Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 6(2007), pp. 195–207.

Minimal surfaces in three dimensional pseudo-Hermitian manifolds

Andrea Malchiodi

Abstract[1]. We consider surfaces immersed in three-dimensional pseudo-Hermitian manifolds. A notion of *pseudo mean curvature* (p-mean curvature for short) is defined, which extends naturally some previous concepts given in the Heisenberg group. We derive this notion in different natural ways, which are all equivalent, and then study the p-minimal surface equation. Of great importance is the study of the *singular set*, which allows us to classify entire graphs in the Heisenberg group. Some applications of this result are then given, and some related issues are discussed.

1. Introduction

The concept of minimal surface plays an important role in several contexts. It appears in physics in the study of phase transitions or in the theory of general relativity, but has also been used in differential geometry to study basic properties of manifolds. We mention for example the two papers [21], [22] where the authors employ minimal surfaces to derive properties of manifolds with positive scalar curvature or to prove the positive mass theorem.

We are interested here in analogous concepts in pseudo-Hermitian manifolds, in order to finding possible similar applications. First of all, we recall some basic notions.

Let M be a three dimensional manifold. A *contact structure* ξ on M is a completely non-integrable two-dimensional distribution, namely the Lie bracket of two (linearly independent) vector fields tangent to ξ is always non parallel to ξ. A *contact form* Θ is a 1-form which annihilates ξ. We will always assume it *oriented*, namely that $d\Theta(u,v) > 0$ if (u,v) is an oriented basis of ξ. The *Reeb vector field* associated to Θ is the unique vector field Θ such that $\Theta(T) = 1$ and such that $d\Theta(T,\cdot) = 0$. A *CR structure* compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -Id$. We assume that also J is oriented, namely that for every non-zero vector field X, the couple (X, JX) is an oriented basis of ξ.

---

[1]Author's address: A. Malchiodi, S.I.S.S.A., Via Beirut 2-4, 34014 Trieste, Italy; e-mail: `malchiod@sissa.it` .

Keywords. Pseudo-Hermitian geometry, minimal surfaces, Heisenberg group.
AMS Subject Classification. 53A10, 53C42, 49Q10.

A *pseudo-Hermitian manifold* is a manifold endowed with a CR structure and with a global contact form Θ. This gives rise to a natural volume form

$$V(\Omega) = \int_\Omega \Theta \wedge d\Theta$$

and to a metric on ξ called *Levi form*

$$L_\Theta(v,w) = d\Theta(v, Jw) \ .$$

Let e_1 be a local section of ξ with unit length, namely satisfying $L_\Theta(e_1,e_1)=1$, and let $e_2 = Je_1$, so (e_1,e_2) is an oriented basis of ξ. Let $\{\Theta, e^1, e^2\}$ be the forms dual to the triple $\{T, e_1, e_2\}$. Then we have the *structure equations*

$$(S1) \qquad d\Theta = 2e^1 \wedge e^2 \ ;$$

$$(S2) \qquad de^1 = -e^2 \wedge \omega \operatorname{mod} \Theta \quad ; \quad de^2 = e^1 \wedge \omega \operatorname{mod} \Theta \ ,$$

for some 1-form ω called *connection form*. The *Tanaka-Webster connection* is defined by

$$\nabla^{p.h.} e_1 = \omega \otimes e_2 \quad , \quad \nabla^{p.h.} e_2 = -\omega \otimes e_1 \ ,$$

while the *Tanaka-Webster curvature* (see [24] and [25]) is given by

$$(1) \qquad d\omega(e_1,e_2) = -2W \ .$$

Given a function f and a vector field V tangent to ξ we define the *subgradient* of f and the *subdivergence* of V as

$$\nabla_b f = (e_1 f)e_1 + (e_2 f)e_2 \quad ; \quad div_b V = L_\Theta(\nabla^{p.h.}_{e_1} V, e_1) + L_\Theta(\nabla^{p.h.}_{e_2} V \ , e_2) \ ,$$

and we have also the *sublaplacian*

$$\Delta_b f = div_b \left(\nabla_b f \right) \ .$$

For the Heisenberg group H^1 we have the standard choices

$$(2) \quad \hat{e}_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \quad , \quad \hat{e}_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z} \quad , \quad \hat{T} = \frac{\partial}{\partial z} \quad , \hat{\Theta} = x\, dy - y\, dx + dz \ .$$

We are now in position to define the p-mean curvature of a two dimensional (regular) surface Σ immersed in M. If $T_p\Sigma \neq \xi(p)$, we let $e_1(p)$ be the unit vector belonging to $T_p\Sigma \cap \xi(p)$ (unique up to the sign), and $e_2(p) = J(p)e_1(p)$. Then we define the (p)-mean curvature H in three equivalent ways

1) As second variation of the volume: if Σ is the boundary of an open set Ω, then taking a variation fe_2 of Σ (if e_1 is well defined) we have $\delta_{fe_2} V(\Omega) = \int_\Sigma f\Theta \wedge e^1$. We call $\Theta \wedge e^1$ the *p-area form* of Σ: taking then a variation of the p-area we obtain a scalar multiple of the p-area itself, and we define H by

$$\delta_{fe_2} \int_\Sigma \Theta \wedge e^1 := - \int_\Sigma fH\Theta \wedge e^1 \ .$$

