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# Multilateral Supervaluationism and Classicality

**Bas Kortenbach**[1] · **Luca Incurvati**[2] · **Julian J. Schloeder**[3]

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## Abstract

Incurvati and Schlöder (*Journal of Philosophical Logic, 51*(6), 1549–1582, 2022) have recently proposed to define supervaluationist logic in a multilateral framework, and claimed that this defuses well-known objections concerning supervaluationism's apparent departures from classical logic. However, we note that the unconventional multilateral syntax prevents a straightforward comparison of inference rules of different levels, across multi- and unilateral languages. This leaves it unclear how the supervaluationist multilateral logics actually relate to classical logic, and raises questions about Incurvati and Schlöder's response to the objections. We overcome the obstacle, by developing a general method for comparisons of strength between multi- and unilateral logics. We apply it to establish precisely on which inferential levels the supervaluationist multilateral logics defined by Incurvati and Schlöder are classical. Furthermore, we prove general limits on how classical a multilateral logic can be while remaining supervaluationistically acceptable. Multilateral supervaluationism leads to sentential logic being classical on the levels of theorems and regular inferences, but necessarily strictly weaker on meta- and higher-levels, while in a first-order language with identity, even some classical theorems and inferences must be forfeited. Moreover, the results allow us to fill in the gaps of Incurvati and Schlöder's strategy for defusing the relevant objections.


**Keywords** Supervaluationism · Multilateral logic · Metainferences · Higher-order vagueness · Global validity

## 1 Introduction

Does supervaluationism imply a departure from classical logic? If so, should this departure be considered a defect?

✉ Bas Kortenbach
bas.kortenbach@gmail.com

1 Department of Philosophy, Scuola Normale Superiore, Pisa, Italy

2 Department of Philosophy & ILLC, University of Amsterdam, Amsterdam, Netherlands

3 Philosophy Department, University of Connecticut, Storrs, CT, USA

In brief, supervaluationism is a theory of vagueness, according to which the phenomenon is due to the semantics of our language being underdetermined. A sentence is only properly true (false) if it is true (false) according to all possible *precisifications*, meaning ways the semantics of the language could be completed. Borderline cases, then, are sentences which come out true on some precisifications but false on others. There is a history of debate over whether supervaluationist logic—that is, the logic that we end up with as a consequence of the supervaluationists account—is classical or not, and insofar as it is not, whether this is undesirable.

The issue might be broken up into several components. First, we need to identify exactly which logic *is* supervaluationist logic. What is the appropriate way to formally represent the supervaluationists understanding of vague language, and to define a consequence relation over it?[1] Call this the *formalization question*. Once we have settled on a definition of the logic, there is the technical task of determining how it relates to classical logic, especially in terms of strength: e.g. does it have the same theorems, valid inferences, valid metarules? Call this the *comparison question*. Finally, when all these results are in, there might still remain room for disagreement on their philosophical consequences—in particular, on the extent to which the deviations from classicality, such as they are, should be considered problematic (e.g. because the relevant aspects of classical logic are taken to codify standard inferential practice).[2] Call this the *acceptability question*.

Incurvati & Schlöder [16] have recently put forth a novel answer to the formalization question. It is based on the multilateral framework developed in [13–15], which treats the speech acts of weak rejection and assertion as primitive alongside the usual strong assertion. Formally, this is done through natural deduction systems which govern inferences between *signed formulae*; formulae prefixed with a force-marker $\ominus$, $\oplus$, or $+$, indicating the three respective speech acts. *Multilateral supervaluationism* is the project of defining supervaluationist logic through such a proof theory. The basic idea behind this is that while definite truths should be strongly asserted, and definite falsehoods strongly rejected (that is, their negation strongly asserted[3]), borderline cases warrant neither, and hence should be both weakly rejected and weakly asserted. Within this general set-up, the authors define three particular logics: the basic propositional $\mathsf{SML}$ (supervaluationist multilateral logic), a weaker $\mathsf{SML}^-$ that can accommodate higher-order vagueness, and a quantified extension $\mathsf{QSML}^-$ of the latter.

[1] For example, Fine [8] and Varzi [32] broadly agree that a supervaluationist model is a set of classical models, which represent the different precisifications. Yet the former takes logical validity over them to be defined 'globally', as preservation of truth-in-all-precisifications, and the latter 'locally' as preservation of truth per individual precisification. Cobreros [4] proposes to add an accessibility relation, and to use it to define consequence 'regionally'. Each of these options would lead to different results at subsequent comparison to classical logic.

[2] For instance, Williamson [34] and Keefe [18] agree on the formalization question, and consequently on the outcomes of the comparison question. But while Williamson argues that the deviations from classicality that fall out of this are unpalatable, Keefe finds them quite appropriate.

[3] Strong rejection can equivalently be treated as a fourth primitive speech act, as in e.g. [30].

Incurvati & Schlöder [16] submit that the approach has several advantages over previously known formalizations of supervaluationism. For one, they contend, $\mathsf{SML}^-$ and $\mathsf{QSML}^-$ offer a natural extension of the supervaluationist treatment of vagueness to higher orders. Moreover, the authors argue, the approach has the resources to defuse a criticism of supervaluationism, due to Graff Fara [11] and Williamson [35], to the following effect: several classically valid metarules, which play a central role in our deductive practices, come out invalid in standard formalizations of supervaluationism. Without cognitively plausible alternatives, supervaluationists cannot provide an account of good deductive reasoning. Yet Incurvati & Schlöder [16] show how all three of the multilateral systems derive restricted multilateral versions of the metarules in question. This suggests at least a partial answer to the acceptability question. However, multilateral supervaluationism currently lacks any sort of precise answer to the comparsion question, or even a clear path towards answering it. The main aim of the present paper is to fill this gap.

A few remarks about this endeavour, before we dive into it. First of all, note that the comparison question is usually the easy part. For more traditional formalizations of supervaluationism, it boils down to putting the set of (schematic) validities at each inferential level next to that of classical logic, and playing 'spot the difference'. But as we will see below, in the multilateral setting things are not so straightforward: the addition of the non-embeddable force-markers to the syntax means that (even schematic) validities cannot be directly compared between a multilateral logic and a unilateral (only treating strong assertion) one. This also means that it is presently not at all clear how the restricted metarules derived in [16] actually relate to the classical rules they are supposed to refine. A significant part of the challenge at hand thus consists in constructing a suitable method for comparing strength between multi- and unilateral logics.

Secondly, whilst investigations into the comparison question have often limited themselves to the levels of formulae, inferences, and metainferences, recent developments around $\mathsf{ST}$ (strict-tolerant logic) show that it can be fruitful and even crucial to compare the strength of logics also with respect to metametainferences, metametametainferences, and so on *ad infinitum*. By constructing an infinite sequence of 'increasingly classical' $\mathsf{ST}$-related logics, Barrio et al. [3] have demonstrated that it is possible for two logics to agree on the first $n$ inferential levels, for any $n$, without thereby agreeing on level $n + 1$. For this reason, it is important to compare our multilateral supervaluationist logics to classical logic taking into account the entire hierarchy of inferential levels.[4]

Finally, let us clarify that we are not just interested in answering the comparison question for the particular systems $\mathsf{SML}$, $\mathsf{SML}^-$ and $\mathsf{QSML}^-$, but for multilateral supervaluationism more broadly. This means that after determining the classicality of each of those logics, we will also explore the possibility of 'improving' on it. That is, we

[4] Porter [23] also studies the relationship between supervaluationism and classical logic at arbitrarily high inferential levels, though starting from a more standard formalization of the former.

will investigate whether there are otherwise plausible variants or modifications, which align with classical logic on strictly more inferential levels.[5]

With all of this in mind, we will proceed as follows. Section 2 will provide the necessary background, by presenting supervaluationism, multilateralism, and the application of the latter to the former. Section 3 is concerned with setting up our complete methodology for measuring agreement with classical logic. This involves importing some definitions from the ST-literature, but above all tackling the aforementioned comparison problem posed by the force-markers. Section 4 contains the results of applying said methodology. This is where we establish the degree of classicality of SML, $\mathsf{SML}^-$ and $\mathsf{QSML}^-$ themselves, and prove several limitations on the idea of doing better. Put together, we take these results to provide a complete answer to the comparison question for multilateral supervaluationism. Finally, Section 5 concludes by circling back to what this means for the acceptability question, as it applies to the multilateral formalization.

## 2 Multilateral Supervaluationism

According to supervaluationism, our language is vague because it is compatible with several ways of making it precise, also known as precisifications. Whether a sentence is true or false is determined by considering its value on all possible ways of making it precise—by *supervaluating* over precisifications. If a sentence is true on all supervaluations (supertrue), it is true *simpliciter*. If a sentence is false on all supervaluations (superfalse), it is false *simpliciter*. If a sentence is true on some supervaluations and false on others, it is *borderline*.

