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PAPER

A scaling limit for Vlasov equations with electrostatic fluctuations

Franco Flandoli[1,*] **and Dejun Luo**[2,3,*]

[1] Scuola Normale Superiore of Pisa, Piazza dei Cavalieri 7, 56124 Pisa, Italy
[2] SKLMS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, Peoples' Republic of China
[3] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China
[*] Authors to whom any correspondence should be addressed.

E-mail: franco.flandoli@sns.it and luodj@amss.ac.cn

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Abstract

We consider a Vlasov equation for a plasma with a given constant magnetic field, and introduce a white noise perturbation of the electric field in the electrostatic approximation, with a discussion of the motivations of such random perturbation. We prove that diffusion in velocity emerges in a suitable scaling limit of the noise, and also discuss the physical relevance of this result.

1. Introduction

Consider the Vlasov-type equation

$$\partial_t f + v \cdot \nabla_x f + (E_\rho + Bv \times \boldsymbol{e}_3) \cdot \nabla_v f = 0,$$

where $f = f(x,v,t)$ is the density of plasma particles in the phase space $\mathbb{T}^3 \times \mathbb{R}^3$, $\rho(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\,\mathrm{d}v$; $B \in \mathbb{R}$ is a constant, $\boldsymbol{e}_3 = (0,0,1)^*$ and $v \times \boldsymbol{e}_3 = (v_2, -v_1, 0)^*$; moreover,

$$E_{\rho(t)}(x) = \int_{\mathbb{T}^3} \nabla G(x-y)\rho(y,t)\,\mathrm{d}y,$$

G being the Green function on $\mathbb{T}^3$, $G(x) \approx -C|x|^{-1}$ for x near the origin. E_ρ stands for the electric field associated to the particle density, while $B\boldsymbol{e}_3$ represents some fixed external magnetic field.

Here we briefly discuss how to introduce noise in the above equation, see section 5 for a more detailed physical explanation. Assume that $f = \bar f + \tilde f$, where $\bar f$ (resp. $\tilde f$) represents the large (resp. small) scale part of the density; accordingly, we have $\rho = \int \bar f \mathrm{d}v + \int \tilde f \mathrm{d}v =: \bar\rho + \tilde\rho$, and the equation for $\bar f$ becomes

$$\partial_t \bar f + v \cdot \nabla_x \bar f + (E_{\bar\rho} + Bv \times \boldsymbol{e}_3) \cdot \nabla_v \bar f + E_{\tilde\rho} \cdot \nabla_v \bar f = 0.$$

Motivated by stochastic model reduction, we approximate the electric field $E_{\tilde\rho}$ corresponding to small scale density by some Gaussian random field

$$W = W(x,t) = \sum_k \sigma_k(x) W_t^k,$$

where $\{\sigma_k\}_k$ are some smooth gradient vector fields on $\mathbb{T}^3$ and $\{W^k\}_k$ are independent standard Brownian motions. We stress a major difference from recent works mentioned below, where the noise is often assumed to be divergence-free in the space variable, while here it is of gradient type since it arises from small scale fluctuations of electric field; a key point is that the noise is divergence-free in the velocity variable.

We can write the above equation as a stochastic partial differential equation with transport noise:

$$\mathrm{d}\bar{f}+v\cdot\nabla_x\bar{f}\mathrm{d}t+(E_{\bar{\rho}}+Bv\times\boldsymbol{e}_3)\cdot\nabla_v\bar{f}\,\mathrm{d}t+\sum_k\sigma_k\cdot\nabla_v\bar{f}\circ\mathrm{d}W_t^k=0.$$

Here, $\circ d$ means the Stratonovich stochastic differential, a choice which is motivated by the Wong–Zakai principle (see [4]). This is the equation treated below and we shall omit the bar in $\bar{f}$ to simplify notation.

Technically speaking, we have been motivated, to introduce the above noise and investigate the scaling limit, on one side by a series of recent works on a new scaling limit based on the Itô–Stratonovich duality (see e.g. [10, 16, 25] and more references at the beginning of section 1.2), and on the other side by the work [2] which contains two very interesting scaling limit results related to turbulent perturbations. Our approach and assumptions are technically very different and therefore complement the study of [2] by a different viewpoint. From the physical side, moreover, we have been strongly motivated by the increasing knowledge and understanding of turbulence in plasma. As an example of works discussing topics and ideas which may connect with our noise model and limit result, let us mention [21, 22] which insist on coherent dynamical structures; our noise is composed of random structures mimicking that intuition. We also mention [9, 27] which discuss in particular the role that turbulence may have in restoring a form of chaoticity typical of molecular collisions, in a framework like non-collisional plasma that *a priori* could then miss any statistical mechanics approach. In section 5.4 we shortly discuss this issue in more detail.

1.1. Weak existence of stochastic Vlasov equations

We will write the time variable t as a subscript to save space. As discussed in the last part, we consider the stochastic Vlasov equation

$$\mathrm{d}f+v\cdot\nabla_x f\mathrm{d}t+(E_\rho+Bv\times\boldsymbol{e}_3)\cdot\nabla_v f\,\mathrm{d}t+\sum_k\sigma_k\cdot\nabla_v f\circ\mathrm{d}W_t^k=0, \tag{1.1}$$

where $\rho_t(x)=\int f_t(x,v)\,\mathrm{d}v$ and

$$E_{\rho_t}(x)=(\nabla G*\rho_t)(x)=\int_{\mathbb{T}^3}\nabla G(x-y)\rho_t(y)\,\mathrm{d}y.$$

For simplicity, we assume the noise $W(x,t)=\sum_k\sigma_k(x)W_t^k$ in (1.1) is translation-invariant and smooth in the space variable, namely, the covariance function

$$Q(x-y)=\sum_k\sigma_k(x)\otimes\sigma_k(y)$$

satisfies $Q\in C_b^\infty(\mathbb{T}^3)$. Moreover, we can assume that

$$Q(0)=\sum_k\sigma_k(x)\otimes\sigma_k(x)=2\kappa I_3 \tag{1.2}$$

for some $\kappa>0$, I_3 being the 3×3 unit matrix; see example 2.4 below for mathematical construction of such noises. Under these assumptions, equation (1.1) can be written in Itô form as

$$\mathrm{d}f+v\cdot\nabla_x f\mathrm{d}t+(E_\rho+Bv\times\boldsymbol{e}_3)\cdot\nabla_v f\,\mathrm{d}t+\sum_k\sigma_k\cdot\nabla_v f\,\mathrm{d}W_t^k=\kappa\Delta_v f\mathrm{d}t, \tag{1.3}$$

where $\kappa\Delta_v$ is the Itô–Stratonovich corrector; we remark that this operator is a 'fake' dissipation term since it is exactly cancelled by the martingale part in energy estimates. We endow the above equation with an initial condition $f_0\in L^1(\mathbb{T}^3\times\mathbb{R}^3,\mathbb{R}_+)$, together with further regularity to be specified later. The meaning of weak solution to (1.3) is defined as usual against test functions $\phi\in C_c^\infty(\mathbb{T}^3\times\mathbb{R}^3)$. The main result of this part is the existence of weak solutions to (1.3) which will be proved in section 3.

Theorem 1.1 (Existence). *Assume that the initial condition $f_0\in(L^1\cap L^3)(\mathbb{T}^3\times\mathbb{R}^3,\mathbb{R}_+)$ has finite kinetic energy*

$$\mathcal{K}(f_0):=\int|v|^2 f_0(x,v)\,\mathrm{d}x\mathrm{d}v<+\infty.$$

Then the stochastic Vlasov equation (1.3) admits a weak solution $f\in L^\infty\big(0,T;(L^1\cap L^3)(\mathbb{T}^3\times\mathbb{R}^3,\mathbb{R}_+)\big)$ such that *(a) for any $p \in [1,3]$, $\mathbb{P}$-a.s. $\|f_t\|_{L^p_{x,v}} \leqslant \|f_0\|_{L^p_{x,v}}$ for all $t \in [0,T]$;*

(b) for all $q \geqslant 1$, it holds $\mathbb{E}\mathcal{K}(f_t)^{2q} \leqslant C_{q,T} < \infty$ for all $t \in [0,T]$.

The L^3-integrability of f_0 looks a little strange; it implies the existence of a sequence of approximate solutions bounded in $L^3(\mathbb{T}^3 \times \mathbb{R}^3)$, which will be used in (3.11) in the proof of lemma 3.6 below.

1.2. Scaling limit for stochastic Vlasov equations

In recent years, there have been intensive studies on the Itô–Stratonovich diffusion limit, starting from the work [16] which shows that, under a suitable rescaling of the space covariance of noises, a sequence of stochastic linear transport equations on the torus converge to the deterministic heat equation. This result was soon extended in [10] to the vorticity form of stochastic 2D Euler equation with transport noise, where the limit equation becomes the deterministic 2D Navier–Stokes equation; see also [25] for similar result on the stochastic 2D inviscid Boussinesq system. The paper [11] extends the results of [16] to stochastic transport-diffusion equations on bounded domains with Dirichlet boundary condition, while [20] and [26] deal with the cases of compact manifold and full space $\mathbb{R}^d$, respectively. We refer to [1, 7, 24, 28] for works treating various models in different setting, and to [12, 31] for quantitative estimates on convergence rates in the scaling limits. We also mention the recent papers [5, 6, 14] dealing with some 3D fluid models where a stretching part of noise is involved. See the lecture notes [15] for more comprehensive introductions to this field.

Motivated by the above works, in this part we show that, under a similar scaling of the noise in (1.3), the stochastic Vlasov equations converge weakly to the deterministic Vlasov equation with an extra dissipation term:

$$\partial_t \bar{f} + v \cdot \nabla_x \bar{f} + (E_{\bar{\rho}} + Bv \times \boldsymbol{e}_3) \cdot \nabla_v \bar{f} = \kappa \Delta_v \bar{f}, \quad \bar{f}|_{t=0} = f_0, \tag{1.4}$$

where $\bar{\rho}_t(x) = \int \bar{f}_t(x,v)\,\mathrm{d}v$.

To this end, assume we are given a sequence of smooth noise

$$W_N = W_N(x,t) = \sum_k \sigma_k^N(x) W_t^k,$$

with covariance functions

$$Q_N(x-y) = \sum_k \sigma_k^N(x) \otimes \sigma_k^N(y), \quad x, y \in \mathbb{T}^3 \tag{1.5}$$

satisfying

$$Q_N(0) = 2\kappa I_3 \ (\forall N \geqslant 1) \quad \text{and} \quad \lim_{N\to\infty} \|Q_N\|_{L^r} = 0 \tag{1.6}$$

for all $r \in [1,\infty)$. We will provide in example 2.4 an explicit choice of noises with the above properties, see also section 5 for more physical discussions. Note that we can regard Q_N as an operator acting on vector fields $A : \mathbb{T}^3 \to \mathbb{R}^3$ defined by

$$(Q_N A)(x) = (Q_N * A)(x) = \int_{\mathbb{T}^3} Q_N(x-y) A(y)\,\mathrm{d}y, \quad x \in \mathbb{T}^3.$$

Here is the second main result of this paper.

Theorem 1.2 (Scaling limit). *Assume that the initial condition f_0 belongs to $(L^1 \cap L^3)(\mathbb{T}^3 \times \mathbb{R}^3)$ with finite kinetic energy*

$$\mathcal{K}(f_0) = \int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f_0(x,v)\,\mathrm{d}x\mathrm{d}v < +\infty.$$

Let W_N be a sequence of smooth noises whose covariance functions Q_N verify the condition (1.6). For any $N \geqslant 1$, let f^N be a weak solution to the stochastic Vlasov equation

$$\mathrm{d}f^N + v \cdot \nabla_x f^N \,\mathrm{d}t + (E_{\rho^N} + Bv \times \boldsymbol{e}_3) \cdot \nabla_v f^N \,\mathrm{d}t + \sum_k \sigma_k^N \cdot \nabla_v f^N \,\mathrm{d}W_t^k = \kappa \Delta_v f^N \,\mathrm{d}t \tag{1.7}$$

with initial condition $f_0^N = f_0$. Then any convergent subsequence of $\{f^N\}_N$ converge weakly, in the topology of $C([0,T], H^{-\epsilon}_{x,v,\mathrm{loc}})$, to weak solutions of (1.4).

We remark that, by theorem 1.1, the approximate solutions $\{f^N\}_N$ satisfy the properties (a) and (b) therein; hence, the weak limits of $\{f^N\}_N$ belong to the same regularity class. However, we are unable to prove uniqueness of solutions to (1.4) under such regularity conditions. Therefore, we cannot claim that the whole sequence $\{f^N\}_N$ is weakly convergent.

This paper is organized as follows. After some preparations in section 2, we provide in section 3 the proof of theorem 1.1 which shows the existence of weak solutions to the stochastic Vlasov equation (1.3). Then we prove the above scaling limit result (theorem 1.2) in section 4. Since the idea is very close to the existence proof, we omit the details and only show why the martingale part vanishes in the limit. We provide in section 5 detailed discussions on the choice of physically well motivated noises satisfying the condition (1.6).

