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Multicomplexes and relative bounded cohomology

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Introduction

Simplicial volume. The simplicial volume is an invariant of manifolds introduced by Gromov in [Gro82]. Let M be an oriented compact connected topological manifold with (possibly empty) boundary, and let $n = \dim(M)$. The *simplicial volume* of M is defined as the infimum of ℓ^1 -norms of real fundamental cycles of M . More precisely, we endow the vector space of real singular n -chains $C_n(M)$ with the ℓ^1 -norm defined by

$$\left\| \sum a_\sigma \cdot \sigma \right\|_1 = \sum |a_\sigma|.$$

This norm descends to a seminorm on the homology $H_n(M, \partial M)$ with real coefficients. The simplicial volume $\|M, \partial M\|$ of M is the ℓ^1 -seminorm of the real fundamental class $[M, \partial M] \in H_n(M, \partial M)$ of M .

Simplicial volume in geometry and topology. Despite being purely homotopic in nature, the simplicial volume carries a lot of information on the geometric structures a manifold can carry. For example, it vanishes for Riemannian manifolds with non-negative Ricci curvature [CG72], and it is known to be positive for negatively curved manifolds [Gro82, IY82]. On the other hand, the simplicial volume of a closed manifold M relates to other geometrically defined invariants, such as its *minimal volume* – the infimum of the volumes of complete Riemannian metrics on M with sectional curvature bounded by 1 in absolute value. A major consequence of Gromov’s *Main Inequality* is that the simplicial volume provides a lower bound of the minimal volume, up to a constant depending only on the dimension [Gro82]. Similar estimates hold for minimal volume entropy, which measures the growth rate of volumes of balls in the universal covering of M [Gro82]. Since then, the simplicial volume has found several applications in understanding the growth of balls and systolic inequalities on Riemannian manifolds [Gut11, BK19, BS21, Alp24, BS24].

The relationship between simplicial volume and geometry becomes especially strong for manifolds with constant sectional curvature. For closed hyperbolic manifolds, a foundational result by Gromov and Thurston shows that $\|M\| = \text{vol}(M)/v_n$, where v_n is a universal constant depending only on the dimension [Thu79, Gro82]. This characterization of hyperbolic volume as a topological invariant plays a central role in Gromov’s proof of Mostow rigidity. More generally, Gromov’s Proportionality Principle states that the ratio $\|M\|/\text{vol}(M)$ remains constant along closed Riemannian manifolds whose universal coverings are isometric.

The relationship between simplicial volume and other topological invariants is both rich and significant, and it gives rise to several long-standing open problems. For example, it was asked by Gromov whether the vanishing of the simplicial volume of a closed aspherical manifold implies the vanishing of its L^2 -Betti numbers – or, in particular, of its Euler characteristic [Gro93, p. 232]. See [LMR22] for further developments and related questions. The vanishing of the Euler characteristic for non-positively curved manifolds has also been investigated in [CRW23] in connection

with another long-standing conjecture of Gromov, which asserts that Riemannian manifolds with non-positive sectional curvature and negative Ricci curvature have positive simplicial volume [Gro82]. On the other hand, the simplicial volume has also been studied in relation to the Chern conjecture, which predicts that the Euler characteristic of closed affine manifolds vanishes [Fri17, BCL18, CM25].

It is known that the simplicial volume of products of closed manifolds satisfies the bounds $\|B\| \cdot \|F\| \leq \|B \times F\| \leq c_{m,n} \|B\| \cdot \|F\|$, where $c_{m,n}$ is a constant depending only on the dimensions $n = \dim(B)$ and $m = \dim(F)$. Since products are special cases of fiber bundles, it is natural to ask whether similar inequalities hold for the total spaces E of fiber bundles with fiber F and base B . The inequality $\|E\| \leq c_{m,n} \|B\| \cdot \|F\|$ cannot hold in general, due to the existence of hyperbolic manifolds that fiber over the circle. On the other hand, it is known that the inequality $\|E\| \geq \|F\| \cdot \|B\|$ holds when F is a surface [HK01, Buc09]. Moreover, the simplicial volume of E vanishes when B is a simply-connected sphere [KR23]. See also [Kas24] for more recent developments in the case where B is simply-connected.

Simplicial volume of manifolds with boundary. Although the definition of simplicial volume applies to both closed manifolds and manifolds with boundary, its behavior in the presence of a non-empty boundary is far less understood. Since every relative fundamental cycle of the manifold induces a fundamental cycle on the boundary, one obtains the elementary inequality

$$\|M, \partial M\| \geq \frac{\|\partial M\|}{(n+1)}.$$

This bound can be improved in the case of 3-manifolds, as shown in [BFP15]. As in the closed case, exact computations of the simplicial volume for manifolds with boundary are very rare. To facilitate computations in certain cases, a variation called the *ideal simplicial volume* has been introduced in [FM21].

It is well known that the simplicial volume of closed manifolds is largely governed by their fundamental group. In particular, simply-connected closed manifolds have vanishing simplicial volume. However, this property fails to hold in the presence of boundary: by the Lickorish–Wallace theorem, every closed 3-manifold bounds a simply connected 4-manifold [Lic62, Wal60]. Consequently, there exist simply connected 4-manifolds whose boundaries are hyperbolic 3-manifolds, and thus have positive simplicial volume. More generally, the simplicial volume of a manifold with boundary depends in a more subtle way on its topology.

Also the interplay between simplicial volume and the geometry of a manifold becomes more delicate when the boundary is non-empty. For example, it is known that, if the interior of M is the compactification of a cusped hyperbolic manifold, then $\|M, \partial M\|$ is proportional to the hyperbolic volume of its interior [Gro82, FM11]. On the other hand, we know that the same equality cannot hold in the case of hyperbolic manifolds with geodesic boundary [Jun97, FP10, BFP17]. Moreover, extending upper bounds for simplicial volume in terms of minimal volume or minimal volume entropy to manifolds with non-empty boundary remains highly non-trivial [Gro82, Section 4.3][Gut11, Section 6].

The simplicial volume of M can be studied via the locally-finite simplicial volume of its interior $\text{int}(M)$. In general, there is an obvious inequality $\|M, \partial M\| \leq \|\text{int}(M)\|^{\text{lf}}$, which is known to be strict, for example, when $\|\text{int}(M)\|^{\text{lf}}$ is infinite. However, it remains open whether equality holds when the right-hand side is finite.

The simplicial volume of manifolds with boundary has also been studied from the perspective of Topological Quantum Field Theory on cobordism categories in [LMR22]. In particular, within the context of aspherical null-cobordisms, it was asked in [LMR22, Question 3.28] whether every closed aspherical manifold with amenable fundamental group can arise as the π_1 -injective boundary of an aspherical manifold whose simplicial volume vanishes.

Bounded cohomology. In order to study the simplicial volume, Gromov introduced in [Gro82] its dual theory of bounded cohomology. Given a real singular cochain $\varphi \in C^n(X)$ on a topological space X , its ℓ^∞ -norm is defined by

$$\|\varphi\|_\infty = \sup\{|\varphi(\sigma)|, \sigma \in S_n(X)\} \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

where $S_n(X)$ denotes the set of singular n -simplices of X . A cochain is called bounded if its ℓ^∞ -norm is finite. Bounded cochains form a subcomplex of $C_b^\bullet(X)$ of $C^\bullet(X)$, and the cohomology of this complex is the bounded cohomology $H_b^\bullet(X)$ of X . The inclusion of complexes $C_b^\bullet(X) \subseteq C^\bullet(X)$ defines the *comparison map*

$$\text{comp}^\bullet: H_b^\bullet(X) \rightarrow H^\bullet(X),$$

which is, in general, neither injective nor surjective. Any topological construction involving only uniformly finitely many simplices preserves bounded cochains. As a result, one can show that bounded cohomology satisfies all the Eilenberg–Steenrod axioms, except for excision. This lack of excision makes bounded cohomology particularly difficult to compute. Nonetheless, the ℓ^1 -seminorm in homology (which underlies the definition of simplicial volume) and the ℓ^∞ -seminorm in cohomology are dual to each other. As a consequence, the vanishing of $H_b^\bullet(X)$ forces the vanishing of the ℓ^1 -seminorm in homology, and hence the vanishing of the simplicial volume when X is a manifold.

A very peculiar aspect of the bounded cohomology of X is that it only depends on the fundamental group of X . In particular, simply-connected spaces have vanishing bounded cohomology. This result holds more generally when the fundamental group of X is amenable, by a result of Johnson [Joh72]. The proof of this fundamental result given by Gromov in [Gro82] is based on the theory of multicomplexes, whose foundations have been established more recently by the work of Frigerio and Moraschini [FM23]. Beside this, multicomplexes have been used in [Kue15, KK15, Str17, Fri22]. Although very transparent from a geometric point of view, the theory of multicomplexes can become quite technical. For this reason, much progress in the theory of bounded cohomology has been achieved through tools from homological algebra. This alternative approach was initiated by Ivanov in [Iva85], and further developed by Burger and Monod in [Mon01, BM02].

Relative bounded cohomology. As in the absolute case, understanding the simplicial volume of manifolds with boundary naturally leads to the study of the relative bounded cohomology of pairs of spaces. The first natural question in this direction is whether the relative bounded cohomology of a pair of spaces (X, A) coincides with the bounded cohomology of their fundamental groups. For instance, when both X and A are path-connected and the inclusion $A \hookrightarrow X$ induces an injective map on fundamental groups, one can associate to (X, A) the group pair $(\pi_1(X), \pi_1(A))$. A straightforward application of the Five Lemma then shows that the $H_b^n(X, A)$ is isomorphic to $H_b^n(\pi_1(X), \pi_1(A))$. However, controlling the natural

semi-norms in bounded cohomology is highly non-trivial. Several efforts have been made to strengthen this isomorphism to an isometric one, using both Gromov’s theory of multicomplexes and Ivanov’s techniques.

The first such attempt is due to Park in [Par03], who described the relative bounded cohomology of (X, A) via the group homomorphism $\pi_1(A) \rightarrow \pi_1(X)$, using the framework of mapping cones. However, the norm studied by Park on relative bounded cohomology is only bi-Lipschitz equivalent to Gromov’s norm and not isometric [FP12, Section 6]. The first evidence of an isometric isomorphism was provided by Frigerio and Pagliantini in [FP12], under additional assumptions on the pair (X, A) . Using a relative version of Thurston’s measure homology [Thu79, Löh06], they proved that $H_b^n(X, A)$ is isometrically isomorphic to $H_b^n(\pi_1(X), \pi_1(A))$ when both X and A are path-connected, the inclusion $A \hookrightarrow X$ is π_1 -injective, and it induces an isomorphism on higher homotopy groups. Building on the theory of fundamental groupoids, Blank extended this result to the case in which A is not necessarily path-connected, under the same assumptions on fundamental groups and higher homotopy groups.

The situation is better understood when every connected component of A has amenable fundamental group. In this case, it was shown in [BBF⁺14] that $H_b^n(X, A)$ is isometrically isomorphic to $H_b^n(X)$ for every $n \in \mathbb{N}$, without requiring any further assumptions on the homotopy type of the pair. An alternative proof of this fact, based on the theory of multicomplexes, has been provided in [KK15].

Finally, the theory of multicomplexes has also been employed by Kuessner in [Kue15] to study relative bounded cohomology. However, Kuessner’s approach appears to rely implicitly on further assumptions concerning the higher homotopy groups of the pair (X, A) . The first goal of this thesis is to adopt the framework of multicomplexes developed by Frigerio and Moraschini in [FM23] in order to study the relative bounded cohomology of topological spaces. We refer the reader to Section 5.4 for a discussion that clarifies the limitations of Kuessner’s approach and explains how it relates to the framework developed in this work.

Multicomplexes. Multicomplexes are simplicial structures introduced by Gromov in [Gro82]. According to [Gro82], a multicomplex is “a set K divided into the union of closed affine simplices Δ_i , $i \in I$, such that the intersection of any two simplices $\Delta_i \cap \Delta_j$ is a (simplicial) subcomplex in Δ_i as well as in Δ_j ”. Informally, a multicomplex can be seen as a generalization of a simplicial complex in which distinct simplices may share arbitrary collections of faces and, in particular, may have the same set of vertices. More formally, multicomplexes are unordered Δ -complexes in which every simplex has distinct vertices, or, equivalently, a symmetric simplicial set in which every non-degenerate simplex has distinct vertices. We refer the reader to Section 1.1 for a precise definition and for a discussion of the relationship between multicomplexes and other classical simplicial structures. As motivated by Gromov’s deep intuition and explained in [FM23, Remark 4.16], multicomplexes provide the preferred simplicial framework for the study of bounded cohomology.

Singular multicomplexes. Let X be a topological space. The *singular multicomplex* $\mathcal{K}(X)$ of X is the multicomplex whose n -simplices are the singular n -simplices with distinct vertices in X , up to affine reparametrization of the standard simplex Δ^n . The geometric realization $|\mathcal{K}(X)|$ is a CW-complex whose 0-skeleton corresponds to the set X itself. Moreover, there is a natural projection

$S_X: |\mathcal{K}(X)| \rightarrow X$, which is a homotopy equivalence when X is a CW-complex [FM23, Theorem 1]. In the theory of multicomplexes, the singular multicomplex plays the same role as the singular complex does in the theory of simplicial sets [Mil57, May92]. One of its key features is the property of *completeness*, which is the analogue of the Kan condition for simplicial sets (see Definition 2.1). This property allows for a purely combinatorial description of the homotopy groups of $\mathcal{K}(X)$, and hence of X [FM23, Theorem 3.9].

As in the theory of simplicial sets, we can reduce the size of the singular multicomplex without changing its homotopy type. To this end, we define a submulticomplex $\mathcal{L}(X)$ of $\mathcal{K}(X)$, called the *minimal multicomplex*, as follows: the 0-skeleton of $\mathcal{L}(X)$ is the same as the one of $\mathcal{K}(X)$ (hence it corresponds to the set X); having already defined the n -skeleton $\mathcal{L}(X)^n$ of $\mathcal{L}(X)$, we define the $(n+1)$ -skeleton by adding to $\mathcal{L}(X)^n$ one $(n+1)$ -simplex for each homotopy class of $(n+1)$ -simplices of $\mathcal{K}(X)$ whose facets are all contained in $\mathcal{L}(X)^n$. The inclusion $|\mathcal{L}(X)| \hookrightarrow |\mathcal{K}(X)|$ is a homotopy equivalence, and although the construction of $\mathcal{L}(X)$ depends on several choices, the resulting submulticomplex is unique up to simplicial isomorphism [FM23, Theorem 3.23].

Finally, we can associate to X an *aspherical multicomplex* $\mathcal{A}(X)$, which is defined as the first Postnikov quotient of $\mathcal{L}(X)$, obtained by identifying simplices that share the same 1-skeleton. There is an obvious simplicial projection $\pi: \mathcal{L}(X) \rightarrow \mathcal{A}(X)$, which restricts to the identity on the 1-skeleton. This quotient map induces an isomorphism on fundamental groups and kills higher homotopy [FM23, Theorem 3.33]. Therefore the topological realization $|\mathcal{A}(X)|$ of $\mathcal{A}(X)$ is a model for the classifying space of the fundamental group of X .

The construction and properties of the singular multicomplexes are, up to this point, entirely analogous to those of the singular simplicial set. However, this analogy breaks down when considering functorial properties. In fact, the singular multicomplex does *not* define a functor from the category of topological spaces to the category of multicomplexes. If $f: X \rightarrow Y$ is a continuous map and σ is a singular simplex of X , then $f \circ \sigma$ is a singular simplex of Y . However, the property of being injective on vertices may be lost under this push-forward. This issue can be avoided by requiring f to be injective, and this is the most important case of interest for our purposes.

The situation for pairs. Let (X, A) be a pair of topological spaces. Since the inclusion $A \hookrightarrow X$ is injective, it induces a simplicial embedding $j_{\mathcal{K}}: \mathcal{K}(A) \hookrightarrow \mathcal{K}(X)$ on singular multicomplexes. Let $\mathcal{L}(A)$ and $\mathcal{L}(X)$ denote the corresponding minimal multicomplexes, and let $i_A: \mathcal{L}(A) \hookrightarrow \mathcal{K}(A)$ and $i_X: \mathcal{L}(X) \hookrightarrow \mathcal{K}(X)$ be their simplicial inclusions. As already noticed in [Kue15, Section 1.3], there is an obvious simplicial map $j_{\mathcal{L}}: \mathcal{L}(A) \rightarrow \mathcal{L}(X)$ that sends each simplex Δ of $\mathcal{L}(A)$ to the unique simplex of $\mathcal{K}(X)$ which is homotopic to Δ in X relative to the 0-skeleton (Proposition 5.1). The map $j_{\mathcal{L}}$ is injective if and only if every connected component of A is π_n -injective for every $n \in \mathbb{N}_{\geq 1}$ (Proposition 5.4).

Let $\pi: \mathcal{L}(X) \rightarrow \mathcal{A}(X)$ denote the simplicial projection identifying simplices sharing the same 1-skeleton. Then $j_{\mathcal{L}}$ factors through π , and thus induces a well-defined simplicial map $j_{\mathcal{A}}: \mathcal{A}(A) \rightarrow \mathcal{A}(X)$. We denote by $\mathcal{A}_X(A)$ the image of $\mathcal{A}(A)$ in $\mathcal{A}(X)$ under $j_{\mathcal{A}}$, so that the pair of multicomplexes $(\mathcal{A}(X), \mathcal{A}_X(A))$ is well-defined. We denote by $q: \mathcal{A}(A) \rightarrow \mathcal{A}_X(A)$ the surjective map induced by $j_{\mathcal{A}}$. Since

simplices of $\mathcal{A}(X)$ are uniquely determined by their 1-skeleton, it follows that $j_{\mathcal{A}}$ is a simplicial embedding if and only if A is π_1 -injective in X (Proposition 5.5). In this case, $\mathcal{A}_X(A)$ is simplicially isomorphic to $\mathcal{A}(A)$. In short, we have the following diagram of simplicial maps

$$\begin{array}{ccccccc} \mathcal{K}(A) & \xleftarrow{i_A} & \mathcal{L}(A) & \xrightarrow{\pi} & \mathcal{A}(A) & \xrightarrow{q} & \mathcal{A}_X(A) \\ \downarrow j_{\mathcal{K}} & & \downarrow j_{\mathcal{L}} & & \downarrow j_{\mathcal{A}} & & \downarrow \\ \mathcal{K}(X) & \xleftarrow{i_X} & \mathcal{L}(X) & \xrightarrow{\pi} & \mathcal{A}(X) & \xlongequal{\quad} & \mathcal{A}(X). \end{array}$$

Isometric isomorphisms. Gromov's fundamental insight is that the n -skeleton of $\mathcal{A}(X)$ is the quotient of the n -skeleton of $\mathcal{L}(X)$ by the action of an amenable group Γ_n . This is a non-trivial consequence of the fact that $\mathcal{A}(X)$ is constructed from $\mathcal{L}(X)$ by killing higher homotopy groups, which are abelian – and therefore amenable. The situation becomes more delicate in the relative setting. In fact, when $\mathcal{L}(A)$ is a submulticomplex of $\mathcal{L}(X)$ – which occurs precisely when A is π_n -injective in X for every $n \in \mathbb{N}_{\geq 1}$ – the action of Γ_n on $\mathcal{L}(X)$ preserves the submulticomplex $\mathcal{L}(A)$ only if A is also π_n -surjective in X for every $n \in \mathbb{N}_{\geq 2}$ (Proposition 3.10).

A pair (X, A) of topological spaces is called *good* if it is a CW-pair such that the inclusion $A \hookrightarrow X$ induces an injective map on fundamental groups and an isomorphism on higher homotopy, i.e. $\pi_1(A \hookrightarrow X, x)$ is injective and $\pi_n(A \hookrightarrow X, x)$ is an isomorphism for every $x \in A$ and every $n \in \mathbb{N}_{\geq 2}$. We refer the reader to Remark 1.14 for an explanation of why we prefer to work within the framework of CW-pairs. We emphasize that our interest lies primarily in the homotopy-theoretic properties of these pairs, rather than in any explicit cell decomposition of the space. Note that we do not assume X or A to be path-connected.

For good pairs, the multicomplex $\mathcal{A}(A)$ naturally embeds as a submulticomplex of $\mathcal{A}(X)$. In this setting, the techniques developed by Frigerio and Moraschini in [FM23] extend to the relative case as well.

THEOREM 1. *Let (X, A) be a good pair. Then, for every $n \in \mathbb{N}$, there is a canonical isometric isomorphism*

$$\Phi^n: H_b^n(\mathcal{A}(X), \mathcal{A}(A)) \rightarrow H_b^n(X, A).$$

Of course, the pair (X, \emptyset) is good for every CW-complex X . Therefore, Theorem 1 naturally generalizes Gromov's result stating that the bounded cohomology of X is isometrically isomorphic to the bounded cohomology of its aspherical multicomplex $\mathcal{A}(X)$ [Gro82][FM23, Theorem 4.23]. This reflects Gromov's original approach to prove that bounded cohomology depends only on the fundamental group and is thus insensitive to higher homotopy. However, all attempts to extend Gromov's results to the relative setting must confront the fact that higher homotopy cannot be entirely ignored if one wishes to maintain control over the norms. In fact the notion of good pairs was already present in the work of Frigerio and Pagliantini [FP12], as well as in Blank's work [Bla16]. It is quite striking that such different approaches seem to require the same additional assumptions on pairs of spaces in order to treat the relative setting.

Thanks to Theorem 1, we can conclude that, within the class of good pairs, relative bounded cohomology depends only on the fundamental groups.

THEOREM 2. *Let $f: (X, A) \rightarrow (Y, B)$ be a continuous map between good pairs which is bijective on the sets of path-connected components. Assume that f induces isomorphisms on fundamental groups on every component of X and every component of A . Then the induced map*

$$H_b^n(f): H_b^n(Y, B) \rightarrow H_b^n(X, A)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

Theorem 2 is stated in [Gro82, pag. 57] without any assumptions on higher homotopy, but it is not clear to us how such assumptions could be removed. In fact, Gromov's statement is more general, since it allows f to induce an epimorphism with amenable kernel on fundamental groups. An analogous statement is proved in Section 5.6 for good pairs. However, in this case, we only obtain a bi-Lipschitz equivalence between the norms, rather than an isometry (Theorem 5.22).

We expect that alternative proofs of Theorem 2 can be obtained within the frameworks developed by Frigerio and Pagliantini [FP12] and by Blank [Bla16].

We know that, when every connected component of A has amenable fundamental group, then $H_b^n(X, A)$ is isometrically isomorphic to $H_b^n(X)$ for every $n \in \mathbb{N}$, without requiring any further assumptions on the homotopy type of the pair [BBF⁺14, Theorem 2] [KK15, Theorem 2.3]. However, the requirement that *every* boundary component of A is amenable – rather than only those one intends to discard – appears somewhat artificial. Thanks to Theorem 1, we can show that, at least within the class of good pairs, it suffices to assume amenability only for the connected components one wishes to disregard.

THEOREM 3. *Let (X, Y) be a good pair and let $A \subseteq Y$ be the union of some connected components of Y . Let $B = Y \setminus A$ be the union of the remaining connected components and let $j: (X, B) \rightarrow (X, Y)$ denote the inclusion. If every connected component of A has an amenable fundamental group, then the map*

$$H_b^n(j): H_b^n(X, Y) \rightarrow H_b^n(X, B)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

The group $\Pi(X, X_0)$. Our proof of Theorem 3 relies on the action on $\mathcal{A}(X)$ of the group $\Pi(X, X_0)$, which was introduced by Gromov in [Gro82]. Let $X_0 \subseteq X$ be a subset of basepoints of X . The elements of $\Pi(X, X_0)$ consist of finite collections of (homotopy classes of) paths in X that induce a permutation of X_0 with finite support.

There is a natural morphism of groupoids from the the fundamental groupoid $\Pi_1(X, X_0)$ of X based at X_0 to the groupoid $\Pi(X, X_0) \rtimes X_0$ associated to the action of $\Pi(X, X_0)$ on X_0 by permutations. This map is universal among groups acting on X_0 by permutations with finite support (Remark 4.3).

The group $\Pi(X, X_0)$ plays a central role in our applications, where geometric information on X is used to detect amenable subgroups of $\Pi(X, X_0)$. For example, if X has amenable fundamental group, then also the group $\Pi(X, X_0)$ is amenable – and, in particular, it has vanishing bounded cohomology (Proposition 4.2). A group with vanishing bounded cohomology with trivial real coefficients is called *boundedly acyclic*. We show that the group $\Pi(X, X_0)$ is boundedly acyclic in much more generality.

THEOREM 4. *Let X be a path-connected topological space with countable fundamental group and let X_0 be an infinite countable subset of X . Then the group $\Pi(X, X_0)$ is boundedly acyclic.*

In Section 4.4 we show that the group $\Pi(X, X_0)$ admits the structure of a permutational wreath product. Theorem 4 then follows from a result of Monod [Mon22]. The class of boundedly acyclic groups has attracted considerable attention in recent years, due to its relevance in new computations of the bounded cohomology of discrete groups [Mon22, MR23, FFLM24, CFFLM24].

Bi-Lipschitz isomorphisms. By removing the assumptions on the higher homotopy of (X, A) , we can only retain a bi-Lipschitz control over the norms. Recall that there is a well-defined simplicial map $j_A: \mathcal{A}(A) \rightarrow \mathcal{A}(X)$. The image of j_A inside $\mathcal{A}(X)$ is denoted by $\mathcal{A}_X(A)$, so that there is a well-defined pair of multicomplexes $(\mathcal{A}(X), \mathcal{A}_X(A))$. Of course, in the case in which A is π_1 -injective in X , then $\mathcal{A}_X(A)$ is simplicially isomorphic to $\mathcal{A}(A)$.

Let N be the kernel of the map $\pi_1(A) \rightarrow \pi_1(X)$ induced by the inclusion. As every normal subgroup of $\pi_1(A)$, the subgroup N determines a normal subgroup \widehat{N} of $\Pi(A, A)$, which then acts on $\mathcal{A}(A)$. In Section 5.1 we show that $\mathcal{A}_X(A)$ is obtained from $\mathcal{A}(A)$ as the quotient by the action of \widehat{N} . This fact implies that (the geometric realization of) $\mathcal{A}_X(A)$ is a model for the classifying space for the group $\pi_1(A)/N$ (Lemma 5.6). Thus, the pair $(\mathcal{A}(X), \mathcal{A}_X(A))$ consists of aspherical multicomplexes.

We prove that, when N is amenable, then the bounded cohomology of (X, A) is bi-Lipschitz isomorphic to the simplicial bounded cohomology of $(\mathcal{A}(X), \mathcal{A}_X(A))$.

THEOREM 5. *Let (X, A) be a CW-pair such that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Then, for every $n \in \mathbb{N}_{\geq 1}$, there is a constant $C_n \in \mathbb{R}_{\geq 1}$ (depending only on the degree) and a bi-Lipschitz isomorphism of vector spaces*

$$\Psi^n: H_b^n(\mathcal{A}(X), \mathcal{A}_X(A)) \rightarrow H_b^n(X, A),$$

such that, for every $\alpha \in H_b^n(\mathcal{A}(X), \mathcal{A}_X(A))$,

$$(n+2)^{-1}(2C_n)^{-2} \cdot \|\alpha\|_\infty \leq \|\Psi^n(\alpha)\|_\infty \leq 2C_n(n+2) \cdot \|\alpha\|_\infty.$$

The constants C_n denote the number of n -simplices in a suitable triangulation of an n -dimensional prism $\Delta^{n-1} \times [0, 1]$ (Definition 2.8). We refer the reader to Section 2.2 for an explicit formula for C_n .

We underline the fact that, unlike the map Φ^n in Theorem 1, the map Ψ^n is *a priori not canonical*, since some choices are required in our construction (see Remark 5.17). The isomorphism Ψ^n is constructed by using the machinery of mapping cones developed by Park in [Par03]. We know that mapping cones do not give in general control over the norms [FP12], therefore we are not able to show that Ψ^n is isometric. Moreover, the bi-Lipschitz constants in Theorem 5 are far from being optimal (see Remark 5.18).

Relative Vanishing Theorems. Let X be a topological space. A subset U of X is called *amenable* if the image of the homomorphism

$$\pi_1(U, x) \rightarrow \pi_1(X, x)$$

induced by the inclusion is an amenable subgroup of $\pi_1(X, x)$ for every $x \in U$. An open cover \mathcal{U} of X is called amenable if it consists of amenable open subsets. Gromov's *Vanishing Theorem* is a powerful vanishing result for the comparison map in presence of amenable open covers of small multiplicity. More precisely, if X admits an amenable open cover \mathcal{U} of multiplicity $\text{mult}(\mathcal{U}) \leq k$, then the comparison map $H_b^n(X) \rightarrow H^n(X)$ vanishes in every degree $n \geq k$ [Gro82, Section 3.1].

Several proofs of this result are available in the literature, whose techniques range between Gromov's theory of multicomplexes [Gro82, Fri22, FM23], sheaf theory [Iva85], equivariant nerves and classifying spaces for families [LS20] and homotopy theory [Rap24]. A generalization of Gromov's result to the relative setting has been provided by Li, Löh and Moraschini in [LLM22] and by Raptis in [Rap24].

In [Cap25] we extended the results of Li, Löh and Moraschini, building on the framework for relative bounded cohomology via multicomplexes. We refer the reader to Remark 6.4 for a detailed discussion on the relationship between our results and the ones in [LLM22]. A pair of topological spaces (X, A) is called *triangulable* if there exists a pair of simplicial complexes (T, S) such that $(X, A) = (|T|, |S|)$.

THEOREM 6 (Relative Vanishing Theorem). *Let (X, A) be a triangulable pair and assume that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Let \mathcal{U} be an amenable cover of X by path-connected open subsets such that:*

- (RC1) *For every $U \in \mathcal{U}$ such that $U \cap A \neq \emptyset$, $U \cap A$ is path-connected;*
- (RC2) *For every $U \in \mathcal{U}$ such that $U \cap A \neq \emptyset$, the inclusion*

$$\text{im}(\pi_1(U \cap A, x) \rightarrow \pi_1(X, x)) \hookrightarrow \text{im}(\pi_1(U, x) \rightarrow \pi_1(X, x))$$

is an isomorphism for every $x \in U \cap A$.

Then the comparison map $\text{comp}^n: H_b^n(X, A; \mathbb{R}) \rightarrow H^n(X, A; \mathbb{R})$ vanishes for every $n \geq \text{mult}(\mathcal{U})$.

Our strategy for proving Theorem 6 follows the same geometric intuition proposed by Gromov [Gro82], and further developed by Frigerio and Moraschini in [FM23]. The amenable open cover \mathcal{U} of X allows us to identify an amenable subgroup G of $\Pi(X, X)$, which thus acts on the aspherical multicomplex $\mathcal{A}(X)$. The regularity conditions (RC1) and (RC2) ensure that the action $G \curvearrowright \mathcal{A}(X)$ has a controlled behavior on the subcomplex $\mathcal{A}_X(A)$ (Lemma 6.14).

Stronger assumptions on the open cover lead to stronger vanishing results for the comparison map. We refer the reader to Section 6.1 for the definitions of *weak convexity*, *convexity*, *relative multiplicity* and *nerve pairs*. We just mention that the relative multiplicity $\text{mult}_A(\mathcal{U})$ provides a lower bound for the standard multiplicity $\text{mult}(\mathcal{U})$. Furthermore, under our assumptions on the open cover, the nerve $N(\mathcal{U}_A)$ of the cover induced on A is naturally a subcomplex of the nerve $N(\mathcal{U})$ of \mathcal{U} . In this context, the relative dimension $\dim(N(\mathcal{U}), N(\mathcal{U}_A))$ of the nerve pair is a lower bound for $\text{mult}_A(\mathcal{U})$.

THEOREM 7. *Let (X, A) be a triangulable pair and assume that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Let \mathcal{U} be a locally-finite amenable cover of X by path-connected open subsets satisfying (RC1) and (RC2).*

- (1) *If \mathcal{U} is weakly-convex on A , then the comparison map comp^n vanishes for every $n \geq \text{mult}_A(\mathcal{U})$.*

- (2) If \mathcal{U} is convex, then for every $n \in \mathbb{N}$ there exists a map Θ^n such that the following diagram commutes

$$\begin{array}{ccc} H_b^n(X, A) & \xrightarrow{\text{comp}^n} & H^n(X, A) \\ \downarrow \Theta^n & & \uparrow H^n(\nu) \\ H^n(N(\mathcal{U}), N(\mathcal{U}_A)) & \xrightarrow{\cong} & H^n(|N(\mathcal{U})|, |N(\mathcal{U}_A)|), \end{array}$$

where $\nu: (X, A) \rightarrow (|N(\mathcal{U})|, |N(\mathcal{U}_A)|)$ is a nerve map.

By duality, Theorem 6 and Theorem 7 yield vanishing results for relative simplicial volume (see Corollary 6.7). As already underlined in [LMR22], these results are however strictly weaker than the main known criterion for the vanishing of relative simplicial volume in presence of amenable open covers [LMR22, Theorem 3.13], which relies on Gromov's Vanishing Theorem for non-compact manifolds [FM23, Corollary 11]. We refer the reader to Section 6.2 for a more detailed discussion on this topic.

Additivity of simplicial volume. Gromov's *Additivity Theorem* provides a powerful tool for computing the simplicial volume of manifolds obtained by gluings. The case of connected sums and gluings along π_1 -injective boundary components with amenable fundamental group was first discussed in [Gro82, Section 3.5]. A complete proof in the case of 3-manifolds was later given by Soma in [Som81], and the general case was established in [BBF⁺14]. Gluings along amenable portions of the boundary are also treated in [Kue15].

Applications of the Additivity Theorem abound in low-dimensional and geometric topology. In [Som81], Soma introduces a knot and link invariant derived from the simplicial volume of their exteriors. In this context, the additivity of simplicial volume under connected sums and gluings along annuli implies that this invariant is additive under disjoint unions and connected sums of knots. On the other hand, additivity along boundary tori is used by Soma to show that every graph manifold has vanishing simplicial volume. Combined with the Geometrization Theorem, this yields that connected sums of graph manifolds are precisely the closed 3-manifolds with vanishing simplicial volume.

We refer the reader to the references in [BBF⁺14] and [Kue15] for further applications of Gromov's Additivity Theorem in various areas of geometric and low-dimensional topology. Without aiming to be exhaustive, let us point out that recently it has been exploited to study rigidity properties of generalized graph manifolds [FLS15, CSS19], to compute the spectrum of simplicial volumes of compact manifolds [HL21] and the simplicial volume of mapping tori of 3-manifolds [BN20]. The additivity of simplicial volume for gluings along amenable boundaries has also been adapted in [LLM22] to the case of boundedly acyclic boundaries.

Gromov's Additivity Theorem, first stated in [Gro82], has been addressed independently in [BBF⁺14] and in [Kue15] with two different approaches. In the former, the authors deduce the additivity of simplicial volume by using techniques from homological algebra developed by Ivanov in [Iva85]. In the latter, the proof relies on the theory of multicomplexes.

However, as already mentioned, Kuessner's approach to bounded cohomology via multicomplexes underestimates some difficulties that may arise in the relative setting. In particular, issues emerge when the manifolds involved are not aspherical

or when gluings are performed along proper submanifolds of boundary components. We refer the reader to Remark 5.21 and Remark 7.9 for a more detailed discussion of these limitations. It is also worth noting that the approach in [BBF⁺14] does not address gluings along proper submanifolds of boundary components.

Building on the framework developed for relative bounded cohomology via multi-complexes, in [Cap24] we provided a complete and self-contained proof of Gromov's Additivity Theorems (see Theorems 8 and 10 below).

THEOREM 8. *Let M_1, \dots, M_k be oriented compact triangulable n -manifolds with π_1 -injective boundary, $n \in \mathbb{N}_{\geq 2}$. Let*

$$(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$$

be a pairing of some oriented compact connected pairwise-disjoint submanifolds of $\sqcup_{j=1}^k \partial M_j$ of codimension 0. Let M be a manifold obtained by gluing M_1, \dots, M_k along the pairing. Assume that the following conditions hold for every $i \in \{1, \dots, h\}$ and every $j \in \{1, \dots, k\}$:

- S_i^\pm has an amenable fundamental group;
- ∂S_i^\pm is path-connected, π_1 -injective in the corresponding ∂M_j and the map $\pi_1(\partial S_i^\pm \hookrightarrow S_i^\pm)$ is an isomorphism.

Then there exists a constant $C_n \in \mathbb{N}_{\geq 1}$ (depending only on the dimension) such that

$$(2C_n)^3(n+2)^3 \cdot \|M, \partial M\| \geq \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|.$$

The constants C_n appearing in Theorem 8 are the same as those appearing in Theorem 5. The hypotheses of Theorem 8 are satisfied, for instance, by boundary connected sums.

COROLLARY 9. *Let M_1 and M_2 be oriented compact triangulable n -manifolds with π_1 -injective boundaries, $n \in \mathbb{N}_{\geq 4}$. Let $M = M_1 \natural M_2$ denote the boundary connected sum of M_1 and M_2 . Then*

$$(2C_n)^3(n+2)^3 \cdot \|M, \partial M\| \geq \|M_1, \partial M_1\| + \|M_2, \partial M_2\|.$$

When gluings are performed along entire boundary components and the manifolds involved are aspherical, we have the following additivity result.

THEOREM 10. *Let M_1, \dots, M_k be oriented compact triangulable aspherical n -manifolds with π_1 -injective aspherical boundary, $n \in \mathbb{N}_{\geq 2}$. Let M be a manifold obtained by gluing M_1, \dots, M_k along (some of) their boundary components. If the glued boundary components have amenable fundamental group, then*

$$\|M, \partial M\| = \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|.$$

The version of Gromov's Additivity Theorem provided in [BBF⁺14] does not cover the case in which there are non-amenable boundary components which remain unglued. Therefore, in the context of aspherical manifolds, Theorem 10 is a stronger result.

Our subadditivity estimates follow from a version of Gromov Equivalence Theorem for good pairs (Theorem 7.21), which in turn relies on the isometric isomorphism provided by Theorem 3. Conversely, superadditivity estimates are derived from extension properties of bounded coclasses with controlled norm (Proposition 7.12). This extension is carried out at the level of the aspherical multicomplexes

associated to the pairs $(M, \partial M)$. The passage from the pair $(M, \partial M)$ to the corresponding pair of aspherical multicomplexes is achieved via the maps Φ^n and Ψ^n , appearing in Theorem 1 and Theorem 5, respectively. The choice between these maps depends on whether the pair $(M, \partial M)$ is good. Different control over the norms for good and non-good pairs translates in different estimates for the simplicial volume in Theorems 10 and 8.

The notion of good pair is very well suited when gluing aspherical spaces. In fact, if a finite CW-complex Z is expressed as a union of two aspherical subcomplexes X and Y , then Z is aspherical provided that the pairs $(X, X \cap Y)$ and $(Y, X \cap Y)$ are good [Whi78][Edm20, Theorem 3.1]. Building on this fact, we deduce Theorem 10 from a more general result regarding good pairs (Proposition 7.26).

The amenability of boundary components ensures also the subadditivity of simplicial volume in presence of gluings of portions of the boundary. The following criterion has been obtained by the author in collaboration with Kevin Li and Clara Löh in [CLL25] in the context of branched coverings.

THEOREM 11. *Let M_1, \dots, M_k be oriented compact n -manifolds, $n \in \mathbb{N}_{\geq 2}$. Let $(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$ be a pairing of some oriented compact connected pairwise-disjoint submanifolds of $\sqcup_{j=1}^k \partial M_j$ of codimension 0. Let M be a manifold obtained by gluing M_1, \dots, M_k along the pairing. We denote by B_j the compact submanifold of ∂M_j of codimension 0 whose interior is the complement in ∂M_j of the gluing loci S_i^\pm . Assume that the following conditions hold for every $i \in \{1, \dots, h\}$ and every $j \in \{1, \dots, k\}$:*

- *Every connected component of ∂M_j and ∂S_i^\pm has amenable fundamental group;*
- *$\|S_i^\pm, \partial S_i^\pm\| = 0$ and $\|B_j, \partial B_j\| = 0$.*

Then $\|M, \partial M\| \leq \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|$.

In the proof of Theorem 11 we explicitly construct an efficient cycle which approximates the simplicial volume of M . This construction relies on the Uniform Boundary Condition of Matsumoto and Morita [MM85], combined with Gromov's Equivalence Theorem [BBF⁺14, Corollary 6]. Our argument does not make use of the theory of multicomplexes and is independent from the rest of this work.

Theorem 11 applies to boundary connected sums of n -manifolds with amenable boundary, provided that $n \in \mathbb{N}_{\geq 4}$.

COROLLARY 12. *Let M_1 and M_2 be oriented compact n -manifolds with non-empty boundaries, $n \in \mathbb{N}_{\geq 4}$. Assume that every connected component of ∂M_1 and ∂M_2 have amenable fundamental group. Let $M = M_1 \natural M_2$ denote the boundary connected sum of M_1 and M_2 . Then*

$$\|M, \partial M\| \leq \|M_1, \partial M_1\| + \|M_2, \partial M_2\|.$$

In combination with Theorem 8, Theorem 11 implies that, for every $n \in \mathbb{N}_{\geq 4}$, the boundary connected sums $M_1 \natural M_2$ of n -manifolds with amenable boundary has vanishing simplicial volume if and only if both M_1 and M_2 have vanishing simplicial volume. The situation is different when $n = 3$, where one cannot expect more than superadditivity (see Section 7.4).

The additivity results discussed so far rely on the amenability of the gluing loci. We conclude this introduction by presenting a vanishing criterion for the simplicial volume of manifolds glued along boundary components that are not necessarily amenable. To this end, the key notion is a variant of the Lusternik-Schnirelmann category – the *amenable category* $\text{cat}_{\text{Am}}(X)$ – which is defined in terms of minimal cardinality of amenable open covers of X . A direct consequence of the Gromov’s Vanishing Theorem is that manifolds with $\text{cat}_{\text{Am}}(M) \leq \dim(M) - 1$ have vanishing simplicial volume. Recent work has shown that techniques inspired by the classical Lusternik–Schnirelmann category can be fruitfully applied to the study of simplicial volume of fiber bundles [LM22], and affine manifolds [CM25].

In Section 7.6, we present a combination theorem for the amenable category of fundamental groups of graph of groups, developed by the author in collaboration with Kevin Li and Clara Löh [CLL25]. This result leads to the following vanishing criterion for simplicial volume. Recall that the geometric dimension of a group is the minimal dimension of a CW-model for its classifying space.

THEOREM 13. *Let M_1, \dots, M_k be oriented compact connected n -manifolds, and let M be a manifold obtained by gluing M_1, \dots, M_k along their boundary components. Assume that M is closed and that, for every $j \in \{1, \dots, k\}$, the following conditions hold:*

- *Every boundary component of M_j is π_1 -injective in M_j ;*
- *Every boundary component of M_j has fundamental group of geometric dimension $\leq n - 2$.*
- $\text{cat}_{\text{Am}}(\pi_1(M_j)) \leq n - 1$.

Then $\text{cat}_{\text{Am}}(M) \leq n - 1$. In particular, the simplicial volume of M vanishes.

While the dimension of the boundaries is $n - 1$, Theorem 13 is useful in situations where the boundaries have fundamental groups of small geometric dimension (Example 7.38). This condition is far from implying amenability — for instance, non-abelian free groups have geometric dimension one but are not amenable (Example 7.38).

Plan of the thesis. Theorem 1 and Theorem 2 are proven in Section 3.4, while Theorem 3 is proven in Section 4.3. Our proof of Theorem 4 is contained in Section 4.4. Section 5.3 contains the proof of Theorem 5. The proof of the Relative Vanishing Theorems (Theorem 6 and Theorem 7) are presented in Section 6.4. Theorem 8 is proven in Section 7.2, while Theorem 10 is proven in Section 7.4. Finally, Theorem 11 and Theorem 13 are proven respectively in Section 7.5 and Section 7.6.

CHAPTER 1

Multicomplexes

The goal of this chapter is to introduce the notion of *multicomplexes*, first defined by Gromov in [Gro82]. Multicomplexes are simplicial structures in which simplices are embedded and may share multiple faces. When organizing various types of simplicial structures according to the degeneracies they allow, multicomplexes occupy a position between simplicial complexes and Δ -complexes. This intermediate nature is already evident in dimension one: a graph that permits both loops and parallel edges can be described as a Δ -complex; a graph without loops or parallel edges is a *simplicial graph*, and a graph allowing parallel edges but no loops is usually referred to as a *multigraph*. This intuition generalizes to higher dimensions: while each simplex in a multicomplex must have distinct vertices, simplices are not uniquely determined by their vertex sets.

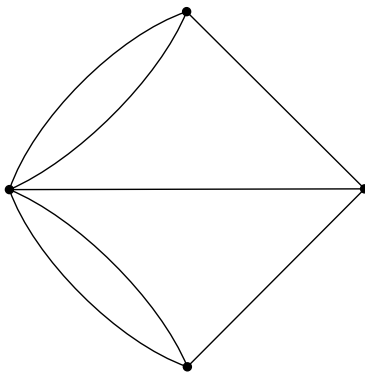


FIGURE 1. A multigraph admits parallel edges, but not loops.

The foundations of the theory of multicomplexes have been developed by Frigerio and Moraschini in [FM23], and in this chapter we closely follow their framework. After introducing the basic definitions and the (relative bounded) cohomology of pairs of multicomplexes, we turn to the central object of the theory: the *singular multicomplex* $\mathcal{K}(X)$ associated to a topological space X (Section 1.3). This construction plays in the theory of multicomplexes the same foundational role that the *singular complex* plays in the theories of Δ -complexes and simplicial sets [EZ50, Mil57, May92, GJ09]. In particular, it shows that every (nice) topological space is weakly homotopy equivalent to the geometric realization of a multicomplex (Proposition 1.11).

1.1. Basic Definitions

A multicomplex can be viewed as a simplicial complex in which distinct simplices are allowed to share any number of common faces and, in particular, may have the same set of vertices. This property is captured in Gromov's definition: according to

[Gro82], a multicomplex is “a set K divided into the union of closed affine simplices Δ_i , $i \in I$, such that the intersection of any two simplices $\Delta_i \cap \Delta_j$ is a (simplicial) subcomplex in Δ_i as well as in Δ_j ”. In what follows, we adopt the combinatorial definition proposed by Frigerio and Moraschini in [FM23].

DEFINITION 1.1. A *multicomplex* K is a triple (V, I, Ω) where:

- (1) V is any set;
- (2) $I = (I_A)_{A \in \mathcal{P}_{\text{fin}}(V)}$ is a family of sets indexed by the set $\mathcal{P}_{\text{fin}}(V)$ of finite subsets of V , i.e. for every $A \in \mathcal{P}_{\text{fin}}(V)$, I_A is a (possibly empty) set;
- (3) if $A = \{v\}$ is a singleton, then I_A is also a singleton;
- (4) Ω is a set of maps $\{\partial_{A,B}: I_A \rightarrow I_B, A, B \in \mathcal{P}_{\text{fin}}(V), A \supseteq B\}$ such that $\partial_{A,A} = \text{id}_A$ for every $A \in \mathcal{P}_{\text{fin}}(V)$, and

$$\partial_{B,C} \circ \partial_{A,B} = \partial_{A,C} \text{ whenever } A \supseteq B \supseteq C.$$

The set V is called the *set of vertices* of K . The set I_A is called the *set of simplices with vertex set* A . The maps $\partial_{A,B}$ are called *boundary maps* of the multicomplex.

In other words, a multicomplex is a contravariant functor from the category $\mathcal{P}_{\text{fin}}(V)$ (with morphisms defined by inclusion) to the category of sets, sending singletons to singletons. For every $A \in \mathcal{P}_{\text{fin}}(V)$, each element $\Delta \in I_A$ represents a simplex with vertices in A . In this way, the element $\partial_{A,B}(\Delta) \in I_B$ represents the face of Δ with vertices in B . We say that $\Delta \in I_A$ is an *n -simplex* if the cardinality of A is $n + 1$. Therefore every n -simplex has $n + 1$ distinct vertices. Moreover, the set of vertices is not assumed to be ordered. It follows that multicomplexes are *not* simplicial sets or Δ -complexes: there are no degenerate simplices and the i -th face of a simplex is in general not defined. From this perspective, multicomplexes more closely resemble (abstract) simplicial complexes, of which they form a natural generalization. In this work a *simplicial complex* is understood as a multicomplex (V, I, Ω) in which every simplex is uniquely determined by its vertices; that is, for every $A \in \mathcal{P}_{\text{fin}}(V)$, I_A is either empty or consists of a single element. This is equivalent to the classical definition (see e.g. [Hat02]).

DEFINITION 1.2. Let $K = (V, I, \Omega)$ and $K' = (V', I', \Omega')$ be multicomplexes. A *simplicial map* from K to K' is given by a function $f: V \rightarrow V'$, and functions $f_A: I_A \rightarrow I'_{f(A)}$, $A \in \mathcal{P}_{\text{fin}}(V)$, such that

$$\partial_{f(A),f(B)}(f_A(\Delta)) = f_B(\partial_{A,B}(\Delta))$$

for every $A \supseteq B$, $A, B \in \mathcal{P}_{\text{fin}}(V)$ and $\Delta \in I_A$. We say that a simplicial map is injective (resp. surjective) if f and each f_A are injective (resp. surjective). An injective simplicial map is also called a *simplicial embedding*.

Simplicial maps are usually denoted by the corresponding map on vertices. In this case, the image $f_A(\Delta)$ is simply denoted by $f(\Delta)$ when $\Delta \in I_A$. Simplicial maps are allowed to decrease the dimension of simplices and, in general, do not preserve their dimension. This behavior is ruled out by the following important notion.

DEFINITION 1.3. Let $K = (V, I, \Omega)$ be a multicomplex. A simplicial map $f: K \rightarrow K'$ is called *non-degenerate* if $f|_A$ is injective whenever $I_A \neq \emptyset$, i.e. every n -simplex of K is mapped to an n -simplex of K' .

A *submulticomplex* of a multicomplex $K = (V, I, \Omega)$ is a triple $L = (W, J, \Theta)$ such that $W \subseteq V$, $J_A \subseteq I_A$, for every $A \in \mathcal{P}_{\text{fin}}(W) \subseteq \mathcal{P}_{\text{fin}}(V)$, and

$$\partial_{A,B}^L = (\partial_{A,B}^K)|_{J_A}$$

for every $A \supseteq B$. A submulticomplex L of K is itself a multicomplex, and there is an obvious simplicial embedding $L \rightarrow K$. Since a submulticomplex is uniquely determined by the simplices it contains, we usually describe submulticomplexes by describing their sets of simplices. If L is a submulticomplex of K , we write $L \subseteq K$. The intersection of two submulticomplexes of K is again submulticomplex. Every simplicial map $f: K \rightarrow K'$ defines a submulticomplex $f(K)$ of K' – which is called the *image* of f – by taking the images of the corresponding maps on simplices.

For every $n \in \mathbb{N}$, the *n-skeleton* $K^n = (V^n, I^n, \Omega^n)$ of $K = (V, I, \Omega)$ is the unique submulticomplex of K such that $V^n = V$ and $I_A^n = I_A$ if A has cardinality at most $n + 1$, while $I_A^n = \emptyset$ otherwise. Every simplicial map induces a unique simplicial map on the n -skeleta.

A *pair of multicomplexes* is a pair (K, L) where L is a submulticomplex of K . A *simplicial map between pairs of multicomplexes* $f: (K, L) \rightarrow (K', L')$ is given by a simplicial map $f: K \rightarrow K'$ such that $f(L)$ is a submulticomplex of L' . We denote by $\text{Aut}(K, L)$ the group of simplicial automorphisms of the pair (K, L) .

REMARK 1.4. Let K_\emptyset be the unique multicomplex whose vertex set is the empty set. Of course the pair (K, K_\emptyset) is well defined for every multicomplex K . In this work, when we refer to the pair (K, K_\emptyset) , we always omit K_\emptyset from the notation. For example, we write $\text{Aut}(K) = \text{Aut}(K, K_\emptyset)$ to denote the group of simplicial automorphisms of K . On the other hand, every statement which is stated for pairs of multicomplexes (K, L) admits an absolute version by taking L to be the empty multicomplex K_\emptyset .