The standard approach is to implement these ideas formally by means of a Kripke-style model theory. We take models to be ordered pairs consisting of a set of precisifications $\mathcal{W}$ and a valuation function $\mathcal{V}$. We then define the propositional connectives as usual, for instance by letting $\neg A$ be true at a precisification just in case $A$ is not true at that precisification. However, the language of standard supervaluationist logic $\mathcal{L}_\Delta$ extends the language of propositional logic by also including an operator $\Delta$, to be read as 'it is definitely the case that'. The intended aim of introducing the $\Delta$ operator is to allow talk of a sentence being true, false, or borderline in the object language. For this reason, it is customary to take $\Delta A$ to be true at a precisification just in case $A$ is true at all precisifications. In the object language, one can then express '$A$ is supertrue' with $\Delta A$, '$A$ is superfalse' with $\Delta\neg A$, and '$A$ is borderline' with $\neg\Delta A \wedge \neg\Delta\neg A$.

The next step is to define validity for $\mathcal{L}_\Delta$. In modal logic, this is standardly done by taking validity to be preservation of truth at a possible world. The *rationale* for this is that validity is typically understood as preservation of truth *simpliciter*, and truth *simpliciter* is truth at a world. However, according to supervaluationism, truth

5 This line of research is also partly inspired by analogy to the ST-sequence, in which each subsequent logic might be seen as a modification which raises classicality with respect to its predecessors, though the outcome in our case will be radically different. Another relevant reference here is [29], which (among other things) conducts a search among certain types of relatives of the most common formalization of supervaluationism, to see if there are any more classical cousins. We will discuss one of [29]'s results briefly in Section 5.

*simpliciter* is supertruth—truth on all precisifications. For this reason, in the context of supervaluationist logics, it is customary to take validity to be preservation of truth at all precisifications. Formally, say that $\Gamma$ superentails $A$ (in symbols: $\Gamma \vDash_{SV} A$) just in case if, for every model $\langle \mathcal{W}, \mathcal{V} \rangle$, if every $C$ in $\Gamma$ is true at every $w$ in $\mathcal{W}$, then $A$ is true at every $w$ in $\mathcal{W}$. Then, supervaluationists typically say that an inference is valid just in case its premises superentail the conclusion.[6]

On this definition, supervaluationist logic validates all valid inferences of classical propositional logic, and indeed all valid inferences of the classical modal logic S5, with the S5 axioms formulated using the $\Delta$ operator. But supervaluationist logic also validates inferences that are known to fail in S5, such as the inference from $A$ to $\Delta A$. This means, however, that several of the standard metainferences of classical logic fail in supervaluationist logic, where a metainference is an inference the premises and conclusion of which are (regular) inferences. The simplest case is perhaps conditional proof. As we have seen, the inference from $A$ to $\Delta A$ is supervaluationistically valid. However, $A \rightarrow \Delta A$ is not a supervaluationistic validity, since it is not true in a model in which $A$ is true at some but not all precisifications. The validity of the inference from $A$ to $\Delta A$ can be similarly exploited to produce counterexamples to the validity of (intuitionistic) *reductio*, proof by cases, and contraposition.

Thus in its common formalization, supervaluationism is classical at the inferential level but not at the metainferential level. This raises (an instance of) the acceptability question. Graff Fara [11] has forcefully argued that the failure of said metainferences is problematic, since it prevents supervaluationists from providing an account of good deductive reasoning. Williamson [35] has subsequently strengthened Graff Fara's arguments, by observing that the failure of the metainferences extends to distinctively first-order metainferences, such as existential instantiation. Graff Fara's and Williamson's arguments have been addressed in recent work by Incurvati & Schlöder [16]. They first provide a natural deduction system which is sound and complete with respect to standard supervaluationist model theory, and secondly a modification, which is sound and complete for an analogously modified model theory, to make room for higher-order vagueness. In addition, the systems include, as derived rules, supervaluationistically licensed versions of the problematic metainferences.

Their approach may be motivated by focusing on the basic doxastic attitudes which, according to supervaluationism, one may take towards simple sentences containing a vague predicates, such as *Alice is tall*. Recall that, according to supervaluationism, *Alice is tall* may be true, false, or borderline. If it is true that Alice is tall, it seems that one ought to believe that Alice is tall, an attitude which can be expressed by asserting *Alice is tall*. If it is false that Alice is tall, it seems that one ought to believe that Alice is not tall, an attitude which can be expressed by asserting *Alice is not tall*. If it is borderline that Alice is tall, it seems that one ought to refrain from believing that Alice is tall and to refrain from disbelieving that Alice is tall. Incurvati and Schlöder call *weak rejection* the speech act expressing refraining from believing [13], and *weak*

[6] This corresponds to defining entailment 'globally', as opposed to 'locally' or 'regionally' (see footnote 1). This distinction should not be confused with the distinction between local and global validity which we will introduce in the next section, which concerns how to generalize the notion of validity from the inferential to the metainferential level. Clearly, standard terminology is not helpful here. For this reason, we speak of 'superentailment', following [2].

*assertion* the speech act expressing refraining from disbelieving [14]. It follows that the attitude one ought to take towards *Alice is tall* if this sentence is borderline may be expressed by both weakly asserting and weakly rejecting *Alice is tall*.

By suitably incorporating (strong) assertion, weak assertion, and weak rejection into the language and the proof theory, Incurvati and Schlöder develop Supervaluationist Multilateral Logic (SML), a deductive system for supervaluationist logic. The language $\mathcal{L}_{\Delta,S}$ of SML is obtained as follows. We extend the language of standard supervaluationist logic $\mathcal{L}_{\Delta}$ with *force-markers* $+$, $\oplus$ and $\ominus$ standing for, respectively, strong assertion, weak assertion, and weak rejection. The *sentences* of SML are formed in the usual way in the language of standard supervaluationist logic $\mathcal{L}_{\Delta}$. A *formula* of SML is either $\perp$ or the result of prefixing a sentence with a force-marker.

The inference rules of SML relate formulae. Thus, the proof theory of SML is *multilateral*, in that it specifies, by means of natural deduction rules, conditions on strong assertion, weak assertion and weak rejection. The proof theory is therefore in the tradition of bilateral systems, natural deduction systems whose rules specify conditions on assertion and rejection. In bilateral systems, rules divide into two sorts: rules which specify the meaning of the logical constants, and rules which specify how the speech acts interact, known as *coordination principles* (in that they coordinate the speech acts). The same is the case for SML.

Let us start with the logical constants. The rules for conjunction are obtained by prefixing each sentence with the strong assertion sign in the standard natural deduction rules.

$$(+\wedge\text{I.})\ \frac{+A \qquad +B}{+A \wedge B} \qquad (+\wedge\text{E.}_1)\ \frac{+A \wedge B}{+A} \qquad (+\wedge\text{E.}_2)\ \frac{+A \wedge B}{+B}$$

The rules for negation tell us that from the weak assertion of a sentence one can infer the weak rejection of its negation and *vice versa*, and that from the weak rejection of a sentence one can infer the weak assertion of its negation, and *vice versa*. In symbols:

$$(\ominus\neg\text{I.})\ \frac{\oplus A}{\ominus\neg A} \qquad (\ominus\neg\text{E.})\ \frac{\ominus\neg A}{\oplus A} \qquad (\oplus\neg\text{I.})\ \frac{\ominus A}{\oplus\neg A} \qquad (\oplus\neg\text{E.})\ \frac{\oplus\neg A}{\ominus A}$$

From a supervaluationist point of view, the first of these rules captures that if a sentence is not false, then it must be either true or borderline, which means that its negation is either false or borderline respectively, hence not true. Similarly for the other three.

