Nonlinear Differ. Equ. Appl.
(2026) 33:74

https://doi.org/10.1007/s00030-026-01215-1

Nonlinear Differential Equations and Applications NoDEA

Ergodicity for stochastic T-monotone parabolic obstacle problems

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Abstract. This work aims to investigate the existence of ergodic invariant measures and its uniqueness, associated with obstacle problems governed by a **T**-monotone operator defined on Sobolev spaces and driven by a multiplicative noise in a bounded domain of $\mathbb{R}^d$ with homogeneous boundary conditions. We show that the solution defines a Markov-Feller semigroup defined on the space of real bounded continuous functions of a convex subset related to the obstacle and we prove the existence of ergodic invariant measures and its uniqueness, under suitable assumptions. Our method relies on a combination of Krylov-Bogoliubov theorem, Krein-Milman theorem and Lewy-Stampacchia inequalities to control the reflection measure.

Mathematics Subject Classification. 35K86, 35R35, 47D07, 60H15.

Keywords. Stochastic PDEs, Obstacle problem, Reflected measure, Invariant measure, Multiplicative noise, Markov process.

Contents

1. Introduction
    1.1. Well-posedness of SPDE with obstacle (state of the art)
    1.2. Long-time behavior and ergodicity
    1.3. Novelty of our contribution
    Organization of the paper
2. Physical motivations for **T**-monotone obstacle problems
    2.1. Non-Newtonian filtration in porous media with saturation
    2.2. Nonlinear elastic membrane with contact to a rigid obstacle
3. Stochastic obstacle problems & main results
    3.1. Notation and function spaces
    3.2. Formulation of the problem
    Assumptions
    3.3. Well-posedness of the stochastic obstacle problem (8)

- 3.4. Main results
- 3.5. Ergodicity and obstacle problems
  - Ergodic invariant measures
  - Ergodicity and uniqueness of the invariant measure
- 4. Proofs
  - 4.1. Construction of solution to (8)
  - 4.2. Feller-Markov process associated with (8)
  - 4.3. Existence of ergodic invariant measure
    - 4.3.1. Existence of an invariant measure
    - i. The case $p \leq 2 < +\infty$
    - ii. The case $\underline{\mathbf{p}} < p < 2$
    - Concentration
    - 4.3.2. Concentration and ergodic invariant measures
  - 4.4. Uniqueness and exponential mixing
- 5. On double obstacles problem
- 6. Appendix: on well-posedness of (8)
    - Proof of Theorem 6.1
    - Penalization
- Acknowledgements
- References

1. Introduction

We are interested in the existence of ergodic invariant measures and its uniqueness, associated with the solution (u, k) to the following obstacle problem

$$\begin{cases} du + [A(u) + \mathbf{F}(u) + k]ds = fds + G(u)dW \\ \langle k, u - \psi\rangle_{V',V} = 0 \ \text{ and } \ \langle -k, \varphi\rangle_{V',V} \geq 0 \quad \forall \varphi \in V, \varphi \geq 0 \\ u \geq \psi, \ \ u(0,\cdot) = u_0 \geq \psi, \end{cases} \tag{1}$$

where A is a nonlinear T-monotone operator defined on a Banach space V with values in V', where V' denotes the dual space of V, $\mathbf{F}$ is a Lipschitz Nemytskii operator, f is an external force, $G(\cdot)$ defines a Hilbert-Schmidt operator and $(W(t))_{t\geq 0}$ is a Q-Wiener process with values in a separable Hilbert space H defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. The *reflection measure* k prevents the solution u from crossing the obstacle ψ. The second line in (1) (we will refer to it as "minimality condition") says that k is pushing away u to cross ψ in minimal sense *i.e.* the support of k is included in the contact set $\{u = \psi\}$ while k should equals 0 in the free set $\{u > \psi\}$ and the free boundary $\partial\{u = \psi\} = \partial\{u > \psi\}$ is part of the unknowns.

Obstacle problems of the form (1) appear in the mathematical modeling of several phenomena including Physics, Biology and Finance. For example, the evolution of damage in a continuous medium taking into account the microscopic description [5], the American option pricing, some questions of flows in porous media, phase transition and some statistical mechanic problems, see *e.g.* [34,39] and references theirin. We also refer to Section 2 for an extensive discussion of some physical motivations. It's worth to mention that a formulation by using variational inequalities is largely used in the deterministic setting, we refer to *e.g.* [20,21,26] and their references for the interested reader.

1.1. Well-posedness of SPDE with obstacle (state of the art)

Before discussing the SPDE with obstacle, it is appropriate to say a few words about the finite-dimensional SDE (stochastic differential equation) with reflection. Without exhaustiveness, we refer to [38], where Skorohod constructed the solution of stochastic differential equations on $\mathbb{R}^+$ with reflection at 0 by using the properties of the space of real continuous functions. Then, the existence and uniqueness have been proven in [9] for SDEs in the presence of a multi-valued maximal monotone operator with normal reflection associated with a closed convex subsets of $\mathbb{R}^d$. In [2], the authors studied the generator of the transition semigroup of SDEs with boundary reflection on a closed and convex subset of $\mathbb{R}^d$. The above works concern a finite-dimensional SDEs. In infinte dimensional setting, without seeking to be exhaustive, let us mention [28] for the existence and uniqueness result about reflected solutions of the heat equation on the spatial interval $[0, 1]$ with Dirichlet boundary conditions, driven by an additive space-time white noise. It is worth mentioning that this can be considered as an extension of one-dimensional SDEs reflected at 0. Their approach relies on some results about deterministic variational inequalities. Then, existence of reflected solutions of the heat equation on the spatial interval $[0, 1]$ with Dirichlet boundary conditions, driven by multiplicative noise, has been proved in [15]. For the formulation as SVI (stochastic variational inequalities) in 1D, see [22]. It seems that the 1D restriction is related to the fact that the expression of the minimality condition requires an appropriate regularity, which is only valid in 1D due to Sobolev embedding theorem. Thus, in a higher dimension, either more in-depth analysis of the regularity of the solution and of the reflected measure is needed to express the minimality condition, see *e.g.* [37] based on parabolic potential theory or some duality arguments, see *e.g.* [36,42] and their references.

It is worth mentioning that the formulation by using either the SVI or multi-valued SPDEs may lead to restrictions in order to transform the problem into random SVI or weakening the notion of solution. We prefer to work with the coupled system (1), which allows to use directly the infinite-dimensional stochastic calculus, namely Itô's formula, as far as we have some control over the reflection measure, thanks to Lewy-Stampacchia's inequalities, see [42]. In addition, the rigorous formulation of stochastic obstacle problems as SVI or differential inclusion, governed by maximal monotone operators, is not always true because it can suffer from strong singularities (reflection) close to the contact set.

Concerning stochastic PDEs with obstacle in different dimensions, let us refer to [13] for an existence and uniqueness result for quasi-linear stochastic PDEs with obstacle, by using the parabolic potential theory. In [46], the author proved the existence and uniqueness of solutions to multivalued stochastic evolution equations involving maximal monotone operators in some Hilbert space and single valued strongly monotone operator. Besides the well posedness, a regularity properties of the reflected stochastic measure was studied in [42], namely stochastic Lewy-Stampacchia's inequalities in the case of stochastic **T**-monotone obstacle problems and then [6,36] for stochastic scalar conservation laws with constraint and pseudomonotone parabolic obstacle problems in the presence of a multiplicative noise. Recently, an existence result based on a comparison theorem and parabolic capacity has been proved in [37], about the solutions to nonlinear, pseudomonotone, stochastic diffusion-convection evolution problem in the presence of additive noise on a bounded domain, with homogeneous boundary conditions and reflection. Finally, the large deviations for stochastic obstacle problems were studied *e.g.* in [25,40,45].

1.2. Long-time behavior and ergodicity

The ergodicity concerns the relation between the "spatial and temporal averages" for a large class of dynamics, the existence of invariant measures is an important factor to investigate the ergodicity. Moreover, it concerns an asymptotic of the dynamic for a large time. There exists many works about the invariant measures and ergodicity for finite and infinite dimensional dynamical systems. First, let us mention the pioneering work [24], where the authors established a procedure to construct an invariant measure for a dynamical system. Concerning some results about the invariant measures and ergodicity for stochastic evolution equations, without exhaustiveness, let us refer to [10–12,17,29,30,32,33,41] and their references. We refer also to [4,19] and their references on the question of ergodicity for the singular stochastic p-Laplace operator in the presence of an additive noise and [18] for stochastic evolution inclusions in Hilbert spaces.

In contrast with unconstrained stochastic parabolic equations, the obstacle constraint introduces a reflection mechanism described by a measure supported on the contact set, leading to a stochastic variational inequality rather than a classical SPDE. This reflection term is intrinsically singular and cannot be handled by perturbative arguments (as a lower-order term), so that the standard ergodic theory for monotone SPDEs does not apply directly. In particular, the presence of reflection obstructs the coercivity and the usual estimates that are central to the existence and uniqueness of an invariant measure. A key contribution of this work is to show that, by combining **T**-monotonicity with the Lewy-Stampacchia inequality, one can obtain an effective control on the reflected measure and restore the coercivity and uniform energy estimates at the level of the variational formulation. This makes it possible to establish ergodicity for stochastic parabolic obstacle problems, thereby extending the scope of the classical theory to a genuinely non-smooth and constrained infinite-dimensional setting.

1.3. Novelty of our contribution

Concerning the invariant measures to stochastic obstacle problems, the invariant measure to the heat equation with an additive white-noise in 1D, with reflection at 0 were studied in [34,43]. In [43], the existence of an explicit symmetrizing invariant measure on $C([0,1])$ were studied and then the relation between the stationary distribution and the Gibbs measure on $C(\mathbb{R})$ were investigated in [34]. In [44], a Harnack inequalities for the semigroup associated with the stochastic heat equation with reflection at 0 were proved. In [23], the author proved the existence of invariant measures for the reflected heat equation with multiplicative noise on $[0,1]$. One notices that all these results are in 1D space setting. In an abstract Hilbertian framework, the authors in [3] studied the well-posedeness and the invariant measures of a stochastic variational inequality on closed convex bounded subsets with nonempty interior and smooth boundary of a Hilbert space in the presence of an additive noise. We refer also to [46] about the existence and uniqueness of invariant measures associated with the solutions of a class of multivalued nonlinear stochastic partial differential equations involving maximal monotone operators in some Hilbert space and single valued strongly monotone operators. In [19], the authors studied the ergodicity for local and nonlocal stochastic stochastic variational inequality governed by p-Laplace operator in the presence of additive noise, where the coercivity is one of the key points of the proof.

Summarizing, the results concern the invariant measures for stochastic obstacle problems are associated either with the stochastic heat equations in 1D, a linear operators in the frame of Hilbert spaces or strongly monotone operator by using formulation based on multivalued SPDEs involving maximal monotone operators in some Hilbert spaces. Moreover, the coercivity is a key factor in the available results. As we have already mentioned, the presence of reflections destroys coercivity and energy estimates may turn out to be incorrect. A key technical ingredient that allows us to overcome this difficulty is the use of the *Lewy-Stampacchia inequalities.* These inequalities play a crucial role in providing bound on the reflected measure. Such control is essential for deriving uniform energy estimates, this step is specific to obstacle problems and does not appear in the unconstrained setting. On the other hand, it is natural to consider (1) in any space dimension in the presence of non linear differential operators, as a natural extension of the problem studied in [15,28]. A typical example we have in mind is $A(u) = -\Delta_p u = -\operatorname{div}[|\nabla u|^{p-2}\nabla u], 1 < p < +\infty$ in (1). The operator $-\Delta_p$ is a natural generalization of the linear Laplace operator and it has interesting features. In particular, it degenerates if $2 < p < +\infty$ and it becomes singular when $1 < p < 2$.

To the best of author's knowledge, the existing papers does not cover constraint setting, reflection or free-boundary phenomena appearing in the dynamics. Namely, there doesn't exist in the literature a result about the existence of invariant measure for stochastic obstacle problems with $\mathbf{T}$-monotone operators (*e.g.* singular Δ_p), including unilateral constraints and reflected measure term. While the existing papers use variational methods for SPDEs, the presence of constraint, reflected measure, free boundaries, and multiplicative noise places our work outside the scope of the earlier frameworks. Moreover, the formulation of the problem in Sobolev spaces was natural but one can see that the approach can be generalized in abstract setting by using **T**-monotone operators, which require only Banach lattice spaces structure.

Our aim, in this contribution, is to propose a results about the existence and uniqueness of ergodic invariant measure associated with an obstacle problem governed by a **T**-monotone operator and in the presence of a multiplicative noise, based on the well-posedness and Lewy-Stampacchia's inequalities proved in [42]. First, we check the well-posedness of (1) in the case $1 < p < 2$, based on [42]. Then, we study the existence and uniqueness of ergodic invariant measure associated with the solution. Our result includes the existence of ergodic invariant measure when $A(u) = -\Delta_p u; 2 \leq p < +\infty$ without any restrictions if $2 < p < +\infty$ and the existence of invariant measure if $A(u) = -\Delta_p u; \max(1, \frac{2d}{d+2}) < p < 2$. In particular, it includes the singular $\Delta_p, 1 < p < 2$ in 1D and 2D. The uniqueness holds under some natural assumptions. We recall that (1) is a free boundary type problem, where the unknowns are u and the reflection measure (Lagrange multiplier) k, see Definition 3.1 for the precise formulation and notion of solution. Our arguments rely on a combination of the unbounded penalization (non-Lipschitz in the case of singular operators [1]), Lewy-Stampacchia inequalities to control the reflection measure, Krylov-Bogoliubov method to construct an invariant measure and Krein-Milman theorem to show that the set of ergodic invariant measures is not empty.

It is worth mentioning that the management of k[2], resulting from the interaction between the solution and the obstacle near to the contact set, is crucial in order to apply infinite dimensional Itô formula to derive some estimates. In our approach, k is an adapted process with values in V' and not in a some Hilbert space. We use Lewy-Stampacchia's inequalities to control it. Hence, we can use directly the classical infinite dimensional Itô formula (see *e.g.* [32]) and derive some estimates in the presence of external forces as well. Thus, using the form (1) and Lewy-Stampacchia's inequalities allow to also consider the case of singular operators. On the other hand, we prove that the semigroup associated with solution of (1) is bounded, Markov-Feller and stochastically continuous in the space of real bounded continuous functions on K_ψ[3], see Lemma 4.2. The existence of an invariant measure follows by using the Krylov-Bogoliubov theorem, after showing the tightness of an average measure constructed using the law of the solution of (1). Then we show that the set of invariant measures is tight to get the existence of an ergodic invariant measure by using Krein-Milman theorem, where Lewy-Stampacchia's inequalities is used to control the reflection measure. Finally, the uniqueness of invariant measure is proved under some natural assumptions.

Organization of the paper. The article is organized as follows: Section 2 present some physical motivation to study obstacle problem with **T**-monotone operators. In Section 3, after giving the hypotheses, we recall the well-posedness

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[1] For example, the case $A(u) = -\Delta_p u; 1 < p < 2$.

[2] It is worth recalling that k depends non-linearly on the solution u.

[3] K_ψ is a convex subset in a Hilbert space related to the obstacle ψ.

result about the stochastic **T**-monotone obstacle problems. Then, we present the main results of this work. Section 4 is devoted to the proof of the main results in the following way: first, we recall the method of construction of the solution *via* penalization, which serves to verify that the semigroup associated with the solution of (8) defines Feller-Markov process. Secondly, we prove that the set of invariant measures is not empty and tight if $2 \leq p < +\infty$. Finally, we prove Theorem 3.5 and Theorem 3.7 about the uniqueness of the invariant measure, under suitable assumptions. Section 6 concerns the well-posedness of a **T**-monotone obstacle problem in the presence of zero order Lipschitz mapping when $1 < p < 2$.

2. Physical motivations for T-monotone obstacle problems

Before addressing the mathematical formulation of the considered obstacle problems, we present examples where such problems arise.

2.1. Non-Newtonian filtration in porous media with saturation

The first example concerns saturation occurring in the mechanics of porous media. Let $D \subset \mathbb{R}^d$ be a bounded porous medium, $T > 0$ a final time and we consider a non-Newtonian fluid infiltrating the medium.

Denote by $u(x,t) \in [0,1]$, $(x,t) \in D \times (0,T)$ the *saturation*, i.e. the volume fraction of pore space occupied by the fluid. Namely, if $u = 0$, then the medium is dry, $0 < u < 1$ is partially saturated and $u = 1$ refers to a fully saturated medium. The bound $u \leq 1$ is a *physical constraint* expressing the finite pore volume, this constraint will give rise to an obstacle problem.

Now, the conservation of mass gives $\partial_t u + \nabla \cdot J = f$, where J is the mass flux and f is a source or sink term. Let v be the seepage velocity of the fluid and recall that the flux is given by $J = uv$. In the case of a heterogeneous medium, complex fluids (*e.g.* non-Newtonian) or turbulence, a generalized, p-power type version of Darcy law (nonlinear relation between velocity and pressure gradient) has been proposed (see *e.g.* [14] and the references therein). Thus we consider generalized Darcy's law: $v = -k\,|\nabla \pi|^{p-2}\nabla \pi, \quad p > 1$, where π is the pressure, $k > 0$ is the permeability constant, and p characterizes the rheology of the fluid[4]. Thus, we get $J = -k\,u\,|\nabla \pi|^{p-2}\nabla \pi$. In local thermodynamic equilibrium, we can consider the constitutive (barotropic equation of state) relation $\pi = G(u, \mathrm{Tem})$, where G is some function and Tem refers to the temperature. In isothermal flow[5], we have $\pi = H(u)$ with some non-decreasing function H (this mean that pressure increases with saturation) and the flux becomes

$$J = -k\,u|H'(u)|^{p-1}\,|\nabla u|^{p-2}\nabla u := -k\mathbf{H}(u)\,|\nabla u|^{p-2}\nabla u.$$

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[4] The Newtonian flow corresponds to $p = 2$, $p > 2$ for shear-thickening fluids, and $1 < p < 2$ for shear-thinning fluids.

[5] The temperature is constant.

Away from dry regions, we assume that saturation variations are moderate and replace u in the mobility by a reference value $u_* > 0$. This yields[6] $J \approx -\kappa\,|\nabla u|^{p-2}\nabla u$ where $\kappa = k\mathbf{H}(u_*)$ and we obtain

$$\partial_t u - \kappa div\big(|\nabla u|^{p-2}\nabla u\big) = f.$$

Concerning the saturation constraint, the porous medium cannot hold more fluid than its pore volume $u(x,t) \leq 1$. This is not a constitutive force but a *geometric constraint* on admissible states. Physically, when the system attempts to violate this constraint, internal pressures develop to prevent further increase in saturation and leads to penalized free energy to enforce the constraint. Mathematically, this interpret as Lagrange multiplier ξ acting only in the contact set $\{u = 1\}$ *i.e.* $\xi(1-u) = 0$ and represents the energetic cost of compressing the fluid beyond saturation, and violations of $u \leq 1$ become forbidden. Finally, we obtain the following system

$$\begin{cases} \partial_t u - \kappa div\big(|\nabla u|^{p-2}\nabla u\big) + \xi = f, \\ u \leq 1, \quad \xi \geq 0, \\ \xi(1-u) = 0. \end{cases}$$

This is an obstacle problem with upper constraint, but with a simple change of variable $w = -u$, we get

$$\begin{cases} \partial_t w - \kappa div\big(|\nabla w|^{p-2}\nabla w\big) = \xi - f, \\ w \geq -1, \quad \xi \geq 0, \\ \xi(w+1) = 0. \end{cases} \tag{2}$$

Due to some random injection or uncertain boundary fluxes, we can consider f to be stochastic (*e.g.* multiplicative noise) and we obtain the stochatic obstacle problem of the form (8), after introducing boundary and initial conditions in (2). It is worth mentioning that the field ξ acts as a *reaction pressure* and acts only in the region $\{w = -1\}$, it plays similar role as pressure in incompressible flows.