The *geometric realization* $|K|$ of a multicomplex $K = (V, I, \Omega)$ is defined by taking one copy of the standard n -simplex Δ^n for every n -simplex of K , and by gluing them according to the boundary maps. Let e_0, \dots, e_n denote the vertices of $\Delta^n \subseteq \mathbb{R}^{n+1}$. Let I^n denote the set of n -simplices of K . We set

$$X^n = I^n \times \Delta^n, \quad X = \sqcup_{n \in \mathbb{N}} X^n.$$

For every $A \in \mathcal{P}_{\text{fin}}(V)$ of cardinality $n + 1$ we fix a bijection between A and the set $\{e_0, \dots, e_n\}$ of vertices of Δ^n , which is equivalent to fixing an ordering of the elements of A . For every $(\sigma_A, x) \in I_A \times \Delta^n$ and $(\sigma_B, y) \in I_B \times \Delta^m$, where $A, B \in \mathcal{P}_{\text{fin}}(V)$ have cardinalities $n + 1$ and $m + 1$ respectively, we set $(\sigma_A, x) \sim (\sigma_B, y)$ if and only if $A \supseteq B$ and $x = \varphi(y)$, where $\varphi: \Delta^m \rightarrow \Delta^n$ is the only affine inclusion preserving the labeling of the vertices. Finally, we set

$$|K| = X / \approx,$$

where \approx denotes the smallest equivalence relation containing \sim . This construction is independent of the choice of the labeling of the vertices. Since every n -simplex has $n + 1$ distinct vertices, for every $\sigma \in I^n$, the map

$$\Delta^n \rightarrow |K|, \quad x \mapsto [(\sigma, x)],$$

is injective. This map is called the *characteristic map* of σ . Let $|\sigma| \subseteq |K|$ denote its image, which may be endowed with the Euclidean topology inherited from Δ^n . We equip $|K|$ with the weak topology induced from its decomposition into simplices:

a subset of $|K|$ is open if and only if its intersection with every $|\sigma|$ is open. In this way $|K|$ is a CW-complex whose closed n -cells are given by the subsets $|\sigma|$, $\sigma \in I^n$. Intuitively speaking, the points of $|\sigma|$ correspond to convex combinations of its vertices. Therefore, if $\sigma \in I_A$ is a simplex with vertices $A = \{a_0, \dots, a_n\}$ and $t_0, \dots, t_n \in [0, 1]$ are such that $t_0 + \dots + t_n = 1$, the point $[(\sigma, t_0e_0 + \dots + t_n e_n)]$ of $|K|$ is usually denoted by

$$(\sigma, t_0a_0 + \dots + t_na_n) \in |K|.$$

This description of points is essentially unique [FM23, Remark 1.6]. Every simplicial map $f: K \rightarrow K'$ induces a continuous map $|f|: |K| \rightarrow |K'|$ such that

$$|f|(\sigma, t_0a_0 + \dots + t_na_n) = (f(\sigma), t_0f(a_0) + \dots + t_nf(a_n)).$$

We underline the fact that this formula makes sense even if f is not injective on vertices, since the symbol $t_0f(a_0) + \dots + t_nf(a_n)$ still represents a convex combination of vertices. We usually say that a continuous map $|K| \rightarrow |K'|$ is *simplicial* if it is induced by a simplicial map $K \rightarrow K'$.

REMARK 1.5. We give a brief overview of the relationship between multicomplexes and other well-known simplicial structures. As already noted, multicomplexes are not simplicial sets or Δ -complexes in general. However, to every multicomplex one can associate a canonical simplicial set or Δ -complex. In fact, the following categories are equivalent: the category of multicomplexes, the category of symmetric simplicial sets in which every non-degenerate simplex has distinct vertices, and the category of unordered Δ -complexes in which every simplex has distinct vertices [FM23, Proposition 1.7]. We refer the reader to [EZ50, Hat02, GJ09] for an introduction to these simplicial structures. It is well-known that the second barycentric subdivision of a Δ -complex is a simplicial complex. From this perspective, multicomplexes lie in between Δ -complexes and simplicial complexes, since the first barycentric subdivision of a Δ -complex is a multicomplex (Figure 2). Moreover, the topological realization of a small category without loops [BH99, Chapter III.C] has the cell structure of a multicomplex.

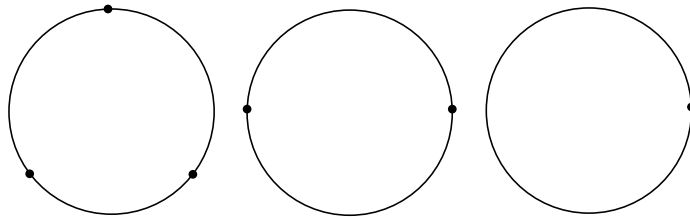


FIGURE 2. The minimal number of vertices required to represent a circle using a simplicial complex, a multicomplex, and a Δ -complex is 3, 2, and 1, respectively.

Let (K, L) be a pair of multicomplexes. Given a group G , a *simplicial group action* $G \curvearrowright (K, L)$ is a group homomorphism $G \rightarrow \text{Aut}(K, L)$. The quotient of a multicomplex by a simplicial action is not a multicomplex in general [FM23, Remark 1.15]. However, the following class of actions allow us to define well-behaved quotients.

DEFINITION 1.6. A simplicial action $G \curvearrowright K$ is called *0-trivial* if G acts trivially on the 0-skeleton of K .

Let $K = (V, I, \Omega)$ be a multicomplex. If an action $G \curvearrowright K$ is 0-trivial, we can define the quotient multicomplex $K/G = (V', I', \Omega')$ as follows: $V' = V$, $(I')^n = I^n/G$ and $\partial_{A,B}[\sigma] = [\partial_{A,B}(\sigma)]$, where $[\sigma]$ denotes the equivalence class of $\sigma \in I_A$ with respect to the action of G .

PROPOSITION 1.7 ([FM23, Proposition 1.14]). *Let $G \curvearrowright K$ be a 0-trivial action. Then the map $\pi: K \rightarrow K/G$, such that $\pi(\sigma) = [\sigma]$, is a non-degenerate simplicial map. Moreover, for every $g \in G$, the following diagram*

$$\begin{array}{ccc} |K| & \xrightarrow{|g|} & |K| \\ & \searrow^{|\pi|} & \swarrow_{|\pi|} \\ & & |K/G| \end{array}$$

is commutative. In particular, $|K/G|$ is homeomorphic to $|K|/G$.

If an action $G \curvearrowright (K, L)$ is 0-trivial on K (hence on L), then the quotient L/G is naturally a submulticomplex of K/G . Moreover, the projection map induces a well-defined map of pairs $(K, L) \rightarrow (K/G, L/G)$.

1.2. Simplicial (Bounded) Cohomology

Let $K = (V, I, \Omega)$ be a multicomplex. Let $n \in \mathbb{N}$. An *algebraic n -simplex* of K is a pair

$$(\Delta, (v_0, \dots, v_n)),$$

where Δ is a k -simplex of K , and $\{v_0, \dots, v_n\}$ is the set of vertices of Δ . Since Δ has exactly $k+1$ vertices, we have that $k \leq n$. Note that there is no requirement for the elements of the tuple (v_0, \dots, v_n) to be pairwise distinct. For every $i \in \{0, \dots, n\}$, the i -th face of σ is the algebraic $(n-1)$ -simplex defined by

$$\partial_n^i \sigma = (\Delta', (v_0, \dots, \hat{v}_i, \dots, v_n)),$$

where $\Delta' = \Delta$, if $\{v_0, \dots, v_n\} = \{v_0, \dots, \hat{v}_i, \dots, v_n\}$, and Δ' is the unique face of Δ with vertices $\{v_0, \dots, \hat{v}_i, \dots, v_n\}$, otherwise. Let R be a ring with unity. We denote by $C_n(K; R)$ the free module generated by the set of algebraic n -simplices. The following boundary operator

$$\partial_n: C_n(K; R) \rightarrow C_{n-1}(K; R), \quad \partial_n = \sum_{i=0}^n (-1)^i \partial_n^i$$

endows $(C_\bullet(K; R), \partial_\bullet)$ with the structure of a chain complex. The complex $(C^\bullet(K; R), \delta^\bullet)$ of simplicial cochains is defined by taking the dual of $(C_\bullet(K; R), \partial_\bullet)$. The simplicial cohomology $H^\bullet(K; R)$ of K is the cohomology of the complex $(C^\bullet(K; R), \delta^\bullet)$.

If the ring R is normed, it induces a natural norm on the space of chains. We are particularly interested in the case in which $R \in \{\mathbb{Z}, \mathbb{R}\}$ with the absolute value norm. If $c = \sum_{i \in I} a_i s_i$ is a chain written in reduced form, where $a_i \in R$ and s_i are algebraic simplices, we define the ℓ^1 -norm of c by

$$\|c\|_1 = \sum_{i \in I} |a_i|.$$

This in turns induces a seminorm in homology, still denoted by $\|\cdot\|_1$, by taking the quotient seminorm from the space of cycles. On the other hand, we endow the module of cochains with the ℓ^∞ -norm: for every $f \in C^n(K; R)$, we set

$$\|f\|_\infty = \sup\{|f(s)|, s \text{ algebraic } n\text{-simplex}\} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

A simplicial cochain f is called *bounded* if $\|f\|_\infty < \infty$. Let $C_b^n(K; R) \subseteq C^n(K; R)$ be the submodule of bounded cochains. This space can be naturally identified with the topological dual of the space $C_n(K; R)$ endowed with the ℓ^1 -norm. Moreover, since the coboundary operator δ^\bullet restricts to bounded cochains, $(C_b^\bullet(K; R), \delta^\bullet)$ is a subcomplex of simplicial cochains, whose cohomology $H_b^\bullet(K; R)$ is called (*simplicial*) *bounded cohomology* of K . The ℓ^∞ -norm on the space of bounded cochains induces a quotient seminorm, still denoted by $\|\cdot\|_\infty$, on $H_b^\bullet(K; R)$.

The definition of simplicial (bounded) cohomology is functorial: simplicial maps between multicomplexes induce chain maps on cochain complexes, hence linear maps in cohomology. Moreover, these maps are norm non-increasing with respect to the ℓ^∞ -norm.

1.2.1. The relative case. Let (K, L) be a pair of multicomplexes. The inclusion $L \subseteq K$ induces a restriction map $C_{(b)}^\bullet(K; R) \rightarrow C_{(b)}^\bullet(L; R)$, which is surjective, and whose kernel is denoted by $C_{(b)}^\bullet(K, L; R)$. In other words, we have a short exact sequence of cochain complexes

$$0 \rightarrow C_{(b)}^\bullet(K, L; R) \rightarrow C_{(b)}^\bullet(K; R) \rightarrow C_{(b)}^\bullet(L; R) \rightarrow 0.$$

We denote by $H_{(b)}^\bullet(K, L; R)$ the cohomology of $(C_{(b)}^\bullet(K, L; R), \delta^\bullet)$. The complex $C_{(b)}^\bullet(K, L; R)$, whose elements are called *relative bounded cochains*, is endowed with the subspace norm from $C_b^\bullet(K; R)$, which in turns induces the quotient seminorm on $H_{(b)}^\bullet(K, L; R)$. The short exact sequence above induces a long exact sequence in cohomology:

$$\cdots \rightarrow H_{(b)}^n(K, L; R) \rightarrow H_{(b)}^n(K; R) \rightarrow H_{(b)}^n(L; R) \rightarrow H_{(b)}^{n+1}(K, L; R) \rightarrow \cdots$$

A classical result in the theory of simplicial complexes and simplicial sets states that their simplicial cohomology coincides with the singular cohomology of their topological realization. The same holds true for multicomplexes. The natural chain inclusion

$$\varphi_\bullet: C_\bullet(K; R) \rightarrow C_\bullet(|K|; R)$$

sending every algebraic n -simplex $(\sigma, (v_0, \dots, v_n)) \in C_n(K; R)$ to the singular simplex

$$\Delta^n \rightarrow |K|, \quad (t_0, \dots, t_n) \mapsto (\sigma, t_0 v_0 + \cdots + t_n v_n),$$

is a chain map which maps algebraic n -simplices of L to singular simplices of $|L|$. Therefore, it induces a well-defined map on relative cochains $\varphi^\bullet: C^\bullet(|K|, |L|; R) \rightarrow C^\bullet(K, L; R)$.

PROPOSITION 1.8 ([FM23, Theorem 1.12]). *For every pair of multicomplexes (K, L) , the homomorphism $H^\bullet(|K|, |L|; R) \rightarrow H^\bullet(K, L; R)$, induced by φ^\bullet , is an isomorphism in every degree.*

Proposition 1.8 does not extend to *bounded* cohomology without additional assumptions on the pair (K, L) . For instance, if K is finite, then every simplicial chain on K is automatically bounded, while $|K|$ may contain singular cohomology classes that do not admit any bounded representative. The key concept of *complete*

multicomplexes, introduced in Chapter 2, allows us to formulate an appropriate version of Proposition 1.8 in the context of bounded cohomology (see Proposition 3.5).

1.2.2. Alternating cochains. Every n -simplex of a multicomplex K defines infinitely many algebraic m -simplices, for $m \geq n$. It is therefore convenient to reduce the number of generators in the simplicial cochain complex. Let (K, L) be a pair of multicomplexes. A cochain $f \in C^n(K; R)$ is called *alternating* if, for every algebraic simplex $(\Delta, (v_0, \dots, v_n))$ of K and every permutation τ of $\{0, \dots, n\}$, it satisfies

$$f(\Delta, (v_0, \dots, v_n)) = \text{sign}(\tau) \cdot f(\Delta, (v_{\tau(0)}, \dots, v_{\tau(n)})).$$

Let $C^n(K; R)_{\text{alt}} \subseteq C^n(K; R)$ be the subspace of alternating cochains, and let

$$C_{(b)}^n(K, L; R)_{\text{alt}} = C_{(b)}^n(K, L; R) \cap C^n(K; R)_{\text{alt}}$$

be the subspace of alternating relative (bounded) cochains. Since the coboundary operator preserves alternating cochains, they define subcomplexes of relative (bounded) cochains. Moreover, $C_{(b)}^n(K, L; R)_{\text{alt}}$ inherits the ℓ^∞ -norm from $C_{(b)}^n(K, L; R)$.

PROPOSITION 1.9. *Let (K, L) be a pair of multicomplexes. The inclusion of complexes $C_{(b)}^n(K, L; R)_{\text{alt}} \hookrightarrow C_{(b)}^n(K, L; R)$ induces an isometric isomorphism in cohomology.*

PROOF. It is well known that these inclusions induce isomorphisms in cohomology for simplicial complexes, and it is not hard to believe that the same argument works for multicomplexes. However, since we want to show that these isomorphisms are isometric, we outline the argument for bounded cochains. Let $\iota: C_{(b)}^\bullet(K, L; R)_{\text{alt}} \hookrightarrow C_{(b)}^\bullet(K, L; R)$ denote the inclusion. Let Σ_{n+1} be the symmetric group of the set $\{0, \dots, n\}$. The alternating map $\text{alt}: C_{(b)}^\bullet(K, L; R) \rightarrow C_{(b)}^\bullet(K, L; R)_{\text{alt}}$, which is defined by the formula

$$\text{alt}(f)(\Delta, (v_0, \dots, v_n)) = \frac{1}{(n+1)!} \sum_{\tau \in \Sigma_{n+1}} f(\Delta, (v_{\tau(0)}, \dots, v_{\tau(n)})),$$

is well defined and provides a left inverse of ι , i.e. $\text{alt} \circ \iota = \text{id}$. Since both alt and ι are norm non-increasing, it readily follows that ι induces an isometric embedding in cohomology. On the other hand, since the composition $\iota \circ \text{alt}$ is homotopic to the identity, ι actually induces an isometric isomorphism in cohomology. \square

The key aspect of the above argument is that any cochain can be made alternating by averaging over a finite group of permutations. This same idea underlies the reduction to G -invariant chains when $G \curvearrowright (K, L)$ is a simplicial action by an amenable group (see Theorem 3.15).

1.3. The Singular Multicomplex

In the theory of multicomplexes, the singular multicomplex plays the same role of the singular simplicial set in the theory of simplicial sets [Mil57, May92]. Both provide convenient frameworks to encode the homotopy type of a space by considering its set of singular simplices. However, since the simplicial structure of a multicomplex is more rigid than that of a simplicial set, additional conditions are required to ensure that this construction yields a valid multicomplex. The singular

multicomplex of a topological space X is the multicomplex whose simplices are singular simplices of X with distinct vertices, considered up to affine parametrization. A more precise definition is the following.

Let X be a topological space and let $n \in \mathbb{N}$. Let $S_n(X)$ denote the set of singular simplices of X . We say that two simplices $\sigma, \sigma' \in S_n(X)$ are equivalent if and only if there exists an affine self-homeomorphism ϑ of Δ^n such that $\sigma \circ \vartheta = \sigma'$. Let $\mathcal{S}_n(X)$ be the set of equivalence classes of singular simplices which are injective on the vertices of Δ^n . The set of vertices of an element $[\sigma] \in \mathcal{S}_n(X)$ is the image of the set of vertices of Δ^n via any of its representatives. By construction, it has cardinality $n+1$. We now define the *singular multicomplex* of X as the triple $\mathcal{K}(X) = (V, I, \Omega)$ such that

- $V = X$;
- for every $A \in \mathcal{P}_{\text{fin}}(X)$ of cardinality $n+1$, the set I_A consists of the elements of $\mathcal{S}_n(X)$ whose set of vertices is A ;
- for every $[\sigma] \in I_A$ and every $B \subseteq A$, the element $\partial_{A,B}([\sigma]) \in I_B$ is the equivalence class of the unique face of σ whose set of vertices is B . This is independent of the choice of representative.

The geometric realization of $|\mathcal{K}(X)|$ is a CW-complex whose 0-skeleton is in bijection with the set of points of X . Moreover, there is a *natural projection*

$$S_X: |\mathcal{K}(X)| \rightarrow X$$

which is defined by the following formula:

$$S_X([\sigma], (t_0x_0 + \dots + t_nx_n)) = \sigma(t_0e_0 + \dots + t_nx_n),$$

where $\{x_0, \dots, x_n\}$ are the vertices of $[\sigma]$ and σ is a representative of $[\sigma]$ which sends the i -th vertex e_i of Δ^n to x_i , for every $i \in \{0, \dots, n\}$. This map is well defined and continuous. Moreover, it is a weak-homotopy equivalence for a suitable class of topological spaces.

DEFINITION 1.10. A topological space X is *nice* if the following conditions hold:

- (1) X is semilocally simply connected; in other words, any path connected component of X admits a universal covering.
- (2) Every non-empty path-connected finite subset of X is a singleton.

PROPOSITION 1.11 ([**FM23**, Theorem 1]). *Let X be a nice topological space. Then the natural projection $S_X: |\mathcal{K}(X)| \rightarrow X$ is a weak homotopy equivalence.*

Nice topological spaces are called *good* in the work of Frigerio and Moraschini [**FM23**, Definition 2.1]. However, since in our setting we use the term *good* to describe pairs of topological spaces with specific control over homotopy (Definition 2.29), we have chosen to adopt different terminology to avoid confusion.

If $X = \emptyset$ is the empty topological space, then it is nice for trivial reasons. In this case, the singular multicomplex $\mathcal{K}(\emptyset)$ is just the empty multicomplex K_\emptyset (Remark 1.4). More interestingly, CW-complexes are nice topological spaces. In particular, the Whitehead Theorem immediately implies the following.

PROPOSITION 1.12 ([**FM23**, Corollary 2.2]). *Let X be a CW-complex. Then the natural projection $S_X: |\mathcal{K}(X)| \rightarrow X$ is a homotopy equivalence.*

Let (X, A) be a pair of topological spaces. Every singular simplex of A which is injective on vertices is, in particular, a singular simplex of X with the same property.

It follows that the singular multicomplex $\mathcal{K}(A)$ of A naturally sits in $\mathcal{K}(X)$ as a submulticomplex. Moreover, the natural projection S_X induces a well-defined map of pairs

$$S_X: (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (X, A).$$

The following is a straightforward generalization of Proposition 1.12 for CW-pairs. Two continuous maps of pairs of topological spaces $f, g: (X, A) \rightarrow (Y, B)$ are *homotopic as maps of pairs* if there exists a continuous homotopy $F: X \times [0, 1] \rightarrow Y$ between f and g such that $F(A \times [0, 1]) \subseteq B$. A map of pairs is a *homotopy equivalence of pairs* if it has a homotopy inverse as a map of pairs.

PROPOSITION 1.13. *Let (X, A) be a CW-pair. Then the natural projection*

$$S_X: (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (X, A)$$

is a homotopy equivalence of pairs.

PROOF. By Proposition 1.12, the horizontal arrows of the following commutative diagram

$$\begin{array}{ccc} |\mathcal{K}(A)| & \xrightarrow{S_A} & A \\ \downarrow & & \downarrow \\ |\mathcal{K}(X)| & \xrightarrow{S_X} & X \end{array}$$

are homotopy equivalences. Since $(|\mathcal{K}(X)|, |\mathcal{K}(A)|)$ and (X, A) are CW-pairs, then the vertical arrows are cofibrations, hence the homotopy equivalences can be promoted to a homotopy equivalence of pairs [**May99**, Chapter 6, Section 5]. \square

REMARK 1.14. In this work we mostly restrict our attention to CW-pairs (X, A) for two main reasons. The first is that we want the natural projections S_X and S_A to induce homotopy equivalences. The second is that, being the inclusion $A \hookrightarrow X$ a cofibration, these homotopy equivalences can be promoted to a homotopy equivalence of pairs, which is crucial to retain some control over the natural norms in bounded cohomology (see Remark 3.3).

Although continuous maps between topological spaces do not, in general, induce simplicial maps between their singular multicomplexes, they nevertheless determine well-defined homotopy classes of maps between the corresponding topological realizations.

LEMMA 1.15. *Let $f: (X, A) \rightarrow (Y, B)$ be a continuous map between CW-pairs. Then there exists a continuous map of pairs*

$$\mathcal{K}(f): (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (|\mathcal{K}(Y)|, |\mathcal{K}(B)|)$$

such that the following diagram

$$\begin{array}{ccc} (|\mathcal{K}(X)|, |\mathcal{K}(A)|) & \xrightarrow{\mathcal{K}(f)} & (|\mathcal{K}(Y)|, |\mathcal{K}(B)|) \\ \downarrow S_X & & \downarrow S_Y \\ (X, A) & \xrightarrow{f} & (Y, B) \end{array}$$

is commutative up to homotopy (of maps of pairs).

PROOF. The argument relies on the functorial properties of the singular multicomplex and is a straightforward adaptation of the one presented in [FM23] for the case $A = \emptyset$, which we include here for the sake of completeness. Let $\mathcal{S}(X)$ denote the singular simplicial set associated to X (whose n -simplices are singular simplices of X). If we endow the singular multicomplex $\mathcal{K}(X)$ with its natural structure of simplicial set (which depends, for example, on an arbitrary ordering on the set X), there is a natural simplicial embedding $\mathcal{K}(X) \rightarrow \mathcal{S}(X)$ (see [FM23, Section 2.1] for the details). In particular, there is well-defined cellular map $\iota_X: |\mathcal{K}(X)| \rightarrow |\mathcal{S}(X)|$. Just as the singular multicomplex, also the singular simplicial set is equipped with a natural projection $j_X: |\mathcal{S}(X)| \rightarrow X$, which is a homotopy equivalence for CW-complexes [Mil57, Theorem 4]. Given a CW-pair (X, A) , there is a commutative diagram of maps of pairs

$$\begin{array}{ccc} (|\mathcal{K}(X)|, |\mathcal{K}(A)|) & \xrightarrow{\iota_X} & (|\mathcal{S}(X)|, |\mathcal{S}(A)|) \\ & \searrow S_X & \swarrow j_X \\ & & (X, A) \end{array}$$

such that both S_X and j_X are homotopy equivalences as maps of pairs (Proposition 1.13). It follows that the inclusion $\iota_X: (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (|\mathcal{S}(X)|, |\mathcal{S}(A)|)$ is also a homotopy equivalence (as map of pairs). Any continuous map $f: (X, A) \rightarrow (Y, B)$ induces a simplicial map $\mathcal{S}(f): (|\mathcal{S}(X)|, |\mathcal{S}(A)|) \rightarrow (|\mathcal{S}(Y)|, |\mathcal{S}(B)|)$. Therefore, we can define $\mathcal{K}(f) = \rho_Y \circ \mathcal{S} \circ \iota_X$, where ρ_Y is any homotopy inverse (as map of pairs) of ι_Y . \square

Homotopy Theory of Complete Multicomplexes

In this chapter we study the homotopy theory of (topological realizations of) multicomplexes, following the framework established in [FM23]. The key objects for this purpose are *complete* multicomplexes, whose defining feature is that their homotopy groups admit a combinatorial description via special spheres (Proposition 2.6). This property serves as the analogue of the Kan condition for simplicial sets [May92, GJ09]. A central result in this context is that the singular multicomplex $\mathcal{K}(X)$ associated with a topological space X is complete (Proposition 2.2). Combined with the fact that the topological realization of $\mathcal{K}(X)$ is weakly homotopy equivalent to X (at least when X is nice), completeness yields a combinatorial model for the homotopy groups of X .

In Section 2.2 we establish straightforward generalizations to pairs of multicomplexes of results from [FM23] about simplicial approximation. We introduce in this way the constants C_n , $n \in \mathbb{N}_{\geq 1}$, which describe the number of n -simplices in the standard multicomplex structure of the prism $\Delta^{n-1} \times [0, 1]$. These constants play a crucial role in the bi-Lipschitz estimates appearing in Theorem 5.

The homotopy information encoded in a complete multicomplex is often highly redundant, as illustrated by the case of the singular multicomplex. In Section 2.3, we describe a classical procedure to reduce the size of a complete multicomplex without changing its homotopy type. This yields a submulticomplex, which is called *minimal*. In order to minimize *simultaneously* a pair of complete multicomplexes, one must impose some compatibility condition on the homotopy of the pair. This leads to the notion of *full* pairs of multicomplexes (Definition 2.15). Given a pair (X, A) of topological spaces, the pair $(\mathcal{K}(X), \mathcal{K}(A))$ is full when A is π_n -injective in X for every $n \in \mathbb{N}_{\geq 1}$ (Proposition 2.19).

To any complete and minimal multicomplex K we can associate an *aspherical* multicomplex having the same fundamental group. This is achieved by taking the first Postnikov quotient of K , which identifies simplices sharing the same 1-skeleton. This construction becomes particularly meaningful when applied to the singular multicomplex $\mathcal{K}(X)$, because it provides an explicit model for the classifying space of the fundamental group of X . Finally, we introduce the notion of *good pair* of topological spaces (Definition 2.29), which plays a central role in this work.

2.1. Complete Multicomplexes

In this section we introduce *complete* multicomplexes and we describe their homotopy groups by means of *special spheres* (Proposition 2.6). Let Δ^n denote the full simplicial complex (hence multicomplex) on a set of $n + 1$ vertices, so that its topological realization $|\Delta^n|$ is homeomorphic to the standard affine n -simplex in \mathbb{R}^{n+1} .

DEFINITION 2.1. A multicomplex K is *complete* if, for every $n \in \mathbb{N}$ and every continuous map $f: |\Delta^n| \rightarrow |K|$ whose restriction to the boundary $f|_{\partial|\Delta^n|}: \partial|\Delta^n| \rightarrow |K|$ is a simplicial embedding, the map f is homotopic relative to $\partial|\Delta^n|$ to a simplicial embedding $f': |\Delta^n| \rightarrow |K|$.

Completeness is a very strong property for a multicomplex. For example, if a complete connected multicomplex is a simplicial complex, then it should be the full simplicial complex on its set of vertices [FM23, Example 3.3]. To our purposes, the most important examples of complete multicomplexes are the singular ones. In fact, every continuous map $f: |\Delta^n| \rightarrow |\mathcal{K}(X)|$ which is a simplicial embedding on the boundary, after the composition with the natural projection $S_X: |\mathcal{K}(X)| \rightarrow X$, identifies a singular simplex of X which is injective on vertices, hence an n -simplex Δ of $\mathcal{K}(X)$. The fact that S_X is a weak homotopy equivalence (when X is a nice topological space) implies that f and the characteristic map of Δ are homotopic.

PROPOSITION 2.2 ([FM23, Theorem 3.7]). *Let X be a nice topological space. Then the singular multicomplex $\mathcal{K}(X)$ is complete.*

The notion of complete multicomplexes is useful in identifying a class of multicomplexes whose homotopy groups can be described combinatorially in terms of parallel faces.

DEFINITION 2.3. Let $n \in \mathbb{N}$. The n -dimensional *special sphere* is an n -dimensional multicomplex $K = (V, I, \Omega)$ defined as follows. The vertex set V is any set of cardinality $n + 1$; if $A = V$, then $I_A = \{\Delta_n, \Delta_s\}$ consists of two elements, called the *northern simplex* Δ_n and the *southern simplex* Δ_s . For every proper subset $A \subsetneq V$, the set I_A consists of a single element. There is a unique choice of boundary maps that makes K into a multicomplex. We denote by \dot{S}^n any multicomplex which is simplicially isomorphic to a n -dimensional special sphere.

The topological realization of a n -dimensional special sphere is homeomorphic to the sphere S^n . Moreover, the CW-structure of S^n induced by the multicomplex structure of the special sphere consists of two n -cells glued along their boundary.

DEFINITION 2.4. Let $K = (V, I, \Omega)$ be a multicomplex. Two n -dimensional simplices Δ_1, Δ_2 of K are *compatible* if their vertex set coincides (i.e. there exists $A \in \mathcal{P}_{\text{fin}}(V)$ such that $\Delta_1, \Delta_2 \in I_A$) and for every proper subset $B \subseteq A$ we have $\partial_{A,B}(\Delta_1) = \partial_{A,B}(\Delta_2)$. Being compatible is an equivalence relation among simplices of K . We denote by $\pi_K(\Delta_1)$ the equivalence class of Δ_1 . Equivalently, two simplices Δ_1 and Δ_2 of K are compatible if and only if $|\Delta_1|$ and $|\Delta_2|$ share the same boundary in $|K|$.

DEFINITION 2.5. Let $K = (V, I, \Omega)$ be a multicomplex. Two n -dimensional simplices Δ_1 and Δ_2 of K are *homotopic* if they are compatible and their characteristic maps $|\Delta^n| \rightarrow |K|$ are homotopic in $|K|$ relative to the boundary. Being homotopic is an equivalence relation among simplices of K .

Let Δ_1, Δ_2 be two compatible simplices of K . We define a (pointed) special sphere inside K in the following way. We consider a special sphere \dot{S}^n . We fix an ordering of the vertices (x_0, \dots, x_n) of Δ_1 and Δ_2 . We fix an ordering of the vertices of \dot{S}^n and we denote by s_0 its minimal vertex. Then

$$\dot{S}^n(\Delta_1, \Delta_2): (\dot{S}^n, s_0) \rightarrow (K, x_0)$$

is the unique simplicial map sending Δ_n to Δ_1 , Δ_s to Δ_2 and preserving the order of the vertices. With an abuse of notation, we denote by $\dot{S}^n(\Delta_1, \Delta_2)$ the induced map on the topological realization, which defines an element in the n -th homotopy group $\pi_n(|K|, x_0)$ of $|K|$ based at x_0 . It is easy to see that two compatible simplices are homotopic if and only if the special sphere $\dot{S}^n(\Delta_1, \Delta_2)$ represents the trivial element in $\pi_n(|K|, x_0)$. The following result is the key feature of completeness.

PROPOSITION 2.6 ([FM23, Theorem 3.9]). *Let K be a complete multicomplex, and let Δ_0 be an n -simplex of K . Fix an ordering of the vertices of Δ_0 , and let x_0 denote the minimal vertex of Δ_0 . The map*

$$\Theta: \pi_K(\Delta_0) \rightarrow \pi_n(|K|, x_0), \quad \Theta(\Delta) = [\dot{S}^n(\Delta_0, \Delta)],$$

is surjective, and $\Theta(\Delta) = \Theta(\Delta')$ if and only if Δ is homotopic to Δ' .

2.2. Simplicial Approximation

It is well known that continuous maps between simplicial complexes can be assumed to be simplicial, up to suitably subdividing the domain. Provided that the target is complete, there is a stronger result that provides a controlled simplicial approximation for maps between multicomplexes. The following is the natural generalization of [FM23, Proposition 3.11] to pairs of multicomplexes.

PROPOSITION 2.7 (Simplicial Approximation for Pairs). *Let (K, K_0) and (L, L_0) be pairs of multicomplexes and assume that both K and K_0 are complete. Let*

$$f: (|L|, |L_0|) \rightarrow (|K|, |K_0|)$$

be a continuous map of pairs that is simplicial on the 0-skeleton of L . Assume that, for every simplex Δ of L , f is injective on the vertices of Δ , and that f is simplicial on a submulticomplex L_1 of L . Then, there exists a non-degenerate simplicial map

$$f': (L, L_0) \rightarrow (K, K_0)$$

such that $|f'|$ is homotopic to f (as maps of pairs) relative to $V(L) \cup L_1$.

PROOF. We construct f' by induction on the skeleta of L . To this end, it is sufficient to define, for every $n \in \mathbb{N}$, continuous maps $f'_n: (|L|, |L_0|) \rightarrow (|K|, |K_0|)$ such that

- (1) f'_n is simplicial on $|L|^n$;
- (2) f'_n is homotopic to f'_{n-1} relative to $L^{n-1} \cup L_1$ (as maps of pairs).

We set $f'_0 = f$ and, assuming that f'_n as above is given, we construct f'_{n+1} . First of all, we set $f'_{n+1} = f'_n$ on $|L|^n$. We extend f'_{n+1} to the $(n+1)$ -skeleton in the following way. Let σ be a $(n+1)$ -simplex of L . If σ is a simplex of L_1 , we define $f'_{n+1}(\sigma) = f(\sigma)$. If σ is a simplex of L_0 , we denote by $\chi: |\Delta^{n+1}| \rightarrow |L_0|$ its characteristic map. Since the restriction of $f'_n \circ \chi$ to $\partial|\Delta^{n+1}|$ is a simplicial embedding into $|K_0|$, by the completeness of K_0 , there exists a simplex σ' of K_0 such that $f'_n \circ \chi$ is homotopic in $|K_0|$ to the characteristic map of σ' . Therefore, we define $f'_{n+1}(\sigma) = \sigma'$. After repeating this operation for every $(n+1)$ -simplex of L_0 , we proceed in the same way with every simplex of L , by making use of the completeness of K when needed. We obtain in this way a simplicial map $f'_{n+1}: (|L|^{n+1}, |L_0|^{n+1}) \rightarrow (|K|, |K_0|)$ which is homotopic (as a map of pairs) to $f'_n|_{|L|^{n+1}}$ relative to $|L|^n \cup |L_1|$. Then, using the homotopy extension property for

CW-pairs, we then extend f'_{n+1} to the whole $|L|$ in such a way that conditions (1) and (2) above hold. \square

One consequence of simplicial approximation is that it allows continuous homotopies to be replaced by simplicial ones. This operation requires a sufficient number of intermediate vertices in the triangulation of a prism [Mun84, Section 19]. Let

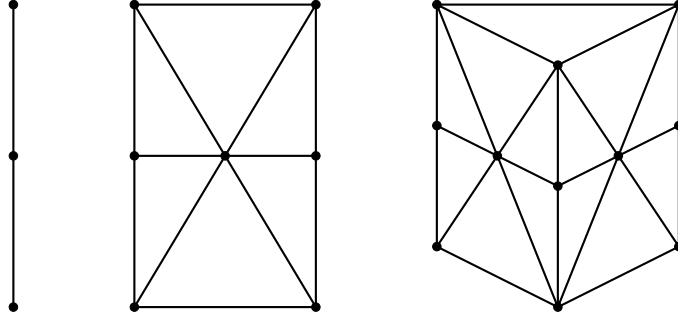


FIGURE 3. The standard multicomplex structure of the prism in low dimensions.

$I = [0, 1]$ and let σ_n denote the standard simplex Δ^n , endowed with its natural structure of simplicial complex (hence multicomplex). We define a structure of multicomplex on $\sigma_n \times I$ such that $\sigma_n \times \{0\}$ and $\sigma_n \times \{1\}$ appear as faces of the triangulation of $\sigma \times I$. We proceed by induction. For $n = 0$ we just take $\sigma_0 \times I$ as the first barycentric subdivision of I (see Figure 3). Assume that $\sigma_{n-1} \times I$ has been triangulated in such a way that $\sigma_{n-1} \times \{0\}$ and $\sigma_{n-1} \times \{1\}$ appear as simplices in the multicomplex structure of $\sigma_{n-1} \times I$. In order to triangulate $\sigma_n \times I$, we consider the following triangulation of the geometric boundary of $\sigma_n \times I$. We triangulate $\partial(\sigma_n \times I)$ by merging $\sigma_n \times \{0\}$, $\sigma_n \times \{1\}$ and the triangulations of $\tau \times I \cong \sigma_{n-1} \times I$, where τ varies among all the facets of σ_n . In conclusion, the triangulation of $\sigma_n \times I$ is obtained by coning the triangulation of $\partial(\sigma_n \times I)$ over an internal point. The number of simplices needed to triangulate $\sigma_{n-1} \times I$ plays an important role in this work.

DEFINITION 2.8. Let $n \in \mathbb{N}_{\geq 1}$ and $I = [0, 1]$. Let σ_n denote the standard simplex Δ^n . The constant C_n denotes the number of n -simplices in $\sigma_{n-1} \times I$.

From the construction above it is easy to deduce the following recursive formula: $C_1 = 2$ and $C_n = 2 + n \cdot C_{n-1}$, from which we obtain the closed formula:

$$C_n = 2 \cdot \sum_{k=1}^n \frac{n!}{k!}.$$

DEFINITION 2.9. Let K be a multicomplex. The product multicomplex $K \times I$ is the multicomplex obtained by gluing a copy of the multicomplex $\sigma \times I$ for every simplex σ of K , according to the boundary maps. The geometric realization $|K \times I|$ of $K \times I$ is homeomorphic to $|K| \times [0, 1]$, and there are simplicial embeddings

$$i_0, i_1: K \rightarrow K \times I,$$

whose images are denoted by $K \times \{0\}$ and $K \times \{1\}$. This construction is functorial. In particular, the inclusion of every submulticomplex L of K induces a natural inclusion of $L \times I$ into $K \times I$.

Let $f, g: (K, L) \rightarrow (K', L')$ be simplicial maps between pairs of multicomplexes. We say that f is *simplicially homotopic to g (as a map of pairs)* if there exists a simplicial map $F: K \times I \rightarrow K'$ such that $F \circ i_0 = f$, $F \circ i_1 = g$ and $F(L \times I) \subseteq L'$.

REMARK 2.10. Simplicially homotopic maps induce the same maps in (bounded) cohomology [**FM23**, Lemma 3.14]. More specifically, for every pair $f_0, f_1: K \rightarrow K'$ of simplicially homotopic maps between multicomplexes, the induced maps $f_i^\bullet: C_b^\bullet(K') \rightarrow C_b^\bullet(K)$, $i \in \{0, 1\}$, on bounded cochains are chain homotopic via a homotopy

$$T^\bullet: C_b^\bullet(K') \rightarrow C_b^{\bullet-1}(K)$$

such that $\|T^n\| \leq C_n$ for every $n \in \mathbb{N}_{\geq 1}$ [**FM23**, Remark 3.15].

The following is the relative version of the Homotopy Lemma [**FM23**, Lemma 3.17]. A multicomplex K is called *large* if every connected component of K contains infinitely many vertices. Of course, the singular multicomplex $\mathcal{K}(X)$ associated to a CW-complex X of positive dimension is large.

LEMMA 2.11 (Homotopy Lemma for Pairs). *Let (K, K_0) and (L, L_0) be pairs of multicomplexes and assume that both K and K_0 are large and complete. Let $f, g: (L, L_0) \rightarrow (K, K_0)$ be non-degenerate simplicial maps such that $|f|$ is homotopic to $|g|$ (as maps of pairs). Then f is simplicially homotopic to g (as maps of pairs) via a non-degenerate simplicial homotopy.*

PROOF. Let $F: |L| \times I \rightarrow |K|$ be a continuous homotopy (as map of pairs) between $|f|$ and $|g|$. By the homotopy extension property for CW-pairs, the largeness of K and the fact that f and g are non-degenerate, we can modify F so that it is injective on the vertex set of every simplex of $L \times I$. Since the multicomplex structure on $L \times I$ is such that no simplex has vertices both on $L \times \{0\}$ and on $L \times \{1\}$, this operation can be performed without altering the behavior of F on $L \times \{0, 1\}$. By Proposition 2.7 (with $f = F$ and $L_1 = L \times \{0, 1\}$), we get a non-degenerate simplicial map $F': L \times I \rightarrow K$ which provides a simplicial homotopy (as a map of pairs) between f and g . \square

REMARK 2.12. The constants C_n appear as part of the Lipschitz constants in Theorem 5. However, given that these bounds are far from being optimal (Remark 5.18), we are not particularly concerned with minimizing C_n as much as possible. To our purposes, the constants C_n could describe the number of n -simplices in every triangulation (as a multicomplex) of the prism $\Delta^{n-1} \times [0, 1]$ such that the following conditions hold: its restriction to $\Delta^{n-1} \times \{0, 1\}$ is the standard triangulation of the simplex Δ^{n-1} ; it is uniform on vertical faces $\Delta^k \times [0, 1]$, $k \in \{0, \dots, n-2\}$; no n -simplex of $\Delta^{n-1} \times [0, 1]$ has vertices both on $\Delta^{n-1} \times \{0\}$ and on $\Delta^{n-1} \times \{1\}$.

2.3. Minimal Multicomplexes

The size of a complete multicomplex can be reduced without changing its homotopy type. In a complete multicomplex, we know that every simplicial embedding of $\partial|\Delta^n|$ which extends continuously to $|\Delta^n|$ can be deformed into the characteristic

map of some simplex. When this deformation is unique, we say that the multicomplex is minimal. It turns out that every complete multicomplex contains a minimal one that captures all its homotopy information (Proposition 2.16). This operation is helpful to avoid the redundancies which are present in the singular multicomplex $\mathcal{K}(X)$, and therefore reduces the number of simplices which are needed to obtain a multicomplex which has the same homotopy type of X .

DEFINITION 2.13. A multicomplex K is *minimal* if, for every $n \in \mathbb{N}$ and every continuous map $f: |\Delta^n| \rightarrow |K|$ whose restriction to the boundary $f|_{\partial|\Delta^n|}: \partial|\Delta^n| \rightarrow |K|$ is a simplicial embedding, the map f is homotopic in $|K|$ relative to $\partial|\Delta^n|$ to *at most one* simplicial embedding $f': |\Delta^n| \rightarrow |K|$. In particular, K is minimal and complete if f is homotopic in $|K|$ relative to $\partial|\Delta^n|$ to *exactly one* simplicial embedding $f': |\Delta^n| \rightarrow |K|$.

Of course, a multicomplex is minimal if and only if it does not contain any pair of distinct homotopic simplices. The following result is an immediate consequence of Proposition 2.6.

PROPOSITION 2.14 ([FM23, Theorem 3.22]). *Let K be a complete minimal multicomplex, and let Δ_0 be an n -simplex of K . Fix an ordering of the vertices of Δ_0 , and let x_0 denote the minimal vertex of Δ_0 . The map*

$$\Theta: \pi_K(\Delta_0) \rightarrow \pi_n(|K|, x_0), \quad \Theta(\Delta) = [\dot{S}^n(\Delta_0, \Delta)],$$

is bijective.

According to Proposition 2.14, the homotopy type of a complete minimal multicomplex is completely determined by the combinatorics of its compatible simplices. It is therefore desirable to replace a complete multicomplex with a minimal one, without altering its homotopy type. At this stage, the first difficulties arise when working with pairs of multicomplexes rather than a single one. Given a pair of complete multicomplexes (K, L) , in order to minimize them *simultaneously* we need to ensure that the homotopy of L injects into that of K in the following way.

DEFINITION 2.15. A pair of multicomplexes (K, L) is *full* if, for every pair of simplices Δ, Δ' of L , if Δ and Δ' are homotopic in K , then they are homotopic in L .

Of course, the pair (K, K_\emptyset) is full for every multicomplex K , where K_\emptyset denotes the empty multicomplex (Remark 1.4). The following result generalizes [FM23, Theorem 3.23] to full pairs of multicomplexes.

PROPOSITION 2.16. *Let (K, L) be a full pair of multicomplexes such that both K and L are complete. Then there exists a submulticomplex M of K such that:*

- (1) M and $M \cap L$ are complete and minimal;
- (2) M has the same vertices of K ;
- (3) There exists a simplicial retraction $r: (K, L) \rightarrow (M, M \cap L)$;
- (4) The geometric realization of r realizes the pair $(|M|, |M \cap L|)$ as a strong deformation retract of the pair $(|K|, |L|)$. In particular the inclusion

$$(|M|, |M \cap L|) \hookrightarrow (|K|, |L|)$$

is a homotopy equivalence of pairs.

Moreover, such a pair of multicomplexes $(M, M \cap L)$ is unique up to simplicial isomorphisms of pairs.

PROOF. We define M inductively on the dimension of simplices. First, we set $M^0 = K^0$. Once M^n has been constructed, we define M^{n+1} by adding to M^n one $(n+1)$ -simplex for every homotopy class of $(n+1)$ -simplices of K whose facets are all contained in M^n . Moreover, if the facets are all contained in $M^n \cap L$ and if there is a simplex of L in the corresponding homotopy class, then we chose this as a representative. In the previous paragraph, when we refer to homotopic simplices, we mean *homotopic in K* , but this is equivalent to require them to be *homotopic in L* because we are assuming (K, L) to be full.

We prove now that the pair $(|M|, |M \cap L|)$ is a strong deformation retract of $(|K|, |L|)$. To this end, it is sufficient to construct, for every $n \in \mathbb{N}$, a map $r_n : (|K|, |L|) \rightarrow (|K|, |L|)$ and a homotopy (of pairs) $h_n : |K| \times I \rightarrow |K|$ between r_n and r_{n+1} such that the following properties hold: r_0 is the identity of $|K|$; $r_n|_{|K^n|}$ is a simplicial retraction of $(|K^n|, |L^n|)$ onto $(|M^n|, |M^n \cap L^n|)$; r_{n+1} and r_n coincide when restricted to $|K^n|$; $h_{n+1}(x, t) = r_n(x)$ for every $x \in |K^n|$ and $t \in I$.

We first set $r_0 = \text{id}_{|K|}$. Assume now we have defined r_n and h_{n-1} . We then set $r_{n+1}|_{|K^n|} = r_n|_{|K^n|}$ and we define r_{n+1} on $(n+1)$ -simplices as follows. Let σ be an $(n+1)$ -simplex of L . Since $r_n|_{\partial|\sigma|}$ is a simplicial embedding, by the completeness of L , there exists a homotopy $h_\sigma : |\sigma| \times I \rightarrow |L|$ between the characteristic map of σ and the characteristic map of some $(n+1)$ -simplex σ' of M . Therefore, we define $r_{n+1}(\sigma) = \sigma'$. After repeating this operation for every $(n+1)$ -simplex of L , we proceed in the same way with every $(n+1)$ -simplex of K , invoking the completeness of K instead. We obtain in this way the desired maps (see [FM23] for the details), and this concludes the construction of the deformation retraction r . Notice that the map r is simplicial.

The minimality of $M \cap L$ is obvious: if two simplices are homotopic in $M \cap L$, then they are homotopic in K , hence they must coincide by definition of M . To show that $M \cap L$ is complete, we need to use that the pair (K, L) is full: if $f : |\Delta^n| \rightarrow |M \cap L|$ is a continuous map with $f|_{\partial|\Delta^n|}$ simplicial, then, by the completeness of L , there exists a simplicial map $f' : |\Delta^n| \rightarrow |L|$ such that f and f' are homotopic in $|L|$ relative to $\partial|\Delta^n|$. By construction of M , there exists a simplicial map $f'' : |\Delta^n| \rightarrow |M \cap L|$ such that f' and f'' are homotopic in $|K|$ relative to $\partial|\Delta^n|$. Since the pair is full, f and f'' are also homotopic in $|L|$, via a homotopy $h : |\Delta^n| \times I \rightarrow |L|$. By composing h with the retraction $r|_{|L|} : |L| \rightarrow |M \cap L|$, we get that f is homotopic in $|M \cap L|$ to a simplicial embedding. In the same way one shows that M is complete and minimal.

Finally, the uniqueness of the pair $(M, M \cap L)$ up to simplicial isomorphisms of pairs is a consequence of Proposition 2.17 below. If M_1 and M_2 are both complete and minimal as in the statement, then composing the inclusion $(M_1, M_1 \cap L) \rightarrow (K, L)$ with the retraction $(K, L) \rightarrow (M_2, M_2 \cap L)$, we get a homotopy equivalence (as map of pairs) which is bijective on the 0-skeleton. \square

PROPOSITION 2.17. *Let (K, K_0) and (L, L_0) be pairs of multicomplexes and assume that K, K_0, L, L_0 are all minimal and complete. Let $g : (K, K_0) \rightarrow (L, L_0)$ be a simplicial map of pairs which is bijective on the 0-skeletons. If the geometric realization $|g| : (|K|, |K_0|) \rightarrow (|L|, |L_0|)$ is a homotopy equivalence of pairs, then g is a simplicial isomorphism of pairs.*

PROOF. The case in which both K_0 and L_0 are empty multicomplexes is established in [FM23, Proposition 3.24]. The relative case directly follows from the absolute one. \square

Our construction of a minimal multicomplex M from a full pair (K, L) of complete multicomplexes requires some choices. However, Proposition 2.16 shows that M is unique up to simplicial isomorphism. We would like to apply Proposition 2.16 to the pair of singular multicomplexes $(\mathcal{K}(X), \mathcal{K}(A))$ associated to a pair (X, A) . However, this pair is in general not full.

SETUP 2.18. Let (X, A) be a pair of nice topological spaces. Assume that, for every $x \in A$ and for every $n \in \mathbb{N}_{\geq 1}$, the map $\pi_n(A \hookrightarrow X, x)$ is injective.

PROPOSITION 2.19. *Let (X, A) be as in Setup 2.18. Then the pair of multicomplexes $(\mathcal{K}(X), \mathcal{K}(A))$ is full.*

PROOF. Let Δ and Δ' be n -simplices of $\mathcal{K}(A)$ that are homotopic in $\mathcal{K}(X)$. Let x be a vertex of Δ (hence a point of A). In order to show that Δ and Δ' are homotopic in $\mathcal{K}(A)$, we consider the following commutative diagram

$$\begin{array}{ccccc} \pi_{\mathcal{K}(A)}(\Delta) & \xrightarrow{\Theta_A} & \pi_n(|\mathcal{K}(A)|, x) & \xrightarrow{\cong} & \pi_n(A, x) \\ \downarrow j_\Delta & & \downarrow j_{\mathcal{K}} & & \downarrow \pi_n(j, x) \\ \pi_{\mathcal{K}(X)}(\Delta) & \xrightarrow{\Theta_X} & \pi_n(|\mathcal{K}(X)|, x) & \xrightarrow{\cong} & \pi_n(X, x) \end{array}$$

where the vertical arrows are induced by inclusions, the horizontal arrows of the left square are surjective (Proposition 2.6) and the horizontal arrows of the right square, which are induced by the natural projections S_X and S_A , are isomorphisms (Proposition 1.11). Since Δ and Δ' are compatible, we have $\Delta' \in \pi_{\mathcal{K}(A)}(\Delta)$. By Proposition 2.6, since Δ and Δ' are homotopic in $\mathcal{K}(X)$, then $\Theta_X(j_\Delta(\Delta)) = \Theta_X(j_\Delta(\Delta'))$, hence, by the commutativity of the diagram, also $j_{\mathcal{K}}(\Theta_A(\Delta)) = j_{\mathcal{K}}(\Theta_A(\Delta'))$. Since $\pi_n(j, x)$ (hence $j_{\mathcal{K}}$) is injective, we have that $\Theta_A(\Delta) = \Theta_A(\Delta')$. We conclude that Δ and Δ' are homotopic in $\mathcal{K}(A)$. \square

Let (X, A) be a pair as in Setup 2.18. By Proposition 2.16, there is a unique (up to simplicial isomorphism) minimal complete submulticomplex $\mathcal{L}(X)$ of $\mathcal{K}(X)$ such that $\mathcal{L}(A) = \mathcal{L}(X) \cap \mathcal{K}(A)$ is a minimal complete submulticomplex of $\mathcal{K}(A)$. Therefore, in the context of Setup 2.18, it makes sense to speak about the pair of complete and minimal multicomplexes $(\mathcal{L}(X), \mathcal{L}(A))$. Since this pair is unique up to simplicial isomorphisms of pairs, we usually refer to $(\mathcal{L}(X), \mathcal{L}(A))$ as the pair of *minimal multicomplexes* associated to (X, A) .

COROLLARY 2.20. *Let (X, A) be a CW-pair as in Setup 2.18 and let*

$$i_X: (|\mathcal{L}(X)|, |\mathcal{L}(A)|) \rightarrow (|\mathcal{K}(X)|, |\mathcal{K}(A)|)$$

denote the inclusion. Then the composition with the natural projection

$$S_X \circ i_X: (|\mathcal{L}(X)|, |\mathcal{L}(A)|) \rightarrow (X, A)$$

is a homotopy equivalence of pairs.

PROOF. This is a direct consequence of Propositions 1.13 and 2.16. \square

Minimal multicomplexes, as well as singular ones, satisfy some functorial properties with respect to continuous maps of pairs.