According to the supervaluationist, $\Delta A$ is intended to express in the object language that $A$ is true *simpliciter*. For this reason, it should be possible to go back and forth from the assertion of $A$ to the assertion of $\Delta A$.

$$(+\Delta\text{I.})\ \frac{+A}{+\Delta A} \qquad (+\Delta\text{E.})\ \frac{+\Delta A}{+A}$$

However, these rules do not suffice to capture the full strength of the model-theoretic definition of $\Delta$, which tells us that $\Delta A$ is true at a precisification just in case, $A$ is true at all precisifications. For the model-theoretic definition of $\Delta$ entails that if $\Delta A$ is true on some precisification, $A$ is true on all precisifications. By contrast, the rules for $\Delta$ presented so far only ensure that $A$ is true on all precisifications if $\Delta A$ is true on *all* precisifications. Accordingly, we lay down an additional elimination rule for $\Delta$, together with the corresponding introduction rule.

$$(\oplus\Delta\mathrm{I.})\ \frac{+A}{\oplus\Delta A} \qquad (\oplus\Delta\mathrm{E.})\ \frac{\oplus\Delta A}{+A}$$

We now turn to the coordination principles. The first set of coordination principles ensures that strong assertion and weak rejection stand in a relation akin to contradictoriness in the standard square of opposition.

$$(\text{Rejection})\ \frac{+A \qquad \ominus A}{\bot} \qquad (\mathrm{SR}_1)\ \frac{\begin{array}{c}[+A]\\ \vdots\\ \bot\end{array}}{\ominus A} \qquad (\mathrm{SR}_2)\ \frac{\begin{array}{c}[\ominus A]\\ \vdots\\ \bot\end{array}}{+A}$$

The second set of coordination principles ensures that weak assertion and strong assertion stand in a relation akin to subalternity. Note that in the subderivation of Weak Inference, all undischarged leafs must be of the form $+C$, which is denoted by $+\vdots$ (see [15, 16] for how the restrictions on Weak Inference are justified).

$$(\text{Assertion})\ \frac{+A}{\oplus A} \qquad (\text{Weak Inference})\ \frac{\oplus A \qquad \begin{array}{c}[+A]\\ +\vdots\\ +B/\bot\end{array}}{\oplus B/\bot}\ \text{if } (+\Delta\mathrm{I.}) \text{ and } (\oplus\Delta\mathrm{I.}) \text{ were not used to derive } +B/\bot.$$

This concludes the presentation of the rules of SML.

Thus defined, SML is straightforwardly a notational variant of Incurvati & Schlöder's [15] Epistemic Multilateral Logic (EML). Therefore we will henceforth adapt facts from [15] about EML to our study of SML without further comment. Moreover, it follows that all results below about the classicality of (variants of) SML apply in parallel to (analogous variants of) EML.

SML bears a precise relation to standard supervaluationist consequence $\vDash_{\mathsf{SV}}$. Let $\vdash_{\mathsf{SML}}$ denote derivability in SML. Then the following holds.

**Theorem 2.1** [16] *For any $\mathcal{L}_\Delta$-sentence $A$ and set of $\mathcal{L}_\Delta$-sentences $\Gamma$, $\{+B \mid B \in \Gamma\} \vdash_{\mathsf{SML}} +A$ iff $\Gamma \vDash_{\mathsf{SV}} A$.*

The theorem shows that the logic of assertion in SML coincides with supervaluationist logic when consequence is defined in terms of superentailment. The theorem is an immediate consequence of the following soundness and completeness result. Let $\tau$ be a translation defined as follows:

$$\tau(\varphi) = \begin{cases} \Delta\psi, \text{ if } \varphi = +\psi \\ \neg\Delta\psi, \text{ if } \varphi = \ominus\psi \\ \neg\Delta\neg\psi, \text{ if } \varphi = \oplus\psi. \end{cases}$$

This provides an embedding of SML into S5.

**Theorem 2.2** [15] *Let $\Gamma$ be a set of $\mathcal{L}_{\Delta,S}$-formulae and $\varphi$ an $\mathcal{L}_{\Delta,S}$-formula. $\Gamma \vdash_{\mathsf{SML}} \varphi$ iff $\tau[\Gamma] \vdash_{\mathsf{S5}} \tau(\varphi)$.*

Before we continue, let us introduce the variants of SML. First, Incurvati & Schlöder [16] point out that SML runs afoul of the paradoxes of higher-order vagueness. To avoid them, they amend this logic to $\mathsf{SML}^-$, obtained from SML by dropping the rules ($\oplus\Delta$I.) and ($\oplus\Delta$E.). It is sound and complete for the following model theory [16]. Let $Prop$ be the set of propositional atoms.

**Definition 2.3** An $\mathsf{SML}^-$ model is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{V}, R \rangle$, where $\mathcal{W}$ is a set of points, $\mathcal{V} : Prop \to \mathcal{P}(\mathcal{W})$ a valuation function, and $R : \mathcal{W} \times \mathcal{L}_\Delta \to \mathcal{P}(\mathcal{W})$ is a function mapping every point and sentence to a set of points, such that $w \in R(w, A)$ for all $w \in \mathcal{W}$ and $A$.

Satisfaction of a formula in a model is defined as follows:

- $\mathcal{M} \Vdash + A$ iff for all $w \in \mathcal{W}$, $\mathcal{M}, w \vDash A$.
- $\mathcal{M} \Vdash \oplus A$ iff for some $w \in \mathcal{W}$, $\mathcal{M}, w \vDash A$.
- $\mathcal{M} \Vdash \ominus A$ iff for some $w \in \mathcal{W}$, $\mathcal{M}, w \nvDash A$.

Where satisfaction of a sentence at a world and model is defined thus:

- $\mathcal{M}, w \vDash p$ iff $w \in \mathcal{V}(p)$.
- Boolean connectives as usual.
- $\mathcal{M}, w \vDash \Delta A$ iff $\mathcal{M}, v \vDash A$ for all $v \in R(w, A)$.

Note that the satisfaction clause for + is the standard definition of supertruth.

Finally, Incurvati & Schlöder show how the treatment of (higher-order) vagueness can be extended to first-order logic. $\mathsf{QSML}^-$, quantified supervaluational multilateral logic, is obtained by adopting the language of first-order modal logic with identity and adding the following rules for the universal quantifier and identity to $\mathsf{SML}^-$ (for discussion of quantifiers and identity in multilateral logics see also, respectively, [1] and [30]).

$$(+\forall\text{I.})\ \frac{+A[a/x]}{+\forall x\, A}\ \text{if } a \text{ is any constant symbol not occurring undischarged leafs used to derive } A[a/x] \qquad (+\forall\text{E.})\ \frac{+\forall x\, A}{+A[a/x]}$$

$$(+=\text{I.})\ \frac{\begin{matrix}[+Fa] \\ \vdots \\ +Fb\end{matrix} \qquad \begin{matrix}[+Fb] \\ \vdots \\ +Fa\end{matrix}}{+a = b}\ \text{where } F \text{ is a predicate symbol not occurring in undischarged leafs}$$

$$(+=\text{E.}_1)\ \frac{+a = b \qquad +Fa}{+Fb}\ \text{where } F \text{ is a predicate symbol} \qquad (+=\text{E.}_2)\ \frac{+a = b \qquad +Fb}{+Fa}\ \text{where } F \text{ is a predicate symbol}$$

$\mathsf{QSML}^-$ is sound and complete for the following extension of the $\mathsf{SML}^-$ model theory [16].

**Definition 2.4** A $\mathsf{QSML}^-$ model is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{D}, \mathcal{I}, R \rangle$, where $\mathcal{W}$ and $R$ are as before, $\mathcal{D}$ is a function mapping each $w \in \mathcal{W}$ to a domain $\mathcal{D}(w)$, and $\mathcal{I}$ is a function mapping each $w$ to a function $\mathcal{I}(w)$ that interprets the non-logical vocabulary in the usual way.

Satisfaction of a signed formula at a model is as above. Satisfaction of a (closed) sentence at a world in a model is defined as follows:

- For closed terms $t_1, \ldots, t_n$: $\mathcal{M}, w \vDash P(t_1, \ldots, t_n)$ iff $(t_1^{\mathcal{I}(w)}, \ldots, t_n^{\mathcal{I}(w)}) \in \mathcal{I}(w)(P)$, where $t^{\mathcal{I}(w)}$ is $t$'s evaluation given the interpretation $\mathcal{I}(w)$ (defined in the usual way).
- Boolean connectives and $\Delta$ as before.
- $\mathcal{M}, w \vDash \forall x\, A$ iff for all closed terms $t$, $\mathcal{M}, w \vDash A[t/x]$.
- For closed terms $t_1, t_2$: $\mathcal{M}, w \vDash t_1 = t_2$ iff $t_1^{\mathcal{I}(w)} = t_2^{\mathcal{I}(w)}$.

Note that QSML$^-$ concerns derivations between closed formulae: sentences with open variables are not treated by the proof system, and receive no satisfaction conditions in the model theory. See [16, pp. 1574-1575] for discussion.