Similarly, we can obtain similar problem for phase transition processes. For example, the stochastic obstacle problem about the evolution of damage in continuum media with damage parameter u, studied in [5]. Namely, consider the free energy functional (instead the one considered in [5])

$$\Psi(u,\nabla u) = w(u) + \frac{1}{p}|\nabla u|^p + I_{[0,1]}(u), \quad 1 < p < \infty,$$

where w is related to the internal cohesion of the material and indicator function of the interval $[0,1]$ restricts the domain of the free energy to the physically admissible values for u, we refer to [5] for $p = 2$ and more details about the resulting obstacle problem. We also mention [31] for obstacle problems for quasi-linear scalar conservation laws associated with Dirichlet boundary conditions with application in petroleum engineering, where the constraint plays the role of preventing the appearance of a new phase.

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[6] We can also consider general Leray-Lions operators *i.e.* $J \approx a(x,\nabla u) = \gamma(x)|\nabla u|^{p-2}\nabla u$ to mimics the heterogeneity of the medium, with certain appropriate assumptions on a.

2.2. Nonlinear elastic membrane with contact to a rigid obstacle

We present a model in contact mechanics. Let $D \subset \mathbb{R}^d$ be a bounded domain representing the median plane of a thin elastic membrane, fixed on its boundary ∂D and $T > 0$ denotes final time.

Let us denote the vertical displacement of the membrane by $u(x,t)$ where $(x,t) \in D \times (0,T)$. Below the membrane lies a rigid obstacle with prescribed profile $\psi : D \to \mathbb{R}$. The membrane cannot penetrate the obstacle, which leads to the constraint

$$u(x,t) \geq \psi(x) \quad \text{for all } (x,t) \in D \times (0,T). \tag{3}$$

Let us describe the dynamics of vertical displacement. First, we recall that the deformation of the membrane is characterized by its gradient ∇u. If we assume that the membrane is made of a nonlinearly elastic material[7], then the total elastic energy is given by $\mathcal{E}(u) = \int_D \frac{1}{p}|\nabla u|^p \, dx, 1 < p < \infty$. Since the elastic force is obtained from the first variation of the energy, a standard computation ensures that the elastic restoring force acting on the membrane is given by $-div(|\nabla u|^{p-2}\nabla u)$, by assuming homogeneous Dirichlet boundary conditions $u = 0$ on ∂D. The vertical motion of the membrane satisfies Newton's second law: inertia + damping + elastic force = external force.

In term of equation, we have $\rho \, \partial_{tt}u + \mathcal{D}(\partial_t u) - div(|\nabla u|^{p-2}\nabla u) = f$, where ρ denotes the mass density, f represents external loads (*e.g.* the gravitational force) and $\mathcal{D}(\partial_t u)$ is a dissipative force. We recall that the dissipation is due to either to internal friction in the material (viscoelastic effects) or interaction with surrounding air or fluid. In the case of linear viscous damping, it is proportional to the velocity and it is reasonable to set $\mathcal{D}(\partial_t u) = \partial_t u$. The next step is to discuss the inertia. In some practical situations, the inertia term is negligible compared to damping and elastic forces. Thus, one assumes $\rho \, \partial_{tt}u \approx 0$, and the balance law reduces to the parabolic equation

$$\partial_t u - div(|\nabla u|^{p-2}\nabla u) = f. \tag{4}$$

On the other hand, the rigid obstacle exerts a reaction force that prevents penetration. This force is nonnegative and acts only when the membrane is in contact with the obstacle, which leads to the complementary condition

$$(\partial_t u - div(|\nabla u|^{p-2}\nabla u) - f)(u - \psi) = 0. \tag{5}$$

From (3), (4) and (5), we get

$$\begin{cases} \partial_t u - div(|\nabla u|^{p-2}\nabla u) = f + \chi \\ u \geq \psi, \quad \chi \geq 0 \\ \chi(u - \psi) = 0, \end{cases} \tag{6}$$

with $u = 0$ on ∂D. Due to imperfect knowledge of forcing, vibrations of the support or turbulence in surrounding fluid for example, we can write a stochastic version of the form (8).

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[7] Notice that the case $p = 2$ corresponds to linear elasticity.

Finally, we used the p-Laplacian to model either Non-Newtonian behavior or non linear elastic energy in the above examples. We can also consider an operator which is also heterogeneous with respect to the spatial variable, such as $A(u) = -\text{div } a(x, \nabla u)$, where a is a Carathéodory function with appropriate assumptions, satisfying H_2 (e.g. monotone Leray-Lions operators). On the other hand, we can also add zero-order terms such as the reaction term in [5] or general Nemytskii operators b, to mimics other physical features of the models. This leads to operator of the form

$$\tilde{A}(u) = -\text{div } a(x, \nabla u) + b(x, u),$$

where b is non-decreasing function with certain conditions of integrability *e.g.* $b(x, u) = \beta(x)|u|^{q-2}u$ where $1 < q < \infty$. The operator $\tilde{A}$ is a general example of a $\mathbf{T}$-monotone operator in Sobolev spaces, this type of operators are covered by the analysis of this work.

3. Stochastic obstacle problems & main results

3.1. Notation and function spaces

Let us denote by $D \subset \mathbb{R}^d, d \geq 1$ a Lipschitz bounded domain, $T > 0$ and consider $\underline{\mathbf{p}} := \max(1, \frac{2d}{d+2}) < p < +\infty$. As usual, $p' = \frac{p}{p-1}$ denotes the conjugate exponent of p, $V = W_0^{1,p}(D)$, the sub-space of elements of $W^{1,p}(D)$ with null trace, endowed with Poincaré's norm, $H = L^2(D)$ is identified with its dual space so that, the corresponding dual spaces to V, V' is $W^{-1,p'}(D)$ and the Lions-Guelfand triple $V \hookrightarrow_d H = L^2(D) \hookrightarrow_d V'$ holds. Denote by $p^* = \frac{pd}{d-p}$ if $p < d$ the Sobolev embedding exponent and remind that (see *e.g* [35, Theorem 1.20])

$$\begin{aligned}
&\text{if } p < d, \quad V \hookrightarrow L^a(D), \ \forall a \in [1, p^*] \text{ and compactly if } a \in [1, p^*),\\
&\text{if } p = d, \quad V \hookrightarrow L^a(D), \ \forall a < +\infty \text{ and compactly,}\\
&\text{if } p > d, \quad V \hookrightarrow C(\overline{D}) \text{ and compactly .}
\end{aligned}$$

Since $\underline{\mathbf{p}} = \max(1, \frac{2d}{d+2}) < p < +\infty$, the compactness of the embeddings hold in Lions-Guelfand triple. The duality bracket for $T \in V'$ and $v \in V$ is denoted by $\langle T, v\rangle$ and the scalar product in H is denoted by $(\cdot, \cdot)$. We recall the existence of $C_D > 0$ such that

$$\|u\|_H^2 \leq C_D \|u\|_V^2, \quad \forall u \in V, \text{ since } V \hookrightarrow H. \tag{7}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space endowed with a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$[8] completed with respect to the measure P. $W(t)$ is a $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Q-Wiener process in H, where Q is non-negative symmetric operator with finite trace *i.e.* $trQ < \infty$. Denote by $\Omega_T = (0, T) \times \Omega$

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[8] For example, $(\mathcal{F}_t)_{t\geq 0}$ is the augmentation of the filtration generated by $\{W(s), 0 \leq s \leq t\}_{0\leq t\leq T}$.

and $\mathcal{P}_T$ the predictable σ-algebra on Ω_T[9]. Set $H_0 = Q^{1/2}H$ and we recall that H_0 is a separable Hilbert space endowed with the inner product $(u,v)_0 = (Q^{-1/2}u, Q^{-1/2}v)$, for any $u,v \in H_0$. The space $(L_2(H_0,H), \|\cdot\|_{L_2(H_0,H)})$ stands for the space of Hilbert-Schmidt operators[10] from H_0 to H and $\mathbb{E}$ stands for the expectation with respect to the probability measure P.

We recall that an element $\xi \in V'$ (resp. $L^{p'}(\Omega_T;V')$) is called *non-negative*, that is $\xi \in (V')^+$(resp. $(L^{p'}(\Omega_T;V'))^+$) if and only if $\langle \xi,\varphi\rangle \geq 0$ (resp. $\mathbb{E}\int_0^T \langle \xi,\varphi\rangle\, dt \geq 0$) holds for all $\varphi \in V$ (resp. $L^p(\Omega_T;V)$) such that $\varphi \geq 0$. In this case, with a slight abuse of notation, we will often write $\xi \geq 0$. Denote by $V^* = (V')^+ - (V')^+ \subsetneq V'$ the order dual as being the difference of two non-negative elements of V', more precisely $h \in V^*$ iff $h = h^+ - h^-$ with $h^+, h^- \in (V')^+$.

3.2. Formulation of the problem

Our aim is to investigate the existence and uniqueness of invariant measures associated with the following problem

$$\begin{cases} du + A(u)ds + \mathbf{F}(u)ds + kds = fds + G(u)dW & \text{in} \quad D\times\Omega_T, \\ \langle k, u-\psi\rangle_{V',V} = 0 \ \text{ and } \ -k \in (V')^+ & \text{a.e. in} \quad \Omega_T, \\ u \geq \psi & \text{in} \quad D\times\Omega_T, \\ u = 0 & \text{on} \quad \partial D\times\Omega_T, \\ u(t=0) = u_0 & \text{in} \quad D, \end{cases} \tag{8}$$

Assumptions. We will consider in the sequel the following assumptions:

H_1 : Let $A : V \to V'$, $G : H \to L_2(H_0,H)$ be measurable such that:

H_2 : there exist $\alpha, \bar{K} > 0, \lambda, \lambda_{\mathbf{T}}, l_1 \in \mathbb{R}$ such that:

- $H_{2,1}$: (Coercivity) for all $v \in V$, $\langle A(v),v\rangle + \lambda\|v\|_H^2 + l_1 \geq \alpha\|v\|_V^p$.
- $H_{2,2}$: (**T**-monotonicity[11]) for all $v_1, v_2 \in V$,

$$\lambda_{\mathbf{T}}(v_1-v_2,(v_1-v_2)^+)_H + \langle A(v_1)-A(v_2),(v_1-v_2)^+\rangle \geq 0.$$

  Note that since $v_1 - v_2 = (v_1-v_2)^+ - (v_2-v_1)^+$, $\lambda_{\mathbf{T}} Id + A$ is also monotone.
- $H_{2,3}$: (Boundedness) for all $v \in V$, $\|A(v)\|_{V'} \leq \bar{K}(\|v\|_V^{p-1}+1)$.
- $H_{2,4}$: (Hemi-continuity) for all $v, v_1, v_2 \in V$: $s \in \mathbb{R} \mapsto \langle A(v_1+sv_2),v\rangle$ is continuous.

H_3 : $\exists L_G, M > 0$ such that: for all $\theta,\sigma \in H$:
$\|G(\theta)-G(\sigma)\|^2_{L_2(H_0,H)} \leq L_G\|\theta-\sigma\|_H^2$ and $\|G(0)\|^2_{L_2(H_0,H)} \leq M$.

H_4 : The obstacle $\psi \in V$ such that $G(\psi) = 0$.

H_5 : The external force $f \in V'$ such that $f - A(\psi) = h \in V^*$.

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[9] $\mathcal{P}_T := \sigma(\{]s,t] \times F_s | 0 \leq s < t \leq T, F_s \in \mathcal{F}_s\} \cup \{\{0\}\times F_0 | F_0 \in \mathcal{F}_0\})$ (see [32, p. 33]). Then, a process defined on Ω_T with values in a given space E is predictable if it is $\mathcal{P}_T$-measurable.

[10] For a linear operator $\mathcal{B}$, we recall $\|\mathcal{B}\|^2_{L_2(H_0,H)} = \sum_{k\in\mathbb{N}}\|\mathcal{B}e_k\|_H^2$ where $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for H_0, see *e.g.* [32, Appendix B].

[11] The "**T**" in **T**-monotone stands for **Truncation**, highlighting that monotonicity is only required in the direction of the truncated difference $(u-v)^+$, which aligns naturally with obstacle and unilateral constraints, see [8] .

H$_6$: The initial datum $u_0 \in H$, and satisfies the constraint, *i.e.* $u_0 \geq \psi$.
H$_7$: Let $\mathbf{F} : \mathbb{R} \to \mathbb{R}$, there exist $L_F, K > 0$ such that

$$\text{for all } x, y \in \mathbb{R} : |\mathbf{F}(x) - \mathbf{F}(y)| \leq L_F|x - y|. \tag{9}$$

Let us make some comments on H$_7$, which plays a role only if $\underline{\mathbf{p}} < p < 2$.

Remark 3.1. Let $v, v_1, v_2 \in H$ and note that

$$|(\mathbf{F}(v_1) - \mathbf{F}(v_2), (v_1 - v_2)^+)_H| = |\int_D [\mathbf{F}(v_1) - \mathbf{F}(v_2)].(v_1 - v_2)1_{\{v_1 - v_2 \geq 0\}} dx|$$
$$\leq L_F \int_D |v_1 - v_2|^2 1_{\{v_1 - v_2 \geq 0\}} dx = L_F \|(v_1 - v_2)^+\|_H^2,$$

and $|(\mathbf{F}(v), v)_H| \leq (L_F + \frac{1}{2})\|v\|_H^2 + \frac{1}{2}\|F(0)\|_H^2 \leq (L_F + \frac{1}{2})\|v\|_H^2 + \frac{CK}{2}$. Furthermore, $\|\mathbf{F}(v)\|_H \leq L_F\|v\|_H + CK$. Denote by $\widehat{A}(u) = A(u) + \mathbf{F}(u)$, then $\widehat{A}$ satisfies $H_1, H_{2,1} - H_{2,4}$ if $p \geq 2$. In the case $1 < p < 2$, $H_{2,3}$ is not satisfied. Finally, thanks to H_4 one has $\mathbf{F}(\psi) \in H$. Thus, by using H$_5$, we get

$$\begin{aligned} f - A(\psi) - \mathbf{F}(\psi) &= [h^+ + (\mathbf{F}(\psi))^-] - [h^- + (\mathbf{F}(\psi))^+] \\ &= \widetilde{h}^+ - \widetilde{h}^- = \widetilde{h} \in V^*. \end{aligned} \tag{10}$$

Notice that *e.g.* the perturbation of $A = -\Delta_p$; $1 < p < 2$, by a zero order Lipschitz mapping is not covered by the well-posedness result [42, Theorem 1], since $H_{2,3}$ is not satisfied. On the other hand, the incorporation of zero order Lipschitz mapping in the operator part is important to study the invariant measures if $\underline{\mathbf{p}} < p < 2$. Therefore, we need to check the well-posedness of (8), which is given in Theorem 3.2. Its proof follows by a cosmetic changes of the one [42, Theorem 1] but for the convenience of the reader, we present the proof in Section 6. We invite the interested reader by more general situations about obstacle problems in the presence of a multiplicative noise to consult the recent work [36], where well-posedness and Lewy-Stampacchia's inequality were proved by using stochastic compactness tools.

Remark 3.2. Considering the different notions of operators, it is useful to make some comments on the relationship between **T**-monotone, maximal monotone, and accretive operators. Let $v_1, v_2 \in V$, recall that $v_1 - v_2 = (v_1 - v_2)^+ - (v_2 - v_1)^+$. Hence, it is not difficult to verify that H$_{2,2}$ implies that A is monotone, *i.e.* for all $v_1, v_2 \in V$,

$$\lambda_{\mathbf{T}}\|v_1 - v_2\|_H^2 + \langle A(v_1) - A(v_2), v_1 - v_2\rangle \geq 0.$$

Since A is also hemicontinuous, then A is maximal monotone in $V \times V'$, see [1, Theorem 2.4]. It is worth mentioning that the notion of **T**-monotone operator was introduced in [8] to study elliptic inequalities and the regularity of its solutions, see also [27]. This is somewhat related to the fact that when solving variational inequalities or obstacle problems, the solution lies in a convex subset and not the full Banach space and one needs to handle non ordered Banach space. Moreover, note also that the monotonicity property is defined from V to its dual space and it is a variational property, whereas the the notion of accretivity is a metric geometric property defined for an operator A from a Banach space X to itself, see *e.g.* [1] for more details.

Remark 3.3. Note that $G(\psi) = 0$ in H_4 ensures that Lewy-Stampacchia inequalities (12) holds, which allows us to manage the reflection measure in appropriate sense and obtain well-posedness result for obstacle problems with larger class of data, we refer to [42]. Moreover, a large class of V-valued obstacle problems can reduce to the question of a positivity obstacle problem with a stochastic reaction term vanishing at 0, see [42, Rmk. 3]. Finally, it is appropriate in many situations as well, from a physical point of view, to consider G vanishes at the obstacle, see *e.g.* [5].

3.3. Well-posedness of the stochastic obstacle problem (8)

Consider the following problem: find (u, k), in the sense of Definition 3.1, solution of (8). Denote by K the convex set of admissible solutions

$$K = \{v \in L^p(\Omega_T; V), \quad v(x, t, \omega) \geq \psi(x) \quad \text{a.e. in } D \times \Omega_T\}.$$

Definition 3.1. The pair (u, k) is a solution to Problem (8) if:

1. $u : \Omega_T \to H$ and $k : \Omega_T \to V'$ are predictable processes.
2. $u \in L^2(\Omega; C([0, T]; H))$, $u \in L^p(\Omega_T; V)$, $u(0, \cdot) = u_0$ and $u \geq \psi$, *i.e.*, $u \in K$.
3. P-a.s, for all $t \in [0, T]$:
$$u(t) + \int_0^t [A(u) + \mathbf{F}(u) + k]ds = u_0 + \int_0^t G(u)dW(s) + \int_0^t f ds \text{ in } V'.$$
4. $-k \in (L^{p'}(\Omega_T, V'))^+$ and $\forall v \in K$, $\langle k, u - v\rangle \geq 0$ a.e. in Ω_T.