LEMMA 2.21. *Let $f: (X, A) \rightarrow (Y, B)$ be a continuous map between CW-pairs as in Setup 2.18. Then there exists a continuous map of pairs*

$$\mathcal{L}(f): (|\mathcal{L}(X)|, |\mathcal{L}(A)|) \rightarrow (|\mathcal{L}(Y)|, |\mathcal{L}(B)|)$$

such that the following diagram

$$\begin{array}{ccc} (|\mathcal{L}(X)|, |\mathcal{L}(A)|) & \xrightarrow{\mathcal{L}(f)} & (|\mathcal{L}(Y)|, |\mathcal{L}(B)|) \\ \downarrow S_X \circ i_X & & \downarrow S_Y \circ i_Y \\ (X, A) & \xrightarrow{f} & (Y, B) \end{array}$$

is commutative up to homotopy (of maps of pairs).

PROOF. We just set $\mathcal{L}(f) = r_Y \circ \mathcal{K}(f) \circ i_X$, where i_X denotes the inclusion of pairs $i_X: (|\mathcal{L}(X)|, |\mathcal{L}(A)|) \rightarrow (|\mathcal{K}(X)|, |\mathcal{K}(A)|)$, $\mathcal{K}(f)$ is the continuous map of pairs from Lemma 1.15 and $r_Y: (|\mathcal{K}(Y)|, |\mathcal{K}(B)|) \rightarrow (|\mathcal{L}(Y)|, |\mathcal{L}(B)|)$ is the strong deformation retraction from Proposition 2.16. \square

REMARK 2.22. Of course, for every nice topological space X , the pair (X, \emptyset) fits into Setup 2.18. Hence the minimal multicomplex $\mathcal{L}(X)$ of X is always defined. In this case, simplices of $\mathcal{L}(X)$ correspond to homotopy classes of singular simplices of X , up to homotopy in X , relative to the vertices.

By Remark 2.22, for every pair (X, A) of nice topological spaces the minimal multicomplexes $\mathcal{L}(X)$ and $\mathcal{L}(A)$ are well-defined. Moreover, there is an obvious simplicial map $j_{\mathcal{L}}: \mathcal{L}(A) \rightarrow \mathcal{L}(X)$, which sends (homotopy classes of) singular simplices of A to (homotopy classes of) the corresponding singular simplex of X . We refer the reader to Section 5.1 for the precise definition of $j_{\mathcal{L}}$. This map, even though is injective on vertices, might not be a simplicial embedding. In fact, it is a simplicial embedding if and only if the pair (X, A) fits into Setup 2.18 (Proposition 5.4).

2.4. Aspherical Multicomplexes

A classical construction of the Postnikov tower of a topological space X involves taking quotients of its singular simplicial set by identifying simplices that share the same k -skeleton [GJ09, Chapter VI, Definition 2.4]. The resulting spaces have the same homotopy groups as X up to degree k , and trivial homotopy groups in higher degrees. This construction can also be carried out at the level of multicomplexes. Such an operation plays a central role in Gromov's proof that bounded cohomology depends only on the fundamental group – showing that higher homotopy groups are invisible to bounded cohomology. For this reason, we are particularly interested in the quotient for $k = 1$, where the quotient yields an aspherical multicomplex.

DEFINITION 2.23. Let A be a multicomplex. We say that A is *aspherical* if its geometric realization $|A|$ is an aspherical topological space.

Complete minimal and aspherical multicomplexes can be characterized in the following way.

PROPOSITION 2.24 ([FM23, Proposition 3.30]). *Let A be a large and connected multicomplex. The A is complete, minimal and aspherical if and only if the following conditions hold:*

- (1) For every pair of distinct vertices v_0 and v_1 of A and every continuous path $\gamma: [0, 1] \rightarrow |A|$ from v_0 to v_1 , there exists a unique simplicial embedding $\gamma': \Delta^1 \rightarrow A$ which is homotopic to γ relative to the endpoints.
- (2) For every $n \in \mathbb{N}_{\geq 2}$ and every simplicial embedding $f: (\Delta^n)^1 \rightarrow A$ such that the restriction of f to each triangular loop is null-homotopic, there exists a unique simplicial embedding $\Delta^n \rightarrow A$ extending f .

Thanks to the previous proposition, one can explicitly describe the simplicial structure of the universal covering of a complete, minimal, and aspherical multicomplex. The following observation is contained in the proof of [FM23, Proposition 3.30] and plays an important role in Chapter 7. The universal covering of (the geometric realization) of a multicomplex K admits a unique multicomplex structure which makes the covering map simplicial and non-degenerate. We denote by \widetilde{K} the corresponding multicomplex and by $p: \widetilde{K} \rightarrow K$ the simplicial projection.

LEMMA 2.25. *Let A be a complete, minimal, aspherical and connected multicomplex. Then \widetilde{A} is a simplicial complex such that $n + 1$ pairwise distinct vertices x_0, \dots, x_n of \widetilde{A} span an n -simplex if and only if the vertices $p(x_0), \dots, p(x_n)$ of A are pairwise distinct.*

PROOF. We first show that \widetilde{A} is a simplicial complex. Let $\widetilde{\Delta}_1, \widetilde{\Delta}_2$ be two n -simplices of \widetilde{A} having the same set of vertices, and assume by contradiction that $\widetilde{\Delta}_1 \neq \widetilde{\Delta}_2$. Let $\Delta_1 = p(\widetilde{\Delta}_1)$ and $\Delta_2 = p(\widetilde{\Delta}_2)$ be the corresponding projections onto A . Since p is locally injective, then $\Delta_1 \neq \Delta_2$. If $n = 1$, then the simplices Δ_1 and Δ_2 would be homotopic in A (being $|A|$ simply connected) and distinct, thus contradicting Proposition 2.24(1). If $n \geq 2$, we can deduce by the same argument above that the 1-skeleton of Δ_1 coincides with the 1-skeleton of Δ_2 , thus contradicting Proposition 2.24(2). It follows that \widetilde{A} is a simplicial complex.

If x_0, \dots, x_n are pairwise-distinct vertices of \widetilde{A} spanning a simplex, then their images $p(x_0), \dots, p(x_n)$ are of course pairwise-distinct, being p a non-degenerate simplicial map. In order to show the converse implication, we consider pairwise-distinct vertices x_0, \dots, x_n of \widetilde{A} whose images $p(x_0), \dots, p(x_n)$ are pairwise-distinct vertices of A . We begin with the case $n = 1$. Since $|A|$ (hence $|\widetilde{A}|$) is connected, there is a continuous path $\gamma: [0, 1] \rightarrow |\widetilde{A}|$ from x_0 to x_1 . By Proposition 2.24(1), the path $|p| \circ \gamma$ is homotopic relative to the vertices to a unique simplex e with endpoints $p(x_0), p(x_1)$. The lift \tilde{e} of e with vertices x_0 and x_1 provides the desired 1-simplex of \widetilde{A} . In general for $n \geq 2$, we know that for every $i \neq j \in \{0, \dots, n\}$ there is a 1-simplex in \widetilde{A} with endpoints x_i and x_j . In other words, there exists a simplicial embedding $\tilde{f}: (\Delta^n)^1 \rightarrow \widetilde{A}$ which maps the vertices of $(\Delta^n)^1$ to $\{x_0, \dots, x_n\}$. Since $|\widetilde{A}|$ is simply connected, the restriction of \tilde{f} to every triangular face is null-homotopic, and the same holds for the composition $p \circ \tilde{f}: (\Delta^n)^1 \rightarrow A$. It follows from Proposition 2.24 that there exists a simplicial embedding $g: \Delta^n \rightarrow A$ extending $p \circ \tilde{f}$. The lift of g whose set of vertices is $\{x_0, \dots, x_n\}$ provides the desired simplex of \widetilde{A} . \square

We can now describe how to construct the *aspherical quotient* A of a complete and minimal multicomplex L . The 1-skeleton of A coincides with the 1-skeleton of L . In particular, A and L have the same set of vertices. Then, if $G \subseteq A^1$ is a subgraph which is isomorphic to the 1-skeleton of the standard simplex Δ^n , then A contains one (and exactly one) n -simplex with 1-skeleton G if and only if there

is some (possibly more) n -simplex of L whose 1-skeleton is G . Equivalently, for every $n \in \mathbb{N}_{\geq 2}$, the set of n -simplices of A is given by the equivalence classes of n -simplices of L , where two n -simplices of L are equivalent if and only if they share the same 1-skeleton. There is an obvious non-degenerate simplicial map $\pi: L \rightarrow A$ which restricts to the identity on $L^1 = A^1$.

PROPOSITION 2.26 ([**FM23**, Theorem 3.31]). *Let L be a complete, minimal and large multicomplex, and let A be the aspherical quotient of L . Then A is complete, minimal and aspherical, and the projection $|\pi|: |L| \rightarrow |A|$ induces an isomorphism on fundamental groups.*

Let (X, A) be a pair of nice topological spaces. We have a well-defined pair of singular multicomplexes $(\mathcal{K}(X), \mathcal{K}(A))$. Moreover, when the inclusion $A \hookrightarrow X$ induces an injective map on homotopy groups (Setup 2.18), there is a complete and minimal submulticomplex $\mathcal{L}(X)$ of $\mathcal{K}(X)$ such that $\mathcal{L}(A) = \mathcal{L}(X) \cap \mathcal{K}(A)$ is also complete and minimal (Proposition 2.16). Let $\mathcal{A}(X)$ be the aspherical quotient of $\mathcal{L}(X)$ and let $\pi: \mathcal{L}(X) \rightarrow \mathcal{A}(X)$ be the simplicial projection which identifies simplices sharing the same 1-skeleton. The image of $\mathcal{L}(A)$ under π is also an aspherical quotient $\mathcal{A}(A)$ of $\mathcal{L}(A)$. Therefore, we have that π induces a well-defined map of pairs

$$\pi: (\mathcal{L}(X), \mathcal{L}(A)) \rightarrow (\mathcal{A}(X), \mathcal{A}(A)).$$

The pair $(\mathcal{A}(X), \mathcal{A}(A))$ is called the pair of *aspherical multicomplexes* associated to (X, A) . The following lemma follows directly from the construction.

LEMMA 2.27. *Let (X, A) be as in Setup 2.18. Then the simplicial projection*

$$\pi: (\mathcal{L}(X), \mathcal{L}(A)) \rightarrow (\mathcal{A}(X), \mathcal{A}(A))$$

induces the following commutative diagram

$$\begin{array}{ccc} \pi_1(|\mathcal{L}(A)|, x) & \longrightarrow & \pi_1(|\mathcal{A}(A)|, x) \\ \downarrow & & \downarrow \\ \pi_1(|\mathcal{L}(X)|, x) & \longrightarrow & \pi_1(|\mathcal{A}(X)|, x) \end{array}$$

for every $x \in A$, where vertical arrows are injective and horizontal arrows are isomorphisms.

The following property is crucial in the sequel, when we show that the pair $(\mathcal{A}(X), \mathcal{A}(A))$ can be obtained as a quotient of $(\mathcal{L}(X), \mathcal{L}(A))$ by the action of a group of simplicial automorphisms (Corollary 3.13).

LEMMA 2.28. *Let (X, A) be as in Setup 2.18. We have that*

$$\pi^{-1}(\mathcal{A}(A)) = \mathcal{L}(A)$$

if and only if the map $\pi_n(A \hookrightarrow X, x)$ is surjective for every $x \in A$ and for every $n \in \mathbb{N}_{\geq 2}$.

PROOF. For every n -simplex Δ of $\mathcal{L}(A)$ and every vertex x of Δ , we have the following commutative diagram

$$(2.1) \quad \begin{array}{ccccc} \pi_{\mathcal{L}(A)}(\Delta) & \xrightarrow{\cong} & \pi_n(|\mathcal{L}(A)|, x) & \xrightarrow{\cong} & \pi_n(A, x) \\ \downarrow j_\Delta & & \downarrow & & \downarrow \pi_n(j, x) \\ \pi_{\mathcal{L}(X)}(\Delta) & \xrightarrow{\cong} & \pi_n(|\mathcal{L}(X)|, x) & \xrightarrow{\cong} & \pi_n(X, x) \end{array}$$

where vertical arrows are induced by the inclusion and horizontal arrows are bijective (Proposition 1.11 and Proposition 2.14). It follows that $\pi_n(j, x)$ is surjective if and only if j_Δ is surjective. Moreover, since j_Δ denotes just the inclusion of simplices, we have that j_Δ is surjective if and only if every simplex of $\mathcal{L}(X)$ which is compatible with Δ is contained in $\mathcal{L}(A)$.

Assume that $\pi^{-1}(\mathcal{A}(A)) = \mathcal{L}(A)$. Let $n \in \mathbb{N}_{\geq 2}$, $x \in A$ and Δ be an n -simplex of $\mathcal{L}(A)$ having x as a vertex (if Δ does not exist, the result is trivially true). We want to show that j_Δ (hence $\pi_n(j, x)$) is surjective. Let Δ' be an n -simplex of $\mathcal{L}(X)$ that is compatible with Δ . Since Δ and Δ' share the same 1-skeleton, we have that $\pi(\Delta') = \pi(\Delta) \in \mathcal{A}(A)$. By assumption, this implies that $\Delta' \in \pi^{-1}(\mathcal{A}(A)) = \mathcal{L}(A)$.

Viceversa, assume that the map $\pi_n(j, x)$ is surjective for every $x \in A$ and every $n \in \mathbb{N}_{\geq 2}$. Since $\mathcal{A}(A) = \pi(\mathcal{L}(A))$, the inclusion $\mathcal{L}(A) \subseteq \pi^{-1}(\mathcal{A}(A))$ trivially holds. We prove the opposite inclusion by induction on the n -skeleton of $\mathcal{L}(X)$. For $n \in \{0, 1\}$, the inclusion is clear, since π induces the identity on the 1-skeleton of $\mathcal{L}(X)$. When it comes to the 2-skeleton, let Δ' be a 2-simplex of $\mathcal{L}(X)$ such that $\pi(\Delta') \in \mathcal{A}(A)$. We want to show that $\Delta' \in \mathcal{L}(A)$. Let $\Delta \in \mathcal{L}(A)$ be such that $\pi(\Delta) = \pi(\Delta') \in \mathcal{A}(A)$. By definition of the aspherical quotient, we have that Δ and Δ' share the same 1-skeleton, hence they are compatible. Consider now diagram (2.1). By assumption $\pi_n(j, x)$ (hence j_Δ) is surjective, and therefore, since Δ' is compatible with Δ in $\mathcal{L}(X)$, it follows that $\Delta' \in \mathcal{L}(A)$. In general, let Δ' be an n -simplex of $\mathcal{L}(X)$ such that $\pi(\Delta') \in \mathcal{A}(A)$, $n \in \mathbb{N}_{\geq 3}$. Let F_0, \dots, F_n be the facets of Δ' . By induction we have that $F_i \in \mathcal{L}(A)$. Moreover, $\pi(F_i)$ are the facets of $\pi(\Delta') \in \mathcal{A}(A)$, and therefore there exists an n -simplex Δ of $\mathcal{L}(A)$ with facets F_0, \dots, F_n . Since Δ and Δ' are compatible, we can repeat the same argument above on diagram (2.1) to deduce that $\Delta' \in \mathcal{L}(A)$. \square

The following notion plays a fundamental role in this work.

DEFINITION 2.29. A CW-pair (X, A) is called *good* if the following conditions hold: for every $x \in A$

- (1) $\pi_1(A \hookrightarrow X, x)$ is injective;
- (2) $\pi_n(A \hookrightarrow X, x)$ is an isomorphism, for every $n \in \mathbb{N}_{\geq 2}$.

Equivalently, (X, A) is good if and only if the relative homotopy groups $\pi_i(X, A)$ vanish for every $i \in \mathbb{N}_{\geq 2}$.

Of course, every good pair fits into Setup 2.18, therefore it makes sense to speak about the pair $(\mathcal{A}(X), \mathcal{A}(A))$ of aspherical multicomplexes associated to a good pair (X, A) . We require (X, A) to be a CW-pair so that the natural projection

$$S: (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (X, A)$$

induces a homotopy equivalence *as map of pairs* (Proposition 1.13).

In the next chapter, we show that the bounded cohomology of a good pair (X, A) is canonically *isometrically* isomorphic to the simplicial bounded cohomology of the pair $(\mathcal{A}(X), \mathcal{A}(A))$. However, a bi-Lipschitz control over the norms in bounded cohomology can be still obtained under milder assumptions on the homotopy of the pair (X, A) . In fact, the pair (X, \emptyset) is good for every CW-complex X . It follows that the minimal multicomplex $\mathcal{L}(X)$ is always defined, hence its aspherical quotient $\mathcal{A}(X)$ also is. As we will see in Section 5.1, there exists a well-defined simplicial map $j_{\mathcal{A}}: \mathcal{A}(A) \rightarrow \mathcal{A}(X)$, which is injective if and only if A is π_1 -injective in X (Proposition 5.5). This construction is used in Chapter 5 to show that the bounded cohomology of any π_1 -injective CW-pair (X, A) is *bi-Lipschitz* isomorphic to the simplicial bounded cohomology of $(\mathcal{A}(X), \mathcal{A}(A))$.

Bounded Cohomology of Good Pairs

It is well-known that the bounded cohomology of a topological space X only depends on its fundamental group. Gromov's original approach to this fact is to prove that the singular bounded cohomology of X is isometrically isomorphic to the simplicial bounded cohomology of the aspherical multicomplex $\mathcal{A}(X)$ associated to X . This approach was rigorously developed by Frigerio and Moraschini in [FM23]. In this chapter, we adapt their framework in the relative setting. To retain isometric control over the norms, we restrict our attention to good pairs (X, A) of topological spaces.

The main goal of this chapter is to show that for every *good* pair (X, A) there is an isometric isomorphism between the singular bounded cohomology of (X, A) and the simplicial bounded cohomology of the pair $(\mathcal{A}(X), \mathcal{A}(A))$ of aspherical multicomplexes (Theorem 1). The first step in this direction is the *Relative Isometry Lemma*, which establishes an isometric isomorphism between the singular and the simplicial bounded cohomology of a pair of complete multicomplexes (Proposition 3.5). We then show that the pair $(\mathcal{A}(X), \mathcal{A}(A))$ can be obtained from the pair $(\mathcal{L}(X), \mathcal{L}(A))$ by taking the quotient with respect to a group Γ_1 of simplicial automorphisms (Corollary 3.13). Using that Γ_1 admits a representation on higher homotopy groups (which are abelian, hence amenable), we show that the restriction of this action to every finite dimensional skeleton of $\mathcal{L}(X)$ is equivalent to the action by an amenable group (Lemma 3.14). Because actions by amenable groups are invisible to bounded cohomology (Proposition 3.15), this will be the key ingredient in order to construct the isometric isomorphism of Theorem 1.

We underline the fact that, under milder assumptions on homotopy groups of the pair (X, A) , one still obtains an isomorphism between the singular cohomology of (X, A) and the simplicial bounded cohomology of a pair of aspherical multicomplexes (Theorem 5). However, in this more general case, only a bi-Lipschitz control over the norms is retained.

We have tried to keep this chapter as self-contained as possible. However, some constructions heavily rely on those introduced by Frigerio and Moraschini in [FM23]. We therefore recommend keeping [FM23] at hand while reading this chapter. In addition, arguments that are straightforward adaptations of the absolute case developed in [FM23] are left to the reader.

3.1. Bounded Cohomology of Topological Spaces

Let (X, A) be a pair of topological spaces, and let R be a normed ring. We are particularly interested in the case in which $R \in \{\mathbb{Z}, \mathbb{R}\}$ with the absolute value norm. We denote by $(C_\bullet(X, A; R), \partial_\bullet)$ and $(C^\bullet(X, A; R), \delta^\bullet)$ the singular chain complex and singular cochain complex respectively. We endow the space of cochains with

the ℓ^∞ -norm. More precisely, for every $f \in C^n(X, A; R)$, we set

$$\|f\|_\infty = \sup\{|f(s)|, s \text{ singular } n\text{-simplex}\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

A singular cochain is called *bounded* if $\|f\|_\infty < \infty$. Since the differential takes bounded cochains to bounded cochains, we can consider the subcomplex of bounded cochains $C_b^\bullet(X, A; R) \subseteq C^\bullet(X, A; R)$. The *bounded cohomology* $H_b^\bullet(X, A; R)$ of X is the cohomology of the complex $(C_b^\bullet(X, A; R), \delta^\bullet)$. Moreover, the ℓ^∞ -norm on $C_b^\bullet(X, A; R)$ descends to a quotient seminorm on $H_b^\bullet(X, A; R)$. The inclusion of complexes $C_b^\bullet(X, A; R) \subseteq C^\bullet(X, A; R)$ induces the so-called *comparison map*

$$\text{comp}^\bullet: H_b^\bullet(X, A; R) \rightarrow H^\bullet(X, A; R).$$

The well known long exact sequence for pairs in ordinary cohomology also holds for bounded cohomology: the short exact sequence of complexes

$$0 \rightarrow C_b^\bullet(X, A; R) \rightarrow C_b^\bullet(X; R) \rightarrow C_b^\bullet(A; R) \rightarrow 0$$

induces the long exact sequence

$$\cdots \rightarrow H_b^{n-1}(A; R) \rightarrow H_b^n(X, A; R) \rightarrow H_b^n(X; R) \rightarrow H_b^n(A; R) \rightarrow \cdots$$

The ℓ^∞ -norm arises as the dual norm of the ℓ^1 -norm on the space of chains. For every $n \in \mathbb{N}_{\geq 0}$ we endow the space $C_n(X, A; R)$ with the ℓ^1 -norm: for every relative singular n -chain $\sum_{i \in I} a_i s_i$ written in reduced form, where $a_i \in R$ and $s_i: \Delta^n \rightarrow X$ are singular n -simplices of X , we set

$$\left\| \sum_{i \in I} a_i s_i \right\|_1 = \sum_{i \in I} |a_i|,$$

This norm induces a seminorm, still denoted by $\|\cdot\|_1$, on the singular homology $H_n(X, A; R)$.

From now on, unless otherwise specified, **all modules will be taken with real coefficients**, and coefficients will be omitted from the notation.

The duality pairing between chains and cochains induces a well-defined bilinear map

$$\langle \cdot, \cdot \rangle: H_b^n(X, A) \times H_n(X, A) \rightarrow \mathbb{R},$$

which is called the *Kronecker product*. The ℓ^∞ -norm in cohomology and the ℓ^1 -norm in homology are the appropriate choices to obtain a Cauchy-Schwarz inequality for the Kronecker pairing. The following result relies on the Hahn-Banach theorem and thus crucially depends on the use of real coefficients.

LEMMA 3.1 ([**Fri17**, Lemma 6.1]). *Let (X, A) be a pair of topological spaces. For every $n \in \mathbb{N}$ and every $\alpha \in H_n(X, A)$ we have*

$$\|\alpha\|_1 = \max\{\langle \beta, \alpha \rangle \mid \beta \in H_b^n(X, A), \|\beta\|_\infty \leq 1\}.$$

Every continuous map $f: (X, A) \rightarrow (Y, B)$ induces norm non-increasing linear maps

$$H_b^n(f): H_b^n(Y, B) \rightarrow H_b^n(X, A), \quad H_n(f): H_n(X, A) \rightarrow H_n(Y, B).$$

Moreover, homotopy equivalences *as maps of pairs* induce isometric isomorphisms in singular homology and in bounded cohomology. The following result follows directly from Proposition 1.13.

PROPOSITION 3.2. *Let (X, A) be a CW-pair and let*

$$S: (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (X, A)$$

be the natural projection. Then the induced map

$$H_b^n(S): H_b^n(X, A) \rightarrow H_b^n(|\mathcal{K}(X)|, |\mathcal{K}(A)|)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

REMARK 3.3. We would like to emphasize the role of homotopy equivalences of pairs in our context. Let $f: (X, A) \rightarrow (Y, B)$ be a map of pairs. Requiring f to be a homotopy equivalence of pairs is of course more restrictive than requiring both f and $f|_A$ to be homotopy equivalences. However, if f is a homotopy equivalence of pairs, it is clear that f induces an *isometric* isomorphism $H_b^n(f): H_b^n(Y, B) \rightarrow H_b^n(X, A)$ for every $n \in \mathbb{N}$. On the contrary, if both f and $f|_A$ are homotopy equivalences, then the Five Lemma implies that $H_b^n(f)$ is an isomorphism, but it is not clear how to show that this isomorphism is isometric.

REMARK 3.4. Milder assumption on the pair (X, A) still allow us to deduce that S_X induces an isomorphism in relative bounded cohomology. We know that, if X is a nice topological space, then the natural projection $S: |\mathcal{K}(X)| \rightarrow X$ is a weak homotopy equivalence (Proposition 1.11). Let now (X, A) be a pair of nice topological spaces. Since weak homotopy equivalences induce isometric isomorphisms in bounded cohomology [Iva17, Corollary 6.4], the Five Lemma implies that $S: (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (X, A)$ induces an isomorphism in relative bounded cohomology. However, it remains unclear whether this isomorphism is isometric.

3.2. The Relative Isometry Lemma

Let K be a multicomplex. Recall that we have a natural chain inclusion

$$\varphi_\bullet: C_\bullet(K) \rightarrow C_\bullet(|K|),$$

sending each algebraic n -simplex $(\sigma, (v_0, \dots, v_n))$ of K to the singular simplex

$$\Delta^n \rightarrow |K|, \quad (t_0, \dots, t_n) \mapsto (\sigma, t_0 v_0 + \dots + t_n v_n).$$

Although φ_\bullet induces an isomorphism in cohomology (Proposition 1.8), we have already observed that the same cannot hold in bounded cohomology. The situation is better for complete multicomplexes: according to Gromov's Isometry Lemma, the bounded cohomology of a large and complete multicomplex is isometrically isomorphic to the bounded cohomology of its geometric realization [Gro82, pag. 43] [FM23, Theorem 2]. This isomorphism has been established for pairs of complete multicomplexes by Kuessner in [Kue15]. Given a pair of multicomplexes (K, L) , since φ_\bullet is natural with respect to the inclusion $L \subseteq K$, we consider the following commutative diagram

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C_b^\bullet(|K|, |L|) & \longrightarrow & C_b^\bullet(|K|) & \longrightarrow & C_b^\bullet(|L|) \longrightarrow 0 \\ & & \downarrow \varphi_{K,L}^\bullet & & \downarrow \varphi_K^\bullet & & \downarrow \varphi_L^\bullet \\ 0 & \longrightarrow & C_b^\bullet(K, L) & \longrightarrow & C_b^\bullet(K) & \longrightarrow & C_b^\bullet(L) \longrightarrow 0, \end{array}$$

where the vertical arrows are induced by the chain inclusion above (hence, they are norm non-increasing).

PROPOSITION 3.5 (Relative Isometry Lemma). *Let (K, L) be a pair of multi-complexes and assume that both K and L are large and complete. Then the map*

$$H_b^n(\varphi_{K,L}^\bullet): H_b^n(|K|, |L|) \rightarrow H_b^n(K, L)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

PROOF. The case with $L = \emptyset$ is addressed in [FM23], where the authors construct a norm non-increasing (partial) chain map

$$\psi_K^\bullet: C_b^\bullet(K) \rightarrow C_b^\bullet(|K|)$$

that induces the inverse of $H_b^n(\varphi_K^\bullet)$ in bounded cohomology. Using the Five Lemma on the long exact sequence in cohomology associated with (3.1), one then deduces that $H_b^n(\varphi_{K,L}^\bullet)$ is also an isomorphism in bounded cohomology. To show that this isomorphism is isometric, it suffices to prove that the inverse of $H_b^n(\varphi_{K,L}^\bullet)$ is likewise induced by ψ_K^\bullet . This, in turn, follows from the naturality of ψ_K^\bullet with respect to the inclusion of submulticomplexes, i.e. $\psi_K^\bullet(C_b^\bullet(L)) \subseteq C_b^\bullet(|L|)$. The original construction of ψ_K^\bullet in [FM23] does not automatically satisfy this naturality condition, but it can be easily adjusted by carefully applying the homotopy extension property for CW-pairs. \square

A major consequence of the Relative Isometry Lemma is the fact that the bounded cohomology of a CW-complex can be isometrically computed via bounded simplicial cochains on the singular multicomplex.

PROPOSITION 3.6. *Let (X, A) be a CW-pair. Then the composition*

$$H_b^n(X, A) \xrightarrow{H_b^n(S)} H_b^n(|\mathcal{K}(X)|, |\mathcal{K}(A)|) \xrightarrow{H_b^n(\varphi^\bullet)} H_b^n(\mathcal{K}(X), \mathcal{K}(A))$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

PROOF. The result follows from Propositions 3.2 and 3.5. \square

Proposition 3.6 holds also for the minimal multicomplexes in presence of further control over the homotopy of the pair (X, A) . We know that, if the inclusion $A \hookrightarrow X$ induces injective maps on homotopy groups (Setup 2.18), then the pair $(\mathcal{K}(X), \mathcal{K}(A))$ retracts onto a well-defined pair $(\mathcal{L}(X), \mathcal{L}(A))$ of complete and minimal multicomplexes. With an abuse of notation, we still denote by

$$S: (|\mathcal{L}(X)|, |\mathcal{L}(A)|) \rightarrow (X, A)$$

the restriction of the natural projection.

PROPOSITION 3.7. *Let (X, A) be a CW-pair as in Setup 2.18. Then the composition*

$$H_b^n(X, A) \xrightarrow{H_b^n(S)} H_b^n(|\mathcal{L}(X)|, |\mathcal{L}(A)|) \xrightarrow{H_b^n(\varphi^\bullet)} H_b^n(\mathcal{L}(X), \mathcal{L}(A))$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

PROOF. The result follows from Corollary 2.20 and Proposition 3.5. \square

3.3. Amenable Groups of Simplicial Automorphisms

We begin with the following definition.

DEFINITION 3.8. Let (K, L) be a pair of multicomplexes. We define $\Gamma(K, L)$ to be the group of all simplicial automorphisms in $\text{Aut}(K, L)$ that are (topologically) homotopic to the identity (as maps of pairs) relative to the 0-skeleton of K . Moreover, for every $i \in \mathbb{N}_{\geq 1}$, we define

$$\Gamma_i(K, L) = \{g \in \Gamma(K, L) \mid g|_{K^i} = \text{id}\}.$$

We want to show that the groups $\Gamma_i(K, L)$ act as transitively as possible on the set of $(i + 1)$ -simplices of *both* K and L . To this end, we introduce the following notion.

DEFINITION 3.9. Let $i \in \mathbb{N}_{\geq 1}$. A pair of multicomplexes (K, L) is *i -coherent* if, for every pair of simplices Δ, Δ' of K sharing the same i -skeleton, one has $\Delta \in L$ if and only if $\Delta' \in L$.

Recall that, for every simplex Δ of K , $\pi_K(\Delta)$ denotes the set of simplices of K that are compatible with Δ . If (K, L) is a pair of multicomplexes, the inclusion $\pi_L(\Delta) \subseteq \pi_K(\Delta)$ holds for every simplex Δ of L . However, if we assume the pair (K, L) to be *i -coherent*, we have that

$$\pi_K(\Delta) = \pi_L(\Delta)$$

for every $(i + 1)$ -simplex Δ of L . Moreover, if (K, L) is *i -coherent*, then it is *n -coherent* for every $n \in \mathbb{N}_{\geq i}$.

Let (X, A) be a CW-pair. We know that, if the inclusion $A \hookrightarrow X$ induces an injective map on homotopy groups (Setup 2.18), then there is a well defined pair of complete minimal multicomplexes $(\mathcal{L}(X), \mathcal{L}(A))$. Moreover, the aspherical quotient induces a map of pairs

$$\pi : (\mathcal{L}(X), \mathcal{L}(A)) \rightarrow (\mathcal{A}(X), \mathcal{A}(A))$$

which identifies simplices sharing the same 1-skeleton. Therefore, the pair

$$(\mathcal{A}(X), \pi^{-1}(\mathcal{A}(A)))$$

is clearly 1-coherent. The same holds for the pair of minimal multicomplexes under further control over the homotopy of (X, A) .

LEMMA 3.10. *Let (X, A) be a good pair, then the pair $(\mathcal{L}(X), \mathcal{L}(A))$ is 1-coherent.*

PROOF. The result follows from the fact that $\mathcal{L}(A) = \pi^{-1}(\mathcal{A}(A))$ for good pairs (Lemma 2.28). \square

It is shown in [FM23, Lemma 4.12] that the groups $\Gamma_i(K)$ act as transitively as possible on $(i + 1)$ -simplices of K . This property may fail to hold when the action is required to preserve a submulticomplex L of K . However, if the pair (K, L) is *i -coherent*, then the groups $\Gamma_i(K, L)$ still act transitively on compatible $(i + 1)$ -simplices *both* of K and L .

LEMMA 3.11. *Let $i \in \mathbb{N}_{\geq 1}$. Let (K, L) be an i -coherent pair of multicomplexes. Assume that both K and L are complete and minimal. Let Δ_0 be an $(i + 1)$ -simplex of K , and let $\Delta \in \pi_K(\Delta_0)$. Let F be a facet of Δ_0 . Then there exists an element*

$g \in \Gamma_i(K, L)$ such that the following statements hold: $g(\Delta_0) = \Delta$, and $g(\Delta') = \Delta'$ for every m -simplex Δ' , $m \geq i$, which does not contain F .

PROOF. The case with $L = \emptyset$ is addressed in [FM23, Lemma 4.12], where an element $g \in \Gamma_i(K)$ with the desired properties is constructed. In our setting, we only need to make sure that g preserves L . To this end, consider the map $\tilde{f}: K^i \cup \Delta_0 \rightarrow K$, which is defined to be the identity on K^i and extends to Δ_0 via a linear isomorphism onto Δ . Since the pair (K, L) is i -coherent, then \tilde{f} preserves the subcomplex L and induces a well-defined map of pairs

$$\tilde{f}: (K^i \cup \Delta_0, (K^i \cup \Delta_0) \cap L) \rightarrow (K, L).$$

Let now F be a facet of Δ_0 , and let $K' \subseteq K^i$ be the submulticomplex of K^i obtained by removing from K^i the interior of F . In [FM23] the authors construct a homotopy

$$h: |K^i \cup \Delta_0| \times [0, 1] \rightarrow |K|$$

between \tilde{f} and the inclusion map, relative to K' . It is straightforward to check that h defines a homotopy of maps of pairs. Then, by the homotopy extension property of the CW-pair $(|K|, |K^i \cup \Delta_0|)$, the homotopy h extends to a homotopy

$$H: |K| \times [0, 1] \rightarrow |K|.$$

Define $\tilde{f}_1 = H_1: |K| \rightarrow |K|$, and let $K'' \subseteq K$ be the subcomplex of K obtained by removing from K the interior of F and all the simplices containing F . By looking at the explicit construction of H in [Hat02, Proposition 0.16], we see that H may be taken to be constant on K'' and such that $H(|L| \times I) \subseteq |L|$. Hence \tilde{f}_1 defines a map of pairs $\tilde{f}_1: (|K|, |L|) \rightarrow (|K|, |L|)$ which restricts to identity on K'' , and is simplicial when restricted on $K'' \cup \Delta_0$. Since K and L are complete, Proposition 2.7 implies that there exists a non-degenerate simplicial map

$$g: (K, L) \rightarrow (K, L)$$

such that \tilde{f}_1 is homotopic to $|g|$ relative to $V(K) \cup K'' \cup \Delta_0$ (as maps of pairs). By construction, g is homotopic to the identity relative to the 0-skeleton, fixes K^i , sends Δ_0 to Δ , and restricts to the identity on K'' . Finally, Proposition 2.17 ensures that g is indeed an isomorphism of pairs of multicomplexes, and thus defines an element of $\Gamma_i(K, L)$. \square

If (K, L) is i -coherent, then it follows from Lemma 3.11 that two $(i+1)$ -simplices of K are compatible if and only if they are in the same $\Gamma_i(K, L)$ orbit. In other words, for every $(i+1)$ -simplex Δ of K , the group $\Gamma_i(K, L)$ acts transitively on $\pi_K(\Delta) = \pi_L(\Delta)$. This observation can be strengthened in the following way.

PROPOSITION 3.12. *Let $i, n \in \mathbb{N}_{\geq 1}$ be such that $n \geq i + 1$. Let (K, L) be an i -coherent pair of multicomplexes and assume that both K and L are complete and minimal. Let Δ, Δ' be n -simplices of K . Then Δ and Δ' are in the same $\Gamma_i(K, L)$ -orbit if and only if they share the same i -skeleton.*

PROOF. The argument is *verbatim* the one given in [FM23, Proposition 4.14]. \square

Let (X, A) be a good pair and let $\Gamma_1 = \Gamma_1(\mathcal{L}(X), \mathcal{L}(A))$. Since the action $\Gamma_1 \curvearrowright (\mathcal{L}(X), \mathcal{L}(A))$ is 0-trivial, the quotient

$$(\mathcal{L}(X), \mathcal{L}(A))/\Gamma_1 = (\mathcal{L}(X)/\Gamma_1, \mathcal{L}(A)/\Gamma_1)$$

is a well-defined pair of multicomplexes (Proposition 1.7). Moreover, since the pair $(\mathcal{L}(X), \mathcal{L}(A))$ is 1-coherent (Lemma 3.10), Proposition 3.12 shows that we can characterize the aspherical quotient $(\mathcal{A}(X), \mathcal{A}(A))$ as the quotient of the simplicial action $\Gamma_1 \curvearrowright (\mathcal{L}(X), \mathcal{L}(A))$.

COROLLARY 3.13. *Let (X, A) be a good pair and let $\Gamma_1 = \Gamma_1(\mathcal{L}(X), \mathcal{L}(A))$. Then the pair $(\mathcal{L}(X), \mathcal{L}(A))/\Gamma_1$ is canonically isomorphic to the pair $(\mathcal{A}(X), \mathcal{A}(A))$.*

Since amenable groups are invisible to bounded cohomology, if Γ_1 were amenable, Corollary 3.13 would imply that the bounded cohomology of $(\mathcal{L}(X), \mathcal{L}(A))$ – and thus of (X, A) , by Proposition 3.7 – is isometrically isomorphic to the bounded cohomology of its aspherical quotient. However, the group Γ_1 is *not* amenable in general. On the other hand, the natural action of the groups Γ_{i-1} on the set of compatible simplices induces a representation of Γ_{i-1} into higher homotopy groups. Using the fact that higher homotopy groups are abelian, it turns out that the quotients Γ_1/Γ_i are solvable (hence amenable) for every $i \in \mathbb{N}_{\geq 1}$ [Gro82, Section 3.3][FM23, Corollary 4.20]. For the sake of completeness, we present the straightforward adaptation of the argument from [FM23] to the relative setting.

Let K be a complete and minimal multicomplex and let $i \in \mathbb{N}_{\geq 2}$. Let $\{\Delta_\alpha\}_{\alpha \in A_i}$ be a set of representatives for the action of $\Gamma_{i-1}(K)$ on the set of i -simplices of K . We fix an ordering of the vertices of Δ_α for every $\alpha \in A_i$, and we denote by x_α the minimal vertex of Δ_α . For every $\alpha \in A_i$ and every $\gamma \in \Gamma_{i-1}(K)$ the simplices Δ_α and $\gamma(\Delta_\alpha)$ are compatible. Therefore, there is a special sphere

$$\dot{S}_\alpha^i: (\dot{S}^i, s_0) \rightarrow (K, x_\alpha), \quad \dot{S}_\alpha^i(\gamma) = \dot{S}^i(\Delta_\alpha, \gamma(\Delta_\alpha)),$$

which identifies an element in the i -th homotopy group of $|K|$ (Proposition 2.6). For every $i \in \mathbb{N}_{\geq 2}$ and every $\alpha \in A_i$, the map

$$\varphi_\alpha^{(i)}: \Gamma_{i-1}(K) \rightarrow \pi_i(|K|, x_\alpha), \quad \gamma \mapsto [\dot{S}_\alpha^i(\gamma)]$$

is a group homomorphism [FM23, Lemma 4.17]. An element $\gamma \in \Gamma_{i-1}(K)$ lies in the kernel of $\varphi_\alpha^{(i)}$ if and only if $\gamma(\Delta_\alpha)$ is homotopic to Δ_α . Since K is minimal, this happens exactly when $\gamma(\Delta_\alpha) = \Delta_\alpha$ (Proposition 2.14). It follows that the kernel of $\varphi_\alpha^{(i)}$ coincides with the stabilizer of Δ_α (hence of any simplex on the orbit of Δ_α). By taking the direct product we obtain a homomorphism

$$\varphi_K^{(i)}: \Gamma_{i-1}(K) \rightarrow \prod_{\alpha \in A_i} \pi_i(|K|, x_\alpha).$$

Moreover, the union of the $\Gamma_{i-1}(K)$ -orbits of Δ_α coincides with the i -skeleton of K . It follows that the kernel of $\varphi_K^{(i)}$ coincides with $\Gamma_i(K)$ [FM23, Lemma 4.18].

Let now L be any submulticomplex of K . By composing $\varphi_K^{(i)}$ with the inclusion $\Gamma_{i-1}(K, L) \leq \Gamma_{i-1}(K)$, we get a homomorphism

$$\varphi_{(K,L)}^{(i)}: \Gamma_{i-1}(K, L) \rightarrow \prod_{\alpha \in A} \pi_i(|K|, x_\alpha),$$

such that

$$\ker \varphi_{(K,L)}^{(i)} = \ker \varphi_K^{(i)} \cap \Gamma_{i-1}(K, L) = \Gamma_i(K) \cap \Gamma_{i-1}(K, L) = \Gamma_i(K, L).$$

Since $i \in \mathbb{N}_{\geq 2}$, the target group of $\varphi_{(K,L)}^{(i)}$ is abelian. Let now Γ_i denote the group $\Gamma_i(K, L)$. It follows that the quotient Γ_{i-1}/Γ_i is abelian and the following chain of

normal subgroups

$$\Gamma_1/\Gamma_i \supseteq \Gamma_2/\Gamma_i \supseteq \cdots \supseteq \Gamma_i/\Gamma_i = \{1\}$$

is a subnormal series with abelian factors. It follows that the quotient Γ_1/Γ_i is solvable. We summarize the discussion above in the following lemma.

LEMMA 3.14. *Let (K, L) be a pair of multicomplexes and assume that K is complete and minimal. Then, for every $i \in \mathbb{N}_{\geq 2}$, the group $\Gamma_1(K, L)/\Gamma_i(K, L)$ is solvable, hence amenable.*

3.4. Isometric Isomorphisms

The goal of this section is to show that the quotient map

$$(\mathcal{L}(X), \mathcal{L}(A)) \rightarrow (\mathcal{A}(X), \mathcal{A}(A))$$

of the aspherical quotient induces an isometric isomorphism in bounded cohomology for every good pair (X, A) . This fact, together with Proposition 3.7, is the last step to show that the simplicial bounded cohomology of $(\mathcal{A}(X), \mathcal{A}(A))$ is isometrically isomorphic to the bounded cohomology of (X, A) (Theorem 1).

We begin with the following result regarding amenable groups of simplicial automorphisms, which is a straightforward generalization of [FM23, Theorem 4.21] to pairs of multicomplexes. If a group G acts simplicially on a pair of multicomplexes (K, L) , then it induces a linear action on the space of algebraic simplices, which in turns induces a linear action $G \curvearrowright C_b^\bullet(K, L)$ on bounded cochains. We denote by $C_b^\bullet(K, L)^G \subseteq C_b^\bullet(K, L)$ the subcomplex of G -invariant bounded cochains. We consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_b^\bullet(K, L)^G & \longrightarrow & C_b^\bullet(K)^G & \longrightarrow & C_b^\bullet(L)^G \longrightarrow 0 \\ & & \downarrow \iota^\bullet & & \downarrow \iota_K^\bullet & & \downarrow \iota_L^\bullet \\ 0 & \longrightarrow & C_b^\bullet(K, L) & \longrightarrow & C_b^\bullet(K) & \longrightarrow & C_b^\bullet(L) \longrightarrow 0, \end{array}$$

where the vertical arrows are inclusions of complexes. The constants C_n , $n \in \mathbb{N}_{\geq 1}$, appearing in the following statement denote the number of n -simplices in the triangulation of $\Delta^{n-1} \times [0, 1]$ with its natural structure of multicomplex (Definition 2.8). Recall that every pair of simplicially homotopic maps induce algebraically homotopic maps on cochains via a homotopy whose norm is bounded from above by C_n (Remark 2.10).

PROPOSITION 3.15. *Let $G \curvearrowright (K, L)$ be a group action on a pair of multicomplexes. For every $i \in \mathbb{N}$, let G_i denote the subgroup of G acting trivially on K^i . Assume that G/G_i is amenable for every $i \in \mathbb{N}$ and that, for every $g \in G$, the simplicial automorphism $g : (K, L) \rightarrow (K, L)$ is simplicially homotopic to the identity as a map of pairs. Then, for every $k \in \mathbb{N}$, there exists a norm non-increasing (partial) chain map*

$$A^i : C_b^i(K, L) \rightarrow C_b^i(K, L)^G, \quad i \in \{0, \dots, k\},$$

such that $A^i \circ \iota^i$ is the identity of $C_b^i(K, L)^G$ and $\iota^i \circ A^i$ is chain homotopic to the identity via a (partial) chain homotopy

$$T^i : C_b^i(K, L) \rightarrow C_b^{i-1}(K, L)$$

such that $\|T^i\| \leq C_i$ for every $i \in \{0, \dots, k\}$. In particular, the inclusion of invariant cochains

$$\iota^\bullet: C_b^\bullet(K, L)^G \hookrightarrow C_b^\bullet(K, L)$$

induces, for every $n \in \mathbb{N}$, an isometric isomorphism

$$H_b^n(\iota^\bullet): H^n(C_b^\bullet(K, L)^G) \rightarrow H_b^n(K, L).$$

PROOF. Let $k \in \mathbb{N}$ and let $G^i = G/G_i$, for every $i \in \{0, \dots, k\}$. Let

$$m_k: \ell^\infty(G^k) \rightarrow \mathbb{R}$$

be a left-invariant mean on G^k . Since the group

$$G^i = G/G_i \cong \frac{G/G_k}{G_i/G_k}$$

is a quotient of G^k , we endow G^i with the left-invariant mean m_i induced by m_k for every $i \in \{0, \dots, k\}$ (see [Fri17, Proposition 3.4]). Since the action of G_i is trivial on the i -skeleton, then G^i acts on $C_i(K, L)$. If s is an algebraic simplex of K and $f \in C_b^n(K)$ is a bounded cochain, then the function

$$G^i \rightarrow \mathbb{R}, \quad \gamma \mapsto f(\gamma^{-1}s)$$

lies in $\ell^\infty(G^i)$. We may therefore define

$$A^i: C_b^i(K, L) \rightarrow C_b^i(K, L)^G,$$

by the formula $A^i(f)(s) = m_i(\gamma \mapsto f(\gamma^{-1}s))$. One can check that our choices for the left-invariant means m_i ensure that A^i is a (partial) chain map. The G -invariance of $A^i(g)$ is a consequence of the invariance of the mean. The fact that $A^i(f)$ is a relative cochain follows from the fact that L is preserved under the action of G on K . Since

$$|A^i(f)(s)| \leq \sup_{\gamma \in G^i} |f(\gamma^{-1}s)| \leq \|f\|_\infty,$$

we have that A^i is norm non-increasing. Moreover, it is straightforward to check that $A^i \circ \iota^i = \text{id}$. This fact is already sufficient to show that ι^\bullet induces an isometric embedding in bounded cohomology. In order to construct a chain homotopy between $\iota^i \circ A^i$ and the identity we proceed as follows. For every $g \in G$, we denote by

$$t_g^i: C_b^i(K, L) \rightarrow C_b^i(K, L)$$

the chain map induced by g . Since $g: (K, L) \rightarrow (K, L)$ is simplicially homotopic to the identity as a map of pairs, then there is a chain homotopy

$$T_g^i: C_b^i(K, L) \rightarrow C_b^{i-1}(K, L)$$

between t_g^i and the identity such that

$$(3.2) \quad \delta^{i-1}T_g^i + T_g^{i+1}\delta^i = t_g^i - \text{id}$$

holds for every $g \in G$. By Remark 2.10, we know that $\|T_g^i\|_\infty \leq C_i$. Let now $f \in C_b^i(K, L)$ and let s be an algebraic $(i-1)$ -simplex. Then the function

$$G^{i-1} \rightarrow \mathbb{R}, \quad \gamma \mapsto T_{\gamma^{-1}}^i(f)(s)$$

lies in $\ell^\infty(G^{i-1})$. We have therefore a linear map

$$T^i: C_b^i(K, L) \rightarrow C_b^{i-1}(K, L)$$

which is defined by the formula $T^i(f)(s) = m_{i-1}(\gamma \mapsto T_{\gamma^{-1}}^i(f)(s))$. By the properties of the means, we have that $|T^i(f)(s)| \leq C_i \|f\|_\infty$, so that T^i is a bounded map with norm

$$\|T^i\| \leq C_i.$$

If we evaluate (3.2) on some bounded cochain $f \in C_b^i(K, L)$ and some algebraic $(i-1)$ -simplex s , we obtain that

$$\delta^{i-1}(T_{\gamma^{-1}}^i(f))(s) + T_{\gamma^{-1}}^{i+1}(\delta^i(f))(s) = t_{\gamma^{-1}}^i(f)(s) - f(s) = f(\gamma^{-1}s) - f(s).$$

We consider both sides of the previous equation as functions over G^i . By averaging both sides with respect to m_i we get

$$\delta^{i-1}(T^i(f))(s) + T^{i+1}(\delta^i(f))(s) = A^i(f)(s) - f(s),$$

which shows that T^i provides a (partial) chain homotopy between $\iota^i \circ A^i$ and the identity. \square

REMARK 3.16. The fact that the action $G \curvearrowright K$ preserves the submulticomplex L is crucial in Proposition 3.15. However, in our applications, we also deal with actions which do *not* preserve a particular submulticomplex. In order to consider invariant cochains also in this case, some regularity of the action $G \curvearrowright K$ on L is required (see Definition 5.19). We refer the reader to Section 5.5 for a detailed discussion on this topic.

We are now ready to prove that the simplicial bounded cohomology of $(\mathcal{A}(X), \mathcal{A}(A))$ *isometrically* computes the bounded cohomology of every good pair (X, A) . The isomorphism below for pairs (X, \emptyset) (which are always good) has been established in [FM23, Theorem 4.23].

THEOREM 1. *Let (X, A) be a good pair. Then, for every $n \in \mathbb{N}$, there is a canonical isometric isomorphism*

$$\Phi^n: H_b^n(\mathcal{A}(X), \mathcal{A}(A)) \rightarrow H_b^n(X, A).$$

PROOF. Let (X, A) be a good pair and let $n \in \mathbb{N}$. We know that the pair $(\mathcal{L}(X), \mathcal{L}(A))$ of complete and minimal multicomplexes is well-defined. Moreover, the composition

$$H_b^n(X, A) \xrightarrow{H_b^n(S)} H_b^n(|\mathcal{L}(X)|, |\mathcal{L}(A)|) \xrightarrow{H_b^n(\varphi^\bullet)} H_b^n(\mathcal{L}(X), \mathcal{L}(A)),$$

is an isometric isomorphism (Proposition 3.7), where $H_b^n(S)$ is induced by the natural projection, and $H_b^n(\varphi^\bullet)$ is the map induced by the inclusion of simplicial chains into singular ones. We denote by Γ_i the group of simplicial automorphisms $\Gamma_i(\mathcal{L}(X), \mathcal{L}(A))$ for every $i \in \mathbb{N}_{\geq 1}$ (Definition 3.8). Recall from Corollary 3.13 that the pair $(\mathcal{A}(X), \mathcal{A}(A))$ is canonically isomorphic to the quotient of $(\mathcal{L}(X), \mathcal{L}(A))$ by the action of Γ_1 . Therefore there is an obvious chain isomorphism between $C_b^\bullet(\mathcal{A}(X), \mathcal{A}(A))$ and $C_b^\bullet(\mathcal{L}(X), \mathcal{L}(A))^{\Gamma_1}$. Under this identification, the projection

$$\pi: (\mathcal{L}(X), \mathcal{L}(A)) \rightarrow (\mathcal{A}(X), \mathcal{A}(A))$$

induces the inclusion $C_b^\bullet(\mathcal{L}(X), \mathcal{L}(A))^{\Gamma_1} \hookrightarrow C_b^\bullet(\mathcal{L}(X), \mathcal{L}(A))$ of invariant cochains. Moreover, the elements of Γ_1 are topologically homotopic to the identity (as maps of pairs), and therefore they are simplicially homotopic to the identity (as maps of pairs) by Lemma 2.11. Since the groups Γ_1/Γ_i are amenable for every $i \in \mathbb{N}_{\geq 1}$ (Lemma 3.14), we deduce from Proposition 3.15 that the projection π induces an

isometric isomorphism in bounded cohomology. The composition of these isomorphisms defines the isometric isomorphism Φ^n . \square

Theorem 1 implies that bounded cohomology only depends on the fundamental group. Let X be a CW-complex. Theorem 1 (with $A = \emptyset$) implies that the bounded cohomology of X is isometrically isomorphic to the simplicial bounded cohomology of $\mathcal{A}(X)$. By Proposition 3.5 this is isometrically isomorphic to the bounded cohomology of its topological realization $|\mathcal{A}(X)|$. Since $|\mathcal{A}(X)|$ is a model of the classifying space of the fundamental group of X , it follows that the bounded cohomology of X is uniquely determined by its fundamental group. We will see in Section 6.3 how to construct an explicit classifying map $X \rightarrow |\mathcal{A}(X)|$ in the case where X is a simplicial complex (see Remark 6.11).