Incurvati & Schlöder [16] demonstrate that SML, SML$^-$ and QSML$^-$ all derive the following variants of the metarules that are classically but not supervaluationistically acceptable. As usual $A \vee B$ abbreviates $\neg(\neg A \wedge \neg B)$ and $A \rightarrow B$ abbreviates $\neg A \vee B$.

$$(+\rightarrow\text{I.})\ \frac{\begin{array}{c}[+A]\\ +\vdots\\ +B\end{array}}{+A \rightarrow B}\ \begin{array}{l}\text{if } (+\Delta\text{I.}) \text{ and } (\oplus\Delta\text{I.})\\ \text{were not used to derive}\\ +B\end{array} \qquad (+\neg\text{E.})\ \frac{\begin{array}{c}[+\neg A]\\ +\vdots\\ \bot\end{array}}{+A}\ \begin{array}{l}\text{if } (+\Delta\text{I.}) \text{ and } (\oplus\Delta\text{I.})\\ \text{were not used to derive}\\ \bot\end{array}$$

$$(+\vee\text{E.})\ \frac{+A \vee B \qquad \begin{array}{c}[+A]\\ +\vdots\\ +C\end{array} \qquad \begin{array}{c}[+B]\\ +\vdots\\ +C\end{array}}{+C}\ \begin{array}{l}\text{if } (+\Delta\text{I.}) \text{ and } (\oplus\Delta\text{I.})\\ \text{were not used to derive}\\ +C\end{array}$$

$$(+\text{Contraposition})\ \frac{\begin{array}{c}[+A]\\ +\vdots\\ +B\end{array} \qquad +\neg B}{+\neg A}\ \begin{array}{l}\text{if } (+\Delta\text{I.}) \text{ and } (\oplus\Delta\text{I.})\\ \text{were not used to derive}\\ +B\end{array}$$

Incurvati & Schlöder claim that this assuages Graff Fara's concerns. However, as anticipated in the introduction, it is not clear in which sense these rules are variants of the classical metarules of conditional proof, reductio, disjunctive syllogism, and contraposition. To illustrate, one may phrase the classical metarule of reductio as follows.

$$\frac{A, \neg B \vdash \bot}{A \vdash B}$$

It is unclear what it means for this rule to be valid in SML, as this is a metarule about inferences between *sentences*, whereas SML-inference relates *formulae* (decorated sentences). It will not do to say that reductio is valid in SML if each metarule obtained by substituting formulae for sentences in reductio is valid, as this also does not deliver a rule about SML-inferences. If, say, we substitute $+p$ for $B$ and $\ominus\Delta\neg p$ for $A$, we obtain the following.

$$\frac{\ominus\Delta\neg p, \neg + p \vdash \bot}{\ominus\Delta\neg p \vdash +p}$$

This is also not a metarule about SML-inferences, since '$\neg + p$' is not an SML-formula. One may think that the obvious solution is to say that a classical metarule

$\mathcal{R}$ is valid in SML if the metarule obtained by prefixing + to every premise and every conclusion in every inference in $\mathcal{R}$ is valid in SML. This is inadequate, since there might be intuitive counterexamples to a metarule that require other force-markers. Consider for example the logic obtained from SML by restricting the language to the one of propositional logic and dropping the rules for $\Delta$. In this logic, the following metarule is valid.

$$\frac{+A, +\neg B \vdash \bot}{+A \vdash +B}$$

However, the following rule is *not* valid in this logic.

$$\frac{\oplus A, +\neg B \vdash \bot}{\oplus A \vdash +B}$$

So it is not at all clear that the validity of the prefixed-with-+ version of reductio in a multilateral logic entails that reductio is valid in this logic. The upshot is that since it is not clear what it means for reductio to be *generally* licensed in a multilateral setting, it cannot be clear in which sense, if any, the derivability of $(+\neg\text{E.})$ shows that reductio is *partially* licensed in the supervaluationist systems.

Similarly, Incurvati & Schlöder [16] claim to adress Williamson's concern. For, abbreviating $\neg\forall x\neg A$ as $\exists x A$, the following version of Existential Instantiation is derivable in QSML$^-$.

$$(+\exists\text{E.})\ \frac{+\exists x\, A \qquad \begin{matrix}[+A[a/x]] \\ +\vdots \\ +B\end{matrix}}{+B}\ \text{if } a \text{ is any constant symbol not occurring in } A, B\text{, or undischarged leafs, and if } (+\Delta\text{I.}) \text{ was not used to derive } +B.$$

However, this prompts an analogue question to the one we asked for reductio. The metarule of Existential Instantiation in classical logic may be put as follows.

$$\frac{\Gamma \vdash \exists x A \qquad \Gamma, A[a/x] \vdash B}{\Gamma \vdash B}\ \text{if } a \text{ is any constant symbol not occurring in } \Gamma \text{ or } B.$$

Under which conditions could we say that this rule is valid in QSML$^-$, and how does the derivability of $(+\exists\text{E.})$ relate to them?

We will see how to settle such questions in the following section.

## 3 Measures of Classicality

We now turn to the question of how to determine the extent to which each of the multilateral logics revises or recovers classical logic. We take agreement with the validities of classical logic as our basic standard for classicality. The present section is concerned with filling in the details of this approach, and in particular with expounding and overcoming the schematic substitution problem flagged in the previous section.

### 3.1 Inferential Levels and Validities

To start, as is known from the case of ST, a complete picture of agreement on validities requires considering not only the regular inferences and metainferences, but the entire infinite sequence of inferential levels.[7] Formally, the stages of concern are as follows.

**Definition 3.1** If $\mathcal{L}$ is a language with $\mathcal{L}^0$ the set of well-formed formulae, for $n > 0$ the set $\mathcal{L}^n$ of level n inferences in $\mathcal{L}$ is defined recursively:

$$\mathcal{L}^{n+1} := \{\langle \Gamma, \Psi \rangle \mid \Gamma \cup \{\Psi\} \subseteq \mathcal{L}^n\}.$$

We will write $\Gamma \Rightarrow^n \Psi$ to denote the level $n$ inference $\langle \Gamma, \Psi \rangle$, adding or omitting brackets and superscripts where it suits readability. To illustrate, consider the propositional language $\mathcal{L}_{PL}$. The following are (random) examples of ((meta)meta)inferences from $\mathcal{L}^1_{PL}$, $\mathcal{L}^2_{PL}$ and $\mathcal{L}^3_{PL}$ respectively.

$$p \wedge q, p \rightarrow r \Rightarrow^1 r$$
$$(p \Rightarrow r, p \Rightarrow q \wedge p) \Rightarrow^2 (\varnothing \Rightarrow r \vee \neg p)$$
$$((p \wedge q \Rightarrow^1 r) \Rightarrow^2 (p \Rightarrow^1 q \wedge p)) \Rightarrow^3 (\varnothing \Rightarrow^2 (p \Rightarrow^1 r \vee \neg p))$$

Furthermore, besides $\mathcal{L}_{PL}$, our naming convention for languages is as follows: the presence or absence of the subscript $_=$ distinguishes the first-order setting with identity from the basic propositional (modal) one, that of subscript $_S$ denotes the presence or absence of force-markers, and that of $_\Delta$ whether the language has $\Delta$. So e.g. $\mathcal{L}^0_\Delta$ is the set of unsigned propositional sentences with $\Delta$, whilst $\mathcal{L}^1_{=,S}$ contains all the inferences between signed formulae from the first-order language with identity but without $\Delta$.

Next, we must decide precisely what it means for a given logic to validate or invalidate a higher-level inference in its language. By a logic $\mathcal{K}$ on $\mathcal{L}$, we mean any $\vdash_\mathcal{K} \subseteq \mathcal{L}^1$, whether defined proof-theoretically or otherwise. We then define inferential validity at higher levels as preservation of validity of the previous level.

**Definition 3.2** Let $\mathcal{K}$ be a logic on $\mathcal{L}$. $\mathcal{K}$-validity of an inference $\Gamma \Rightarrow^n \Psi \in \mathcal{L}^n$, written $\vDash_\mathcal{K} \Gamma \Rightarrow^n \Psi$, is defined recursively as follows:

- $\vDash_\mathcal{K} \Gamma \Rightarrow^1 \Psi$ iff$_{Def}$ $\Gamma \vdash_\mathcal{K} \Psi$.
- $\vDash_\mathcal{K} \Gamma \Rightarrow^{n+1} \Psi$ iff$_{Def}$ if $\vDash_\mathcal{K} \gamma$ for every $\gamma \in \Gamma$, then $\vDash_\mathcal{K} \Psi$.

Whilst natural, this definition is neither the only choice nor the obvious one. There are several competing validity criteria available at level 2 (and hence from there on up), the main contenders being so-called global and local validity [12]. In brief, the former comes down to preservation of validity, whereas the latter demands preservation of satisfaction per individual model. There has been plenty of discussion about the advantages and disadvantages of these different options [3, 6, 7, 10, 19, 31]. The local criterion is typically argued to be preferable, and has become the standard option in the literature on ST. Nevertheless, our notion of choice, as per Definition 3.2, is a higher-level generalization of level 2 global validity—though notably not equivalent to the generalization assumed by e.g. [9, 28], which amounts to generalized preservation of satisfaction-at-every-model.