Remark 3.4. The point (4) in Definition 3.1 can be replaced by $k \in L^{p'}(\Omega_T; V')$ with

$$-k \in (L^{p'}(\Omega_T; V'))^+ \text{ and } \mathbb{E}\int_0^T \langle k, u - \psi\rangle_{V',V}\, dt = 0 \tag{11}$$

Condition (11) can be understood as a minimality condition on k in the sense that k vanishes on the set $\{u > \psi\}$. Moreover, (11) implies that, for all $v \in K$, $\mathbb{E}\int_0^T \langle k, u - v\rangle\, dt \geq 0$.

Let us present the following result about the solution to (8). We distinguish between the cases $2 \leq p < +\infty$ and $\underline{\mathbf{p}} < p < 2$. By using [42, Theorem 1], we have the following result.

Theorem 3.1. *($2 \leq p < +\infty$) Under the assumptions H_1-H_7, there exists a unique solution (u, k) to (8) in the sense of Definition 3.1. Moreover, the following Lewy-Stampacchia inequality holds*

$$0 \leq \partial_t\Big(u - \int_0^{\cdot} G(u)dW\Big) + A(u) - f = -k \leq h^- = (f - A(\psi))^-. \tag{12}$$

The next theorem concerns the well-posedness of (8) if $1 < p < 2$, see Section 6.

Theorem 3.2. ($\underline{\mathbf{p}} < p < 2$) *Assume* $H_1 - H_7$ *hold and assume moreover that* $\widetilde{h}^-$ *is a non negative element of* $L^{p'}(D)$, *see* (10). *Then, there exists a unique solution* $(\widetilde{u}, \widetilde{k})$ *to* (8) *in the sense of Definition 3.1. Moreover,* $-\widetilde{k} \in \big((L^{p'}(\Omega_T; L^{p'}(D))\big)^+$ *and the following holds*

$$\mathbb{E} \sup_{t\in[0,T]} \|(\widetilde{u}_\epsilon - \widetilde{u})(t)\|_H^2 \leq C\epsilon^{\frac{1}{p-1}} \quad \text{for all } \epsilon > 0, \tag{13}$$

where $\widetilde{u}_\epsilon$ *is the unique solution of* (58).

Remark 3.5. We wish to draw the reader's attention to the fact that the extra regularity of $\widetilde{h}^-$ if $\underline{\mathbf{p}} < p < 2$, namely $\widetilde{h}^- \in (L^{p'}(D))^+$ can be removed and consider $\widetilde{h}^- \in (V')^+$ and prove the well-posedness of (58). First, one proves Lewy-Stampacchia inequality with $\widetilde{h}^- \in (L^{p'}(D))^+$ by repeating the same arguments of [42, Subsection 3.2]. Then, use Lewy-Stampacchia inequality to extend the well-posedness to the case $\widetilde{h}^- \in (V')^+$, see [42, Subsection 3.3]. Therefore, the results can be extended to the case of $\widetilde{h}^- \in (V')^+$ by similar arguments.

Remark 3.6. Notice that the unique solution to (8) in the cases $2 \leq p < +\infty$ and $\underline{\mathbf{p}} < p < 2$ satisfies the same properties of Definition 3.1. Therefore, we do not distinguish between them in the rest of the paper unless it is necessary to understand the argument.

3.4. Main results

In this subsection, we collect the results of this work. First, let us recall the following: $L^2(H, \mu)$ denotes the Hilbert space of all equivalance classes of Borel square integrable real functions on H with respect to the Gaussian measure μ, $B_b(H)$ denotes the set of bounded Borel functions from H to $\mathbb{R}$ and denote by $\mathcal{C}_b(H)$ the set of bounded continuous function from H to $\mathbb{R}$. We recall that

$$(\varphi, \Psi)_{L^2(H,\mu)} = \int_H \varphi\Psi d\mu, \quad \varphi, \Psi \in L^2(H, \mu),$$

we refer *e.g.* to [10, Chapter 9] for more details.

Definition 3.2. Let $(P_t)_t$ be a Markov semigroup on H.

- A probability measure ν is said to be invariant for P_t if:

$$\int_H P_t\varphi d\nu = \int_H \varphi d\nu, \quad \forall \varphi \in B_b(H) \text{ and } t \geq 0.$$

- Let ν be an invariant measure for P_t.
    i. ν is ergodic if:

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T P_t\varphi dt = \int_H \varphi(x)\nu(dx), \quad \forall \varphi \in L^2(H, \nu).$$

    ii. P_t is strongly mixing if:

$$\lim_{t\to+\infty} P_t\varphi(x) = \int_H \varphi(x)\nu(dx) \text{ in } L^2(H, \nu) \text{ where } x \in H.$$

Now, denote by $K_\psi = \{h \in H, \quad h \geq \psi\}$ and note that K_ψ is a closed convex set in H. Denote by π_ψ the projection from H onto the closed convex set K_ψ *i.e.* $\pi_\psi : H \to K_\psi$. It is known that π_ψ is 1-Lipschitz continuous. We denote by $\mathcal{C}_b(K_\psi)$ the space of all $\varphi : K_\psi \to \mathbb{R}$ being bounded and uniformly continuous with respect to $\|\cdot\|_H$. Thus, we can identify $\mathcal{C}_b(K_\psi)$ with a subspace of $\mathcal{C}_b(H)$ by using the embedding $\mathcal{C}_b(K_\psi) \to \mathcal{C}_b(H); \varphi \mapsto \varphi \circ \pi_\psi$.

For $t \geq 0$, we define the following Markov Feller semigroup $(P_t)_{t\geq 0}$ (see Lemma 4.2).

$$P_t\varphi : K_\psi \to \mathbb{R}; \quad (P_t\varphi)(\eta) := \mathbb{E}[\varphi(u(t;\eta))], \quad \eta \in K_\psi \quad \forall \varphi \in \mathcal{C}_b(K_\psi). \tag{14}$$

where $u(t;\eta), t \geq 0$ be the unique strong solution to (8) starting from $\eta \in K_\psi$. Let us state our main results, we distinguish between the cases: $2 \leq p < \infty$ and $\underline{\mathbf{p}} < p < 2$. Before that, let us introduce the following assumption

$$\text{There exists } \mathbf{K} > 0 : \forall \sigma \in H : \|G(\sigma)\|^2_{L_2(H_0,H)} \leq \mathbf{K}. \tag{15}$$

The following result concerns a moment estimates of the set of invariant measures. Denote by Λ the set of invariant measures for the semigroup $(P_t)_t$ defined by (14), see propositions 4.6 and 4.4 for the proof.

Proposition 3.3. *Assume that H_1-H_7 hold. If $2 \leq p < +\infty$, then there exists $\widetilde{\mathbf{K}} > 0$ such that $\int_H \|x\|_V^p \mu(dx) \leq \widetilde{\mathbf{K}}, \forall \mu \in \Lambda$. If $\underline{\mathbf{p}} < p < 2$, then there exists $\bar{\mathbf{K}} > 0$ such that $\int_H \|x\|_H^p \mu(dx) \leq \bar{\mathbf{K}}, \forall \mu \in \Lambda$.*

Now, we present the results about the existence and uniqueness of invariant measures.

Theorem 3.4. (case $2 \leq p < \infty$) *Assume that H_1-H_6 hold and $\mathbf{F} \equiv 0$ in H_7*[12].

1. *Let $2 < p < +\infty$, there exists an ergodic invariant measure μ for $(P_t)_t$ associated with (8). Moreover, μ is concentrated in $V \cap K_\psi$ and satisfying $\int_H \|x\|_V^p \mu(dx) < +\infty$.*
2. *Let $K > 0$ and $p = 2$ and assuming that there exists $\bar{\alpha} > 0$ such that*
$$(1-\delta)\alpha - C_D[\lambda + \frac{L_G(1+K^2)}{2K^2}]^+ \geq \bar{\alpha} \text{ for some } \delta \in]0,1[, \tag{16}$$
then, there exists an ergodic invariant measure μ for $(P_t)_t$ associated with (8), concentrated in $V \cap K_\psi$ and satisfying $\int_H \|x\|_V^2 \mu(dx) < +\infty$.
3. *Let $p = 2$, the same result holds if one replaces (16) by $\lambda \leq 0$ in $H_{2,1}$ and (15).*

Theorem 3.5. *Under the assumptions of Theorem 3.4.*

1. *If $\dfrac{L_G}{2} + \lambda_{\mathbf{T}} < 0$ in the case (1) and (2). Then there exists a unique ergodic and strongly mixing invariant measure μ and the following convergence to equilibrium holds.*
$$\left| P_t\varphi(x) - \int_H \varphi(y)\mu(dy) \right| \leq \mathcal{G}\|\varphi\|_{1,\infty} e^{[\frac{L_G}{2}+\lambda_{\mathbf{T}}]t}, \quad \forall \varphi \in \mathcal{C}_b^1(H). \tag{17}$$

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[12] In this case, $\mathbf{F} \equiv 0$ is not a restriction, since one needs only to change λ by $\lambda + L_F$.

2. *If $\lambda_{\mathbf{T}} < 0$ in the case (3). Then there exists a unique ergodic and strongly mixing invariant measure μ and the following convergence to equilibrium holds.*

$$\Big|P_t\varphi(x) - \int_H \varphi(y)\mu(dy)\Big| \leq \|\varphi\|_{1,\infty}(\mathcal{G} + \sqrt{\mathbf{K}t})e^{\lambda_{\mathbf{T}}t} \quad \forall \varphi \in \mathcal{C}_b^1(H), \tag{18}$$

where $\mathcal{G}$ depends only on $\widetilde{\mathbf{K}}$ and $\bar{\mathbf{K}}$.

Concerning the case $\underline{\mathbf{p}} < p < 2$, we have

Theorem 3.6. *Let $\underline{\mathbf{p}} < p < 2$, under the assumptions H_1-H_7, let $K > 0$, the following holds.*

1. *Under the assumptions $\frac{L_G(1+K^2)}{2K^2} + \lambda + L_F \leq 0$. There exists an invariant measure μ for $(P_t)_t$ associated with (8).*
2. *Assume that $\lambda + L_F \leq 0$ and G satisfies (15). Then there exists an invariant measure μ for $(P_t)_t$ associated with (8).*

Moreover, μ is concentrated in $V \cap K_\psi$ and satisfying $\int_H \|x\|_V^p \mu(dx) < +\infty$.

Theorem 3.7. *Under the assumptions Theorem 3.6.*

1. *If $\dfrac{L_G}{2} + \lambda_{\mathbf{T}} + L_F < 0$ in the case (1). Then there exists a unique ergodic and strongly mixing invariant measure μ and the following convergence to equilibrium holds.*

$$\Big|P_t\varphi(x) - \int_H \varphi(y)\mu(dy)\Big| \leq \mathcal{G}\|\varphi\|_{1,\infty} e^{[\frac{L_G}{2} + \lambda_{\mathbf{T}} + L_F]t}, \quad \forall \varphi \in \mathcal{C}_b^1(H). \tag{19}$$

2. *If $\lambda_{\mathbf{T}} + L_F < 0$ in the case (2). Then there exists a unique ergodic and strongly mixing invariant measure μ and the following convergence to equilibrium holds.*

$$\Big|P_t\varphi(x) - \int_H \varphi(y)\mu(dy)\Big| \leq \|\varphi\|_{1,\infty}(\mathcal{G} + \sqrt{\mathbf{K}t})e^{\lambda_{\mathbf{T}}t} \quad \forall \varphi \in \mathcal{C}_b^1(H), \tag{20}$$

where $\mathcal{G}$ depends only on $\widetilde{\mathbf{K}}$ and $\bar{\mathbf{K}}$.

Remark 3.7. Since we are interested in the existence of invariant measures to a Markov semigroup associated with non linear SPDEs in the presence of a multiplicative noise, where the solution takes values in the Sobolev space V, one expects a restriction on the Lipschitz constant L_G, see H_3 to obtain the existence of invariant measures. We distinguish two cases. If the dissipation generated by the **T**-monotone operator is strong enough, then the restriction on the Lipschitz constant L_G is not needed, see Theorem $3.4_{(1)}$. Otherwise, a restriction on L_G is needed, see (16) and Theorem 3.6.

3.5. Ergodicity and obstacle problems

It is worth saying few words about the ergodicity in our setting. First, recall that the existence of an invariant measure guarantees only a stationary statistical state, but not necessarily an equilibrium state. In particular, invariant measures may correspond to non-equilibrium steady states when, for example, reversibility fails. In the case of obstacle problem, the reflected measure may introduce irreversible dynamics, since it generates some dissipation near to the contact set and also the constraint may destroy the symmetry of the generator (the system is pushed away from the obstacle, but never symmetrically pushed back). Bearing this in mind, one would understand the equilibrium state, which can be represented by an ergodic invariant measure.

Ergodic invariant measures. From Definition 3.2, one sees that an *ergodic* invariant measure ensures that time averages coincide with ensemble averages for a large class of observables φ and the system forgets its initial condition. In the presence of constraint, the obstacle reshapes the equilibrium distribution by excluding forbidden configurations near the contact set. On the other hand, an ergodic invariant measure cannot be written as a nontrivial convex combination of other invariant measures (this can be also seen form Krein-Milman theorem about extreme points) and an ergodic invariant measure represents a pure stationary phase. Regarding the special features of obstacle problems (in particular the constraint and reflected measure) the dynamic explores all typical configurations allowed by the constraint (the non linear state space K_ψ) and the long time dynamics does not get trapped in different *"contact regimes"*. In other words, there is no measurable decomposition of K_ψ into invariant subsets such as *"contact set* $\{u = \psi\}$*"* and *"free set* $\{u > \psi\}$*"* that remain dynamically isolated. If a measurable decomposition exists, then the system falls into different invariant *"contact phases"* with positive probability, each with its own stationary law.

Ergodicity and uniqueness of the invariant measure. In our setting, ergodicity corresponds to the uniqueness and convergence to equilibrium, presented by the ergodic and strongly mixing invariant measure. Namely, the system reaches a unique universal long-time statistics consistent with the imposed physical constraint (the phase space K_ψ). In obstacle problems, one might think about different long-time regimes: states mostly pinned to the obstacle, states mostly free or states with intermittent contact. Uniqueness rules this out: there is only one stationary phase compatible with the noise and the obstacle. Moreover, uniqueness of the invariant measure means that all initial configurations consistent with the constraint (*i.e.* starts touching the obstacle everywhere, starts far above it, or touches it only on small regions) lead in the long-time limit, to the same stationary statistical law and the system completely forgets how it was prepared (loss of memory), despite the presence of the obstacle. This excludes fixed metastable configurations and invariant subsets defined by permanent contact with the obstacle. Furthermore, the uniqueness implies that the *contact set* $\{u = \psi\}$ and the reflected measure k possess unique stationary statistics. If there were multiple invariant measures, the system would exhibit phase coexistence, depending on initial data.

Furthermore, the uniqueness of invariant measure and the ergodicity of the system indicate that one long experiment is statistically equivalent to many independent experiments. This is crucial in constrained systems where the obstacle could otherwise trap the dynamics. Ergodicity also guarantees that there is no non-trivial measurable splitting of K_ψ. If ergodicity fails, one may observe multiple invariant measures (phase coexistence), dependence on initial contact configurations and persistent pinning or localization near the obstacle. Physically, this corresponds to phenomena such as depinning transitions, metastability, or broken symmetries.

Consider the following example.

Example 3.1. Let $\underline{\mathbf{p}} < p < \infty$, $T > 0$, and $(u, -k) \in (L^2(\Omega; C([0,T]; H)) \cap L^p(\Omega_T; V)) \times (L^{p'}(\Omega_T, V'))^+$ such that $u \geq 0$, be the unique solution of the following problem

$$\begin{cases} du - \operatorname{div}[|\nabla u|^{p-2}\nabla u]ds + \kappa u ds + k ds = f ds + \mathbf{c} u d\beta_s & \text{in} \quad D \times \Omega_T, \\ u(\cdot, 0) = u_0 \in H, \quad u_0 \geq 0; \quad f(x) = \sin(x) & \text{in} \quad D, \\ \langle k, u \rangle_{V', V} = 0 \ \text{ and } \ -k \in (V')^+ & \text{a.e. in} \quad \Omega_T, \end{cases} \tag{21}$$

where $\kappa, \mathbf{c} \in \mathbb{R}$, β_t is a one dimensional brownian motion. Indeed, we can verify that $A(u) = -\operatorname{div}[|\nabla u|^{p-2}\nabla u]$ satisfies $H_1, H_{2,1}, H_{2,3}$ and $H_{2,4}$ by repeating those used in [30, Example 4.1.9]. We need only to verify $H_{2,2}$, the T-monotonicity. Denote by $1_{\{u>v\}}$ the indicator function of the set $\{x \in D; u(x) \geq v(x)\}$. Let $u, v \in V$ and note that, for any $p > 1$, we have

$$\begin{aligned} &\langle A(u) - A(v), (u-v)^+ \rangle = -\langle \operatorname{div}[|\nabla u|^{p-2}\nabla u] - \operatorname{div}[|\nabla v|^{p-2}\nabla v], (u-v)^+ \rangle \\ &= \int_D [|\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x)] \cdot \nabla(u(x) - v(x)) 1_{\{u>v\}}(x) dx \\ &\geq \int_D (|\nabla u(x)|^{p-1} - |\nabla v(x)|^{p-1}) \cdot (|\nabla u(x)| - |\nabla v(x)|) 1_{\{u>v\}}(x) dx \\ &\geq 0, \ \text{since } \mathbb{R}_+ \ni s \mapsto s^{p-1} \text{ is an increasing function.} \end{aligned}$$

Hence $H_{2,2}$ is satisfied. An application of Theorems 3.4, 3.5, 3.6 and 3.7 yields

- The solution u of (21) defines a homogeneous Feller-Markov process.
- If $p > 2$: there exists an ergodic invariant measure μ for $(P_t)_t$ associated with (21), for any $\kappa, \mathbf{c} \in \mathbb{R}$. If moreover $\dfrac{\mathbf{c}^2}{2} < \kappa$, there exists a unique ergodic and strongly mixing invariant measure μ for $(P_t)_t$ associated with (21).
- If $p = 2$ and $2\mathbf{c}^2 \leq \kappa$: there exists a unique ergodic and strongly mixing invariant measure μ for $(P_t)_t$ associated with (21).
- If $\underline{\mathbf{p}} < p < 2$ and $2\mathbf{c}^2 < \kappa$: there exists a unique ergodic and strongly mixing invariant measure μ for $(P_t)_t$ associated with (21).

Remark 3.8. It's worth to draw the reader attention to the following: in 1D and 2D cases, Theorem 3.6 ensures the existence of an invariant measure for a stochastic obstacle problems governed by the singular p-Laplace operator, when $1 < p < 2$.

4. Proofs

4.1. Construction of solution to (8)

Let us recall here the main steps to construct the solution to (8). We refer to [42] for the detailed proofs, see also Section 6.