2. Useful lemmas and example of noises

In the proof below we will use some classical lemmas which can be found in [18, 19], where the results are proved on the phase space $\mathbb{R}^N \times \mathbb{R}^N$ with $N \geqslant 2$. In the sequel, ω_N is the surface area of $N-1$ dimensional sphere, and λ^N is the Lebesgue measure on $\mathbb{R}^N$.

The first result is taken from [19, lemma 3.2], see also [18, lemma 5.5].

Lemma 2.1. *Assume $1 < p \leqslant \infty$, $f \in L^p(\mathbb{R}^{2N})$, $f \geqslant 0$, $\mathcal{K}(f) := \int_{\mathbb{R}^{2N}} |v|^2 f(x,v)\,\mathrm{d}x\mathrm{d}v < \infty$. Let $\rho(x) = \int f(x,v)\,\mathrm{d}v$ and*

$$r = r(p) := \begin{cases} (2p + N(p-1))/(2 + N(p-1)), & \text{if } p < \infty, \\ (2+N)/N, & \text{if } p = \infty. \end{cases}$$

Then $\rho \in L^r(\mathbb{R}^N)$ and

$$\|\rho\|_{L^r} \leqslant C \|f\|_{L^p}^{2p'/(N+2p')} \mathcal{K}(f)^{N/(N+2p')}$$

with $C = 1 + (\omega_N/N)^{1/p'}$.

The next estimate can be proved by Hölder's inequality.

Lemma 2.2. *Let $p \in [1,\infty]$, $f \in L^p(\mathbb{R}^{2N})$, $K \subset \mathbb{R}^N$ compact, $\rho_K(x) := \int_K f(x,v)\,\mathrm{d}v$, $x \in \mathbb{R}^N$. Then $\rho_K \in L^p(\mathbb{R}^N)$ and $\|\rho_K\|_{L^p} \leqslant (\lambda^N(K))^{1/p'} \|f\|_{L^p}$.*

The following estimates can be found in [19, lemma 3.5].

Lemma 2.3. *Let $p \in (1,\infty]$, $f_1, f_2 \in L^p(\mathbb{R}^{2N})$, $a \in [0,N)$.*

(i) *If $p < \infty$, $\frac{1}{p} + \frac{1}{r(p)} + \frac{a}{N} = 2$ and $\mathcal{K}(|f_2|) < +\infty$, then*

$$\int_K \int_{\mathbb{R}^{2N}} \frac{1}{|x-y|^a} |f_1(x,v)|\,|f_2(y,w)|\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w \leqslant C(\lambda^{2N}(K))^{1/p'} \|f_1\|_{L^p} \|f_2\|_{L^p}^{2p'/(N+2p')} \mathcal{K}(|f_2|)^{N/(N+2p')}$$

for all compact sets $K \subset \mathbb{R}^{2N}$.

(ii) *If $\frac{2}{r(p)} + \frac{a}{N} = 2$ and $\mathcal{K}(|f_1|) \vee \mathcal{K}(|f_2|) < +\infty$, then*

$$\int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} \frac{1}{|x-y|^a} |f_1(x,v)|\,|f_2(y,w)|\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w \leqslant C\big[\|f_1\|_{L^p}\|f_2\|_{L^p}\big]^{2p'/(N+2p')} \big[\mathcal{K}(|f_1|)\mathcal{K}(|f_2|)\big]^{N/(N+2p')}.$$

We finish this section by giving an example of noises verifying the properties (1.6).

Example 2.4. Let $\mathbb{Z}_0^3 := \mathbb{Z}^3 \setminus \{0\}$ be the set of nonzero lattice points, with the partition $\mathbb{Z}_0^3 = \mathbb{Z}_+^3 \cup \mathbb{Z}_-^3$ satisfying $\mathbb{Z}_+^3 = -\mathbb{Z}_-^3$. Set $a_k = \frac{k}{|k|}$ for $k \in \mathbb{Z}_+^3$ and $a_k = -\frac{k}{|k|}$ for $k \in \mathbb{Z}_-^3$. Now we define the gradient vector fields on $\mathbb{T}^3$ by $\sigma_k(x) = a_k \mathrm{e}^{2\pi\,\mathrm{i}k\cdot x}$, $x \in \mathbb{T}^3$, $k \in \mathbb{Z}_0^3$; the family $\{\sigma_k\}_{k\in\mathbb{Z}_0^3}$ constitutes a complete orthonormal system of the space of square integrable and gradient fields on $\mathbb{T}^3$ with zero mean. Let

$$W_N(x,t) = \sqrt{3\kappa} \sum_k \Gamma_k^N \sigma_k(x) W_t^k,$$

where the coefficients $\Gamma^N = (\Gamma^N_k)_k \in \ell^2(\mathbb{Z}^3_0)$ are radial symmetric in $k \in \mathbb{Z}^3_0$ and satisfy $\|\Gamma^N\|_{\ell^2} = 1$, $\{W^k\}_{k\in\mathbb{Z}^3_0}$ is a family of complex Brownian motions with quadratic variation

$$\left[W^k, W^l\right]_t = 2\delta_{k,-l}\, t.$$

Then, for any $x, y \in \mathbb{T}^3$, the covariance function verifies

$$Q_N(x-y) = 6\kappa \sum_k \left(\Gamma^N_k\right)^2 \sigma_k(x) \otimes \sigma_{-k}(y) = 6\kappa \sum_k \left(\Gamma^N_k\right)^2 \frac{k\otimes k}{|k|^2} \mathrm{e}^{2\pi\,\mathrm{i}k\cdot(x-y)}.$$

As in [13, 16], we can take $\{\Gamma^N\}_N \subset \ell^2(\mathbb{Z}^3_0)$ such that

$$\left\|\Gamma^N\right\|_{\ell^2} = 1 \quad \text{for all } N \geqslant 1, \text{but} \quad \left\|\Gamma^N\right\|_{\ell^\infty} \to 0 \quad \text{as } N \to \infty.$$

Now it is not difficult to show that $Q_N(0) = 2\kappa I_3$, see [13, (2.3)]; moreover,

$$|Q_N(x)| \leqslant 6\kappa \sum_k \left(\Gamma^N_k\right)^2 = 6\kappa \quad \text{for all } x \in \mathbb{T}^3,$$

and

$$\|Q_N\|_{L^2} = 6\kappa \left[\sum_k \left(\Gamma^N_k\right)^4 \left|\frac{k\otimes k}{|k|^2}\right|^2\right]^{1/2} \leqslant 6\kappa \left\|\Gamma^N\right\|_{\ell^\infty}.$$

Fix any $r \in [1,\infty)$; if $r \in [1,2]$, then

$$\|Q_N\|_{L^r} \leqslant \|Q_N\|_{L^2} \leqslant 6\kappa \left\|\Gamma^N\right\|_{\ell^\infty},$$

while if $r \in (2,\infty)$, then by interpolation with parameters $\frac{1}{r} = \frac{2/r}{2} + \frac{1-2/r}{\infty}$,

$$\|Q_N\|_{L^r} \leqslant \|Q_N\|_{L^2}^{2/r} \|Q_N\|_{L^\infty}^{1-2/r} \leqslant 6\kappa \left\|\Gamma^N\right\|_{\ell^\infty}^{2/r}$$

which vanishes as $N \to \infty$ due to the choice of Γ^N.

3. Proof of theorem 1.1

We devote this section to the proof of theorem 1.1. In the first subsection we introduce smooth approximations of the stochastic Vlasov equation (1.3) and prove some basic estimates, then in the second part we show the weak convergence of approximate solutions by using the compactness method.

3.1. Smooth approximations and estimates

Note that the Green function satisfies $\nabla G(-x) = -\nabla G(x)$ for all $x \in \mathbb{T}^3 = \left[-\frac{1}{2}, \frac{1}{2}\right]^3$; moreover, $\nabla G(x) \sim C\frac{x}{|x|^3}$ as $|x| \to 0$. Due to the singularity of ∇G near the origin, the existence of weak solutions to (1.3) is not straightforward, thus we introduce a regularization of G. For $\delta \in (0,1/2)$, let $G^\delta \in C^\infty(\mathbb{T}^3)$ be a symmetric function such that $G^\delta(x) = G(x)$ for $|x| \geqslant \delta$, and $|G^\delta(x)| \lesssim \delta^{-1}$. Now for fixed $\delta > 0$, we consider the approximating stochastic Vlasov equations

$$\mathrm{d}f^\delta + v\cdot\nabla_x f^\delta\,\mathrm{d}t + (E_{\rho^\delta} + Bv\times \boldsymbol{e}_3)\cdot\nabla_v f^\delta\,\mathrm{d}t + \sum_k \sigma_k\cdot\nabla_v f^\delta \circ \mathrm{d}W^k_t = 0, \tag{3.1}$$

where

$$E_{\rho^\delta_t}(x) = (\nabla G^\delta * \rho^\delta_t)(x) = \int_{\mathbb{T}^3} \nabla G^\delta(x-y)\rho^\delta_t(y)\,\mathrm{d}y. \tag{3.2}$$

We supplement (3.1) with an initial condition $f^\delta_0 \in C^\infty_c(\mathbb{T}^3\times\mathbb{R}^3, \mathbb{R}_+)$.

Proposition 3.1. *Assume that $f_0^\delta \in C_c^\infty(\mathbb{T}^3 \times \mathbb{R}^3, \mathbb{R}_+)$. Then for any $\delta > 0$, the approximate equation (3.1) admits a unique probabilistically strong solution f^δ, represented as*

$$f_t^\delta(x,v) = f_0^\delta\big(\Phi_t^{\delta,-1}(x,v)\big),$$

where $\{\Phi_t^{\delta,-1}\}_{t\geqslant 0}$ is the inverse flow associated to the stochastic equations

$$\begin{cases} \mathrm{d}X_t^\delta = V_t^\delta\,\mathrm{d}t, \\ \mathrm{d}V_t^\delta = \big[E_{\rho_t^\delta}(X_t^\delta) + BV_t^\delta \times \boldsymbol{e}_3\big]\,\mathrm{d}t + \sum_k \sigma_k(X_t^\delta) \circ \mathrm{d}W_t^k \end{cases} \tag{3.3}$$

with initial condition $(X_0^\delta, V_0^\delta) = (x,v) \in \mathbb{T}^3 \times \mathbb{R}^3$. Moreover, $\|f_t^\delta\|_{L^p_{x,v}} = \|f_0^\delta\|_{L^p_{x,v}}$ for any $p \geqslant 1$.

Proof. This result can be proved by following the method in [8, theorem 13]. Note that, since $\nabla G^\delta \in C_b^\infty(\mathbb{T}^3, \mathbb{R}^3)$, the electric field in (3.2) is smooth on $\mathbb{T}^3$ for any $\rho_t^\delta \in L^1(\mathbb{T}^3)$. Combined with our assumptions on the noise, we see that the stochastic equations (3.3) generate a stochastic flow of diffeomorphisms, see [23]. The preservation of $L^p_{x,v}$-norm is due to the measure-preserving property of the stochastic flow $\{\Phi_t^\delta\}_{t\geqslant 0}$, a fact following from the stochastic version of Liouville theorem. □

Remark 3.2. Since $d\big(\sigma_k(X_t^\delta)\big) = V_t^\delta \cdot \nabla\sigma_k(X_t^\delta)\,\mathrm{d}t$, one has $d\big[\sigma_k(X^\delta), W^k\big]_t = 0$, thus the second equation in (3.3) has the equivalent Itô form

$$\mathrm{d}V_t^\delta = \big[E_{\rho_t^\delta}(X_t^\delta) + BV_t^\delta \times \boldsymbol{e}_3\big]\,\mathrm{d}t + \sum_k \sigma_k(X_t^\delta)\,\mathrm{d}W_t^k.$$

As a result, the quadratic variation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[V^\delta, V^\delta\big]_t = \sum_k \big|\sigma_k(X_t^\delta)\big|^2 = \mathrm{Tr}(Q(0)) = 6\kappa,$$

where the last step is due to (1.2).

Introduce the potential energy

$$\begin{aligned} \mathcal{V}_\delta(f_0) &= -\int_{\mathbb{T}^3}\int_{\mathbb{T}^3} G^\delta(x-y)\rho_0(x)\rho_0(y)\,\mathrm{d}x\mathrm{d}y \\ &= -\int_{\mathbb{T}^3\times\mathbb{R}^3}\int_{\mathbb{T}^3\times\mathbb{R}^3} G^\delta(x-y) f_0(x,v) f_0(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w \\ &= -\int_{\mathbb{T}^6}\int_{\mathbb{R}^6} G^\delta(x-y) f_0(x,v) f_0(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w. \end{aligned}$$

Note that $-G^\delta$ is uniformly bounded from below; thus for any probability density $f_0 : \mathbb{T}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$, $\mathcal{V}_\delta(f_0) \geqslant C$ for some constant independent of $\delta > 0$.