2) Viewing Σ as a level surface: if $\Sigma = \{\psi = 0\}$, then we set

$$H = -div_b \left(\frac{\nabla_b \psi}{|\nabla_b \psi|} \right) \ .$$

3) Using the Tanaka-Webster connection: as for curves of unit velocity in the plane, whose curvature is perpendicular to the tangent vector, in this case we have that $\nabla^{p.h.}_{e_1} e_1$ is a scalar multiple of e_2, so we define

$$\nabla^{p.h.}_{e_1} e_1 = He_2 \ .$$

The points for which $T\Sigma = \xi$ are called *singular* and at these the vector e_1 is not well-defined. For the Heisenberg group the first two definitions coincide with those given in [3], [8], and [18]. The p-area element $\Theta \wedge e^1$ coincides with the three-dimensional Hausdorff measure of Σ, considered in [1] and [9]. In particular these notions, especially in the framework of geometric measure theory, have been used to study existence or regularity properties of minimizers for the relative perimeter or extremizers of isoperimetric inequalities (see, e.g., [8], [10], [14], [15], and [17]).

2. Minimal graphs in the Heisenberg group

Let $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let Σ be the graph of u

$$\Sigma = \left\{(x, y, u(x, y)) \mid (x, y) \in \mathbb{R}^2\right\} .$$

Recall that e_1 is the unique (up to a sign) vector in $T\Sigma \cap \xi$ which is unitary with respect to the Levi metric. By the pseudo-Hermitian structure of the Heisenberg group, which is determined by (2), for a graph we have that

$$e_1 = \frac{1}{D}\left[-(u_y + x)\begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} + (u_x - y)\begin{pmatrix} 0 \\ 1 \\ -x \end{pmatrix}\right] ,$$

where

$$D = \left[(u_x - y)^2 + (u_y + x)^2\right]^{1/2} .$$

One also finds that

$$H = \frac{1}{D^3}\{(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy}\} ,$$

so a graph is p-minimal if and only if the following equation holds

$$(*) \qquad (u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy} = 0 .$$

The *singular points* of Σ (or of u) are given by

$$S(u) = \{(x, y) \ : \ u_x - y = u_y + x = 0\} ,$$

while the p-area of the graph of u over Ω can be written in parametric form as

$$\mathcal{F}(u) = \int_\Omega \underbrace{\sqrt{(u_x - y)^2 + (u_y + x)^2}}_{\text{p-area}} \, dx\, dy .$$

This section is devoted to one of the main results in [6] (see also [11]), which is the following.

Theorem A. *The only entire C^2 solutions to $(*)$ are of the following two forms*

$$(3) \qquad u = ax + by + c ;$$

$$(4) \qquad u = -abx^2 + (a^2 - b^2)xy + aby^2 + g(-bx + ay) ,$$

for some real constants a, b, c with $a^2 + b^2 = 1$ and for some function $g : \mathbb{R} \to \mathbb{R}$ of class C^2.

We give some sketch of the proof, referring to [6] for complete details. We begin with the following geometric characterization of p-minimal graphs (which holds for more general p-minimal surfaces as well) in the three dimensional Heisenberg group.

Lemma 1. *Every minimal graph in H^1 is locally a ruled surface.*

Proof. Given a non-singular point of Σ, denote the projection of $-e_2$ ($-e_1$, respectively) onto the xy-plane by $N(u)$ or simply N ($N^\perp(u)$ or simply $N^\perp$, respectively). Recall that $e_1 = [-(u_y + x)\hat{e}_1 + (u_x - y)\hat{e}_2]/D$ where $D = [(u_x - y)^2 + (u_y + x)^2]^{1/2}$, so $N^\perp = [(u_y + x)\partial_x - (u_x - y)\partial_y]/D$. Write $(u_y + x)D^{-1} = \sin\theta$, $(u_x - y)D^{-1} = \cos\theta$ for some local function θ. Then we have that

$$N = (\cos\theta)\partial_x + (\sin\theta)\partial_y \,, \tag{5}$$

$$N^\perp = (\sin\theta)\partial_x - (\cos\theta)\partial_y \,. \tag{6}$$

Using the second definition of p-mean curvature one finds

$$0 = div\, N = (\cos\theta)_x + (\sin\theta)_y = -(\sin\theta)\theta_x + (\cos\theta)\theta_y \,. \tag{7}$$

Now using (6) and the last equation we deduce that

$$(N^\perp)^2 = \sin^2\theta\, \partial_x^2 - 2\sin\theta\cos\theta\, \partial_x\, \partial_y + \cos^2\theta\, \partial_y^2 \,.$$

Below, we will call *characteristic curves* both the integral curves of e_1 on Σ and their projections onto the xy plane, namely the integral curves of $N^\perp$. Along a characteristic curve $(x(s), y(s))$ (in the plane), where s is a unit-speed parameter, we have the equations

$$\frac{dx}{ds} = \sin\theta \quad , \quad \frac{dy}{ds} = -\cos\theta \tag{8}$$

by (6). Noticing that $u_x = (\cos\theta)D + y$, $u_y = (\sin\theta)D - x$, we have

$$\frac{du}{ds} = u_x\,\frac{dx}{ds} + u_y\,\frac{dy}{ds} = [(\cos\theta)D + y]\sin\theta + [(\sin\theta)D - x](-\cos\theta) = \tag{9}$$

$$= x\cos\theta + y\sin\theta \,,$$

$$\frac{d\theta}{ds} = \theta_x\,\frac{dx}{ds} + \theta_y\,\frac{dy}{ds} = \theta_x\sin\theta - \theta_y\cos\theta = 0 \tag{10}$$

by (7). From (8), (9) and (10) we find that

$$\frac{d^2x}{ds^2} = 0 \quad , \quad \frac{d^2y}{ds^2} = 0 \quad , \quad \frac{d^2u}{ds^2} = 0 \,. \tag{11}$$

From the above formulas it follows that $s \mapsto (x(s), y(s), u(s))$ parameterizes a straight line in H^1 (identified with $\mathbb{R}^3$), and $(x(s), y(s))$ is a plane curve parameterized with unit speed. This concludes the proof.