The situation for relative bounded cohomology is more delicate, because of the role of higher homotopy in the definition of Φ^n . The following result is suggested in [FM23, Section 4.5] without assumptions on the higher homotopy of the pair. A continuous map of pairs $f: (X, A) \rightarrow (Y, B)$ is *bijective on the sets of path-connected components* if it induces bijective maps

$$\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y), \quad \pi_0(f|_A): \pi_0(A) \rightarrow \pi_0(B).$$

THEOREM 2. *Let $f: (X, A) \rightarrow (Y, B)$ be a continuous map between good pairs which is bijective on the sets of path-connected components. Assume that f induces isomorphisms on fundamental groups on every component of X and every component of A , i.e. for every $x \in X$ and every $a \in A$, the maps*

$$\pi_1(f, x): \pi_1(X, x) \rightarrow \pi_1(Y, f(x)), \quad \pi_1(f|_A, a): \pi_1(A, a) \rightarrow \pi_1(B, f(a)),$$

are isomorphisms. Then the induced map

$$H_b^n(f): H_b^n(Y, B) \rightarrow H_b^n(X, A)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

PROOF. We assume without loss of generality X and Y to be path-connected. We know by Lemma 2.21 that there exists a continuous map of pairs

$$\mathcal{L}(f): (|\mathcal{L}(X)|, |\mathcal{L}(A)|) \rightarrow (|\mathcal{L}(Y)|, |\mathcal{L}(B)|)$$

such that the following diagram

$$\begin{array}{ccc} (|\mathcal{L}(X)|, |\mathcal{L}(A)|) & \xrightarrow{\mathcal{L}(f)} & (|\mathcal{L}(Y)|, |\mathcal{L}(B)|) \\ \downarrow S_X \circ i_X & & \downarrow S_Y \circ i_Y \\ (X, A) & \xrightarrow{f} & (Y, B) \end{array}$$

is commutative up to homotopy (of maps of pairs). Since the natural projections $S_X \circ i_X$ and $S_Y \circ i_Y$ are homotopy equivalences (of maps of pairs), it follows that also $\mathcal{L}(f)$ induces isomorphisms on fundamental groups and is bijective on the sets of path connected components. In order to conclude, it is enough to show that $\mathcal{L}(f)$ induces isometric isomorphisms in relative bounded cohomology. We consider now the aspherical quotients $\pi_X: (\mathcal{L}(X), \mathcal{L}(A)) \rightarrow (\mathcal{A}(X), \mathcal{A}(A))$ and $\pi_Y: (\mathcal{L}(Y), \mathcal{L}(B)) \rightarrow (\mathcal{A}(Y), \mathcal{A}(B))$. We know that aspherical quotients induce isomorphisms on fundamental groups, i.e. for every $x \in A$ the horizontal arrows of

the following commutative diagram

$$\begin{array}{ccc} \pi_1(|\mathcal{L}(A)|, x) & \longrightarrow & \pi_1(|\mathcal{A}(A)|, x) \\ \downarrow & & \downarrow \\ \pi_1(|\mathcal{L}(X)|, x) & \longrightarrow & \pi_1(|\mathcal{A}(X)|, x) \end{array}$$

are isomorphisms (Lemma 2.27), and the same holds for the pair (Y, B) . It follows that $\mathcal{L}(f)$ induces the following commutative diagram on fundamental groups

$$(3.3) \quad \begin{array}{ccc} \pi_1(|\mathcal{A}(A)|, x) & \longrightarrow & \pi_1(|\mathcal{A}(B)|, y) \\ \downarrow & & \downarrow \\ \pi_1(|\mathcal{A}(X)|, x) & \longrightarrow & \pi_1(|\mathcal{A}(Y)|, y) \end{array}$$

where $x \in A$ and $y = \pi_Y \circ \mathcal{L}(f)(x)$. Without loss of generality we can assume $\mathcal{L}(f)$ to be cellular on the 1-skeleton. Since aspherical quotients share the same 1-skeleton of the corresponding minimal multicomplexes, diagram (3.3) is induced by the restriction of $\mathcal{L}(f)$ to the 1-skeleton. Since $|\mathcal{A}(Y)|$ and $|\mathcal{A}(B)|$ are aspherical, we can invoke Proposition 3.17 below to obtain a map $\mathcal{A}(f): (|\mathcal{A}(X)|, |\mathcal{A}(A)|) \rightarrow (|\mathcal{A}(Y)|, |\mathcal{A}(B)|)$ such that the following diagram

$$\begin{array}{ccc} (|\mathcal{L}(X)|, |\mathcal{L}(A)|) & \xrightarrow{\mathcal{L}(f)} & (|\mathcal{L}(Y)|, |\mathcal{L}(B)|) \\ \downarrow \pi_X & & \downarrow \pi_Y \\ (|\mathcal{A}(X)|, |\mathcal{A}(A)|) & \xrightarrow{\mathcal{A}(f)} & (|\mathcal{A}(Y)|, |\mathcal{A}(B)|) \end{array}$$

commutes up to homotopy (of maps of pairs). Since $\mathcal{L}(f)$ induces isomorphisms on fundamental groups, $\mathcal{A}(f)$ does as well. Moreover, as $|\mathcal{A}(X)|$ and $|\mathcal{A}(A)|$ are aspherical, the restriction of $\mathcal{A}(f)$ to $|\mathcal{A}(X)|$ and $|\mathcal{A}(A)|$ is a homotopy equivalence [Hat02, Proposition 1B.9]. Since the inclusion $|\mathcal{A}(A)| \hookrightarrow |\mathcal{A}(X)|$ is a cofibration, it follows that $\mathcal{A}(f)$ is a homotopy equivalence as a map of pairs [May99, Chapter 6, Section 5], and thus induces an isometric isomorphism in bounded cohomology. In addition, the quotient maps π_X and π_Y induce isometric isomorphisms in bounded cohomology (Proposition 3.13). We conclude that $\mathcal{L}(f)$ (hence f) induces an isometric isomorphism in bounded cohomology. \square

PROPOSITION 3.17. *Let (X, A) and (Y, B) be π_1 -injective CW-pairs such that X, Y are path-connected and Y, B are aspherical. Assume there is a bijection $\xi: \pi_0(A) \rightarrow \pi_0(B)$ such that for every connected component C of A there exists a commutative diagram of groups*

$$(3.4) \quad \begin{array}{ccc} \pi_1(A, x) & \longrightarrow & \pi_1(B, y) \\ \downarrow & & \downarrow \\ \pi_1(X, x) & \longrightarrow & \pi_1(Y, y) \end{array}$$

for some $x \in C$ and $y \in \xi(C)$. Assume moreover that there exists a continuous map

$$g: (X^1, A^1) \rightarrow (Y^1, B^1)$$

defined on the 1-skeleton which induces the diagrams (3.4) Then there exists a continuous map $f: (X, A) \rightarrow (Y, B)$ extending g and inducing the diagram (3.4). Moreover, f is unique up to homotopy (as maps of pairs).

PROOF. The same argument of [Hat02, Proposition 1B.9] applies: by using the asphericity of Y and B , it is possible to extend g to higher dimensional cells and to prove uniqueness. \square

In fact, a stronger result than Theorem 2 holds in absolute bounded cohomology: if a continuous map $f: X \rightarrow Y$ induces epimorphisms with amenable kernel on fundamental groups, then it induces an isometric isomorphisms in bounded cohomology [FM23, Theorem 5] [Fri17, Corollary 5.11]. In Section 5.6 we show that a similar result holds for the relative bounded cohomology of good pairs, although in this case only a bi-Lipschitz control over the norms is obtained (Theorem 5.22).

CHAPTER 4

The Group $\Pi(X, X_0)$

In this chapter, we introduce the group $\Pi(X, X_0)$, defined by Gromov in [Gro82], and its action on the aspherical multicomplex $\mathcal{A}(X)$. The elements of $\Pi(X, X_0)$ consists of finite collections of (homotopy classes of) paths in X that induce a permutation with finite support of a subset $X_0 \subseteq X$ of basepoints. There is a natural morphism of groupoids from the the fundamental groupoid $\Pi_1(X, X_0)$ of X based at X_0 to the groupoid $\Pi(X, X_0) \rtimes X_0$ associated to the action of $\Pi(X, X_0)$ on X_0 by permutations. This map is universal among groups acting on X_0 by permutations with finite support (Remark 4.3). The group $\Pi(X, X_0)$ admits a natural action on the 1-skeleton of $\mathcal{A}(X)$ by conjugating its edges with elements of $\Pi(X, X_0)$. Since $\mathcal{A}(X)$ is aspherical, the action extends canonically to the whole $\mathcal{A}(X)$. This action is a key tool in Section 4.3, where it is used to show that amenable subsets are invisible to bounded cohomology (Theorem 3). More generally, it plays a central role in our applications to open covers and gluings of manifolds, where geometric information on X is used to detect amenable subgroups of $\Pi(X, X_0)$.

In Section 4.4 we show that the group $\Pi(X, X_0)$ admits the structure of a permutational wreath product. As a consequence of a result of Monod [Mon22], we deduce that $\Pi(X, X_0)$ is boundedly acyclic (Theorem 4).

4.1. The Group $\Pi(X, X_0)$

Let X be a topological space. Given two paths $\gamma, \gamma': [0, 1] \rightarrow X$ such that $\gamma(1) = \gamma'(0)$, we denote by $\gamma * \gamma'$ their concatenation (where, as usual, the path γ comes before the path γ'). We denote by $\bar{\gamma}: [0, 1] \rightarrow X$ the path $\bar{\gamma}(t) = \gamma(1 - t)$.

DEFINITION 4.1. Let X be a topological space and let X_0 be a subset of X . Let $\Omega(X, X_0)$ be the set whose elements are families of paths $(\gamma_x)_{x \in X_0}$ satisfying the following conditions:

- (1) for every $x \in X_0$, $\gamma_x: [0, 1] \rightarrow X$ is a continuous path such that $\gamma_x(0) = x$ and $\gamma_x(1) \in X_0$;
- (2) the path γ_x is constant for all but finitely many $x \in X_0$;
- (3) the map

$$X_0 \rightarrow X_0, \quad x \mapsto \gamma_x(1),$$

is a bijection (hence a permutation of X_0 with finite support).

The following concatenation of paths endows $\Omega(X, X_0)$ with the structure of a semi-group: given two elements $(\gamma_x)_{x \in X_0}$ and $(\gamma'_x)_{x \in X_0}$ of $\Omega(X, X_0)$, their concatenation is $(\gamma_x * \gamma'_{\gamma_x(1)})_{x \in X_0}$. Two elements $(\gamma_x)_{x \in X_0}$ and $(\gamma'_x)_{x \in X_0}$ of $\Omega(X, X_0)$ are said to be *homotopic* if γ_x is homotopic to γ'_x in X relative to the endpoints for every $x \in X_0$. The set of homotopy classes of elements of $\Omega(X, X_0)$ is denoted by $\Pi(X, X_0)$, which is a group under the operation induced by the concatenation above.

Only a finite number of paths in a fixed element of $\Omega(X, X_0)$ is non-constant. Therefore, we usually denote elements of $\Pi(X, X_0)$ just by specifying the list of homotopically non-trivial paths $\{\gamma_1, \dots, \gamma_n\}$ in one of its representatives.

If $X_0 = \{x_0\}$ consists of a single point, then $\Pi(X, X_0) = \pi_1(X, x_0)$ is the fundamental group of X at the basepoint x_0 . In general, there is an injective group homomorphism

$$\bigoplus_{x \in X_0} \pi_1(X, x) \hookrightarrow \Pi(X, X_0),$$

whose image is the kernel of the obvious map from $\Pi(X, X_0)$ to the group $\Sigma_{\text{fin}}(X_0)$ of permutations of X_0 with finite support. In fact, every class $[(\gamma_x)_{x \in X_0}] \in \Pi(X, X_0)$ defines a permutation $x \mapsto \gamma_x(1)$ in $\Sigma_{\text{fin}}(X_0)$. In this way, we obtain a map $\Pi(X, X_0) \rightarrow \Sigma_{\text{fin}}(X_0)$, which is an *antihomomorphism*, i.e. it becomes an homomorphism after composing with the inversion in $\Sigma_{\text{fin}}(X_0)$. We denote by $q: \Pi(X, X_0) \rightarrow \Sigma_{\text{fin}}(X_0)$ the induced homomorphism.

PROPOSITION 4.2 ([**FM23**, Proposition 6.5]). *Let X be a topological space and let $X_0 \subseteq X$ be any subset. Then the following sequence of groups*

$$1 \longrightarrow \bigoplus_{x \in X_0} \pi_1(X, x) \longrightarrow \Pi(X, X_0) \xrightarrow{q} \Sigma_{\text{fin}}(X_0)$$

is exact. Moreover, if X_0 is contained in a path-connected component of X , then q is surjective.

Since every finitely generated subgroup of $\Sigma_{\text{fin}}(X_0)$ is finite, then $\Sigma_{\text{fin}}(X_0)$ is locally amenable, hence amenable [**Fri17**, Proposition 3.4]. On the other hand, direct sums of amenable groups is also amenable. It follows from Proposition 4.2 that, if every connected component of X has amenable fundamental group, then also $\Pi(X, X_0)$ is amenable. This result holds more generally for subspaces of X whose contribution at the level of the fundamental groups is amenable. Let (X, A) be a pair of topological spaces. The inclusion of pairs $(A, A) \hookrightarrow (X, X)$ induces a group homomorphism

$$\Pi(A, A) \hookrightarrow \Pi(X, X),$$

and we denote by $\Pi_X(A)$ its image. If A is π_1 -injective in X , then this map is injective and $\Pi_X(A)$ is isomorphic to $\Pi(A, A)$. It follows that, if A is π_1 -injective in X and A has amenable fundamental group, then $\Pi_X(A)$ is an amenable subgroup of $\Pi(X, X)$.

In Chapter 6, we consider the case in which subsets are not necessarily π_1 -injective. In that context, a subset is called *amenable* if its contribution to the fundamental group of X is amenable (see Definition 6.1). Such amenable subsets identify amenable subgroups of $\Pi(X, X)$ (Lemma 6.10).

REMARK 4.3. There is a natural groupoid homomorphism from the fundamental groupoid $\Pi_1(X, X_0)$ of X based at X_0 to the groupoid $\Pi(X, X_0) \rtimes X_0$ associated to the action of $\Pi(X, X_0)$ on X_0 by permutations. A groupoid is understood to be a small category where each morphism is invertible. The fundamental groupoid $\Pi_1(X, X_0)$ of X based at X_0 is the category whose objects are points of X_0 and whose morphisms are homotopy classes of paths in X between points of X_0 . The groupoid $\Pi(X, X_0) \rtimes X_0$ is the category whose objects are points of X_0 and whose morphisms are defined by

$$\text{Hom}(x, y) = \{g \in \Pi(X, X_0) \mid g \cdot x = y\},$$

where $x, y \in X_0$ and the action of $\Pi(X, X_0)$ on X_0 is by permutations. There is an obvious groupoid homomorphism

$$\Xi: \Pi_1(X, X_0) \rightarrow \Pi(X, X_0) \rtimes X_0,$$

which is the identity on objects and is defined on arrows in the following way: the (homotopy class of a) path γ is sent to $g = \{\gamma\} \in \Pi(X, X_0)$, if γ is a loop, and to the element $g = \{\gamma, \bar{\gamma}\}$ of $\Pi(X, X_0)$, otherwise. Moreover, it is easy to check that Ξ is universal in the following way: if G is a group acting on X_0 by permutations with finite support, then every groupoid homomorphism $\Pi_1(X, X_0) \rightarrow G \rtimes X_0$ factors through Ξ .

4.2. The Action of $\Pi(X, X)$ on $\mathcal{A}(X)$

Let X be a nice topological space. Our goal is to define a simplicial action of $\Pi(X, X)$ on the aspherical multicomplex $\mathcal{A}(X)$ of X . Recall that the 1-skeleton of $\mathcal{A}(X)$ coincides with the 1-skeleton of $\mathcal{L}(X)$. Therefore edges of $\mathcal{A}(X)$ correspond to homotopy classes of paths relative to the endpoints. In particular, the 0-skeleton of $\mathcal{A}(X)$ coincides with the set X itself. The action of $\Pi(X, X)$ on the 1-skeleton on $\mathcal{A}(X)$ is defined by conjugating edges of $\mathcal{A}(X)$ with elements of $\Pi(X, X)$. More precisely, let $g \in \Pi(X, X_0)$, and let $(\gamma_x)_{x \in X_0}$ be a representative of g . On the 0-skeleton, we simply define the action of g as the permutation (with finite support) induced by g on $X = \mathcal{A}(X)^0$. Let now e be a 1-simplex of $\mathcal{A}(X)$ with vertices v_0 and v_1 . We fix a representative $\gamma_e: [0, 1] \rightarrow X$ of e from v_0 to v_1 . We define the edge $g \cdot e$ of $\mathcal{A}(X)$ as the homotopy class (relative to the endpoints) of the following concatenation of paths

$$\bar{\gamma}_{v_0} * \gamma_e * \gamma_{v_1}.$$

Observe that the endpoints of $g \cdot e$ correspond to the image of the endpoints of e under the permutation induced by g , hence the action on edges extends the action on vertices. Of course, this definition does not depend on the chosen representatives, neither on the numbering of the vertices of e . Since simplices of $\mathcal{A}(X)$ are uniquely determined by their 1-skeleton, this action extends to the whole $\mathcal{A}(X)$ in the following way. Every n -simplex σ of $\mathcal{A}(X)$ defines a simplicial embedding $f: (\Delta^n)^1 \rightarrow \mathcal{A}(X)$ of the 1-skeleton of Δ^n , which is null-homotopic when restricted to each triangular loop. It is easy to check that the composition $g \circ f: (\Delta^n)^1 \rightarrow \mathcal{A}(X)$ is also null-homotopic when restricted to each triangular loop (see [FM23, Section 5.2] for the details). It follows from Proposition 2.24 that there exists a unique simplex $g \cdot \sigma$ whose characteristic map $\Delta^n \rightarrow \mathcal{A}(X)$ extends $g \circ f$. Thus the action of g on $\mathcal{A}(X)^1$ extends uniquely to a non-degenerate simplicial map $\psi(g): \mathcal{A}(X) \rightarrow \mathcal{A}(X)$.

PROPOSITION 4.4 ([FM23, Theorem 5.3]). *Let X be a nice topological space, and let*

$$\psi: \Pi(X, X) \rightarrow \text{Aut}(\mathcal{A}(X))$$

be the action described above. Then $\psi(g)$ is simplicially homotopic to the identity for every $g \in \Pi(X, X)$.

PROOF. Let $g \in \Pi(X, X)$ and let $(\gamma_x)_{x \in X}$ be one of its representatives. We may assume that X is path-connected and contains at least two distinct points, so that the multicomplex $\mathcal{A}(X)$ is large. In particular, there are two points x_0 and x_1 of X such that γ_{x_0} and γ_{x_1} are homotopic to the constant path. It follows

that g acts trivially on every edge having x_0 and x_1 as endpoints. Since every element of $\pi_1(|\mathcal{A}(X)|, x_0)$ can be represented by a pair of edges with x_0 and x_1 as endpoints (Proposition 2.14), it follows that g acts trivially on $\pi_1(|\mathcal{A}(X)|, x_0)$. Since the topological realization of $\mathcal{A}(X)$ is a model for the classifying space of the fundamental group of X , it follows that $\psi(g)$ is topologically homotopic to the identity [Hat02, Proposition 1B.9]. In conclusion, Lemma 2.11 implies that $\psi(g)$ is also simplicially homotopic to the identity. \square

Let (X, A) be a pair of nice topological spaces. We know that, if the inclusion $A \hookrightarrow X$ induces an injective map on homotopy groups (Setup 2.18), then the pair $(\mathcal{A}(X), \mathcal{A}(A))$ of aspherical multicomplexes is well-defined. The action of $\Pi(X, X)$ on $\mathcal{A}(X)$ does not preserve the submulticomplex $\mathcal{A}(A)$. However, the subgroup $\Pi_X(A)$ of $\Pi(X, X)$, whose elements are represented by families of paths supported on A (up to homotopy in X), induces a well-defined simplicial action on the pair $(\mathcal{A}(X), \mathcal{A}(A))$.

PROPOSITION 4.5. *Let (X, A) be a pair of topological spaces as in Setup 2.18, and let*

$$\psi: \Pi_X(A) \rightarrow \text{Aut}(\mathcal{A}(X), \mathcal{A}(A))$$

be the simplicial action described above. Then $\psi(g)$ is simplicially homotopic (as a map of pairs) to the identity for every $g \in \Pi_X(A)$.

PROOF. Proposition 4.4 implies that the maps

$$|\mathcal{A}(X)| \rightarrow |\mathcal{A}(X)|, \quad |\mathcal{A}(A)| \rightarrow |\mathcal{A}(A)|,$$

induced by $\psi(g)$ are both homotopic to the identity. Since the inclusion $|\mathcal{A}(A)| \hookrightarrow |\mathcal{A}(X)|$ is a cofibration, this is indeed enough to conclude that $|\psi(g)|$ is homotopic to the identity as a map of pairs [May99, Chapter 6, Section 5]. Finally, Lemma 2.11 implies that $\psi(g)$ is simplicially homotopic to the identity as a map of pairs. \square

We consider now 0-trivial actions on $\mathcal{A}(X)$ induced by the group $\Pi(X, X)$. Let X be a topological space. By Proposition 4.2, we know that there is an injective homomorphism

$$\bigoplus_{x \in X} \pi_1(X, x) \hookrightarrow \Pi(X, X),$$

hence we can view $\bigoplus_{x \in X} \pi_1(X, x)$ as a subgroup of $\Pi(X, X)$. We fix now some notation. For every $x \in X$ we set $G_x = \pi_1(X, x)$. We fix a basepoint $\bar{x} \in X$ and we set $G = G_{\bar{x}}$. Let N be a normal subgroup of G . We know that there exists an isomorphism $G_x \cong G$ for every $x \in X$, which is canonical up to conjugation. Since N is normal, there exists a well-defined isomorphic image N_x of N inside G_x , which is a normal subgroup of G_x . We set

$$\widehat{N} = \bigoplus_{x \in X} N_x.$$

The group \widehat{N} naturally sits into $\Pi(X, X)$ as a subgroup. Moreover, since it is contained in $\bigoplus_{x \in X} \pi_1(X, x)$, its action on $\mathcal{A}(X)$ is 0-trivial (Definition 1.6). It follows that the quotient $\mathcal{A}(X)/\widehat{N}$ is a well-defined multicomplex whose set of vertices is canonically identified with the set of vertices of $\mathcal{A}(X)$. We denote by $q: \mathcal{A}(X) \rightarrow \mathcal{A}(X)/\widehat{N}$ the quotient map. We will use the following property of the quotient map. For the sake of completeness, we include the argument from [FM23].

LEMMA 4.6 ([**FM23**, Lemma 5.5]). *Let e, e' be edges of $\mathcal{A}(X)$ sharing the same endpoints and let $e * e'$ be the loop in X obtained as one of the possible concatenations of e and e' . Then $q(e) = q(e')$ if and only if $e * e'$ lies in N .*

PROOF. Let x_0 and x_1 be the endpoints of e and e' . Let γ_e (resp. $\gamma_{e'}$) be a path in X representing the edge e (resp. e') such that $\gamma_e(0) = x_0$ and $\gamma_e(1) = x_1$ (resp. $\gamma_{e'}(0) = x_0$ and $\gamma_{e'}(1) = x_1$). We want to show that $q(e) = q(e')$ if and only if the composition $\gamma_e * \bar{\gamma}_{e'}$ is N_{x_0} . By definition, $q(e) = q(e')$ if and only if there exists an element $(\varepsilon_x)_{x \in X}$ of \widehat{N} such that $\bar{\varepsilon}_{x_0} * \gamma_e * \varepsilon_{x_1}$ is homotopic relative to the endpoints to $\gamma_{e'}$, or, in other words, if and only if the path $\bar{\varepsilon}_{x_0} * \gamma_e * \varepsilon_{x_1} * \bar{\gamma}_{e'}$ is null-homotopic relative to x_0 . This is in fact equivalent to require the composition $\gamma_e * \bar{\gamma}_{e'}$ to be in N_{x_0} . \square

It is shown in [**FM23**] that the (topological realization of the) multicomplex $\mathcal{A}(X)/\widehat{N}$ is a model of the classifying space for the group G/N .

PROPOSITION 4.7 ([**FM23**, Theorem 5.9]). *The multicomplex $\mathcal{A}(X)/\widehat{N}$ is complete, minimal and aspherical. Let x_0 be a vertex of $\mathcal{A}(X)$. Then the projection $q: \mathcal{A}(X) \rightarrow \mathcal{A}(X)/\widehat{N}$ induces an epimorphism*

$$q_*: \pi_1(|\mathcal{A}(X)|, x_0) \rightarrow \pi_1(|\mathcal{A}(X)/\widehat{N}|, x_0)$$

with kernel $\ker(q_*) = N_{x_0}$. In particular, the CW-complex $|\mathcal{A}(X)/\widehat{N}|$ is a model of the classifying space of G/N .

Proposition 4.7 is the key step of the proof presented in [**FM23**] of Gromov's Mapping Theorem, according to which continuous maps inducing surjective morphisms with amenable kernel at the level of the fundamental group induce isometric isomorphisms in bounded cohomology [**FM23**, Theorem 5]. We refer the reader to Section 5.6 for a discussion on the extension of the Mapping Theorem in the relative setting (Theorem 5.22).

4.3. A Crucial Observation

As already suggested by Gromov [**Gro82**, pag. 57], the following simple observation is crucial.

LEMMA 4.8 ([**Kue15**, Observation 1]). *Let (X, A) be a pair of topological spaces as in Setup 2.18, and let $G = \Pi_X(A)$. Let $n \in \mathbb{N}$ and let $z \in C_b^n(\mathcal{A}(X))_{\text{alt}}^G$ be a G -invariant alternating cochain. Let $(\Delta, (x_0, \dots, x_n))$ be an algebraic n -simplex of $\mathcal{A}(X)$ and assume that Δ has at least one edge in $\mathcal{A}(A)$. Then $z(\Delta, (x_0, \dots, x_n)) = 0$.*

PROOF. If $x_h = x_k$ for some $h \neq k$, then $z(\Delta, (x_0, \dots, x_n)) = 0$, since z is alternating. Therefore, we can assume that $x_h \neq x_k$ for every $h \neq k$. Let e be an edge of Δ which is contained in $\mathcal{A}(A)$. Without loss of generality, let x_0 and x_1 be its endpoints. We know that e represents a homotopy class (relative to endpoints) of paths in A joining x_0 and x_1 . Let $\gamma: [0, 1] \rightarrow A$ be a representative of e and let $g = \{\gamma, \bar{\gamma}\} \in \Pi_X(A)$. We have that $g \cdot \Delta = \Delta$, $g \cdot x_0 = x_1$, $g \cdot x_1 = x_0$ and $g \cdot x_i = x_i$

for every $l \in \{2, \dots, n\}$. Since z is alternating and G -invariant, it follows that

$$\begin{aligned} z(\Delta, (x_0, x_1, \dots, x_n)) &= z(g \cdot (\Delta, (x_0, x_1, \dots, x_n))) \\ &= z(\Delta, (x_1, x_0, \dots, x_n)) \\ &= -z(\Delta, (x_0, x_1, \dots, x_n)), \end{aligned}$$

hence $z(\Delta, (x_0, \dots, x_n)) = 0$. \square

The first important consequence of Lemma 4.8 is the following result, which motivates a version of Gromov Equivalence Theorem (Theorem 7.21) that plays a fundamental role in our proof of the additivity of simplicial volume for aspherical manifolds (Theorem 10).

THEOREM 3. *Let (X, Y) be a good pair and let $A \subseteq Y$ be the union of some connected components of Y . Let $B = Y \setminus A$ be the union of the remaining connected components and let $j: (X, B) \rightarrow (X, Y)$ denote the inclusion. If every connected component of A has an amenable fundamental group, then the map*

$$H_b^n(j): H_b^n(X, Y) \rightarrow H_b^n(X, B)$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

PROOF. We consider the following commutative diagram

$$\begin{array}{ccc} H_b^n(X, Y) & \xrightarrow{H_b^n(j)} & H_b^n(X, B) \\ \uparrow \Phi^n & & \uparrow \Phi^n \\ H_b^n(\mathcal{A}(X), \mathcal{A}(Y)) & \longrightarrow & H_b^n(\mathcal{A}(X), \mathcal{A}(B)) \\ \uparrow & & \uparrow \\ H^n(C_b^\bullet(\mathcal{A}(X), \mathcal{A}(Y))_{\text{alt}}^G) & \longrightarrow & H^n(C_b^\bullet(\mathcal{A}(X), \mathcal{A}(B))_{\text{alt}}^G), \end{array}$$

where:

- The horizontal arrows are induced by inclusions.
- Since the pair (X, Y) is good, also the pairs (X, A) and (X, B) are good. The upper vertical arrows denote the isometric isomorphisms of Theorem 1.
- We know that $G = \Pi_X(A)$ is an amenable group. Since A is a collection of path connected components of Y , the action $G \curvearrowright \mathcal{A}(X)$ induces well defined actions $G \curvearrowright (\mathcal{A}(X), \mathcal{A}(Y))$ and $G \curvearrowright (\mathcal{A}(X), \mathcal{A}(B))$. These actions consist of elements that are simplicially homotopic to the identity as maps of pairs (Proposition 4.5). It follows from Proposition 3.15 that the lower vertical arrows are isometric isomorphisms.

In order to conclude, it is sufficient to show that the map

$$\alpha^n: C_b^n(\mathcal{A}(X), \mathcal{A}(Y))_{\text{alt}}^G \rightarrow C_b^n(\mathcal{A}(X), \mathcal{A}(B))_{\text{alt}}^G$$

induces an isometric isomorphism in cohomology. But since every bounded alternating G -invariant cochain vanishes on simplices supported in $\mathcal{A}(A) \subseteq \mathcal{A}(Y)$ (Lemma 4.8), it follows that α^n is bijective. Thus α^n itself is an isometric isomorphism and the statement follows. \square

4.4. The Group $\Pi(X, X_0)$ is Boundedly Acyclic

The goal of this section is to show that the group $\Pi(X, X_0)$ is boundedly acyclic, i.e. it has vanishing bounded cohomology with trivial real coefficients. This section is independent from the rest of the work.

Let X be a path-connected topological space and let X_0 be any subset of X . By Proposition 4.2, there is a short exact sequence

$$1 \longrightarrow \bigoplus_{x \in X_0} \pi_1(X, x) \longrightarrow \Pi(X, X_0) \xrightarrow{q} \Sigma_{\text{fin}}(X_0) \longrightarrow 1,$$

where $\Sigma_{\text{fin}}(X_0)$ is the group of permutations of X_0 with finite support. Since $\Sigma_{\text{fin}}(X_0)$ is locally finite, then it is locally amenable, hence amenable [Fri17, Proposition 3.4]. Since the class of amenable group is closed under extensions, if X has amenable fundamental group, then also the group $\Pi(X, X_0)$ is amenable. Amenable groups are known to be boundedly acyclic. We show that the group $\Pi(X, X_0)$ is boundedly acyclic in much more generality, even without the assumption that X has amenable fundamental group.

THEOREM 4. *Let X be a path-connected topological space with countable fundamental group and let X_0 be an infinite countable subset of X . Then the group $\Pi(X, X_0)$ is boundedly acyclic.*

In order to prove Theorem 4, we show that the projection $q: \Pi(X, X_0) \rightarrow \Sigma_{\text{fin}}(X_0)$ admits a section, endowing the group $\Pi(X, X_0)$ with a semidirect-product structure. We already underline that this semidirect-product structure is not canonical, but depends on the choice of some *system of paths* from any point of X_0 to a fixed basepoint $\bar{x} \in X$. From this, we deduce that $\Pi(X, X_0)$ is isomorphic to a permutational wreath product $G \wr_{X_0} A$, where $A = \Sigma_{\text{fin}}(X_0)$ and $G = \pi_1(X, \bar{x})$. The conclusion that $\Pi(X, X_0)$ is boundedly acyclic follows then from a general result by Monod about permutational wreath products [Mon22, Section 4.2] (see also [CFFLM24, Section 3]).

The proof of Theorem 4 proceeds as follows. Let X be a path-connected topological space and let X_0 be an infinite countable subset of X . We choose a basepoint $\bar{x} \in X$ and a *system of paths* $\Theta = \{\vartheta_x \mid x \in X_0\}$, where every $\vartheta_x: [0, 1] \rightarrow X$ is a continuous path from x to \bar{x} . The notion of system of paths has also been studied in [BM22, BM25] in the context of transformation groups.

In the following, in order to ease the notation, we usually denote homotopy classes (relative to the endpoints) by writing one of their representatives. Moreover, we denote by $*$ both the usual concatenation of paths and the induced operation on $\Pi(X, X_0)$.

LEMMA 4.9. *The map*

$$\psi_\Theta: \Sigma_{\text{fin}}(X_0) \rightarrow \Pi(X, X_0), \quad \sigma \mapsto (\vartheta_x * \bar{\vartheta}_{\sigma(x)})_{x \in X_0}$$

is an antihomomorphism such that $q \circ \psi_\Theta = \text{id}$.

PROOF. It is easy to check that ψ_Θ is well-defined and is a set-theoretic section of q . Moreover, since

$$\begin{aligned}\psi_\Theta(\sigma \circ \tau) &= (\vartheta_x * \bar{\vartheta}_{\sigma \circ \tau(x)})_{x \in X_0} \\ &= (\vartheta_x * \bar{\vartheta}_{\tau(x)} * \vartheta_{\tau(x)} * \bar{\vartheta}_{\sigma \circ \tau(x)})_{x \in X_0} \\ &= (\vartheta_x * \bar{\vartheta}_{\tau(x)})_{x \in X_0} * (\vartheta_x * \bar{\vartheta}_{\sigma(x)})_{x \in X_0} \\ &= \psi_\Theta(\tau) * \psi_\Theta(\sigma),\end{aligned}$$

it follows that ψ_Θ is an antihomomorphism. \square

It follows from Lemma 4.9 that the group $\Pi(X, X_0)$ can be described as a semidirect product. More specifically, the map

$$\varphi_\Theta: \Sigma_{\text{fin}}(X_0) \rightarrow \text{Aut}\left(\bigoplus_{x \in X_0} \pi_1(X, x)\right),$$

defined by $\varphi_\Theta(\sigma)((\varepsilon_x)_{x \in X_0}) = (\vartheta_x * \bar{\vartheta}_{\sigma^{-1}(x)} * \varepsilon_{\sigma^{-1}(x)} * \vartheta_{\sigma^{-1}(x)} * \bar{\vartheta}_x)_{x \in X_0}$, is a group homomorphism such that the group

$$\bigoplus_{x \in X_0} \pi_1(X, x) \rtimes_{\varphi_\Theta} \Sigma_{\text{fin}}(X_0)$$

and $\Pi(X, X_0)$ are isomorphic via the maps

$$\xi_\Theta: \bigoplus_{x \in X_0} \pi_1(X, x) \rtimes_{\varphi_\Theta} \Sigma_{\text{fin}}(X_0) \rightarrow \Pi(X, X_0),$$

defined by $\xi_\Theta((\varepsilon_x)_{x \in X_0}, \sigma) = (\vartheta_x * \bar{\vartheta}_{\sigma(x)} * \varepsilon_{\sigma(x)})_{x \in X_0}$, and

$$\chi_\Theta: \Pi(X, X_0) \rightarrow \bigoplus_{x \in X_0} \pi_1(X, x) \rtimes_{\varphi_\Theta} \Sigma_{\text{fin}}(X_0),$$

such that $\chi_\Theta((\gamma_x)_{x \in X_0}) = ((\vartheta_x * \bar{\vartheta}_{\sigma^{-1}(x)} * \gamma_{\sigma^{-1}(x)})_{x \in X_0}, \sigma)$, where σ is the permutation associated to $(\gamma_x)_{x \in X_0}$. It follows that any choice of basepoint $\bar{x} \in X$ and system of paths Θ induces a semidirect product structure on $\Pi(X, X_0)$. This structure is *not canonical* a priori. From this description of $\Pi(X, X_0)$ we want to recover the structure of a permutational wreath product.

DEFINITION 4.10. Let Γ and A be groups and let Y be an A -set (i.e. a set endowed with an action by A). The *permutational wreath product* $\Gamma \wr_Y A$ is defined as the semidirect product

$$\left(\bigoplus_{y \in Y} \Gamma\right) \rtimes A,$$

where A acts on the subgroup $\bigoplus_{y \in Y} \Gamma$ by permuting the factors. The case in which $Y = A$ with the left-multiplication action recovers the usual *regular wreath product*, denoted by $\Gamma \wr A$.

We denote by $G = \pi_1(X, \bar{x})$ the fundamental group of X based at \bar{x} . Similarly, we set $G_x = \pi_1(X, x)$ for every $x \in X_0$. The system of paths Θ can be used to get isomorphisms $\zeta_x: G_x \rightarrow G$ sending each path ε based at x to the composition $\bar{\vartheta}_x * \varepsilon * \vartheta_x$. Summing up these isomorphisms we obtain an isomorphism $\zeta_\Theta: \bigoplus_{x \in X_0} G_x \rightarrow$

$\bigoplus_{x \in X_0} G$. It is easy to check that the following diagram

$$\begin{array}{ccc} \bigoplus_{x \in X_0} G_x & \xrightarrow{\zeta_\Theta} & \bigoplus_{x \in X_0} G \\ \downarrow \varphi_\Theta(\sigma) & & \downarrow \pi(\sigma) \\ \bigoplus_{x \in X_0} G_x & \xrightarrow{\zeta_\Theta} & \bigoplus_{x \in X_0} G \end{array}$$

is commutative for every $\sigma \in \Sigma_{\text{fin}}(X_0)$, where $\pi(\sigma)$ denotes the permutation of the factors: $\pi(\sigma)(\varepsilon_x)_{x \in X_0} = (\varepsilon_{\sigma^{-1}(x)})_{x \in X_0}$. As a consequence, the map

$$\alpha_\Theta: \bigoplus_{x \in X_0} G_x \rtimes_{\varphi_\Theta} \Sigma_{\text{fin}}(X_0) \rightarrow \bigoplus_{x \in X_0} G \rtimes_{\pi} \Sigma_{\text{fin}}(X_0)$$

defined by $\alpha(\varepsilon, \sigma) = (\zeta_\Theta(\varepsilon), \sigma)$ is an isomorphism of groups. We conclude that the group $\Pi(X, X_0)$ is isomorphic to the permutational wreath product $G \wr_{X_0} \Sigma_{\text{fin}}(X_0)$.

PROPOSITION 4.11 ([**Mon22**, Section 4.2], [**CFFLM24**, Corollary 3.4]). *Let Γ and A be countable groups and let Y be a countable A -set such that all A -orbits are infinite. Assume that A is amenable. Then the permutation wreath product $\Gamma \wr_Y A$ is boundedly acyclic.*

In order to conclude that $\Pi(X, X_0)$ is boundedly acyclic, we just need to observe that $\Sigma_{\text{fin}}(X_0)$ is a countable amenable group acting transitively on X_0 . Therefore, if the set X_0 is countable infinite, all the assumptions of Proposition 4.11 are satisfied by $G \wr_{X_0} \Sigma_{\text{fin}}(X_0)$, which is in turn isomorphic to $\Pi(X, X_0)$. This concludes the proof of Theorem 4.

Mapping Cones

We know that every nice topological space X admits a complete and minimal multicomplex $\mathcal{L}(X)$, whose simplices correspond to homotopy classes of singular simplices of X relative to the vertices. Given a pair (X, A) of nice topological spaces, there is therefore a natural non-degenerate simplicial map $j_{\mathcal{L}}: \mathcal{L}(A) \rightarrow \mathcal{L}(X)$ which sends (homotopy classes of) singular simplices in A to (homotopy classes of) the corresponding singular simplex of X . This map factors through aspherical quotients, thus inducing a non-degenerate simplicial map $j_{\mathcal{A}}: \mathcal{A}(A) \rightarrow \mathcal{A}(X)$, which is *not* injective in general (Proposition 5.5). Let $\mathcal{A}_X(A)$ denote the image of $j_{\mathcal{A}}$ inside $\mathcal{A}(X)$. Using the action of $\Pi(A, A)$ on $\mathcal{A}(A)$, we can show that $\mathcal{A}_X(A)$ is also complete, minimal and aspherical (Lemma 5.6). This provides a well-defined pair $(\mathcal{A}(X), \mathcal{A}_X(A))$ of complete minimal and aspherical multicomplexes. However, since neither $j_{\mathcal{L}}$ nor $j_{\mathcal{A}}$ is injective in general, this situation falls outside the standard framework of pairs of multicomplexes. To address this, we employ the machinery of *mapping cones*, which has already proven effective in the theory of bounded cohomology (see for instance [Par03, Löh08, BBF⁺14]).

In Section 5.3, we show that there exists an isomorphism Ψ between the singular bounded cohomology of (X, A) and the simplicial bounded cohomology of $(\mathcal{A}(X), \mathcal{A}_X(A))$, provided that the inclusion $A \hookrightarrow X$ induces a homomorphism with amenable kernel at the level of fundamental groups. As already mentioned in the introduction, mapping cones do not allow for isometric control of the norm in bounded cohomology. Consequently, we are only able to conclude that Ψ is bi-Lipschitz (Theorem 5).

In Section 5.5, we introduce a regularity condition for simplicial actions. This condition plays a crucial role in passing to invariant chains in the relative setting when the group action does not preserve a submulticomplex (Definition 5.19). This framework is then applied in Section 5.6 to establish a bi-Lipschitz version of the Relative Mapping Theorem for good pairs (Theorem 5.22).

5.1. Beyond Goodness

We have seen in Chapter 2 how to construct a pair of aspherical multicomplexes $(\mathcal{A}(X), \mathcal{A}(A))$ from every pair (X, A) of nice topological spaces such that the inclusion $A \hookrightarrow X$ induces injective maps on homotopy groups (Setup 2.18). In this section we want to construct a well-defined pair of aspherical multicomplexes $(\mathcal{A}(X), \mathcal{A}_X(A))$ in a much greater generality.

Let (X, A) be a pair of nice topological spaces. In general, we do not assume X or A to be path-connected. Being injective, the inclusion map $j: A \hookrightarrow X$ naturally induces a simplicial embedding $j_{\mathcal{K}}: \mathcal{K}(A) \hookrightarrow \mathcal{K}(X)$ of singular multicomplexes. If (X, A) is a CW-pair, we know that the natural projection

$$S: (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (X, A)$$

is a homotopy equivalence of pairs (Proposition 1.13). In general, the vertical arrows of following commutative diagram

$$\begin{array}{ccc} |\mathcal{K}(A)| & \xrightarrow{|\mathcal{j}_{\mathcal{K}}|} & |\mathcal{K}(X)| \\ \downarrow S_A & & \downarrow S_X \\ A & \xrightarrow{j} & X \end{array}$$

are weak homotopy equivalences (Proposition 1.11).

We know that the minimal multicomplexes $\mathcal{L}(A)$ and $\mathcal{L}(X)$ are well-defined (Remark 2.22). We underline the fact that the minimization procedure is performed separately on $\mathcal{K}(X)$ and $\mathcal{K}(A)$, hence we do not need to require the pair $(\mathcal{K}(X), \mathcal{K}(A))$ to be full (see Proposition 2.19). We denote by

$$i_A: \mathcal{L}(A) \hookrightarrow \mathcal{K}(A), \quad i_X: \mathcal{L}(X) \hookrightarrow \mathcal{K}(X),$$

the simplicial inclusions. In Proposition 5.1 we describe how to construct a simplicial map $j_{\mathcal{L}}$ from $\mathcal{L}(A)$ to $\mathcal{L}(X)$ which sends each simplex Δ of $\mathcal{L}(A)$ to the unique simplex of $\mathcal{L}(X)$ which is homotopic to Δ relative to the 0-skeleton. Recall that every simplex of $\mathcal{K}(X)$ (hence of $\mathcal{L}(X)$) is the equivalence class of singular simplices of X which are injective on vertices, up to affine parametrization. A simplex Δ of $\mathcal{K}(X)$ (hence of $\mathcal{L}(X)$) is therefore represented by a singular simplex $|\Delta| \rightarrow X$. The same holds for simplices of $\mathcal{L}(A)$.

PROPOSITION 5.1 ([Kue15, Section 1.3]). *There is a non-degenerate simplicial map $j_{\mathcal{L}}: \mathcal{L}(A) \rightarrow \mathcal{L}(X)$ such that, for every simplex Δ of $\mathcal{L}(A)$, there is a homotopy $H_{\Delta}: |\Delta| \times [0, 1] \rightarrow X$ relative to the vertices between a representative $\sigma: |\Delta| \rightarrow A$ of Δ and a representative of $j_{\mathcal{L}}(\Delta)$. Moreover, for every submulticomplex Δ' of Δ , the restriction of H_{Δ} to $|\Delta'| \times [0, 1]$ coincides with $H_{\Delta'}$.*

PROOF. We define $j_{\mathcal{L}}$ inductively on the skeleta. On the 0-skeleton, $j_{\mathcal{L}}$ is simply defined as the map induced by the inclusion $j: A \hookrightarrow X$. In this case, for every vertex v of $\mathcal{L}(A)$, we set H_v to be the constant map on v . Assume we have defined $j_{\mathcal{L}}$ on the n -skeleton of $\mathcal{L}(A)$ and that, for every k -simplex Δ' of $\mathcal{L}(A)$, $k \in \{0, \dots, n\}$, we have a homotopy $H_{\Delta'}$ as in the statement. We define $j_{\mathcal{L}}$ on the $(n+1)$ -skeleton as follows. Let Δ be an $(n+1)$ -simplex of $\mathcal{L}(A)$ and let $\sigma: |\Delta| \rightarrow X$ denote a parametrization of Δ . We want to extend σ to a continuous map $\bar{\sigma}: |\Delta| \rightarrow X$ whose facets are all contained in $\mathcal{L}(X)^n$. To this end, we exploit the homotopies $H_{\Delta'}$, where $\Delta' \subset \Delta$ is a proper face of Δ , as follows. We consider the diagram

$$\begin{array}{ccc} |\Delta| \sqcup \left(\bigsqcup_{\Delta' \subset \Delta} |\Delta'| \times [0, 1] \right) & \xrightarrow{f} & X \\ \downarrow & & \parallel \\ |\Delta| & \xrightarrow{\bar{\sigma}} & X, \end{array}$$

where f is induced by σ and the homotopies $H_{\Delta'}$, respectively, while the vertical map is the quotient map induced by the obvious identification of faces. By the compatibility condition of the homotopies with respect to the inclusion of faces, it follows that f factors through the quotient map. Therefore, it defines a singular simplex $\bar{\sigma}$ of X , whose facets, by construction, represent simplices in $\mathcal{L}(X)$. By

completeness and minimality of $\mathcal{L}(X)$, there exists a unique $(n+1)$ -simplex $j_{\mathcal{L}}(\Delta)$ of $\mathcal{L}(X)$ which is homotopic to $\bar{\sigma}$ in X relative to the boundary. Then, by construction, we have a homotopy

$$H_{\Delta}: |\Delta| \times [0, 1] \rightarrow X$$

between Δ and $j_{\mathcal{L}}(\Delta)$ which satisfies the compatibility conditions in the statement. \square

It is desirable that the map $j_{\mathcal{L}}$ constructed in Lemma 5.1 commutes with the inclusion $j_{\mathcal{K}}$. However, this is not true *a priori*. In fact, even if the multicomplexes $\mathcal{L}(A)$ and $\mathcal{L}(X)$ are unique up to simplicial isomorphism [FM23, Theorem 3.23], we need to make some choices in order to make the following diagram commutative.

$$\begin{array}{ccc} \mathcal{K}(A) & \xleftarrow{j_{\mathcal{K}}} & \mathcal{K}(X) \\ i_A \uparrow & & i_X \uparrow \\ \mathcal{L}(A) & \xrightarrow{j_{\mathcal{L}}} & \mathcal{L}(X) \end{array}$$

SETUP 5.2. Let (X, A) be a pair of nice topological spaces. Assume that $\mathcal{L}(A)$ has been constructed. We construct $\mathcal{L}(X)$ so that, if the facets of a simplex are all contained in A and if there is a simplex of $\mathcal{L}(A)$ in the same homotopy class, then we choose this simplex as a representative.

In this work, whenever we construct a pair of multicomplexes from a pair of nice topological spaces (X, A) , we always adhere to Setup 5.2.

LEMMA 5.3. *In the situation of Setup 5.2, we have that $j_{\mathcal{K}} \circ i_A = i_X \circ j_{\mathcal{L}}$.*

The map $j_{\mathcal{L}}$ is not injective in general: to this end one needs to have more control on the homotopy of the pair (X, A) . This happens when the inclusion $A \hookrightarrow X$ induces injective maps on homotopy groups (Setup 2.18), hence when the pair $(\mathcal{K}(X), \mathcal{K}(A))$ is full (Proposition 2.19) and the minimization procedure can be performed simultaneously on both $\mathcal{K}(X)$ and $\mathcal{K}(A)$ (Proposition 2.16).

PROPOSITION 5.4. *Let $n \in \mathbb{N}_{\geq 1}$. The simplicial map $j_{\mathcal{L}}$ is injective on the n -skeleton if and only if the map $\pi_k(A \hookrightarrow X, x)$ is injective for every $x \in A$ and every $k \in \{1, \dots, n\}$.*

PROOF. For every $x \in A$ and every k -simplex Δ of $\mathcal{K}(A)$ having x as a vertex, we can consider the following commutative diagram

$$(5.1) \quad \begin{array}{ccccc} \pi_{\mathcal{L}(A)}(\Delta) & \xrightarrow{\Theta_A} & \pi_k(|\mathcal{L}(A)|, x) & \xrightarrow{\cong} & \pi_k(A, x) \\ \downarrow j_{\Delta} & & \downarrow \pi_k(|j_{\mathcal{L}}|, x) & & \downarrow \pi_k(j, x) \\ \pi_{\mathcal{L}(X)}(\Delta) & \xrightarrow{\Theta_X} & \pi_k(|\mathcal{L}(X)|, x) & \xrightarrow{\cong} & \pi_k(X, x), \end{array}$$

where vertical arrows are induced either by $j_{\mathcal{L}}$ or by the inclusion j , horizontal arrows of the left squares are bijective (Proposition 2.14) and horizontal arrows of the right square, induced by S_X and S_A , are isomorphisms (Proposition 1.11). By using the homotopies $H_{\Delta'}$ in Proposition 5.1, one can show that the diagram commutes. Therefore, we have that $\pi_k(j, x)$ is injective if and only if j_{Δ} is injective.

Assume now that $j_{\mathcal{L}}$ is injective on the n -skeleton. Let $k \in \mathbb{N}_{\leq n}$, $x \in A$ and let Δ_0 be a k -simplex of $\mathcal{K}(A)$ having x as a vertex. It follows from our assumptions that j_{Δ} is injective, hence also $\pi_k(j, x)$ is so.

On the other hand, assume that $\pi_k(A \hookrightarrow X, x)$ is injective for every $x \in A$ and every $k \in \{1, \dots, n\}$. We show that $j_{\mathcal{L}}$ is injective by induction of the skeleta. This is always true on the 0-skeleton. In general, let Δ and Δ' be $(k+1)$ -simplices of $\mathcal{L}(A)$ such that $j_{\mathcal{L}}(\Delta) = j_{\mathcal{L}}(\Delta')$. By the inductive hypothesis, we deduce that Δ and Δ' are compatible. By considering diagram (5.1), we have that j_{Δ} is injective. Since $j_{\Delta}(\Delta') = j_{\Delta}(\Delta)$, it follows that $\Delta = \Delta'$. \square

Let $\pi: \mathcal{L}(X) \rightarrow \mathcal{A}(X)$ denote the simplicial projection which identifies simplices sharing the same 1-skeleton. The map $j_{\mathcal{L}}$ factors to a well-defined simplicial map $j_{\mathcal{A}}: \mathcal{A}(A) \rightarrow \mathcal{A}(X)$. We denote by $\mathcal{A}_X(A)$ the image of $\mathcal{A}(A)$ in $\mathcal{A}(X)$ via $j_{\mathcal{A}}$, so that the pair of multicomplexes $(\mathcal{A}(X), \mathcal{A}_X(A))$ is well-defined, and we still denote by $j_{\mathcal{A}}: \mathcal{A}(A) \rightarrow \mathcal{A}_X(A)$ the surjective map induced by $j_{\mathcal{A}}$. In short, we have the following commutative diagram of simplicial maps

$$\begin{array}{ccccccc} \mathcal{K}(A) & \xleftarrow{i_A} & \mathcal{L}(A) & \xrightarrow{\pi} & \mathcal{A}(A) & \xrightarrow{j_{\mathcal{A}}} & \mathcal{A}_X(A) \\ & & \downarrow j_{\mathcal{K}} & & \downarrow j_{\mathcal{A}} & & \downarrow \\ \mathcal{K}(X) & \xleftarrow{i_X} & \mathcal{L}(X) & \xrightarrow{\pi} & \mathcal{A}(X) & \xrightarrow{\text{id}} & \mathcal{A}(X) \end{array}$$

Since simplices of $\mathcal{A}(X)$ are uniquely determined by their 1-skeleton, the following observation can be easily deduced from Proposition 5.4.