In view of the assumptions on f_0 in theorem 1.1, we make the following assumptions on the initial data $\{f_0^\delta\}_{\delta>0}$ of equation (3.1):

$$\sup_{\delta>0}\big[\|f_0^\delta\|_{L^1_{x,v}\cap L^3_{x,v}} \vee \mathcal{K}(f_0^\delta)\big] \leqslant C_0 < \infty. \tag{3.4}$$

Recall that $|G^\delta(x)| \lesssim |x|^{-1}$ uniformly in $\delta > 0$; applying lemma 2.3-(ii) with $a = 1$, $N = 3$ and $p = 9/7$ (hence $r(p) = 6/5$), we obtain the estimate

$$\big|V_\delta(f_0^\delta)\big| \lesssim \|f_0^\delta\|_{L^{9/7}}^{3/2} \mathcal{K}(f_0^\delta)^{1/2}.$$

Therefore, condition (3.4) implies

$$\sup_{\delta>0}\big|V_\delta(f_0^\delta)\big| \leqslant C_0' < \infty.$$

In order to derive a relation between $\mathcal{K}(f_t^\delta)$ and $\mathcal{V}_\delta(f_t^\delta)$, we first prove the following estimate which is not uniform in $\delta > 0$.

Lemma 3.3. *For any $\delta > 0$, there is a constant $C_{\delta,\kappa} > 0$ such that for any $t \geqslant 0$, it holds*

$$\mathbb{E}\mathcal{K}(f_t^\delta) \leqslant \mathrm{e}^{C_{\delta,\kappa} t}\big[\mathcal{K}(f_0^\delta) + \|f_0^\delta\|_{L^1_{x,v}}\big].$$

Proof. We have, by proposition 3.1,

$$\mathcal{K}(f_t^\delta) = \int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f_0^\delta\big(\Phi_t^{\delta,-1}(x,v)\big)\,\mathrm{d}x\mathrm{d}v = \int_{\mathbb{T}^3\times\mathbb{R}^3} \big|V_t^\delta(x,v)\big|^2 f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v, \tag{3.5}$$

where in the last step we have used the measure-preserving property of the stochastic flow $\Phi_t^\delta = \big(X_t^\delta, V_t^\delta\big)$, $t \geqslant 0$. In the sequel, we omit sometimes the initial condition (x,v) in $\Phi_t^\delta(x,v) = \big(X_t^\delta(x,v), V_t^\delta(x,v)\big)$.

By Itô's formula and remark 3.2,

$$\begin{aligned} d\big|V_t^\delta\big|^2 &= 2V_t^\delta\cdot \mathrm{d}V_t^\delta + d\big[V^\delta, V^\delta\big]_t \\ &= 2V_t^\delta \cdot E_{\rho_t^\delta}\big(X_t^\delta\big)\,\mathrm{d}t + 2\sum_k V_t^\delta\cdot\sigma_k\big(X_t^\delta\big)\,\mathrm{d}W_t^k + 6\kappa\,\mathrm{d}t, \end{aligned} \tag{3.6}$$

where we have used $V_t^\delta\cdot\big(V_t^\delta\times \boldsymbol{e}_3\big) = 0$. Note that $|\nabla G^\delta(x)| \leqslant C_\delta < +\infty$, thus

$$\big|E_{\rho_t^\delta}(x)\big| \leqslant \int |\nabla G^\delta(x-y)|\rho_t^\delta(y)\,\mathrm{d}y \leqslant C_\delta\|\rho_t^\delta\|_{L_x^1} = C_\delta\|f_t^\delta\|_{L_{x,v}^1} = C_\delta\|f_0^\delta\|_{L_{x,v}^1} \leqslant C_\delta'.$$

Let $N_t = N_t(x,v) = 2\sum_k\int_0^t V_s^\delta\cdot\sigma_k\big(X_s^\delta\big)\,\mathrm{d}W_s^k$ be the martingale part, with quadratic variation

$$[N]_t = 4\sum_k\int_0^t \big(V_s^\delta\cdot\sigma_k\big(X_s^\delta\big)\big)^2\,\mathrm{d}s = 8\kappa\int_0^t \big|V_s^\delta\big|^2\,\mathrm{d}s,$$

where the last step follows from (1.2). Introduce a sequence of stopping times $\tau_n = \tau_n(x,v) = \inf\big\{t\geqslant 0 : \big|V_t^\delta(x,v)\big| \geqslant n\big\}$ which are increasing in $n \geqslant 1$; then $N_{t\wedge\tau_n}$ are square integrable martingales. Thus, by (3.6),

$$\begin{aligned} \mathbb{E}\big|V_{t\wedge\tau_n}^\delta\big|^2 &\leqslant |v|^2 + 2\,\mathbb{E}\int_0^{t\wedge\tau_n}\big(C_\delta'\big|V_s^\delta\big| + 3\kappa\big)\,\mathrm{d}s \\ &\leqslant |v|^2 + C_{\delta,\kappa}\int_0^t \mathbb{E}\big(\big|V_{s\wedge\tau_n}^\delta\big|^2 + 1\big)\,\mathrm{d}s. \end{aligned}$$

Gronwall's inequality implies

$$\mathbb{E}\big|V_{t\wedge\tau_n}^\delta\big|^2 \leqslant (|v|^2+1)\mathrm{e}^{C_{\delta,\kappa}t}.$$

Letting $n\to\infty$ and by Fatou's lemma, we obtain

$$\mathbb{E}\big|V_t^\delta(x,v)\big|^2 \leqslant (|v|^2+1)\mathrm{e}^{C_{\delta,\kappa}t}.$$

The desired result follows by combining it with (3.5). □

Now we can derive the relation between $\mathcal{K}(f_t^\delta)$ and $\mathcal{V}_\delta(f_t^\delta)$.

Proposition 3.4 (Energy evolution). *Fix $\delta > 0$. For any $t > 0$, it holds*

$$\mathcal{K}(f_t^\delta) + \mathcal{V}_\delta(f_t^\delta) = \mathcal{K}(f_0^\delta) + \mathcal{V}_\delta(f_0^\delta) + 6\kappa t\|f_0^\delta\|_{L_{x,v}^1} + M_t, \tag{3.7}$$

where M_t is a square integrable martingale with quadratic variation

$$[M]_t \leqslant 8\kappa\|f_0^\delta\|_{L_{x,v}^1}\int_0^t \mathcal{K}(f_s^\delta)\,\mathrm{d}s.$$

Proof. By (3.5) and (3.6),

$$\begin{aligned} \mathcal{K}(f_t^\delta) = \mathcal{K}(f_0^\delta) + 6\kappa t\|f_0^\delta\|_{L_{x,v}^1} &+ 2\int_0^t\int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta\cdot E_{\rho_s^\delta}\big(X_s^\delta\big) f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}s \\ &+ 2\sum_k\int_0^t\int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta\cdot\sigma_k\big(X_s^\delta\big) f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}W_s^k. \end{aligned} \tag{3.8}$$

It remains to analyse the last two terms in (3.8). Recalling (3.2),

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta\cdot E_{\rho_s^\delta}\big(X_s^\delta\big) f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v$$

$$= \int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta(x,v)\cdot\left[\int_{\mathbb{T}^3\times\mathbb{R}^3}\nabla G^\delta\left(X_s^\delta(x,v)-y\right)f_s^\delta(y,w)\,\mathrm{d}y\mathrm{d}w\right]f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v$$
$$= \int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta(x,v)\cdot\left[\int_{\mathbb{T}^3\times\mathbb{R}^3}\nabla G^\delta\left(X_s^\delta(x,v)-X_s^\delta(y,w)\right)f_0^\delta(y,w)\,\mathrm{d}y\mathrm{d}w\right]f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v,$$

where the last step is due to proposition 3.1. Exchanging variables $(x,v)\leftrightarrow(y,w)$, we have

$$\begin{aligned}
&2\int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta\cdot E_{\rho_s^\delta}\left(X_s^\delta\right)f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\\
&\quad=\int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta(x,v)\cdot\left[\int_{\mathbb{T}^3\times\mathbb{R}^3}\nabla G^\delta\left(X_s^\delta(x,v)-X_s^\delta(y,w)\right)f_0^\delta(y,w)\,\mathrm{d}y\mathrm{d}w\right]f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\\
&\qquad+\int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta(y,w)\cdot\left[\int_{\mathbb{T}^3\times\mathbb{R}^3}\nabla G^\delta\left(X_s^\delta(y,w)-X_s^\delta(x,v)\right)f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\right]f_0^\delta(y,w)\,\mathrm{d}y\mathrm{d}w\\
&\quad=\int_{\mathbb{T}^6}\int_{\mathbb{R}^6}\left(V_s^\delta(x,v)-V_s^\delta(y,w)\right)\cdot\nabla G^\delta\left(X_s^\delta(x,v)-X_s^\delta(y,w)\right)f_0^\delta(y,w)f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w,
\end{aligned}$$

where we have used the property $\nabla G^\delta(-z)=-\nabla G^\delta(z)$.

Next, letting M_t be the martingale part in (3.8), then

$$\begin{aligned}
d[M]_t &= 4\sum_k\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta\cdot\sigma_k\left(X_s^\delta\right)f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\right)^2\mathrm{d}t\\
&\leqslant 4\|f_0^\delta\|_{L^1_{x,v}}\sum_k\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|V_s^\delta\cdot\sigma_k\left(X_s^\delta\right)\right|^2 f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}t\\
&\leqslant 8\kappa\|f_0^\delta\|_{L^1_{x,v}}\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|V_s^\delta\right|^2 f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}t,
\end{aligned}$$

where the last step is due to (1.2). By (3.5) and the assertion of lemma 3.3, we see that $\{M_t\}_{t\geqslant 0}$ is a square integrable martingale whose quadratic variation has the desired estimate.

Finally, we consider the potential energy:

$$\begin{aligned}
\mathcal{V}_\delta(f_t^\delta) &= -\int_{\mathbb{T}^6}\int_{\mathbb{R}^6} G^\delta(x-y)f_t^\delta(x,v)f_t^\delta(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\\
&= -\int_{\mathbb{T}^6}\int_{\mathbb{R}^6} G^\delta\left(X_t^\delta(x,v)-X_t^\delta(y,w)\right)f_0^\delta(y,w)f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w.
\end{aligned}$$

By the first equation in (3.3), we have

$$\begin{aligned}
&G^\delta\left(X_t^\delta(x,v)-X_t^\delta(y,w)\right)\\
&\quad= G^\delta(x-y)+\int_0^t\nabla G^\delta\left(X_s^\delta(x,v)-X_s^\delta(y,w)\right)\cdot\left(V_s^\delta(x,v)-V_s^\delta(y,w)\right)\mathrm{d}s,
\end{aligned}$$

as a result,

$$\mathcal{V}_\delta(f_t^\delta)=\mathcal{V}_\delta(f_0^\delta)-\int_0^t\int_{\mathbb{T}^6}\int_{\mathbb{R}^6}\nabla G^\delta\left(X_s^\delta(x,v)-X_s^\delta(y,w)\right)\cdot\left(V_s^\delta(x,v)-V_s^\delta(y,w)\right)\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\mathrm{d}s.$$

Combining this identity with the computations below (3.8), we arrive at the desired result. □

The next estimate will play an important role in the proof of tightness of laws of $\{f^\delta\}_{\delta>0}$.