□

Remark 1. More in general, the p-mean curvature of a graph Σ coincides with the ordinary curvature of the xy projection of the characteristic curves of Σ.

We characterize next the singular points of Σ with the next proposition, which corresponds to Theorem B in [6]. We do not give the proof here for reasons of brevity, and we refer to the aforementioned paper for details.

Proposition 1. *Let Ω be a domain in the xy-plane. Let $u \in C^2(\Omega)$ be such that $div\, N(u) = H$ in $\Omega \setminus S(u)$. Suppose $|H(p)| \leq C(1/r(p))$ (where $r(p) = |p - p_0|$) for a positive constant C and for $p \in \Omega \setminus S(u)$ near a singular point $p_0 \in S(u)$.*

Then either p_0 is isolated in $S(u)$ or there exists a small neighborhood of p_0 which intersects $S(u)$ in exactly a C^1 smooth curve through p_0. Setting

$$U = \begin{bmatrix} u_{xx} & u_{xy}-1 \\ u_{xy}+1 & u_{yy} \end{bmatrix}, \tag{U}$$

p_0 is isolated in $S(u)$ if and only if $det(U)(p_0) \neq 0$.

It is also crucial to understand the behavior of characteristic curves near the singular points. We have the following two results (which are not proved here) concerning isolated and non-isolated singular points respectively.

Lemma 2. *Suppose p_0 is an isolated singular point of a minimal graph Σ of class C^2. Then in a neighborhood of p_0 we have*

$$N^{\perp}(p) = \frac{p-p_0}{|p-p_0|} \quad ; \quad p \neq p_0 \,.$$

Moreover we have that $u_{xx} = u_{xy} = u_{yy} = 0$ at p_0.

We consider next the case of a non-isolated singular point. We let B denote a small ball centered at $p_0 \in S(u)$ which is contained in the neighborhood given in Proposition 1. Since we are assuming p_0 to be not isolated, $S(u) \cap B$ consists of a C^1 curve. We let B_+ and B_- denote the subsets of B which are divided by $S(u)$. We have then the following result.

Lemma 3. *Let u be of class C^2, and suppose $p_0 \in S(u)$ is a non isolated singular point. Then both the limits $N(u)(p_0^+) \equiv \lim_{p\in B^+ \to p_0} N(u)(p)$ and $N(u)(p_0^-) \equiv \lim_{p\in B^- \to p_0} N(u)(p)$ exist. Moreover $N(u)(p_0^+) = -N(u)(p_0^-)$. Moreover the characteristic curves near p_0 intersect $S(u)$ transversally.*

Proof of Theorem A. First of all we claim that if there exists an isolated singular point, then this must be unique. In fact, assuming we have two such points $p_1, p_2 \in S(u)$, by Lemma 2 and Lemma 3, since the characteristic curves in the xy plane are straight lines (see the proof of Lemma 1), there exist two distinct straight lines passing through p_1, p_2 respectively and intersecting at a third point $q \notin S(u)$. But then at q $N^{\perp}(u)$ would have two values, which is a contradiction.

On the other hand, assuming that there are no isolated singular points, we get that the projections onto the xy plane of the characteristic curves are parallel. In fact, since all the singular points (if any) are non-isolated, by Lemma 3 we know that the limit of N (or of $N^{\perp}$) exists through singular curves, and therefore two characteristic lines cannot intersect.

In conclusion, we have the following two cases.

Case 1: $S(u)$ contains one isolated singular point. Let p_0 be the singular point, and let r, ϑ denote the polar coordinates with center p_0. We can write $\pm\check{N}^{\perp}(u) = \partial/\partial r$ by Lemma 2 and Lemma 3. By the last equation in (11) we have

$$u_{rr} = \frac{\partial^2 u}{\partial r^2} = 0 \tag{12}$$

on the whole xy-plane except for p_0. Integrating, it follows that $u = rf(\vartheta) + g(\vartheta)$ for some C^2 functions f, g. Since u is continuous at $p_0 = (x_0, y_0)$ (where $r = 0$), $u(x_0, y_0) = g(\vartheta)$ for all ϑ, so we deduce that g is a constant function, say $g \equiv c$.