PROPOSITION 5.5 ([Kue15, Section 1.3]). *Let (X, A) be a pair of nice topological spaces such that A is π_1 -injective in X . Then*

$$j_{\mathcal{A}}: \mathcal{A}(A) \rightarrow \mathcal{A}(X)$$

is a simplicial embedding and the submulticomplex $\mathcal{A}_X(A)$ of $\mathcal{A}(X)$ is canonically isomorphic to $\mathcal{A}(A)$.

The multicomplex $\mathcal{A}_X(A)$ admits the following equivalent characterization. We consider the action on $\mathcal{A}(A)$ of $\bigoplus_{x \in A} \pi_1(A, x)$, understood as a subgroup of $\Pi(A, A)$ (see Section 4.2). For every $x \in A$, we set $\Gamma_x = \pi_1(A, x)$. Let $\{A_i \mid i \in I\}$ be the set of path-connected components of A . For every $i \in I$, we fix a basepoint $\bar{x}_i \in A_i$ and we set $\Gamma_i = \Gamma_{\bar{x}_i}$. Let N_i be the kernel of the morphism $\pi_1(A \hookrightarrow X, \bar{x}_i)$. Being N_i normal in Γ_i , we can find a well-defined isomorphic image N_x of N_i inside Γ_x , so that $N_{\bar{x}_i} = N_i$. We set

$$\widehat{N} = \bigoplus_{i \in I} \bigoplus_{x \in A_i} N_x.$$

We know that the quotient $\mathcal{A}(A)/\widehat{N}$ is a complete, minimal and aspherical multicomplex (Proposition 4.7). We denote by $q: \mathcal{A}(A) \rightarrow \mathcal{A}(A)/\widehat{N}$ the (simplicial) quotient map.

LEMMA 5.6. *The multicomplex $\mathcal{A}_X(A)$ is simplicially isomorphic to the multicomplex $\mathcal{A}(A)/\widehat{N}$. In particular, it is complete, minimal and aspherical and its topological realization $|\mathcal{A}_X(A)|$ is π_1 -injective in $|\mathcal{A}(X)|$.*

PROOF. We first show that $\mathcal{A}_X(A)$ is simplicially isomorphic to $\mathcal{A}(A)/\widehat{N}$. Let e, e' be edges of $\mathcal{A}(A)$ sharing the same endpoints. We denote by $e * e'$ the loop obtained as one of the possible concatenations of e and e' . By construction, $j_{\mathcal{A}}(e) = j_{\mathcal{A}}(e')$ if and only if $e * e'$ is in N . By Lemma 4.6, this is equivalent to have $q(e) = q(e')$. It follows that $j_{\mathcal{A}}$ factorizes through q on the 1-skeleton. Since $\mathcal{A}(X)$

and $\mathcal{A}(A)/\widehat{N}$ are complete, minimal and aspherical, then the factorization can be uniquely extended to a simplicial isomorphism

$$\widehat{j}_{\mathcal{A}}: \mathcal{A}(A)/\widehat{N} \rightarrow \mathcal{A}_X(A)$$

such that $\widehat{j}_{\mathcal{A}} \circ q = j_{\mathcal{A}}$. The second statement is a direct consequence of Proposition 4.7. \square

5.2. Mapping Cones

A *normed cochain complex* $(M^\bullet, \delta_M^\bullet)$ is a cochain complex consisting of normed real vector spaces where coboundary operators are bounded linear operators. To our purposes, the main examples are the complex of singular bounded cochains $(C_b^\bullet(X, A), \delta^\bullet)$ of a pair of topological spaces (X, A) and the complex of simplicial bounded cochains $(C_b^\bullet(K, L), \delta^\bullet)$ of a pair of multicomplexes (K, L) , both endowed with the ℓ^∞ -norm.

DEFINITION 5.7. Let $(M^\bullet, \delta_M^\bullet)$ and $(N^\bullet, \delta_N^\bullet)$ be normed cochain complexes and let $\varepsilon^\bullet: M^\bullet \rightarrow N^\bullet$ be a chain map between them. The *mapping cone complex* $(\text{Cone}(\varepsilon)^\bullet, \delta_C^\bullet)$ of ε^\bullet is the normed cochain complex defined in the following way. For every $n \in \mathbb{N}$, we set

$$\text{Cone}(\varepsilon)^n = M^n \oplus N^{n-1}$$

and we define a boundary operator $\delta_C^n: \text{Cone}(\varepsilon)^n \rightarrow \text{Cone}(\varepsilon)^{n+1}$ by

$$\delta_C^n(u, v) = (\delta_M^n(u), -\varepsilon^n(u) - \delta_N^{n-1}(v)).$$

We equip $\text{Cone}^n(\varepsilon)$ with the norm $\|(u, v)\| = \max\{\|u\|, \|v\|\}$, which induces a seminorm in cohomology.

Every commutative diagram of cochain maps

$$(5.2) \quad \begin{array}{ccc} \overline{M}^\bullet & \xrightarrow{\overline{\varepsilon}^\bullet} & \overline{N}^\bullet \\ \downarrow \eta_M^\bullet & & \downarrow \eta_N^\bullet \\ M^\bullet & \xrightarrow{\varepsilon^\bullet} & N^\bullet \end{array}$$

induces a cochain map

$$\eta^n: \text{Cone}(\overline{\varepsilon})^n \rightarrow \text{Cone}(\varepsilon)^n, \quad (u, v) \mapsto (\eta_M^n(u), \eta_N^{n-1}(v)).$$

Let ΣN^\bullet denote the *suspension* of N^\bullet , i.e. the complex obtained by setting $(\Sigma N)^n = N^{n-1}$. Consider the short exact sequence of complexes

$$0 \longrightarrow \Sigma N^\bullet \xrightarrow{\iota} \text{Cone}(\varepsilon)^\bullet \xrightarrow{\xi} M^\bullet \longrightarrow 0,$$

where $\iota(v) = (0, v)$ and $\xi(u, v) = u$. Since $H^n(\Sigma N^\bullet) = H^{n-1}(N^\bullet)$, the corresponding long exact sequence in cohomology is given by

$$\dots \rightarrow H^n(\text{Cone}(\varepsilon)^\bullet) \rightarrow H^n(M^\bullet) \rightarrow H^n(N^\bullet) \rightarrow H^{n+1}(\text{Cone}(\varepsilon)^\bullet) \rightarrow \dots$$

By the naturality of the long exact sequence, the following diagram

$$\begin{array}{ccccccccc} H^{n-1}(\overline{M}^\bullet) & \longrightarrow & H^{n-1}(\overline{N}^\bullet) & \longrightarrow & H^n(\text{Cone}(\overline{\varepsilon})^\bullet) & \longrightarrow & H^n(\overline{M}^\bullet) & \longrightarrow & H^n(\overline{N}^\bullet) \\ \downarrow H^{n-1}(\eta_M^\bullet) & & \downarrow H^{n-1}(\eta_N^\bullet) & & \downarrow H^n(\eta^\bullet) & & \downarrow H^n(\eta_M^\bullet) & & \downarrow H^n(\eta_N^\bullet) \\ H^{n-1}(M^\bullet) & \longrightarrow & H^{n-1}(N^\bullet) & \longrightarrow & H^n(\text{Cone}(\varepsilon)^\bullet) & \longrightarrow & H^n(M^\bullet) & \longrightarrow & H^n(N^\bullet) \end{array}$$

is commutative and has exact rows. A straightforward application of the Five Lemma implies the following.

LEMMA 5.8. *If η_M^\bullet and η_N^\bullet induce isomorphisms in cohomology, then also η^\bullet does.*

In order to retain further control over the norms we need to make the maps at the level of cochains as explicit as possible. For this reason we introduce the following property.

DEFINITION 5.9. Let (K_n) , $n \in \mathbb{N}_{\geq 1}$, be a sequence of positive real numbers and let $\eta_M^\bullet: \overline{M}^\bullet \rightarrow M^\bullet$ be a norm non-increasing chain map. We say that η_M^\bullet has a (K_n) -chain homotopy inverse if there exists a norm non-increasing chain map $A_M^\bullet: M^\bullet \rightarrow \overline{M}^\bullet$ such that the following conditions hold:

- (i) $A_M^\bullet \circ \eta_M^\bullet = \text{id}$;
- (ii) The composition $\eta_M^\bullet \circ A_M^\bullet$ is chain homotopic to the identity via a chain homotopy $T^\bullet: M^\bullet \rightarrow M^{\bullet-1}$ such that $\|T^n\| \leq K_n$ for every $n \in \mathbb{N}_{\geq 1}$. In particular, for every $u \in M^n$ such that $\delta_M^n(u) = 0$, there exists $u' \in M^{n-1}$ such that

$$\eta_M^n \circ A_M^n(u) - u = \delta_M^{n-1}(u'), \quad \|u'\| \leq K_n \cdot \|u\|.$$

Let G be an amenable group acting on a multicomplex K by automorphisms which are simplicially homotopic to the identity. It follows from Proposition 3.15 that the inclusion $C_b^\bullet(K)^G \hookrightarrow C_b^\bullet(K)$ of invariant cochains admits a (C_n) -chain homotopy inverse, where the sequence (C_n) is introduced in Definition 2.8. Another important example of a chain map admitting a (C_n) -chain homotopy inverse is given by the map $C_b^\bullet(\mathcal{K}(X)) \rightarrow C_b^\bullet(\mathcal{L}(X))$ induced by the inclusion $\mathcal{L}(X) \hookrightarrow \mathcal{K}(X)$ for every CW-complex X (see Lemma 5.14 below). These examples play a fundamental role in our proof of Theorem 5.

LEMMA 5.10. *Consider the following commutative diagram of cochain maps and assume that η_M^\bullet , η_N^\bullet , ε^\bullet and $\bar{\varepsilon}^\bullet$ are norm non-increasing.*

$$\begin{array}{ccc} \overline{M}^\bullet & \xrightarrow{\bar{\varepsilon}^\bullet} & \overline{N}^\bullet \\ \downarrow \eta_M^\bullet & & \downarrow \eta_N^\bullet \\ M^\bullet & \xrightarrow{\varepsilon^\bullet} & N^\bullet \end{array}$$

If there exists a sequence (K_n) , $n \in \mathbb{N}_{\geq 1}$, of positive real numbers such that the following conditions hold:

- (1) η_M^\bullet has a (K_n) -chain homotopy inverse A_M^\bullet ;
- (2) η_N^\bullet has a (K_n) -chain homotopy inverse A_N^\bullet ;

then η^\bullet induces a bi-Lipschitz isomorphism in cohomology such that

$$(2K_n)^{-1} \cdot \|\alpha\| \leq \|H^n(\eta^\bullet)(\alpha)\| \leq \|\alpha\|,$$

for every $n \in \mathbb{N}$ and every $\alpha \in H^n(\text{Cone}(\bar{\varepsilon})^\bullet)$.

PROOF. Conditions (1) and (2) clearly imply that η_M^\bullet and η_N^\bullet induce isometric isomorphisms in cohomology. Moreover, if η_M^\bullet and η_N^\bullet are norm non-increasing, then also η^\bullet is norm non-increasing, hence also the induced isomorphism in cohomology is so. In order to retain further control over the norms, we proceed as follows. We take a class in $H^n(\text{Cone}(\varepsilon)^\bullet)$ and we consider a cocycle $(u, v) \in \text{Cone}(\varepsilon)^n$ representing

it. Since (u, v) is a cocycle, by the definition of the differential of $\text{Cone}(\varepsilon)^\bullet$, we have that

$$\delta_M^n(u) = 0, \quad \varepsilon^n(u) + \delta_N^{n-1}(v) = 0.$$

Since η_M^\bullet has a (K_n) -chain homotopy inverse, there exists $u' \in M^{n-1}$ such that

$$\eta_M^n \circ A_M^n(u) - u - \delta_M^{n-1}(u') = 0, \quad \|u'\| \leq K_n \cdot \|u\|.$$

By applying ε^n and A_N^n on both sides of the equality, we get

$$\begin{aligned} 0 &= \varepsilon^n \circ \eta_M^n \circ A_M^n(u) - \varepsilon^n(u) - \varepsilon^n \circ \delta_M^{n-1}(u'), \\ 0 &= \eta_N^n \circ \bar{\varepsilon}^n \circ A_M^n(u) + \delta_N^{n-1}(v) - \delta_N^{n-1} \circ \varepsilon^{n-1}(u'), \\ 0 &= A_N^n \circ \eta_N^n \circ \bar{\varepsilon}^n \circ A_M^n(u) + A_N^n \circ \delta_N^{n-1}(v) - A_N^n \circ \delta_N^{n-1} \circ \varepsilon^{n-1}(u'), \\ 0 &= \bar{\varepsilon}^n \circ A_M^n(u) + \delta_N^{n-1} \circ A_N^{n-1}(v) - \delta_N^{n-1} \circ A_N^{n-1} \circ \varepsilon^{n-1}(u'), \\ 0 &= \bar{\varepsilon}^n \circ A_M^n(u) + \delta_N^{n-1} \circ A_N^{n-1}(v - \varepsilon^{n-1}(u')), \end{aligned}$$

where $\delta_{\bar{N}}^{n-1}$ denotes the differential of the cochain complex \bar{N}^\bullet . It readily follows that

$$(z, w) = (A_M^n(u), A_N^{n-1}(v - \varepsilon^{n-1}(u'))) \in \text{Cone}(\bar{\varepsilon})^n$$

is a cocycle. Moreover, we have that

$$\begin{aligned} \eta^n(z, w) - (u, v) &= (\eta_M^n \circ A_M^n(u), \eta_N^{n-1} \circ A_N^{n-1}(v - \varepsilon^{n-1}(u'))) - (u, v) \\ &= (\delta_M^{n-1}(u'), \eta_N^{n-1} \circ A_N^{n-1}(v - \varepsilon^{n-1}(u'))) - (v - \varepsilon^{n-1}(u')) - \varepsilon^{n-1}(u') \\ &= (0, x) + \delta_C^{n-1}(u', 0), \end{aligned}$$

where $x = \eta_N^{n-1} \circ A_N^{n-1}(v - \varepsilon^{n-1}(u')) - (v - \varepsilon^{n-1}(u'))$. Since both $\eta^n(z, w)$ and (u, v) are cocycles, it follows that also $(0, x)$ is a cocycle, hence $\delta_N^{n-1}(x) = 0$. Since η_N^\bullet has a (K_n) -chain homotopy inverse, there exists $x' \in N^{n-1}$ such that

$$\eta_N^{n-1} \circ A_N^{n-1}(x) - x = \delta_N^{n-2}(x').$$

It is easy to check that $\eta_N^{n-1} \circ A_N^{n-1}(x) = 0$, hence $x = -\delta_N^{n-2}(x')$. It follows that

$$\begin{aligned} \eta^n(z, w) - (u, v) &= (0, x) + \delta_C^{n-1}(u', 0) \\ &= \delta_C^{n-1}(0, x') + \delta_C^{n-1}(u', 0) = \delta_C^{n-1}(u', x'), \end{aligned}$$

which implies that $\eta^n(z, w)$ and (u, v) represent the same class in cohomology. The conclusion follows from the following chains of inequalities:

$$\begin{aligned} \|(z, w)\| &= \max \{\|z\|, \|w\|\} \\ &= \max \left\{ \|A_M^n(u)\|, \|A_N^{n-1}(\varepsilon^{n-1}(u') - v)\| \right\} \\ &\leq \max \left\{ \|u\|, \|\alpha^{n-1}(u') - v\| \right\} \\ &\leq \max \left\{ \|u\|, \|u'\| + \|v\| \right\} \\ &\leq \max \left\{ \|u\|, K_n \cdot \|u\| + \|v\| \right\} \\ &= K_n \cdot \|u\| + \|v\| \\ &\leq K_n \cdot (\|u\| + \|v\|) \\ &\leq 2K_n \cdot \max \{\|u\|, \|v\|\} = 2K_n \cdot \|(u, v)\|. \end{aligned}$$

□

REMARK 5.11. In the assumptions of Lemma 5.10 we are *not* requiring that A_M^\bullet and A_N^\bullet commute with the chain maps ε^\bullet and $\bar{\varepsilon}^\bullet$, i.e. $A_N^\bullet \circ \varepsilon^\bullet = \bar{\varepsilon}^\bullet \circ A_M^\bullet$. If this were the case, it would be easy to exhibit a norm non-increasing chain map which is a left inverse of η^\bullet , thus obtaining that η^\bullet induces an isometric isomorphism in cohomology.

Mapping cones are a useful tool for moving beyond the realm of pairs of multi-complexes. This construction is essential for handling the map $j_{\mathcal{L}}: \mathcal{L}(A) \rightarrow \mathcal{L}(X)$ between minimal multicomplexes, which is not injective in general (Proposition 5.4). Let $f: L \rightarrow K$ be a simplicial map between multicomplexes. We denote by

$$C_b^\bullet(f: L \rightarrow K)$$

the mapping cone complex associated to the cochain map $C_b^\bullet(K) \rightarrow C_b^\bullet(L)$ induced by f , and we denote by $H_b^\bullet(f: L \rightarrow K)$ its cohomology. The fact that mapping cones provide bi-Lipschitz estimates on relative bounded cohomology is already present in the work of Park [Par03, Theorem 3.19].

LEMMA 5.12. *Let (K, L) be a pair of multicomplexes, and let $L \hookrightarrow K$ be the inclusion map. The chain map*

$$\beta^n: C_b^n(K, L) \rightarrow C_b^n(L \hookrightarrow K), \quad \beta^n(u) = (u, 0),$$

induces, for every $n \in \mathbb{N}$, a bi-Lipschitz isomorphism of vector spaces

$$H^n(\beta^\bullet): H_b^n(K, L) \rightarrow H_b^n(L \hookrightarrow K),$$

such that, for every $\alpha \in H_b^n(K, L)$,

$$(n+2)^{-1} \cdot \|\alpha\|_\infty \leq \|H_b^n(\beta^\bullet)(\alpha)\|_\infty \leq \|\alpha\|_\infty.$$

PROOF. For sake of completeness, we adapt the argument in [Par03] to our context (see also [Fri17, Proposition 6.15]). We denote by δ^\bullet and $\bar{\delta}^\bullet$ the differentials of $C_b^\bullet(K, L)$ and $C_b^\bullet(L \hookrightarrow K)$ respectively. For every $v \in C_b^\bullet(L)$, we denote by $v' \in C_b^\bullet(K)$ the extension of v which vanishes on algebraic simplices which are not in L . We first show that β^n induces an isomorphism in cohomology. Let $(u, v) \in C_b^n(L \hookrightarrow K)$ be a cocycle. Since $\bar{\delta}^n(u, v) = 0$, we have that $u + \delta^{n-1}v' \in C_b^n(K, L)$ is a relative cocycle. Moreover, the fact that

$$\beta^n(u + \delta^{n-1}v') - (u, v) = -\bar{\delta}^{n-1}(v', 0)$$

shows that β^n induces a surjective map in cohomology. On the other hand, let $u \in C_b^n(K, L)$ such that $\beta^n(u) = \bar{\delta}^{n-1}(\alpha, \beta)$. Then $\alpha + \delta^{n-1}\beta' \in C_b^{n-1}(K, L)$ is a relative cocycle such that $\delta^{n-1}(\alpha + \delta^{n-1}\beta') = u$, showing that β^n induces an injective map in cohomology. Since β^\bullet is norm non-increasing, it follows that also $H_b^n(\beta^\bullet)$ is norm non-increasing. The other inequality in the statement follows from the following estimate: for every cocycle $(u, v) \in C_b^n(L \hookrightarrow K)$,

$$\begin{aligned} \|u + \delta^{n-1}v'\| &\leq \|u\| + \|\delta^{n-1}\| \cdot \|v'\| \\ &= \|u\| + (n+1) \cdot \|v\| \\ &\leq (n+2) \cdot \max\{\|u\|, \|v\|\} \\ &= (n+2) \cdot \|(u, v)\|. \end{aligned}$$

□

5.3. Bi-Lipschitz Isomorphisms

The goal of this section is to prove the following result.

THEOREM 5. *Let (X, A) be a CW-pair such that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Then, for every $n \in \mathbb{N}_{\geq 1}$, there is a constant $C_n \in \mathbb{R}_{\geq 1}$ (depending only on the degree) and a bi-Lipschitz isomorphism of vector spaces*

$$\Psi^n: H_b^n(\mathcal{A}(X), \mathcal{A}_X(A)) \rightarrow H_b^n(X, A),$$

such that, for every $\alpha \in H_b^n(\mathcal{A}(X), \mathcal{A}_X(A))$,

$$(n+2)^{-1}(2C_n)^{-2} \cdot \|\alpha\|_\infty \leq \|\Psi^n(\alpha)\|_\infty \leq 2C_n(n+2) \cdot \|\alpha\|_\infty.$$

The constants C_n in the statement of Theorem 5 denote the number of n -simplices in the natural structure of multicomplex of the n -dimensional prism $\Delta^{n-1} \times [0, 1]$ (see Definition 2.8).

REMARK 5.13. Of course, Theorem 5 holds also for $n = 0$, since the path-connected components of $\mathcal{A}(X)$ (resp. $\mathcal{A}_X(A)$) bijectively correspond to the path-connected components of X (resp. A).

Let (X, A) be a CW-pair such that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Unlike in Theorem 1, we are *not* requiring the pair (X, A) to be good. In Section 5.1 we have constructed a well-defined pair of aspherical multicomplexes $(\mathcal{A}(X), \mathcal{A}_X(A))$. Moreover, we have the following commutative diagram of simplicial maps

$$(5.3) \quad \begin{array}{ccccccc} \mathcal{K}(A) & \xleftarrow{i_A} & \mathcal{L}(A) & \xrightarrow{\pi} & \mathcal{A}(A) & \xrightarrow{q} & \mathcal{A}_X(A) \\ & & \downarrow j_{\mathcal{L}} & & \downarrow j_{\mathcal{A}} & & \downarrow \\ \mathcal{K}(X) & \xleftarrow{i_X} & \mathcal{L}(X) & \xrightarrow{\pi} & \mathcal{A}(X) & \xrightarrow{\text{id}} & \mathcal{A}(X), \end{array}$$

where the left square is commutative under the choices described in Setup 5.2, which we assume throughout the whole work (see Remark 5.17).

Our strategy to prove Theorem 5 is to pass through mapping cones and invoke Lemma 5.10 for every square of diagram (5.3). To this end, we first show that the maps i_X , i_A , π and q have (C_n) -chain homotopy inverses (see Definition 5.9).

LEMMA 5.14. *Let X be a CW-complex. Then the map induced by $i_X: \mathcal{L}(X) \hookrightarrow \mathcal{K}(X)$ on bounded cochains has a (C_n) -chain homotopy inverse.*

PROOF. We know that there exists a simplicial retraction

$$r_X: \mathcal{K}(X) \rightarrow \mathcal{L}(X)$$

such that $r_X \circ i_X$ is the identity of $\mathcal{L}(X)$ (Proposition 2.16). We claim that the chain map $r_X^\bullet: C_b^\bullet(\mathcal{L}(X)) \rightarrow C_b^\bullet(\mathcal{K}(X))$ induced by r_X provides a (C_n) -chain homotopy inverse of i_X^\bullet . In fact, the geometric realization of r_X realizes $|\mathcal{L}(X)|$ as a strong deformation retract of $|\mathcal{K}(X)|$. Since $\mathcal{K}(X)$ is large and complete, it follows from the Homotopy Lemma 2.11 that $i_X \circ r_X$ is simplicially homotopic to the identity. Therefore, by Remark 2.10, there exists a chain homotopy $T^\bullet: C_b^\bullet(\mathcal{K}(X)) \rightarrow C_b^{\bullet-1}(\mathcal{K}(X))$ between $r_X^\bullet \circ i_X^\bullet$ and the identity such that $\|T^n\| \leq C_n$, for every $n \in \mathbb{N}$. \square

LEMMA 5.15. *Let X be a CW-complex. Then the map induced by $\pi: \mathcal{L}(X) \rightarrow \mathcal{A}(X)$ on bounded cochains has a (C_n) -chain homotopy inverse.*

PROOF. By Corollary 3.13, we know that $\mathcal{A}(X) = \mathcal{L}(X)/\Gamma_1$, where Γ_1 denotes the group $\Gamma_1(\mathcal{L}(X))$ (see Definition 3.8). It follows that the projection $\pi: \mathcal{L}(X) \rightarrow \mathcal{A}(X)$ induces the inclusion

$$\iota^\bullet: C_b^\bullet(\mathcal{L}(X))^{\Gamma_1} \hookrightarrow C_b^\bullet(\mathcal{L}(X))$$

of invariant cochains. We know that the groups Γ_1/Γ_m , $m \in \mathbb{N}$, are amenable (Lemma 3.14). Moreover Γ_1 acts by definition on $\mathcal{L}(X)$ by automorphisms which are simplicially homotopic to the identity. Therefore, we obtain by Proposition 3.15 that ι^\bullet admits a (C_n) -chain homotopy inverse. \square

LEMMA 5.16. *Let (X, A) be a CW-pair such that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Then the map induced by $q: \mathcal{A}(A) \rightarrow \mathcal{A}_X(A)$ on bounded cochains has a (C_n) -chain homotopy inverse.*

PROOF. By Lemma 5.6, we know that the projection map q can be described as the quotient of $\mathcal{A}(A)$ by the action of the group

$$\widehat{N} = \bigoplus_{i \in I} \bigoplus_{x \in A_i} N_x \leq \Pi(A, A),$$

where $\{A_i \mid i \in I\}$ denotes the set of connected components of A . Here N_x denotes the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$, which is amenable by assumption. Since direct products of amenable groups is amenable, we have that \widehat{N} is amenable. Moreover, we know that the action $\widehat{N} \curvearrowright \mathcal{A}(A)$ is by automorphisms which are simplicially homotopic to the identity (Proposition 4.4). Hence we conclude by Proposition 3.15. \square

We are now ready to prove Theorem 5. We say that a linear map $f: V \rightarrow W$ between normed spaces is *bi-Lipschitz of type (K, L)* (for short, *(K, L) -bi-Lipschitz*) if

$$K \cdot \|v\| \leq \|f(v)\| \leq L \cdot \|v\|$$

for every $v \in V$. For example, isometric isomorphisms are $(1, 1)$ -bi-Lipschitz maps. If $f: V \rightarrow W$ is (K, L) -bi-Lipschitz and $f': U \rightarrow W$ is (K', L') -bi-Lipschitz, then their composition $f \circ f'$ is (KK', LL') -bi-Lipschitz.

PROOF OF THEOREM 5. Let (X, A) be a CW-pair such that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. We know that the natural projection

$$S: (|\mathcal{K}(X)|, |\mathcal{K}(A)|) \rightarrow (X, A)$$

induces a homotopy equivalence of pairs (Proposition 1.13), hence the chain map

$$S^\bullet: C_b^\bullet(X, A) \rightarrow C_b^\bullet(|\mathcal{K}(X)|, |\mathcal{K}(A)|)$$

induces an isometric isomorphism in bounded cohomology. Moreover, by the Relative Isometry Lemma (Proposition 3.5), we know that the natural chain inclusion $\varphi_\bullet: C_\bullet(\mathcal{K}(X), \mathcal{K}(A)) \rightarrow C_\bullet(|\mathcal{K}(X)|, |\mathcal{K}(A)|)$ induces an isometric isomorphism in bounded cohomology. By Lemma 5.12, the chain map

$$\beta^\bullet: C_b^\bullet(\mathcal{K}(X), \mathcal{K}(A)) \rightarrow C_b^\bullet(j_{\mathcal{K}}: \mathcal{K}(A) \hookrightarrow \mathcal{K}(X))$$

induces a $((n+2)^{-1}, 1)$ -bi-Lipschitz isomorphism in degree n . By Lemma 5.14, Lemma 5.15 and Lemma 5.16 above, we can invoke Lemma 5.10 for every square

of diagram (5.3). Therefore, the following chain maps

$$\begin{aligned} i^\bullet &: C_b^\bullet(j_{\mathcal{K}}: \mathcal{K}(A) \hookrightarrow \mathcal{K}(X)) \rightarrow C_b^\bullet(j_{\mathcal{L}}: \mathcal{L}(A) \rightarrow \mathcal{L}(X)), \\ \pi^\bullet &: C_b^\bullet(j_{\mathcal{A}}: \mathcal{A}(A) \rightarrow \mathcal{A}(X)) \rightarrow C_b^\bullet(j_{\mathcal{L}}: \mathcal{L}(A) \rightarrow \mathcal{L}(X)), \\ q^\bullet &: C_b^\bullet(\mathcal{A}_X(A) \hookrightarrow \mathcal{A}(X)) \rightarrow C_b^\bullet(j_{\mathcal{A}}: \mathcal{A}(A) \rightarrow \mathcal{A}(X)) \end{aligned}$$

induce $((2C_n)^{-1}, 1)$ -bi-Lipschitz isomorphisms in bounded cohomology in degree n . Since $\mathcal{A}_X(A)$ is a subcomplex of $\mathcal{A}(X)$, we can apply Lemma 5.12 to get a chain map

$$\beta^\bullet: C_b^\bullet(\mathcal{A}(X), \mathcal{A}_X(A)) \rightarrow C_b^\bullet(\mathcal{A}_X(A) \hookrightarrow \mathcal{A}(X)),$$

which induces a $((n+2)^{-1}, 1)$ -bi-Lipschitz isomorphism in cohomology in degree n .

In conclusion, both the maps $\zeta^\bullet = i^\bullet \circ \beta^\bullet \circ \varphi^\bullet \circ S^\bullet$ and $\vartheta^\bullet = \pi^\bullet \circ q^\bullet \circ \beta^\bullet$ induce bi-Lipschitz isomorphisms in cohomology, where $H_b^n(\zeta^\bullet)$ is bi-Lipschitz of type

$$((n+2)^{-1}(2C_n)^{-1}, 1),$$

and $H_b^n(\vartheta^\bullet)$ is bi-Lipschitz of type

$$((n+2)^{-1}(2C_n)^{-2}, 1).$$

For every $n \in \mathbb{N}$, the map Ψ^n is then defined as the composition $H^n(\zeta^\bullet)^{-1} \circ H^n(\vartheta^\bullet)$. It follows that Ψ^n is bi-Lipschitz of type

$$((n+2)^{-1}(2C_n)^{-2}, 2C_n(n+2)),$$

which is the estimate in the statement of Theorem 5. \square

REMARK 5.17. The isomorphism Ψ^n of Theorem 5 is *a priori not canonical*, since the choices described in Setup 5.2 are relevant in our construction.

REMARK 5.18. The bi-Lipschitz constants appearing in Theorem 5 are far from being optimal. For example, if A is π_1 -injective in X , then we know that $\mathcal{A}_X(A)$ is simplicially isomorphic to $\mathcal{A}(A)$, hence it embeds into $\mathcal{A}(X)$ (Proposition 5.5). In this case the passage through mapping cones in the rightmost square of diagram (5.3) is unnecessary. It follows that Ψ^n actually satisfies the following inequalities: for every $\alpha \in H_b^n(\mathcal{A}(X), \mathcal{A}(A))$,

$$(2C_n)^{-1}(n+2)^{-1} \cdot \|\alpha\|_\infty \leq \|\Psi^n(\alpha)\|_\infty \leq 2C_n(n+2) \cdot \|\alpha\|_\infty.$$

5.4. The Role of Higher Homotopy

The goal of this section is to discuss the role of higher homotopy in our setting. This discussion aims to clarify the connection between our framework and Kuessner's work [Kue15] on relative bounded cohomology via multicomplexes.

In fact, [Kue15, Proposition 3] states that, for every pair of topological spaces (X, A) and every $n \in \mathbb{N}$, we have that $H_b^n(X, A)$ is isometrically isomorphic to $H_b^n(\mathcal{A}(X), \mathcal{A}(A))$, provided that the map $\pi_1(A \hookrightarrow X, x)$ is injective for every $x \in A$. This statement is clearly stronger than both Theorem 1 and Theorem 5, and was used by Kuessner to deduce similar conclusions of our Theorem 10 without assuming the manifolds to be aspherical. However, in the proof of [Kue15, Proposition 3], Kuessner seems to implicitly exploit some further assumptions, concerning the higher homotopy of the pair (X, A) . In order to address these issues, which were initially highlighted by Moraschini in his Ph.D. thesis [Mor18, Section 4.4], we first recall Kuessner's setting. Unfortunately, our notation differs from Kuessner's

one: our multicomplex $\mathcal{L}(X)$ is denoted by $\widehat{K}(X)$ in [Kue15] and the aspherical multicomplex $\mathcal{A}(X)$ is denoted by $K(X)$.

We fix a CW-pair (X, A) such that A is π_1 -injective in X . Since A is π_1 -injective in X , we can write $\mathcal{A}(A) \subseteq \mathcal{A}(X)$ (Proposition 5.5). However, without any further assumptions on the higher homotopy of the pair (X, A) , we do *not* have in general an inclusion of $\mathcal{L}(A)$ inside $\mathcal{L}(X)$ (Proposition 5.4). Let $\pi: \mathcal{L}(X) \rightarrow \mathcal{A}(X)$ be the quotient map which identifies simplices having the same 1-skeleton. We consider the submulticomplex $\pi^{-1}(\mathcal{A}(A))$ of $\mathcal{L}(X)$ and we denote by

$$\iota: \pi^{-1}(\mathcal{A}(A)) \hookrightarrow \mathcal{L}(X)$$

the inclusion map. In the proof of [Kue15, Proposition 3] the following statements appear:

- (a) The multicomplex $\pi^{-1}(\mathcal{A}(A))$ is complete. This is in fact required to invoke the Relative Isometry Lemma [Kue15, Proposition 1] (see also our Proposition 3.5).
- (b) The map $S_X: |\mathcal{L}(X)| \rightarrow X$ maps $|\pi^{-1}(\mathcal{A}(A))|$ to A .

We want to show that, if we assume (a) and (b), then $\pi_n(A \hookrightarrow X, x)$ is an isomorphism for every $x \in A$ and every $n \in \mathbb{N}_{\geq 2}$, which is indeed the assumption we are making in Theorem 1.

$$\begin{array}{ccccc}
 \pi_{\mathcal{L}(A)}(\Delta) & \xrightarrow{\Theta_A} & \pi_n(|\mathcal{L}(A)|, x) & \xrightarrow{\pi_n(S_A, x)} & \pi_n(A, x) \\
 \downarrow \alpha & & \downarrow \pi_n(|j_{\mathcal{L}}|, x) & & \downarrow \text{id} \\
 \pi_{\pi^{-1}(\mathcal{A}(A))}(j_{\mathcal{L}}(\Delta)) & \xrightarrow{\Theta} & \pi_n(|\pi^{-1}(\mathcal{A}(A))|, x) & \longrightarrow & \pi_n(A, x) \\
 \downarrow \beta & & \downarrow \pi_n(|\iota|, x) & & \downarrow \pi_n(j, x) \\
 \pi_{\mathcal{L}(X)}(j_{\mathcal{L}}(\Delta)) & \xrightarrow{\Theta_X} & \pi_n(|\mathcal{L}(X)|, x) & \xrightarrow{\pi_n(S_X, x)} & \pi_n(X, x),
 \end{array}$$

FIGURE 4. The surjectivity of the map Θ follows from (a), while the commutativity of the bottom-right square follows from (b).

Let $n \in \mathbb{N}_{\geq 2}$ and let $x \in A$. Let Δ be an n -simplex of $\mathcal{L}(A)$ having x as a vertex. We consider the commutative diagram in Figure 4, where we need to make the following clarifications:

- The maps Θ , Θ_A and Θ_X are defined in Proposition 2.6. Moreover, since $\mathcal{L}(A)$ and $\mathcal{L}(X)$ are minimal, then Θ_A and Θ_X are bijective (Proposition 2.14).
- Since S_A and S_X are homotopy equivalences (Proposition 1.12), then both $\pi_n(S_A, x)$ and $\pi_n(S_X, x)$ are isomorphisms.
- The map α is induced by $j_{\mathcal{L}}$, whose image is contained in $\pi^{-1}(\mathcal{A}(A))$, while β denotes just the inclusion of simplices.

Of course, the pair $(\mathcal{L}(X), \pi^{-1}(\mathcal{A}(A)))$ is 1-coherent, hence n -coherent (see Definition 3.9). It follows that β is a bijective map. Therefore, since $\Theta_X \circ \beta = \pi_n(|\iota|, x) \circ \Theta$ is bijective, we can deduce that Θ is injective (hence bijective). Similarly, being $\pi_n(S_A, x)$ an isomorphism, we can deduce that $\pi_n(|j_{\mathcal{L}}|, x)$ is injective, and hence also α is injective. It follows from these observations that $\pi_n(j, x)$ is an isomorphism.

5.5. Regularity of Actions on Pairs of Multicomplexes

Let (K, L) be a pair of multicomplexes and let G be an amenable group acting simplicially on K . If the action $G \curvearrowright K$ preserves L , we know by Proposition 3.15 that every relative bounded coclass in $H_b^\bullet(K, L)$ can be represented by a G -invariant cocycle. The goal of this section is obtain a similar result in the case in which the action $G \curvearrowright K$ does *not* preserve L . To this end, we introduce a regularity condition of the action $G \curvearrowright K$ on L (Definition 5.19). Unlike in Proposition 3.15, in this setting we are only able to retain a bi-Lipschitz control over the norms.

Given an action $G \curvearrowright K$, we consider the induced action $G \curvearrowright C_b^\bullet(K)$ on bounded cochains. If H is a subgroup of G whose action on K preserves a submulticomplex L of K , there is an obvious restriction map

$$r_{G,H}^\bullet: C_b^\bullet(K)^G \rightarrow C_b^\bullet(L)^H,$$

whose kernel is $C_b^\bullet(K, L)^G = C_b^\bullet(K)^G \cap C_b^\bullet(K, L)$. Unlike the case described in Proposition 3.15, H could be a proper subgroup of G .

DEFINITION 5.19. Let (K, L) be a pair of multicomplexes. Let $G \curvearrowright K$ be a simplicial action and let H be a subgroup of G which induces an action $H \curvearrowright (K, L)$. We say that the action $G \curvearrowright K$ *has orbits in L induced by H* if, for every algebraic simplex s of L and every $g \in G$ such that $g \cdot s$ is an algebraic simplex of L , there exists $h \in H$ such that $h \cdot s = g \cdot s$.

Simplicial actions as defined in Definition 5.19 arise in various geometric contexts — for instance, in applications to the Relative Mapping Theorem (Lemma 5.25), to amenable open covers (Lemma 6.14), and to simplicial volume of gluings (Lemma 7.10). The following result shows that, under the regularity condition above, the restriction map $r_{G,H}^\bullet$ is surjective, and will allow us, under suitable circumstances, to take G -invariant cocycles in every relative coclass of $H_b^\bullet(K, L)$. The constant C_n appearing in the following is described in Definition 2.8.

LEMMA 5.20. *Let (K, L) be a pair of multicomplexes. Let $G \curvearrowright K$ be a simplicial action and let H be a subgroup of G which induces an action $H \curvearrowright (K, L)$. Assume that the orbits of G in L are induced by H . Then the rows of the following commutative diagram are both exact*

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_b^\bullet(K, L)^G & \longrightarrow & C_b^\bullet(K)^G & \xrightarrow{r_{G,H}^\bullet} & C_b^\bullet(L)^H & \longrightarrow & 0 \\ & & \downarrow \iota^\bullet & & \downarrow \iota_G^\bullet & & \downarrow \iota_H^\bullet & & \\ 0 & \longrightarrow & C_b^\bullet(K, L) & \longrightarrow & C_b^\bullet(K) & \xrightarrow{r^\bullet} & C_b^\bullet(L) & \longrightarrow & 0 \end{array}$$

where the vertical arrows are inclusions of cochains. Moreover, if G is amenable and the actions $G \curvearrowright K$ and $H \curvearrowright L$ are by automorphisms which are simplicially homotopic to the identity, then $\iota^\bullet: C_b^\bullet(K, L)^G \rightarrow C_b^\bullet(K, L)$ induces a bi-Lipschitz isomorphism in cohomology such that

$$(n+2)^{-1}(2C_n)^{-1} \cdot \|\alpha\|_\infty \leq \|H_b^n(\iota^\bullet)(\alpha)\|_\infty \leq \|\alpha\|_\infty,$$

for every $n \in \mathbb{N}$ and every $\alpha \in H^n(C_b^\bullet(K, L)^G)$.

PROOF. First of all, we show that we can extend H -invariant chains on L to G -invariant chains on K , i.e. $r_{G,H}^\bullet$ is surjective. To this end our conditions on the

action $G \curvearrowright K$ are crucial. Let $n \in \mathbb{N}$ and let $z \in C_b^n(L)^H$. We define a cochain $z' \in C_b^n(K)^G$ in the following way: for every algebraic n -simplex s of K , we set $z'(s) = z(\hat{s})$, if $s = g \cdot \hat{s}$ for some $g \in G$ and $\hat{s} \in C_n(L)$, and $z'(s) = 0$, otherwise. Since the action $G \curvearrowright K$ has orbits in L induced by H and z is H -invariant, it is easy to check that z' is indeed well-defined. Moreover it is clear from the construction that z' is G -invariant and satisfies $r_{G,H}^n(z') = z$, thus the rows of the diagram are both exact.

Assume now that G (hence H) is amenable and the actions $G \curvearrowright K$ and $H \curvearrowright L$ are by automorphisms which are simplicially homotopic to the identity. It follows that ι_G^\bullet and ι_H^\bullet induce isometric isomorphisms in cohomology (Proposition 3.15). Therefore, as an application of the Five Lemma, we can already deduce that ι^\bullet induces an isomorphism in relative cohomology. Moreover, $H_b^n(\iota^\bullet)$ is of course norm non-increasing. In order to retain further control over the norms, we need to pass through mapping cones in the following way. We consider the mapping cone complex $\text{Cone}(r_{G,H})^\bullet$ associated to the restriction map $r_{G,H}^\bullet$, and we denote by $\bar{\delta}^\bullet$ its differential. Similarly, we consider the mapping cone complex

$$\text{Cone}(r)^\bullet = C_b^\bullet(L \hookrightarrow K)$$

associated to the restriction map $r^\bullet: C_b^\bullet(K) \rightarrow C_b^\bullet(L)$. There is a natural chain map induced by the inclusion of invariant cochains:

$$\varphi^n: \text{Cone}(r_{G,H})^n \rightarrow \text{Cone}(r)^n, \quad (u, v) \rightarrow (\iota_G^n(u), \iota_H^{n-1}(v)).$$

We consider the following commutative diagram

$$\begin{array}{ccc} C_b^n(K, L)^G & \xleftarrow{\iota^n} & C_b^n(K, L) \\ \downarrow \beta_{G,H}^n & & \downarrow \beta^n \\ \text{Cone}(r_{G,H})^n & \xrightarrow{\varphi^n} & \text{Cone}(r)^n, \end{array}$$

where β^n was defined in Lemma 5.12 and $\beta_{G,H}^n$ is defined by the formula

$$\beta_{G,H}^n(u) = (u, 0).$$

Of course, if β^\bullet , $\beta_{G,H}^\bullet$ and φ^\bullet induce bi-Lipschitz isomorphisms in cohomology, then also ι^\bullet does. By Lemma 5.12, we know that β^\bullet induces in cohomology a bi-Lipschitz isomorphism of type $((n+2)^{-1}, 1)$. The same argument of Lemma 5.12 applies *verbatim* to the map $\beta_{G,H}^\bullet$. It follows that $\beta_{G,H}^\bullet$ induces a bi-Lipschitz isomorphism in cohomology such that

$$(n+2)^{-1} \cdot \|\alpha\|_\infty \leq \|H_b^n(\beta_{G,H}^\bullet)(\alpha)\|_\infty \leq \|\alpha\|_\infty,$$

for every $\alpha \in H^n(C_b^\bullet(K, L)^G)$. On the other hand, since G is amenable and the action $G \curvearrowright K$ is by automorphisms which are simplicially homotopic to the identity, we know that ι_G^\bullet admits a (C_n) -chain homotopy inverse (Proposition 3.15). The same holds for the action $H \curvearrowright L$, so that ι_H^\bullet admits a (C_n) -chain homotopy inverse. It follows from Lemma 5.10 that φ^\bullet induces a bi-Lipschitz isomorphism in cohomology such that

$$(2C_n)^{-1} \cdot \|\alpha\| \leq \|H^n(\varphi^\bullet)(\alpha)\| \leq \|\alpha\|,$$

for every $\alpha \in H^n(\text{Cone}(r_{G,H})^\bullet)$. In conclusion, since β^\bullet , $\beta_{G,H}^\bullet$ and φ^\bullet induce bi-Lipschitz isomorphisms in cohomology, then also ι^\bullet does and the estimate over the norms in the statement easily follows. \square

REMARK 5.21. Let (K, L) be a pair of multicomplexes and let G be a group acting on K . Assume that the action $G \curvearrowright K$ does *not* preserve the submulticomplex L . The set GL of G -translates of L , is a submulticomplex of K and there is a well-defined action $G \curvearrowright (K, GL)$. In Kuessner's work on multicomplexes (see, for example, [Kue15, Corollary 2]) the following statement is claimed to be true. If G is amenable and it acts on K by automorphisms which are simplicially homotopic to the identity, then the relative bounded cohomology $H_b^n(K, L)$ can be isometrically computed via the resolution $C_b^\bullet(K, GL)^G$ of G -invariant cochains on K relative to GL . We believe that this approach underestimates some difficulties, which could arise in geometric contexts, if we do not require some regularity of the action $G \curvearrowright K$ on L . See Remark 6.15 and Remark 7.9 for more details.

5.6. The Relative Mapping Theorem

Gromov's Mapping Theorem states that continuous maps between topological spaces induces an isometric isomorphisms in bounded cohomology, provided that the induced map on the fundamental groups is surjective with amenable kernel. The situation for relative bounded cohomology is more delicate. The aim of this section is to apply the regularity condition for simplicial actions introduced in Section 5.5 to establish the following version of the Relative Mapping Theorem.

THEOREM 5.22 (Relative Mapping Theorem). *Let $f: (X, A) \rightarrow (Y, B)$ be a continuous map between good pairs which is bijective on the sets of path-connected components. Assume that f induces epimorphisms with amenable kernel on fundamental groups of every component of X and every component of A , i.e. for every $x \in X$ and every $a \in A$, the maps*

$$\pi_1(f, x): \pi_1(X, x) \rightarrow \pi_1(Y, f(x)), \quad \pi_1(f|_A, a): \pi_1(A, a) \rightarrow \pi_1(B, f(a)),$$

are surjective with amenable kernel. Then the induced map in bounded cohomology

$$H_b^n(f): H_b^n(Y, B) \rightarrow H_b^n(X, A)$$

is a bi-Lipschitz isomorphism such that

$$(n + 2)^{-1}(2C_n)^{-1} \cdot \|\alpha\|_\infty \leq \|H_b^n(f)(\alpha)\|_\infty \leq \|\alpha\|_\infty,$$

for every $n \in \mathbb{N}$ and every $\alpha \in H_b^n(Y, B)$.

REMARK 5.23. Theorem 5.22 differs from the version of the Relative Mapping Theorem suggested by Gromov in [Gro82, Section 4.1], where no assumptions on higher homotopy are made and an isometric control over the norms is claimed. It is unclear to us how to remove the higher homotopy assumptions or how to achieve a stronger control over the norms. Moreover, we expect that alternative proofs of Theorem 5.22 may be formulated within the framework developed by Park in [Par03].

Our proof of Theorem 5.22 follows the proof of the Mapping Theorem presented in [FM23]. We can assume without loss of generality that X and Y are path-connected. Let $\{A_i \mid i \in I\}$ be the set of path connected components of A , and take a basepoint $x_i \in A_i$ for every $i \in I$. From the assumptions of Theorem 5.22, we

have a commutative diagram of groups

$$(5.4) \quad \begin{array}{ccccccc} 1 & \longrightarrow & N_i & \longrightarrow & \pi_1(A, x_i) & \longrightarrow & \pi_1(B, f(x_i)) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & M_i & \longrightarrow & \pi_1(X, x_i) & \longrightarrow & \pi_1(Y, f(x_i)) \longrightarrow 1 \end{array}$$

where the rows are exact, the vertical maps are injective and the groups M_i, N_i are amenable. The group $\pi_1(X, x_i)$ (resp. $\pi_1(A, x_i)$) is canonically isomorphic to $\pi_1(X, x)$ (resp. $\pi_1(A, a)$) for every $x \in X$ (resp. $a \in A_i$), up to conjugation. Since M_i is normal in $\pi_1(X, x_i)$, there exist well-defined isomorphic images $(M_i)_x$ of M_i inside $\pi_1(X, x)$ for every $x \in X$. Similarly, there exist well-defined isomorphic images $(N_i)_a$ of N_i inside $\pi_1(A, a)$ for every $a \in A_i$. We set

$$H_i = \bigoplus_{a \in A_i} (N_i)_a, \quad G_i = \bigoplus_{x \in X} (M_i)_x.$$

Since X is assumed to be connected, we have that, for every $i \neq j \in I$, G_i and G_j identify the same subgroup of $\Pi(X, X)$, which is denoted by G . On the other hand, the groups H_i and H_j , understood as subgroups of $\Pi(A, A)$, commute with each other, as they act on disjoint sets of vertices of $\mathcal{A}(A)$. Since A is π_1 -injective in X , then $\Pi(A, A)$ embeds into $\Pi(X, X)$. It follows that the group

$$H = \bigoplus_{i \in I} H_i$$

is naturally a subgroup of $G \leq \Pi(X, X)$. Since the class of amenable groups is closed under direct sums, both G and H are amenable. Since the pair (X, A) is good, there is a well-defined pair of aspherical multicomplexes $(\mathcal{A}(X), \mathcal{A}(A))$. The group G acts on the multicomplex $\mathcal{A}(X)$ as described in Section 4.2. Moreover, the subgroup $H \leq G$ induces a well-defined action $H \curvearrowright (\mathcal{A}(X), \mathcal{A}(A))$. The first part of the following lemma is [FM23, Lemma 5.4]. The second part, which follows directly from the construction, is the key step in showing that the action $G \curvearrowright \mathcal{A}(X)$ has orbits in $\mathcal{A}(A)$ induced by H (Lemma 5.25).