1. Let $\epsilon > 0$ and $u_0 \geq \psi$, consider the following penalized problem:

$$u_\epsilon(t) + \int_0^t (A(u_\epsilon) - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} - f)ds = u_0 + \int_0^t \widetilde{G}(u_\epsilon)dW(s) \quad (22)$$

where $\tilde{q} = \min(p, 2)$ and $\widetilde{G}(u_\epsilon, \cdot) = G(\max(u_\epsilon, \psi))$. By [32, Thm. 4.2.4], we get

Lemma 4.1. *For all $\epsilon > 0$, there exists a unique solution $u_\epsilon \in L^p(\Omega_T; V)$ predictable such that $u_\epsilon \in L^2(\Omega; C([0,T], H))$ and satisfying (22) P-a.s. in Ω for all $t \in [0,T]$.*

2. (A priori estimates) According to [42, Lemma 2], we have $(u_\epsilon)_{\epsilon>0}$ and $(A(u_\epsilon))_{\epsilon>0}$ are bounded in $L^p(\Omega_T, V) \cap C([0,T], L^2(\Omega, H))$ and $L^{p'}(\Omega_T, V')$ respectively.
3. (Convergence with regular data) From [42, Lemma 3 & Lemma 4], under the assumption h^- non negative element of $L^{\tilde{q}'}(D)$, we have
   - $(\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1})_{\epsilon>0}$ is bounded in $L^{\tilde{q}'}(\Omega_T \times D)$.
   - $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in the space $L^2(\Omega; C([0,T], H))$. Moreover

$$\mathbb{E} \sup_{t\in[0,T]} \|(u_\epsilon - u_\delta)(t)\|_H^2 \leq C(T)\epsilon \quad \text{for } 1 > \epsilon \geq \delta > 0. \quad (23)$$

   - From [42, Lemma 5 & Theorem 2], there exists a unique predictable stochastic process $(u, k) \in L^p(\Omega_T; V) \times L^{\tilde{q}'}(\Omega_T; L^{\tilde{q}'}(D))$ and $(u_\epsilon)_{\epsilon>0}$ converges strongly in $L^2(\Omega, C([0,T], H))$ to u, where u satisfying
     - i. $u \in L^2(\Omega; C([0,T], H)) \cap K$, $u(0) = u_0$.
     - ii. $k \leq 0$ and $\forall v \in K$, $\langle k, u - v\rangle \geq 0$ a.e. in Ω_T.
     - iii. P-a.s, for all $t \in [0,T]$:

$$u(t) + \int_0^t [A(u) + k]ds = u_0 + \int_0^t G(u)dW(s) + \int_0^t fds.$$

     - iv. The Lewy-Stampacchia inequality holds, *i.e.*

$$0 \leq -k \leq h^- = (f - A(\psi))^- .$$

4. (Convergence with general data) Following [42, Subsection 3.3], under the assumption $h^- \in (V')^+$, there exists $h_n \in L^{\tilde{q}'}(\Omega_T; L^{\tilde{q}'}(D))$ predictable and non negative such that $h_n \longrightarrow h^- \quad \text{in} \quad L^{p'}(\Omega_T; V')$. Denote by (u_n, k_n) the sequence of solutions given by (2) where h^- is replaced by h_n and recall that
   - Associated with h_n, denote the following f_n by, $f_n = A(\psi) + h^+ - h_n, h^+ \in (V')^+$. Note that $f_n \in V'$ and f_n converges strongly to f in V'.

- By Lewy-Stampacchia inequality, one has $0 \le -k_n \le h_n$.
- $(u_n)_n$ is a Cauchy sequence in the space $L^2(\Omega; C([0,T],H))$ and
$$\mathbb{E}\sup_{t\in[0,T]}\|(u_n-u_m)(t)\|_H^2 \le C\|f_n-f_m\|_{L^{p'}(\Omega_T;V')} \le C\epsilon \tag{24}$$
for large n and m.
- $(u_n)_{n>0}$ converges strongly in $L^2(\Omega; C([0,T],H))$ to u, solution of to (8) with $\mathbf{F}\equiv 0$ in the sense of Definition 3.1.

From (23) and (24), we are able to infer
$$\mathbb{E}\sup_{t\in[0,T]}\|(u_\epsilon-u)(t)\|_H^2 \le C\epsilon \quad \text{for all } \epsilon>0, \tag{25}$$
where u_ϵ and u, respectively are the unique solution of (22) and (8) with $\mathbf{F}\equiv 0$, respectively. From Theorem 6.1, we have
$$\mathbb{E}\sup_{t\in[0,T]}\|(\widetilde{u}_\epsilon-\widetilde{u})(t)\|_H^2 \le C\epsilon^{\frac{1}{p-1}} \quad \text{for all } \epsilon>0, \tag{26}$$
where $\widetilde{u}_\epsilon$ and $\widetilde{u}$, respectively are the unique solution of (58) and (8), respectively.

4.2. Feller-Markov process associated with (8)

Let $\epsilon>0$ and $u_\epsilon(t,s;\xi), t\ge s\ge 0$ be the unique strong solution to (22) starting at time s from $\mathcal{F}_s$-measurable initial data $\xi\in H$. Let $T>0$, we have
$$\mathbb{E}\|u_\epsilon(t,s;\xi)-u_\epsilon(t,s;\theta)\|_H^2 \le e^{[L_G+2\lambda_{\mathbf{T}}](t-s)}\mathbb{E}\|\xi-\theta\|_H^2 \quad \forall 0\le s\le t\le T. \tag{27}$$

Indeed, by using (22) and applying Itô's formula we get
$$\begin{aligned}
&\frac12\|u_\epsilon(t,s;\xi)-u_\epsilon(t,s;\theta)\|_H^2-\frac12\|\xi-\theta\|_H^2\\
&+\int_s^t\langle A(u_\epsilon(r,s;\xi))-A(u_\epsilon(r,s;\theta)), u_\epsilon(r,s;\xi)-u_\epsilon(r,s;\theta)\rangle dr\\
&-\frac{1}{\epsilon}\int_s^t\langle[(u_\epsilon(r,s;\xi)-\psi)^-]^{\tilde q-1}-[(u_\epsilon(r,s;\theta)-\psi)^-]^{\tilde q-1}, u_\epsilon(r,s;\xi)-u_\epsilon(r,s;\theta)\rangle dr\\
&=\int_s^t\langle(\widetilde{G}(u_\epsilon(r,s;\xi))-\widetilde{G}(u_\epsilon(r,s;\theta))), u_\epsilon(r,s;\xi)-u_\epsilon(r,s;\theta)\rangle dW(r)\\
&\quad+\frac12\int_s^t\|\widetilde{G}(u_\epsilon(r,s;\xi))-\widetilde{G}(u_\epsilon(r,s;\theta))\|_{L_2(H_0,H)}^2 dr.
\end{aligned}$$

By taking the expectation, using that $x\mapsto -(x)^{\tilde q-1}$ is non-decreasing, $H_{2,2}$ and $H_{3,1}$ we get
$$\begin{aligned}
&\mathbb{E}\|u_\epsilon(t,s;\xi)-u_\epsilon(t,s;\theta)\|_H^2\\
&\le \mathbb{E}\|\xi-\theta\|_H^2+(L_G+2\lambda_{\mathbf{T}})\mathbb{E}\int_s^t\|u_\epsilon(r,s;\xi)-u_\epsilon(r,s;\theta)\|_H^2 dr,
\end{aligned}$$
and (27) follows from Grönwall's lemma.

Remark 4.1. *($\underline{\mathbf{p}}<p<2$)* It's clear that (27) holds for the solution to (58), denoted by $\widetilde{u}_\epsilon$, with $\lambda_{\mathbf{T}}$ replaced by $\lambda_{\mathbf{T}}+L_F$.

Let $\varphi \in \mathcal{C}_b(H)$, we define the operators

$$P_t^\epsilon \varphi : H \to \mathbb{R}; \quad (P_t^\epsilon \varphi)(\xi) := \mathbb{E}[\varphi(u_\epsilon(t;\xi))], \quad \xi \in H,$$

where $u_\epsilon(t;\xi), t \geq 0$ is the unique strong solution to (22) (or (58) if $\underline{\mathbf{p}} < p < 2$) starting from $\xi \in H$. Thanks to Lemma 4.1(see Lemma 6.2 for the case of (58)) and by using a similair arguments to the one used in [29, Section 9.6](see also [32, Section 4.3]), we obtain the following properties of $(P_t^\epsilon)_t$:

$$P_{s,t}^\epsilon = P_{0,t-s}^\epsilon \quad \forall 0 \leq s \leq t < \infty \text{ and } P_t^\epsilon := P_{0,t}^\epsilon, \tag{28}$$

$$\mathbb{E}[\varphi(u_\epsilon(t+s;\eta))|\mathcal{F}_t] = (P_s^\epsilon \varphi)(u_\epsilon(t;\eta)) \quad \forall \varphi \in \mathcal{C}_b(H), \forall \eta \in H, \forall t,s > 0, \tag{29}$$

$$(P_{t+s}^\epsilon \varphi)(\eta) = (P_t^\epsilon(P_s^\epsilon \varphi))(\eta) \quad \forall \varphi \in \mathcal{C}_b(H), \forall \eta \in H, \forall t,s > 0, \tag{30}$$

where $P_{s,t}^\epsilon \varphi(\xi) := \mathbb{E}[\varphi(u_\epsilon(t,s;\xi))], \quad \xi \in H$. Therefore, $u_\epsilon(\cdot;\xi), \xi \in H$ is a family of Markov processes on the state space H with respect to the filtration $(\mathcal{F}_t)_t$, and the transition semi-group $(P_t^\epsilon)_{t\geq 0}$ is Feller semigroup thanks to (27). Thus, (22) defines a homogeneous Feller-Markov process.

Now, Let us show that (8) defines a homogeneous Feller-Markov process. Let $\eta \in K_\psi$ and $u(t;\eta), t \geq 0$ be the unique strong solution to (8) and recall that $u(t;\eta) \in K_\psi$ P-a.s.

In the following we focus on the case $\mathbf{F} \equiv 0$ and the case $\mathbf{F} \neq 0$ follows in the same way. Denote by $u(t,s;\xi); t \geq s \geq 0$ the unique strong solution to (8) with $\mathbf{F} \equiv 0$ starting at time s from an $\mathcal{F}_s$-measurable initial data $\xi \in K_\psi$. Thanks to (25) and (26), we can pass to the limit as $\epsilon \to 0$ in (27) and deduce

$$\mathbb{E}\|u(t,s;\xi) - u(t,s;\theta)\|_H^2 \leq e^{[L_G + 2\lambda_{\mathbf{T}}](t-s)} \mathbb{E}\|\xi - \theta\|_H^2 \quad \forall 0 \leq s \leq t \leq T, \tag{31}$$

Next, we present the following result about $(P_t)_t$ defined in (14).

Lemma 4.2. *$(P_t)_t$ is bounded on $\mathcal{C}_b(K_\psi)$, Markov-Feller and stochastically continuous semigroup on $\mathcal{C}_b(K_\psi)$. Secondly, for $\varphi \in \mathcal{C}_b(K_\psi), \eta \in K_\psi$, and $t,s \geq 0$, it holds that*

$$\lim_{\epsilon \to 0} (P_t^\epsilon \varphi)(\eta) = (P_t \varphi)(\eta), \tag{32}$$

$$(P_{t+s}\varphi)(\eta) = (P_t(P_s\varphi))(\eta). \tag{33}$$

Moreover, $P_{t+s,t} = P_{s,0}$ where $P_{t,s}\varphi(\eta) := \mathbb{E}[\varphi(u(t,s;\eta))]$ for $0 \leq s \leq t < \infty$, and $P_t := P_{0,t}$.

Proof. Let $\varphi \in \mathcal{C}_b(K_\psi), \eta \in K_\psi$, then

$|P_t\varphi(\eta)| \leq \|\varphi\|_\infty < +\infty$ and therfore $(P_t)_t$ is bounded. Secondly, we recall that $(P_t^\epsilon \varphi)(\eta) = \mathbb{E}[\varphi(u_\epsilon(t;\eta))]$ and $(P_t\varphi)(\eta) = \mathbb{E}[\varphi(u(t;\eta))]$. Then, Lebesgue dominated convergence theorem with (25) and (26) ensure (32).

From (30), we recall $(P_{t+s}^\epsilon \varphi)(\eta) = (P_t^\epsilon(P_s^\epsilon \varphi))(\eta) \quad \forall t,s > 0$. We show that we can pass to the limit as $\epsilon \to 0$. Indeed, by using (32) we get that $(P_{t+s}^\epsilon \varphi)(\eta)$ converges to $(P_{t+s}\varphi)(\eta)$, $(P_s^\epsilon \varphi)_\epsilon$ is uniformly bounded and equicontinuous on H thanks to (27). Moreover, by (32) one has $\lim\limits_{\epsilon \to 0} (P_s^\epsilon \varphi)(\eta) =$

$(P_s\varphi)(\eta)$. On the other hand, denote by $(\mu_\epsilon^t)_\epsilon$ the laws of $(u_\epsilon(t;\eta))_\epsilon$ and μ^t the law of $u(t;\eta)$ on H and K_ψ respectively, then (25) and (26) ensure that

$$\lim_{\epsilon\to 0}\int_H \phi d\mu_\epsilon^t = \int_H \phi d\mu^t, \quad \forall \phi \in \mathcal{C}_b(K_\psi).$$

Moreover, recall that $\mu^t(K_\psi) = 1$. Next, [43, Lemma 1] gives

$$\lim_{\epsilon\to 0}\int_H P_s^\epsilon \phi d\mu_\epsilon^t = \int_{K_\psi} P_s\phi d\mu^t, \quad \forall \phi \in \mathcal{C}_b(K_\psi). \tag{34}$$

Finally, we obtain

$$\lim_{\epsilon\to 0}(P_t^\epsilon(P_s^\epsilon\varphi))(\eta) = \lim_{\epsilon\to 0}\int_H P_s^\epsilon\varphi d\mu_\epsilon^t = \int_{K_\psi} P_s\varphi d\mu^t = (P_t(P_s\varphi))(\eta).$$

Thus $(P_{t+s}\varphi)(\eta) = (P_t(P_s\varphi))(\eta)$. The last claim follows similarly to (32), by using (25), (26) and (28). □

4.3. Existence of ergodic invariant measure

In this part, we prove Theorem 3.4 and Theorem 3.6.

4.3.1. Existence of an invariant measure. Let $\eta \in K_\psi$ and denote by μ_t^η the law of $u(t;\eta), t \geq 0$. We recall that

$$\begin{aligned} P_t\varphi(\eta) &= \mathbb{E}[\varphi(u(t;\eta))] = \int_{K_\psi} \varphi(y)\mu_t^\eta(dy) \\ &= \langle\langle \varphi, \mu_t^\eta\rangle\rangle = \langle\langle P_t\varphi, \delta_\eta\rangle\rangle = \langle\langle \varphi, P_t^*\delta_\eta\rangle\rangle, \quad \forall \varphi \in \mathcal{C}_b(K_\psi), \end{aligned}$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the duality between $\mathcal{C}_b(H)$ and $\mathcal{P}(H)$, the space of all Borel probability measures on H. Recall that $(P_t^*)_t$ is the adjoint semigroup defined on $\mathcal{P}(H)$ by

$$P_t^*\mu(\Gamma) = \int_H P_t(x,\Gamma)\mu(dx) \text{ with } P_t(\eta,\Gamma) := P(u(t,\eta) \in \Gamma) \text{ for any } \Gamma \in \mathcal{B}(H). \tag{35}$$

We use *"Krylov-Bogoliubov Theorem"* (see *e.g.* [12, Theorem 3.1.1]) to construct at least one invariant measure. Thus, it is sufficient (see *e.g.* [12, Corollary 3.1.2]) to show the following.

Proposition 4.3. *The set* $\{\nu_n := \frac{1}{t_n}\int_0^{t_n} \mu_s^\eta ds;\ n \in \mathbb{N}^*\}$ *is tight on* H.

Proof. Let $\eta \in K_\psi$ and $t \in [0,T]$, we distinguish two cases.

i. The case $p \leq 2 < +\infty$. By using Itô's formula to the solution of (8) with $\mathbf{F} \equiv 0$, we get

$$\begin{aligned} \frac{1}{2}\|u(t)\|_H^2 + \int_0^t \langle A(u), u\rangle ds &= \frac{1}{2}\|\eta\|_H^2 + \int_0^t \langle -k, u\rangle ds + \int_0^t \langle f, u\rangle ds \\ &\quad + \int_0^t \langle G(u)dW(s), u\rangle + \frac{1}{2}\int_0^t \|G(u)\|_{L_2(H_0,H)}^2 ds. \end{aligned} \tag{36}$$

By using $H_{2,1}$, $H_{3,2}$ and taking the expectation, we obtain for any $K > 0$

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|u(t)\|_H^2 + \alpha\int_0^t \mathbb{E}\|u\|_V^p ds\\
&\leq \lambda\int_0^t \mathbb{E}\|u\|_H^2 ds + l_1 t + \frac{1}{2}\|\eta\|_H^2 + \mathbb{E}\int_0^t |\langle f-k,u\rangle| ds\\
&\quad + \frac{M(1+K^2)}{2}t + \frac{L_G(1+K^2)}{2K^2}\mathbb{E}\int_0^t \|u\|_H^2 ds.
\end{aligned}$$

Let $0<\delta<1$, thanks to Lewy-Stampacchia inequality (12), H_5 and by using Young inequality, we obtain

$$\begin{aligned}
\mathbb{E}\int_0^t |\langle f-k,u\rangle| ds &\leq \mathbb{E}\int_0^t [\|k\|_{V'} + \|f\|_{V'}]\|u\|_V ds\\
&\leq \delta\alpha\int_0^t \mathbb{E}\|u\|_V^p ds + \frac{C(p,f,\psi)}{(\alpha\delta)^{\frac{1}{p-1}}}t.
\end{aligned}$$

Thus, the following inequality holds

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|u(t)\|_H^2 + (1-\delta)\alpha\int_0^t \mathbb{E}\|u\|_V^p ds\\
&\leq \frac{1}{2}\|\eta\|_H^2 + [\frac{C(p,f,\psi)}{(\alpha\delta)^{\frac{1}{p-1}}} + l_1 + M]t + [\lambda + \frac{L_G(1+K^2)}{2K^2}]\int_0^t \mathbb{E}\|u\|_H^2 ds. \qquad (37)
\end{aligned}$$

We distinguish the following cases:

1. **The case $p>2$:** since $V \hookrightarrow H$ and thanks to Young inequality we get

   $$|\lambda + L_G|\|u\|_H^2 \leq \frac{\alpha}{4}\|u\|_V^p + C(\alpha,p,\lambda,L_G)$$

   and (37) ensures with $\delta = \frac{1}{4}$

   $$\int_0^t \mathbb{E}\|u\|_V^p ds \leq \frac{1}{\alpha}[\|\eta\|_H^2 + C(f,\psi,l_1,\alpha,p,\lambda,L_G,\delta)t]. \qquad (38)$$

2. **The case $p=2$:** since $V \hookrightarrow H$ and thanks to (7), we obtain

   $$[\lambda + \frac{L_G(1+K^2)}{2K^2}]^+\int_0^t \mathbb{E}\|u\|_H^2 ds \leq C_D[\lambda + \frac{L_G(1+K^2)}{2K^2}]^+\int_0^t \mathbb{E}\|u\|_V^2 ds,$$

   where $v^+ = \max(0,v); v\in\mathbb{R}$. Now (37) and (16) with $p=2$ ensure

   $$\int_0^t \mathbb{E}\|u\|_V^2 ds \leq \frac{1}{2\bar{\alpha}}[\|\eta\|_H^2 + C(f,\psi,l_1,\alpha,p,\lambda,L_G,\delta)t]; \quad \forall\delta\in]0,1[. \qquad (39)$$

3. **The case $p=2$ and bounded multiplicative noise:** it follows from (36) after using (15) and $\lambda\leq 0$ and arguments already detailed that

$$\frac{1}{2}\mathbb{E}\|u(t)\|_H^2 + (1-\delta)\alpha\int_0^t \mathbb{E}\|u\|_V^p ds \leq \frac{1}{2}\|\eta\|_H^2 + [\frac{C(p,f,\psi)}{(\alpha\delta)^{\frac{1}{p-1}}} + l_1 + \frac{\mathbf{K}}{2}]t. \qquad (40)$$

Consequently, in all the cases, we have for $\delta \in]0,1[$

$$\mathbb{E}\|u(t)\|_H^2 + \int_0^t \mathbb{E}\|u\|_V^p ds \leq C(\alpha)[\|\eta\|_H^2 + C(f,\psi,l_1,\alpha,p,\lambda,L_G)t]. \tag{41}$$

Let $B_R := \{v \in V : \|v\|_V \leq R\}; R \in \mathbb{N}$ and note that B_R is relatively compact in H. Let $t_n > 0, R > 0$. By using Chebyshev-Markov inequality, we write

$$\begin{aligned}\nu_n(B_R^c) = \frac{1}{t_n}\int_0^{t_n} \mu_s^\eta(B_R^c)ds &= \frac{1}{t_n}\int_0^{t_n} P(\|u(s;\eta)\|_V > R)ds \\ &\leq \frac{1}{t_n R^p}\int_0^{t_n} \mathbb{E}\|u(s;\eta)\|_V^p ds.\end{aligned}$$

Let $\varrho > 0$, by using (41), we are able to deduce

$$\nu_n(B_R^c) \leq \frac{C(\alpha)}{R^p}[\frac{\|\eta\|_H^2}{t_n} + C(f,\psi,l_1,\alpha,p,\lambda,L)] \leq \varrho, \tag{42}$$

provided R is chosen large enough, which ensures the existence of an invariant measure.

ii. The case $\underline{\mathrm{p}} < p < 2$.