Corollary 3.5. *Under (3.4), for any $q\geqslant 1$, we have*

$$\sup_{\delta>0}\sup_{t\in[0,T]}\mathbb{E}\mathcal{K}(f_t^\delta)^{2q}\leqslant C_{q,T}<+\infty.$$

Proof. We know that $\mathcal{V}_\delta(f_t^\delta)\geqslant -C_1>-\infty$, thus by (3.4) and proposition 3.4, we have

$$\mathcal{K}(f_t^\delta)\leqslant C_1+C_0(2+3\kappa t)+M_t.$$

As a result, for any $q\geqslant 1$,

$$\mathcal{K}(f_t^\delta)^{2q}\leqslant 2^{2q-1}\left[C_\kappa^{2q}(1+t)^{2q}+M_t^{2q}\right].$$

Introducing stopping times $\tau_n := \inf\{t > 0 : \mathcal{K}(f_t^\delta) \geqslant n\}$ which almost surely tend to infinity by lemma 3.3; then,

$$\mathcal{K}(f^\delta_{t\wedge\tau_n})^{2q} \leqslant 2^{2q-1}\big[C_\kappa^{2q}(1+t)^{2q} + M_{t\wedge\tau_n}^{2q}\big]. \tag{3.9}$$

By Burkholder-Davis-Gundy inequality and Cauchy's inequality,

$$\begin{aligned}\mathbb{E}|M_{t\wedge\tau_n}|^{2q} &\leqslant C_q\,\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\sum_k\bigg(\int_{\mathbb{T}^3\times\mathbb{R}^3} V_s^\delta\cdot\sigma_k\big(X_s^\delta\big)f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\bigg)^2\mathrm{d}s\bigg]^q\\ &\leqslant C_q\|f_0^\delta\|_{L^1_{x,v}}^q\,\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\sum_k\int_{\mathbb{T}^3\times\mathbb{R}^3}\big[V_s^\delta\cdot\sigma_k\big(X_s^\delta\big)\big]^2 f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}s\bigg]^q\\ &\leqslant C_q'\kappa^q\|f_0^\delta\|_{L^1_{x,v}}^q\,\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\int_{\mathbb{T}^3\times\mathbb{R}^3}\big|V_s^\delta\big|^2 f_0^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}s\bigg]^q,\end{aligned}$$

where the last step follows from (1.2). The definition of the kinetic energy implies

$$\begin{aligned}\mathbb{E}|M_{t\wedge\tau_n}|^{2q} &\leqslant C_q'\kappa^q\|f_0^\delta\|_{L^1_{x,v}}^q\,\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\mathcal{K}(f_s^\delta)\,\mathrm{d}s\bigg]^q\\ &\leqslant C_q'\kappa^q\|f_0^\delta\|_{L^1_{x,v}}^q\,\mathbb{E}\bigg[\int_0^{t}\mathcal{K}(f_{s\wedge\tau_n}^\delta)\,\mathrm{d}s\bigg]^q\\ &\leqslant C_{q,\kappa}t^{q-1}\mathbb{E}\int_0^t\mathcal{K}(f^\delta_{s\wedge\tau_n})^q\,ds\\ &\leqslant \frac{C_{q,\kappa}t^{q-1}}{2}\bigg(t+\int_0^t\mathbb{E}\,\mathcal{K}(f^\delta_{s\wedge\tau_n})^{2q}\,\mathrm{d}s\bigg).\end{aligned}$$

Combining this estimate with (3.9), and using Gronwall's inequality, we obtain

$$\mathbb{E}\,\mathcal{K}(f^\delta_{t\wedge\tau_n})^{2q} \leqslant C_{1,q,T}\mathrm{e}^{C_{2,q,T}} \quad \text{for all } t\in[0,T].$$

By Fatou's lemma, letting $n\to\infty$ leads to the desired result. □

3.2. Tightness argument and weak convergence of approximate solutions

Next we turn to proving the tightness of laws of the family $\{f^\delta\}_{\delta>0}$ in $C\big([0,T],H^{-\epsilon}_{x,v,\mathrm{loc}}\big)$. By proposition 3.1 and the assumption (3.4), we already know that $\mathbb{P}$-a.s. $f^\delta\in L^\infty\big(0,T;L^2_{x,v}\big)$; thus by Simon's compactness embedding result [30], it suffices to show that $\{f^\delta\}_{\delta>0}$ is bounded in probability in $C^\alpha\big([0,T],H^{-5}_{x,v}\big)$ for some $\alpha\in(0,1/2)$.

First, we recall the definition of Sobolev space $H^a_{x,v}=H^a(\mathbb{T}^3\times\mathbb{R}^3)$ for $a\in\mathbb{R}$. Let $f\in L^1(\mathbb{T}^3\times\mathbb{R}^3,\mathbb{R})$ with Fourier transform

$$\hat f(l,\xi)=\mathcal{F}(f)(l,\xi)=\int_{\mathbb{T}^3\times\mathbb{R}^3} f(x,v)\mathrm{e}^{-\mathrm{i}l\cdot x-\mathrm{i}\xi\cdot v}\,\mathrm{d}x\mathrm{d}v,\quad l\in\mathbb{Z}^3,\xi\in\mathbb{R}^3.$$

We say that $f\in H^a_{x,v}$ if

$$\|f\|^2_{H^a_{x,v}}=\sum_l\int(1+|l|^2+|\xi|^2)^a|\hat f(l,\xi)|^2\,\mathrm{d}\xi<+\infty.$$

Lemma 3.6. *For all $0\leqslant s<t\leqslant T$, it holds that*

$$\mathbb{E}\|f_t^\delta-f_s^\delta\|^4_{H^{-5}_{x,v}}\leqslant C|t-s|^2.$$

Proof. By the equation for f_t^δ, we have

$$\begin{aligned}f_t^\delta-f_s^\delta=&-\int_s^t v\cdot\nabla_x f_r^\delta\,\mathrm{d}r-\int_s^t\big(E_{\rho_r^\delta}+Bv\times\boldsymbol{e}_3\big)\cdot\nabla_v f_r^\delta\,\mathrm{d}r\\ &+\kappa\int_s^t\Delta_v f_r^\delta\,\mathrm{d}r-\sum_k\int_s^t\sigma_k\cdot\nabla_v f_r^\delta\,\mathrm{d}W_r^k.\end{aligned}$$

We denote the four terms on the right-hand side by I_i, $i=1,2,3,4$. First,

$$\|I_1\|^4_{H^{-5}_{x,v}}\leqslant|t-s|^3\int_s^t\|v\cdot\nabla_x f_r^\delta\|^4_{H^{-5}_{x,v}}\,dr.$$

By definition,

$$\mathcal{F}(v\cdot\nabla_x f_r^\delta)(l,\xi) = \mathrm{i}\int_{\mathbb{T}^3\times\mathbb{R}^3}(v\cdot l)f_r^\delta(x,v)\mathrm{e}^{-\mathrm{i}l\cdot x-\mathrm{i}\xi\cdot v}\,\mathrm{d}x\mathrm{d}v;$$

Cauchy's inequality implies

$$\big|\mathcal{F}(v\cdot\nabla_x f_r^\delta)(l,\xi)\big|^2 \leqslant \bigg(\int_{\mathbb{T}^3\times\mathbb{R}^3}|v\cdot l f_r^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\bigg)^2 \leqslant |l|^2\|f_r^\delta\|_{L^1_{x,v}}\mathcal{K}(f_r^\delta).$$

As a result,

$$\|v\cdot\nabla_x f_r^\delta\|_{H^{-5}_{x,v}}^2 \leqslant \sum_l\int(1+|l|^2+|\xi|^2)^{-5}|l|^2\|f_r^\delta\|_{L^1_{x,v}}\mathcal{K}(f_r^\delta)\,\mathrm{d}\xi \leqslant C\|f_0^\delta\|_{L^1_{x,v}}\mathcal{K}(f_r^\delta),$$

where C is a dimensional constant. Therefore, by (3.4), we have

$$\mathbb{E}\|I_1\|_{H^{-5}_{x,v}}^4 \lesssim |t-s|^3\int_s^t \mathbb{E}\,\mathcal{K}(f_r^\delta)^2\,\mathrm{d}r \lesssim_T |t-s|^4, \tag{3.10}$$

where the last step is due to corollary 3.5.

Concerning the term I_2, we first consider the part $I_{2,1}$ involving $E_{\rho_r^\delta}$:

$$\|I_{2,1}\|_{H^{-5}_{x,v}}^4 \leqslant |t-s|^3\int_s^t \big\|E_{\rho_r^\delta}\cdot\nabla_v f_r^\delta\big\|_{H^{-5}_{x,v}}^4\,\mathrm{d}r.$$

We have

$$\mathcal{F}\big(E_{\rho_r^\delta}\cdot\nabla_v f_r^\delta\big)(l,\xi) = \mathrm{i}\int_{\mathbb{T}^3\times\mathbb{R}^3}\big(E_{\rho_r^\delta}(x)\cdot\xi\big)f_r^\delta(x,v)\mathrm{e}^{-\mathrm{i}l\cdot x-\mathrm{i}\xi\cdot v}\,\mathrm{d}x\mathrm{d}v,$$

thus

$$\begin{aligned}\big|\mathcal{F}\big(E_{\rho_r^\delta}\cdot\nabla_v f_r^\delta\big)(l,\xi)\big| &\leqslant |\xi|\int_{\mathbb{T}^3\times\mathbb{R}^3}\big|E_{\rho_r^\delta}(x)\big|f_r^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\\ &= |\xi|\int_{\mathbb{T}^3\times\mathbb{R}^3}\big|\nabla G^\delta*\rho_r^\delta(x)\big|f_r^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\\ &\leqslant |\xi|\int_{(\mathbb{T}\times\mathbb{R})^6}\frac{C}{|x-y|^2}f_r^\delta(y,w)f_r^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w,\end{aligned}$$

where C is independent of $\delta>0$. Applying lemma 2.3(ii) with $p=3$, we obtain

$$\big|\mathcal{F}\big(E_{\rho_r^\delta}\cdot\nabla_v f_r^\delta\big)(l,\xi)\big| \lesssim |\xi|\,\|f_r^\delta\|_{L^3_{x,v}}\mathcal{K}(f_r^\delta) = |\xi|\,\|f_0^\delta\|_{L^3_{x,v}}\mathcal{K}(f_r^\delta). \tag{3.11}$$

By (3.4), we have

$$\big\|E_{\rho_r^\delta}\cdot\nabla_v f_r^\delta\big\|_{H^{-5}_{x,v}}^2 \lesssim \sum_l\int(1+|l|^2+|\xi|^2)^{-5}|\xi|^2\mathcal{K}(f_r^\delta)^2\,\mathrm{d}\xi \lesssim \mathcal{K}(f_r^\delta)^2.$$

As a consequence,

$$\mathbb{E}\|I_{2,1}\|_{H^{-5}_{x,v}}^4 \lesssim |t-s|^3\mathbb{E}\int_s^t\mathcal{K}(f_r^\delta)^4\,\mathrm{d}r \lesssim_T |t-s|^4,$$

where the last step is again due to corollary 3.5. The estimate of the part involving $v\times\boldsymbol{e}_3$ in I_2 is similar, indeed, we have

$$\big|\mathcal{F}\big((v\times\boldsymbol{e}_3)\cdot\nabla_v f_r^\delta\big)(l,\xi)\big| \leqslant |\xi|\,\|f_0^\delta\|_{L^1_{x,v}}^{1/2}\mathcal{K}(f_r^\delta)^{1/2}$$

and thus $\big\|(v\times\boldsymbol{e}_3)\cdot\nabla_v f_r^\delta\big\|_{H^{-5}_{x,v}}^2 \lesssim \mathcal{K}(f_r^\delta)$, which immediately gives us the desired estimate.

Next, the treatment of I_3 is very easy; in fact, one can easily show that $\big\|\Delta_v f_r^\delta\big\|_{H^{-5}_{x,v}}^2 \leqslant \|f_0^\delta\|_{L^1_{x,v}}$ which, combined with (3.4), implies $\mathbb{P}$-a.s.

$$\|I_3\|_{H^{-5}_{x,v}}^4 \lesssim |t-s|^4.$$

Finally, we deal with the stochastic term I_4:

$$\mathbb{E}\|I_4\|_{H_{x,v}^{-5}}^4 \lesssim \mathbb{E}\bigg(\int_s^t \sum_k \|\sigma_k\cdot\nabla_v f_r^\delta\|_{H_{x,v}^{-5}}^2\,\mathrm{d}r\bigg)^2.$$

One has $\big|\mathcal{F}\big(\sigma_k\cdot\nabla_v f_r^\delta\big)(l,\xi)\big| \lesssim |\xi|\int|\sigma_k(x)|f_r^\delta(x,v)\,\mathrm{d}x\mathrm{d}v$, and thus by Cauchy's inequality,

$$\big|\mathcal{F}\big(\sigma_k\cdot\nabla_v f_r^\delta\big)(l,\xi)\big|^2 \lesssim |\xi|^2\|f_r^\delta\|_{L_{x,v}^1}\int|\sigma_k(x)|^2 f_r^\delta(x,v)\,\mathrm{d}x\mathrm{d}v.$$

Consequently,

$$\begin{aligned}\sum_k \|\sigma_k\cdot\nabla_v f_r^\delta\|_{H_{x,v}^{-5}}^2 &\lesssim \sum_k\sum_l\int(1+|l|^2+|\xi|^2)^{-5}|\xi|^2\int|\sigma_k(x)|^2 f_r^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}\xi\\ &\leqslant 6\kappa\sum_l\int(1+|l|^2+|\xi|^2)^{-4}\int f_r^\delta(x,v)\,\mathrm{d}x\mathrm{d}v\mathrm{d}\xi\\ &\lesssim \kappa\|f_0^\delta\|_{L_{x,v}^1}\leqslant C\kappa\end{aligned}$$

by assumption (3.4). Therefore, $\mathbb{E}\|I_4\|_{H_{x,v}^{-5}}^4 \lesssim \kappa^2|t-s|^2$. Summarizing the above estimates, we complete the proof. □

Thanks to lemma 3.6 and the discussions at the beginning of section 3.2, we deduce from Simon's compactness theorem [30] that the family of laws of $\{f^\delta\}_{\delta>0}$ is tight in

$$\mathcal{X} := C\big([0,T],H_{x,v,\mathrm{loc}}^{-\epsilon}\big)$$

for any $\epsilon>0$. By Prohorov's theorem, we can find a subsequence $\delta_n\to0$ as $n\to\infty$, such that the laws η_n of f^{δ_n} converges weakly, in the topology of $\mathcal{X}$, to some probability measure $\eta\in\mathcal{P}(\mathcal{X})$. By Skorohod's representation theorem, there exist a new probability space $\big(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}}\big)$, a sequence of stochastic processes $\big\{\tilde f^n\big\}_{n\geqslant1}$ and a limit process $\tilde f$, such that

(i) for any $n\geqslant1$, $\tilde f^n$ has the same law η_n as f^{δ_n};
(ii) $\tilde{\mathbb{P}}$-a.s. $\tilde f^n$ converges in the topology of $\mathcal{X}$ to $\tilde f$.