LEMMA 5.24. *Let e, e' be edges of $\mathcal{A}(X)$ such that there exists $g \in G$ such that $g \cdot e = e'$. In particular, e and e' have the same endpoints x_0 and x_1 . Then, for every $g_0 \in M_{x_0}$ there exists a unique element $g_1 \in M_{x_1}$ such that $(g_0, g_1) \cdot e = e'$. Moreover, if e and e' are both edges of $\mathcal{A}(A)$, and $g_0 \in N_{x_0}$, then $g_1 \in N_{x_1}$. In particular, $(g_0, g_1) \in H$.*

PROOF. Let γ_e (resp. $\gamma_{e'}$) be a path in X representing the edge e (resp. e') such that $\gamma_e(0) = x_0$ and $\gamma_e(1) = x_1$ (resp. $\gamma_{e'}(0) = x_0$ and $\gamma_{e'}(1) = x_1$). The fact that $g \cdot e = e'$ implies that the concatenation $\bar{\gamma}_e * \gamma_{e'}$ is in M_{x_1} (Lemma 4.6). Let α_0 be a loop based at x_0 representing g_0 . We consider the loop α_1 based at x_1 given by the concatenation

$$\alpha_1 = \bar{\gamma}_e * \alpha_0 * \gamma_{e'},$$

and we denote by g_1 the element of $\pi_1(X, x_1)$ represented by α_1 . It follows directly from the construction that $g_1 \in M_{x_1}$ is the unique element such that $(g_0, g_1) \cdot e = e'$. On the other hand, if e, e' are edges of $\mathcal{A}(A)$ and $g_0 \in N_{x_0}$, then $\gamma_e, \gamma_{e'}$ and α_0 can be all assumed to have values in A , hence also α_1 does. Since $\pi_1(A, x_1) \cap M_{x_1} = N_{x_1}$, we have that $g_1 \in N_{x_1}$. \square

LEMMA 5.25. *The action $G \curvearrowright \mathcal{A}(X)$ has orbits in $\mathcal{A}(A)$ induced by H .*

PROOF. Let $s = (\Delta, (x_0, \dots, x_n))$ be an algebraic n -simplex of $\mathcal{A}(A)$ and let $g \in G$ be such that $g \cdot s$ is an algebraic n -simplex of $\mathcal{A}(A)$. We need to show that there exists an element $h \in H$ such that $h \cdot s = g \cdot s$. Let $\{v_0, \dots, v_k\}$ be the vertices of Δ , where $k \leq n$. We denote by $e_{i,j}$ the edge of Δ between the vertices v_i and v_j , for $i < j$, and we set $e'_{i,j} = g \cdot e_{i,j}$. Let $\gamma_{i,j}: [0, 1] \rightarrow A$ (resp. $\gamma'_{i,j}: [0, 1] \rightarrow A$) be a path from v_i to v_j representing $e_{i,j}$ (resp. $e'_{i,j}$). Of course, we have that $\gamma_{i,j} * \gamma_{j,k} \simeq \gamma_{i,k}$ and $\gamma'_{i,j} * \gamma'_{j,k} \simeq \gamma'_{i,k}$ for every $i < j < k$ (where \simeq means that the paths are homotopic in A relative to the endpoints). By repeatedly using Lemma 5.24 (with $h_0 = 1$), for every $i \in \{0, \dots, k\}$ there is $h_i \in N_{v_i}$ such that $(h_{i-1}, h_i) \cdot e_{i-1,i} = e'_{i-1,i}$. This means that, for every loop α_i representing h_i , we have that

$$\bar{\alpha}_{i-1} * \gamma_{i-1,i} * \alpha_i \simeq \gamma'_{i-1,i}.$$

Let $h = \bigoplus_{j=0}^k h_j$. We want to show that $h \cdot e_{i,k} = e'_{i,k}$ for every $i < k$. We know that this holds if $|i - k| = 1$. If $|i - k| > 1$, we can take $j \in \{0, \dots, k\}$ such that $i < j < k$. Therefore, by induction on $|i - k|$, we deduce that

$$\begin{aligned} \bar{\alpha}_i * \gamma_{i,k} * \alpha_k &\simeq \bar{\alpha}_i * \gamma_{i,j} * \gamma_{j,k} * \alpha_k \\ &\simeq \bar{\alpha}_i * \gamma_{i,j} * \alpha_j * \bar{\alpha}_j * \gamma_{j,k} * \alpha_k \\ &\simeq \gamma'_{i,j} * \gamma'_{j,k} \simeq \gamma'_{i,k}, \end{aligned}$$

hence $h \cdot e_{i,k} = e'_{i,k}$. It follows that $h \cdot e = g \cdot e$, for every edge e of Δ , hence $g \cdot \Delta^1 = h \cdot \Delta^1$, where Δ^1 denotes the 1-skeleton of Δ . Since $\mathcal{A}(X)$ is an aspherical multicomplex, this is indeed sufficient to conclude (Proposition 2.24). \square

Since the actions $G \curvearrowright \mathcal{A}(X)$ and $H \curvearrowright \mathcal{A}(A)$ are 0-trivial, there are well-defined quotient multicomplexes $\mathcal{A}(X)/G$ and $\mathcal{A}(A)/H$ (Proposition 1.7). Moreover, being H a subgroup of G , the inclusion $\mathcal{A}(A) \hookrightarrow \mathcal{A}(X)$ descends to the quotient, i.e. there exists a simplicial map

$$\hat{j}: \mathcal{A}(A)/H \rightarrow \mathcal{A}(X)/G$$

such that the following diagram

$$\begin{array}{ccc} \mathcal{A}(A) & \hookrightarrow & \mathcal{A}(X) \\ \downarrow q_N & & \downarrow q_M \\ \mathcal{A}(A)/H & \xrightarrow{\hat{j}} & \mathcal{A}(X)/G \end{array}$$

is commutative, where the vertical arrows are quotient maps. Using the same argument of Lemma 5.25, it is easy to see that, for every pair of edges e, e' of $\mathcal{A}(A)$ such that $g \cdot e = e'$ for some $g \in G$, there exists $h \in H$ such that $h \cdot e = e'$. It follows that the simplicial map \hat{j} is, in fact, a simplicial embedding. Therefore there exists a well-defined map of pairs

$$q: (\mathcal{A}(X), \mathcal{A}(A)) \rightarrow (\mathcal{A}(X)/G, \mathcal{A}(A)/H).$$

PROOF OF THEOREM 5.22. As in our proof of Theorem 2, we have the following diagram

$$\begin{array}{ccc}
(X, A) & \xrightarrow{f} & (Y, B) \\
S_X \circ i_X \uparrow & & S_Y \circ i_Y \uparrow \\
(|\mathcal{L}(X)|, |\mathcal{L}(A)|) & \xrightarrow{\mathcal{L}(f)} & (|\mathcal{L}(Y)|, |\mathcal{L}(B)|) \\
\downarrow \pi_X & & \downarrow \pi_Y \\
(|\mathcal{A}(X)|, |\mathcal{A}(A)|) & \xrightarrow{\mathcal{A}(f)} & (|\mathcal{A}(Y)|, |\mathcal{A}(B)|)
\end{array}$$

which is commutative up to homotopy (of maps of pairs) and whose vertical arrows induce isometric isomorphisms in bounded cohomology. The map induced by the quotient map q on bounded cochains coincides with the inclusion

$$C_b^\bullet(\mathcal{A}(X), \mathcal{A}(A))^G \hookrightarrow C_b^\bullet(\mathcal{A}(X), \mathcal{A}(A))$$

of invariant cochains. Since the action $G \curvearrowright \mathcal{A}(X)$ has orbits in $\mathcal{A}(A)$ induced by H , it follows that the map induced by q in relative bounded cohomology is a bi-Lipschitz isomorphism (Proposition 5.20). We want to construct a homotopy equivalence of pairs h such that the following diagram

$$\begin{array}{ccc}
(|\mathcal{A}(X)|, |\mathcal{A}(A)|) & \xrightarrow{\mathcal{A}(f)} & (|\mathcal{A}(Y)|, |\mathcal{A}(B)|) \\
\searrow |q| & & \nearrow h \\
& (|\mathcal{A}(X)/G|, |\mathcal{A}(A)/H|) &
\end{array}$$

is commutative up to homotopy. To this end, we want to invoke Proposition 3.17. We know that (every connected component of) $|\mathcal{A}(X)/G|$ and $|\mathcal{A}(A)/H|$ are classifying spaces for the groups $\pi_1(X, x)/M_x$ and $\pi_1(A, a)/N_a$ for every $x \in X$ and every $a \in A$ (Proposition 4.7). Diagram (5.4) induces a commutative diagram

$$(5.5) \quad \begin{array}{ccc}
\pi_1(|\mathcal{A}(A)/H|, x) & \longrightarrow & \pi_1(|\mathcal{A}(B)|, y) \\
\downarrow & & \downarrow \\
\pi_1(|\mathcal{A}(X)/G|, x) & \longrightarrow & \pi_1(|\mathcal{A}(Y)|, y)
\end{array}$$

whose vertical arrows are injective and whose horizontal arrows are isomorphisms, for some $x \in X$ and some $y \in Y$. In order to invoke Proposition 3.17 we need to show that (5.5) is induced by a map on the 1-skeleton. Without loss of generality, we can assume $\mathcal{A}(f)$ to be cellular on the 1-skeleton. Let now e, e' be edges of $\mathcal{A}(X)$ such that $q(e) = q(e')$ (in particular they have the same endpoints). Let $\chi_e: [0, 1] \rightarrow |e| \subseteq |\mathcal{A}(X)|$ and $\chi_{e'}: [0, 1] \rightarrow |e'| \subseteq |\mathcal{A}(X)|$ be the characteristic maps of e and e' . Then $\mathcal{A}(f) \circ \chi_e$ and $\mathcal{A}(f) \circ \chi_{e'}$ identify two continuous paths in $|\mathcal{A}(Y)|$ with the same endpoints whose concatenation is homotopically trivial (being $e * e'$ in M by Lemma 4.6). Completeness and minimality of $\mathcal{A}(Y)$ imply that $\mathcal{A}(f) \circ \chi_e$ and $\mathcal{A}(f) \circ \chi_{e'}$ are homotopic (relative to the vertices) to the same simplicial embedding. This fact, together with the fact that the quotient map q has a section on the 1-skeleton, guarantees that (5.5) is induced by a continuous map on the 1-skeleton. We can therefore invoke Proposition 3.17 to obtain a map $h: (|\mathcal{A}(X)/G|, |\mathcal{A}(A)/H|) \rightarrow (|\mathcal{A}(Y)|, |\mathcal{A}(B)|)$ such that $h \circ |q|$ is homotopic (as a map of pairs) to $\mathcal{A}(f)$. Moreover, as $|\mathcal{A}(X)/G|$ and $|\mathcal{A}(A)/H|$ are aspherical, the

maps induced by h on $|\mathcal{A}(X)/G|$ and $|\mathcal{A}(A)/H|$ are homotopy equivalences [**Hat02**, Proposition 1B.9]. Since the inclusion $|\mathcal{A}(A)/H| \hookrightarrow |\mathcal{A}(X)/G|$ is a cofibration, it follows that h is a homotopy equivalence of pairs [**May99**, Chapter 6, Section 5], and thus induces an isometric isomorphism in bounded cohomology. This concludes the proof of Theorem 5.22. The bi-Lipschitz constants in the statement are the same appearing in Lemma 5.20. \square

The Relative Vanishing Theorem

Gromov’s Vanishing Theorem is a powerful vanishing criterion for the comparison map in presence of amenable open covers of small multiplicity. A generalization of Gromov’s result to the relative setting has been provided by Li, Löh and Moraschini in [LLM22] and by Raptis in [Rap24]. In this chapter, we present the results of [Cap25], which extend those of Li, Löh, and Moraschini in two directions, following Gromov’s approach via multicomplexes. Our arguments build on the same geometric construction developed in the absolute case in [FM23, Chapter 6].

6.1. Amenable Covers and Relative Vanishing Theorems

We begin with the fundamental notion of amenable subset of a topological space.

DEFINITION 6.1. Let X be a topological space. A subset U of X is *amenable* if, for every $x \in U$, the image of the map

$$\pi_1(U, x) \rightarrow \pi_1(X, x),$$

induced by the inclusion $U \hookrightarrow X$, is an amenable subgroup of $\pi_1(X, x)$. Amenable subsets are not required to be path-connected.

Let X be a topological space. An *open cover* \mathcal{U} of X is a set of open subsets of X such that $\bigcup_{U \in \mathcal{U}} U = X$. An open cover \mathcal{U} of X is called *amenable* if every element of \mathcal{U} is an amenable subset of X . The *multiplicity* $\text{mult}(\mathcal{U})$ of \mathcal{U} is the supremum of the set of natural numbers $k \in \mathbb{N}_{\geq 1}$ such that

$$U_1 \cap \cdots \cap U_k \neq \emptyset$$

for some pairwise distinct $U_1, \dots, U_k \in \mathcal{U}$. If such a number does not exist, we set $\text{mult}(\mathcal{U}) = \infty$. We say that \mathcal{U} is *locally finite* if it has finite multiplicity.

THEOREM 6.2 (Vanishing Theorem). *Let X be a topological space and let \mathcal{U} be an amenable open cover of X . Then the comparison map $\text{comp}^n: H_b^n(X) \rightarrow H^n(X)$ vanishes for every $n \geq \text{mult}(\mathcal{U})$.*

As already mentioned in the introduction, several proofs of the Vanishing Theorem are available in the literature. In this chapter we adapt the proof given by Frigerio and Moraschini in [FM23] to the relative setting. A pair of topological spaces (X, A) is called *triangulable* if there exists a pair of simplicial complexes (T, S) such that $(X, A) = (|T|, |S|)$.

THEOREM 6 (Relative Vanishing Theorem). *Let (X, A) be a triangulable pair and assume that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Let \mathcal{U} be an amenable cover of X by path-connected open subsets such that:*

(RC1) *For every $U \in \mathcal{U}$ such that $U \cap A \neq \emptyset$, $U \cap A$ is path-connected;*

(RC2) For every $U \in \mathcal{U}$ such that $U \cap A \neq \emptyset$, the inclusion

$$\mathrm{im}(\pi_1(U \cap A, x) \rightarrow \pi_1(X, x)) \hookrightarrow \mathrm{im}(\pi_1(U, x) \rightarrow \pi_1(X, x))$$

is an isomorphism for every $x \in U \cap A$.

Then the comparison map $\mathrm{comp}^n: H_b^n(X, A; \mathbb{R}) \rightarrow H^n(X, A; \mathbb{R})$ vanishes for every $n \geq \mathrm{mult}(\mathcal{U})$.

Our proof of Theorem 6 is presented in Section 6.4. The regularity condition (RC1) and (RC2) were introduced by Li, Löh and Moraschini in [LLM22]. Of course, (RC1) and (RC2) are empty if $A = \emptyset$. Moreover, (RC2) is empty when \mathcal{U} consists of π_1 -contractible subsets, i.e. when the map $\pi_1(U, x) \rightarrow \pi_1(X, x)$ is trivial for every $U \in \mathcal{U}$ and every $x \in U$.

REMARK 6.3. Let S be an oriented compact connected surface of genus one with one boundary component. It is well known that the relative simplicial volume $\|S, \partial S\|$ of S is positive [Gro82], hence the comparison map $\mathrm{comp}^2: H_b^2(S, \partial S) \rightarrow H^2(S, \partial S)$ is non-zero by duality (Lemma 3.1). Moreover, the boundary ∂S is π_1 -injective in S . Therefore, the open covers of S in Figure 5 show that the regularity assumptions (RC1) and (RC2) of Theorem 6 are optimal.

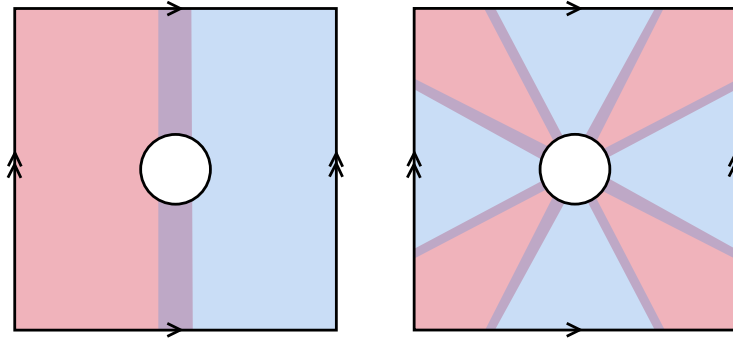


FIGURE 5. These open covers of S are amenable and have multiplicity 2. The one on the left satisfies (RC1), but not (RC2), the one on the right satisfies (RC2), but not (RC1).

To every open cover \mathcal{U} of X , one can associate a simplicial complex in the following way. The *nerve* $N(\mathcal{U})$ of \mathcal{U} is the simplicial complex whose set of vertices is the set \mathcal{U} itself and pairwise-different sets $U_0, \dots, U_k \in \mathcal{U}$ span a k -simplex in $N(\mathcal{U})$ if and only if

$$U_0 \cap \dots \cap U_k \neq \emptyset.$$

Of course, $\dim(N(\mathcal{U})) = \mathrm{mult}(\mathcal{U}) - 1$. If X is paracompact (hence admits partitions of unity subordinated to any open cover), there is a well defined *nerve map* $\nu: X \rightarrow |N(\mathcal{U})|$. Given a partition of unity $\Theta = (\vartheta_U)_{U \in \mathcal{U}}$ subordinated to the cover \mathcal{U} , the map

$$\nu_\Theta: X \rightarrow |N(\mathcal{U})|, \quad \nu_\Theta(x) = \sum_{U \in \mathcal{U}} \vartheta_U(x) \cdot U$$

is well-defined and continuous. Such a map is unique up to homotopy: if Θ' is another partition of unity subordinated to \mathcal{U} , then the convex combination

$$t\nu_\Theta + (1-t)\nu_{\Theta'}: X \rightarrow |N(\mathcal{U})|$$

is well-defined for every $t \in [0, 1]$ and defines a homotopy between ν_Θ and $\nu_{\Theta'}$. We say that \mathcal{U} is *convex* if, for every $k \in \mathbb{N}$ and every $U_1, \dots, U_k \in \mathcal{U}$, the intersection $U_1 \cap \dots \cap U_k$ is path-connected.

Let (X, A) be a pair of topological spaces and let \mathcal{U} be an open cover of X . We denote by

$$\mathcal{U}_A = \{U \cap A \mid U \in \mathcal{U}\}$$

the open cover of A induced by \mathcal{U} . If \mathcal{U} consists of path-connected sets and satisfies (RC1), then there is an obvious simplicial inclusion $N(\mathcal{U}_A) \rightarrow N(\mathcal{U})$, which is given on vertices by $U \cap A \mapsto U$. Therefore, in this case the pair of simplicial complexes $(N(\mathcal{U}), N(\mathcal{U}_A))$ is well-defined and every nerve map $\nu: X \rightarrow |N(\mathcal{U})|$ induces a map of pairs

$$\nu: (X, A) \rightarrow (|N(\mathcal{U})|, |N(\mathcal{U}_A)|).$$

The following notions were introduced in [LLM22]. An open cover \mathcal{U} of X is called *weakly-convex on A* if, for every $k \in \mathbb{N}$ and every $U_1, \dots, U_k \in \mathcal{U}$ with

$$U_1 \cap \dots \cap U_k \cap A \neq \emptyset,$$

then each path-connected component of $U_1 \cap \dots \cap U_k$ intersects A . Convex open covers are also weakly convex. The *relative multiplicity* $\text{mult}_A(\mathcal{U})$ of \mathcal{U} (with respect to A) is defined as the supremum of the set of natural numbers $k \in \mathbb{N}_{\geq 1}$ such that

$$U_1 \cap \dots \cap U_k \neq \emptyset, \quad U_1 \cap \dots \cap U_k \cap A = \emptyset,$$

for some pairwise distinct $U_1, \dots, U_k \in \mathcal{U}$. Of course, $\text{mult}_A(\mathcal{U})$ is a lower bound of $\text{mult}(\mathcal{U})$. The following result, proven in Section 6.4, shows that imposing stronger assumptions on the open cover yields stronger vanishing conclusions for the comparison map.

THEOREM 7. *Let (X, A) be a triangulable pair and assume that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Let \mathcal{U} be a locally-finite amenable cover of X by path-connected open subsets satisfying (RC1) and (RC2).*

- (1) *If \mathcal{U} is weakly-convex on A , then the comparison map comp^n vanishes for every $n \geq \text{mult}_A(\mathcal{U})$.*
- (2) *If \mathcal{U} is convex, then for every $n \in \mathbb{N}$ there exists a map Θ^n such that the following diagram commutes*

$$\begin{array}{ccc} H_b^n(X, A) & \xrightarrow{\text{comp}^n} & H^n(X, A) \\ \downarrow \Theta^n & & \uparrow H^n(\nu) \\ H^n(N(\mathcal{U}), N(\mathcal{U}_A)) & \xrightarrow{\cong} & H^n(|N(\mathcal{U})|, |N(\mathcal{U}_A)|), \end{array}$$

where $\nu: (X, A) \rightarrow (|N(\mathcal{U})|, |N(\mathcal{U}_A)|)$ is a nerve map.

Let $\dim(N(\mathcal{U}), N(\mathcal{U}_A))$ denote the dimension of the relative simplicial complex $(N(\mathcal{U}), N(\mathcal{U}_A))$, i.e. the maximum dimension of simplices of $N(\mathcal{U})$ which are not contained in $N(\mathcal{U}_A)$. By definition, we have that $\dim(N(\mathcal{U}), N(\mathcal{U}_A)) = \text{mult}_A(\mathcal{U}) - 1$. Since $\dim(N(\mathcal{U}), N(\mathcal{U}_A)) \leq \text{mult}_A(\mathcal{U}) \leq \text{mult}(\mathcal{U})$, the conclusion of Theorem 7(2) is stronger than that of Theorem 7(1), which in turn is stronger than the conclusion of Theorem 6. The price to pay for this increased generality is the need for stronger assumptions on the open cover.

REMARK 6.4. Theorem 7 is a natural extensions of the Relative Vanishing Theorem by Li, Löh and Moraschini [LLM22, Theorem 1.1]. Their result in fact deduces the same conclusions (1) and (2) of our Theorem 7, under the assumptions that (X, A) is a CW-pair such that A is π_1 -injective in X and \mathcal{U} is a (not necessarily locally-finite) boundedly acyclic open cover by path-connected subsets satisfying (RC1) and (RC2). Of course, triangulable pairs are in particular CW-pairs. However, not every CW-pair can be triangulated [FP90, Corollary 4.6.12]. Moreover, amenable open covers are in particular boundedly acyclic, and it is not clear whether our results can be extended to this case. On the other hand, our assumption which requires the kernel of the map $\pi_1(A \hookrightarrow X)$ to be amenable is clearly more flexible than π_1 -injectivity. We also underline the fact that, with respect to [LLM22], we need to assume \mathcal{U} to be locally finite.

The statement of Theorem 6 differs from that of [LLM22, Theorem 1.1] in that it is formulated using the absolute multiplicity rather than the relative one. However, following the announcement of the results in [Cap25], we learned through private communication with the authors of [LLM22] that the statement of Theorem 6 can also be derived within their framework.

We conclude this section with another formulation of the Relative Vanishing Theorem, as presented in the work of Raptis [Rap24]. Recall that a k -simplex σ of $N(\mathcal{U})$ is a set $\{U_0, \dots, U_k\}$ of pairwise distinct elements of \mathcal{U} whose intersection is non-empty. For every simplex σ of $N(\mathcal{U})$, we denote by $\mathcal{U}_\sigma = U_0 \cap \dots \cap U_k$ the common intersection of its vertices, and we consider the map

$$j_\sigma: \pi_0(\mathcal{U}_\sigma \cap A) \rightarrow \pi_0(\mathcal{U}_\sigma)$$

induced by inclusion. Observe that requiring \mathcal{U} to be weakly convex on A is equivalent to requiring that j_σ is surjective for every simplex σ of $N(\mathcal{U})$. Curiously, the injectivity of j_σ also implies the vanishing of the comparison map in bounded cohomology.

THEOREM 6.5 ([Rap24, Application 5.3]). *Let (X, A) be a pair of topological spaces and let \mathcal{U} an amenable open cover of X . Assume that \mathcal{U}_A is an amenable open cover of A . If j_σ is injective for every simplex σ of $N(\mathcal{U})$, then the comparison map*

$$\text{comp}^n: H_b^n(X, A) \rightarrow H^n(X, A)$$

vanishes for every $n \geq \text{mult}(\mathcal{U})$.

Even though condition (RC1) is implicitly included in the assumptions of Theorem 6.5 via the injectivity of j_v for every vertex v of $N(\mathcal{U})$, there is no trace of condition (RC2).

REMARK 6.6. Requiring \mathcal{U}_A to be amenable is strictly weaker than requiring A to be π_1 -injective in X , or the kernel of the morphism $\pi_1(A \hookrightarrow X)$ to be amenable, which is the assumption we do in Theorems 6 and 7. For example, if A is π_1 -injective in X , then the open cover \mathcal{U}_A remains amenable, because amenability is preserved under passing to subgroups. The same holds if the kernel of the morphism $\pi_1(A \hookrightarrow X)$ is amenable. This follows from a more general fact about pull-back covers [CLM22, Remark 2.9]. In fact, for every continuous map $f: X \rightarrow Y$ with amenable kernel and every amenable subset U of Y , then the pull-back $f^{-1}(U)$ is

an amenable subset of X . In fact, for every $x \in f^{-1}(U)$, we have a commutative diagram of groups

$$\begin{array}{ccc} \pi_1(f^{-1}(U), x) & \longrightarrow & \pi_1(U, f(x)) \\ \downarrow & & \downarrow \\ \pi_1(X, x) & \xrightarrow{\pi_1(f, x)} & \pi_1(Y, f(x)) \end{array}$$

whose vertical arrows are induced by inclusions. It follows that

$$\text{im}(\pi_1(f^{-1}(U) \hookrightarrow X, x))$$

is an extension of $\text{im}(\pi_1(U \hookrightarrow Y, f(x)))$ by a subgroup of $\ker(\pi_1(f, x))$. Since extensions of amenable groups is amenable, we conclude that $f^{-1}(U)$ is amenable in X .

6.2. Applications to Simplicial Volume

Via duality (Lemma 3.1), Theorem 6 implies the following vanishing result for the simplicial volume of manifolds with boundary.

COROLLARY 6.7. *Let M be an oriented compact connected triangulable manifold with non-empty boundary. Assume that the kernel of the morphism $\pi_1(\partial M \hookrightarrow M, x)$ is amenable for every $x \in \partial M$. Assume moreover that M admits an amenable open cover \mathcal{U} by path-connected open subsets satisfying conditions (RC1) and (RC2) (with $X = M$ and $A = \partial M$). If $\text{mult}(\mathcal{U}) \leq \dim(M)$, then $\|M, \partial M\| = 0$.*

We want to compare Corollary 6.7 with the main relative vanishing result available for simplicial volume, which is based on Gromov's vanishing theorem for non-compact manifolds [Gro82] [FM23, Corollary 11].

THEOREM 6.8 ([LMR22, Theorem 3.13]). *Let M be an oriented compact connected manifold with non-empty boundary that admits an amenable open cover \mathcal{U} with the following properties:*

- (1) *The multiplicity of \mathcal{U} is at most $\dim(M)$;*
- (2) *The multiplicity of $\mathcal{U}_{\partial M}$ is at most $\dim(\partial M)$;*
- (3) *The open cover $\mathcal{U}_{\partial M}$ of ∂M is amenable.*

Then $\|M, \partial M\| = 0$.

Since the previous result has no assumptions on the pair $(M, \partial M)$, it is of course more general than Corollary 6.7. However, by making further assumptions on the pair $(M, \partial M)$, one can give simplified proofs of Theorem 6.8 based on relative versions of Gromov's Vanishing Theorem. One such argument is presented in [LLM22, Section 6.4]. In the particular case where M is triangulable and the kernel of the morphism $\pi_1(\partial M \hookrightarrow M)$ is amenable, Theorem 6.8 also follows from Corollary 6.7. For the sake of completeness, we include the proof of this fact, which follows a classical strategy for modifying open covers while controlling the multiplicity [FM23, Theorem 11.8] [LS09, Theorem 5.3].

PROOF OF THEOREM 6.8. (FOR TRIANGULABLE MANIFOLDS WITH $\ker(\pi_1(\partial M \hookrightarrow M))$ AMENABLE). Let M be an oriented compact connected triangulable manifold with non-empty boundary such that the kernel of the morphism $\pi_1(\partial M \hookrightarrow M, x)$ is amenable for every $x \in \partial M$. Let \mathcal{U} be an amenable open cover of M such that

$\text{mult}(\mathcal{U}) \leq \dim(M)$ and $\text{mult}(\mathcal{U}_{\partial M}) \leq \dim(\partial M)$. Observe that under our assumption $\mathcal{U}_{\partial M}$ is automatically an amenable open cover of ∂M . We want to construct an amenable open cover \mathcal{U}' of M by path-connected subsets satisfying (RC1) and (RC2) on ∂M such that $\text{mult}(\mathcal{U}') \leq \dim(M)$. This implies that $\|M, \partial M\| = 0$ by Corollary 6.7.

By compactness and after possibly splitting connected components, we may assume that \mathcal{U} is finite and consists of path-connected subsets, say $\mathcal{U} = \{U_1, \dots, U_k\}$. Since ∂M admits a collar neighborhood in M , we may write

$$M \cong M' = M \cup (\partial M \times [0, 1])$$

so that $\partial M' = \partial M \times \{1\}$. Let $a_0 = 0$ and $a_{k+1} = 1$. We take $\varepsilon \in \mathbb{R}_{>0}$ and $a_1, \dots, a_k \in [0, 1]$ such that $a_{i-1}, a_{i+1} \notin (a_i - \varepsilon, a_i + \varepsilon)$ for every $i \in \{1, \dots, k\}$. Let $\mathcal{U}(i)$ be the set of path-connected components of $U_i \cap \partial M$. For every $i \in \{1, \dots, k\}$ such that $U_i \cap \partial M \neq \emptyset$, we set

$$U'_i = U_i \cup ((U_i \cap \partial M) \times [0, a_i + \varepsilon]) \subseteq M'$$

and

$$\mathcal{U}'(i) = \{U \times (a_i - \varepsilon, 1] \mid U \in \mathcal{U}(i)\}.$$

If $U_i \cap \partial M = \emptyset$, we just set $U'_i = U_i$ and $\mathcal{U}'(i) = \emptyset$. We obtain in this way a new open cover

$$\mathcal{U}' = \{U'_1, \dots, U'_k\} \cup \mathcal{U}'(1) \cup \dots \cup \mathcal{U}'(k)$$

of M' . One can check that \mathcal{U}' is an amenable open cover such that

$$\text{mult}(\mathcal{U}') \leq \max\{\text{mult}(\mathcal{U}), \text{mult}(\mathcal{U}_{\partial M}) + 1\},$$

hence $\text{mult}(\mathcal{U}') \leq \dim(M)$. Since only sets of the form $U \times (a_i - \varepsilon, 1]$ touch the boundary $\partial M'$, it is easy to check that both (RC1) and (RC2) are satisfied. It is shown in [LLM22, Section 6.4] that \mathcal{U}' is also weakly-convex on M' , therefore also Theorem 7(1) applies. \square

Another application of Theorem 6 is the following. Let $p: E \rightarrow B$ be a locally-trivial fiber bundle, where E and B are manifolds, and let F denote the fiber. As already mentioned in the introduction, it is still an open question whether the inequality $\|E\| \geq \|B\| \cdot \|F\|$ holds in full generality [HK01, Buc09]. We want to investigate under which conditions the vanishing of $\|E\|$ implies the vanishing of $\|B\|$. By the long exact sequence of homotopy groups of fibrations, we know that pair (E, F) satisfies the assumptions of Theorem 6, i.e. for every $x \in F$ the morphism $\pi_1(F \hookrightarrow E, x)$ has amenable kernel. Assume that p has a section s . If E admits an amenable open cover \mathcal{U} of multiplicity $\text{mult}(\mathcal{U}) \leq \dim(B)$ (which of course implies that $\|E\| = 0$ by Gromov's Vanishing Theorem), then the pull-back cover $\{s^{-1}(U) \mid U \in \mathcal{U}\}$ is an amenable open cover of B (Remark 6.6) and its multiplicity is at most $\text{mult}(\mathcal{U}) \leq \dim(B)$, hence $\|B\| = 0$. The following corollary shows that, under a cohomological condition which is weaker than having a section, one can get the same conclusion by requiring some regularity of \mathcal{U} over the fibers.

COROLLARY 6.9. *Let $p: E \rightarrow B$ be a smooth locally trivial fiber bundle with fiber F , where E , B and F are oriented closed connected smooth manifolds. Let $n = \dim(B) \geq 2$ and assume that*

$$H^n(p): H^n(B) \rightarrow H^n(E)$$

is injective (which occurs, for example, when p admits a section). Assume moreover that E admits an amenable open cover \mathcal{U} by path-connected open subsets satisfying (RC1) and (RC2) on F . If $\text{mult}(\mathcal{U}) \leq \dim(B)$, then $\|B\| = 0$.

PROOF. Let $b_0 \in B$ be a basepoint such that $F = p^{-1}(b_0)$. We consider the following commutative diagram

$$\begin{array}{ccccc} H_b^n(B) & \xrightarrow{\cong} & H_b^n(B, \{b_0\}) & \xrightarrow{H_b^n(p)} & H_b^n(E, F) \\ \downarrow \text{comp}_B^n & & \downarrow & & \downarrow \text{comp}^n \\ H^n(B) & \xrightarrow{\cong} & H^n(B, \{b_0\}) & \xrightarrow{H^n(p)} & H^n(E, F) \end{array}$$

where the vertical arrows are comparison maps. Under our assumptions, we know by Theorem 6 that $\text{comp}^n = 0$. Moreover, the long exact sequence of the pairs (E, F) and $(B, \{b_0\})$ implies that

$$H^n(p): H^n(B, \{b_0\}) \rightarrow H^n(E, F)$$

is injective if and only if $H^n(p): H^n(B) \rightarrow H^n(E)$ is injective. It follows that $\text{comp}_B^n = 0$, hence $\|B\| = 0$ by duality (Lemma 3.1). \square

6.3. Amenable subgroups of $\Pi(X, X)$

Before entering into the proofs of Theorem 6 and Theorem 7 we need some preliminary results. We refer the reader to Section 4.2 for the definition of the group $\Pi(X, X)$ and its action on the aspherical multicomplex $\mathcal{A}(X)$.

Amenable subsets identify amenable subgroups of $\Pi(X, X)$, which in turn act on the aspherical multicomplex $\mathcal{A}(X)$. If $V \subseteq U \subseteq X$, then the inclusion $(U, V) \hookrightarrow (X, X)$ induces a group homomorphism

$$\Pi(U, V) \rightarrow \Pi(X, X),$$

whose image is denoted by $\Pi_X(U, V)$.

LEMMA 6.10 ([FM23, Lemma 6.6]). *Let U be an amenable subset of X and let $V \subseteq U$ be any subset. Then the subgroup $\Pi_X(U, V) \leq \Pi(X, X)$ is amenable.*

PROOF. The exact sequence from Proposition 4.2 induces the exact sequence

$$1 \rightarrow \bigoplus_{x \in V} \text{im}(\pi_1(U \hookrightarrow X, x)) \rightarrow \Pi_X(U, V) \rightarrow \Sigma_{\text{fin}}(V)$$

where $\Sigma_{\text{fin}}(V)$ denotes the group of permutations of V with finite support (which is locally amenable, hence amenable). It follows that the group $\Pi_X(U, V)$ is an extension of an amenable group by an amenable group, hence it is amenable. \square

Let (X, A) be a triangulable pair, and let (T, S) be any pair of simplicial complexes such that $(X, A) = (|T|, |S|)$. Recall from Section 5.1 that we have a commutative diagram of simplicial maps

$$\begin{array}{ccccccc} \mathcal{K}(A) & \xleftarrow{i_A} & \mathcal{L}(A) & \xrightarrow{\pi} & \mathcal{A}(A) & \xrightarrow{q} & \mathcal{A}_X(A) \\ \downarrow j_{\mathcal{K}} & & \downarrow j_{\mathcal{L}} & & \downarrow j_{\mathcal{A}} & & \downarrow \\ \mathcal{K}(X) & \xleftarrow{i_X} & \mathcal{L}(X) & \xrightarrow{\pi} & \mathcal{A}(X) & \xrightarrow{\text{id}} & \mathcal{A}(X) \end{array}$$

where $j_{\mathcal{K}}$ is the inclusion of singular multicomplexes, i_X and i_A are the inclusions of minimal multicomplexes, $j_{\mathcal{L}}$ is the map described in Proposition 5.1, π is the

quotient map which identifies simplices sharing the same 1-skeleton and $j_{\mathcal{A}}$ is the map induced by $j_{\mathcal{L}}$ on the quotient. We know that $j_{\mathcal{L}}$ and $j_{\mathcal{A}}$ are not injective in general (Proposition 5.4 and Proposition 5.5). The multicomplex $\mathcal{A}_X(A)$ is the image of $j_{\mathcal{A}}$ inside $\mathcal{A}(X)$.

There is a natural way to construct a copy of T inside $\mathcal{A}(X)$ [FM23, Section 6.2]. Inside the singular multicomplex $\mathcal{K}(X)$ we can find a submulticomplex $\mathcal{K}_T(X) \cong T$ whose simplices correspond to equivalence classes of affine parametrizations of simplices of T . Moreover, we can assume that $\mathcal{K}_T(X) \subseteq \mathcal{L}(X)$, by choosing simplices of $\mathcal{K}_T(X)$ as representatives in their homotopy class. Since T is a simplicial complex, every simplex in $\mathcal{K}_T(X)$ is uniquely determined by its 0-skeleton, hence the quotient map $\pi: \mathcal{L}(X) \rightarrow \mathcal{A}(X)$ is injective on $\mathcal{K}_T(X)$. Therefore we can construct a copy of T inside $\mathcal{A}(X)$. The same construction applies to S , so that $\mathcal{K}_S(A) \subseteq \mathcal{L}(A)$. As before, both $\pi: \mathcal{L}(A) \rightarrow \mathcal{A}(A)$ and $q: \mathcal{A}(A) \rightarrow \mathcal{A}_X(A)$ are injective on $\mathcal{K}_S(A)$. It follows that S sits inside $\mathcal{A}_X(A) \cap T$. Moreover, the choices which ensure that $\mathcal{K}_T(X) \subseteq \mathcal{L}(X)$ and $\mathcal{K}_S(A) \subseteq \mathcal{L}(A)$ are clearly compatible with the ones described in Setup 5.2, so that both the squares of the following diagram are commutative

$$(6.1) \quad \begin{array}{ccccc} \mathcal{K}_S(A) & \hookrightarrow & \mathcal{L}(A) & \xleftarrow{i_A} & \mathcal{K}(A) \\ \downarrow & & \downarrow j_{\mathcal{L}} & & \downarrow j_{\mathcal{K}} \\ \mathcal{K}_T(X) & \hookrightarrow & \mathcal{L}(X) & \xleftarrow{i_A} & \mathcal{K}(X). \end{array}$$

REMARK 6.11. Since $\mathcal{A}(X)$ is a model for the classifying space of the fundamental group of X , the inclusion $T \cong \mathcal{K}_T(X) \hookrightarrow \mathcal{A}(X)$ gives an explicit classifying map when X is triangulable.

We want to invoke Theorem 5 to get an isomorphism of vector spaces

$$\Psi^n: H_b^n(\mathcal{A}(X), \mathcal{A}_X(A)) \rightarrow H_b^n(X, A).$$

To this end, it is crucial that the kernel of the map $\pi_1(A \hookrightarrow X)$ is amenable. This is the only step in our proof in which we use this assumption.

LEMMA 6.12. *Let (X, A) be a triangulable pair, and let (T, S) be any pair of simplicial complexes such that $(X, A) = (|T|, |S|)$. Assume that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Let $z \in C_b^n(\mathcal{A}(X), \mathcal{A}_X(A))$ be a bounded cocycle which vanishes on $C_n(T) \subseteq C_n(\mathcal{A}(X))$. Then $\text{comp}^n \circ \Psi^n([z]) = 0$ in $H^n(X, A)$.*

PROOF. The inclusion of pairs $(T, S) \subseteq (\mathcal{A}(X), \mathcal{A}_X(A))$ induces a restriction map $r_{\mathcal{A}}^n: C_b^n(\mathcal{A}(X), \mathcal{A}_X(A)) \rightarrow C_b^n(T, S)$. Similarly, the leftmost square of diagram (6.1) induces a restriction map $r_{\mathcal{L}}^n: C_b^n(\mathcal{L}(A) \rightarrow \mathcal{L}(X)) \rightarrow C_b^n(S \hookrightarrow T)$ on mapping cones. We consider the following diagram

$$\begin{array}{ccccccc} C_b^n(\mathcal{A}(X), \mathcal{A}_X(A)) & \xrightarrow{r_{\mathcal{A}}^n} & C_b^n(T, S) & \hookrightarrow & C^n(T, S) & \xlongequal{\quad} & C^n(T, S) \\ \downarrow \zeta^n & & \downarrow \beta^n & & \downarrow \beta^n & & \uparrow \varphi^n \\ C_b^n(\mathcal{L}(A) \rightarrow \mathcal{L}(X)) & \xrightarrow{r_{\mathcal{L}}^n} & C_b^n(S \hookrightarrow T) & \hookrightarrow & C^n(S \hookrightarrow T) & & \\ \uparrow \vartheta^n & & & & & & \\ C_b^n(X, A) & \xrightarrow{\quad} & & & & & C^n(X, A). \end{array}$$

Here hooked arrows denote inclusion of chains, which induce comparison maps in cohomology. The maps ζ^n and ϑ^n , introduced in the proof of Theorem 5, induce isomorphisms in cohomology whose composition defines Ψ^n . The maps β^n , defined in Lemma 5.12, and the map φ^n , induced by the chain inclusion $C_n(T) \rightarrow C_n(|T|) = C_n(X)$, also induce isomorphisms in cohomology, by Lemma 5.12 and Proposition 1.8 respectively. Since z vanishes on $C_n(T)$, we have that $r_{\mathcal{A}}^n(z) = 0$. Since the diagram is commutative and every vertical arrow induces an isomorphism in cohomology, the statement easily follows. \square

Let $\mathcal{U} = \{U_i \mid i \in I\}$ be a locally finite amenable open cover of X by path-connected subsets. We denote by V the set of vertices of T . For every vertex $v \in V$, the *closed star* of v in T is the subcomplex of T containing all the simplices having v as a vertex. By suitably subdividing T , we can assume that, for every $v \in V$, there exists $i(v) \in I$ such that the closed star of v in T is contained in $U_{i(v)}$ [Mun84, Theorem 16.4]. Of course the choice of $i(v)$ may not be unique. For every $i \in I$, we set $V_i = \{v \in V(T) \mid i(v) = i\}$. We consider the group

$$G = \bigoplus_{i \in I} \Pi_X(U_i, V_i).$$

Since U_i is amenable in X , then $\Pi_X(U_i, V_i)$ is an amenable group by Lemma 6.10. The direct sum of amenable groups is amenable, hence also G is amenable. Moreover, since $V_i \cap V_j = \emptyset$ for every $i \neq j$, elements of $\Pi_X(U_i, V_i)$ commute with elements of $\Pi_X(U_j, V_j)$. It follows that there is a well-defined map from G to $\Pi(X, X)$. This map is injective because the intersection $\Pi_X(U_i, V_i) \cap \langle \Pi_X(U_j, V_j) : j \neq i \rangle$ is trivial for every $i \in I$. In fact, non-trivial elements of the former group are represented by paths with endpoints in V_i , while non-trivial elements of the latter by paths with endpoints in $\bigcup_{j \neq i} V_j$, which is disjoint from V_i . Therefore G can be identified with a subgroup of $\Pi(X, X)$ and acts on $\mathcal{A}(X)$ accordingly. Notice that this action does not preserve the submulticomplex $\mathcal{A}_X(A)$. The group

$$H = \bigoplus_{i \in I} \Pi_X(U_i \cap A, V_i \cap A)$$

identifies a subgroup of G inducing an action on the pair $(\mathcal{A}(X), \mathcal{A}_X(A))$. Being a subgroup of an amenable group, H is amenable. The regularity conditions (RC1) and (RC2) are used in our proof via the following lemma.

LEMMA 6.13. *Let (X, A) be a pair of topological spaces and let \mathcal{U} be an cover of X by path-connected open subsets. Then \mathcal{U} satisfies the following conditions:*

- (RC1) *For every $U \in \mathcal{U}$ such that $U \cap A \neq \emptyset$, $U \cap A$ is path-connected;*
- (RC2) *For every $U \in \mathcal{U}$ such that $U \cap A \neq \emptyset$, the inclusion*

$$\text{im}(\pi_1(U \cap A, x) \rightarrow \pi_1(X, x)) \hookrightarrow \text{im}(\pi_1(U, x) \rightarrow \pi_1(X, x))$$

is an isomorphism for every $x \in U \cap A$;

if and only if, for every $U \in \mathcal{U}$ and for every path $\gamma: [0, 1] \rightarrow U$ with endpoints in $U \cap A$, there exists a path $\lambda: [0, 1] \rightarrow U \cap A$ such that λ is homotopic in X to γ relative to the endpoints.

PROOF. Assume that \mathcal{U} satisfies (RC1) and (RC2). Given $U \in \mathcal{U}$ and γ as above, by (RC1) there exists a path $\varepsilon: [0, 1] \rightarrow U \cap A$ from $\gamma(0)$ to $\gamma(1)$. The homotopy class of the loop $\gamma * \bar{\varepsilon}$ identifies an element of $\text{im}(\pi_1(U \hookrightarrow X, \gamma(0)))$,

hence, by (RC2), there exists a loop $\delta: [0, 1] \rightarrow U \cap A$ which is homotopic in X to $\gamma * \bar{\varepsilon}$ relative to the endpoints. Therefore, we can set $\lambda = \delta * \varepsilon$. The converse implication is straightforward. \square

The regularity conditions (RC1) and (RC2) satisfied by the open cover are crucial to show that the action $G \curvearrowright \mathcal{A}(X)$ has orbits in $\mathcal{A}_X(A)$ induced by H .

LEMMA 6.14. *Assume that \mathcal{U} satisfies (RC1) and (RC2). Then the action $G \curvearrowright \mathcal{A}(X)$ has orbits in $\mathcal{A}_X(A)$ induced by H .*

PROOF. Let $\sigma = (\Delta, (x_0, \dots, x_n))$ be an algebraic n -simplex of $\mathcal{A}_X(A)$ and let $g \in G$ be such that $g \cdot \sigma$ is an algebraic n -simplex of $\mathcal{A}_X(A)$. We need to show that there exists an element $h \in H$ such that $h \cdot \sigma = g \cdot \sigma$. Let $\{v_0, \dots, v_k\}$ be the vertices of Δ , where $k \leq n$, and let $(\gamma_x)_{x \in X}$ be any representative of g . Since v_j and $g \cdot v_j$ are both vertices of $\mathcal{A}_X(A)$ (hence points of A), by the definition of the action $G \curvearrowright \mathcal{A}(X)$, we have that γ_{v_j} has both endpoints in A . Moreover, by the definition of G , we have that γ_{v_j} is supported in some element of \mathcal{U} . Hence there are $i_0, \dots, i_k \in I$ such that $\gamma_{v_j}: [0, 1] \rightarrow U_{i_j}$ has both endpoints in $U_{i_j} \cap A$. Since the open cover \mathcal{U} satisfies (RC1) and (RC2), by Lemma 6.13 there exist paths $\lambda_{v_j}: [0, 1] \rightarrow U_{i_j} \cap A$, $j \in \{0, \dots, k\}$, such that λ_{v_j} is homotopic to γ_{v_j} in X relative to the endpoints. Therefore, if we set $h_j = \{\lambda_j\}$ if λ_j is a loop, and $h_j = \{\lambda_j, \bar{\lambda}_j\}$ otherwise, then $h = \bigoplus_{j=0}^k h_j$ defines an element of H . It is easy to check that $h \cdot e = g \cdot e$, for every edge e of Δ , hence $g \cdot \Delta^1 = h \cdot \Delta^1$, where Δ^1 denotes the 1-skeleton of Δ . Since $\mathcal{A}(X)$ is an aspherical multicomplex, this is indeed sufficient to conclude (Proposition 2.24). \square

6.4. Proof of Relative Vanishing Theorems

We proceed now with the proof of Theorem 6 and Theorem 7. Let (X, A) be a triangulable pair such that the kernel of the morphism $\pi_1(A \hookrightarrow X, x)$ is amenable for every $x \in A$. Let \mathcal{U} be an amenable open cover of X by path-connected subsets satisfying (RC1) and (RC2). We may assume without loss of generality that \mathcal{U} is locally-finite: this is explicitly required in Theorem 7, while the conclusion of Theorem 6 is empty if \mathcal{U} is not locally-finite (since in that case $\text{mult}(\mathcal{U})$ is infinite). We keep the notation from the previous sections.

PROOF OF THEOREM 6. Let $n \geq \text{mult}(\mathcal{U})$ and let $\alpha \in H_b^n(X, A)$. We want to show that $\text{comp}^n(\alpha) = 0$. Since Ψ^n is an isomorphism, we can take a bounded co-class $\beta \in H_b^n(\mathcal{A}(X), \mathcal{A}_X(A))$ such that $\Psi^n(\beta) = \alpha$. Let now $z \in C_b^n(\mathcal{A}(X), \mathcal{A}_X(A))$ be a cocycle representing β . By Proposition 1.9, we can assume that z is alternating. The actions $G \curvearrowright \mathcal{A}(X)$ and $H \curvearrowright \mathcal{A}_X(A)$ are both by amenable groups and by automorphisms which are simplicially homotopic to the identity (Proposition 4.4). It follows from these two facts that the inclusion maps $C_b^\bullet(\mathcal{A}(X))^G \hookrightarrow C_b^\bullet(\mathcal{A}(X))$ and $C_b^\bullet(\mathcal{A}_X(A))^H \hookrightarrow C_b^\bullet(\mathcal{A}_X(A))$ induce isomorphisms in cohomology (Proposition 3.15). Moreover, since the action $G \curvearrowright \mathcal{A}(X)$ has orbits in $\mathcal{A}_X(A)$ induced by H , by Lemma 5.20 we can assume that z is alternating and G -invariant. By Lemma 6.12, in order to show that $\text{comp}^n(\alpha) = 0$, it is enough to show that z vanishes on every algebraic n -simplex of $C_n(T) \subseteq C_n(\mathcal{A}(X))$.

Let $(\Delta, (x_0, \dots, x_n))$ be an algebraic n -simplex in $C_n(T) \subseteq C_n(\mathcal{A}(X))$. If $x_h = x_k$, for some $h \neq k$, then $z(\Delta, (x_0, \dots, x_n)) = 0$, since z is alternating. We assume therefore that the points x_0, \dots, x_n are pairwise distinct. By construction, the cover

of X by the closed stars of vertices of T is a refinement of \mathcal{U} . Since Δ is an n -simplex of T and since $n \geq \text{mult}(\mathcal{U})$, there are at least two vertices of Δ belonging to the same V_i . In other words, there exist $h \neq k \in \{0, \dots, n\}$ such that $i(x_h) = i(x_k)$. Let e denote the edge of Δ joining x_h with x_k . By assumption, the closed stars of x_k and x_h are both contained in U_i , therefore the edge e of $\mathcal{A}(X)$ (which is also an edge of $\mathcal{L}(X)^1 = \mathcal{A}(X)^1$, hence of $\mathcal{K}(X)$) projects via $S_X: |\mathcal{K}(X)| \rightarrow X$ to a path $\gamma: [0, 1] \rightarrow U_i$ with endpoints x_h and x_k . As a consequence, if we consider $g = \{\gamma, \bar{\gamma}\} \in \Pi_X(U_i, V_i) < G$, it is easy to check that $g \cdot \Delta = \Delta$, $g \cdot x_h = x_k$, $g \cdot x_k = x_h$ and $g \cdot x_j = x_j$, for every $j \neq h, k$. Since z is G -invariant, we obtain

$$z(\Delta, (x_0, \dots, x_n)) = z(g \cdot (\Delta, (x_0, \dots, x_n))),$$

while, since z is alternating, we have

$$z(\Delta, (x_0, \dots, x_n)) = -z(g \cdot (\Delta, (x_0, \dots, x_n))).$$

Therefore $z(s, (x_0, \dots, x_n)) = 0$ and this concludes the proof of Theorem 6. \square

REMARK 6.15. We discuss the role of conditions (RC1) and (RC2) in our context. In fact, in light of Lemma 6.14, the only reason to introduce these hypothesis is that, under these regularity assumptions, one can show that the action of G on $\mathcal{A}(X)$ has orbits in $\mathcal{A}_X(A)$ induced by H . This in turns allows to consider G -invariant cochains in the relative setting (Lemma 5.20). In fact, since the action of G on $\mathcal{A}(X)$ does not preserve $\mathcal{A}_X(A)$, some care is needed when passing to invariant cochains. Assume, on the contrary, that one could work with G -invariant chains regardless of any regularity condition of the action of G on $\mathcal{A}_X(A)$. This assumptions seem implicit in Kuessner's work on relative bounded cohomology via multicomplexes (see for example [Kue15, Corollary 2]). In this case, using the same argument above, one could prove that the comparison map $\text{comp}^n: H_b^n(X, A) \rightarrow H^n(X, A)$ vanishes for every $n \geq \text{mult}(\mathcal{U})$, where \mathcal{U} is *any* amenable open cover of X . This is obviously false: if $(X, A) = (M, \partial M)$ for some smooth m -manifold M with non-empty boundary, then we know that M is homotopy equivalent a subcomplex of dimension at most $m - 1$, therefore it admits a contractible (hence amenable) open cover of cardinality at most m [CLM22, Remark 2.8]; by duality (Lemma 3.1), it would follow that $\|M, \partial M\| = 0$. This gives a contradiction, for example, if the interior of M admits a complete finite-volume hyperbolic metric [Gro82].

PROOF OF THEOREM 7(1). We assume that \mathcal{U} is *weakly-convex* on A . Using the fact that \mathcal{U} is locally finite, up to subdividing T , we may suppose that $V_i \cap A \neq \emptyset$ for every $i \in I$ such that $U_i \cap A \neq \emptyset$. Let $n \geq \text{mult}_A(\mathcal{U})$. We need to show that the comparison map $\text{comp}^n: H_b^n(X, A) \rightarrow H^n(X, A)$ vanishes. Let $\alpha \in H_b^n(X, A)$. As above, we can take a bounded coclass $\beta \in H_b^n(\mathcal{A}(X), \mathcal{A}_X(A))$ such that $\Psi^n(\beta) = \alpha$. Let $z \in C_b^n(\mathcal{A}(X), \mathcal{A}_X(A))$ be a cocycle representing β . As in the proof of Theorem 6, we can assume that z is alternating and G -invariant. By Lemma 6.12, we need to show that z vanishes on every algebraic n -simplex of $C_n(T) \subseteq C_n(\mathcal{A}(X))$.