1. **The case of Lipschitz multiplicative noise:** it follows from Corollary 6.7 the existence of $\mathcal{K} > 0$ such that

$$\frac{1}{\alpha}\mathbb{E}\|u(t)\|_H^2 + \mathbb{E}\int_0^t \|u(s)\|_V^p ds \leq \frac{4}{\alpha}[\|y_0\|_H^2 + \mathcal{K}(t+1)], \tag{43}$$

   for $\frac{L_G(1+K^2)}{2K^2} + \lambda + L_F \leq 0$ and any $t \in [0,T]$ and any $K > 0$.
2. **The case of bounded multiplicative noise:** it follows from (72) after using (15), $\lambda + L_F \leq 0$ and arguments already detailed in Corollary 6.7 the existence of $C > 0$ such that

$$\frac{1}{\alpha}\mathbb{E}\|u(t)\|_H^2 + \mathbb{E}\int_0^t \|u(s)\|_V^p ds \leq \frac{4}{\alpha}[\|y_0\|_H^2 + C(t+1)]. \tag{44}$$

Arguments already detailed ensure the existence of an invariant measure in this case. □

Concentration. Let $n \in \mathbb{N}^*$, thanks to (41) we obtain

$$\nu_n(\|\cdot\|_V^p) = \frac{1}{n}\int_0^n \mathbb{E}\|u(s;\eta)\|_V^p ds \leq C(\alpha)[\frac{\|\eta\|_H^2}{n} + C(f,\psi,l_1,\alpha,p,\lambda,L_G)] \leq C,$$

where $C > 0$ independent of n. Since μ is the weak limit of a subsequence of ν_n, it follows that $\mu(\|\cdot\|_V^p) \leq C$. That is $\mu(\|\cdot\|_V^p) = \int_H \|x\|_V^p \mu(dx) \leq C$.

Since $\nu_n(K_\psi) := \dfrac{1}{t_n}\displaystyle\int_0^{t_n} \mu_s^\eta(K_\psi)ds = 1$, it follows that $\mu(K_\psi) = 1$.

4.3.2. Concentration and ergodic invariant measures. In this part, we prove Proposition 3.3. Namely, we have

Proposition 4.4. *Let* $p \geq 2$, [13] *there exists* $\widetilde{\mathbf{K}} > 0$ *such that*

$$\int_H \|x\|_V^p \mu(dx) \leq \widetilde{\mathbf{K}}. \tag{45}$$

for any invariant measure μ *for the semigroup* $(P_t)_t$ *defined by* (14)*.*

Proof. Let $\eta \in K_\psi$ and consider the following function $f_\epsilon : x \mapsto \dfrac{x}{1+\epsilon x}, \epsilon > 0$. Note that $f_\epsilon \in C_b^2(\mathbb{R}_+)$ satisfying

$$x \in \mathbb{R}_+ : \quad f_\epsilon'(x) = \frac{1}{(1+\epsilon x)^2} > 0, \quad f_\epsilon''(x) = -\frac{2\epsilon}{(1+\epsilon x)^3} < 0.$$

Let $\epsilon > 0$, by applying Itô's formula to the process $\|u\|_H^2$ given by (36) and the function f_ϵ, we get

$$\begin{aligned} &f_\epsilon(\|u(t)\|_H^2) - f_\epsilon(\|\eta\|_H^2) + 2\int_0^t f_\epsilon'(\|u(s)\|_H^2)\langle A(u), u\rangle ds \qquad (46)\\ &= 2\int_0^t f_\epsilon'(\|u(s)\|_H^2)\langle f-k, u\rangle ds + 2\int_0^t f_\epsilon'(\|u(s)\|_H^2)\langle G(u)dW(s), u\rangle \\ &+ \int_0^t f_\epsilon'(\|u(s)\|_H^2)\|G(u)\|_{L_2(H_0,H)}^2 ds + 2\int_0^t f_\epsilon''(\|u(s)\|_H^2)\|(G(u),u)\|_{L_2(H_0,\mathbb{R})}^2 ds. \end{aligned}$$

Since $f_\epsilon'' < 0$, the last term is non positive. On the other hand, by using that $f_\epsilon' > 0$, $H_{2,1}$, H_3, we obtain for any $K > 0$

$$\begin{aligned} &f_\epsilon(\|u(t)\|_H^2) + 2\alpha\int_0^t f_\epsilon'(\|u(s)\|_H^2)\|u\|_V^p ds \\ &\leq f_\epsilon(\|\eta\|_H^2) + 2\lambda\int_0^t f_\epsilon'(\|u(s)\|_H^2)\|u\|_H^2 ds + 2l_1 t \\ &+ 2\int_0^t f_\epsilon'(\|u(s)\|_H^2)|\langle f-k, u\rangle| ds + 2\int_0^t f_\epsilon'(\|u(s)\|_H^2)\langle G(u)dW(s), u\rangle \\ &\quad + M(1+K^2)t + \frac{L_G(1+K^2)}{K^2}\int_0^t f_\epsilon'(\|u(s)\|_H^2)\|u\|_H^2 ds. \end{aligned} \tag{47}$$

Recall that $f_\epsilon' \leq 1$, which ensures that the stochastic integral is $(\mathcal{F}_t)$-martingale. Hence, by taking the expectation and using arguments already

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[13] We recall that $\mathbf{F} \equiv 0$ in (8) in this case.

detailed, one has

$$\begin{aligned}
&\mathbb{E}f_\epsilon(\|u(t)\|_H^2) + 2\alpha\mathbb{E}\int_0^t f_\epsilon'(\|u(s)\|_H^2)\|u\|_V^p ds \\
&\leq f_\epsilon(\|\eta\|_H^2) + 2\lambda\mathbb{E}\int_0^t f_\epsilon'(\|u(s)\|_H^2)\|u\|_H^2 ds + 2l_1 t \\
&\quad + 2\mathbb{E}\int_0^t f_\epsilon'(\|u(s)\|_H^2)|\langle f-k, u\rangle| ds + M(1+K^2)t \\
&\quad + \frac{L_G(1+K^2)}{K^2}\mathbb{E}\int_0^t f_\epsilon'(\|u(s)\|_H^2)\|u\|_H^2 ds.
\end{aligned}$$

Let $0<\delta<1$, thanks to Lewy-Stampacchia inequality (12), H_5 and by using Young inequality, we obtain

$$\mathbb{E}\int_0^t f_\epsilon'(\|u(s)\|_H^2)|\langle f-k, u\rangle| ds \leq \delta\alpha\int_0^t \mathbb{E}f_\epsilon'(\|u(s)\|_H^2)\|u\|_V^p ds + \frac{C(p,f,\psi)}{(\alpha\delta)^{\frac{1}{p-1}}}t.$$

Thus, the following inequality holds

$$\begin{aligned}
&\mathbb{E}f_\epsilon(\|u(t)\|_H^2) + 2(1-\delta)\alpha\int_0^t \mathbb{E}f_\epsilon'(\|u(s)\|_H^2)\|u\|_V^p ds \\
&\qquad \leq f_\epsilon(\|\eta\|_H^2) + \mathbf{Y}t + 2\mathbf{R}\int_0^t \mathbb{E}f_\epsilon'(\|u(s)\|_2^2)\|u\|_H^2 ds, \qquad (48)
\end{aligned}$$

where $\mathbf{Y} = [2\frac{C(p,f,\psi)}{(\alpha\delta)^{\frac{1}{p-1}}} + 2l_1 + M(1+K^2)], \mathbf{R} = \lambda + \frac{L_G(1+K^2)}{2K^2}$.

1. **The case $p>2$:** choose $\delta = \frac{1}{2}$ in (48), since $V \hookrightarrow H$ and thanks to Young inequality we get $2\mathbf{R}\|u\|_H^2 \leq \frac{\alpha}{2}\|u\|_V^p + C(\alpha,\mathbf{R})$. Thus
$$\mathbb{E}f_\epsilon(\|u(t)\|_H^2) + \frac{\alpha}{2}\int_0^t \mathbb{E}f_\epsilon'(\|u(s)\|_H^2)\|u\|_V^p ds \leq f_\epsilon(\|\eta\|_H^2) + (\mathbf{Y}+C(\alpha,\mathbf{R}))t.$$
2. **The case $p=2$:** since $V \hookrightarrow H$ and thanks to (7), we obtain
$$2[\lambda + \frac{L_G(1+K^2)}{2K^2}]^+\int_0^t \mathbb{E}\|u\|_H^2 ds \leq 2C_D[\lambda + \frac{L_G(1+K^2)}{2K^2}]^+\int_0^t \mathbb{E}\|u\|_V^2 ds,$$
where $v^+ = \max(0,v); v\in\mathbb{R}$. Now (48) and (16) with $p=2$ ensure
$$\mathbb{E}f_\epsilon(\|u(t)\|_H^2) + 2\bar{\alpha}\int_0^t \mathbb{E}f_\epsilon'(\|u(s)\|_H^2)\|u\|_V^2 ds \leq f_\epsilon(\|\eta\|_H^2) + \mathbf{Y}t.$$
3. **The case $p=2$ and bounded multiplicative noise:** it follows from (46) after using (15) and arguments already detailed that
$$\begin{aligned}
&\mathbb{E}f_\epsilon(\|u(t)\|_H^2) + 2(1-\delta)\alpha\int_0^t \mathbb{E}f_\epsilon'(\|u(s)\|_H^2)\|u\|_V^2 ds \\
&\leq f_\epsilon(\|\eta\|_H^2) + [2\frac{C(p,f,\psi)}{(\alpha\delta)^{\frac{1}{p-1}}} + 2l_1 + \mathbf{K}]t.
\end{aligned}$$

Therefore, in all the cases, we have for $\delta \in]0,1[$, there exist $\mathbf{C} := \mathbf{Y} + \mathbf{K} + C(\alpha, \mathbf{R})$ and $c_{\alpha,\bar{\alpha}} > 0$ such that

$$\mathbb{E} f_\epsilon(\|u(t)\|_H^2) + c_{\alpha,\bar{\alpha}} \int_0^t \mathbb{E} f'_\epsilon(\|u(s)\|_H^2)\|u\|_V^p ds \leq f_\epsilon(\|\eta\|_H^2) + \mathbf{C}t, \quad p \geq \underline{\mathbf{p}}.$$

Let $p \geq 2$ and recall that $V \hookrightarrow H$, thus $\frac{1}{C_D}\|u\|_H^2 \leq \|u\|_V^2 \leq \|u\|_V^p + C_p$.

Therefore, for any $\epsilon > 0$

$$\mathbb{E} f_\epsilon(\|u(t)\|_H^2) + \frac{c_{\alpha,\bar{\alpha}}}{C_D} \int_0^t \mathbb{E} \frac{\|u(s)\|_H^2}{(1+\epsilon\|u(s)\|_H^2)^2} ds \leq f_\epsilon(\|\eta\|_H^2) + (\mathbf{C} + C_p c_{\alpha,\bar{\alpha}})t.$$

For $y \in H$, set $F_\epsilon(y) = f_\epsilon \circ \|y\|_H^2$ and note that $F_\epsilon \in C_b(H)$. We recall that $P_t F_\epsilon(\eta) = \mathbb{E} F_\epsilon(u(t))$. Let μ be an invariant measure for $(P_t)_t$, by the definition of invariant measure for the semigroup $(P_t)_t$, we obtain after integrating with respect to μ

$$\frac{c_{\alpha,\bar{\alpha}}}{C_D} \int_H \int_0^t \mathbb{E} \frac{\|u(s)\|_H^2}{(1+\epsilon\|u(s)\|_H^2)^2} ds d\mu \leq (\mathbf{C} + C_p c_{\alpha,\bar{\alpha}})t.$$

Let $g_\epsilon(x) = \dfrac{x}{(1+\epsilon x)^2}$ and $G_\epsilon = g_\epsilon \circ \|\cdot\|_H^2 \in C_b(H)$. Hence

$$\mathbb{E} \frac{\|u(s)\|_H^2}{(1+\epsilon\|u(s)\|_H^2)^2} := P_s G_\epsilon(\eta).$$

Tonelli theorem and the invariance of μ ensure

$$\begin{aligned} \frac{c_{\alpha,\bar{\alpha}}}{C_D} \int_H \int_0^t \mathbb{E} \frac{\|u(s)\|_H^2}{(1+\epsilon\|u(s)\|_H^2)^2} ds d\mu &= \frac{c_{\alpha,\bar{\alpha}}}{C_D} \int_0^t \int_H P_s G_\epsilon(\eta) d\mu ds \\ &= t \frac{c_{\alpha,\bar{\alpha}}}{C_D} \int_H G_\epsilon(\eta) d\mu \leq (\mathbf{C} + C_p c_{\alpha,\bar{\alpha}})t. \end{aligned}$$

Finally, by letting $\epsilon \to 0$ and using monotone convergence theorem we get

$$\int_H \|y\|_H^2 \mu(dy) \leq \frac{(\mathbf{C} + C_p c_{\alpha,\bar{\alpha}})C_D}{c_{\alpha,\bar{\alpha}}} \leq \widetilde{\mathbf{K}}_1. \tag{49}$$

Next, we use the last inequality (49) to show (45). Define the following non decreasing sequence

$$n \in \mathbb{N}: \quad F_n : H \to \mathbb{R}_+ \cup \{+\infty\}; \quad u \mapsto \begin{cases} \|u\|_V^p & \text{if } \|u\|_V \leq n; \\ n^p & \text{else.} \end{cases} \tag{50}$$

Note that F_n converges to $F_V := \sup_n F_n$, where

$$F_V : H \to \mathbb{R}_+ \cup \{+\infty\}; \quad u \mapsto \begin{cases} \|u\|_V^p \text{ if } u \in V; \\ +\infty \text{ if } u \in H \setminus V. \end{cases}$$

It is clear that $F_n \in B_b(H)$[14] for every $n \in \mathbb{N}$ and $F_n(u) \le \|u\|_V^p$. By using the invariance of μ, we are able to infer

$$\int_H F_n d\mu = \int_0^1\int_H P_s F_n d\mu ds = \int_0^1\int_H \mathbb{E}F_n(u(s))d\mu ds = \int_H\int_0^1 \mathbb{E}F_n(u(s))dsd\mu.$$

By using (41), one has

$$\begin{aligned}&\int_0^T \mathbb{E}F_n(u(s))ds\\ &\le \int_0^T \mathbb{E}\|u\|_V^p ds \le C(\alpha)[\|\eta\|_H^2 + C(f,\psi,l_1,\alpha,p,\lambda,L_G)T].\end{aligned} \tag{51}$$

Set $T = 1$ and integrate with respect to μ the last inequality, one has

$$\begin{aligned}\int_H F_n(\eta)d\mu &= \int_H\int_0^1 P_s F_n(\eta)dsd\mu = \int_H\int_0^1 \mathbb{E}F_n(u(s))dsd\mu\\ &\le C(\alpha)[\int_H \|\eta\|_H^2 d\mu + C(f,\psi,l_1,\alpha,p,\lambda,L_G)].\end{aligned}$$

Consequently, the monotone convergence theorem and (49) give

$$\int_H \|x\|_V^p \mu(dx) \le \int_H F_V(\eta)d\mu \le C(\alpha)[\widetilde{\mathbf{K}}_1 + C(f,\psi,l_1,\alpha,p,\lambda,L_G)]] \le \widetilde{\mathbf{K}}.$$

□

Proposition 4.5. (Ergodicity) *Let $p \ge 2$, then there exists at least one ergodic invariant measure for the semigroup $(P_t)_t$ defined by (14).*

Proof. Denote by Λ, the set of all invariant measures for the Markov semigroup $(P_t)_t$ defined by (14). From Subsection 4.3, Λ is nonempty convex subset of $(C_b(H))'$ and (45) ensures that Λ is tight, since $2 \le p < \infty$. Therefore, Krein-Milman theorem (see *e.g.* [7, Theorem 1.13]) ensures that the set of extreme points is non empty and then any extremal point of Λ is an ergodic invariant measure, since the set of all invariant ergodic measures of $(P_t)_t$ coincides with the set of all extremal points of Λ, see [10, Theorem 5.18]. □

Concerning the case $\underline{\mathbf{p}} < p < 2$, we prove only that the p^{th}-moment is bounded.

Proposition 4.6. *Let $\underline{\mathbf{p}} < p < 2$, there exists $\bar{\mathbf{K}} > 0$ such that*

$$\int_H \|x\|_H^p \mu(dx) \le \bar{\mathbf{K}}. \tag{52}$$

for any invariant measure μ for the semigroup $(P_t)_t$ defined by (14).

Proof. Let $\underline{\mathbf{p}} < p < 2$, we have

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[14] $B_b(H)$ denotes the set of bounded Borel functions on H.