By item (i), we know that for all $n\geqslant1$ and $p\in[1,3]$,

$$\tilde{\mathbb{P}}\text{-a.s.,}\quad \big\|\tilde f_t^n\big\|_{L_{x,v}^p} = \big\|f_t^{\delta_n}\big\|_{L_{x,v}^p} = \big\|f_0^{\delta_n}\big\|_{L_{x,v}^p}\quad\text{for all } t\in[0,T],$$

and for all $q\geqslant1$,

$$\tilde{\mathbb{E}}\mathcal{K}\big(\tilde f_t^n\big)^{2q} = \mathbb{E}\mathcal{K}\big(f_t^{\delta_n}\big)^{2q}\leqslant C_{q,T}\quad\text{for all } t\in[0,T].$$

Thanks to (3.7), passing to a further subsequence if necessary, we can deduce related bounds for the limit process $\{\tilde f_t\}_{t\in[0,T]}$; namely, for all $p\in[1,3]$,

$$\tilde{\mathbb{P}}\text{-a.s.,}\quad \big\|\tilde f_t\big\|_{L_{x,v}^p}\leqslant C_0<\infty\quad\text{for all } t\in[0,T]$$

and for all $q\geqslant1$,

$$\tilde{\mathbb{E}}\mathcal{K}\big(\tilde f_t\big)^{2q}\leqslant C_{q,T}\quad\text{for all } t\in[0,T].$$

Therefore, the properties (a) and (b) in theorem 1.1 are satisfied.

With slightly more effort, we can also find a sequence of Brownian motions $\big\{\tilde B^{n,k}\big\}_{n,k\geqslant1}$ and limit Brownian motions $\{\tilde B^k\}_{k\geqslant1}$ on $\tilde\Omega$, such that

$$\big(\tilde f^n,(\tilde B^{n,k})_{k\geqslant1}\big) \overset{\mathcal{L}}{\sim} \big(f^{\delta_n},(W^k)_{k\geqslant1}\big)\quad\text{for all } n\geqslant1, \tag{3.12}$$

and $\tilde{\mathbb{P}}$-a.s. for any $k\geqslant1$, $\tilde B^{n,k}$ converges in $C([0,T],\mathbb{R})$ to $\tilde B^k$. We omit the details here, cf [10, section 2].

Now we start proving that $(\tilde f, \{\tilde B^k\}_{k\geqslant 1})$ is a weak solution to (1.3). Let $\langle\cdot,\cdot\rangle$ be the inner product in $L^2(\mathbb{T}^3\times\mathbb{R}^3)$. First, for any $\phi\in C_c^\infty(\mathbb{T}^3\times\mathbb{R}^3)$, by (3.12) and the Itô formulation of (3.1), we know that $\tilde{\mathbb{P}}$-a.s. for all $t\geqslant 0$, it holds

$$\langle \tilde f^n_t,\phi\rangle = \langle f^{\delta_n}_0,\phi\rangle + \int_0^t \langle \tilde f^n_s, v\cdot\nabla_x\phi\rangle\, ds + \int_0^t \big\langle \tilde f^n_s, \big(E_{\tilde\rho^n_s} + B(v\times \boldsymbol{e}_3)\big)\cdot\nabla_v\phi\big\rangle\, ds \\ + \sum_k \int_0^t \langle \tilde f^n_s, \sigma_k\cdot\nabla_v\phi\rangle\, d\tilde B^{n,k}_s + \kappa\int_0^t \langle \tilde f^n_s, \Delta_v\phi\rangle\, ds,$$

where $\tilde\rho^n_s(x) = \int \tilde f^n_s(x,v)\,\mathrm{d}v$. Thanks to the above discussions, as $n\to\infty$, all the terms converge to the corresponding ones involving the limit process $\tilde f$, except the nonlinear part

$$J_n := \int_0^t \langle \tilde f^n_s, E_{\tilde\rho^n_s}\cdot\nabla_v\phi\rangle\, ds.$$

Thus we focus on showing the convergence of J_n, which can be further expressed as

$$J_n := \int_0^t\int_{(\mathbb{T}\times\mathbb{R})^6} \nabla G^{\delta_n}(x-y)\cdot\nabla_v\phi(x,v)\tilde f^n_s(x,v)\tilde f^n_s(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\mathrm{d}s,$$

where $(\mathbb{T}\times\mathbb{R})^6 = (\mathbb{T}^3\times\mathbb{R}^3)\times(\mathbb{T}^3\times\mathbb{R}^3)$. Let

$$J := \int_0^t\int_{(\mathbb{T}\times\mathbb{R})^6} \nabla G(x-y)\cdot\nabla_v\phi(x,v)\tilde f_s(x,v)\tilde f_s(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\mathrm{d}s,$$

then

$$\begin{aligned} J_n - J &= \int_0^t\int_{(\mathbb{T}\times\mathbb{R})^6} \big(\nabla G^{\delta_n}(x-y) - \nabla G(x-y)\big)\cdot\nabla_v\phi(x,v)\tilde f^n_s(x,v)\tilde f^n_s(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\mathrm{d}s \\ &\quad + \int_0^t\int_{(\mathbb{T}\times\mathbb{R})^6} \nabla G(x-y)\cdot\nabla_v\phi(x,v)\big(\tilde f^n_s(x,v) - \tilde f_s(x,v)\big)\tilde f^n_s(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\mathrm{d}s \\ &\quad + \int_0^t\int_{(\mathbb{T}\times\mathbb{R})^6} \nabla G(x-y)\cdot\nabla_v\phi(x,v)\tilde f_s(x,v)\big(\tilde f^n_s(y,w) - \tilde f_s(y,w)\big)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\mathrm{d}s \\ &=: J^1_n + J^2_n + J^3_n. \end{aligned}$$

First, letting $\tilde\rho^{n,\phi}_s(x) = \int_{\mathbb{R}^3}\nabla_v\phi(x,v)\tilde f^n_s(x,v)\,\mathrm{d}v$ where the integral is in fact on a compact subset in $\mathbb{R}^3$, then

$$J^1_n = \int_0^t\int_{\mathbb{R}^3} \tilde\rho^{n,\phi}_s(x)(\nabla G^{\delta_n} - \nabla G)*\tilde\rho^n_s(x)\, dxds;$$

by Cauchy's inequality and Young's inequality,

$$\begin{aligned} |J^1_n| &\leqslant \int_0^t \|\tilde\rho^{n,\phi}_s\|_{L^2}\|(\nabla G^{\delta_n} - \nabla G)*\tilde\rho^n_s\|_{L^2}\, ds \\ &\leqslant \int_0^t \|\tilde\rho^{n,\phi}_s\|_{L^2}\|\nabla G^{\delta_n} - \nabla G\|_{L^q}\|\tilde\rho^n_s\|_{L^{r(2)}}\, ds, \end{aligned}$$

where the parameters satisfy $1+\frac12 = \frac1q + \frac{1}{r(2)}$ and, by lemma 2.1, $r(2) = \frac75$; therefore, $q = \frac{14}{11} < \frac32$. Using lemmas 2.2 and 2.1, we have

$$\|\tilde\rho^{n,\phi}_s\|_{L^2} \leqslant C_\phi\|\tilde f^n_s\|_{L^2}, \quad \|\tilde\rho^n_s\|_{L^{r(2)}} \leqslant C'\|\tilde f^n_s\|_{L^2}^{4/7}\mathcal{K}(\tilde f^n_s)^{3/7},$$

hence,

$$\begin{aligned} |J^1_n| &\leqslant C'_\phi\|\nabla G^{\delta_n} - \nabla G\|_{L^{14/11}}\int_0^t \|\tilde f^n_s\|_{L^2}^{11/7}\mathcal{K}(\tilde f^n_s)^{3/7}\, ds \\ &= C'_\phi\|\nabla G^{\delta_n} - \nabla G\|_{L^{14/11}}\|f^{\delta_n}_0\|_{L^2}^{11/7}\int_0^t \mathcal{K}(\tilde f^n_s)^{3/7}\,\mathrm{d}s. \end{aligned}$$

By the properties of G, we know that $\left\|\nabla G^{\delta_n}-\nabla G\right\|_{L^{14/11}}\to 0$ as $n\to\infty$; combining with (3.4) and corollary 3.5, we conclude that

$$\lim_{n\to\infty}\tilde{\mathbb{E}}|J_n^1|=0.$$

Next, J_n^2 can be rewritten as

$$\begin{aligned}J_n^2&=\int_0^t\int_{\mathbb{T}^3\times\mathbb{R}^3}\big(\tilde f_s^n(x,v)-\tilde f_s(x,v)\big)\nabla_v\phi(x,v)\cdot\left[\int_{\mathbb{T}^3}\nabla G(x-y)\tilde\rho_s^n(y)\mathrm{d}y\right]\mathrm{d}x\mathrm{d}v\mathrm{d}s\\&=\int_0^t\int_{\mathbb{T}^3\times\mathbb{R}^3}\big(\tilde f_s^n(x,v)-\tilde f_s(x,v)\big)\nabla_v\phi(x,v)\cdot E_{\tilde\rho_s^n}(x)\,\mathrm{d}x\mathrm{d}v\mathrm{d}s.\end{aligned}$$

Note that $\nabla_v\phi$ has compact support, thus we can regard $\tilde f_s^n-\tilde f_s$ as a compactly supported distribution, and

$$|J_n^2|\leqslant\int_0^t\|\tilde f_s^n-\tilde f_s\|_{H^{-\epsilon}_{x,v,\mathrm{loc}}}\|\nabla_v\phi\cdot E_{\tilde\rho_s^n}\|_{H^{\epsilon}_{x,v}}\mathrm{d}s.\tag{3.13}$$

We have

$$\begin{aligned}\|\nabla_v\phi\cdot E_{\tilde\rho_s^n}\|^2_{H^{\epsilon}_{x,v}}&=\sum_l\int(1+|l|^2+|\xi|^2)^{\epsilon}\big|\mathcal{F}\big(\nabla_v\phi\cdot E_{\tilde\rho_s^n}\big)(l,\xi)\big|^2\mathrm{d}\xi\\&=\sum_l\int(1+|l|^2+|\xi|^2)^{\epsilon}\big|\mathcal{F}_x\big(E_{\tilde\rho_s^n}\cdot\mathcal{F}_v(\nabla_v\phi(x,\cdot))(\xi)\big)(l)\big|^2\mathrm{d}\xi,\end{aligned}$$

where $\mathcal{F}_v(\nabla_v\phi(x,\cdot))(\xi)$ means the Fourier transform of $\nabla_v\phi(x,\cdot)$ with respect to the velocity variable v, similarly for $\mathcal{F}_x$. Therefore,

$$\begin{aligned}\|\nabla_v\phi\cdot E_{\tilde\rho_s^n}\|^2_{H^{\epsilon}_{x,v}}&\leqslant\int(1+|\xi|^2)^{\epsilon}\sum_l(1+|l|^2)^{\epsilon}\big|\mathcal{F}_x\big(E_{\tilde\rho_s^n}\cdot\mathcal{F}_v(\nabla_v\phi(x,\cdot))(\xi)\big)(l)\big|^2\mathrm{d}\xi\\&=\int(1+|\xi|^2)^{\epsilon}\|E_{\tilde\rho_s^n}\cdot\mathcal{F}_v(\nabla_v\phi(x,\cdot))(\xi)\|^2_{H^{\epsilon}_x}\mathrm{d}\xi\\&\lesssim_{\epsilon}\int(1+|\xi|^2)^{\epsilon}\|E_{\tilde\rho_s^n}\|^2_{H^{2\epsilon}_x}\|\mathcal{F}_v(\nabla_v\phi(x,\cdot))(\xi)\|^2_{H_x^{3/2-\epsilon}}\mathrm{d}\xi,\end{aligned}$$

where in the last step we have used the product rule of Sobolev functions. As a result,

$$\begin{aligned}\|\nabla_v\phi\cdot E_{\tilde\rho_s^n}\|^2_{H^{\epsilon}_{x,v}}&\leqslant\|E_{\tilde\rho_s^n}\|^2_{H^{2\epsilon}_x}\int(1+|\xi|^2)^{\epsilon}\sum_l(1+|l|^2)^{3/2-\epsilon}\big|\mathcal{F}_x\big(\mathcal{F}_v(\nabla_v\phi(x,\cdot))(\xi)\big)(l)\big|^2\mathrm{d}\xi\\&\leqslant\|E_{\tilde\rho_s^n}\|^2_{H^{2\epsilon}_x}\sum_l\int(1+|l|^2+|\xi|^2)^{3/2}\big|\mathcal{F}_{x,v}(\nabla_v\phi)(l,\xi)\big|^2\mathrm{d}\xi\\&=\|E_{\tilde\rho_s^n}\|^2_{H^{2\epsilon}_x}\|\nabla_v\phi\|^2_{H^{3/2}_{x,v}}.\end{aligned}$$