Also $f(\vartheta) = f(\vartheta + 2\pi)$ implies that we can write $f(\vartheta) = \tilde{f}(\cos\vartheta, \sin\vartheta)$ where $\tilde{f}$ is C^2 in $\alpha = \cos\vartheta$ and $\beta = \sin\vartheta$. By direct computation we have $u_x = u_r r_x + u_\vartheta \vartheta_x = \alpha\tilde{f} + \beta^2\tilde{f}_\alpha - \alpha\beta\tilde{f}_\beta$ in which $\tilde{f}_\alpha = \partial\tilde{f}/\partial\alpha$, $\tilde{f}_\beta = \partial\tilde{f}/\partial\beta$, etc. and we have used $\vartheta_x = -(\sin\vartheta)/r$. Similarly we obtain $u_y = \beta\tilde{f} + \alpha^2\tilde{f}_\beta - \alpha\beta\tilde{f}_\alpha$. Since u_x and u_y are continuous at (x_0, y_0), we immediately have the following identities

$$\beta^2\tilde{f}_\alpha - \alpha\beta\tilde{f}_\beta + \alpha\tilde{f} = a \tag{13}$$

$$-\alpha\beta\tilde{f}_\alpha + \alpha^2\tilde{f}_\beta + \beta\tilde{f} = b \tag{14}$$

for all α, β. Here $a = u_x(x_0, y_0), b = u_y(x_0, y_0)$. Multiplying the last two equations by α, β, respectively and adding the resulting identities, we obtain $(\alpha^2 + \beta^2)\tilde{f} = a\alpha + b\beta$. It follows that $\tilde{f} = a\alpha + b\beta$ since $\alpha^2 + \beta^2 = 1$. We have shown that $u(x, y) = r(a\cos\vartheta + b\sin\vartheta) + c = a(x - x_0) + b(y - y_0) + c_0 = ax + by + (c - ax_0 - by_0 = ax + by + c$. In fact $(x_0, y_0) = (-b, a)$ from the definition of a singular point and the plane $\{(x, y, u(x, y)\}$ is just the contact plane passing through (x_0, y_0).

Case 2. $S(u)$ **contains no isolated singular points.** By the arguments at the beginning of the proof we can find a rotation $\tilde{x} = ax + by$, $\tilde{y} = -bx + ay$ with $a^2 + b^2 = 1$ such that

$$\check{N}^\perp(u) = \pm\frac{\partial}{\partial\tilde{x}} \ . \tag{15}$$

By the third equation in (11) our equation reads $\tilde{u}_{\tilde{x}\tilde{x}} = 0$ where $\tilde{u}(\tilde{x}, \tilde{y}) = u(x, y)$. Integrating, it follows that

$$\tilde{u} = \tilde{x}f(\tilde{y}) + g(\tilde{y}) \ , \tag{16}$$

for some C^2 smooth functions f, g. From (15) we know that $N(u) = (0, \pm 1)$, and by the definition of $N(u)$ we obtain $\tilde{u}_{\tilde{x}} - \tilde{y} = 0$, so $f(\tilde{y}) = \tilde{y}$. Substituting this into (16) gives $\tilde{u} = \tilde{x}\tilde{y} + g(\tilde{y})$, and hence $u = -abx^2 + (a^2 - b^2)xy + aby^2 + g(-bx + ay)$. The proof is therefore concluded.

□

Remark 2. More general p-minimal surfaces in H^1, not necessarily graphs, have been considered in [5]. The authors classify all surfaces of helicoid type, namely for which the intersection with a family of parallel planes foliating H^1 consists of contact lines.

3. The Dirichlet problem

In this section we consider the problem of finding a graph over a domain Ω which satisfies the p-minimal surface equation and a prescribed boundary condition φ on $\partial\Omega$. Some solutions can be found using the direct methods of the calculus of variations, trying to solve the following minimization problem, where the integrand (see the previous section) represents the p-area element of the graph of u

$$\min\left\{F(u) := \int_\Omega \sqrt{(u_x - y)^2 + (u_y + x)^2}\, dx\, dy \ : u = \varphi \quad \text{on } \partial\Omega\right\} .$$

If u is of class C^2, then the Euler equation for F coincides with $(*)$. There is a more general notion of solution to $(*)$ which is given by the following definition (see Section 3 in [7]).

Definition 1. Let Ω be a bounded domain in $\mathbb{R}^2$. A function $u \in W^{1,1}(\Omega)$ is said to be a *weak solution* of $(*)$ if and only if for any $\varphi \in C_0^\infty(\Omega)$ there holds

$$\int_{S(u)} |\nabla \varphi| \, dx + \int_{\Omega \setminus S(u)} N(u) \cdot \nabla \varphi \, dx \geq 0 \, .$$

We observe that the first integral in the above formula is evaluated with respect to the standard Lebesgue measure, and therefore heuristically one expects the integral to vanish since the Hausdorff measure of $S(u)$ is small in general (see for example [1] or Lemma 5.4 in [6]). The minimizers of F of class $W^{1,1}$ in Ω (satisfying the prescribed given boundary condition) are characterized by the following result (see Theorem 3.3 in [7]).

Theorem B. *Let $u \in W^{1,1}(\Omega)$. Then u is a minimizer for F if and only if u is a weak solution of $(*)$.*

We mention also the following result from [7], concerning existence of solutions to the above problem.

Theorem C. *Let Ω be a parabolically convex bounded domain in $\mathbb{R}^2$, with $\partial\Omega$ of class $C^{2,\alpha}$, and suppose $\varphi \in C^{2,\alpha}(\partial\Omega)$. Then there exists a (unique) Lipschitz continuous minimizer $u \in C^{0,1}(\Omega)$ for F which coincides with φ on $\partial\Omega$.*

We refer to the original paper for the definition of parabolically convex domain. A different version of the above result was given in [18] where the author, assuming that the boundary data satisfies the *bounded slope condition*, proved the existence of a $W^{1,p}$ minimizer for every $p > 1$. The result in [7] also extends to higher dimensions.