1. **The case of bounded multiplicative noise:** if $\lambda + L_F \leq 0$, it follows from Corollary 6.7 the existence of $S > 0$ such that

$$\begin{aligned}&\mathbb{E} f_\epsilon(\|u(t)-\psi\|_H^2) + \frac{\alpha}{2}\mathbb{E}\int_0^t f'_\epsilon(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds\\ &\quad\leq f_\epsilon(\|u_0-\psi\|_H^2) + (S+\mathbf{K})t.\end{aligned}$$

2. **The case of Lipschitz noise with** $\dfrac{L_G(1+K^2)}{2K^2} + \lambda + L_F \leq 0$: it follows from Corollary 6.7 the existence of $S > 0$ such that

$$\mathbb{E} f_\epsilon(\|u(t)-\psi\|_H^2) + \frac{\alpha}{2}\mathbb{E}\int_0^t f'_\epsilon(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds \leq f_\epsilon(\|u_0-\psi\|_H^2) + St.$$

Let $\eta \in K_\psi$, there exists $\mathbf{C} > 0$ in both cases such that, after using $V \hookrightarrow H$, one has

$$\begin{aligned}&2\mathbb{E} f_\epsilon(\|u(t)-\psi\|_H^2) + \frac{\alpha}{C_D^{p/2}}\int_0^t \mathbb{E}\frac{\|u(s)\|_H^p}{(1+\epsilon\|u(s)-\psi\|_H^2)^2}ds\\ &\quad\leq 2f_\epsilon(\|\eta-\psi\|_H^2) + \mathbf{C}t, \quad \epsilon > 0.\end{aligned}$$

Since $\psi \in V$, by repeating the same arguments as above, namely the definition of invariant measure one has

$$\frac{\alpha}{C_D^{p/2}}\int_H\int_0^t \mathbb{E}\frac{\|u(s)\|_H^p}{(1+\epsilon\|u(s)-\psi\|_H^2)^2}dsd\mu \leq \mathbf{C}t.$$

Let $\widetilde{G}_\epsilon(\xi) = \dfrac{\|\xi\|_H^p}{(1+\epsilon\|\xi-\psi\|_H^2)^2} \in C_b(H), \xi \in H$. Hence

$$\mathbb{E}\frac{\|u(s)\|_H^p}{(1+\epsilon\|u(s)-\psi\|_H^2)^2} := P_s\widetilde{G}_\epsilon(\eta).$$

Let μ be an invariant measure for $(P_t)_t$. Tonelli theorem and the invariance of μ ensure

$$\begin{aligned}\int_H\int_0^t \mathbb{E}\frac{\|u(s)\|_H^p}{(1+\epsilon\|u(s)-\psi\|_H^2)^p}dsd\mu &= \int_0^t\int_H P_s\widetilde{G}_\epsilon(\eta)d\mu ds\\ &= t\int_H \widetilde{G}_\epsilon(\eta)d\mu \leq \frac{C_D^{p/2}}{\alpha}\mathbf{C}t.\end{aligned}$$

Finally, by letting $\epsilon \to 0$ and using monotone convergence theorem we get

$$\int_H \|y\|_H^p\mu(dy) \leq \frac{C_D^{p/2}}{\alpha}\mathbf{C} \leq \widetilde{\mathbf{K}}_1. \tag{53}$$

□

Remark 4.2. Let $\underline{\mathbf{p}} < p < 2$, from (52), we have $\int_H \|y\|_H^p\mu(dy)$ is uniformly bounded for any invariant measure $\mu \in \Lambda$. In order to prove that the set of invariant measures is tight in the same way as (51) we need $(\int_H \|y\|_H^2\mu(dy))_{\mu\in\Lambda}$ is bounded which is not *a priori* satisfied.

4.4. Uniqueness and exponential mixing

In this part, we prove Theorem 3.5 and Theorem 3.7. Namely, let us prove the uniqueness of invariant measures. It is well known that one only needs to show the uniqueness of ergodic invariant measures, see *e.g.* [10, Theorem 5.16]. For that, let μ_1 and μ_2 be two ergodic invariant measures, then for any bounded Lipschitz function φ

$$\begin{aligned}\lim_{T\to+\infty}\frac{1}{T}\int_0^T P_t\varphi dt &= \int_H \varphi(x)\mu_1(dx) \text{ in } L^2(H,\mu_1),\\ \lim_{T\to+\infty}\frac{1}{T}\int_0^T P_t\varphi dt &= \int_H \varphi(y)\mu_2(dy) \text{ in } L^2(H,\mu_2).\end{aligned}$$

Hence

$$\begin{aligned}&\Big|\frac{1}{T}\int_0^T P_t\varphi(x)dt-\frac{1}{T}\int_0^T P_t\varphi(y)dt\Big|\\ &\le \frac{1}{T}\int_0^T \Big|P_t\varphi(x)-P_t\varphi(y)\Big|dt \le \frac{\|\varphi\|_{Lip}}{T}\int_0^T \mathbb{E}\|u(t;x)-u(t;y)\|_H dt. \qquad (54)\end{aligned}$$

On the other hand, let (u_1,k_1) and (u_2,k_2) be two solutions to (8) with $\mathbf{F}\equiv 0$, corresponding to two initial data x and y respectively. P-a.s, for any $t\in[0,T]$ we have

$$\begin{aligned}u_1(t)-u_2(t)+\int_0^t[k_1-k_2]ds+\int_0^t[A(u_1)-A(u_2)]ds\\ =\int_0^t[G(u_1)-G(u_2)]dW(s).\end{aligned}$$

Applying Ito's formula with $F(t,v)=\frac{1}{2}\|v\|_H^2$, one gets for any $t\in[0,T]$

$$\begin{aligned}&\frac{1}{2}\|(u_1-u_2)(t)\|_H^2+\int_0^t\langle A(u_1)-A(u_2),u_1-u_2\rangle ds\\ &+\int_0^t\langle k_1-k_2,u_1-u_2\rangle ds=\frac{1}{2}\|x-y\|_H^2+\int_0^t\langle[G(u_1)-G(u_2)]dW(s),u_1-u_2\rangle\\ &\qquad\qquad+\frac{1}{2}\int_0^t\|G(u_1)-G(u_2)\|^2_{L_2(H_0,H)}ds.\end{aligned}$$

Recall that $\int_0^t\langle k_1-k_2,u_1-u_2\rangle ds\ge 0$, see Definition 3.1. Since $\lambda_{\mathbf{T}}Id+A$ is $\mathbf{T}$-monotone (see $H_{2,2}$), we have

$$\int_0^t\langle A(u_1)-A(u_2),u_1-u_1\rangle ds\ge -\lambda_{\mathbf{T}}\int_0^t\|u_1-u_2\|_H^2 ds.$$

We distinguish two cases.

1. **The case of Lipschitzian noise:** by taking the expectation and using $H_{3,1}$ we get

$$\mathbb{E}\|(u_1-u_2)(t)\|_H^2\le\|x-y\|_H^2+2[\frac{L_G}{2}+\lambda_{\mathbf{T}}]\int_0^t\mathbb{E}\|u_1-u_2\|_H^2 ds.$$

Therefore $\mathbb{E}\|(u_1-u_2)(t)\|_H^2 \le \|x-y\|_H^2 e^{2[\frac{L_G}{2}+\lambda_{\mathbf{T}}]t}$, which gives

$$\mathbb{E}\|(u_1-u_2)(t)\|_H \le \|x-y\|_H e^{[\frac{L_G}{2}+\lambda_{\mathbf{T}}]t}. \tag{55}$$

Hence (54) and (55) give

$$\begin{aligned}\Big|\frac{1}{T}\int_0^T P_t\varphi(x)dt - \frac{1}{T}\int_0^T P_t\varphi(y)dt\Big| &\le \frac{\|\varphi\|_{Lip}}{T}\int_0^T \mathbb{E}\|u(t;x)-u(t;y)\|_H dt\\ &\le \|x-y\|_H g(T),\end{aligned}$$

where $g(T) := \frac{\|\varphi\|_{Lip}}{T}\dfrac{2}{L_G+2\lambda_{\mathbf{T}}}(e^{[\frac{L_G+2\lambda_{\mathbf{T}}}{2}]T}-1) \to 0$ as $T \uparrow +\infty$, provided that $\dfrac{L_G}{2}+\lambda_{\mathbf{T}}<0$.

2. **The case of Lipschitzian and bounded noise:** if G satisfies (15), it is not difficult to see that we get

$$\mathbb{E}\|(u_1-u_2)(t)\|_H^2 \le \|x-y\|_H^2 + 2\lambda_{\mathbf{T}}\int_0^t \mathbb{E}\|u_1-u_2\|_H^2 ds + \mathbf{K}t.$$

Thus $\mathbb{E}\|(u_1-u_2)(t)\|_H^2 \le (\|x-y\|_H^2+\mathbf{K}t)e^{2\lambda_{\mathbf{T}}t} \le (\|x-y\|_H+\sqrt{\mathbf{K}t})^2 e^{2\lambda_{\mathbf{T}}t}$. Hence

$$\begin{aligned}&\Big|\frac{1}{T}\int_0^T P_t\varphi(x)dt - \frac{1}{T}\int_0^T P_t\varphi(y)dt\Big| \le \frac{\|\varphi\|_{Lip}}{T}\int_0^T \mathbb{E}\|u(t;x)-u(t;y)\|_H dt\\ &\le \frac{\|\varphi\|_{Lip}}{T}\int_0^T (\|x-y\|_H+\sqrt{\mathbf{K}t})e^{\lambda_{\mathbf{T}}t}dt\\ &\le \frac{\|\varphi\|_{Lip}\|x-y\|_H}{\lambda_{\mathbf{T}}T}(e^{\lambda_{\mathbf{T}}T}-1) + \frac{\|\varphi\|_{Lip}\sqrt{\mathbf{K}}}{T}\int_0^{+\infty}\sqrt{t}e^{\lambda_{\mathbf{T}}t}dt := H(T).\end{aligned}$$

A standard calculation ensures $\int_0^{+\infty}\sqrt{t}e^{\lambda_{\mathbf{T}}t}dt < +\infty$ and $H(T)\to 0$ as $T\uparrow +\infty$.

Therefore, the uniqueness holds, namely

$$\int_H \varphi d\mu_1 = \int_H \varphi d\mu_2 \text{ for any bounded Lipschitz function } \varphi \text{ on } H.$$

Finally, let μ be an invariant measure for (P_t), for $\varphi \in \mathcal{C}_b^1(H)$ we have

$$\begin{aligned}&\Big|P_t\varphi(x) - \int_H \varphi(y)\mu(dy)\Big|\\ &\le \Big|\int_H [P_t\varphi(x)-P_t\varphi(y)]\mu(dy) \le \|\varphi\|_{1,\infty}\int_H \mathbb{E}\|u(t;x)-u(t;y)\|\mu(dy)\\ &\le \|\varphi\|_{1,\infty}e^{[\frac{L_G}{2}+\lambda_{\mathbf{T}}]t}\int_H \|x-y\|\mu(dy) \le \mathcal{G}\|\varphi\|_{1,\infty}e^{[\frac{L_G}{2}+\lambda_{\mathbf{T}}]t},\end{aligned} \tag{56}$$

where $\mathcal{G} := \int_H \|x-y\|\mu(dy) < +\infty$, thanks to (49) and (53). If moreover G satisfies (15), we have instead

$$\Big|P_t\varphi(x) - \int_H \varphi(y)\mu(dy)\Big| \le \|\varphi\|_{1,\infty}(\mathcal{G}+\sqrt{\mathbf{K}t})e^{\lambda_{\mathbf{T}}t}. \tag{57}$$

Moreover, since $\mathcal{C}_b^1(H)$ is dense in $L^2(H,\mu)$, it follows

$$\forall \varphi \in L^2(H,\mu): \quad \lim_{t\to+\infty} P_t\varphi(x) = \int_H \varphi d\mu, \quad x \in H,$$

provided that $\frac{L_G}{2}+\lambda_{\mathbf{T}} < 0$ and $\lambda_{\mathbf{T}} < 0$ in (56) and (57) respectively. Therefore, μ is ergodic and strongly mixing.

Remark 4.3. If $\underline{\mathbf{p}} < p < 2$, one obtains similarly $\lambda_{\mathbf{T}} + L_F$ instead of $\lambda_{\mathbf{T}}$ in (56) and (57).

5. On double obstacles problem

We are interested in this subsection to say a few words about the invariant measures associated with a double obstacle problem *i.e.* find (u,k) in a space defined straight after, solution to

$$\begin{cases} du + A(u)ds + kds = fds + G(u)dW & in \quad D\times\Omega_T,\\ k = k_2 - k_1, \langle k_i, u-\psi_i\rangle = 0, (k_i)_{i=1}^{i=2} \in (V')^+ & in \quad \Omega_T,\\ \psi_2 \geq u \geq \psi_1 & in \quad D\times\Omega_T,\\ u = 0 & on \quad \partial D\times\Omega_T,\\ u(t=0) = u_0 & \text{in } H,\ a.s. \end{cases}$$

Let us precise the assumptions on the obstacles. Namely, we replace H$_4$-H$_6$ by the following:

- H_4^* : The obstacles $\psi_i \in V$, $\psi_1 \leq \psi_2$ such that $G(\psi_i) = 0$ for $i = 1, 2$.
- H_5^* : The external force $f \in V'$ such that $f - A(\psi_i) = h_i = h_i^+ - h_i^- \in V^*$ for $i = 1, 2$.
- H_6^* : The initial datum $u_0 \in H$, and satisfies $\psi_1 \leq u_0 \leq \psi_2$.

Denote by K the convex set of admissible functions

$$K_{\psi_1}^{\psi_2} = \{v \in L^p(\Omega_T, V), \quad \psi_2(x) \geq v(x,t,\omega) \geq \psi_1(x) \quad \text{a.e. in } D\times\Omega_T\}.$$

Following [42, Theorem 3], we have

Theorem 5.1. *Assume that H_1-H_3 and H_4^*-H_6^* hold, assume moreover that h_1^-, h_2^+ are non negative elements of $L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$*[15]*. Then, there exists a unique predictable stochastic process*

$$(u,\rho_1,\rho_2) \in L^p(\Omega_T,V)\times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))\times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$$

such that:

i. $u \in L^2(\Omega, \mathcal{C}([0,T],H)) \cap K_{\psi_1}^{\psi_2}$, $u(0) = u_0$.
ii. $-\rho_1, \rho_2 \geq 0$ *and* $\forall v \in K_{\psi_1}^{\psi_2}$, $\langle \rho_i, u - v\rangle \geq 0, \quad i = 1,2$ *a.e. in* Ω_T.
iii. *P-a.s, for all* $t \in [0,T]$,

$$u(t) + \int_0^t (\rho_1+\rho_2)ds + \int_0^t A(u,\cdot)ds = u_0 + \int_0^t G(u,\cdot)dW(s) + \int_0^t fds.$$

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[15] $\tilde{q} = \min(p,2)$.

iv. *The following Lewy-Stampacchia's inequality holds:*

$$-h_2^+ = -\left(f - A(\psi_2)\right)^+ \leq \partial_t(u - \int_0^{\cdot} G(u)dW) + A(u) - f$$
$$\leq h_1^- = \left(f - A(\psi_1)\right)^- .$$

It is not difficult to see that the extension of the results to the case of double obstacles is straightforward, it follows by repeating the same argument of Section 4 and using Theorem 5.1.

6. Appendix: on well-posedness of (8)

Let $1 < p < 2$ and denote by $V = W_0^{1,p}(D) \cap L^2(D), V' = W^{-1,p'}(D) + L^2(D)$.

Theorem 6.1. *Assume that* H_1*-*H_7 *hold and* $\widetilde{h}^- \in L^{p'}(D)$ *in* (10)*, there exists a unique* $(u, \rho) \in L^p(\Omega_T; V) \times L^{p'}(\Omega_T; L^{p'}(D))$*, solution to* (8)*, both predictable, satisfying:*

- $u \in L^2(\Omega; C([0,T], H)) \cap K$ *and* $\rho \leq 0$.
- *P-a.s. for any* $t \in [0,T]$*:*
$$u(t) + \int_0^t (A(u) + \mathbf{F}(u) + \rho - f)ds = u_0 + \int_0^t G(u)dW(s).$$
- $\langle \rho, u - \psi \rangle = 0$ *a.e. in* Ω_T *and, for all* $v \in K$, $\langle \rho, u - v \rangle \geq 0$ *a.e. in* Ω_T.

Moreover, we have $\mathbb{E} \sup_{t \in [0,T]} \|(u_\epsilon - u)(t)\|_H^2 \leq C\epsilon^{\frac{1}{p-1}}$ *for all* $\epsilon > 0$*, where* u_ϵ *is the unique solution of* (58).

Remark 6.1. Let $t \in [0,T]$ and y_1 and y_2 be two solution to (8). Note that

$$\left|\int_0^t (\mathbf{F}(y_1) - \mathbf{F}(y_2), y_1 - y_2)ds\right| \leq L_F \int_0^t \|y_1 - y_2\|_H^2 ds,$$

and the uniqueness of the solution to (8) follows by using arguments similar to that of [42, Lemma 1].

Proof of Theorem 6.1.

Penalization. Let $\epsilon > 0$ and $u_0 \geq \psi$. Consider the following approximating problem:

$$u_\epsilon(t) + \int_0^t (A(u_\epsilon) + \mathbf{F}(u_\epsilon) - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{p-1} - f)ds = u_0 + \int_0^t \widetilde{G}(u_\epsilon)dW(s) \tag{58}$$

where $\widetilde{G}(u_\epsilon) = G(\max(u_\epsilon, \psi))$. Note that $\widetilde{G}$ satisfies also H_1 and H_3 and H_4 since $\widetilde{G}(\psi) = G(\psi) = 0$. Indeed, For any $u, v \in V$,

$$\|\widetilde{G}(u) - \widetilde{G}(v)\|_{L_2(H_0,H)}^2 \leq L_G \|\max(u,\psi) - \max(v,\psi)\|_H^2 \leq L_G \|u - v\|_H^2,$$
$$\|\widetilde{G}(u)\|_{L_2(H_0,H)}^2 \leq M + 2L_G\|\psi\|_H^2 + 2L_G\|u\|_H^2 = \widehat{M} + L_G\|u\|_H^2,$$

where $\widehat{M} > 0$ is a $L^\infty(\Omega_T)$-predictable element, depending only on the data.