[7] Recently, [20, 23, 28] have also treated transfinite-level inferences. However, we leave these out of the present paper, because it is questionable whether such inferences play any role in even highly idealized deductive reasoning, and moreover the phenomena we wish to clarify all limit themselves to the finite levels.

For a detailed defense of the particular version of global validity we adopt, we refer the reader to [19]. Let us highlight here only the key advantage that this notion can be defined purely in terms of a proof system, whereas both of the other options (the local and the alternative generalization of global) are always relative to a specific model theory. This is crucial for our present purposes, because the multilateral logics we are concerned with are partially motivated from an inferentialist perspective. With [15, 16], we take it that the meaning of the logical vocabulary is determined first and foremost by the proof rules that govern it. Since we want higher level validity to be determined in turn by the meaning of that vocabulary, we want it to be fully determined by the proof theory alone. This desideratum would not be met if we adopted either of the other criteria; one proof system may be sound and complete for several different model theories, and each of those model theories may produce a different set of metainferential validities on the other definitions.[8] Only global validity, as generalized in Definition 3.2, is uniquely determined by a natural deduction system.

With these definitions in place, one observation can immediately be made that will serve us well repeatedly in the remainder. It is that any inference of level $n$ can equivalently be thought of as a level $n+1$ inference with empty premises.

**Observation 3.3** Let $\mathcal{K}$ be a logic on $\mathcal{L}$. Take arbitrary $\Gamma \Rightarrow^n \Psi \in \mathcal{L}^n$.

$$\vDash_{\mathcal{K}} \Gamma \Rightarrow^n \Psi \ \text{ iff } \vDash_{\mathcal{K}} \varnothing \Rightarrow^{n+1} (\Gamma \Rightarrow^n \Psi).$$

In this sense, each inference of a given level is a limit case of every higher level. This has the convenient effect that, when comparing the strength of two logics on a language, a difference between them on any level $n$ immediately brings about a corresponding difference on all levels above $n$. Thus, the question on which levels they agree is reduced to the question *up to* which level they agree, if not on every level.[9]

### 3.2 Schematic Validity

The setup so far suffices to define basic classicality: logics $\mathcal{K}$ and $\mathcal{H}$ agree up to level $n$ just in case they agree on the validity of every inference in $\mathcal{L}^n$, and a logic is classical up to level $n$ just in case it agrees with classical logic on level $n$. However, this notion of agreement assumes that the logics are both formulated in a common language $\mathcal{L}$. The resulting notion of classicality therefore only applies to logics in the basic propositional or first-order languages, since this is where we have classical logics available for direct comparison. In cases where additional syntax is present, some further machinery is required.

When it comes to the multilateral logics under consideration, such additional syntax comes in two forms: there is the operator $\Delta$, and there are the force-markers $+$, $\oplus$ and $\ominus$.

[8] One example of this is ST itself. Any standard natural deduction system for classical propositional logic (CPL) is also sound and complete with respect to the model theory of ST (where the latter is formulated without constants). Yet the level 2 local validities of ST are very different from those of CPL, as are the level 3 global validities on the other generaliztion of global validity.

[9] This is not, however, an argument in favour of global validity, as the same points would apply given the local criterion.

Operators are easily dealt with by moving to the comparison of general inference rules, expressed by schemas, as opposed to individual inferences. It's worth spelling this out a bit, in order to see why it does not quite suffice to handle the force-markers. To this end, let us first focus on the basic propositional and modal logics, leaving the additional complications of the first-order setting to be dealt with below. Fix once and for all a countably infinite set of metalinguistic *sentence variables* $\mathcal{A} = \{A_1, A_2, A_3, ...\}$, and let $\mathcal{A}^P$ be the closure of $\mathcal{A}$ under $\neg$ and $\wedge$. We can then define schemas as follows.[10]

**Definition 3.4** The set $UBS^n$ of level $n$ unilateral Boolean schemas is defined recursively for $n > 0$:

$$UBS^1 := \{\langle \Lambda, \Omega \rangle \mid \Lambda \cup \{\Omega\} \subseteq \mathcal{A}^P\}.$$
$$UBS^{n+1} := \{\langle \Lambda, \Omega \rangle \mid \Lambda \cup \{\Omega\} \subseteq UBS^n\}.$$

As before, we will denote $\langle \Lambda, \Omega \rangle \in UBS^n$ by $\Lambda \Rightarrow^n \Omega$. A schema is valid according to a logic just in case all uniform substitutions instances of formulae for the metalinguistic variables are. But the variables can range over the formulae of different languages when validity is assessed with respect to different logics. Thus we can investigate whether a logic agrees with classical propositional logic on the validity of all inference rules, no matter which connectives or operators are included among its logical constants. Such variance will simply mean that each schema has a different set of instances with respect to the different logics.

Formally, an instance in a language $\mathcal{L}$ corresponds to a substitution function $\sigma : \mathcal{A} \to \mathcal{L}^0$. We write $\sigma[X]$ for the result of simultaneously substituting $\sigma(A)$ for every $A \in \mathcal{A}$ appearing in $X$. Provided that $\mathcal{L}^0$ is closed under $\neg$ and $\wedge$, each $\sigma[\Lambda \Rightarrow^n \Omega]$ will be an element of $\mathcal{L}^n$, meaning (recall) an individual level $n$ inference of the target language. This allows for the following definition.

**Definition 3.5** Let $\mathcal{K}$ be a logic in $\mathcal{L}$ such that $\mathcal{L}^0$ is closed under $\neg$ and $\wedge$. $\mathcal{K}$ validates $\Lambda \Rightarrow^n \Omega \in UBS^n$ (written $\vDash_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$) iff$_{Def}$

$$\vDash_{\mathcal{K}} \sigma[\Lambda \Rightarrow^n \Omega] \text{ for every } \sigma : \mathcal{A} \to \mathcal{L}^0.$$

Naturally, classicality per level $n$ can then be defined as agreement with CPL on the validity of every element of $UBS^n$.

However, at this point the problem posed by the force-markers becomes explicit. It is precisely the requirement that the well-formed formulae of the language be closed under the Boolean connectives. For given that the force-markers can only appear as prefixes, rather than embedded under connectives, the multilateral languages do not enjoy this property. To illustrate, consider a unilateral Boolean schema as simple as $A_1 \wedge A_2 \Rightarrow^1 A_1$, expressing the rule of Conjunction Elimination. The validity in e.g. SML of this schema would depend on the validity of ungrammatical instances such as $(\oplus p) \wedge (\oplus q) \Rightarrow^1 \oplus p$.

It might seem like the mistake here was simply to insist that the metalinguistic variables range over the target language's well-formed formulae, since in the case of

10 According to this definition, many obviously co-expressing schemas are strictly speaking distinct, e.g. $A_1 \Rightarrow A_1$ and $A_2 \Rightarrow A_2$ if $A_1 \neq A_2$. Yet we will treat such cases as identical. To be precise, we identify each schema with the equivalence class it represents, under the equivalence relation of having the same set of instances (alternatively, of being obtainable from one another by uniformly substituting elements of $\mathcal{A}$).

multilateral languages these are signed. Yet if we let them range over sentences instead, Conjunction Elimination has multilateral instances such as $p \wedge q \Rightarrow^1 p$. But this is an inference between unsigned sentences, therefore multilateral logics have nothing to say on its validity as such. We could try to prefix force-markers to all sentences after substitution has taken place, thereby finally obtaining inference instances in the multilateral target language. But there are many ways to assign force-markers to a given unsigned inference (e.g. three markers and two sentences means nine possible assignments for $p \wedge q \Rightarrow^1 p$). So the question arises: which such 'signings' of any given substitution are considered to be proper instances, so as to be relevant to the validity of the schema? There are several possible answers. All of them fall short, but it is illustrative to explore why, as this will help us design our eventual alternative.

First of all, we might simply count *all* signings of uniform substitutions to be among the instances of the schema. So e.g. a multilateral logic would validate Conjunction Elimination just in case it validates $\bigcirc A_1 \wedge A_2 \Rightarrow^1 \odot A_1$ for all sentences $A_1, A_2$ and every $\bigcirc, \odot \in \{+, \oplus, \ominus\}$. But this approach leads to unreasonable results. For example, it means that all the multilateral logics discussed invalidate the Boolean schema of Reflexivity, $A \Rightarrow A$, because they invalidate $+p \Rightarrow \ominus p$, which now comes out as an instance. But this inference is not relevant to the general rule that Reflexivity expresses with respect to CPL, namely that everything entails itself (or, proof-theoretically, that every zero-step derivation is permitted). That rule *does* hold for the multilateral logics. It is clear that if we count all signings as instances, schemas like Reflexivity place demands for their validity on multilateral logics which are no longer analogous to the demands they place on unilateral logics. The method is therefore not suitable for evaluating classicality.