The proof is done by considering the ε-regularization

$$\min \left\{ F(u) := \int_\Omega \sqrt{\varepsilon^2 + (u_x - y)^2 + (u_y + x)^2} \, dx \, dy \ : u = \varphi \quad \text{on } \partial\Omega \right\} ,$$

whose Euler equation is

$$\left[(u_y + x)^2 + \varepsilon^2\right] u_{xx} - 2(u_y + x)(u_x - y) u_{xy} + \left[(u_x - y)^2 + \varepsilon^2\right] u_{yy} = 0 \, . \tag{17}$$

For any fixed $\varepsilon > 0$, the last equation is uniformly elliptic, and therefore solutions can be found using standard arguments in elliptic theory. Under the assumptions of Theorem C, it can be shown that the solutions $(u_\varepsilon)_\varepsilon$ of (17) satisfy uniform C^1 bounds on Ω, so by the Ascoli theorem they converge to a $C^{0,1}$ function u which is a minimizer of the functional F.

It is not known whether $C^{0,1}$ is the optimal regularity for a smooth boundary data. From (4) one sees that in general equation $(*)$ does not gain regularity at the interior of Ω.

However, there are cases in which the boundary data is C^∞ smooth but the minimizer is not C^2. For example, consider the boundary curve

$$(\cos\theta, \sin\theta, \cos^2\theta + \sin\theta\cos\theta) \quad , \quad \theta \in [0, 2\pi] \, . \tag{18}$$

It was proved in [7] (see Example 7.3 there), that the corresponding minimizer is of class $C^{1,1}$ only. This example was indeed considered by S. Pauls in [18], where he showed that the above boundary data spans two different p-minimal surfaces which are of class C^2, x^2+xy and $xy+1-y^2$. Since by Theorem C the minimizer is unique, we deduce that none of these C^2 solutions can be also a minimizer for F. Indeed, there is a criterion for local minimality of C^2 solutions, which is given by the following result (Proposition 6.2 in [7], see also Theorem C in [19]).

Proposition 2. *Let u be a weak solution of $(*)$ in B, and assume u is of class C^2 in $B \setminus \Gamma$, where Γ is a smooth curve in the closure of B which divides B into two parts B_+ and B_-. Assume $N(u)$ is smooth in both the closures of B_+ and B_-. Then, letting N_+ and N_- be the values of $N(u)$ in these sets and letting ν be the normal to the curve Γ we have the following relation*

$$(N_+(u) - N_-(u)) \cdot \nu = 0$$

on Γ.

Proof. Using the divergence theorem and letting ν_+, ν_- denote the values of ν in B_+ and B_- respectively, for any given smooth test function φ we have that

$$\int_{B\setminus\Gamma} N(u) \cdot \nabla\varphi = \int_{B_+} N(u) \cdot \nabla\varphi + \int_{B_-} N(u) \cdot \nabla\varphi =$$

$$= \int_{\partial B_+} \varphi N_+(u) \cdot \nu_+ \int_{\partial B_-} \varphi N_-(u) \cdot \nu_- = \int_{\Gamma\cap B} \varphi(N_+(u) - N_-(u)) \cdot \nu_+ \,.$$

In this equalities we have used the fact that $\nu_- = -\nu_+$ and that $div\, N(u) = 0$ in both B_+ and B_- (see (7)). Since $S(u)$ is contained in Γ, the Lebesgue measure of $S(u) \cap B$ is bounded by the measure of $\Gamma \cap B$, which is zero by our assumptions on Γ. Therefore, from Definition 1 and from the last formula (also replacing φ by $-\varphi$) we derive that u is a weak solution if and only if u satisfies

$$\int_{\Gamma\cap B} \varphi(N_+(u) - N_-(u)) \cdot \nu_+ = 0$$

for every test function φ, which is the desired conclusion.

□

Regarding example (18), the vector field $N(u)$ is given respectively by

$$N(u) = \begin{cases} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) & \text{for } x > 0 \,; \\ \\ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) & \text{for } x < 0 \,, \end{cases} \qquad \text{if } u = x^2 + xy \,, \tag{19}$$

and

$$N(u) = \begin{cases} (1,0) & \text{for } x > y \,; \\ \\ (-1,0) & \text{for } x < y \,, \end{cases} \qquad \text{if } u = xy + 1 - y^2 \,. \tag{20}$$

Therefore by Proposition 2 one checks that none of these solutions can be a minimizer for the above Dirichlet problem. The explicit expression of the minimizer for the above boundary condition is given in [7], Example 7.3.

The above example shows that not only minimizers of problems with smooth boundary data might not be smooth, but also that in general pointwise solutions (C^2 regular) of $(*)$ might not be unique and do not satisfy in general a comparison principle. Indeed, a maximum principle holds for C^2 solutions, provided we have some control on the Hausdorff dimension of their singular sets, see [6].

Theorem D. *For a bounded domain Ω in $\mathbb{R}^2$, let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $div\, N(u) \geq div\, N(v)$ in $\Omega \setminus S$ and $u \leq v$ on $\partial\Omega$ where $S = S(u) \cup S(v)$. Suppose $\mathcal{H}_1(\bar{S})$, the 1-dimensional Hausdorff measure of $\bar{S}$, vanishes. Then $u \leq v$ in Ω.*

As an immediate consequence of Theorem D, we have the following uniqueness result for the Dirichlet problem of $(*)$. The result does not apply of course to the functions $x^2 + xy$ and $xy + 1 - y^2$ in the unit disk, since for these cases the singular sets consist of one-dimensional curves, see (19), (20).

