44}$$

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The condensate (5.1) does not break it.

The associated currents are

$$J^{\mu}_{\psi\chi} = i\left\{\tfrac{N-2}{N^*}\,\bar{\psi}\bar{\sigma}^{\mu}\psi - \tfrac{N+2}{N^*}\,\bar{\chi}\bar{\sigma}^{\mu}\chi\right\} \ , \qquad \partial_{\mu}J^{\mu}_{\psi\chi} = 0 \ , \tag{7.45}$$

$$\tilde{J}^{\mu} = i\left\{2\,\bar{\psi}\bar{\sigma}^{\mu}\psi - 2\,\bar{\chi}\bar{\sigma}^{\mu}\chi - \bar{\eta}^{a}\bar{\sigma}^{\mu}\eta^{a}\right\} \ , \qquad \partial_{\mu}\tilde{J}^{\mu} = 0 \ , \tag{7.46}$$

$$J^{\mu}_{\rm an} = i\bar{\psi}\bar{\sigma}^{\mu}\psi - i\bar{\chi}\bar{\sigma}^{\mu}\chi \ , \qquad \partial_{\mu}\, J^{\mu}_{\rm an} = \frac{2g^2}{32\pi^2}G_{\mu\nu}\tilde{G}^{\mu\nu} \ . \tag{7.47}$$

Among $U(1)_{an}$, $\tilde{U}(1)\,U(1)_{\psi\chi}$, only $U(1)_{\psi\chi}$ is broken by the condensate, one expects a single NG oson, π. At this point one can start to write down an effective Lagrangian

$$\mathcal{L}^{(eff)} = \mathcal{L}(\psi,\eta,A^{(i)}_{\mu}) + \mathcal{L}(\pi) - \mathcal{V}(\pi,\psi,\eta) + \dots \ , \tag{7.48}$$

with π, the photons, A^k, $k = 1,...,N-1$, and the massless fermions, ψ^{ii},η^{A}_{i}. Now we can use $U(1)_{an}$ and $U(1)_{\psi\chi}$ to learn as much as possible about $\mathcal{L}^{(eff)}$.

$U(1)_{an}$ is an (anomalously) broken symmetry, therefore the IR lagrangian must break it. In terms of the UV fields, the anomalous conservation equation leads to the 't Hooft vertices,

$$\epsilon_{a_1a_2\dots a_8}\underbrace{(\psi\chi)^{i}_{j}(\psi\chi)^{j}_{k}\dots(\psi\chi)^{p}_{i}}_{N-2}\underbrace{(\psi\eta^{a_1}\eta^{a_2})\dots(\psi\eta^{a_7}\eta^{a_8})}_{4} \ , \tag{7.49}$$

where a possible (certainly not unique) way to contract the color $SU(N)$ and the flavor $SU(8)$ indices in an invariant way is shown. By defining $U(x)$

$$U(x) = (\psi\chi)^{1}_{1}(x) = \text{const.}\,\Lambda^3\,e^{i\pi(x)/F} \tag{7.50}$$

one can capture this contribution in the IR effective theory by including

$$\mathcal{V}(\pi,\psi,\eta)(x) \sim U(x)^{N-2}\,\underbrace{\psi\cdots\psi}_{4}\,\underbrace{\eta\,\eta\cdots\eta}_{8}+h.c. = \text{const.}\,\pi\pi\dots\pi\ \ \underbrace{\psi\cdots\psi}_{4}\,\underbrace{\eta\,\eta\cdots\eta}_{8}+h.c. \ , \tag{7.51}$$

with any number of pions, four ψ's and eight η's. This term is invariant under the full UV symmetries $SU(N)_{\rm c}\times SU(8)_{\rm f}\times\tilde{U}(1)\times U(1)_{\psi\chi}$, and *a fortiori* with the unbroken symmetries $\prod_{\ell=1}^{N-1}U(1)_{\ell}\times SU(8)_{\rm f}\times\tilde{U}(1)$, but clearly not under the anomalous $U(1)_{an}$.

The effect of the $U(1)_{an}$ anomaly, however, is not exhausted in the explicit breaking of $U(1)_{an}$ symmetry in $\mathcal{V}(x)$. As the $U(1)_{an}$ charge of the low-energy, massless fermions is well defined,

$$\psi^{ii}\to e^{i\gamma}\psi^{ii} \ , \quad \eta^{A}_{i}\to\eta^{A}_{i} \ , \tag{7.52}$$

it manifests itself also through the massless ψ fermion loops,

$$J^{\mu}_{\rm an} = i\sum_{i=1}^{N}\bar{\psi}_{ii}\bar{\sigma}^{\mu}\psi^{ii} \ , \qquad \partial_{\mu}J^{\mu}_{\rm an} = \frac{1}{16\pi^2}\sum_{j=1}^{N-1}e_j^2\,F^{(j)}_{\mu\nu}\tilde{F}^{(j)\,\mu\nu} \ . \tag{7.53}$$

Such an anomaly has a natural interpretation as a remnant of the original strong anomaly (7.47) in the UV theory. The original strong anomaly divergence equation has turned into the anomalous divergences due to the weak $U(1)^{N-1}$ gauge interactions of the low-energy theory.