Let $(\Delta, (x_0, \dots, x_n))$ be an algebraic n -simplex in $C_n(T)$. Since z is alternating, we can assume that the points x_0, \dots, x_n are pairwise-distinct. We set $i_j = i(x_j)$ for every $j \in \{0, \dots, n\}$. If there exist $h \neq k \in \{0, \dots, n\}$ such that $i_h = i_k$, then we can argue as in the previous section to show that $z(\Delta, (x_0, \dots, x_n)) = 0$. We assume therefore that i_0, \dots, i_n are pairwise-distinct.

Since $n \geq \text{mult}_A(\mathcal{U})$ and since by construction $\bigcap_{j=0}^n U_{i_j} \neq \emptyset$, we deduce that $U_{i_0} \cap \cdots \cap U_{i_n} \cap A$ is nonempty. Hence, since \mathcal{U} is weakly-convex on A , the path-connected component of $U_{i_0} \cap \cdots \cap U_{i_n}$ containing Δ intersects A . Therefore, if x denotes the barycenter of Δ , there exists $x' \in U_{i_0} \cap \cdots \cap U_{i_n} \cap A$ and a path

$$\lambda: [0, 1] \rightarrow U_{i_0} \cap \cdots \cap U_{i_n}$$

from x' to x . Let e_{hk} denote the oriented 1-simplex of Δ from x_h to x_k . We denote by

$$\delta_j: [0, 1] \rightarrow U_{i_0} \cap \cdots \cap U_{i_n}$$

a path from x_j to x supported in Δ such that $\delta_h * \bar{\delta}_k$ is homotopic in X relative to the endpoints to a parametrization of e_{hk} . Since $V_{i_j} \cap A \neq \emptyset$, for every $j \in \{0, \dots, n\}$ there exists $x'_j \in V_{i_j} \cap A$. Moreover, by (RC1), $U_{i_j} \cap A$ is path-connected, hence we can find continuous paths

$$\xi_j: [0, 1] \rightarrow U_{i_j} \cap A$$

from x'_j to x' . For $0 \leq h < k \leq n$, we then set $\xi_{hk} = \xi_h * \bar{\xi}_k: [0, 1] \rightarrow A$. Since x'_0, \dots, x'_n are pairwise-distinct, by definition of $\mathcal{A}_X(A)$, there exists a unique oriented 1-simplex e'_{hk} of $\mathcal{A}_X(A)$ whose projection is a path in A which is homotopic to ξ_{hk} in X relative to the endpoints.

LEMMA 6.16. *There exists a simplex Δ' of $\mathcal{A}_X(A)$ whose 1-skeleton is given by the union of the e'_{hk} .*

PROOF. Notice that the e'_{hk} can be considered to be 1-simplices of $\mathcal{A}(A)$. Of course, the loop $\xi_{jh} * \xi_{hk} * \bar{\xi}_{jk}$ is null-homotopic in A (hence in X). By Proposition 2.26 the concatenation of oriented simplices $e'_{jh} * e'_{hk} * \bar{e}'_{jk}$ is null-homotopic in $|\mathcal{A}(A)|$. By Proposition 2.24, it follows that there exists a unique n -simplex Δ'' of $\mathcal{A}(A)$ whose 1-skeleton is the union of the e'_{hk} . In conclusion, Δ' denotes just the image of Δ'' in $\mathcal{A}_X(A)$. \square

For every $j \in \{0, \dots, n\}$ we set $\gamma_j = \xi_j * \lambda * \bar{\delta}_j$. By construction γ_j joins x'_j with x_j and is supported on U_{i_j} . Therefore, if we set $g_j = \{\gamma_j\}$ if $x_j = x'_j$, and $g_j = \{\gamma_j, \bar{\gamma}_j\}$ if $x_j \neq x'_j$, then $g = \bigoplus_{j=0}^n g_j$ defines an element of G . It is clear that $g \cdot e'_{hk} = e_{hk}$. Since simplices of $\mathcal{A}(X)$ are determined by their 1-skeleton (Proposition 2.24), we obtain $g \cdot \Delta' = \Delta$. Hence we have that $z(\Delta, (x_0, \dots, x_n)) = z(\Delta', (x'_0, \dots, x'_n))$, since z is G -invariant, and $z(\Delta', (x'_0, \dots, x'_n)) = 0$, since z vanishes on simplices of $\mathcal{A}_X(A)$. This concludes the proof of 7(1). \square

PROOF OF THEOREM 7(2). Assume that \mathcal{U} is *convex*. We denote by $N(\mathcal{U})$ the nerve of \mathcal{U} and by $N(\mathcal{U}_A)$ the nerve of the open cover of A induced by \mathcal{U} . This is a well-defined subcomplex of $N(\mathcal{U})$ under our assumptions. Given a partition of unity subordinated to \mathcal{U} , one can construct a nerve map $\nu: X \rightarrow |N(\mathcal{U})|$, which is unique up to homotopy and which induces a well-defined map of pairs $\nu: (X, A) \rightarrow (|N(\mathcal{U})|, |N(\mathcal{U}_A)|)$. By assumption, we know that \mathcal{U} is locally-finite. Therefore, up to taking further subdivisions of T , we may suppose that $V_i \neq \emptyset$, for every $i \in I$, and $V_i \cap A \neq \emptyset$ for every $i \in I$ such that $U_i \cap A \neq \emptyset$. We need to show that there exists a map $\Theta^n: H_b^n(X, A) \rightarrow H^n(N(\mathcal{U}), N(\mathcal{U}_A))$ such that the following diagram

is commutative:

$$(6.2) \quad \begin{array}{ccc} H_b^n(X, A) & \xrightarrow{\text{comp}^n} & H^n(X, A) \\ \downarrow \Theta^n & & \uparrow H^n(\nu) \\ H^n(N(\mathcal{U}), N(\mathcal{U}_A)) & \xrightarrow{\cong} & H^n(|N(\mathcal{U})|, |N(\mathcal{U}_A)|). \end{array}$$

Recall from the previous sections that the relative bounded cohomology of $(\mathcal{A}(X), \mathcal{A}_X(A))$ may be computed by the complex $C_b^\bullet(\mathcal{A}(X), \mathcal{A}_X(A))_{\text{alt}}^G$ of G -invariant alternating cochains which vanish on simplices supported on $\mathcal{A}_X(A)$ (see Lemma 5.20 and Lemma 6.14). Therefore, in order to define Θ^n , we first construct chain maps Ω_X^\bullet and Ω_A^\bullet such that the following diagram is commutative

$$(6.3) \quad \begin{array}{ccc} C_b^\bullet(\mathcal{A}(X))_{\text{alt}}^G & \longrightarrow & C_b^\bullet(\mathcal{A}_X(A))_{\text{alt}}^H \\ \downarrow \Omega_X^\bullet & & \downarrow \Omega_A^\bullet \\ C_b^\bullet(N(\mathcal{U})) & \longrightarrow & C_b^\bullet(N(\mathcal{U}_A)), \end{array}$$

where the horizontal arrows are restriction maps. The map Ω_X^n can be constructed as follows.

Let $z \in C_b^n(\mathcal{A}(X))_{\text{alt}}^G$ and let $(i_0, \dots, i_n) \in N(\mathcal{U})$. If there exists $h \neq k \in \{0, \dots, n\}$ such that $i_h = i_k$, we set $\Omega_X^n(z)(i_0, \dots, i_n) = 0$. Otherwise, by definition of the nerve $N(\mathcal{U})$, we have that $U_{i_0} \cap \dots \cap U_{i_n} \neq \emptyset$, hence we may choose a point $q \in U_{i_0} \cap \dots \cap U_{i_n}$. Moreover, since $V_i \neq \emptyset$ for every $i \in I$, we can pick a point $v_{i_j} \in V_{i_j}$. Since U_{i_j} is path-connected, there exists a path $\alpha_j: [0, 1] \rightarrow U_{i_j}$ from v_{i_j} to q . For $0 \leq h < k \leq n$, we set $\alpha_{hk} = \alpha_h * \bar{\alpha}_k$. Since v_{i_0}, \dots, v_{i_n} are pairwise distinct, there exists a unique oriented 1-simplex e_{hk} of $\mathcal{A}(X)$ whose projection on X is a path which is homotopic to α_{hk} in X relative to the endpoints. Therefore, by Proposition 2.24 there exists a unique n -simplex Δ of $\mathcal{A}(X)$ whose 1-skeleton is given by the union of the e_{hk} . We then set $\Omega_A^n(z)(i_0, \dots, i_n) = z(\Delta, (v_{i_0}, \dots, v_{i_n}))$. One can check that Ω_X^n is well-defined *verbatim* as in [FM23, Section 6.4], using the fact that z is G -invariant. A similar argument can be found in the following to show that Ω_A^n is well-defined.

The construction of Ω_A^n goes as follows. We identify the set of vertices of $N(\mathcal{U}_A)$ with those indices $i \in I$ such that $U_i \cap A \neq \emptyset$. Let $z \in C_b^n(\mathcal{A}_X(A))_{\text{alt}}^H$ and let $(i_0, \dots, i_n) \in N(\mathcal{U}_A)$. If there exists $h \neq k \in \{0, \dots, n\}$ such that $i_h = i_k$, we set $\Omega_A^n(z)(i_0, \dots, i_n) = 0$. Otherwise, by definition of the nerve $N(\mathcal{U}_A)$, we have that $U_{i_0} \cap \dots \cap U_{i_n} \cap A \neq \emptyset$, hence we may choose a point $q \in U_{i_0} \cap \dots \cap U_{i_n} \cap A$. Moreover, since $V_{i_j} \cap A \neq \emptyset$ for every $j \in \{0, \dots, n\}$, we can pick a point $v_{i_j} \in V_{i_j} \cap A$. We know by (RC1) that $U_{i_j} \cap A$ is path-connected. Hence there exists a path $\alpha_j: [0, 1] \rightarrow U_{i_j} \cap A$ from v_{i_j} to q . For $0 \leq h < k \leq n$, we set $\alpha_{hk} = \alpha_h * \bar{\alpha}_k$. Since v_{i_0}, \dots, v_{i_n} are pairwise distinct, by definition of $\mathcal{A}_X(A)$, there exists a unique oriented 1-simplex e_{hk} of $\mathcal{A}_X(A)$ whose projection on A is a path which is homotopic to α_{hk} in X relative to the endpoints. Using the same argument of Lemma 6.16, there exists a unique n -simplex Δ of $\mathcal{A}_X(A)$ whose 1-skeleton is given by the union of the e_{hk} . We then set $\Omega_A^n(z)(i_0, \dots, i_n) = z(\Delta, (v_{i_0}, \dots, v_{i_n}))$.

We need to show that $\Omega_A^n(z)$ is well-defined, i.e. different choices in the construction lead to the same value for $z(\Delta, (v_{i_0}, \dots, v_{i_n}))$. Let $q' \in U_{i_0} \cap \dots \cap U_{i_n} \cap A$, let $v'_{i_j} \in V_{i_j} \cap A$, $j \in \{0, \dots, n\}$, and let $\alpha'_j: [0, 1] \rightarrow U_{i_j} \cap A$ be a path from v'_{i_j} to

q' . We set $\alpha'_{hk} = \alpha'_h * \bar{\alpha}'_k$ and we take e'_{hk} to be the unique edge of $\mathcal{A}_X(A)$ whose projection is a path in A which is homotopic to α'_{hk} in X relative to the endpoints. Finally, we take Δ' to be the n -simplex of $\mathcal{A}_X(A)$ whose 1-skeleton is given by the e'_{hk} . Since $U_{i_0} \cap \cdots \cap U_{i_n}$ is path-connected (by convexity of \mathcal{U}), there exists a path $\beta: [0, 1] \rightarrow \bigcap_{j=0}^n U_{i_j}$ from q to q' . By construction, the path γ_j , defined by

$$\gamma_j = \alpha_j * \beta * \bar{\alpha}_j,$$

is supported on U_{i_j} and joins v_{i_j} with v'_{i_j} , which are both points of A . By the regularity conditions (RC1) and (RC2), we deduce from Lemma 6.13 that, for every $j \in \{0, \dots, n\}$, there exists a path $\lambda_j: [0, 1] \rightarrow U_{i_j} \cap A$ which is homotopic to γ_j in X relative to the endpoints. Therefore, if we set $h_j = \{\lambda_j\}$ if λ_j is a loop, and $h_j = \{\lambda_j, \bar{\lambda}_j\}$ otherwise, then $h = \bigoplus_{j=0}^n h_j$ defines an element of H . It is straightforward to check that $h \cdot e'_{hk} = e_{hk}$ for every $h \neq k$, which implies that $h \cdot \Delta' = \Delta$. Since z is H -invariant, it follows that $z(\Delta, (v_{i_0}, \dots, v_{i_n})) = z(\Delta', (v'_{i_0}, \dots, v'_{i_n}))$, i.e. that Ω_A^n is indeed well-defined.

It is easy to check that (6.3) is commutative and that Ω_A^\bullet defines a chain map. From (6.3), we get a chain map

$$\Omega^\bullet: C_b^\bullet(\mathcal{A}(X), \mathcal{A}_X(A))_{\text{alt}}^G \rightarrow C^\bullet(N(\mathcal{U}), N(\mathcal{U}_A)),$$

whose induced map in cohomology, precomposed with the inverse of Ψ^n , defines Θ^n . The commutativity of (6.2) can be checked *verbatim* as in the absolute case (see [FM23, Section 6.4]). This concludes the proof of Theorem 7(2). \square

Additivity of Simplicial Volume

Gromov Additivity Theorem can be used to compute the simplicial volume of manifolds that are obtained by gluing smaller manifolds along (parts of) their boundaries. Originally stated in [Gro82], this result has been addressed independently in [BBF⁺14] and [Kue15], following two fundamentally different approaches. In [BBF⁺14], the authors derive the additivity of simplicial volume using homological algebra techniques developed by Ivanov in [Iva85], while in [Kue15], the proof is based on the theory of multicomplexes. In the approach developed by Kuessner in [Kue15], however, some problems arise especially in the case when the manifolds involved are not aspherical (Section 5.4) and the gluings are performed along proper submanifolds of the boundary (Remark 5.21).

In this chapter, we build on the framework developed in the previous chapters to provide a complete and self-contained proof of Gromov Additivity Theorems (see Theorems 8 and 10), as announced by the author in [Cap24]. Finally, in Section 7.6, we present a result obtained in collaboration with Kevin Li and Clara Löh [CLL25], which establishes vanishing criteria for the simplicial volume of manifolds glued along boundary components that are not necessarily amenable.

7.1. Statement of the Main Results

Let M_1, \dots, M_k be oriented compact n -dimensional manifolds, and let

$$(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$$

be a pairing of some oriented compact connected pairwise disjoint submanifolds of $\sqcup_{j=1}^k \partial M_j$ of codimension 0. In particular, S_i^\pm are compact manifolds of dimension $n - 1$ (with possibly empty boundary) for every $i \in \{1, \dots, h\}$. We say that M is obtained by *gluing* M_1, \dots, M_k *along the pairing* if there exist orientation-reversing homeomorphisms $f_i: S_i^+ \rightarrow S_i^-$ such that M is obtained by taking the quotient of the disjoint union $M_1 \sqcup \dots \sqcup M_k$ by the equivalence relation generated by $x \sim f_i(x)$ for every $x \in S_i^\pm$. The constants C_n appearing in the following statement are described in Definition 2.8.

THEOREM 8. *Let M_1, \dots, M_k be oriented compact triangulable n -manifolds with π_1 -injective boundary, $n \in \mathbb{N}_{\geq 2}$. Let*

$$(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$$

be a pairing of some oriented compact connected pairwise-disjoint submanifolds of $\sqcup_{j=1}^k \partial M_j$ of codimension 0. Let M be a manifold obtained by gluing M_1, \dots, M_k along the pairing. Assume that the following conditions hold for every $i \in \{1, \dots, h\}$ and every $j \in \{1, \dots, k\}$:

- (i) S_i^\pm has an amenable fundamental group;

(ii) ∂S_i^\pm is path-connected, π_1 -injective in the corresponding ∂M_j and the map $\pi_1(\partial S_i^\pm \hookrightarrow S_i^\pm)$ is an isomorphism.

Then there exists a constant $C_n \in \mathbb{N}_{\geq 1}$ (depending only on the dimension) such that

$$(2C_n)^3(n+2)^3 \cdot \|M, \partial M\| \geq \|M_1, \partial M_1\| + \cdots + \|M_k, \partial M_k\|.$$

The proof of Theorem 8 is presented in Section 7.2. The various hypotheses of Theorem 8 are verified for example by boundary connected sums.

COROLLARY 9. *Let M_1 and M_2 be oriented compact triangulable n -manifolds with π_1 -injective boundaries, $n \in \mathbb{N}_{\geq 4}$. Let $M = M_1 \natural M_2$ denote the boundary connected sum of M_1 and M_2 . Then*

$$(2C_n)^3(n+2)^3 \cdot \|M, \partial M\| \geq \|M_1, \partial M_1\| + \|M_2, \partial M_2\|.$$

REMARK 7.1. The boundary connected sum $M = M_1 \natural M_2$ of M_1 and M_2 can be constructed by attaching a 1-handle to the disjoint union $N = M_1 \sqcup M_2$. More precisely, M is obtained from N by attaching a 1-handle $\mathbb{D}^1 \times \mathbb{D}^{n-1}$ along some homeomorphism $\mathbb{S}^0 \times \mathbb{D}^{n-1} \rightarrow \partial N$, where the two boundary components of $\mathbb{S}^0 \times \mathbb{D}^{n-1}$ are mapped to ∂M_1 and ∂M_2 , respectively. Naturally, Corollary 9 extends more generally to arbitrary 1-handle attachments.

For boundary connected sums with $n = 3$ we do not expect more than super-additivity: the boundary connected sum of two solid tori is homeomorphic to a handlebody of genus 2. In this case, the glued manifold has positive simplicial volume [BFP15], while the solid tori have vanishing simplicial volume (because they admit maps of high degree). The situation is different for $n \in \mathbb{N}_{\geq 4}$ when the boundary components of the pieces have amenable fundamental group (see Corollary 12 below).

When gluings are performed along entire boundary components, one of the most widely applicable versions of Gromov Additivity Theorem is the one established in [BBF⁺14].

THEOREM 7.2 ([BBF⁺14, Theorem 3]). *Let M_1, \dots, M_k be oriented compact n -manifolds, $n \in \mathbb{N}_{\geq 2}$. Assume that the fundamental group of every boundary component of M_j is amenable, and let M be a manifold obtained by gluing M_1, \dots, M_k along (some of) their boundary components. Then*

$$\|M, \partial M\| \leq \|M_1, \partial M_1\| + \cdots + \|M_k, \partial M_k\|.$$

Moreover, if the gluings are performed along π_1 -injective boundary components, then

$$\|M, \partial M\| = \|M_1, \partial M_1\| + \cdots + \|M_k, \partial M_k\|.$$

An immediate consequence of Theorem 7.2 is that simplicial volume is additive with respect to connected sums. However, Theorem 7.2 has the technical limitation of not covering the case where non-amenable boundary components are left unglued. In the context of triangulable aspherical manifolds, we have the following stronger result, which was stated initially in [Kue15] without any assumption on higher homotopy.

THEOREM 10. *Let M_1, \dots, M_k be oriented compact triangulable aspherical n -manifolds with π_1 -injective aspherical boundary, $n \in \mathbb{N}_{\geq 2}$. Let M be a manifold*

obtained by gluing M_1, \dots, M_k along (some of) their boundary components. If the glued boundary components have amenable fundamental group, then

$$\|M, \partial M\| = \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|.$$

In order to obtain additivity, Theorem 7.2 assumes that every boundary component has an amenable fundamental group, while only the glued components are required to be π_1 -injective. In contrast, Theorem 10 assumes that every boundary component is π_1 -injective, while only the glued ones are required to have amenable fundamental group. Our proof of Theorem 10 is presented in Section 7.4.

REMARK 7.3. Theorem 7.2 was extended in [LLM22] to the case of gluings along boundedly acyclic boundary components, although only control over the vanishing behavior was obtained. Even in this broader context, one must still require every boundary component (other than the glued ones) to be boundedly acyclic [LLM22, Remark 8.2]. However, extending our additivity result to the setting of boundedly acyclic boundary components does not appear to be straightforward.

The proof of Theorem 8 and Theorem 10 is based on the duality between simplicial volume and bounded cohomology (Lemma 3.1). Subadditivity is established via a version of Gromov Equivalence Theorem for good pairs (Theorem 7.21), while superadditivity is derived from extension properties of bounded coclasses with controlled norm (Proposition 7.12). This extension is carried out at the level of the aspherical multicomplexes associated to the pairs $(M, \partial M)$. To this end, we make use of the maps Φ^n and Ψ^n , appearing in Theorem 1 and Theorem 5, respectively, depending on whether the pair $(M, \partial M)$ is good or not. Different control over the norms for good and non-good pairs translates in different estimates for the simplicial volume in Theorem 10 and Theorem 8.

REMARK 7.4. The notion of good pair is very well suited when gluing aspherical spaces. In fact, given a finite CW-complex Z , expressed as a union of two aspherical subcomplexes X and Y , we have that Z is aspherical if the pairs $(X, X \cap Y)$ and $(Y, X \cap Y)$ are good [Whi78]. Building on this fact, we deduce Theorem 10 from a more general result regarding good pairs (Proposition 7.26).

REMARK 7.5. The requirement that the manifolds $(M, \partial M)$ are triangulable reflects the fact that the isomorphisms Φ^n and Ψ^n are defined only when $(M, \partial M)$ is a CW-pair.

The amenability of boundary components ensures also the subadditivity of simplicial volume in presence of partial gluings. The following criterion, whose proof is contained in Section 7.5, has been obtained by the author in collaboration with Kevin Li and Clara Löh in [CLL25] in the context of branched coverings.

THEOREM 11. *Let M_1, \dots, M_k be oriented compact n -manifolds, $n \in \mathbb{N}_{\geq 2}$. Let $(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$ be a pairing of some oriented compact connected pairwise-disjoint submanifolds of $\sqcup_{j=1}^k \partial M_j$ of codimension 0. Let M be a manifold obtained by gluing M_1, \dots, M_k along the pairing. We denote by B_j the compact submanifold of ∂M_j of codimension 0 whose interior is the complement in ∂M_j of the gluing loci. Assume that the following conditions hold for every $i \in \{1, \dots, h\}$ and every $j \in \{1, \dots, k\}$:*

- *Every connected component of ∂M_j and ∂S_i^\pm has amenable fundamental group;*

- $\|S_i^\pm, \partial S_i^\pm\| = 0$ and $\|B_j, \partial B_j\| = 0$.

Then $\|M, \partial M\| \leq \|M_1, \partial M_1\| + \cdots + \|M_k, \partial M_k\|$.

It follows from Theorem 11 that the simplicial volume is subadditive under boundary connected sums of n -manifolds with amenable boundary, provided that $n \in \mathbb{N}_{\geq 4}$. The same conclusion clearly holds for general 1-handle attachments (see Remark 7.1).

COROLLARY 12. *Let M_1 and M_2 be oriented compact n -manifolds with non-empty boundaries, $n \in \mathbb{N}_{\geq 4}$. Assume that every connected component of ∂M_1 and ∂M_2 has amenable fundamental group. Let $M = M_1 \natural M_2$ denote the boundary connected sum of M_1 and M_2 . Then*

$$\|M, \partial M\| \leq \|M_1, \partial M_1\| + \|M_2, \partial M_2\|.$$

PROOF. For boundary connected sums, the gluing pairing (S_1^+, S_1^-) is given by embedded $(n-1)$ -balls in ∂M_1 and ∂M_2 , respectively. It readily follows that $\|S_1^\pm, \partial S_1^\pm\| = 0$ (because, for example, they admit self-maps of high degree). On the other hand, B_j is obtained from ∂M_j by removing an open ball, hence ∂B_j is an $(n-2)$ -sphere, $j \in \{1, 2\}$. Since ∂M_j has dimension at least 3, the fundamental groups of ∂M_j and B_j are isomorphic. It follows that both B_j and ∂B_j have amenable fundamental group, hence they have vanishing bounded cohomology in positive degrees. By considering the long exact sequence of the pair $(B_j, \partial B_j)$, it follows that the comparison map $\text{comp}^n: H_b^n(B_j, \partial B_j) \rightarrow H^n(B_j, \partial B_j)$ vanishes, hence $\|B_j, \partial B_j\| = 0$ by duality (Lemma 3.1). All the assumptions of Theorem 11 are therefore satisfied. \square

The additivity results discussed so far rely on the amenability of the gluing loci. In Section 7.6 we present a vanishing criterion for the simplicial volume of manifolds glued along boundary components that are not necessarily amenable (Theorem 13). The central concept here is the *amenable category* $\text{cat}_{\text{Am}}(M)$ of a manifold M , which is defined as the minimal cardinality of an amenable open cover of M . We introduce a combination theorem for the amenable category, developed by the author in collaboration with Kevin Li and Clara Löh [CLL25], which leads to a vanishing criterion for the simplicial volume of glued manifolds.

7.2. Superadditivity

The goal of this section is to establish Theorem 8. We begin by introducing some notation.

SETUP 7.6. Let $n \in \mathbb{N}_{\geq 2}$. Let M_1, \dots, M_k be oriented compact connected triangulable n -manifolds with non-empty boundary. Let

$$(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$$

be a pairing of some oriented compact connected pairwise-disjoint submanifolds of $\sqcup_{j=1}^k \partial M_j$ of codimension 0. Let $f_i: S_i^+ \rightarrow S_i^-$ be an orientation-reversing homeomorphism, $i \in \{1, \dots, h\}$. Let M be the manifold obtained by gluing M_1, \dots, M_k along f_1, \dots, f_h and assume that M is connected. For every $j \in \{1, \dots, k\}$, let $q_j: M_j \rightarrow M$ denote the quotient map (which is an embedding, provided that M_j does not admit self-gluing). We still denote by M_j the image of M_j in M via q_j . We

denote by \mathcal{S} the union of the manifolds S_i^\pm , $i \in \{1, \dots, h\}$, understood as a subset of M . Let $B_j = q_j^{-1}(\partial M)$, understood a subset of ∂M_j , so that $B_j = \partial M \cap M_j$.

We first deduce Theorem 8 from the following slightly more general result, which is proved later in this section. The constants C_n in the statement are described in Definition 2.8.

PROPOSITION 7.7. *In the situation of Setup 7.6, assume that the following conditions hold for every $i \in \{1, \dots, h\}$ and $j \in \{1, \dots, k\}$:*

- (1) *The kernel of $\pi_1(\partial M \hookrightarrow M, x)$ is amenable for every $x \in \partial M$;*
- (2) *The kernel of $\pi_1(B_j \hookrightarrow M_j, x)$ is amenable for every $x \in B_j$;*
- (3) *M_j is π_1 -injective in M ;*
- (4) *S_i^\pm is π_1 -injective in M_j and has an amenable fundamental group;*
- (5) *∂S_i^\pm is path-connected and the map $\pi_1(\partial S_i^\pm \hookrightarrow S_i^\pm, x)$ is an isomorphism for one (whence every) $x \in \partial S_i^\pm$.*

Then

$$(7.1) \quad (2C_n)^4(n+2)^3 \cdot \|M, \partial M\| \geq \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|.$$

Moreover, if ∂M and B_j are π_1 -injective in M , then

$$(7.2) \quad (2C_n)^3(n+2)^3 \cdot \|M, \partial M\| \geq \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|.$$

REMARK 7.8. In Proposition 7.7 we can always reduce to the case when all the maps $q_j: M_j \rightarrow M$ are embeddings. Problems arise only in presence of self-gluing. In this case, instead of considering the self-gluing $f_i: S_i^+ \rightarrow S_i^-$ of M_j , we add another piece

$$N_i = S_i^+ \times [0, 1],$$

which is glued to M_j via the maps

$$\begin{aligned} S_i^+ \times \{0\} &\rightarrow S_i^+, & (x, 0) &\mapsto x, \\ S_i^+ \times \{1\} &\rightarrow S_i^-, & (x, 1) &\mapsto f_i(x). \end{aligned}$$

The manifold N obtained in this way is homeomorphic to M . Hence, if inequalities (7.1) or (7.2) hold for N , then they also hold for M . Notice that N is constructed by gluing embedded pieces, which satisfy the assumptions of Proposition 7.7. **As a consequence, throughout the sequel, we always assume that M_1, \dots, M_k are embedded in M .**

PROOF OF THEOREM 8 ASSUMING PROPOSITION 7.7. Assume that the manifolds M_1, \dots, M_k have π_1 -injective boundary. We need to check that conditions (1) - (5) of Proposition 7.7 are satisfied under the assumptions of Theorem 8, which we recall for the convenience of the reader:

- (i) S_i^\pm has an amenable fundamental group;
- (ii) ∂S_i^\pm is path-connected, π_1 -injective in the corresponding ∂M_j and the map $\pi_1(\partial S_i^\pm \hookrightarrow S_i^\pm)$ is an isomorphism.

We first observe that S_i^\pm and B_j are π_1 -injective in ∂M_j . In fact, by the Seifert-Van Kampen theorem, $\pi_1(\partial M_j)$ can be described as the fundamental group of a graph of groups with vertex groups $\pi_1(S_i^\pm)$, $\pi_1(B_j)$ and edge groups $\pi_1(\partial S_i^\pm)$. Under our assumptions, the edge morphisms are injective. Consequently, we know that the vertex groups embeds into $\pi_1(\partial M_j)$ [**Ser03**].

By assumption, we know that S_i^\pm is connected with connected boundary for every $i \in \{1, \dots, h\}$. It follows that for every connected component of $\mathcal{S} \cap M_j$ there exists a unique connected component of $\partial(\mathcal{S} \cap M_j) = \partial B_j$ contained in it. In other words, ∂B_j has the same number of connected components of $\mathcal{S} \cap M_j$. An easy application of the Mayer-Vietoris exact sequence shows that B_j has the same number of connected components as ∂M_j .

In order to get (1), we prove that ∂M is π_1 -injective in M . To this end, we observe that $\pi_1(M)$ can be described as the fundamental group of a graph of groups with vertex groups $\pi_1(M_j)$ and edge groups $\pi_1(S_i^\pm)$, while $\pi_1(\partial M)$ can be described as the fundamental group of a graph of groups with vertex groups $\pi_1(B_j)$ and edge groups $\pi_1(\partial S_i^\pm)$. Since the underlying graphs are isomorphic (here we use that B_j has the same number of connected components of ∂M_j) and the inclusion $\partial M \hookrightarrow M$ induces injective morphisms at the level of vertex and edge groups, it follows that ∂M is π_1 -injective in M . In order to get (2), we just recall that B_j is π_1 -injective in ∂M_j , which is in turn π_1 -injective in M_j . In order to get (3), we observe that $\pi_1(M)$ is the fundamental group of a graph of groups with injective edge morphisms, therefore the vertex groups $\pi_1(M_j)$ embed into $\pi_1(M)$. Conditions (4) and (5) clearly follow from (i) and (ii). We conclude that inequality (7.2) holds. \square

REMARK 7.9. Our general strategy to get the estimate in Proposition 7.7 can be summarized as follows. In order to obtain a fundamental cocycle of M with controlled norm, we average some fundamental cocycles of the manifolds M_1, \dots, M_k on the gluing locus \mathcal{S} . To this end, we need to consider the action of the group $G = \Pi_M(\mathcal{S})$ on the multicomplex $\mathcal{A}(M)$, thus passing to G -invariant cochains. This averaging procedure is possible when the group G is amenable, e.g. when the connected components of \mathcal{S} have amenable fundamental group. If $\partial\mathcal{S}$ is non-empty, i.e. when gluings are performed along proper submanifolds of the boundary, the action $G \curvearrowright \mathcal{A}(M)$ does *not* preserve the submulticomplex $\mathcal{A}(\partial M)$. Since the boundary ∂M is also obtained by gluing the remaining boundaries of M_1, \dots, M_k along $\partial\mathcal{S}$ (unless we are gluing entire boundary components), the averaging fails to be relative unless we assume some regularity of the action of G on $\partial\mathcal{S}$. Therefore, in order to consider invariant chains in the relative setting, our solution is to show that the action $G \curvearrowright \mathcal{A}(M)$ has orbits in $\mathcal{A}(\partial M)$ induced by $H = \Pi_M(\partial\mathcal{S})$ (Lemma 7.10). This translates in the assumption (5) of Proposition 7.7. When we are gluing along entire boundary components, passing to invariant chains is more straightforward. This case is presented in Section 7.4 to prove Theorem 10.

7.2.1. Proof of Proposition 7.7. We define nested pairs of multicomplexes from the gluing data. To this end, we implicitly adhere the choices described in Setup 5.2, according to which inclusions of topological spaces translate into inclusions of the corresponding aspherical multicomplexes. For every $j \in \{1, \dots, k\}$, we consider the following pairs of multicomplexes

$$(K, K') = (\mathcal{A}(M), \mathcal{A}_M(\partial M)),$$

$$(L_j, L'_j) = (\mathcal{A}(M_j), \mathcal{A}_{M_j}(B_j)),$$

where $\mathcal{A}_M(\partial M)$ (resp. $\mathcal{A}_{M_j}(B_j)$) denotes the image of $\mathcal{A}(\partial M)$ (resp. $\mathcal{A}(B_j)$) inside $\mathcal{A}(M)$ (resp. $\mathcal{A}(M_j)$). Since M_j is π_1 -injective in M , we have that $\mathcal{A}(M_j)$ is naturally isomorphic to $\mathcal{A}_M(M_j)$ and $\mathcal{A}_{M_j}(B_j)$ is naturally isomorphic to $\mathcal{A}_M(B_j)$

(Proposition 5.5). It follows that there is a natural inclusion of pairs

$$(L_j, L'_j) \subseteq (K, K').$$

Let A denote the multicomplex $\mathcal{A}(\mathcal{S})$. Since \mathcal{S} is π_1 -injective in M , we have that A naturally sits in K as a submulticomplex (Proposition 5.5). We set

$$G = \Pi_M(\mathcal{S}).$$

Since every connected component of \mathcal{S} has amenable fundamental group, we know that G is an amenable group. Moreover, G acts on $K = \mathcal{A}(M)$ as described in Section 4.2. Since both M_j and \mathcal{S} are π_1 -injective in M , it follows that $A \cap L_j$ can be identified with the aspherical multicomplex $\mathcal{A}(M_j \cap \mathcal{S})$ associated to the collection of path-connected components of \mathcal{S} contained in M_j .

Since \mathcal{S} is the disjoint union of $\mathcal{S} \cap M_j$, for every $j \in \{1, \dots, k\}$, the group G splits into the direct sum

$$G \cong \bigoplus_{j=1}^k \Pi_M(M_j \cap \mathcal{S}).$$

Moreover, being M_j π_1 -injective in M , the groups $\Pi_M(M_j \cap \mathcal{S})$ and $\Pi_{M_j}(M_j \cap \mathcal{S})$ are isomorphic. It follows that the action of G on K preserves the submulticomplex $L_j = \mathcal{A}(M_j)$, and thus defines an action of G on the pair (K, L_j) , where G acts on L_j as $\Pi_{M_j}(M_j \cap \mathcal{S})$ does.

By Proposition 4.5, the action $G \curvearrowright (K, L_j)$ is by automorphisms that are simplicially homotopic (as maps of pairs) to the identity. The actions $G \curvearrowright K$ and $G \curvearrowright L_j$ do *not* preserve the submulticomplexes K' and L'_j respectively. However, by setting

$$H = \Pi_M(\partial\mathcal{S}),$$

which is a subgroup of G in our assumptions, we have that these actions have orbit induced by H on those multicomplexes (see Definition 5.19). In order to show this, the assumption (5) in Proposition 7.7 is crucial.

LEMMA 7.10. *The action $G \curvearrowright K$ (resp. $G \curvearrowright L_j$) has orbits in K' (resp. L'_j) induced by H .*

PROOF. We prove the statement for the action $G \curvearrowright K$, since the argument is the same for the other action. Let $\sigma = (\Delta, (x_0, \dots, x_n))$ be an algebraic n -simplex of $K' = \mathcal{A}(\partial M)$ and let $g \in G$ be such that $g \cdot \sigma$ is an algebraic n -simplex of $\mathcal{A}(\partial M)$. We need to show that there exists an element $h \in H$ such that $h \cdot \sigma = g \cdot \sigma$. Let $\{v_0, \dots, v_k\}$ be the vertices of Δ , with $k \leq n$. Since v_j and $g \cdot v_j$ are both vertices of $\mathcal{A}(\partial M)$ (hence points of ∂M), then, for every $j \in \{0, \dots, k\}$, there is a representative $(\gamma_x)_{x \in X}$ of g such that $\gamma_{v_j}: [0, 1] \rightarrow \mathcal{S}$ has both endpoints in $\partial M \cap \mathcal{S} = \partial\mathcal{S}$. By assumption (5), we can invoke Lemma 7.11 below to deduce that there exist some paths

$$\lambda_{v_j}: [0, 1] \rightarrow \partial\mathcal{S}, \quad j \in \{0, \dots, k\},$$

such that λ_{v_j} is homotopic to γ_{v_j} in M relative to the endpoints. Therefore $h = \{\lambda_0, \dots, \lambda_k\}$ defines an element of H . By construction we have that $h \cdot e = g \cdot e$, for every edge e of Δ , hence $g \cdot \Delta^1 = h \cdot \Delta^1$, where Δ^1 denotes the 1-skeleton of Δ . Since $K = \mathcal{A}(M)$ is an aspherical multicomplex, this is indeed sufficient to conclude (Proposition 2.24). \square

LEMMA 7.11. *For every path $\gamma: [0, 1] \rightarrow \mathcal{S}$ with endpoints in $\partial\mathcal{S}$, there exists a path $\lambda: [0, 1] \rightarrow \partial\mathcal{S}$ such that λ is homotopic in M to γ relative to the endpoints.*

PROOF. The path-connected components of \mathcal{S} can be identified with S_i^\pm , for $i \in \{1, \dots, k\}$. Since ∂S_i^\pm is path-connected, there exists a path $\varepsilon: [0, 1] \rightarrow \partial\mathcal{S}$ from $\gamma(0)$ to $\gamma(1)$. The homotopy class of the loop $\gamma * \bar{\varepsilon}$ identifies an element of $\pi_1(S_i^\pm, \gamma(0))$. Since $\pi_1(\partial S_i^\pm \hookrightarrow S_i^\pm, \gamma(0))$ is an isomorphism, there exists a loop $\delta: [0, 1] \rightarrow \partial S_i^\pm$ which is homotopic in S_i^\pm (hence in M) to $\gamma * \bar{\varepsilon}$ relative to the endpoints. It follows that $\lambda = \delta * \varepsilon$ satisfies the statement. \square

$$\begin{array}{ccc}
H_b^n(M, \partial M) & \xrightarrow{\oplus H_b^n(q_j)} & \bigoplus_{j=1}^k H_b^n(M_j, \partial M_j) \\
\parallel & & \oplus H_b^n(\alpha_j) \downarrow \\
H_b^n(M, \partial M) & \xrightarrow{\oplus H_b^n(q_j)} & \bigoplus_{j=1}^k H_b^n(M_j, B_j) \\
\uparrow \Psi^n & & \oplus \Psi_j^n \uparrow \\
H_b^n(K, K') & \xrightarrow{\oplus l_j} & \bigoplus_{j=1}^k H_b^n(L_j, L'_j) \\
\uparrow I^n & & \oplus I_j^n \uparrow \\
H^n(C_b^\bullet(K, K')_{\text{alt}}^G) & \longrightarrow & \bigoplus_{j=1}^k H^n(C_b^\bullet(L_j, L'_j)_{\text{alt}}^G),
\end{array}$$

FIGURE 6. Proof of Theorem 8. Horizontal maps in the diagram are induced by inclusions. The maps $\alpha_j: (M_j, B_j) \rightarrow (M_j, \partial M_j)$ denote the inclusions of pairs.

Let us now consider the diagram in Figure 6, and point out the following observations. By conditions (1) and (2) of Proposition 7.7, we can invoke Theorem 5 to get bi-Lipschitz isomorphisms Ψ^n and Ψ_j^n of type

$$((n+2)^{-1}(2C_n)^{-2}, 2C_n(n+2)).$$

Even if these maps are *a priori* not canonical, the choices described in Setup 5.2 for the pairs $(M, \partial M)$ and (M_j, B_j) guarantee that the middle square is commutative. The maps I^n and I_j^n are induced by the inclusions of complexes

$$C_b^\bullet(K, K')_{\text{alt}}^G \hookrightarrow C_b^\bullet(K, K'), \quad C_b^\bullet(L_j, L'_j)_{\text{alt}}^G \hookrightarrow C_b^\bullet(L_j, L'_j).$$

By Proposition 1.9, the inclusion of alternating chains induces an isometric isomorphism in bounded cohomology. The same holds for invariant chains, because G (hence H) is amenable, and because the actions $G \curvearrowright (K, L_j)$, $H \curvearrowright K'$ and $H \curvearrowright L'_j$ are by automorphisms that are simplicially homotopic to the identity (Proposition 4.4). It follows from Lemma 5.20 that I^n and I_j^n are bi-Lipschitz isomorphisms of type

$$((n+2)^{-1}(2C_n)^{-1}, 1).$$

Moreover, the lower square is commutative. Our proof of Proposition 7.7 is based on the following extension property for bounded invariant coclasses.

PROPOSITION 7.12. *Let $\varepsilon > 0$ and take for every $j \in \{1, \dots, k\}$ an element*

$$\varphi_j \in H^n(C_b^\bullet(L_j, L'_j)_{\text{alt}}^G).$$

Then there exists a coclass $\varphi \in H_b^n(K, K')$ such that $l_j(\varphi) = I_j^n(\varphi_j)$ for every $j \in \{1, \dots, k\}$ and

$$\|\varphi\|_\infty \leq \max\{\|\varphi_j\|_\infty \mid j \in \{1, \dots, k\}\} + \varepsilon.$$

We first show how Proposition 7.12 can be used to conclude the proof of Proposition 7.7. By duality (Lemma 3.1), for every $j \in \{1, \dots, k\}$ we can take an element $\gamma_j \in H_b^n(M_j, \partial M_j)$ such that

$$\|M_j, \partial M_j\| = \langle \gamma_j, [M_j, \partial M_j] \rangle, \quad \|\gamma_j\|_\infty \leq 1.$$

Let $\gamma'_j \in H^n(C_b^\bullet(L_j, L'_j)_{\text{alt}}^G)$ be such that

$$\Psi_j^n \circ I_j^n(\gamma'_j) = H_b^n(\alpha_j)(\gamma_j).$$

Since $H_b^n(\alpha_j)$ is norm non-increasing and Ψ_j^n and I_j^n are bi-Lipschitz isomorphisms, we deduce from the estimates in Lemma 5.20 and Theorem 5 that

$$\|\gamma'_j\|_\infty \leq (2C_n)^3(n+2)^2 \cdot \|\gamma_j\|_\infty \leq (2C_n)^3(n+2)^2.$$

Let $\varepsilon > 0$. By Proposition 7.12, there exists $\gamma' \in H_b^n(K, K')$ such that

$$\|\gamma'\|_\infty \leq (2C_n)^3(n+2)^2 + \varepsilon, \quad l_j(\gamma') = I_j^n(\gamma'_j), \quad j \in \{1, \dots, k\}.$$

Let $\gamma = \Psi^n(\gamma') \in H_b^n(M, \partial M)$. Since Ψ^n is a bi-Lipschitz isomorphism, again from the estimates in Lemma 5.20 and Theorem 5 we deduce that

$$\|\gamma\|_\infty \leq 2C_n(n+2) \cdot \|\gamma'\|_\infty \leq 2C_n(n+2) \cdot ((2C_n)^3(n+2)^2 + \varepsilon).$$

We proceed with the proof of Proposition 7.7 by constructing a fundamental cycle of M that is well behaved with respect to the gluings. We do not claim that this cycle is efficient in computing the simplicial volume of M . In the following, with an abuse of notation, we identify any chain in M_j and \mathcal{S} with the corresponding chain in M via the corresponding inclusions.

LEMMA 7.13. *There exists a real relative fundamental cycle $z \in C_n(M)$ of M such that $z = z_1 + \dots + z_k$, where $z_j \in C_n(M_j) \subseteq C_n(M)$ is a real relative fundamental cycle of M_j , for every $j \in \{1, \dots, k\}$.*

PROOF. Let $w_j \in C_n(M_j)$ be a real chain representing a real fundamental class of M_j and let $w = w_1 + \dots + w_k \in C_n(M)$. We need to modify w in order to obtain the desired fundamental cycle for M .

For every $i \in \{1, \dots, h\}$, let $\sigma_i^+ \in C_{n-1}(S_i^+)$ be a real relative fundamental cycle of S_i^+ . We denote by

$$\sigma_i^- = -(f_i)_*(\sigma_i^+)$$

the fundamental cycle of S_i^- induced by the gluing map f_i . Let $\beta_j \in C_{n-1}(B_j)$ be a real relative fundamental cycle of B_j , $j \in \{1, \dots, k\}$.

We denote by σ_j the sum of σ_i^\pm where $i \in \{1, \dots, h\}$ is such that $S_i^\pm \subseteq M_j$. Since $\partial\sigma_j$ and $\partial\beta_j$ represent a fundamental class of ∂B_j with opposite orientation, it follows that there exists $d_j \in C_{n-1}(\partial B_j) \subseteq C_{n-1}(B_j)$ such that $\partial d_j = \partial\sigma_j + \partial\beta_j$. Therefore $b_j = d_j - \beta_j$ is a real relative fundamental cycle of B_j . We set $c_j =$

$\sigma_j - b_j \in C_{n-1}(\partial M_j)$. Since both c_j and ∂w_j represent a fundamental class of ∂M_j , there exists $u_j \in C_n(\partial M_j)$ such that $c_j - \partial w_j = \partial u_j$. Therefore

$$z_j = w_j + u_j \in C_n(M_j)$$

is a real relative fundamental cycle of $(M_j, \partial M_j)$ such that $\partial z_j = \sigma_j - b_j$. In conclusion, since f is orientation reversing, we have that

$$z = z_1 + \cdots + z_k \in C_n(M)$$

is a real relative fundamental cycle of $(M, \partial M)$. \square

Let $c \in C_n(M)$ be any cycle representing γ . By using Lemma 7.13 and the fact that the diagram in Figure 6 is commutative, we deduce that

$$\begin{aligned} \langle \gamma, [M, \partial M] \rangle &= c(z) = \sum_{j=1}^k c(z_j) = \sum_{j=1}^k \langle \gamma, [z_j] \rangle \\ &= \sum_{j=1}^k \langle H_b^n(q_j)(\gamma), [z_j] \rangle = \sum_{j=1}^k \langle H_b^n(\alpha_j)(\gamma_j), [z_j] \rangle \\ &= \sum_{j=1}^k \langle \gamma_j, [z_j] \rangle = \sum_{j=1}^k \langle \gamma_j, [M_j, \partial M_j] \rangle, \end{aligned}$$

where, with an abuse of notation, $[z_j]$ represents the class represented by the same chain $z_j \in C_n(M_j) \subseteq C_n(M)$, either in $H_n(M, \partial M)$, $H_n(M_j, B_j)$ or $H_n(M_j, \partial M_j)$. In conclusion, we get

$$\begin{aligned} \sum_{j=1}^k \|M_j, \partial M_j\| &= \sum_{j=1}^k \langle \gamma_j, [M_j, \partial M_j] \rangle \\ &= \langle \gamma, [M, \partial M] \rangle \\ &\leq \|\gamma\|_\infty \cdot \|M, \partial M\| \\ &\leq 2C_n(n+2) \cdot ((2C_n)^3(n+2)^2 + \varepsilon) \cdot \|M, \partial M\|, \end{aligned}$$

and, since ε is arbitrary, we obtain (7.1).

REMARK 7.14. If ∂M is π_1 -injective in M and B_j is π_1 -injective in M_j , we can improve the bi-Lipschitz constants of Ψ^n and Ψ_j^n according to Remark 5.18. It follows that we can take

$$\|\gamma\|_\infty \leq 2C_n(n+2) \cdot \|\gamma'\|_\infty \leq 2C_n(n+2) \cdot ((2C_n)^2(n+2)^2 + \varepsilon)$$

in the construction above, from which we deduce inequality (7.2).

In order to conclude the proof of Proposition 7.7 we are left to prove Proposition 7.12. To this end, we need the following preparation. Let $p: \widetilde{K} \rightarrow K$ denote the universal covering (simplicial) map of K . By Lemma 2.25 \widetilde{K} is a simplicial complex such that an $(n+1)$ -tuple (x_0, \dots, x_n) of pairwise distinct vertices of \widetilde{K} spans an n -simplex if and only if the vertices $p(x_0), \dots, p(x_n)$ of K are pairwise distinct. Consider the submulticomplex

$$\cup L_j = L_1 \cup \cdots \cup L_k$$

of K . Since every $L_j = \mathcal{A}(M_j)$ is connected and $K = \mathcal{A}(M)$ is connected by assumption, it follows that also $\cup L_j$ is connected. Let $\widehat{L}_j = p^{-1}(L_j)$ and $\widehat{A} =$

$p^{-1}(A)$. The intersection $\widehat{L}_i \cap \widehat{L}_j$ consists of a collection of connected components of \widehat{A} , for every $i \neq j$. Consider the multicomplex

$$\cup \widehat{L}_j = p^{-1}(\cup L_j) = \widehat{L}_1 \cup \cdots \cup \widehat{L}_k.$$

Let $p_M: \widetilde{M} \rightarrow M$ denote the universal covering map of the manifold M . We describe the structure of \widetilde{M} as a *tree of spaces*, following the construction in [Fri17, Chapter 9]. We define a tree T' as follows. We pick a vertex for every connected component of $p_M^{-1}(M_j)$, $j \in \{1, \dots, k\}$, and we join two vertices if the corresponding connected components intersect (along a connected component of $p_M^{-1}(\mathcal{S})$). Since T' is a deformation retract of \widetilde{M} , then it is connected and simply connected, i.e. a tree.

Similarly, we can describe the structure of $\cup \widehat{L}_j$ as a *tree of multicomplexes*. We construct a tree T as follows. We pick a vertex for every connected component of \widehat{L}_j , $j \in \{1, \dots, k\}$, and we join two vertices if the corresponding components intersect (along a connected component of \widehat{A}). Therefore, edges of T correspond to the connected components of \widehat{A} . Since the connected components of \widehat{L}_j and \widehat{A} bijectively correspond to the connected components of $p_M^{-1}(M_j)$ and $p_M^{-1}(\mathcal{S})$, respectively, it follows that T and T' are simplicially isomorphic. Hence also T is a tree. We denote by $V(T)$ the set of vertices of T and, for every $v \in V(T)$, we denote by \widetilde{L}_v the connected component of \widehat{L}_j corresponding to v .

LEMMA 7.15. *For every $v \in V(T)$ there exists $j(v) \in \{1, \dots, k\}$ and a universal covering (simplicial) map $p_v: \widetilde{L}_v \rightarrow L_{j(v)}$ such that the following diagram commutes*

$$\begin{array}{ccc} \widetilde{L}_v & \xrightarrow{p_v} & L_{j(v)} \\ & \searrow p|_{\widetilde{L}_v} & \downarrow \\ & & K. \end{array}$$

Moreover, for every path connected component C of \widehat{A} , there exists a universal covering (simplicial) map $p_C: C \rightarrow p(C)$ such that the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{p_C} & p(C) \\ & \searrow p|_C & \downarrow \\ & & K. \end{array}$$

PROOF. The aspherical multicomplexes $K = \mathcal{A}(M)$, $L_j = \mathcal{A}(M_j)$ and $A = \mathcal{A}(\mathcal{S})$ are classifying spaces for the fundamental groups of M , M_j and \mathcal{S} respectively. By assumption we know that M_j and \mathcal{S} are π_1 -injective in M . It follows that also L_j and $p(C)$ are π_1 -injective in K . The statement now follows from standard covering theory. \square

Let \bar{x} be a vertex of K , hence, a point of M . We denote by Γ the fundamental group of M based at \bar{x} . By Lemma 2.27, there are canonical isomorphisms

$$\Gamma = \pi_1(M, \bar{x}) \cong \pi_1(K, \bar{x}).$$

Since $\cup \widehat{L}_j$ is Γ -invariant, we can consider the simplicial action $\Gamma \curvearrowright \cup \widehat{L}_j$ by deck transformations. Let Γ_v denote the stabilizer of \widetilde{L}_v in Γ .