Corollary 1. *For a bounded domain Ω in the xy-plane, let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $div\, N(u) = div\, N(v) = 0$ in $\Omega \setminus S$ and $u = v$ on $\partial\Omega$ where $S = S(u) \cup S(v)$. Suppose $\mathcal{H}_1(\bar{S})$, the 1-dimensional Hausdorff measure of $\bar{S}$, vanishes. Then $u = v$ in Ω.*

In [19] some criteria for the regularity of solutions, depending on the boundary data are given.

4. Some extensions and applications

In this section we collect some related results and consequences of the above arguments. We discuss first some extensions of the singular set analysis with applications to (non)existence results for p-minimal surfaces in three dimensional pseudo-Hermitian manifolds and to the isoperimetric problem in H^1. We turn then to the second variation formula for the p-area and we exhibit some sufficient condition for ensuring minimality based on calibration methods.

4.1. **Consequences of the singular set analysis.** It can be shown (see Section 7 in [6]) that the structure of the singular set in a surface Σ of class C^2 embedded with bounded p-mean curvature in a three dimensional pseudo-Hermitian manifold M has the same structure as the one described in Proposition 1. This means that the singular set consists of isolated points and C^1 curves, through which the vector field e_1 extends continuously up to the sign. Near isolated singular points, similarly to Lemma 2, the vector field e_1 has index 1. It follows that the line field of e_1 (e_1 is defined up to the sign) has only isolated singular points of index 1, and therefore the total sum of the indices is non-negative. By the Hopf index theorem, see for example [23], it follows that the total index coincides with the Euler characteristic of Σ, which therefore is also non-negative. We obtain then the following result.

Theorem E. *Let M be a pseudo-Hermitian 3-manifold. Let Σ be a closed, connected surface, C^2 smoothly immersed in M with bounded p-mean curvature. Then the genus of Σ is less than or equal to 1. In particular, there are no constant p-mean curvature or p-minimal surfaces Σ of genus greater than one in M.*

The above result specializes to the case when M is the standard pseudo-Hermitian 3-sphere, for which the foliation of p-minimal surfaces consists of Legendrian great circles (see again Section 7 in [6]).

Corollary 2. *There are no closed, connected, C^2 smoothly immersed constant p-mean curvature or p-minimal surfaces of genus ≥ 2 in the standard pseudo-Hermitian 3-sphere.*

The latter results are in striking contrast with the riemannian case, where for example there exist many closed C^∞ minimal surfaces of genus ≥ 2 ([13]) smoothly embedded in the standard three sphere, see also [22].

As we noticed in Remark 1, the p-mean curvature coincides with the ordinary curvature of the projection of integral curves of the line field e_1 and in particular, when H is constant, these projections are simply circles. This case is of particular interest since boundaries of isoperimetric sets in H^1 have constant p-mean curvature, see for example [4], [14], [15], [16], [20] and references therein for more details. Under some extra assumptions on the isoperimetric set, it can be shown that (up to a Heisenberg translation) the lifting in H^1 of the circles are geodesics with endpoints on the z axis. We mention a result from [20] in this spirit.

Theorem F. *Suppose Σ is a closed surface of class C^2 bonding an isoperimetric set in H^1. Then, up to a translation, Σ is foliated (through rotations) by geodesics with endpoints on the z-axis.*

The above result was obtained in [14] under the assumption that Σ is rotationally symmetric. A variant of this classification result has been given in [16], where the regularity assumption of Σ is replaced by the convexity of the isoperimetric set.

4.2. **Second variation formula and area-minimizing property.** In this section we will derive the second variation formula for the p-area functional and examine the p-mean curvature H from the viewpoint of calibration geometry, see [12]. As a result we can prove the area-minimizing property for a p-minimal graph in H_1.

We follow the notation in the introduction. We assume that the surface Σ is p-minimal. Let f, g be functions with compact support away from the singular set and the boundary of Σ. We compute the second variation of the p-area $\Theta \wedge e^1$ with respect to a variation $V = fe_2 + gT$ of Σ

$$\delta_V^2 \int_\Sigma \Theta \wedge e^1 = \int_\Sigma L_V^2(\Theta \wedge e^1) = \int_\Sigma i_V \circ d\{i_V \circ d(\Theta \wedge e^1)\} \,. \tag{21}$$

Here we have used the Stokes theorem, the formula $L_V = i_V \circ d + d \circ i_V$ and the fact that $d^2 = 0$. From $H = \omega(e_1)$ (see our third definition of H), we get

$$d(\Theta \wedge e^1) = -H\Theta \wedge e^1 \wedge e^2 \,. \tag{22}$$

We define locally a function α on $\Sigma \setminus S_\Sigma$ such that $\alpha e_2 + T \in T\Sigma$. Observe that $\{\alpha e_2 + T, e_1\}$ is a basis of $T(\Sigma \setminus S_\Sigma)$. Therefore on $\Sigma \setminus S_\Sigma$ we have

$$e^2 \wedge e^1 = \alpha\Theta \wedge e^1 \,. \tag{23}$$

From (22) it is easy to see that $i_V \circ d(\Theta \wedge e^1) = gHe^2 \wedge e^1 - fH\Theta \wedge e^1$. Then applying $i_V \circ d$ to this expression and making use of $(S1)$, $(S2)$, the last formula and $H = 0$ on Σ, we obtain