The Reflexivity example suggests that perhaps we should restrict ourselves and only count the signings with all +'s. However, this means completely ignoring everything but the strongly asserted fragment of the language when assessing a multilateral logic's classicality. This tends to undergenerate instances. E.g. while it helpfully rules out $+p \Rightarrow^1 \ominus p$ as an instance of Reflexivity, it also rules out $\ominus p \Rightarrow^1 \ominus p$. Yet if a multilateral logic invalidates $\ominus p \Rightarrow^1 \ominus p$, then unlike classical logic its derivation relation simply is not reflexive, and it does not validate the rule that Reflexivity is supposed to express. So this approach is entirely blind to many potential deviations from classicality.

This in turn suggests that we should lift the restriction slightly, and consider all 'homogeneous' signings: those obtained by prefixing the same force-marker to every sentence in the inference. Then a multilateral logic would validate Reflexivity just in case it validates $+A \Rightarrow^1 +A$, $\oplus A \Rightarrow^1 \oplus A$ and $\ominus A \Rightarrow^1 \ominus A$ for all sentences $A$, which is exactly what we want. Yet the homogeneity approach runs into problems elsewhere. To start, it still undergenerates instances in some cases. Consider the classically valid metaschema of Transitivity: $\{A_1 \Rightarrow^1 A_2,\ A_2 \Rightarrow^1 A_3\} \Rightarrow^2 A_1 \Rightarrow^1 A_3$. Like Reflexivity, Transitivity expresses a structural feature that should concern weak assertions and rejections just as well as strong assertions. But on the homogeneous approach, e.g. $\{+p \Rightarrow^1 \ominus q,\ \ominus q \Rightarrow^1 \oplus r\} \Rightarrow^2 +p \Rightarrow^1 \oplus r$ would not be included as an instance of Transitivity. Yet any multilateral logic invalidating this inference is not transitive, and in violation of the metarule expressed by Transitivity. Moreover, regarding other schemas the homogeneous approach overgenerates. E.g. while $+(p \wedge q) \Rightarrow^1 \ominus p$ is thankfully not counted as an instance of Conjunction Elimination, $\ominus(p \wedge q) \Rightarrow^1 \ominus p$ still is. In our multilateral logics, rejecting a conjunction does not commit one to rejecting the first conjunct in particular; it may be merely the second conjunct that is doubted. But again, this is not a departure from classicality, as it does not violate the rule that Conjunction Elimination expresses with respect to CPL. Rejecting $p \wedge q$ can be weaker than rejecting $p$, precisely because a conjunction can be strictly stronger than its first conjunct, which is something that classical logic certainly allows for.

Thus all of these options fail. Before moving on, it is worth briefly dwelling on why it is that $\ominus(p \wedge q) \Rightarrow^1 \ominus p$ and $+(p \wedge q) \Rightarrow^1 \ominus p$ should not be relevant to the validity of Conjunction Elimination, whilst $\ominus p \Rightarrow^1 \ominus p$ and $\{+p \Rightarrow^1 \ominus q, \ominus q \Rightarrow^1 \oplus r\} \Rightarrow^2 +p \Rightarrow^1 \oplus r$ are clearly required for the validity of Reflexivity and Transitivity respectively. The former observation is explained as follows: in general, it is not a sensible demand of classicality for multilateral logics that connectives should display their classical behaviour with respect to every force-marker. Under strong assertion they should, for from the multilateral perspective, classical logic is just a logic of strong assertion. Hence a schema expressing a rule about how operators behave in classical logic is really expressing a rule about how the operators behave *under strong assertion*, and so if a schema is to express the analogous rule for multilateral logics, it should only concern strongly asserted signings.

Reflexivity and Transitivity, on the other hand, do not express a rule about the treatment of content of any particular logical form, as witnessed by the absence of operator manipulation over the course of these schemas. Hence the fact that different force-markers treat particular forms of content differently from strong assertion does not lead us to expect that they fail Reflexivity or Transitivity. These schemas instead state rules about the structure of commitment preservation in general, and so should be respected by all force-markers equally, if classicality is to be maintained.

The takeaway is as follows: regarding classicality for multilateral logics, those classical inference rules which involve operator manipulation should be required to hold only under the scope of +, whereas those which don't should apply to all speech acts. Note that there are cases which combine these two types. Recall for instance Reductio, $(A_1, \neg A_2 \Rightarrow^1 \bot) \Rightarrow^2 (A_1 \Rightarrow^1 A_2)$, where $A_2$ is subjected to operator manipulation, but $A_1$ is a static background assumption.[11] Thus this schema seems to tell us something about how negation behaves under the scope of strong assertion, given *any* context.

### 3.3 Multilateral Schemas

We've reached the following situation: we need a way to assess whether a multilateral logic agrees with the classically valid inference rules of different levels. Unilateral Boolean schemas capture the notion of a classically valid (propositional) inference

[11] Technically, $\bot$ itself cannot appear directly in schemas according to our definitions. Yet we will occasionally use it in such contexts, as an abbreviation of $A_1 \wedge \neg A_1$ (or of $+A_1 \wedge \neg A_1$, in case of the multilateral schemas to be introduced below).

rule. But attempting to directly evaluate these schemas for validity in a multilateral logic results in grave distortions of the demands that the schemas originally expressed. It seems we would do better to formulate an independent schematic framework to capture the notion of an valid multilateral inference rule, so that it is well suited to accommodate the quirks of multilateralism. If we can couple this with a systematic way to pair these schemas with unilateral ones expressing the same rule, we will be in position to compare multilateral logics to classical logic. This is what we will undertake now.

The preceding discussion brought to light how certain inference rules in multilateral logic can and should hold only with respect to specific speech acts. So our schematic framework should allow for the specification of force-markers, where necessary, to express these rules. So we let $\mathcal{A}^{\Delta}$ be the closure of $\mathcal{A}$ under the operators $\neg$, $\wedge$ and $\Delta$, and we take $\mathcal{A}^{S}$ to be the set of $\mathcal{A}^{\Delta}$ elements prefixed with $+$, $\oplus$ or $\ominus$. On the other hand, there are structural rules like Reflexivity and Transitivity, where the interest lies precisely in their holding across all speech acts. So it should be possible to leave force-markers out of the schemas in some places, in which case the metalinguistic variables must range over signed formulae instead of sentences. Thus we additionally fix a countably infinite set of *signed formula variables* $\Phi = \{\varphi_1, \varphi_2, ...\}$. These are of course not closed under the connectives, to avoid ungrammaticality after substitution.

**Definition 3.6** The set $MMS^n$ of level $n$ multilateral modal schemas is defined recursively for $n > 0$:

$$MMS^1 := \{\langle\Lambda, \Omega\rangle \mid \Lambda \cup \{\Omega\} \subseteq \Phi \cup \mathcal{A}^{S}\}.$$
$$MMS^{n+1} := \{\langle\Lambda, \Omega\rangle \mid \Lambda \cup \{\Omega\} \subseteq MMS^n\}.$$

**Definition 3.7** Let $\mathcal{K}$ be a logic on $\mathcal{L}_{\Delta,S}$. $\mathcal{K}$ validates $\Lambda \Rightarrow^n \Omega \in MMS^n$ (written as $\vDash_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$) iff$_{Def}$

$$\vDash_{\mathcal{K}} \mu[\Lambda \Rightarrow^n \Omega] \text{ for every } \mu = \mu_{\Phi} \cup \mu_{\mathcal{A}}, \text{ with } \mu_{\Phi} : \Phi \to \mathcal{L}^0_{\Delta,S} \text{ and } \mu_{\mathcal{A}} : \mathcal{A} \to \mathcal{L}^0_{\Delta}.$$

Multilateral schemas are able to express structural rules like Reflexivity as $\varphi \Rightarrow^1 \varphi$, but also rules about interactions between particular force-markers and operators, as in $\ominus\Delta(A_1 \to A_2) \Rightarrow^1 +A_1 \wedge \neg A_2$. Furthermore, we can express rules which combine the two, describing the interaction of a particular force-marker with some operator, but dictating that it should hold in the context of any type of speech act. For instance, Multilateral Reductio, $(\varphi, +\neg A \Rightarrow^1 \bot) \Rightarrow^2 (\varphi \Rightarrow^1 +A)$, expresses that if strong assertion of $\neg A$ is incoherent given *some* other attitude, that attitude commits one to strongly asserting $A$. This schema allows us to finally capture in formal terms one of the observations motivating this project: that SML fails *reductio ad absurdum*, due to metainferences such as $(\oplus p, +\neg p \Rightarrow^1 \bot) \Rightarrow^2 (\oplus p \Rightarrow^1 +p)$ being SML invalid. For this is an instance of Multilateral Reductio, namely such that $\mu_{\Phi}(\varphi) = \oplus p$ and $\mu_{\mathcal{A}}(A) = p$.[12]

[12] We define Multilateral Reductio (and similarly Reductio itself) as including one context premise $\varphi$, rather than e.g. as simply $(+\neg A \Rightarrow^1 \bot) \Rightarrow^2 (\varnothing \Rightarrow^1 +A)$, or as $(\varphi_1, ..., \varphi_n, +\neg A \Rightarrow^1 \bot) \Rightarrow^2 (\varphi_1, ..., \varphi_n, \Rightarrow^1 +A)$ for any other $n$. This is because (i) all the versions with at least one context premise are equivalent to each other, for all multilateral logics that meet some basic conditions, and (ii) the version without any context premise is strictly weaker; unlike all the others it is valid in SML and its variants.