Recall that $K' = \mathcal{A}(\partial M)$ and $L'_j = \mathcal{A}(B_j)$ are not necessarily connected. Let $\widehat{K}' = p^{-1}(K')$ and $\widehat{L}'_j = p^{-1}(L'_j)$. By Lemma 5.6, the morphisms $\pi_1(L'_j \hookrightarrow K)$ and $\pi_1(K' \hookrightarrow K)$ are injective for every choice of basepoint. Therefore, inside \widehat{K}' (resp. \widehat{L}'_j) we can find several copies of the universal covering of K' (resp. of L'_j). For every $v \in V(T)$, we set

$$\widehat{L}'_v = \widehat{L}'_{j(v)} \cap \widetilde{L}_v.$$

7.2.2. Proof of Proposition 7.12. Let $\varepsilon > 0$. For every $j \in \{1, \dots, k\}$, we are given a coclass $\varphi_j \in H^n(C_b^\bullet(L_j, L'_{j,\text{alt}}))^G$. Let $f_j \in C_b^n(L_j, L'_{j,\text{alt}})^G$ be a cocycle representing φ_j such that

$$\|f_j\|_\infty \leq \|\varphi_j\|_\infty + \varepsilon.$$

For every $v \in V(T)$, we denote by $f_v \in C_b^n(\widetilde{L}_v)$ the pull-back of f_j via the covering $p_v: \widetilde{L}_v \rightarrow L_{j(v)}$. Of course, also f_v is a cocycle. Moreover, since f_j vanishes on simplices supported on L'_j , it follows that f_v vanishes on simplices supported on \widehat{L}'_v , hence

$$f_v \in C_b^n(\widetilde{L}_v, \widehat{L}'_v)^{\Gamma_v}.$$

In order to prove Proposition 7.12, it is sufficient to find a bounded cocycle

$$f \in C_b^n(\widetilde{K}, \widehat{K}')^\Gamma$$

which restricts to f_v on each $C_n(\widetilde{L}_v)$ and is such that

$$\|f\|_\infty \leq \max\{\|f_j\|_\infty \mid j \in \{1, \dots, k\}\}.$$

Our construction is based on the following lemma, inspired by the notion of *special cochains* from [Fri17, Definition 5.15]. Since \widetilde{K} is a simplicial complex, its simplices are uniquely characterized by their vertices. By Lemma 2.25, the vertices x_0, \dots, x_n span a simplex σ of \widetilde{K} if and only if $p(x_0), \dots, p(x_n)$ are pairwise distinct vertices of $K = \mathcal{A}(M)$.

LEMMA 7.16. *For every $v \in V(T)$, the cochain $f_v \in C_b^n(\widetilde{L}_v, \widehat{L}'_v)$ satisfies the following properties:*

- f_v is alternating and Γ_v -invariant;
- Let $(\sigma, (x_0, \dots, x_n))$ and $(\sigma', (x'_0, \dots, x'_n))$ be algebraic n -simplices of \widetilde{L}_v such that, for every $i \in \{0, \dots, n\}$, either $x_i = x'_i$ or x_i and x'_i both lie in the same connected component of $\widetilde{L}_v \cap \widehat{A}$. Then

$$f_v(\sigma, (x_0, \dots, x_n)) = f_v(\sigma', (x'_0, \dots, x'_n)).$$

In particular, f_v vanishes on every simplex having two vertices on the same connected component of $\widetilde{L}_v \cap \widehat{A}$.

PROOF. Let $j = j(v) \in \{1, \dots, k\}$, so that f_v is the pullback of f_j via the universal covering map p . Since f_j is alternating, then f_v is alternating and Γ_v -invariant. Let now $(\sigma, (x_0, \dots, x_n))$ and $(\sigma', (x'_0, \dots, x'_n))$ be algebraic n -simplices of \widetilde{L}_v as in the statement. Let $\Delta = p(\sigma)$ and $\Delta' = p(\sigma')$. By definition of pullback, we have that

$$\begin{aligned} f_v(\sigma, (x_0, \dots, x_n)) &= f_j(\Delta, (p(x_0), \dots, p(x_n))), \\ f_v(\sigma', (x'_0, \dots, x'_n)) &= f_j(\Delta', (p(x'_0), \dots, p(x'_n))). \end{aligned}$$

We now exhibit an element $g \in G$ such that $g \cdot \Delta = \Delta'$ and $g \cdot x_i = x'_i$, for every $i \in \{0, \dots, n\}$. Since f_j is G -invariant, this is enough to conclude. For every $i \in \{0, \dots, n\}$, we proceed as follows. If $x_i = x'_i$, we set g_i to be the identity of G . If x_i and x'_i are in the same connected component C of $\widehat{A} \cap \widetilde{L}_v$, there are two cases to consider.

If $p(x_i) = p(x'_i)$, then there exists a unique element $a_i \in \pi_1(\mathcal{S}, p(x_i))$ such that $a_i \cdot x_i = x'_i$, where $\pi_1(\mathcal{S}, p(x_i))$ acts on C by deck transformations. Let τ_i be any loop representing a_i . We set $g_i = \{\tau_i\} \in G$. On the other hand, if $p(x_i)$ and $p(x'_i)$ are two different points of A , by Lemma 2.25, there exists a unique edge e between x_i and x'_i in $\widehat{A} \cap \widetilde{L}_v$. In this case, let τ_i be a parametrization of the edge $p(e)$, i.e. a path in \mathcal{S} from $p(x_i)$ to $p(x'_i)$. We set $g_i = \{\tau_i, \bar{\tau}_i\} \in G$. Since they act on distinct points of K , the elements g_0, \dots, g_n pairwise commute in G . We then set $g = g_0 \oplus \dots \oplus g_n \in G$. By construction, we have that $g \cdot \Delta = \Delta'$. \square

Let $s = (\sigma, (x_0, \dots, x_n)) \in C_n(\widetilde{K})$ be an algebraic n -simplex. Since we work with alternating cochains, we can assume without loss of generality that $x_0, \dots, x_n \in \widetilde{K}^0$ are the vertices of σ , i.e. they are pairwise distinct and project onto pairwise different points of K (Lemma 2.25). Moreover σ is uniquely determined by its vertices. Observe that \widetilde{K} and $\cup \widehat{L}_j$ have the same 0-skeleton.

DEFINITION 7.17. Let $v_0, v_1 \in (\cup \widehat{L}_j)^0$. A *simplicial path* in $\cup \widehat{L}_j$ from v_0 to v_1 is a 1-dimensional submulticomplex of $\cup \widehat{L}_j$ whose topological realization realizes a continuous path in $|\cup \widehat{L}_j|$ from v_0 to v_1 .

In the previous definition, it is important that we do not allow generic edges in \widetilde{K} , where every pair of vertices projecting to distinct vertices of K can be connected by a unique edge (by Lemma 2.25). Since x_0, \dots, x_n are vertices of \widetilde{K} , hence of $\cup \widehat{L}_j$, we can consider simplicial paths in $\cup \widehat{L}_j$ between them.

DEFINITION 7.18. A vertex $v \in V(T)$ is a *barycenter* of σ if, for every $i, j \in \{0, \dots, n\}$, $i \neq j$, every simplicial path in $\cup \widehat{L}_j$ from x_i to x_j has an edge e such that e is contained in \widetilde{L}_v and e is not contained in \widehat{A} .

The proof of the following lemma is based on the description of \widetilde{K} as a tree of multicomplexes.

LEMMA 7.19. *Let σ be an n -simplex of \widetilde{K} , where $n \in \mathbb{N}_{\geq 2}$. Then σ has at most one barycenter. If σ is supported in \widetilde{L}_v , for some $v \in V(T)$, then v is the barycenter of σ if and only if every component of \widehat{A} contains at most one vertex of σ . If this is not the case, then σ does not admit a barycenter.*

PROOF. The argument follows closely the one presented in [Fri17, Lemma 9.5]. Assume by contradiction that σ admits two distinct barycenters v_1 and v_2 . The multicomplex $\cup \widehat{L}_j \setminus \widetilde{L}_{v_1}$ has two connected components. We denote by C the connected component of $\cup \widehat{L}_j \setminus \widetilde{L}_{v_1}$ containing $\widetilde{L}_{v_2} \setminus \widehat{A}$. Since v_1 is a barycenter of σ , at most one vertex of σ is contained in C . On the other hand, the multicomplex $\cup \widehat{L}_j \setminus C$ is connected and disjoint from $\widetilde{L}_{v_2} \setminus \widehat{A}$. Since v_2 is a barycenter of σ , then at most one vertex of σ is contained in $\cup \widehat{L}_j \setminus C$. Since the set of vertices of \widetilde{K} and $\cup \widehat{L}_j$ coincides, it would follow that σ has at most two vertices, which contradicts our assumption that $n \geq 2$. \square

Let $v \in V(T)$ be fixed. We associate (quite arbitrarily) to σ a n -simplex σ_v with vertices x'_0, \dots, x'_n in such a way that the following conditions hold:

- σ_v is a n -simplex of \tilde{L}_v ;
- if $x_i \in \tilde{L}_v$, then $x'_i = x_i$;
- if $x_i \notin \tilde{L}_v$, then there exists a unique component C of $\tilde{L}_v \cap \hat{A}$ such that every simplicial path in $\cup \tilde{L}_j$ joining x_i with \tilde{L}_v intersects C ; in this case, we choose x'_i to be any point of C .

In order to obtain a simplex of \tilde{L}_v we need to choose the points x'_i so that they project to pairwise distinct points of $L_{j(v)}$ – which is always possible. The simplex σ_v is called a *projection of σ on \tilde{L}_v* , and plays the role of the *central simplex* of Kuessner's work [Kue15]. The projection is unique once we fix its vertices, since \tilde{K} is a simplicial complex (Lemma 2.25). We denote by $s_v = (\sigma_v, (x'_0, \dots, x'_n))$ the algebraic simplex associated to σ_v . We now set, for every $v \in V(T)$,

$$\hat{f}_v(s) = f_v(s_v).$$

By Lemma 7.16, $\hat{f}_v(s)$ is indeed well defined i.e. it does not depend on the choice of s_v . The following lemma rephrases in our setting the crucial observation in Lemma 4.8.

LEMMA 7.20. *If v is not a barycenter of σ , then $\hat{f}_v(s) = 0$.*

PROOF. If v is not a barycenter of σ , then there exists a connected component C of $\hat{A} \cap \tilde{L}_v$ which contains at least two vertices of σ_v , say x_i and x_j , with $i, j \in \{0, \dots, n\}$ (Lemma 7.19). The conclusion follows from Lemma 7.16. \square

We are ready to define the cochain h of \tilde{K} as follows: for every algebraic simplex $s = (\sigma, (x_0, \dots, x_n)) \in C_n(\tilde{K})$ we set

$$h(s) = \sum_{v \in V(T)} \hat{f}_v(s).$$

By Lemma 7.19 and Lemma 7.20, the sum on the right hand side is either empty or consists of a single term. From this we already deduce that

$$\|h\|_\infty \leq \max\{\|f_v\|_\infty \mid v \in V(T)\} = \max\{\|f_j\|_\infty \mid j \in \{1, \dots, k\}\}.$$

Since f_v is alternating for every $v \in V(T)$, it follows that also h is alternating. Moreover, it is clear from the construction that h is Γ -invariant. It follows that h projects to a well-defined chain $f \in C_b^n(K)$ such that $\|f\| \leq \|h\|$. To conclude, it remains to show that f is a cocycle which vanishes on chains supported on K' and which restricts to f_j on chains supported on L_j , for every $j \in \{1, \dots, k\}$. We show equivalently that h is a cocycle which vanishes on chains supported on \hat{K}' and which restricts to f_v on chains supported on \tilde{L}_v , for every $v \in V(T)$.

Assume that the simplex σ is supported on \tilde{L}_v , for some $v \in V(T)$. Then we can set $s_v = s$, and, from Lemma 7.19 and Lemma 7.20, we deduce that $h(s) = f_v(s)$. Therefore, we have that h indeed restricts to f_v on simplices supported on \tilde{L}_v . In order to show that h defines a cocycle, let $s = (\sigma, (x_0, \dots, x_{n+1}))$ be an algebraic $(n+1)$ -simplex, where σ is a $(n+1)$ -simplex of \tilde{K} . Let σ_v denote the projection of σ on \tilde{L}_v , according to the same procedure that was described for n -simplices. It

readily follows from the construction that $\partial_i(\sigma_v)$ is a projection of $\partial_i\sigma$ on \tilde{L}_v , hence

$$\begin{aligned}\widehat{f}_v(\partial s) &= \sum_{i=0}^{n+1} (-1)^i \widehat{f}_v((\partial_i\sigma), (x_0, \dots, \widehat{x}_i, \dots, x_{n+1})) \\ &= \sum_{i=0}^{n+1} (-1)^i f_v((\partial_i\sigma)_v, (x'_0, \dots, \widehat{x}'_i, \dots, x'_{n+1})) \\ &= \sum_{i=0}^{n+1} (-1)^i f_v(\partial_i(\sigma_v), (x'_0, \dots, \widehat{x}'_i, \dots, x'_{n+1})) \\ &= f_v(\partial s_v) = 0,\end{aligned}$$

where the last equality follows from the fact that f_v is a cocycle. Therefore we have that

$$h(\partial s) = \sum_{v \in V(T)} \widehat{f}_v(\partial s) = 0,$$

so h is a cocycle. It remains to show that h vanishes on simplices supported on \widehat{K}' . Let $s = (\sigma, (x_0, \dots, x_n)) \in C_n(\widehat{K}')$. By Lemma 7.20, it is sufficient to prove that $f_v(s_v) = 0$, where $v \in V(T)$ is the barycenter of σ . We have that σ is contained in some connected component C of \widehat{K}' . Observe now that C either is contained \tilde{L}_v , or it does intersect some connected components of \widehat{A} . In the first case, since v is the barycenter of σ , we can take $\sigma = \sigma_v$. It follows that $\sigma_v \in \widehat{K}' \cap \tilde{L}_v \subseteq \widehat{L}'_v$. In the second case, we can clearly construct a projection σ_v of σ on \widehat{L}'_v . In both cases, since f_v vanishes on simplices supported on \widehat{L}'_v , we get $f_v(s_v) = 0$. This concludes the proof of Proposition 7.12, hence of Theorem 8.

7.3. Gromov Equivalence Theorem for Good Pairs

In this section we establish a version of Gromov Equivalence Theorem for good pairs (Theorem 7.21), which is based on the isometric isomorphism of Theorem 3. This is the key step in our proof of the subadditivity of simplicial volume for aspherical manifolds (Theorem 10).

Let (X, Y) be a pair of topological spaces. Let $A \subseteq Y$ be the union of some connected components of Y . We start by introducing the following one-parameter family of seminorms on $H_n(X, Y)$, depending on A (see [Thu79, Section 6.5]). Let $\vartheta \in [0, \infty)$. We identify $C_n(X, Y)$ with the quotient of $C_n(X)$ by its subspace $C_n(Y)$. We want to define a norm depending on ϑ and A on the space of cycles $Z_n(X, Y) \subseteq C_n(X, Y)$. Let $c \in Z_n(X, Y)$. Since $\partial c = 0$, it follows that every cycle $b \in C_n(X)$ representing c satisfies $\partial b \in C_{n-1}(Y)$. There is a canonical projection $C_{n-1}(Y) \rightarrow C_{n-1}(A)$, and we denote by $\partial b|_A \in C_{n-1}(A)$ the image of ∂b under this projection. We set

$$\|c\|_1^A(\vartheta) = \inf \left\{ \|b\|_1 + \vartheta \|\partial b|_A\|_1 \mid b \in C_n(X) \text{ is a representative of } c \right\}.$$

This indeed defines a norm on $Z_n(X, Y)$, which is equivalent to the usual norm

$$\|\cdot\|_1 = \|\cdot\|_1^A(0),$$

and induces a quotient seminorm on $H_n(X, Y)$, still denoted by $\|\cdot\|_1^A(\vartheta)$.

The goal of this section is to give a proof of the following version of Gromov Equivalence Theorem [Gro82, Section 4.1][BBF⁺14, Theorem 5]. This was originally stated in [Kue15] without assumptions on higher homotopy.

THEOREM 7.21 (Gromov Equivalence Theorem). *Let (X, Y) be a good pair and let $A \subseteq Y$ be the union of some connected components of Y . If every connected component of A has an amenable fundamental group, then the seminorms $\|\cdot\|_1^A(\vartheta)$ on $H_n(X, Y)$ coincide for every $\vartheta \in [0, \infty)$.*

The previous result admits the following equivalent formulation, which is inspired by [Thu79, Section 6.5], and plays a fundamental role in our proof of Theorem 10.

COROLLARY 7.22. *Let (X, Y) be a good pair of topological spaces. Let $A \subseteq Y$ be the union of some connected components of Y and assume that every connected component of A has an amenable fundamental group. Let $\alpha \in H_n(X, Y)$, $n \in \mathbb{N}_{\geq 2}$. Then, for every $\varepsilon \geq 0$, there exists an element $c \in C_n(X)$ with $\partial c \in C_{n-1}(Y)$ such that $[c] = \alpha \in H_n(X, Y)$, $\|c\|_1 \leq \|\alpha\|_1 + \varepsilon$ and $\|\partial c|_A\|_1 < \varepsilon$.*

PROOF. The argument is the same of [BBF⁺14, Corollary 6], which we present here for the sake of completeness. Let $\vartheta = (\|\alpha\|_1 + \varepsilon)/\varepsilon$. By Theorem 7.21, the norm $\|\cdot\|_1^A(\vartheta)$ induces the same norm as $\|\cdot\|_1$ in homology. It follows that there exists a representative $c \in C_n(X)$ of α such that

$$\|c\|_1^A(\vartheta) = \|c\|_1 + \vartheta \|\partial c|_A\|_1 \leq \|\alpha\|_1 + \varepsilon,$$

which implies that $\|c\|_1 \leq \|\alpha\|_1 + \varepsilon$ and $\|\partial c|_A\|_1 < \varepsilon$. \square

Our proof of Theorem 7.21 uses the same construction of the mapping cone complex of Section 5.2. We denote by $i_n: C_n(Y) \rightarrow C_n(X)$ the map induced by the inclusion $i: Y \hookrightarrow X$. We consider the complex $(C_\bullet(Y \hookrightarrow X), \bar{d}_\bullet)$, where

$$C_n(Y \hookrightarrow X) = C_n(X) \oplus C_{n-1}(Y), \quad \bar{d}_n(u, v) = (\partial_n u + i_{n-1}(v), -\partial_{n-1}(v)).$$

We denote by $Z_\bullet(Y \hookrightarrow X)$ the corresponding cocycles and by $H_\bullet(Y \hookrightarrow X)$ its homology. For every $n \in \mathbb{N}$ and $\vartheta \in [0, \infty)$, we endow $C_n(Y \hookrightarrow X)$ with the norm

$$\|(u, v)\|_1(\vartheta) = \|u\|_1 + \vartheta \|v\|_1.$$

In a similar way, we endow $Z_n(Y \hookrightarrow X)$ with the norm

$$\|(u, v)\|_1^A(\vartheta) = \|u\|_1 + \vartheta \|v|_A\|_1,$$

which induces a seminorm, still denoted by $\|\cdot\|_1^A(\vartheta)$, on $H_\bullet(Y \hookrightarrow X)$. Consider the chain map

$$\beta_n: C_n(Y \hookrightarrow X) \rightarrow C_n(X, Y), \quad \beta_n(u, v) = [u].$$

The very same argument of [BBF⁺14, Lemma 5.1] can be used to prove the following.

LEMMA 7.23. *The map*

$$H_n(\beta_n): (H_n(Y \hookrightarrow X), \|\cdot\|_1^A(\vartheta)) \rightarrow (H_n(X, Y), \|\cdot\|_1^A(\vartheta))$$

is an isometric isomorphism for every $\vartheta \in [0, \infty)$.

The dual chain complex of $(C_\bullet(Y \hookrightarrow X), \|\cdot\|_1(\vartheta))$ is the mapping cone complex associated to the chain map $i^\bullet: C_b^\bullet(X) \rightarrow C_b^\bullet(Y)$ (see Section 5.2), i.e. the complex $(C_b^\bullet(Y \hookrightarrow X), \bar{\delta}^\bullet)$, where

$$C_b^n(Y \hookrightarrow X) = C_b^n(X) \oplus C_b^{n-1}(Y), \quad \bar{\delta}^n(f, g) = (\delta^n f, -i^n(f) - \delta^{n-1}g).$$

In our setting, $C_b^n(Y \hookrightarrow X)$ is endowed with the norm

$$\|(f, g)\|_\infty(\vartheta) = \max\{\|f\|_\infty, \vartheta^{-1}\|g\|_\infty\}.$$

We denote by $Z_b^\bullet(Y \hookrightarrow X)$ the corresponding cocycles and by $H_b^\bullet(Y \hookrightarrow X)$ the corresponding cohomology. Similarly, the dual chain complex of $(Z_\bullet(Y \hookrightarrow X), \|\cdot\|_1^A(\vartheta))$ is $(Z_b^\bullet(Y \hookrightarrow X), \bar{\delta}^\bullet)$ equipped with the norm

$$\|(f, g)\|_\infty^A(\vartheta) = \max\{\|f\|_\infty, \vartheta^{-1}\|g|_A\|_\infty\}.$$

As above, this norm induces a seminorm, still denoted by $\|\cdot\|_\infty^A(\vartheta)$, on $H_b^\bullet(Y \hookrightarrow X)$. Consider the chain map

$$\beta^n: C_b^n(X, Y) \rightarrow C_b^n(Y \hookrightarrow X), \quad \beta^n(f) = (f, 0).$$

In presence of good pairs, we can prove the following.

LEMMA 7.24. *Assume that (X, Y) is a good pair and that every component of A has an amenable fundamental group. Then the map*

$$H^n(\beta^n): (H_b^n(X, Y), \|\cdot\|_\infty) \rightarrow (H_n(Y \hookrightarrow X), \|\cdot\|_\infty^A(\vartheta))$$

is an isometric isomorphism for every $\vartheta \in [0, \infty)$.

PROOF. By the very same argument of Lemma 5.12 one can deduce that $H^n(\beta^n)$ is an isomorphism. In order to show that it is isometric, we need to exploit the fact that every connected component of A has an amenable fundamental group. We denote by $B = Y \setminus A$ be the union of the remaining connected components of Y . Consider the chain map

$$\gamma^\bullet: C_b^\bullet(Y \hookrightarrow X) \rightarrow C_b^\bullet(X), \quad (f, g) \rightarrow f,$$

and the chain map

$$h^\bullet: C_b^\bullet(Y \hookrightarrow X) \rightarrow C_b^\bullet(B \hookrightarrow X),$$

induced by the inclusion $j: (X, B) \rightarrow (X, Y)$. It is easy to verify that the composition $\gamma^n \circ h^n \circ \beta^n: Z_b^n(X, Y) \rightarrow Z_b^n(X)$ actually takes values in $Z_b^n(X, B) \subseteq Z_b^n(X)$ and in fact it coincides with the chain map

$$j^n: Z_b^n(X, Y) \rightarrow Z_b^n(X, B),$$

induced by j . Since the map j induces an isometric isomorphism in bounded cohomology (Theorem 3) and since $H^n(\beta^n)$, $H^n(h^n)$ and $H^n(\gamma^n)$ are norm non-increasing, we can conclude that the isomorphism $H^n(\beta^n)$ is isometric for every $n \in \mathbb{N}_{\geq 2}$. \square

PROOF OF THEOREM 7.21. We know that $(C_b^\bullet(X, Y), \|\cdot\|_\infty)$ is the topological dual complex of

$$(C_\bullet(X, Y), \|\cdot\|_1) = (C_\bullet(X, Y), \|\cdot\|_1^A(0))$$

and $(C_b^\bullet(Y \hookrightarrow X), \|\cdot\|_\infty^A(\vartheta))$ is the topological dual complex of $(C_\bullet(Y \hookrightarrow X), \|\cdot\|_1^A(\vartheta))$. Therefore, from Lemma 7.24 and Löh Translation Principle [Löh08, Theorem 1.1], we deduce that the map

$$H_n(\beta_n): (H_n(Y \hookrightarrow X), \|\cdot\|_1^A(\vartheta)) \rightarrow (H_n(X, Y), \|\cdot\|_1)$$

is an isometric isomorphism. In conclusion, we know from Lemma 7.23 that also the map

$$H_n(\beta_n): (H_n(Y \hookrightarrow X), \|\cdot\|_1^A(\vartheta)) \rightarrow (H_n(X, Y), \|\cdot\|_1^A(\vartheta))$$

is an isometric isomorphism. We have thus shown that the seminorms $\|\cdot\|_1^A(\vartheta)$ coincide on $H_n(X, Y)$ for every $\vartheta \in [0, \infty)$, which concludes the proof. \square

7.4. Additivity for Aspherical Manifolds

The goal of this section is to prove Theorem 10, which states that the simplicial volume is additive under gluings of aspherical manifolds. We begin by setting up some notation.

SETUP 7.25. Let $n \in \mathbb{N}_{\geq 2}$. Let M_1, \dots, M_k be oriented compact triangulable n -manifolds. Let $(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$ be a pairing of pairwise disjoint boundary components of M_1, \dots, M_k . Let M be the manifold obtained by gluing M_1, \dots, M_k along the pairing. Let $q_j: M_j \rightarrow M$ denote the quotient map. We still denote by M_j the image of M_j in M via q_j . We denote by \mathcal{S} the union of the manifolds S_i^\pm , $i \in \{1, \dots, h\}$, understood as a subset of M . For all $i \in \{1, \dots, h\}$, let $j^\pm(i) \in \{1, \dots, k\}$ be the index such that S_i^\pm is a boundary component of $M_{j^\pm(i)}$.

We deduce Theorem 10 from a more general result stated in terms of good pairs. Recall that the union of two aspherical CW-complexes X and Y along a common subcomplex Z is again aspherical, provided that Z is itself aspherical and π_1 -injective in X and Y [Whi78][Edm20, Theorem 3.1]. Of course, in this case both the pairs (X, Z) and (Y, Z) are good. Building on this fact, it is immediate to deduce Theorem 10 from the following.

PROPOSITION 7.26. *In the situation of Setup 7.25, assume that the following conditions hold for every $j \in \{1, \dots, k\}$:*

- (1) $(M, \partial M)$, (M, M_j) and $(M_j, \partial M_j)$ are good pairs;
- (2) Every connected component of \mathcal{S} has amenable fundamental group.

Then $\|M, \partial M\| = \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|$.

We subdivide the proof of Proposition 7.26 in two steps. In the first step, the inequality

$$(7.3) \quad \|M, \partial M\| \leq \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|$$

is deduced as an application of Gromov Equivalence Theorem for good pairs (Corollary 7.22) and the Uniform Boundary Condition of Matsumoto and Morita [MM85]. Our proof follows closely the argument in [BBF⁺14, Remark 6.2]. In the second step, we show how the inequality

$$(7.4) \quad \|M, \partial M\| \geq \|M_1, \partial M_1\| + \dots + \|M_k, \partial M_k\|,$$

can be deduced by adapting the argument presented in Section 7.2.

PROOF OF (7.3). Since the pairs $(M_j, \partial M_j)$, $j \in \{1, \dots, k\}$, are good, by Corollary 7.22, we can choose a real fundamental cycle $c_j \in C_n(M_j, \partial M_j)$ of M_j such that

$$\|c_j\|_1 \leq \|M_j, \partial M_j\| + \varepsilon, \quad \|\partial_n c_j|_{\mathcal{S}_j}\|_1 < \varepsilon.$$

Let $c = c_1 + \dots + c_k \in C_n(M)$, where we identify every chain in M_j with its image in M . Since ∂c_j is the sum of real fundamental cycles of the boundary components of M_j , and since the gluing maps are orientation-reversing, it follows that $\partial c = \partial b + z$ for some $b \in C_n(\mathcal{S})$ and $z \in C_{n-1}(\partial M)$. Moreover, every connected component of \mathcal{S} has amenable fundamental group, hence the singular chain complex of \mathcal{S} satisfies the Uniform Boundary Condition in every positive degree [MM85, Theorem 2.8]. Therefore, we can assume that the following inequalities

$$\|b\|_1 \leq \alpha \|\partial c - z\|_1 \leq \alpha k \varepsilon$$

hold for some universal constant α . The chain $c' = c - b$ is a relative fundamental cycle for $(M, \partial M)$. It follows that

$$\|M, \partial M\| \leq \|c'\|_1 \leq \|c\|_1 + \|b\|_1 \leq \sum_{j=1}^k \|M_j, \partial M_j\| + k\varepsilon + \alpha k \varepsilon,$$

which implies (7.3), since ε is arbitrary. \square

PROOF OF (7.4). We show how to adapt the argument presented in Section 7.2 to get the desired inequality. In fact, under our assumptions, we know that the pairs $(M, \partial M)$ and (M_j, B_j) are good, for every $j \in \{1, \dots, k\}$. Therefore, in the proof of Proposition 7.7, we can replace the bi-Lipschitz isomorphisms Ψ^n and Ψ_j^n in the diagram in Figure 6 with the *isometric* isomorphisms

$$\Phi^n: H_b^n(\mathcal{A}(M), \mathcal{A}(\partial M)) \rightarrow H_b^n(M, \partial M),$$

$$\Phi_j^n: H_b^n(\mathcal{A}(M_j), \mathcal{A}(B_j)) \rightarrow H_b^n(M_j, B_j),$$

from Theorem 1. Moreover, since the action of the group $G = \Pi_M(\mathcal{S})$ on $\mathcal{A}(M)$ (resp. $\mathcal{A}(M_j)$) preserves the subcomplex $\mathcal{A}(\partial M)$ (resp. $\mathcal{A}(B_j)$), we can replace the bi-Lipschitz isomorphisms I^n and I_j^n in the diagram in Figure 6 with the *isometric* isomorphisms

$$H^n(C_b^\bullet(\mathcal{A}(M), \mathcal{A}(\partial M))_{\text{alt}}^G) \rightarrow H_b^n(\mathcal{A}(M), \mathcal{A}(\partial M)),$$

$$H^n(C_b^\bullet(\mathcal{A}(M_j), \mathcal{A}(B_j))_{\text{alt}}^G) \rightarrow H_b^n(\mathcal{A}(M_j), \mathcal{A}(B_j)),$$

from Proposition 3.15. The same construction then applies and the inequality (7.4) follows. \square

REMARK 7.27. In [BBF⁺14] the authors show that the inequality (7.4) is true under the additional assumption that *every* boundary component of M_1, \dots, M_k has amenable fundamental group. Their proof is based on the following result on bounded cohomology: given a CW-pair (X, A) such that every connected component of A has amenable fundamental group, then the map $H_b^n(X, A) \rightarrow H_b^n(X)$ is an isometric isomorphism for every $n \in \mathbb{N}_{\geq 2}$ [BBF⁺14, Theorem 2]. If we restrict our attention to *good* pairs, their argument can be adapted *verbatim* to prove inequality (7.4) in our setting, by using Theorem 3 when needed.

7.5. Subadditivity for Partial Gluings

The goal of this section is to prove Theorem 11, which establishes the subadditivity of simplicial volume under partial gluings, provided that every boundary component involved in the gluing is amenable, and that the simplicial volume of the gluing loci (and their complements) vanishes.

We consider the situation of Setup 7.6, which we recall briefly. Let $n \in \mathbb{N}_{\geq 2}$. Let M_1, \dots, M_k be oriented compact connected n -manifolds with non-empty boundary. Let

$$(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$$

be a pairing of some oriented compact connected pairwise-disjoint submanifolds of

$$\partial M_1 \sqcup \dots \sqcup \partial M_k$$

of codimension 0. Let M be the manifold obtained by gluing M_1, \dots, M_k along some orientation-reversing homeomorphisms $f_i: S_i^+ \rightarrow S_i^-$, $i \in \{1, \dots, h\}$. We denote by \mathcal{S} the union of the manifolds S_i^\pm , $i \in \{1, \dots, h\}$, understood as a subset of M . Let B_j be the compact $(n-1)$ -submanifold of M_j whose interior is the complement in ∂M_j of the gluing loci. We have that $\partial B_j = \partial \mathcal{S} \cap \partial M_j$. We recall for the convenience of the reader the assumptions of Theorem 11:

- Every connected component of ∂M_j has amenable fundamental group;
- Every connected component of $\partial \mathcal{S}$ has amenable fundamental group;
- $\|S_i^\pm, \partial S_i^\pm\| = 0$ and $\|B_j, \partial B_j\| = 0$.

Our strategy to prove Theorem 11 is to construct an efficient cycle computing the ℓ^1 -norm of the fundamental class of M by suitable modifying fundamental classes of M_j . The construction closely follows the one of Lemma 7.13. The amenability of the boundaries is key in this operation to use the Uniform Boundary Condition of Matsumoto and Morita [MM85] and the version of Gromov Equivalence Theorem established in [BBF⁺14].

PROOF OF THEOREM 11. In the following, with an abuse of notation, we identify any chain in M_j and \mathcal{S} with the corresponding chain in M via the corresponding inclusions. Let $w_j \in C_n(M_j)$ be relative fundamental cycles of M_j , and let $w = w_1 + \dots + w_k \in C_n(M)$. We need to modify w in order to get a fundamental cycle of M with controlled norm. Let $\varepsilon \in \mathbb{R}_{>0}$. Since every connected component of M_j has amenable fundamental group, by Gromov Equivalence Theorem [BBF⁺14, Corollary 6], we can assume without loss of generality that

$$\|w_j\|_1 \leq \|M_j, \partial M_j\| + \varepsilon, \quad \|\partial w_j\|_1 \leq \varepsilon.$$

Let now $\sigma_i^+ \in C_{n-1}(S_i^+)$ be a relative fundamental cycle of S_i^+ , and let

$$\sigma_i^- = -(f_i)_*(\sigma_i^+) \in C_{n-1}(S_i^-)$$

be the corresponding fundamental cycle of S_i^- . Let $\beta_j \in C_{n-1}(B_j)$ be a relative fundamental cycle of B_j . We denote by σ_j the sum of the σ_i^\pm where $i \in \{1, \dots, h\}$ is such that $S_i^\pm \subseteq M_j$. Since $\|S_i^\pm, \partial S_i^\pm\|$ and $\|B_j, \partial B_j\|$ vanish by assumption for every $i \in \{1, \dots, h\}$ and every $j \in \{1, \dots, k\}$, we can assume without loss of generality that

$$\|\beta_j\|_1 \leq \varepsilon, \quad \|\sigma_j\|_1 \leq \varepsilon,$$

which in turns imply that $\|\partial \beta_j\|_1 \leq n\varepsilon$ and $\|\partial \sigma_j\|_1 \leq n\varepsilon$. Moreover, the amenability of ∂B_j ensures that the chain complex $C_\bullet(\partial B_j)$ satisfies the Uniform Boundary

Condition in every positive degree [MM85]. Let $K \in \mathbb{R}_{>0}$ be the UBC-constant for $C_\bullet(\partial B_j)$ in degree $n-2$. Since $\partial\sigma_j$ and $\partial\beta_j$ represent opposite classes in $H_{n-2}(\partial B_j)$, it follows that there exists $d_j \in C_{n-1}(\partial B_j)$ such that

$$\partial d_j = \partial\sigma_j + \partial\beta_j, \quad \|d_j\|_1 \leq K\|\partial d_j\|_1 \leq 2nK\varepsilon.$$

Therefore $b_j = d_j - \beta_j$ is a fundamental cycle of B_j , and $c_j = \sigma_j - b_j \in C_{n-1}(\partial M_j)$ is a fundamental cycle of ∂M_j . Since every connected component of ∂M_j has amenable fundamental group, then also the chain complex $C_\bullet(\partial M_j)$ satisfies the Uniform Boundary Condition in every positive degree. Let $L \in \mathbb{R}_{>0}$ be the UBC-constant for $C_\bullet(\partial M_j)$ in degree $n-1$. Since ∂w_j and c_j represent the same class in homology, there exists $u_j \in C_n(\partial M_j)$ such that

$$\partial u_j = c_j - \partial w_j, \quad \|u_j\|_1 \leq L\|\partial u_j\|_1 \leq 2nKL\varepsilon + 3L\varepsilon.$$

We set now $z_j = w_j + u_j \in C_n(M_j)$, which is a fundamental cycle of M_j such that $\partial z_j = \sigma_j - b_j$. It follows from the construction then $z = z_1 + \dots + z_k$ is a fundamental cycle of M such that

$$\|z\|_1 \leq \sum_{j=1}^k \|z_j\|_1 \leq \sum_{j=1}^k \|M_j, \partial M_j\| + k\varepsilon(1 + 3L + 2nKL).$$

Since ε was arbitrary, we conclude that

$$\|M, \partial M\| \leq \sum_{j=1}^k \|M_j, \partial M_j\|.$$

□

REMARK 7.28. In the Proof of Theorem 11 we use the version of Gromov's Equivalence Theorem established in [BBF⁺14]. To this end, we need to assume the technical assumption from [BBF⁺14] that also the boundary components which remain unglued need to have amenable fundamental group. One could avoid this by using the version of Gromov's Equivalence Theorem for good pairs (Theorem 7.21) instead. The price to pay are stronger assumptions on higher homotopy.

7.6. Beyond Amenability

Most additivity results for the simplicial volume of manifolds obtained by gluings require some amenability condition on the gluing loci. The goal of this section is to present a combination theorem for generalized Lusternik-Schnirelmann categories which has been obtained by the author in collaboration with Kevin Li and Clara Löh in [CLL25]. As an application, we discuss vanishing results for the simplicial volume of manifolds obtained by gluings along not necessarily amenable boundaries.

In Gromov's Vanishing Theorem, as well as in its relative versions (Theorem 6), the vanishing of the comparison map is typically expressed in terms of the *multiplicity* of some amenable open cover. However, the condition on multiplicity can equivalently be reformulated in terms of the *cardinality* of the open cover (Proposition 7.32). This reformulation is inspired by the classical notion of the *Lusternik-Schnirelmann category*, which is defined as the minimal cardinality of an open cover by contractible subsets.

DEFINITION 7.29. Let X be a path-connected topological space. The *amenable category* $\text{cat}_{\text{Am}}(X)$ of X is defined as the minimum $n \in \mathbb{N}$ such that there exists an open cover $\{U_0, U_1, \dots, U_n\}$ of X by amenable subsets. If such an integer does not exist, we set $\text{cat}_{\text{Am}}(X) = \infty$.

Let G be a group. We define the amenable category of G as

$$\text{cat}_{\text{Am}}(G) = \text{cat}_{\text{Am}}(BG),$$

where BG is a classifying space for G . If there exists a π_1 -isomorphism $X \rightarrow Y$, then pulling back open covers shows $\text{cat}_{\text{Am}}(X) \leq \text{cat}_{\text{Am}}(Y)$. In particular, $\text{cat}_{\text{Am}}(G)$ is well-defined and we have

$$\text{cat}_{\text{Am}}(X) \leq \text{cat}_{\text{Am}}(\pi_1(X)).$$

We have $\text{cat}_{\text{Am}}(G) = 0$ if and only if G is amenable. Moreover, $\text{cat}_{\text{Am}}(G) = 1$ if and only if G is not amenable and G is the fundamental group of a graph of groups whose vertex groups (and hence also edge groups) are amenable [CLM22, Proposition 5.3]. In the literature [CLM22, LM22] also a different normalization is used, producing values that are larger by 1 than ours.

REMARK 7.30. We have that $\text{cat}_{\text{Am}}(X) \leq \dim(X)$ for every simplicial complex X . Indeed, the cover of X by the open stars of the first barycentric subdivision yields an open cover by contractible (hence amenable) subsets, whose cardinality is equal to $\dim(X)$. Since every CW-complex is homotopy equivalent to a simplicial complex, and since cat_{Am} is homotopy invariant, then the same inequality holds for CW-complexes. In particular, for every group G , we have that

$$\text{cat}_{\text{Am}}(G) \leq \text{gd}(G),$$

where $\text{gd}(G)$ is the minimal dimension of a CW-model for the classifying space BG .

Of course, every open cover of cardinality $n \in \mathbb{N}$ has multiplicity at most n . On the other hand, the following lemma shows that also the converse implication holds, at least for topological spaces admitting partitions of unity subordinate to every open cover.

LEMMA 7.31 ([CLOT03, Lemma A.4]). *Let X be a topological space and let $n \in \mathbb{N}$. Let \mathcal{U} be an open cover of X of multiplicity $n + 1$ with a partition of unity subordinate to the cover. Then there exist an index set B and an open covering*

$$\{V_{i\beta} \mid i \in \{1, \dots, n + 1\}, \beta \in B\}$$

of X refining \mathcal{U} such that $V_{i\beta} \cap V_{i\beta'} = \emptyset$ for all $\beta \neq \beta'$.

It follows from Lemma 7.31 that Gromov's Vanishing Theorem (Theorem 6.2) admits the following equivalent formulation in terms of the amenable category.

PROPOSITION 7.32. *Let X be a CW-complex. Then the comparison map*

$$\text{comp}^n: H_b^n(X) \rightarrow H^n(X)$$

vanishes for every $n \geq \text{cat}_{\text{Am}}(X)$.

We prove a combination theorem for the amenable category of groups. Our argument follows the classical strategy to modify open covers while controlling the multiplicity (see the proof of Theorem 6.8).

THEOREM 7.33 ([**CLL25**, Corollary 1.5]). *Let G be the fundamental group of a finite graph of groups with vertex groups $(G_v)_{v \in V}$ and edge groups $(G_e)_{e \in E}$. Then*

$$\text{cat}_{\text{Am}}(G) \leq \max \left\{ \sup_{v \in V} \text{cat}_{\text{Am}}(G_v), \sup_{e \in E} (\text{gd}(G_e) + 1) \right\}.$$

PROOF. We present the argument for amalgamated products. The same strategy can be easily adapted to the case of HNN extensions, and more generally, to arbitrary graphs of groups. Let G denote the amalgamated product of G_1 and G_2 along a common subgroup G_0 . Then a model for the classifying space BG can be constructed as

$$BG = BG_1 \cup (BG_0 \times [0, 1]) \cup BG_2,$$

where $BG_0 \times \{0\}$ is glued to BG_1 and $BG_0 \times \{1\}$ is glued to BG_2 via the map induced by the subgroup inclusion. Let $\mathcal{U}_1 = \{U_{i_1} \mid i_1 \in I_1\}$ and $\mathcal{U}_2 = \{U_{i_2} \mid i_2 \in I_2\}$ be open covers of BG_1 and BG_2 , respectively, by amenable subsets and of minimal cardinality. The index sets I_1 and I_2 should be understood to be disjoint.

After realizing BG_0 as a simplicial complex of dimension $n = \text{gd}(G_0)$, we may take iterated barycentric subdivisions so that the open cover of BG_0 by open stars is a refinement of both the open covers induced by \mathcal{U}_1 and \mathcal{U}_2 on BG_0 . More precisely, there exists an open cover $\mathcal{V} = \{V_j \mid j \in J\}$ of BG_0 by contractible subsets such that, for every $j \in J$ there exists $i_1(j) \in I_1$ and $i_2(j) \in I_2$ such that $V_j \times \{0\} \subseteq U_{i_1(j)}$ and $V_j \times \{1\} \subseteq U_{i_2(j)}$. Moreover, the multiplicity of \mathcal{V} is at most n . By Lemma 7.31, up to taking further refinements of \mathcal{V} , there exists a map

$$c: J \rightarrow \{0, \dots, n\}$$

such that $c(j) \neq c(j')$ whenever $V_j \cap V_{j'} \neq \emptyset$. We consider positive real numbers ε and $a_k, k \in \{0, \dots, n\}$, such that

$$0 < \varepsilon < a_0 < \dots < a_n < 1 - \varepsilon < 1$$

and $a_{k-1}, a_{k+1} \notin (a_k - \varepsilon, a_k + \varepsilon)$. For every $i_1 \in I_1$, we set

$$J(i_1) = \{j \in J \mid i_1(j) = i_1\},$$

and, in the same way, we define $J(i_2)$. For every $i_1 \in I_1$, we set

$$W_{i_1} = U_{i_1} \cup ((U_{i_1} \cap (BG_0 \times \{0\})) \times [0, \varepsilon]) \cup \bigcup_{j \in J(i_1)} V_{c(j)} \times [0, a_{c(j)} + \varepsilon),$$

and, for every $i_2 \in I_2$, we set

$$W_{i_2} = \bigcup_{j \in J(i_2)} V_{c(j)} \times (a_{c(j)} - \varepsilon, 1] \cup ((U_{i_2} \cap (BG_0 \times \{1\})) \times (1 - \varepsilon, 1]) \cup U_{i_2}.$$

It is easy to check that $\mathcal{W} = \{W_{i_1} \mid i_1 \in I_1\} \cup \{W_{i_2} \mid i_2 \in I_2\}$ is an open cover of BG by amenable subsets. Moreover, the multiplicity of \mathcal{W} is bounded from above by

$$\max \{ \text{cat}_{\text{Am}}(G_1), \text{cat}_{\text{Am}}(G_2), \text{gd}(G_0) + 1 \},$$

hence the same estimate holds for the amenable category of G by Lemma 7.31. \square

REMARK 7.34. Theorem 7.33 is deduced in [**CLL25**] from the following more general statement. For every group G and every (non-equivariantly) contractible G -CW-complex X , we have that

$$(7.5) \quad \text{cat}_{\text{Am}}(G) \leq \max \left\{ \sup_{v \in \Sigma_0} \text{cat}_{\text{Am}} G_v, \sup_{\sigma \in \Sigma_{\geq 1}} (\text{gd}(G_\sigma) + \dim(\sigma)) \right\},$$

where Σ_0 and $\Sigma_{\geq 1}$ denote the sets of all 0-cells and all positive dimensional cells of X , respectively, and G_σ denotes the stabilizer of the cell σ . Theorem 7.33 clearly follows from (7.5) by considering the action on the Bass-Serre tree. Instead of constructing open covers explicitly, as done in our proof of Theorem 7.33, the estimate (7.5) is derived in [CLL25] using an equivalent characterization of the amenable category, formulated in terms of the dimension of G -CW-complexes with amenable isotropy [CLM22, Lemma 7.6], together with dimensional estimates for G -pushouts arising in the construction of classifying spaces. We also underline the fact that the upper bound (7.5), hence Theorem 7.33, works more generally for generalized Lusternik-Schnirelmann categories with more general constraints on the fundamental group.

We now turn our attention on applications of Theorem 7.33 to the vanishing of simplicial volume of manifolds obtained by gluings (Theorem 13). We first establish a relative version of the Vanishing Theorem in presence of control over the geometric dimension of boundary components.

LEMMA 7.35. *Let M be an oriented compact connected n -manifold with non-empty boundary. Assume that:*

- *Every boundary component of M is π_1 -injective in M ;*
- *Every boundary component of M has fundamental group of geometric dimension $\leq n - 2$;*
- $\text{cat}_{\text{Am}}(\pi_1(M)) \leq n - 1$.

Then $\|M, \partial M\| = 0$.

PROOF. By duality (Lemma 3.1), the simplicial volume of $(M, \partial M)$ is zero if and only if the comparison map

$$H_b^n(M, \partial M) \rightarrow H^n(M, \partial M)$$

vanishes. Let S_1, \dots, S_h be the boundary components of M . Let $\pi_1(\mathcal{S})$ denote the collection $(\pi_1(S_i))_i$ of subgroups of $\pi_1(M)$. By the Mapping Theorem [FM23, Theorem 5] and the Five Lemma, it suffices to show that the comparison map for the group pair $(\pi_1(M), \pi_1(\mathcal{S}))$ vanishes in degree n . We consider the following diagram

$$\begin{array}{ccccc} \bigoplus_{i=1}^h H_b^{n-1}(\pi_1(S_i)) & \longrightarrow & H_b^n(\pi_1(M), \pi_1(\mathcal{S})) & \longrightarrow & H_b^n(\pi_1(M)) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{i=1}^h H^{n-1}(\pi_1(S_i)) & \longrightarrow & H^n(\pi_1(M), \pi_1(\mathcal{S})) & \longrightarrow & H^n(\pi_1(M)) \end{array}$$

where the rows are portions of the long exact sequences of pairs and the vertical maps are comparison maps. Since $\text{gd}(\pi_1(S_i)) \leq n - 2$ for all $i \in \{1, \dots, h\}$, the lower left module is trivial. Since $\text{cat}_{\text{Am}}(\pi_1(M)) \leq n - 1$, the right vertical map vanishes by the Vanishing Theorem (Proposition 7.32). It follows that the middle vertical map vanishes, as desired. \square

We consider now the situation described in Setup 7.25, where some oriented compact connected n -manifolds M_1, \dots, M_k are glued along a pairing

$$(S_1^+, S_1^-), \dots, (S_h^+, S_h^-)$$

of some of their boundary components. We denote by M the glued manifold. We know that, if all boundary components are amenable, then the simplicial volume is

subadditive (Theorem 7.2). In particular, the simplicial volume of the gluing is zero if the simplicial volume of all pieces is zero. We prove a similar vanishing result. While our assumptions on the pieces are stronger than the vanishing of simplicial volume, so is our conclusion on the gluing. The advantage of our result is that the boundary components need not be amenable. Theorem 13 directly follows from the following slightly more general result.

THEOREM 7.36. *In the situation of Setup 7.25, suppose that the following hold:*

- *For all $i \in \{1, \dots, h\}$, the subspace S_i^\pm is π_1 -injective in $M_{j^\pm(i)}$;*
- *For all $i \in \{1, \dots, h\}$, we have $\text{gd}(\pi_1(S_i^+)) \leq n - 2$;*
- *For all $j \in \{1, \dots, k\}$, we have $\text{cat}_{\text{Am}}(\pi_1(M_j)) \leq n - 1$.*

Then $\text{cat}_{\text{Am}}(M) \leq n - 1$. Moreover, if every boundary component of the M_j is π_1 -injective and has fundamental group of geometric dimension $\leq n - 2$, then $\|M, \partial M\| = 0$.

PROOF. Since $\text{cat}_{\text{Am}}(M) \leq \text{cat}_{\text{Am}}(\pi_1(M))$, we will prove the stronger statement $\text{cat}_{\text{Am}}(\pi_1(M)) \leq n - 1$. Since M is constructed as a gluing along π_1 -injective boundary components, $\pi_1(M)$ is the fundamental group of a finite graph of groups with vertex groups $\pi_1(M_j)$ and edge groups $\pi_1(S_i^+)$. Then the claim follows from Theorem 7.33. The vanishing of simplicial volume follows from Lemma 7.35. \square

We give a concrete application of our results in the case of twisted doubles. Let $(M, \partial M)$ be an oriented compact connected manifold with non-empty boundary. Let $f: \partial M \rightarrow \partial M$ be an orientation-preserving self-homeomorphism. The *twisted double* $D_f(M)$ of M along f is the oriented closed connected manifold defined as

$$D_f(M) = M \cup_f -M,$$

where $-M$ is a copy of M with the opposite orientation. For the identity $f = \text{id}$ on ∂M , one obtains the (*untwisted*) *double* $D_{\text{id}}(M)$ of M . In this case, since the gluing is performed via the identity, any relative fundamental cycle of M can be juxtaposed with its reversed copy (which is a fundamental cycle of $-M$), thus yielding a fundamental cycle of $D_{\text{id}}(M)$. It follows that the inequality

$$(7.6) \quad \|D_f(M)\| \leq 2 \cdot \|M, \partial M\|$$

holds when $f = \text{id}$. Moreover, by Theorem 7.2, if every boundary component of ∂M has amenable fundamental group, then (7.6) holds for every gluing map f . In this case, if $\|M, \partial M\| = 0$, then the simplicial volume of $D_f(M)$ vanishes. However, (7.6) does not hold in general [**BBF**⁺14, Remark 6.4]. As an application of Theorem 7.36, we obtain the following vanishing criterion for the simplicial volume of twisted doubles.

COROLLARY 7.37. *Let M be an oriented compact connected n -manifold with non-empty boundary. Suppose that the following hold:*

- *Every boundary component of M is π_1 -injective in M ;*
- *Every boundary component of M has fundamental group of geometric dimension $\leq n - 2$;*
- $\text{cat}_{\text{Am}}(\pi_1(M)) \leq n - 1$.

Then the simplicial volume of $D_f(M)$ vanishes for every orientation-preserving self-homeomorphism $f: \partial M \rightarrow \partial M$.

EXAMPLE 7.38. Let $n \in \mathbb{N}$ with $n \geq 4$. Let V be an oriented compact $(n - 1)$ -manifold obtained by removing an open ball from an aspherical closed connected $(n - 1)$ -manifold. The boundary ∂V is an $(n - 2)$ -sphere. Consider the n -manifold that is the boundary connected sum

$$M = (V \times S^1) \natural (V \times S^1)$$

of two copies of $V \times S^1$. The boundary ∂M is homeomorphic to the connected sum $(S^{n-2} \times S^1) \# (S^{n-2} \times S^1)$ and ∂M is π_1 -injective in M . Since $\pi_1(\partial M)$ is a free group of rank 2, we have $\text{gd}(\pi_1(\partial M)) = 1$. Moreover, we have $\text{cat}_{\text{Am}}(\pi_1(M)) \leq n - 1$ by Theorem 7.33 and a standard estimate for cat_{Am} of products [CLM22, Proposition 2.7]. Then Corollary 7.37 shows that the simplicial volume of every twisted double of M is zero.

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