$$\begin{aligned} i_V \circ d\{i_V \circ d(\Theta \wedge e^1)\} &= (g\alpha - f)(gT + fe_2)(H)\Theta \wedge e^1 = \\ &= -(g\alpha - f)^2 e_2(H)\Theta \wedge e^1 \end{aligned} \tag{24}$$

on Σ. For the last equality we have used $T(H) = -\alpha e_2(H)$ since $\alpha e_2 + T \in T\Sigma$ and $H = 0$ on Σ. Expanding the left-hand side of (1) gives

$$e_2(H) = 2W + e_1(\omega(e_2)) + 2\omega(T) + (\omega(e_2))^2 \; . \tag{25}$$

Here we have used the fact that $[e_1, e_2] = -2T - \omega(e_1)e_1 - \omega(e_2)e_2$ and $\omega(e_1) = H = 0$ on Σ. The surfaces $\varphi_t(\Sigma \backslash S_\Sigma)$ are the level sets of a defining function ρ. Here $\dot{\varphi}_t = fe_2 + gT$. It follows that $(fe_2 + gT)(\rho) = 1$. On the other hand, $(\alpha e_2 + T)(\rho) = 0$ from the definition of α. So $T(\rho) = -\alpha e_2(\rho)$ and $e_2(\rho) = (f - \alpha g)^{-1}$ (where $f - \alpha g \neq 0$). Applying the operator $[e_1, e_2]$ and $[e_1, T] = (\mathrm{Re}\, A_{11})e_1 - ((\mathrm{Im}\, A_{11}) + \omega(T))e_2$ (where A_{11} is the *torsion* of the Tanaka-Webster connection, see the appendix of [6]) to the function ρ, and using the above formulas, we obtain

$$\omega(e_2) = h^{-1}e_1(h) + 2\alpha \; , \tag{26}$$

$$\omega(T) = e_1(\alpha) - \alpha h^{-1} e_1(h) - \mathrm{Im}\, A_{11} \tag{27}$$

where $h = f - \alpha g$. Now substituting the last two equations into (25), we get

$$\begin{aligned} e_2(H) = {} & 2W - 2\mathrm{Im}\, A_{11} + 4e_1(\alpha) + 4\alpha^2 + \\ & + h^{-1}e_1^2(h) + 2\alpha h^{-1} e_1(h) \; . \end{aligned} \tag{28}$$

Observing that $e_1(e_1(h^2))\Theta \wedge e^1 = \Theta \wedge d(e_1(h^2)) = -d(e_1(h^2)\Theta) + 2e_1(h^2)\alpha e^1 \wedge \Theta$ on Σ by $(S1)$ and (23), we integrate $(1/2)e_1(e_1(h^2)) = (e_1(h))^2 + he_1^2(h)$ to obtain

$$-\int_\Sigma he_1^2(h)\Theta \wedge e^1 = \int_\Sigma [(e_1(h))^2 + 2\alpha h e_1(h)]\Theta \wedge e^1 \; .$$

With some substitutions, we finally reach the following second variation formula.

Proposition 3. *Suppose the surface Σ is p-minimal as defined in Section 2. Let f, g be functions with compact support away from the singular set and the boundary of Σ. Then*

$$\begin{aligned} & \delta^2_{fe_2+gT} \int_\Sigma \Theta \wedge e^1 = \\ & = \int_\Sigma \{(e_1(f - \alpha g))^2 + (f - \alpha g)^2[-2W + 2\mathrm{Im}\, A_{11} - 4e_1(\alpha) - 4\alpha^2]\}\Theta \wedge e^1 \; . \end{aligned} \tag{29}$$

Note that the Webster-Tanaka curvature W and the torsion A_{11} are geometric quantities of the ambient pseudo-Hermitian 3-manifold M.

In Riemannian (three dimensional) geometry, to construct a calibrating form one considers the inner product of the volume form with a vector field orthogonal to a family of surfaces, see [12]. This 2-form restricts to the surfaces, and its exterior differentiation equals the mean curvature times the volume form along a surface. We have analogous results here. Suppose M is foliated by a family of surfaces Σ_t, $-\varepsilon < t < \varepsilon$. Let e_1 be a vector field which is characteristic along each surface Σ_t. We are assuming the Σ_t's to have no singular points. Let $e_2 = Je_1$ denote the Legendrian normal along each Σ_t. Then the 2-form $\Phi = (1/2)\, i_{e_2}(\Theta \wedge d\Theta)$ satisfies the following properties. First, a direct computation shows that $\Phi = \Theta \wedge e^1$, the p-area form, from formula $(S1)$. Secondly, $d\Phi = -H\Theta \wedge e^1 \wedge e^2$ by (22). So $\{\Sigma_t\}$ are p-minimal surfaces if and only if $d\Phi = 0$. Now suppose this is the case and Σ' is a deformed surface with no singular points near a p-minimal surface $\Sigma = \Sigma_0$

having the same boundary. Also suppose the Poincaré lemma holds, namely there is a 1-form Ψ such that $\Phi = d\Psi$. Then by Stokes' theorem, we have

$$\text{p-Area}(\Sigma) = \int_{\Sigma} \Phi = \int_{\partial\Sigma} \Psi = \int_{\partial\Sigma'} \Psi = \int_{\Sigma'} \Phi \, . \tag{30}$$