To furthermore capture in which sense this marks a departure from classicality, however, we need to relate Multilateral Reductio to a unilateral schema expressing *reductio*. When it comes to such relations, we should observe that many multilateral modal schemas state rules which cannot be interpreted by unilateral Boolean schemas at all. This includes all those containing modal operators, such as things like $+\Diamond\Diamond A \Rightarrow^1 +\Diamond A$. This is not a defect, however; the question whether the rule expressed by this schema is valid for, say, CPL simply does not make sense, and so we cannot hope to compare supervaluationist or otherwise modal logics to classical logic on these types of inference rules. A similar point applies to multilateral schemas with built-in $\oplus$ or $\ominus$ signs. No unilateral schema can express anything analogous to e.g. $\ominus\neg A_1 \Rightarrow^1 \oplus A_1$. But as we have noted, CPL is just a logic of strong assertion, and so does not have anything to say on how negation interacts with weak assertion and rejection. This means that all such schemas can and should be ignored when comparing a multilateral to a unilateral logic. So we recall $\mathcal{A}^P$, the closure of $\mathcal{A}$ under the operators $\neg$ and $\wedge$, and let $\mathcal{A}^+$ be the set of $\mathcal{A}^P$ elements prefixed with $+$.

**Definition 3.8** The set $CMS^n$ of level $n$ classically expressible multilateral modal schemas is defined recursively for $n > 0$:

$$CMS^1 := \{\langle\Lambda, \Omega\rangle \mid \Lambda \cup \{\Omega\} \subseteq \Phi \cup \mathcal{A}^+\}.$$
$$CMS^{n+1} := \{\langle\Lambda, \Omega\rangle \mid \Lambda \cup \{\Omega\} \subseteq CMS^n\}.$$

Of course $CMS^n \subseteq MMS^n$, so Definition 3.7 for validity carries over.

Since unilateral logic can be seen as a logic of strong assertion, we can obtain a unilateral schema expressing the same rule as some classically expressible schema by simply removing the $+$'s and the distinction between the two variable types. Thus, given some $\Lambda \Rightarrow^n \Omega \in CMS^n$, let $U(+\chi)$ be $\chi$, for every $+\chi \in \mathcal{A}^+$ that appears in $\Lambda \Rightarrow^n \Omega$, and pick fresh and distinct $U(\varphi) \in \mathcal{A}$, for every signed formulae variable $\varphi$.

**Definition 3.9** The unilateralization of $\Lambda \Rightarrow^n \Omega \in CMS^n$ is $U[\Lambda \Rightarrow^n \Omega]$.

The result of unilateralization is a unilateral schema expressing the analogue of what the multilateral schema expressed. For example, Multilateral Reductio is classically expressible, and its unilateralization is Reductio: $(A_1, \neg A_2 \Rightarrow^1 \bot) \Rightarrow^2 (A_1 \Rightarrow^1 A_2)$. The fact that SML and its variants fail Multilateral Reductio whilst CPL validates its unilateralization thus marks a difference between the metainferential rules validated by the logics. This, then, is the precise sense in which e.g. SML is not classical on the metalevel.

Note that every unilateral schema is the unilateralization of at least some classically expressible multilateral one. Therefore, we can formulate what it means for a multilateral logic to validate more/less/the same inference rules as CPL at some level $n$, by quantifying over $CMS^n$.

**Definition 3.10** Let $\mathcal{K}$ be a logic on $\mathcal{L}_{\Delta,S}$.

- $\mathcal{K}$ is subclassical on level $n$ iff$_{Def}$

$$\text{For all } \Lambda \Rightarrow^n \Omega \in CMS^n, \text{ if } \vDash_{\mathcal{K}} \Lambda \Rightarrow^n \Omega, \text{ then } \vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^n \Omega].$$

- $\mathcal{K}$ is superclassical on level $n$ iff$_{Def}$

$$\text{For all } \Lambda \Rightarrow^n \Omega \in CMS^n\text{, if } \vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^n \Omega]\text{, then } \vDash_{\mathcal{K}} \Lambda \Rightarrow^n \Omega.$$

We furthermore say that, on a level $n$, a logic is *classical* if it is both subclassical and superclassical, *incomparable to classical logic* if it is neither, and *strictly sub-/superclassical* if it is one but not the other. This completes our method for measuring classicality in multilateral logics, at least for the modal propositional language.

Before moving on to the first-order setting, there is a potential objection to consider at this point. Since multiple non-equivalent multilateral schemas can unilateralize to the same schema, it seems that this resulting schema cannot express a rule that is equivalent to what *all* of the original multilateral schemas expressed, since the originals were not equivalent to each other. For example, $\{\varphi_1 \Rightarrow^1 \varphi_2, \varphi_2 \Rightarrow^1 \varphi_3\} \Rightarrow^2 \varphi_1 \Rightarrow^1 \varphi_3$ expresses general transitivity, whilst $\{+A_1 \Rightarrow^1 +A_2,\ +A_2 \Rightarrow^1 +A_3\} \Rightarrow^2 +A_1 \Rightarrow^1 +A_3$ only demands transitivity of strong assertion. Meanwhile $\{\varphi_1 \Rightarrow^1 +A,\ +A \Rightarrow^1 \varphi_2\} \Rightarrow^2 \varphi_1 \Rightarrow^1 \varphi_2$ expresses something in between. But the unilateralization of any of them is just Transitivity: $\{A_1 \Rightarrow^1 A_2,\ A_2 \Rightarrow^1 A_3\} \Rightarrow^2 A_1 \Rightarrow^1 A_3$. So which of them is supposed to express the true multilateral analogue to Transitivity?

However, all such cases are of the same sort, which is such that it does not undermine our approach. They only occur because $\varphi$'s and non-embedded $+A$'s both become non-embedded $A$'s after unilateralization. So take an arbitrary unilateral schema $\Omega \in UBS^n$ and consider the class of all multilateral ones that unilateralize to it: $\mathcal{C} = \{\Omega' \in CMS^n \mid U[\Omega'] = \Omega\}$. Any two elements of $\mathcal{C}$ can be obtained from each other by uniformly substituting appearances of $\varphi_i \in \Phi$ for non-embedded appearances of $+A_i$, and/or *vice versa* (whilst keeping the metalinguistic variables appropriately fresh). $\mathcal{C}$ has a particular element $\mathbb{C}$ in which no non-embedded $+A$'s appear, as all these positions are taken up by $\varphi$'s. This is the proper multilateral analogue of $\Omega$. For as argued above, when some $A_i$ appears only non-embedded in a unilateral schema, its behaviour according to the rule does not concern the interaction of strong assertion with content of a particular form, but rather a structural property of commitment altogether. Hence we identified $\{\varphi_1 \Rightarrow^1 \varphi_2, \varphi_2 \Rightarrow^1 \varphi_3\} \Rightarrow^2 \varphi_1 \Rightarrow^1 \varphi_3$ as the proper multilateral expression of Transitivity. But putting a fresh $\varphi$ in place of a $+A$ can only make the demands of the schema stronger, as it strictly enlarges the set of instances. Thus $\mathbb{C}$ is also the strongest element of $\mathcal{C}$, in the sense that $\mathbb{C}$ is valid for some logic $\mathcal{K}$ iff every element of $\mathcal{C}$ is valid for $\mathcal{K}$. Therefore, the fact that other elements of $\mathcal{C}$ also happen to unilateralize to $\Omega$ does not interfere with the conditions for (sub- or super)classicality at level $n$, as per Definition 3.10.

### 3.4 First-Order Schemas

Finally, the task remains to generalize the whole comparison method to the first-order case. Of course, when it comes to multilateral predicate logics such as $\mathsf{QSML}^-$, we are not just content to know whether they agree with all of the classical propositional and structural validities. We want to compare them to classical first-order logic, to discover whether their treatment of quantifiers, identity, and so on is classical as well. Boolean schemas will not suffice to this end, so to start we will need schemas that can capture the classically valid first-order inference rules.