Substituting this estimate into (3.13) leads to

$$|J_n^2|\lesssim\int_0^t\|\tilde f_s^n-\tilde f_s\|_{H^{-\epsilon}_{x,v,\mathrm{loc}}}\|E_{\tilde\rho_s^n}\|_{H^{2\epsilon}_x}\|\nabla_v\phi\|_{H^{3/2}_{x,v}}\mathrm{d}s.$$

Since $\int E_{\tilde\rho_s^n}(x)\,\mathrm{d}x=0$, we have

$$\|E_{\tilde\rho_s^n}\|^2_{H^{2\epsilon}_x}\lesssim\sum_{l\neq0}|l|^{4\epsilon}\big|\mathcal{F}(\nabla G*\tilde\rho_s^n)(l)\big|^2\lesssim\sum_{l\neq0}|l|^{4\epsilon-2}\big|\mathcal{F}(\tilde\rho_s^n)(l)\big|^2\lesssim\|\tilde\rho_s^n\|^2_{H^{2\epsilon-1}_x}.$$

By Sobolev embedding, $L_x^{7/5}\subset H_x^{-9/14}\subset H_x^{2\epsilon-1}$ for $\epsilon\in(0,5/28]$, thus

$$\|E_{\tilde\rho_s^n}\|_{H^{2\epsilon}_x}\lesssim\|\tilde\rho_s^n\|_{L_x^{7/5}}\lesssim\|\tilde f_s^n\|^{4/7}_{L_x^2}\mathcal{K}(\tilde f_s^n)^{3/7},$$

where the last step follows from lemma 2.1. Summarizing these discussions, we obtain

$$|J_n^2|\leqslant\|\nabla_v\phi\|_{H^{3/2-\epsilon}_{x,v}}\|f_0^{\delta_n}\|^{4/7}_{L_x^2}\int_0^t\|\tilde f_s^n-\tilde f_s\|_{H^{-\epsilon}_{x,v,\mathrm{loc}}}\mathcal{K}(\tilde f_s^n)^{3/7}\mathrm{d}s.$$

By assumption (3.4), corollary 3.5 and item (ii) above, we deduce that

$$\lim_{n\to\infty} \tilde{\mathbb{E}}|J_n^2| = 0.$$

Finally, the convergence of J_n^3 follows from the weak convergence of $\tilde{f}^n$ to $\tilde{f}$, up to a further subsequence, thus we complete the proof of weak existence of solutions to (1.3).

4. Proof of theorem 1.2

By theorem 1.1, the solutions $\{f^N\}_N$ to (1.7) verify the following bounds, uniformly in N:

(a′) for any $N \geqslant 1$ and $p \in [1,3]$, $\mathbb{P}$-a.s. $\|f_t^N\|_{L^p_{x,v}} \leqslant \|f_0\|_{L^p_{x,v}}$ for all $t \in [0,T]$;

(b′) for any $N \geqslant 1$ and $q \geqslant 1$, $\mathbb{E}\mathcal{K}(f_t^N)^{2q} \leqslant C_{q,T}$ for all $t \in [0,T]$.

Note that the solutions $\{f^N\}_N$ might be defined on different probability spaces $\big(\Omega^N, \mathcal{F}^N, \mathbb{P}^N\big)$, but we do not distinguish the notations $\mathbb{P}^N$ or $\mathbb{E}^N$ for simplicity of notation.

Thanks to the above bounds, using the equation (1.7) and following the computations in the proof of lemma 3.6, we can show that the laws of the family $\{f^N\}_N$ are tight on

$$\mathcal{X} := C\big([0,T], H^{-\epsilon}_{x,v,\mathrm{loc}}\big).$$

Therefore, we can repeat the arguments at the end of the last section to show that, along a subsequence, $\{f^N\}_N$ converges weakly, in the topology of $\mathcal{X}$, to some limit $\bar{f}$. To show that $\bar{f}$ solves (1.4) in the weak sense, we use the weak formulation of (1.7): for any test function $\phi \in C_c^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$,

$$\begin{aligned}\langle f_t^N, \phi\rangle = \langle f_0, \phi\rangle &+ \int_0^t \langle f_s^N, v\cdot\nabla_x\phi\rangle\, ds + \int_0^t \big\langle f_s^N, \big(E_{\rho_s^N} + B(v\times \boldsymbol{e}_3)\big)\cdot\nabla_v\phi\big\rangle\, \mathrm{d}s \\ &+ \sum_k \int_0^t \langle f_s^N, \sigma_k^N\cdot\nabla_v\phi\rangle\, \mathrm{d}W_s^k + \kappa\int_0^t \langle f_s^N, \Delta_v\phi\rangle\, \mathrm{d}s,\end{aligned}$$

where $\rho_t^N(x) = \int f_t^N(x,v)\,\mathrm{d}v$. Since all the terms, except the martingale part, can be treated similarly as in section 3.2, here we only show that the martingale part

$$\bar{M}_t := \sum_k \int_0^t \langle f_s^N, \sigma_k^N\cdot\nabla_v\phi\rangle\, \mathrm{d}W_s^k$$

vanishes in a suitable sense. By the Itô isometry,

$$\begin{aligned}\mathbb{E}\bar{M}_t^2 &= \sum_k \mathbb{E}\int_0^t \langle f_s^N, \sigma_k^N\cdot\nabla_v\phi\rangle^2\, \mathrm{d}s \\ &= \sum_k \mathbb{E}\int_0^t\int_{(\mathbb{T}\times\mathbb{R})^6} \big(f_s^N\nabla_v\phi\big)(x,v)^*\sigma_k^N(x)\otimes\sigma_k^N(y)\big(f_s^N\nabla_v\phi\big)(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\mathrm{d}s \\ &= \mathbb{E}\int_0^t\int_{(\mathbb{T}\times\mathbb{R})^6} \big(f_s^N\nabla_v\phi\big)(x,v)^* Q_N(x-y)\big(f_s^N\nabla_v\phi\big)(y,w)\,\mathrm{d}x\mathrm{d}v\mathrm{d}y\mathrm{d}w\mathrm{d}s,\end{aligned}$$

where $*$ means transposition of vectors and the last step is due to (1.5). Letting

$$A_s^N(x) := \int_{\mathbb{R}^3} \big(f_s^N\nabla_v\phi\big)(x,v)\,\mathrm{d}v, \quad (s,x)\in[0,T]\times\mathbb{T}^3,$$

then by Hölder's inequality and Young's inequality,

$$\begin{aligned}\mathbb{E}\bar{M}_t^2 &= \mathbb{E}\int_0^t\int_{\mathbb{T}^3} A_s^N(x)^*\,\big(Q_N * A_s^N\big)(x)\, dxds \\ &\leqslant \mathbb{E}\int_0^t \big\|A_s^N\big\|_{L_x^{7/5}}\big\|Q_N * A_s^N\big\|_{L_x^{7/2}}\, ds \\ &\leqslant \|Q_N\|_{L_x^{7/4}}\,\mathbb{E}\int_0^t \big\|A_s^N\big\|^2_{L_x^{7/5}}\, ds.\end{aligned}$$

Note that

$$\|A_s^N\|_{L_x^{7/5}} \leqslant \|\nabla_v \phi\|_{L_{x,v}^\infty} \|\rho_s^N\|_{L_x^{7/5}} \lesssim_\phi \|f_s^N\|_{L_x^2}^{4/7} \mathcal{K}(f_s^N)^{3/7},$$

where we have used lemma 2.1 in the last step. As a result,

$$\begin{aligned} \mathbb{E}\bar{M}_t^2 &\lesssim_\phi \|Q_N\|_{L_x^{7/4}} \mathbb{E}\int_0^t \|f_s^N\|_{L_x^2}^{8/7} \mathcal{K}(f_s^N)^{6/7} \,\mathrm{d}s \\ &\leqslant \|Q_N\|_{L_x^{7/4}} \|f_0\|_{L_x^2}^{8/7} \int_0^t \mathbb{E}\mathcal{K}(f_s^N)^{6/7} \,\mathrm{d}s \\ &\lesssim_T \|Q_N\|_{L_x^{7/4}} \to 0 \end{aligned}$$

as $N \to \infty$, thanks to (1.6).

Remark 4.1. As mentioned before, we cannot prove that the limit equation (1.4) admits a unique weak solution in the class of functions satisfying (a′) and (b′), therefore, we do not know whether the whole sequence $\{f^N\}_N$ converges to the same limit.

5. A particular noise in Vlasov equations

In this section we describe the origin and motivation for introducing a noise $\partial_t W(x,t)$, with certain properties, in the Vlasov equation

$$\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v((E + v \times B)f) + \mathrm{div}_v(\partial_t W(x,t) \circ f) = 0. \tag{5.1}$$

A noise in that particular transport form may come either from fluctuations of the magnetic field, or fluctuations of the electric field (or both). We concentrate our attention on the electric field only. In the case of the magnetic field, it could be reasonable to assume that the noise is divergence free (similarly to many recent investigations of transport noise in fluid mechanic models, see e.g. [10, 16]). But being related to the electric field, it is a gradient in the space variable, hence we cannot assume it to be divergence free.

Before we enter into some detail, let us explain a few elements of the idea. The noise comes from the electric field

$$E(x,t) = \int_{\mathbb{T}^3} \nabla G(x-y)\,\rho(y,t)\,\mathrm{d}y,$$

where G is the Green kernel on $\mathbb{T}^3$. We may think that $\rho(y,t)$ is composed of a slowly varying part $\overline{\rho}(y,t)$ plus a rapidly fluctuating one $\rho'(y,t)$, so that the same holds for the electric field:

$$E(x,t) = \int_{\mathbb{T}^3} \nabla G(x-y)\,\overline{\rho}(y,t)\,\mathrm{d}y + \int_{\mathbb{T}^3} \nabla G(x-y)\,\rho'(y,t)\,\mathrm{d}y.$$

Thus it would be natural to assume that $\rho'(y,t)$ is a noise. However, a noise is typically a zero-mean (the mean of the signal being incorporated into $\overline{\rho}(y,t)$). In the particular case of the space density $\rho(y,t)$, it is not so clear that its natural fluctuations are zero mean. A superficial reason is that $\rho(y,t) \geqslant 0$, thus decomposing it into $\overline{\rho}(y,t) \geqslant 0$ and $\rho'(y,t)$ would impose a complicated constraint on $\rho'(y,t)$. More deeply, the model of $\rho'(y,t)$ should correspond to the information we have on the coherent structures of electrons and ions, information that is accumulating in the physical literature, see for instance [21, 22], and may lead us to consider non-zero-mean perturbations $\rho'(y,t)$. Therefore, we develop a scheme which does not require $\rho'(y,t)$ to be zero mean. The key remark is that, assuming $\rho'(y,t)$ space-homogeneous, the term

$$E'(x,t) = \int_{\mathbb{T}^3} \nabla G(x-y)\,\rho'(y,t)\,\mathrm{d}y$$

is zero mean and thus may have the properties of a noise.

We divide the section into four subsections. The first one 5.1 illustrates how a single perturbation of non-zero-average blob generates a zero-mean electric field $E'(x,t)$, and we compute its covariance operator, giving also an example of a Gaussian random field with the same covariance operator. Section 5.2 shows how such objects may be used as building blocks of a white-noise-in-time Gaussian process, that will be our noise $W(x,t)$ in equation (5.1). Next, we construct in section 5.3 a sequence of noises satisfying the conditions needed in the scaling limit argument. We finish the paper with a brief discussion on the physical relevance of our results.

5.1. Preliminaries on random fields

Assume we are on the torus $\mathbb{T}^3$. The Coulomb Green kernel, up to constants, is $G(x) = -\frac{1}{|x|} + h(x)$ for a smooth function h, and the associated electric field kernel is $\nabla G(x) = \frac{x}{|x|^3} + \nabla h(x)$. One has the following properties, beside integrable:

$$\nabla G(-x) = -\nabla G(x) \quad \text{and} \quad \nabla G \, \text{periodic}.$$

The Fourier transforms are

$$\widehat{G}(k) = \frac{1}{|k|^2}, \qquad \widehat{\nabla G}(k) = \frac{ik}{|k|^2}, \qquad k \in \mathbb{Z}_0^3,$$

where $\mathbb{Z}_0^3$ are the non-zero elements of $\mathbb{Z}^3$.