Consider $\bar{A}(u_\epsilon) = A(u_\epsilon) + \mathbf{F}(u_\epsilon) - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{p-1} - f$ and note that:

- By construction, $\bar{A}$ is an operator defined on V with values in V'.
- Since $x \mapsto -x^-$ is non-decreasing, $(\lambda_{\mathbf{T}} + L_F)Id + \bar{A}$ is $\mathbf{T}$-monotone.
- The structure of the penalization operator yields that $\bar{A}$ satisfies $\mathrm{H}_{2,4}$.
- (Coercivity): Note that for any $\delta > 0$, there exists $C_{\delta,\epsilon} > 0$ such that: $\forall v \in V$,
$$\begin{aligned}\langle f, v\rangle \leq & C_\delta \|f\|_{V'}^{p'} + \delta \|v\|_V^p \quad ,\\ \langle -\frac{1}{\epsilon}[(v-\psi)^-]^{p-1}, v\rangle \geq & \langle -\frac{1}{\epsilon}[(v-\psi)^-]^{p-1}, \psi\rangle \geq -\delta C\|v\|_V^p - C_{\delta,\epsilon}\|\psi\|_{L^p(D)}^p,\end{aligned}$$
where C is related to the continuous embedding of V in $L^p(D)$. Denote by $\tilde{l}_1 = l_1 + C_\delta\|f\|_{V'}^{p'} + C_{\delta,\epsilon}\|\psi\|_{L^p(D)}^p + \frac{CK}{2}$. It is a $L^\infty(\Omega_T)$ predictable element thanks to the assumptions on f and ψ. Therefore, by a convenient choice of δ, $\bar{A}$ satisfies $H_{2,1}$ by considering $\tilde{l}_1$ instead of l_1 and $\lambda - L_F - \frac{1}{2} \in \mathbb{R}$ instead of λ.
- $\forall v \in V$, we have
$$\begin{aligned}\| -\frac{1}{\epsilon}[(v-\psi)^-]^{p-1}\|_{L^{p'}(D)} &= \frac{1}{\epsilon}\|(v-\psi)^-\|_{L^p(D)}^{p-1}\\ &\leq C_\epsilon\left(\|v\|_{L^p(D)}^{p-1} + \|\psi\|_{L^p(D)}^{p-1}\right)\end{aligned}$$
Now, since the embeddings of $L^{p'}(D)$ in V' and of V in $L^p(D)$ are continuous,
$$\| -\frac{1}{\epsilon}[(v-\psi)^-]^{p-1}\|_{V'} \leq C_\epsilon\left(\|v\|_V^{p-1} + \|\psi\|_V^{p-1}\right)$$
and Assumption $\mathrm{H}_{2,3}$ is satisfied with $\bar{K}$ replaced by $\bar{K} + C_\epsilon$ and $\bar{K}$ by $\widetilde{g} = \bar{K} + C_\epsilon\|\psi\|_V^{p-1} + \|f\|_{V'}$ which is a predictable element of $L^\infty(\Omega_T)$. Moreover, $\|\mathbf{F}(v)\|_{V'} \leq C\|\mathbf{F}(v)\|_H \leq L_F\|v\|_H + CK$. Therefore
$$\|\bar{A}(v)\|_{V'}^{p'} \leq L_F\|v\|_H^{p'} + \tilde{g} + CK + C_\epsilon\|v\|_V^p.$$
By [32, Th. 5.1.3 p.125] with $\beta = p'$, we get

Lemma 6.2. *For all $\epsilon > 0$, there exists a unique solution $u_\epsilon \in L^p(\Omega_T; V)$ predictable such that $u_\epsilon \in L^2(\Omega; C([0,T], H))$ and satisfying (58) P-a.s. in Ω for all $t \in [0,T]$. Moreover, $(u_\epsilon(t))_{t\in[0,T]}$ (solution to (58)) is a Markov process.*

Lemma 6.3. *$(u_\epsilon)_{\epsilon>0}$ and $(A(u_\epsilon))_{\epsilon>0}$ are bounded in $L^p(\Omega_T; V)\cap C([0,T], L^2(\Omega, H))$ and $L^{p'}(\Omega_T; V')$, respectively.*

Proof. Let $\epsilon > 0$ and note that
$$\begin{aligned}u_\epsilon(t) - \psi + \int_0^t \Big(A(u_\epsilon) + \mathbf{F}(u_\epsilon) - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{p-1}\Big)ds\\ = u_0 - \psi + \int_0^t f ds + \int_0^t \widetilde{G}(u_\epsilon)dW(s).\end{aligned}$$

Itô's stochastic energy (see *e.g.* [16, Thm. 2.3]) yields

$$\|u_\epsilon-\psi\|_H^2(t)+2\int_0^t\langle A(u_\epsilon)+\mathbf{F}(u_\epsilon),u_\epsilon-\psi\rangle ds+\frac{2}{\epsilon}\int_0^t\int_D[(u_\epsilon-\psi)^-]^p dxds$$
$$=\|u_0-\psi\|_H^2+2\int_0^t\langle f,u_\epsilon-\psi\rangle ds+2\int_0^t\left(u_\epsilon-\psi,\widetilde{G}(u_\epsilon)dW(s)\right)_H$$
$$+\int_0^t\|\widetilde{G}(u_\epsilon)\|^2_{L_2(H_0,H)}ds.$$

Note that $\langle A(u_\epsilon),u_\epsilon-\psi\rangle\geq\alpha\|u_\epsilon\|_V^p-\lambda\|u_\epsilon\|_H^2-l_1-\langle A(u_\epsilon),\psi\rangle$. Thus

$$\begin{aligned}\langle A(u_\epsilon),u_\epsilon-\psi\rangle\geq&\alpha\|u_\epsilon\|_V^p-\lambda\|u_\epsilon\|_H^2-l_1-\bar{K}\|u_\epsilon\|_V^{p-1}\|\psi\|_V-\bar{K}\|\psi\|_V\\ \geq&\frac{\alpha}{2}\|u_\epsilon\|_V^p-\lambda\|u_\epsilon\|_H^2-l_1-\bar{K}\|\psi\|_V-C(\bar{K},\alpha)\|\psi\|_V^p.\end{aligned}\quad(59)$$

Additionally, $\langle\mathbf{F}(u_\epsilon),u_\epsilon-\psi\rangle\geq-(L_F+1)\|u_\epsilon\|_H^2-C\mathbf{F}(0)^2-\langle\mathbf{F}(u_\epsilon),\psi\rangle$.

By using (9), we get

$$\langle\mathbf{F}(u_\epsilon),u_\epsilon-\psi\rangle\geq-(L_F+1)\|u_\epsilon\|_H^2-C\mathbf{F}(0)^2-L_F^2\|u_\varepsilon\|_H^2-\|\psi\|_H^2,$$

hence

$$\langle\mathbf{F}(u_\epsilon),u_\epsilon-\psi\rangle\geq-(2L_F^2+C)\|u_\epsilon\|_H^2-[C\mathbf{F}(0)^2+\|\psi\|_H^2].\quad(60)$$

Thus, for any positive γ, Young's inequality yields the existence of a positive constant C_γ such that

$$\begin{aligned}&\mathbb{E}\|u_\epsilon-\psi\|_H^2(t)+2\mathbb{E}\int_0^t\frac{\alpha}{2}\|u_\epsilon\|_V^p(s)ds+\frac{2}{\epsilon}\mathbb{E}\int_0^t\int_D[(u_\epsilon-\psi)^-]^p dxds\\ &\leq\mathbb{E}\|u_0-\psi\|_H^2+C(\lambda+L_F^2+1)\mathbb{E}\int_0^t\|u_\epsilon\|_H^2(s)ds\\ &\quad+2t[l_1+\bar{K}\|\psi\|_V+C(\bar{K},\alpha)\|\psi\|_V^p+C\mathbf{F}(0)^2+\|\psi\|_H^2+C_\gamma\|f\|_{V'}^{p'}]\\ &\quad+\gamma\mathbb{E}\int_0^t\|u_\epsilon-\psi\|_V^p(s)ds+\mathbb{E}\int_0^t\|\widetilde{G}(u_\epsilon)-\widetilde{G}(\psi)\|^2_{L_2(H_0,H)}ds.\\ &\leq C\mathbb{E}\int_0^t\|u_\epsilon-\psi\|_H^2(s)ds+\frac{\alpha}{2}\mathbb{E}\int_0^t\|u_\epsilon\|_V^p(s)ds+\mathbf{C}t,\end{aligned}\quad(61)$$

where $\mathbf{C}=2[l_1+\bar{K}\|\psi\|_V+C(\bar{K},\alpha)\|\psi\|_V^p+C\mathbf{F}(0)^2+C\|\psi\|_H^2+C_\gamma\|f\|_{V'}^{p'}]$ and $C>0$ is a generic constant, after a suitable choice of γ. Then, the first part of the lemma is proved by Gronwall's lemma, and the second one by adding $H_{2,3}$ to the first estimate. □

Notice that (61) does not guarantee the necessary bound on the penalized term, we will therefore prove the following lemma.

Lemma 6.4. *Assume that* $\widetilde{h}^-\in L^{p'}(D)$*, then* $(\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1})_{\epsilon>0}$ *is bounded in* $L^{p'}(\Omega_T\times D)$ *and* $(u_\epsilon)_{\epsilon>0}$ *is a Cauchy sequence in the space* $L^2(\Omega;C([0,T],H))$*.*

Proof. Let $\delta > 0$ and consider the following approximation

$$F_\delta(r) = \begin{cases} r^2 - \dfrac{\delta^2}{6} & if \;\; r \leq -\delta, \\ -\dfrac{r^4}{2\delta^2} - \dfrac{4r^3}{3\delta} & if \;\; -\delta \leq r \leq 0, \\ 0 & if \;\; r \geq 0. \end{cases} \tag{62}$$

Set $\varphi_\delta(v) = \int_D F_\delta(v(x))dx; v \in H$ and denote by S the set $\{u_\epsilon \leq \psi\}$. Applying Ito's formula to the process $u_\epsilon - \psi$, one gets for any $t \in [0, T]$

$$\begin{aligned} &\varphi_\delta(u_\epsilon(t) - \psi(t)) + \int_0^t \langle A(u_\epsilon) - A(\psi), F_\delta'(u_\epsilon - \psi)\rangle ds \\ &+ \int_0^t \langle \mathbf{F}(u_\epsilon) - \mathbf{F}(\psi), F_\delta'(u_\epsilon - \psi)\rangle ds - \frac{1}{\epsilon}\int_0^t \langle [(u_\epsilon - \psi)^-]^{p-1}, F_\delta'(u_\epsilon - \psi)\rangle ds \\ &= \overbrace{\varphi_\delta(u_\epsilon(0) - \psi(0))}^{=0} + \overbrace{\int_0^t \langle \widetilde{h}^+, F_\delta'(u_\epsilon - \psi)\rangle ds}^{\leq 0} - \int_0^t \langle \widetilde{h}^-, F_\delta'(u_\epsilon - \psi)\rangle ds \\ &+ \int_0^t (\widetilde{G}(u_\epsilon)dW_s, F_\delta'(u_\epsilon - \psi)) + \frac{1}{2}\int_0^t Tr(F_\delta''(u_\epsilon - \psi)\widetilde{G}(u_\epsilon)Q\widetilde{G}(u_\epsilon)^*)ds. \end{aligned}$$

Since $\widetilde{G}(u_\epsilon) = \widetilde{G}(\psi) = 0$ on the set S, we deduce

$$\frac{1}{2}\int_0^t Tr(F_\delta''(u_\epsilon - \psi)\widetilde{G}(u_\epsilon)Q\widetilde{G}(u_\epsilon)^*)ds = 0.$$

Taking the expectation, one has

$$\begin{aligned} &\mathbb{E}\varphi_\delta(u_\epsilon(t) - \psi(t)) + \mathbb{E}\int_0^t \langle A(u_\epsilon) - A(\psi), F_\delta'(u_\epsilon - \psi)\rangle ds \\ &+ \mathbb{E}\int_0^t \langle \mathbf{F}(u_\epsilon) - \mathbf{F}(\psi), F_\delta'(u_\epsilon - \psi)\rangle ds - \frac{1}{\epsilon}\mathbb{E}\int_0^t \langle [(u_\epsilon - \psi)^-]^{p-1}, F_\delta'(u_\epsilon - \psi)\rangle ds \\ &\leq \mathbb{E}\int_0^t \langle -\widetilde{h}^-, F_\delta'(u_\epsilon - \psi)\rangle ds. \end{aligned}$$

Recall that $F_\delta'(u_\epsilon - \psi)$ converges to $-2(u_\epsilon - \psi)^-$ in V a.e. $t \in [0, T]$ and P-a.s. (see [42, Proof of Lemma 3]). Thus, dominated convergence theorem ensures that for any $t \in [0, T]$

$$\begin{aligned} &\mathbb{E}\|(u_\epsilon - \psi)^-(t)\|_{L^2(D)}^2 + \frac{2}{\epsilon}\mathbb{E}\int_0^t \|(u_\epsilon - \psi)^-(s)\|_{L^p(D)}^p ds \\ &\leq 2\mathbb{E}\int_0^t \langle \widetilde{h}^-(s), (u_\epsilon - \psi)^-(s)\rangle ds + 2(\lambda_{\mathbf{T}} + L_F)\mathbb{E}\int_0^t \|(u_\epsilon - \psi)^-(s)\|_H^2 ds. \end{aligned} \tag{63}$$

Recall that $1 < p < 2$, from Grönwall's lemma applied to (63), one gets

$$\begin{aligned} \frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|_{L^p(\Omega_T \times D)}^p &= \frac{1}{\epsilon}\mathbb{E}\int_0^T \|(u_\epsilon - \psi)^-(s)\|_{L^p(D)}^p ds \\ &\leq C_T \|\widetilde{h}^-\|_{L^{p'}(\Omega_T \times D)} \|(u_\epsilon - \psi)^-\|_{L^p(\Omega_T \times D)}, \end{aligned}$$

hence $\frac{1}{\epsilon}\|(u_\epsilon-\psi)^-\|^{p-1}_{L^p(\Omega_T\times D)} \le C_T\|h^-\|_{L^{p'}(\Omega_T\times D)}$. On the other hand, we have

$$\begin{aligned}\|\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}\|_{L^{p'}(\Omega_T\times D)} &= \frac{1}{\epsilon}(\mathbb{E}\int_0^T\int_D[(u_\epsilon-\psi)^-]^{(p-1)p'}dxds)^{\frac{1}{p'}}\\ &= \frac{1}{\epsilon}\|(u_\epsilon-\psi)^-\|^{p-1}_{L^p(\Omega_T\times D)}.\end{aligned}$$

Consequently, $(\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1})_{\epsilon>0}$ is bounded in $L^{p'}(\Omega_T\times D)$.

- Let $1>\epsilon\ge\delta>0$ and consider the process $u_\epsilon-u_\delta$, which satisfies the following

$$\begin{aligned}&u_\epsilon(t)-u_\delta(t)+\int_0^t[(A(u_\epsilon)-A(u_\delta))+(\mathbf{F}(u_\epsilon)-\mathbf{F}(u_\delta))\\ &+(-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}+\frac{1}{\delta}[(u_\delta-\psi)^-]^{p-1})]ds=\int_0^t(\widetilde{G}(u_\epsilon)-\widetilde{G}(u_\delta))dW(s).\end{aligned}$$

Applying Ito's formula with $F(t,v)=\frac{1}{2}\|v\|^2_H$, one gets for any $t\in[0,T]$

$$\begin{aligned}&\frac{1}{2}\|(u_\epsilon-u_\delta)(t)\|^2_H+\int_0^t[\langle A(u_\epsilon)-A(u_\delta),u_\epsilon-u_\delta\rangle+(\mathbf{F}(u_\epsilon)-\mathbf{F}(u_\delta),u_\epsilon-u_\delta)]ds\\ &+\int_0^t\langle-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}+\frac{1}{\delta}[(u_\delta-\psi)^-]^{p-1},u_\epsilon-u_\delta\rangle ds\\ &=\int_0^t\langle(\widetilde{G}(u_\epsilon)-\widetilde{G}(u_\delta))dW(s),u_\epsilon-u_\delta\rangle+\frac{1}{2}\int_0^t\|\widetilde{G}(u_\epsilon)-\widetilde{G}(u_\delta)\|^2_{L_2(H_0,H)}ds.\end{aligned}$$

Recall that $\int_0^t|(\mathbf{F}(u_\epsilon)-\mathbf{F}(u_\delta),u_\epsilon-u_\delta)|ds\le L_F\int_0^t\|(u_\epsilon-u_\delta)(s)\|^2_H ds$. We argue as in the proof of [42, Lemma 4] to estimate the other terms. Thus, we deduce

$$\mathbb{E}\sup_{t\in[0,T]}\|(u_\epsilon-u_\delta)(t)\|^2_H\le C\epsilon^{\frac{1}{p-1}}+C\int_0^T\mathbb{E}\sup_{\tau\in[0,s]}\|(u_\epsilon-u_\delta)(\tau)\|^2_H ds$$

and Grönwall's lemma ensures that $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in the space $L^2(\Omega;C([0,T],H))$. □

From Lemma 6.3 and Lemma 6.4, we deduce the following result.

Lemma 6.5. *There exist $u\in L^p(\Omega_T;V)\cap L^2(\Omega;C([0,T],H))\cap\mathcal{N}^2_W(0,T,H)$[16] and $(\rho,\chi)\in L^{p'}(\Omega_T;L^{p'}(D))\times L^{p'}(\Omega_T;V')$, each one predictable, such that the following convergences hold (as $\epsilon\to0$), up to sub-sequences denoted by the*

---

[16] $\mathcal{N}^2_W(0,T,H)$ denotes the space of all predictable process of $L^2(\Omega_T;H)$ (see [32, p. 36]).

same way,

$$u_\epsilon \rightharpoonup u \quad in \quad L^p(\Omega_T;V), \tag{64}$$

$$u_\epsilon \to u \quad in \quad L^2(\Omega;C([0,T],H)), \tag{65}$$

$$A(u_\epsilon) \rightharpoonup \chi \quad in \quad L^{p'}(\Omega_T;V'), \tag{66}$$

$$-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1} \rightharpoonup \rho, \quad \rho \le 0 \quad in \quad L^{p'}(\Omega_T\times D). \tag{67}$$

Moreover, $u(0)=u_0$ *and* $u\in K$. *Finally, we get*

$$\int_0^\cdot \widetilde{G}(u_\epsilon)dW(s) \to \int_0^\cdot G(u)dW(s) \text{ and } \mathbf{F}(u_\epsilon)\to\mathbf{F}(u) \text{ in } L^2(\Omega;C([0,T],H)). \tag{68}$$

Proof. By compactness with respect to the weak topology in the spaces $L^p(\Omega_T;V)$, $L^{p'}(\Omega_T;V')$ and $L^{p'}(\Omega_T\times D)$, there exist $u\in L^p(\Omega_T;V)$, $\chi\in L^{p'}(\Omega_T;V')$ and $\rho\in L^{p'}(\Omega_T\times D)$ such that (64), (66) and (67) hold (for subsequences). Thanks to Lemma 6.4, we get the strong convergence of u_ϵ to u in $L^2(\Omega;C([0,T],H))\hookrightarrow L^2(\Omega_T\times D)$. Moreover, since $u_\epsilon\in\mathcal{N}_W^2(0,T,H)$ and $(A(u_\epsilon))_\epsilon$ is predictable with values in V', the same applies to $u\in\mathcal{N}_W^2(0,T,H)$ and χ. Furthermore, $-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}$ is a predictable process with values in $L^{p'}(D)$. Hence ρ is a predictable process with values in $L^{p'}(D)$ and $\rho\le 0$ since the set of non positive functions of $L^{p'}(\Omega_T\times D)$ is a closed convex subset of $L^{p'}(\Omega_T\times D)$. In addition, since u_ϵ converges to u in $L^2(\Omega;C([0,T],H))$ then $u_\epsilon(0)=u_0$ converges to $u(0)$ in $L^2(\Omega;H)$ and $u(0)=u_0$ in $L^2(\Omega;H)$. Thanks to Lemma 6.4, we deduce that $(u_\epsilon-\psi)^-\to(u-\psi)^-=0$ in $L^p(\Omega_T\times D)$ and $u\in K$. Finally, note that

$$\mathbb{E}\sup_{t\in[0,T]}\|\int_0^t(\mathbf{F}(u_\epsilon)-\mathbf{F}(u))ds\|_H^2\le L_F^2\mathbb{E}\int_0^T\|u_\epsilon-u\|_H^2ds.$$

By Burkhölder-Davis-Gundy's inequality and H_3, one gets

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|\int_0^t(\widetilde{G}(u_\epsilon)-\widetilde{G}(u))dW(s)\|_H^2 &\le 3\mathbb{E}\int_0^T\|\widetilde{G}(u_\epsilon)-\widetilde{G}(u)\|_{L_2(H_0,H)}^2ds\\ &\le 3L_G\mathbb{E}\int_0^T\|u_\epsilon-u\|_H^2ds.\end{aligned}$$

Since $u_\epsilon\to u$ in $L^2(\Omega,C([0,T],H))$ and $u\in K$, one deduces (68). □

Lemma 6.6. $\rho(u-\psi)=0$ *a.e. in* $\Omega_T\times D$ *and,* $\forall v\in K$, $\rho(u-v)\ge 0$ *a.e. in* $\Omega_T\times D$.