For Σ', we have corresponding $e'_1, e'_2, e^{1\prime}, e^{2\prime}$. There is a function α' such that $T + \alpha' e'_2$ is tangent to Σ'. Applying $\Phi = \Theta \wedge e^1$ to the basis $(T + \alpha' e'_2, e'_1)$ of $T\Sigma'$, we obtain $e^1(e'_1)$. It follows that $\Phi = e^1(e'_1)\Theta \wedge e^{1\prime}$ when restricted to Σ'. So we have

$$\begin{aligned} \int_{\Sigma'} \Phi &= \int_{\Sigma'} e^1(e'_1)\Theta \wedge e^{1\prime} \leq \\ &\leq \int_{\Sigma'} \Theta \wedge e^{1\prime} = \text{p-Area}(\Sigma') \qquad (\text{since } e^1(e'_1) \leq 1) \, . \end{aligned} \tag{31}$$

From the last two formulas we have shown that

$$\text{p-Area}(\Sigma) \leq \text{p-Area}(\Sigma') \, . \tag{32}$$

Let us summarize the above arguments in the following proposition.

Proposition 4. *Suppose we can foliate an open neighborhood of a p-minimal surface Σ by a family of p-minimal surfaces with no singular points, and in this neighborhood the Poincaré lemma holds* (*i.e., any closed* 2*-form is exact*)*. Then Σ has the local p-area-minimizing property. In other words, if Σ' is a deformed surface with no singular points near Σ having the same boundary, then* (32) *holds.*

We remark that a p-minimal surface in H_1 with no singular points, which is a graph over the xy-plane, satisfies the assumption in Proposition 4. Note that a translation of such a p-minimal graph in the z-axis is still p-minimal (notice that $u + c$ is again a solution if $u = u(x, y)$ is a solution to $(*)$). Also a vertical plane in H_1 (i.e. perpendicular to the xy-plane) satisfies the assumption in Proposition 4. Note that a vertical plane is a p-minimal surface with no singular points, and a family of parallel such surfaces surely foliates an open neighborhood of a given one. Such planes are indeed the only entire *X-minimal graphs*, according to a Bernstein-type result in [2] (to which we refer also for the terminology).

References

[1] Z. Balogh, *Size of characteristic sets and functions with prescribed gradient*, J. Reine Angew. Math., 564(2003) 63–83.

[2] V. Barone Adesi, F. Serra Cassano & D. Vittone, *THe Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations*, Preprint.

[3] L. Capogna, D. Danielli & N. Garofalo, *The geometric Sobolev embedding fir vector fields and the isoperimetric inequality*, Comm. Anal. Geom., 2(1994) 203–215.

[4] L. Capogna, D. Danielli, S. Pauls & T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Preprint.

[5] J.-H. Cheng & J.-F. Hwang, *Properly embedded and immersed minimal surfaces in the Heisenberg group*, Bull. Austr. Math. Soc., (3)70(2004), 507–520.

[6] J.-H. Cheng, J.-F. Hwang, A. Malchiodi & P. Yang, *Minimal surfaces in pseudo-Hermitian geometry and the Bernstein problem in the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci.(5),(1)4(2005), 129–177.

[7] J.-H. Cheng, J.-F. Hwang & P. Yang, *Existence and uniqueness for p-area minimizers in the Heisenberg group*, Math. Ann., to appear.

[8] D. Danielli, N. Garofalo & D.-M. Nhieu, *Minimal surfaces, surfaces of constant mean curvature and isoperimetry in Carnot groups*, Preprint.

[9] B. Franchi, R. Serapioni & F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann., 321(2001) 479–531.

[10] N. Garofalo & D.-M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Caratheodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math., 49(1996) 1081–1144.

[11] N. Garofalo & S. Pauls, *The Bernstein problem in the Heisenberg group*, Preprint, 2004.

[12] R. Harvey & H.B. Jr. Lawson, *Calibrated geometries*, Acta Math., 148(1982) 47–157.

[13] H.B. Jr. Lawson, *Complete minimal surfaces in S^3*, Ann. Math., 92(1970) 335–374.

[14] G.P. Leonardi & S. Masnou, *On the isoperimetric problem in the Heisenberg group H^n*, Ann. Mat. Pura Appl., (IV)(4)184(2005), 533–553.

[15] G.P. Leonardi & S. Rigot, *Isoperimetric sets on Carnot groups*, Houston J. Math., (3)29(2003), 609–637.

[16] R. Monti & M. Rickly, *Convex isoperimetric sets in the Heisenberg group*, Preprint.

[17] P. Pansu, *Une inegalite isoperimetrique sur le groupe de Heisenberg*, C. R. Acad. Sci. Paris Sér. I Math., (2)295(1982), 127–130.

[18] S.D. Pauls, *Minimal surfaces in the Heisenberg group*, Geometria Dedicata, 104(2004) 201–231.

[19] S.D. Pauls, *H-minimal graphs of low regularity in H^1*, Comment. Math. Helv., (2)81 (2006), 337–381.

[20] M. Ritore & C. Rosales, *Area stationary surfaces in the Heisenberg group $\mathbb{H}^1$*, Preprint.

[21] R. Schoen & S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys., 65(1979) 45–76.

[22] R. Schoen & S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math., (1-2)110(1979), 127–142.

[23] M. Spivak, *A Comprehensive introduction to differential geometry*, vol. 3, Publish or Perish, Inc., Boston, 1975.

[24] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Kinokuniya Co. Ltd., Tokyo, 1975.

[25] S.M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Diff. Geom., 13(1978), 25–41.