Consider a perturbation $\rho'(x)$ (at some time t) of the space-density $\rho(x,t)$ of the form

$$\rho'(x) = r\theta_\ell(x - x_0),$$

where $r > 0$, $x_0 \in \mathbb{T}^3$, and $\theta_\ell(x) = \ell^{-3}\theta\left(\ell^{-1}x\right)$ for some square integrable periodic function θ. It is a paradigmatic (idealized) choice for modelling a blob, a concentration of density, possibly surrounded by an annulus of negative values (a Mexican hat form of θ); a blob localized around x_0, with size (space-scale) ℓ and intensity r. We may randomize all parameters, r, ℓ, x_0, and consider a random blob of the form

$$\rho'(x) = R\theta_L(x - X),$$

where R, L and X are independent random variables, X is uniform in $\mathbb{T}^3$, L is positive with density f, R is real valued with square average $\mathbb{E}\left[R^2\right] = \sigma^2$.

Assume, in addition to θ being square integrable and f a density, that

$$\sum_{k \in \mathbb{Z}_0^3} \frac{1}{|k|^2} \int_0^\infty \left|\widehat{\theta_\ell^T}(k)\right|^2 f(\ell)\, \mathrm{d}\ell < \infty,$$

where $\theta_\ell^T(x) = \theta_\ell(-x)$ for all $x \in \mathbb{T}^3$. This condition will be trivially satisfied by our special choice of f (compact support disjoined from zero) in the scaling limit, which is our main concern in the paper.

Define the random vector field

$$E(x) = R\nabla(G * \theta_L)(x - X).$$

As it is clear from the computations below, under the assumption just made we have

$$\mathbb{E}\left[\langle E, \phi\rangle^2\right] < \infty$$

for every $\phi \in L^2\left(\mathbb{T}^3, \mathbb{R}^3\right)$, hence it is well defined at least as a generalized random field.

Lemma 5.1. *For every* $\phi \in L^2\left(\mathbb{T}^3, \mathbb{R}^3\right)$,

$$\mathbb{E}\left[\langle E, \phi\rangle\right] = 0.$$

Proof. By definition,

$$\begin{aligned}
\mathbb{E}\left[\langle E, \phi\rangle\right] &= \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \phi(x) \cdot \nabla G(x - y)\, \mathbb{E}\left[R\left(\int_{\mathbb{T}^3} \theta_L(y - z)\, dz\right)\right] \mathrm{d}y\mathrm{d}x \\
&= \|\theta\|_{L^1} \mathbb{E}[R] \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \phi(x) \cdot \nabla G(x - y)\, \mathrm{d}y\mathrm{d}x \\
&= \|\theta\|_{L^1} \mathbb{E}[R] \int_{\mathbb{T}^3} \phi(x) \cdot \int_{\mathbb{T}^3} \nabla G(-y)\, \mathrm{d}y\mathrm{d}x = 0.
\end{aligned}$$

□

Lemma 5.2. *For every $\phi, \psi \in L^2\left(\mathbb{T}^3, \mathbb{R}^3\right)$,*

$$\mathbb{E}\left[\langle E, \phi\rangle \langle E, \psi\rangle\right] = \sigma^2 \sum_{k\in\mathbb{Z}_0^3} \frac{\chi^2(k)}{|k|^2} \left\langle \frac{k\otimes k}{|k|^2}\widehat{\phi}(k), \overline{\widehat{\psi}(k)}\right\rangle \tag{5.2}$$

with

$$\chi(k) = \sqrt{\int_0^\infty \left|\widehat{\theta_\ell^T}(k)\right|^2 f(\ell)\,\mathrm{d}\ell}.$$

Namely, the covariance function is

$$Q(x-y) = \sigma^2 \sum_{k\in\mathbb{Z}_0^3} \frac{\chi^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2} \mathrm{e}^{2\pi\,\mathrm{i}k\cdot(x-y)}.$$

Proof. We have

$$\begin{aligned}\langle E, \phi\rangle &= R\int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \phi(x)\cdot\nabla G(x-y)\,\theta_L(y-X)\,\mathrm{d}y\mathrm{d}x\\ &= R\left(\theta_L^T * (\nabla G)^T * \phi\right)(X),\end{aligned}$$

where $(\nabla G)^T(x) = \nabla G(-x)$. Hence

$$\begin{aligned}\mathbb{E}\left[\langle E, \phi\rangle\langle E, \psi\rangle\right] &= \sigma^2\mathbb{E}\int_{\mathbb{T}^3}\left(\theta_L^T * (\nabla G)^T * \phi\right)(x)\left(\theta_L^T * (\nabla G)^T * \psi\right)(x)\,\mathrm{d}x\\ &= \sigma^2\sum_{k\in\mathbb{Z}_0^3}\mathbb{E}\left[\left|\widehat{\theta_L^T}(k)\right|^2\right]\frac{1}{|k|^2}\left\langle\frac{k\otimes k}{|k|^2}\widehat{\phi}(k), \overline{\widehat{\psi}(k)}\right\rangle.\end{aligned}$$

Using the density function for L, we obtain the desired result. □

Consider the space $L_0^2\left(\mathbb{T}^3, \mathbb{R}^3\right)$ (mean zero periodic L^2-fields), defined on the complex field. A complete orthonormal system is $\left(\mathrm{e}^{2\pi\,\mathrm{i}k\cdot x}\right)_{k\in\mathbb{Z}_0^3}$.

Corollary 5.3. *The centred random vector field E (random element of $L_0^2\left(\mathbb{T}^3, \mathbb{R}^3\right)$) has the same covariance function as the Gaussian centred random vector field*

$$F(x) = \sigma\sum_{k\in\mathbb{Z}_0^3}\frac{\chi(k)}{|k|}\frac{k}{|k|}\mathrm{e}^{2\pi\mathrm{i}k\cdot x}Z_k,$$

where $\{Z_k\}_k$ are i.i.d. standard normal random variables.

Proof. One has

$$\begin{aligned}\mathbb{E}\left(\langle F, \phi\rangle\overline{\langle F, \psi\rangle}\right) &= \sigma^2\sum_{k\in\mathbb{Z}_0^3}\frac{\chi^2(k)}{|k|^2}\frac{1}{|k|^2}\langle k\mathrm{e}^{2\pi\,\mathrm{i}k\cdot x}, \phi\rangle\overline{\langle k\mathrm{e}^{2\pi\,\mathrm{i}k\cdot x}, \psi\rangle}\\ &= \sigma^2\sum_{k\in\mathbb{Z}_0^3}\frac{\chi^2(k)}{|k|^2}\frac{1}{|k|^2}k\cdot\widehat{\phi}(k)\,k\cdot\overline{\widehat{\psi}(k)}\\ &= \sigma^2\sum_{k\in\mathbb{Z}_0^3}\frac{\chi^2(k)}{|k|^2}\left\langle\frac{k\otimes k}{|k|^2}\widehat{\phi}(k), \overline{\widehat{\psi}(k)}\right\rangle,\end{aligned}$$

which is the same as the right-hand side of (5.2). □

Let us go back to $L_0^2\left(\mathbb{T}^3, \mathbb{R}^3\right)$ considered on the field of real numbers. Split $\mathbb{Z}_0^3$ in two parts, called $\mathbb{Z}_+^3, \mathbb{Z}_-^3$, as in example 2.4. Define

$$F_{\mathrm{real}}(x) = \sqrt{2\sigma^2}\sum_{k\in\mathbb{Z}_+^3}\frac{\chi(k)}{|k|}\frac{k}{|k|}\cos(2\pi k\cdot x)Z_k$$

$$+\sqrt{2\sigma^2}\sum_{k\in\mathbb{Z}^3_-}\frac{\chi(k)}{|k|}\frac{k}{|k|}\sin(2\pi k\cdot x)Z_k$$

where, again, $\{Z_k\}_k$ are i.i.d. standard normal random variables. We have

$$\begin{aligned}\mathbb{E}[F(x)F(y)] &= 2\sigma^2\sum_{k\in\mathbb{Z}^3_+}\frac{\chi^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2}\cos(2\pi k\cdot x)\cos(2\pi k\cdot y)\\ &\quad+2\sigma^2\sum_{k\in\mathbb{Z}^3_-}\frac{\chi^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2}\sin(2\pi k\cdot x)\sin(2\pi k\cdot y)\\ &= 2\sigma^2\sum_{k\in\mathbb{Z}^3_+}\frac{\chi^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2}\cos(2\pi k\cdot(x-y)),\end{aligned}$$

hence

$$\begin{aligned}\mathbb{E}[F(x)F(y)] &= 2\sigma^2\sum_{k\in\mathbb{Z}^3_+}\frac{\chi^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2}\left(\frac{\mathrm{e}^{2\pi\mathrm{i}k\cdot(x-y)}+\mathrm{e}^{-2\pi\mathrm{i}k\cdot(x-y)}}{2}\right)\\ &= \sigma^2\sum_{k\in\mathbb{Z}^3_0}\frac{\chi^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2}\mathrm{e}^{2\pi\mathrm{i}k\cdot(x-y)}\end{aligned}$$

which gives the same result as the complex case.

5.2. The noise and Donsker type argument

Based on the previous subsection and the Donsker type argument described below, we shall choose in equation (5.1) the following space-dependent Brownian motion

$$\begin{aligned}W(x,t) &= \sqrt{2\tau\sigma^2}\sum_{k\in\mathbb{Z}^3_+}\frac{\chi(k)}{|k|}\frac{k}{|k|}\cos(2\pi k\cdot x)W^k_t\\ &\quad+\sqrt{2\tau\sigma^2}\sum_{k\in\mathbb{Z}^3_-}\frac{\chi(k)}{|k|}\frac{k}{|k|}\sin(2\pi k\cdot x)W^k_t,\end{aligned}$$

where τ is described below and $\{W^k_t\}_{k\in\mathbb{Z}^3_0}$ is a family of independent standard Brownian motions. Its covariance function is

$$Q(x-y)=2\tau\sigma^2\sum_{k\in\mathbb{Z}^3_+}\frac{\chi^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2}\cos(2\pi k\cdot(x-y)).$$

Let us motivate this choice of noise. We start from a space density $\rho(x,t)$, and we assume that there is a slowly varying background density $\overline{\rho}(x,t)$, a sequence of increasing times $(t_n)_{n=0,1,\ldots}$ and perturbations $(\rho_n(x))_{n=0,1,\ldots}$ such that $\rho(x,t)$ in the time interval $[t_n,t_{n+1})$ is equal to $\overline{\rho}(x,t)+\rho_n(x)$; namely

$$\rho(x,t)=\overline{\rho}(x,t)+\sum_n\rho_n(x)1_{[t_n,t_{n+1})}(t).$$

Suppose for simplicity that there is $\tau>0$ such that $t_{n+1}-t_n=\tau$ (but the conclusion is similar if, for instance, we assume that the intertimes $t_{n+1}-t_n$ are independent and exponentially distributed random variables with mean τ). We assume that $\rho_n(x)$ are independent random fields of the form

$$\rho_n(x)=R_n\theta_{L_n}(x-X_n),$$

where all the random variables $(R_n,L_n,X_n)_{n=0,1,\ldots}$ are independent, X_n are uniformly distributed in $\mathbb{T}^3$, R_n and L_n are positive with probability density functions $g(r)$ and $f(\ell)$ respectively, θ is an integrable function (not necessarily positive), $\theta_\ell(x)=\ell^{-3}\theta(\ell^{-1}x)$. The intuition is that the perturbations have the form $R\theta_\ell(x-x_0)$, where the size R, the space-scale ℓ and the centre x_0 are randomly distributed.

The corresponding electric potential $\phi(x,t)$ is given by

$$\phi(x,t)=\overline{\phi}(x,t)+\phi'(x,t)\quad\text{with}$$

$$\overline{\phi}(\cdot,t) = (-\Delta)^{-1}\rho(\cdot,t),$$
$$\phi'(\cdot,t) = \sum_n (-\Delta)^{-1}\rho_n 1_{[t_n,t_{n+1})}(t).$$

Namely, $\phi(x,t)$ is the sum of a background slowly varying electric potential $(-\Delta)^{-1}\rho(\cdot,t)$ plus a fast varying perturbation. For the link with experiments and physical arguments, it may be possible that the measured fluctuations are for the electric potential, instead of the space density.