Let us now turn our attention to $U(1)_{\psi\eta}$. This is an exact symmetry of the UV theory, so it must be (although non-linearly realized) symmetry in the IR theory. In particular, in IR, this becomes the shift symmetry for the NGB,

$$\pi(x) \to \pi(x) - \frac{4F}{N^*}\beta \;, \qquad U(x) \to e^{-\frac{4i\beta}{N^*}}\, U(x) \;, \tag{7.54}$$

but still act non-trivially on ψ^{ii}, $i = 1, 2, \ldots, N$:

$$\psi^{ii} \to e^{i\frac{N-2}{N^*}\beta}\psi^{ii} \;. \tag{7.55}$$

But now the anomaly due to the ψ^{ii} loops,

$$\Delta\mathcal{L}^{eff} = \frac{N-2}{N^*}\beta\frac{1}{16\pi^2}\sum_{j=1}^{N-1} e_j^2\, F_{\mu\nu}^{(j)}\,\tilde{F}^{(j)\,\mu\nu} \tag{7.56}$$

is not cancelled, leading to the apparent paradox: $U(1)_{\psi\chi}$ is a nonanomalous (exact) symmetry of the system, but in the low-energy effective theory it seems to be broken by anomaly! The answer to this puzzle is that the low-energy effective Lagrangian (7.48) contains an axion-like term

$$\mathcal{L}(\pi, A_\mu^{(i)}) = \pi(x)\,\frac{N-2}{4F}\,\frac{1}{16\pi^2}\sum_{j=1}^{N-1} e_j^2\, F_{\mu\nu}^{(j)}\,\tilde{F}^{(j)\,\mu\nu} \tag{7.57}$$

which transforms under (7.54) as

$$\Delta\mathcal{L}(\pi, A_\mu^{(i)}) = -\frac{N-2}{N^*}\beta\frac{1}{16\pi^2}\sum_{j=1}^{N-1} e_j^2\, F_{\mu\nu}^{(j)}\,\tilde{F}^{(j)\,\mu\nu} \tag{7.58}$$

cancelling exactly the anomaly due to the ψ^{ii} loops, (7.56), ensuring the $U_{\psi\chi}(1)$ invariance of the system.

8. Conclusion

We considered in this work a large class of simple SU(N) chiral gauge theories which admit large N limit, some of which are relatively unexplored. We discussed possible IR effective theories, realization of symmetries and matching of anomalies. In some limited class of models, conventional 't Hooft anomaly matching constraints alone, appears to suggest that it is possible that the system flows into a confinement phase, without global symmetry breaking, i.e., with no condensates forming. However, the considerations of tighter constraints following from the generalized symmetries and mixed anomalies [4, 5], as well as those based on the strong anomaly [6], clearly disfavor such a dynamical possibility. Dynamical Higgs mechanism, instead, is a very natural solution, and appears to be consistent. It is also consistent with the mixed-anomaly

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matching as well as with the strong anomaly constraints. A particular variation of dynamical Higgs phase, which might occur in some chiral gauge theories with composite bifermion scalar field in the adjoint representation of $SU(N)$, is dynamical Abelianizatio, discussed in a separate section (Sec. 5) above.

Which is the "real" channel of condensation is a hard dynamical question. A maximally attractive channel (MAC) idea suggests that the condensate form in the most strongly attractive bi-fermion composite state. But such a naïve consideration based on the one-gluon exchange picture cannot tell which is the right answer, when there are several possible competing condensates having similar attractive strength. Dynamical arguments would be required to go further.[8] Another line of development is the adiabatic continuity to a semiclassical regime using compactification. This was started for chiral theories in [19] and very recently applied to the $\psi\chi\eta$ model [38]. The result [38] is consistent with dynamical Abelianization [8].

We considered effective actions of massless fermions and NB bosons. We studied the different types of realization of strong anomaly in the IR. There are peculiar features of chiral theories regarding discrete symmetries and generalized symmetries we have not discussed here (see [7] for a review).

An interesting open problem for the future is to write the effective action for all cases, and especially for the ones with non-Abelian symmetry breaking patterns in IR.

An interesting line for future developments is to study the soliton in the effective Lagrangian. In large N QCD, the Skyrmions of the chiral Lagrangian are identified with the baryon. Their mass scales like N and the WZW provides the baryon charge to the soliton. The same happens in (S)QCD and (A)QCD with heavy baryons made of $\frac{N(N\pm1)}{2}$ quarks [51]. Large N solitons and heavy baryons should appear also in these chiral theories, and hopefully they would be related in some way to the effective actions we constructed.

The hope is that some of these theories will turn out to be useful in the context of realistic model building. The Glashow-Weinberg-Salam (GWS) $SU(2)_L \times U(1)_Y$ theory (as well as its GUT generalizations) is a weakly coupled theory and, as such, is well defined and well understood within the perturbation theory framework. But this also means that the theory should be regarded, at best, as a very good low-energy effective theory. In particular, it is unlikely that the gauge symmetry breaking sector described by a potential term for the Higgs scalar, though phenomenologically quite successful, is a self-consistent, fundamental description. Nevertheless, attempts to replace it by new, QCD-like strongly-coupled gauge theories (technicolor, extended technicolor, walking technicolor, etc.) have not been fully successful so far. At the same time, our rather limited understanding of *strongly-coupled chiral gauge theories* has been hindering us from making a concrete progress by using them so far. We hope that our new proposals involving the low-energy effective non-Abelian gauge symmetries could find a phenomenological application in near future.

[8] Some attempts have been made recently, by starting from a supersymmetric version of the models, and by adding a small susy-breaking term as perturbation. In some cases, they show that a sort of dynamical Higgs phase is realized [43, 49, 50]. It is however not easy to extend these results when supersymmetry breaking becomes large. In particular, the most bifermion condensates appearing in the dynamical Higgs phase, discussed in Sec. 4 here, are forbidden by supersymmetry, meaning that the dynamical Higgs phases in these chiral theories cannot be reached from the supersymmetric version of the models, by susy-breaking perturbation.

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