Proof. On one hand, by Lemma 6.4, we have

$$\begin{aligned}0&\le-\frac{1}{\epsilon}\mathbb{E}\int_0^t\langle[(u_\epsilon-\psi)^-]^{p-1},u_\epsilon-\psi\rangle ds\\ &=\frac{1}{\epsilon}\mathbb{E}\int_0^t\|(u_\epsilon-\psi)^-(s)\|_{L^p}^pds\le C\epsilon^{p'-1}\to 0.\end{aligned}$$

On the other hand, by Lemma 6.5 $-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1} \rightharpoonup \rho$ in $L^{p'}(\Omega_T\times D)$ and $u_\epsilon-\psi\to u-\psi$ in $L^p(\Omega_T\times D)$. Hence $\mathbb{E}\int_0^T\int_D \rho(u-\psi)dxdt=0$ and $\rho(u-\psi)=0$ a.e. since the integrand is always non-positive. One finishes the proof by noticing that if $v\in K$, one has a.e. in Ω_T that, $\langle\rho,u-v\rangle = \underbrace{\langle\rho,u-\psi\rangle}_{=0}+\underbrace{\langle\rho,\psi-v\rangle}_{\geq 0}\geq 0$. □

We have for any $t\in[0,T]$,

$$u_\epsilon(t)+\int_0^t(A(u_\epsilon)-\mathbf{F}(u_\epsilon)-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}-f)ds=u_0+\int_0^t\widetilde{G}(u_\epsilon)dW(s),$$

and $u(t)+\int_0^t(\chi-\mathbf{F}(u)+\rho-f)ds=u_0+\int_0^t\widetilde{G}(u)dW(s)$, after passing to the limit as $\epsilon\downarrow 0$ in (58). Our aim now is to prove that $A(u)=\chi$. Note that

$$\begin{aligned}u_\epsilon(t)-u(t)+\int_0^t((A(u_\epsilon)-\chi)-(\mathbf{F}(u_\epsilon)-\mathbf{F}(u))+(-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}-\rho))ds\\ =\int_0^t(\widetilde{G}(u_\epsilon)-\widetilde{G}(u))dW(s).\end{aligned}$$

Note that $(A(u_\epsilon)-\chi)+(-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}-\rho)\in L^{p'}(\Omega_T;V')$ and $\mathbf{F}(u_\epsilon)-\mathbf{F}(u)\in L^2(\Omega_T;H)$ are predictables and that $\int_0^t(\widetilde{G}(u_\epsilon)-\widetilde{G}(u))dW(s)$ is a square integrable $\mathcal{F}_t$−martingale. Thus, we can apply Itô's formula (*e.g.* [16, Thm. 2.3]) to the process $u_\epsilon-u$ with $F(v)=\frac{1}{2}\|v\|_H^2$ to get

$$\begin{aligned}&\frac{1}{2}\|(u_\epsilon-u)(t)\|_H^2+\overbrace{\int_0^t(\mathbf{F}(u_\epsilon)-\mathbf{F}(u),u_\epsilon-u)ds}^{I_0}+\overbrace{\int_0^t\langle A(u_\epsilon)-\chi,u_\epsilon-u\rangle ds}^{I_1}\\ &+\overbrace{\int_0^t\langle-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}-\rho,u_\epsilon-u\rangle ds}^{I_2}\\ &=\overbrace{\int_0^t\langle(\widetilde{G}(u_\epsilon)-\widetilde{G}(u))dW(s),u_\epsilon-u\rangle}^{I_3}+\overbrace{\frac{1}{2}\int_0^t\|\widetilde{G}(u_\epsilon)-\widetilde{G}(u)\|^2_{L_2(H_0,H)}ds}^{I_4}.\end{aligned}$$

Let us consider in the sequel a given $v\in L^p(\Omega_T;V)\cap L^2(\Omega;C[0,T],H))$ and $t\in]0,T]$.

- Note that $I_1=\int_0^t\langle A(u_\epsilon),u_\epsilon\rangle ds-\int_0^t\langle A(u_\epsilon),u\rangle ds-\int_0^t\langle\chi,u_\epsilon-u\rangle ds$ and

$$\int_0^t\langle A(u_\epsilon),u_\epsilon\rangle ds\geq\int_0^t\langle A(v),u_\epsilon-v\rangle ds+\int_0^t\langle A(u_\epsilon),v\rangle ds-\lambda_{\mathbf{T}}\int_0^t\|u_\epsilon-v\|_H^2ds.$$

- $$\begin{aligned}\mathbb{E}(I_2) &= \mathbb{E}\int_0^t \langle -\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}, u_\epsilon - u\rangle ds - \mathbb{E}\int_0^t \langle \rho, u_\epsilon - u\rangle ds\\ &\geq \mathbb{E}\int_0^t \langle -\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}, \psi - u\rangle ds - \mathbb{E}\int_0^t \langle \rho, u_\epsilon - u\rangle ds.\end{aligned}$$

Since I_3 is a $\mathcal{F}_t$-square integrable martingale then $\mathbb{E}(I_3)=0$ and thanks to H_3 we have $\mathbb{E}(I_4) \leq L_G\mathbb{E}\int_0^t \|u_\epsilon(s)-u(s)\|_H^2 ds$. Furthermore, (9) gives $|I_0| \leq L_F\int_0^t \|u_\epsilon - u\|_H^2 ds$. By gathering the previous computation and taking the expectation, one has for any $t \in]0,T]$

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\|(u_\epsilon-u)(t)\|_H^2 + \mathbb{E}\int_0^t \langle A(v), u_\epsilon - v\rangle ds + \mathbb{E}\int_0^t \langle A(u_\epsilon), v-u\rangle ds\\ &- \mathbb{E}\int_0^t \langle \chi, u_\epsilon - u\rangle ds + \mathbb{E}\int_0^t \langle -\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{p-1}, \psi-u\rangle ds - \mathbb{E}\int_0^t \langle \rho, u_\epsilon - u\rangle ds\\ &\leq (L_G+L_F)\mathbb{E}\int_0^t \|u_\epsilon(s)-u(s)\|_H^2 ds + \lambda_{\mathbf{T}}\mathbb{E}\int_0^t \|u_\epsilon - v\|_H^2 ds.\end{aligned}$$

By passing to the limit as $\epsilon \to 0$, thanks to Lemmas 6.5 and 6.6, we get

$$\mathbb{E}\int_0^T \langle A(v)-\chi, u-v\rangle ds \leq {\lambda_{\mathbf{T}}}^+\mathbb{E}\int_0^T \|u-v\|_H^2 ds + \overbrace{\mathbb{E}\int_0^T \langle \rho, u-\psi\rangle ds}^{=0}.$$

We are now in a position to use "Minty's trick" *e.g.* [35, Lemma 2.13 p.35]. Indeed, set $v = u+\delta\varphi, \varphi \in L^p(\Omega_T;V)\cap L^2(\Omega; C([0,T],H)), \delta > 0$ to conclude

$$\mathbb{E}\int_0^T \langle A(u+\delta\varphi)-\chi, \varphi\rangle ds \geq {\lambda_{\mathbf{T}}}^+\delta\mathbb{E}\int_0^T \|\varphi\|_H^2 ds \to_{\delta\to 0} 0.$$

By using $H_{2,4}$, we get

$$\mathbb{E}\int_0^T \langle A(u)-\chi, \varphi\rangle ds = 0, \forall \varphi \in L^p(\Omega_T;V)\cap L^2(\Omega; C([0,T],H)),$$

which yields $\chi = A(u)$ and ends the proof of Theorem 6.1.

Corollary 6.7. *Let $K>0$, there exists $\mathcal{K} := C(f,\psi,l_1,\bar{K},\alpha,p,\gamma,M,K) > 0$ such that*

$$\frac{1}{\alpha}\mathbb{E}\|u(t)\|_H^2 + \mathbb{E}\int_0^t \|u(s)\|_V^p ds \leq \frac{4}{\alpha}[\|y_0\|_H^2 + \mathcal{K}(t+1)], \tag{69}$$

for $\frac{L_G(1+K^2)}{2K^2} + \lambda + L_F \leq 0$ and any $t \in [0,T]$. Moreover, there exists $S>0$ depending only on the data such that

$$\begin{aligned}&\mathbb{E}f_\epsilon(\|u(t)-\psi\|_H^2) - f_\epsilon(\|u_0-\psi\|_H^2) + \frac{\alpha}{2}\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds\\ &\leq (2\lambda + 2L_F + \frac{L_G(1+K^2)}{K^2})\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_H^2 ds + St. \qquad (70)\end{aligned}$$

If G satisfies (15), it holds

$$\mathbb{E}f_\epsilon(\|u(t)-\psi\|_H^2)+\frac{\alpha}{2}\mathbb{E}\int_0^t f'_\epsilon(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds$$
$$\le f_\epsilon(\|u_0-\psi\|_H^2)+(2\lambda+2L_F)\mathbb{E}\int_0^t f'_\epsilon(\|u(s)-\psi\|_H^2)\|u(s)\|_H^2 ds+(S+\mathbf{K})t. \tag{71}$$

Proof. We have

$$u(t)-\psi+\int_0^t \Big(A(u)+\mathbf{F}(u)+\rho)ds=u_0-\psi+\int_0^t f ds+\int_0^t G(u)dW(s).$$

Itô's stochastic energy (*e.g.* [16, Thm. 2.3]) yields

$$\|u-\psi\|_H^2(t)+2\int_0^t\langle A(u),u-\psi\rangle ds+2\int_0^t\langle \mathbf{F}(u),u-\psi\rangle ds$$
$$+2\int_0^t\int_D \rho(u-\psi)dxds=\|u_0-\psi\|_H^2+2\int_0^t\langle f,u-\psi\rangle ds$$
$$+2\int_0^t\Big(u-\psi,G(u)dW(s)\Big)_H+\int_0^t\|G(u)\|_{L_2(H_0,H)}^2 ds. \tag{72}$$

i. By taking the expectation and using Lemma 6.6, we obtain for any $K>0$

$$\frac{1}{2}\mathbb{E}\|u(t)-\psi\|_H^2+\mathbb{E}\int_0^t\langle A(u),u-\psi\rangle ds+\mathbb{E}\int_0^t\langle \mathbf{F}(u),u-\psi\rangle ds$$
$$\le\frac{1}{2}\|u_0-\psi\|_H^2+\mathbb{E}\int_0^t\langle f,u-\psi\rangle ds$$
$$+\frac{M(1+K^2)}{2}t+\frac{L_G(1+K^2)}{2K^2}\mathbb{E}\int_0^t\|u(s)\|_H^2 ds.$$

Similarly to (59), we get

$$\langle A(u),u-\psi\rangle\ge\frac{\alpha}{2}\|u\|_V^p-\lambda\|u\|_H^2-l_1-\bar{K}\|\psi\|_V-C(\bar{K},\alpha)\|\psi\|_V^p. \tag{73}$$

By using $V\hookrightarrow H\hookrightarrow V'$ and Young's inequality, we get

$$\mathbb{E}\int_0^t\langle\mathbf{F}(u),u-\psi\rangle ds\le L_F\mathbb{E}\int_0^t\|u\|_H^2 ds++\mathbb{E}\int_0^t\langle\mathbf{F}(u),\psi\rangle ds$$
$$\le L_F\mathbb{E}\int_0^t\|u\|_H^2 ds+(L_F\|\psi\|_V+\|\mathbf{F}(0)\|_H)\mathbb{E}\int_0^t\|u\|_H ds+t\|\mathbf{F}(0)\|_H\|\psi\|_V$$
$$\le L_F\mathbb{E}\int_0^t\|u\|_H^2 ds+\frac{\alpha}{8}\mathbb{E}\int_0^t\|u\|_V^p ds+C(\alpha,p,\|\mathbf{F}(0)\|_H,\|\psi\|_V,L_F)t, \tag{74}$$

and

$$\mathbb{E}\int_0^t|\langle f,u-\psi\rangle|ds\le\frac{\alpha}{8}\mathbb{E}\int_0^t\|u\|_V^p ds+t[C(\alpha,p)\|f\|_{V'}^{p'}+\|f\|_{V'}\|\psi\|_V]. \tag{75}$$

Thus, we obtain for $K > 0$, after using the above estimates

$$\begin{aligned}
&\mathbb{E}\|u(t)\|_H^2 + \alpha\mathbb{E}\int_0^t \|u(s)\|_V^p ds \\
&\leq 4\|u_0\|_H^2 + \mathcal{K}(t+1) + 4[\frac{L_G(1+K^2)}{2K^2} + \lambda + L_F)]\mathbb{E}\int_0^t \|u(s)\|_H^2 ds, \qquad (76)
\end{aligned}$$

where $\mathcal{K} > 0$ depending only on the data. If $\frac{L_G(1+K^2)}{2K^2} + \lambda + L_F \leq 0$, one has

$$\frac{1}{\alpha}\mathbb{E}\|u(t)\|_H^2 + \mathbb{E}\int_0^t \|u(s)\|_V^p ds \leq \frac{1}{\alpha}[4\|u_0\|_H^2 + \mathcal{K}(t+1)].$$

ii. Let us consider the following function $f_\epsilon : x \mapsto \dfrac{x}{1+\epsilon x}, \epsilon > 0$ and recall that $f_\epsilon \in C_b^2(\mathbb{R}_+)$. By using (72), Lemma 6.6 and applying Itô's formula to the process $\|u-\psi\|_H^2$, we obtain

$$\begin{aligned}
&f_\epsilon(\|u(t)-\psi\|_H^2) - f_\epsilon(\|u_0-\psi\|_H^2) + 2\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\langle A(u), u-\psi\rangle ds \\
&+ 2\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\langle \mathbf{F}(u), u-\psi\rangle ds \\
&= \int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)[\langle 2f, u-\psi\rangle + \|G(u)\|_{L_2(H_0,H)}^2] ds \\
&\quad + 2\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\langle G(u)dW(s), u-\psi\rangle \\
&\quad + 2\int_0^t f_\epsilon''(\|u(s)\|_H^2)\|(G(u), u-\psi)\|_{L_2(H_0,\mathbb{R})}^2 ds. \qquad (77)
\end{aligned}$$

Since $f_\epsilon'' < 0$, then the last term is non positive. In addition, recall that $f_\epsilon' \leq 1$ and the stochastic integral is a $(\mathcal{F}_t)$-martingale. On the other hand, by using that $f_\epsilon' > 0$, $H_{2,1}$, H_3, (73), (74) and (75) we obtain for any $K > 0$

$$\begin{aligned}
&\mathbb{E}f_\epsilon(\|u(t)-\psi\|_H^2) + \alpha\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds \\
&\leq (l_1 + \bar{K}\|\psi\|_V + C(\bar{K},\alpha)\|\psi\|_V^p + C(\alpha,p)\|f\|_{V'}^{p'} + \|f\|_{V'}\|\psi\|_V)t \\
&+ f_\epsilon(\|u_0-\psi\|_H^2) + 2\lambda\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_H^2 ds \\
&+ 2\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)|(\mathbf{F}(u), u-\psi)| ds \\
&+ \frac{\alpha}{4}\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds \\
&+ M(1+K^2)t + \frac{L_G(1+K^2)}{K^2}\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_H^2 ds.
\end{aligned}$$

By using (74), we have

$$\begin{aligned}&\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)|(\mathbf{F}(u),u-\psi)|ds\\&\le L_F\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u\|_H^2ds\\&+\frac{\alpha}{8}\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u\|_V^p ds+C(\alpha,p,\|\mathbf{F}(0)\|_H,\|\psi\|_V,L_F)t,\end{aligned}$$

thus, we get

$$\begin{aligned}&\mathbb{E}f_\epsilon(\|u(t)-\psi\|_H^2)+\frac{\alpha}{2}\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds\\&\le (l_1+\bar{K}\|\psi\|_V+C(\bar{K},\alpha)\|\psi\|_V^p+C(\alpha,p)\|f\|_{V'}^{p'}+\|f\|_{V'}\|\psi\|_V)t\\&+f_\epsilon(\|u_0-\psi\|_H^2)+(2\lambda+2L_F+\frac{L_G(1+K^2)}{K^2})\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_H^2ds\\&+M(1+K^2)t+C(\alpha,p,\|\mathbf{F}(0)\|_H,\|\psi\|_V,L_F)t.\end{aligned}$$

In conclusion, there exists $S>0$ depending only on the data such that

$$\begin{aligned}&\mathbb{E}f_\epsilon(\|u(t)-\psi\|_H^2)-f_\epsilon(\|u_0-\psi\|_H^2)+\frac{\alpha}{2}\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds\\&\le (2\lambda+2L_F+\frac{L_G(1+K^2)}{K^2})\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_H^2ds+St.\end{aligned}$$

Similarly, if G satisfies (15), one obtains

$$\begin{aligned}&\mathbb{E}f_\epsilon(\|u(t)-\psi\|_H^2)-f_\epsilon(\|u_0-\psi\|_H^2)+\frac{\alpha}{2}\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_V^p ds\\&\le (2\lambda+2L_F)\mathbb{E}\int_0^t f_\epsilon'(\|u(s)-\psi\|_H^2)\|u(s)\|_H^2ds+(S+\mathbf{K})t.\end{aligned}$$

□

Acknowledgements

This work is funded partially by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications). Finally, the author would like to thank the anonymous referees for valuable comments made on the initially submitted manuscript.

Author contributions Single author contribution.

Funding Open access funding provided by Scuola Normale Superiore within the CRUI-CARE Agreement.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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Received: 30 January 2025.
Revised: 2 January 2026.
Accepted: 3 March 2026.