The corresponding electric field $E(x,t)$ is composed of a background slowly varying electric field

$$\overline{E}(\cdot,t) = \nabla(-\Delta)^{-1}\rho(\cdot,t)$$

plus a fast varying perturbation

$$E'(\cdot,t) = \sum_n \nabla(-\Delta)^{-1}\rho_n 1_{[t_n,t_{n+1})}(t).$$

In our preliminary modelling phase (still only heuristic), we replace $E'(\cdot,t)$ by the time derivative of a space-dependent Brownian motion:

$$E'(\cdot,t) \longrightarrow \sqrt{\tau}\partial_t W(\cdot,t)$$

multiplied by a factor $\sqrt{\tau}$ which gives us the correct unit of measure and keeps a trace of the correlation time of the original random fluctuations. The heuristic procedure for this replacement is that, given $T>0$ of the form $T=N\tau$, dropping x for notational simplicity,

$$\begin{aligned}\frac{1}{\sqrt{\tau}}\int_0^T E'(t)\,\mathrm{d}t &= \frac{1}{\sqrt{\tau}}\sum_{t_{n+1}\leqslant T}\nabla(-\Delta)^{-1}\rho_n(t_{n+1}-t_n) + \frac{1}{\sqrt{\tau}}\nabla(-\Delta)^{-1}\rho_{n+1}(T-t_{n+1})\\ &= \sqrt{\frac{T}{N}}\sum_{n\leqslant N-1}\nabla(-\Delta)^{-1}\rho_n\end{aligned}$$

which converges (by the Central Limit theorem) to a Gaussian random variable for given T; and (by Donsker theorem) to a Brownian motion W_T when T varies, with covariance operator $\mathbb{Q}$ in $L^2(\mathbb{T}^3,\mathbb{R}^3)$ given by

$$\langle \mathbb{Q}v,w\rangle_{L^2(\mathbb{T}^3,\mathbb{R}^3)} = \mathbb{E}\left[\left\langle\nabla(-\Delta)^{-1}\rho',v\right\rangle_{L^2(\mathbb{T}^3,\mathbb{R}^3)}\left\langle\nabla(-\Delta)^{-1}\rho',w\right\rangle_{L^2(\mathbb{T}^3,\mathbb{R}^3)}\right],$$

where ρ' is distributed as ρ_n. Essential for the result is the property

$$\mathbb{E}\left[\nabla(-\Delta)^{-1}\rho'\right] = 0,$$

otherwise we would get, in the scaling limit, an infinite contribution from the mean.

5.3. Choice for the scaling limit

We construct now an example of a sequence of Brownian motions which will be used in the diffusive scaling limit. We prescribe a sequence of random variables R_N with $\mathbb{E}\left[R_N^2\right]=\sigma_N^2$, and a sequence of functions f_N (densities of a sequence L_N) of the form

$$f_N(\ell) = \ell_N^{-1}1_{[\ell_N,2\ell_N]}(\ell),$$

with $\ell_N\to 0$ as $N\to\infty$. The intuition behind this choice is that we have blob-like fluctuations at space-scale ℓ_N, uniformly and densely distributed on $\mathbb{T}^3$, with small intensity σ_N^2. The noise has the form

$$\begin{aligned}W_N(x,t) = \sqrt{2\tau\sigma_N^2}&\sum_{k\in\mathbb{Z}^3_+}\frac{\chi_N(k)}{|k|}\frac{k}{|k|}\cos(2\pi k\cdot x)W_t^k\\ &+\sqrt{2\tau\sigma_N^2}\sum_{k\in\mathbb{Z}^3_-}\frac{\chi_N(k)}{|k|}\frac{k}{|k|}\sin(2\pi k\cdot x)W_t^k,\end{aligned}$$

with

$$\chi_N(k) = \sqrt{\ell_N^{-1}\int_{\ell_N}^{2\ell_N}\left|\widehat{\theta_\ell^T}(k)\right|^2\mathrm{d}\ell}$$

and covariance function

$$Q_N(x-y) = 2\tau\sigma_N^2 \sum_{k\in\mathbb{Z}^3_+} \frac{\chi_N^2(k)}{|k|^2} \frac{k\otimes k}{|k|^2} \cos\left(2\pi k\cdot(x-y)\right).$$

In view of the next computation, where $\chi_N(k)$ becomes close to a constant, notice that for finite N this expression is well defined, just because θ is assumed to be square integrable.

In the sequel, we assume that θ is radially symmetric and has compact support on $\mathbb{T}^3$, and thus we can also regard θ as a function on $\mathbb{R}^3$ which vanishes outside $\left(-\frac{1}{2},\frac{1}{2}\right)^3$.

Lemma 5.4. *We have*

$$\lim_{N\to\infty} \chi_N(k) = \|\theta\|_{L^1}$$

for every $k\in\mathbb{Z}^3_0$.

Proof. By definition,

$$\begin{aligned}\widehat{\theta_\ell^T}(k) &= \int_{\mathbb{T}^3} \theta_\ell^T(x)\mathrm{e}^{-2\pi\,\mathrm{i}k\cdot x}\,\mathrm{d}x = \int_{\mathbb{T}^3} \ell^{-3}\theta(-\ell^{-1}x)\mathrm{e}^{-2\pi\,\mathrm{i}k\cdot x}\,\mathrm{d}x \\ &= \int_{\mathbb{T}^3_{\ell^{-1}}} \theta(y)\mathrm{e}^{-2\pi\,\mathrm{i}\ell k\cdot y}\,\mathrm{d}y = \int_{\mathbb{T}^3} \theta(y)\mathrm{e}^{-2\pi\,\mathrm{i}\ell k\cdot y}\,\mathrm{d}y = \widehat{\theta}(\ell k),\end{aligned} \tag{5.3}$$

where $\mathbb{T}^3_{\ell^{-1}} = \left[-\frac{1}{2\ell},\frac{1}{2\ell}\right]^3$ and we have used the fact that θ has compact support in $\left(-\frac{1}{2},\frac{1}{2}\right)^3$. The desired limit follows immediately from the fact that $\widehat{\theta}(\ell k)\to\widehat{\theta}(0) = \|\theta\|_{L^1}$ as $\ell\to 0$. □

Lemma 5.5. *Assume*

$$\lim_{N\to\infty} 2\sigma_N^2 \sum_{k\in\mathbb{Z}^3_+} \frac{\chi_N^2(k)}{|k|^2} = k_T^2 > 0,$$

then (I_3 being the 3×3 unit matrix)

$$Q_N(0) \to \frac{1}{3}\tau k_T^2 I_3.$$

Remark 5.6. The series $\sum_{k\in\mathbb{Z}^3_+} \frac{\chi_N^2(k)}{|k|^2}$ is well defined (finite numbers) and diverges to infinity as $N\to\infty$. The lemma states that we have to take σ_N^2 going to zero as the reciprocal of this sequence, in order to have a finite non zero limit of the diffusion term.

Proof of lemma 5.5. Recall that we assume $\theta\in L^1(\mathbb{R}^3)$ to be radially symmetric, hence so is the function $\xi\mapsto\left|\widehat{\theta}(\xi)\right|$; by (5.3) and the definition of $\chi_N(k)$, we see that the latter is also a radial function in k. As a result,

$$Q_N(0) = 2\tau\sigma_N^2 \sum_{k\in\mathbb{Z}^3_+} \frac{\chi_N^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2} = \tau\sigma_N^2 \sum_{k\in\mathbb{Z}^3_0} \frac{\chi_N^2(k)}{|k|^2}\frac{k\otimes k}{|k|^2} = \frac{1}{3}\tau\sigma_N^2 \sum_{k\in\mathbb{Z}^3_0} \frac{\chi_N^2(k)}{|k|^2} I_3.$$

The last equality is classical; we present the proof here for completeness. For any $i,j\in\{1,2,3\}$ with $i\neq j$, say $i=1, j=2$,

$$Q_N^{1,2}(0) = \tau\sigma_N^2 \sum_{k\in\mathbb{Z}^3_0} \frac{\chi_N^2(k)}{|k|^2}\frac{k_1k_2}{|k|^2};$$

computing the sum over the four points (k_1,k_2,k_3), $(k_1,-k_2,k_3)$, $(-k_1,k_2,k_3)$ and $(-k_1,-k_2,k_3)$, we see that $Q_N^{1,2}(0) = 0$; the same is true for other values of $i\neq j$. Next, by symmetry, one has

$$Q_N^{1,1}(0) = Q_N^{2,2}(0) = Q_N^{3,3}(0) = \frac{1}{3}\tau\sigma_N^2 \sum_{k\in\mathbb{Z}^3_0} \frac{\chi_N^2(k)}{|k|^2}\frac{k_1^2+k_2^2+k_3^2}{|k|^2} = \frac{1}{3}\tau\sigma_N^2 \sum_{k\in\mathbb{Z}^3_0} \frac{\chi_N^2(k)}{|k|^2}.$$

This completes the proof. □

Lemma 5.7. *Under the assumption in lemma 5.5, the covariance function Q_N satisfies the conditions in (1.6).*

Proof. The first property is satisfied by taking $\kappa = \frac{1}{6}\tau k_T^2$. To show the second property, let

$$\Gamma_k^N = \sigma_N \frac{\chi_N(k)}{|k|}, \quad k \in \mathbb{Z}_0^3;$$

we only need to prove that $\|\Gamma^N\|_{\ell^\infty} \to 0$. Indeed, by (5.3), one has for any k,

$$\chi_N^2(k) = \ell_N^{-1} \int_{\ell_N}^{2\ell_N} \left|\widehat{\theta}(\ell k)\right|^2 \mathrm{d}\ell \leqslant \|\theta\|_{L^1}^2,$$

thus $\|\Gamma^N\|_{\ell^\infty} \leqslant \sigma_N \|\theta\|_{L^1}$. It remains to show that $\sigma_N \to 0$ as $N \to \infty$. By definition,

$$\sigma_N \lesssim \left(\sum_{k\in\mathbb{Z}_+^3} \frac{\chi_N^2(k)}{|k|^2} \right)^{-1/2} \leqslant \left(\sum_{k\in\mathbb{Z}_+^3, |k|\leqslant M} \frac{\chi_N^2(k)}{|k|^2} \right)^{-1/2}$$

for any $M \geqslant 1$. By lemma 5.4, letting $N \to \infty$ in the above inequality yields

$$\limsup_{N\to\infty} \sigma_N \lesssim \left(\sum_{k\in\mathbb{Z}_+^3, |k|\leqslant M} \frac{\|\theta\|_{L^1}^2}{|k|^2} \right)^{-1/2} = \|\theta\|_{L^1}^{-1} \left(\sum_{k\in\mathbb{Z}_+^3, |k|\leqslant M} \frac{1}{|k|^2} \right)^{-1/2};$$

the last quantity vanishes as $M \to \infty$. □

5.4. Remarks on collisions, turbulence and statistical mechanics

A complicated open problem in the rigorous mathematical analysis of interacting charged particles is whether a collision term should appear or not in the kinetic description by a Vlasov type equation; as well as the consequences that this fact may have on the statistical properties of particles. Let us briefly discuss this topic, in order to illustrate the potential interest of our result in this connection.

Charged particle interaction is long range, mediated by the electric and magnetic fields, hence a mean field term must appear in the kinetic equation. The difficult question is whether a collision term should also appear. Indeed, the interaction kernel is singular at the origin, hence the pairwise interaction of two very close particles is similar to a collision, or at least more similar to a collision than to the infinitesimal interaction that cumulates in a mean field way in the case of smooth long range interactions. Therefore, some form of collision operator could appear, maybe depending on the degree of sparseness of the system. In the regime considered in the outstanding work of S. Serfaty [29] (the first rigorous work covering the true Coulomb potential), the collision operator does not arise, the final equation is purely mean field; but this work adds the mean field factor $\frac{1}{N}$ in front of the interactions, N being the particle number, factor that does not explicitly appear in a traditional physical model of interacting ions and electrons.

When collisions exist, like in ordinary uncharged atoms or molecules of a gas, in the macroscopic description of the particle density it is natural to find a Laplacian describing particle diffusion. Each interaction, typically very short range, produces a quite unpredictable (in the sense of very unstable) change of direction of the motion of the particle. The accumulation of these almost random changes leads to a Brownian behaviour and to a diffusion property at the density level. This is rigorously formalized in a number of works, see in particular [3].

For ions and electrons, we do not know whether 'Coulomb collisions' (namely the relatively strong change of direction of motion occurring when particles are very close) may give rise to a true diffusive behaviour.

Our result proves that a diffusive behaviour occurs due to small scale turbulence. We rigorously identify a set of assumptions under which one can prove that diffusion (a Laplacian) arises in the phase density. It is diffusion in velocity, namely a form of randomness in the velocity that may be a surrogate of the result of collisions.

The diffusive behaviour, in turn, may have consequences on statistical mechanics properties. Uncharged particles usually satisfy a Maxwellian local equilibrium, which quantifies the wide randomness of velocity directions of particles, due in particular to collisions. In plasmas, the validity of the Maxwellian distribution is controversial, since collisions are less efficient; it is presumably more correct for electrons, possibly less for ions. The work [9] and references quoted therein discuss this topic. The main point of [9] is that turbulence acts as a substitute of collisions, producing the randomness needed to establish a form of statistical equilibrium, which however is not Maxwellian. We hope to clarify in future works if the diffusion in velocity proved to exist in our work may have consequences on the statistical equilibrium, like the one investigated in [9].

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Data availability statement

No new data were created or analysed in this study.

ORCID iD

Dejun Luo 0000-0002-6944-3200

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