Setting up schemas for predicate logic is non-trivial. As will be seen below, there is a certain degree of expressivity which cannot, as far as we know, be achieved without making the definitions overly complex and cumbersome. Moreover, in our setting, some care must be taken to prevent sentences with open variables from ending up in the instances, since as noted in Section 2, such sentences aren't treated by $\mathsf{QSML}^{-}$. For our purposes, it suffices to ensure the absence of open variables more or less by force, and to settle for a workably simple set-up, even if at a minor cost in expressivity. Let $At$ be the set of atomic sentences of $\mathcal{L}^0_=$. We let $\mathcal{A}^\forall$ be the closure of $\mathcal{A} \cup At$ under $\neg$, $\wedge$ and $\forall$, but minus all elements which contain open variables.

**Definition 3.11** The set $UQS^n$ of level $n$ unilateral quantified schemas is defined recursively for $n > 0$:

$$UQS^1 := \{\langle \Lambda, \Omega\rangle \mid \Lambda \cup \{\Omega\} \subseteq \mathcal{A}^\forall\}.$$
$$UQS^{n+1} := \{\langle \Lambda, \Omega\rangle \mid \Lambda \cup \{\Omega\} \subseteq UQS^n\}.$$

**Definition 3.12** Let $\mathcal{K}$ be a logic in $\mathcal{L}_=$. $\mathcal{K}$ validates $\Lambda \Rightarrow^n \Omega \in UQS^n$ (written $\vDash_\mathcal{K} \Lambda \Rightarrow^n \Omega$) iff$_{Def}$

$\vDash_\mathcal{K} \sigma[\Lambda \Rightarrow^n \Omega]$ for every $\sigma : \mathcal{A} \to \mathcal{L}^0_=$ such that $\sigma[\Lambda \Rightarrow^n \Omega]$ features no sentences with open variables.

There are a number of differences from the Boolean case to discuss here. The first is that we are now of course closing under quantifiers as well as under the propositional connectives. However, only those elements of the closure which don't contain open object-language variables are actually admitted into our schemas. In the validity condition, on the other hand, we simply let the metalinguistic variables range over arbitrary sentences, not restricting to those without open variables. However, we only count as instances those inferences which don't have open variables *after* substitution. To see why, consider for example the following schema: $\forall x(A_1 \wedge A_2) \Rightarrow^1 \forall x A_1 \wedge \forall x A_2$. We want this to express that universal quantification distributes over conjunction. However, if we let $A_1$ and $A_2$ range merely over closed sentences, the schema only expresses that *vacuous* universal quantification does so. Instead, we want them to range over all sentences except those where variables other than $x$ occur open, which is what the present definition does in effect.

Perhaps most notably, we are not just taking the closure of the set of sentence variables $\mathcal{A}$, but we add the atoms first. This is in any case necessary to do for identity, in order to schematically express identity-specific principles, such as e.g. Symmetry in $\varnothing \Rightarrow^1 \forall x \forall y(x = y \to y = x)$. The predicate atoms are added as well, in order to express for example cases of universal instantiation: $\forall x Px \Rightarrow^1 Pa$. This would not be possible with only $\mathcal{A}$—the closest we could get is $\forall x A \Rightarrow^1 A$—because the predicate has to be applied to two different terms in the schema.[13]

Admittedly, this does suggest that what we would really want is to express something like general universal instantiation: $\forall x A(x) \Rightarrow^1 A(a)$, understood as applying

[13] Thus the key difference between propositional and predicate atoms, which explains why the latter but not the former can be fruitfully included alongside $\mathcal{A}$ in schemas, is that predicate atoms are (despite their name) composite.

to any unary sentence $A(x)$. However, incorporating such schemas would require something like the following: start with disjoint sets of metalinguistic variables $\mathcal{A}_n$ for each arity $n$, and let each $A \in \mathcal{A}_n$ appear in schemas only next to a list of $n$ many terms $\langle a_1, ..., a_n \rangle$. When assessing validity, let $A \in \mathcal{A}^n$ range over a set of formula each paired with a list of $n$ many terms appearing in them, so that when substitution takes place, what ends up in the instance is that formula, but with all those terms replaced by $\langle a_1, ..., a_n \rangle$ respectively. But even this would not really suffice. For one, because in order for something like $\forall x A(x) \Rightarrow^1 A(a)$ to express fully general universal instantiation, we actually have to read it as being about all terms, rather than about some $a$ in particular. This would require including yet another type of metalinguistic variable, which range over the closed terms of the object language. The point of all this is to illustrate the complexity involved in implementing such schemas *formally*—which we would presently need to do if they are to be included at all, since we aim to prove quantified statements about schemas and their validities in different logics—despite the ease with which we intuitively read and use them. Hence we opt to settle for expressing things like universal instantiation not in general, but with a separate schema for each predicate formula and constant. This much can be achieved by inclusion of the atoms.

Having settled on these definitions, the rest of the comparison method can be straightforwardly adapted. Multilateral schemas and their validity are defined exactly as one would expect. Let $\mathcal{A}^{\Delta,\forall}$ be the closure of $\mathcal{A} \cup At$ under the operators $\neg$, $\wedge$, $\Delta$ and $\forall$, but minus all elements which contain open variables. Let $\mathcal{A}^{\circ,\forall}$ be the set of $\mathcal{A}^{\Delta,\forall}$ elements prefixed with $+$, $\oplus$ or $\ominus$.

**Definition 3.13** The set $MQS^n$ of level $n$ multilateral quantified modal schemas is defined recursively for $n > 0$:

$$MQS^1 := \{\langle \Lambda, \Omega \rangle \mid \Lambda \cup \{\Omega\} \subseteq \Phi \cup \mathcal{A}^{\circ,\forall}\}.$$
$$MQS^{n+1} := \{\langle \Lambda, \Omega \rangle \mid \Lambda \cup \{\Omega\} \subseteq MQS^n\}.$$

**Definition 3.14** Let $\mathcal{K}$ be a logic on $\mathcal{L}_{=,\Delta,S}$. $\mathcal{K}$ validates $\Lambda \Rightarrow^n \Omega \in MQS^n$ (written $\vDash_{\mathcal{K}} \Lambda \Rightarrow^n \Omega$) iff$_{Def}$

$\vDash_{\mathcal{K}} \mu[\Lambda \Rightarrow^n \Omega]$ for every $\mu = \mu_\Phi \cup \mu_{\mathcal{A}}$ with $\mu_\Phi : \Phi \to \mathcal{L}^0_{=,\Delta,S}$ and $\mu_{\mathcal{A}} : \mathcal{A} \to \mathcal{L}^0_{=,\Delta}$ such that $\mu[\Lambda \Rightarrow^n \Omega]$ features no formulae with open variables.

As before, only those multilateral schemas without modal operators or weak assertion or rejection can be sensibly compared to classical logic. So let $\mathcal{A}^{+,\forall}$ be the set of $\mathcal{A}^{\forall}$ elements prefixed with $+$.

**Definition 3.15** The set $CQS^n$ of level $n$ classically expressible multilateral quantified modal schemas is defined recursively for $n > 0$:

$$CQS^1 := \{\langle \Lambda, \Omega \rangle \mid \Lambda \cup \{\Omega\} \subseteq \Phi \cup \mathcal{A}^{+,\forall}\}.$$
$$CQS^{n+1} := \{\langle \Lambda, \Omega \rangle \mid \Lambda \cup \{\Omega\} \subseteq CQS^n\}.$$

The remaining schemas are once more paired with unilateral ones by simply removing $+$'s and turning $\varphi$'s into $A$'s, so we keep the definition of $U$ precisely the same.

**Definition 3.16** The unilateralization of $\Lambda \Rightarrow^n \Omega \in CQS^n$ is $U[\Lambda \Rightarrow^n \Omega]$.

Naturally, super- and subclassicality at level $n$ are analysed as agreement with classical first-order logic (CFOL) on pairs of elements of $CQS^n$ and their unilateralizations.

**Definition 3.17** Let $\mathcal{K}$ be a logic on $\mathcal{L}_{=,\Delta,S}$.

- $\mathcal{K}$ is subclassical on level $n$ iff$_{Def}$

$$\text{For all } \Lambda \Rightarrow^n \Omega \in CQS^n, \text{ if } \vDash_{\mathcal{K}} \Lambda \Rightarrow^n \Omega, \text{ then } \vDash_{\mathsf{CFOL}} U[\Lambda \Rightarrow^n \Omega].$$

- $\mathcal{K}$ is superclassical on level $n$ iff$_{Def}$

$$\text{For all } \Lambda \Rightarrow^n \Omega \in CQS^n, \text{ if } \vDash_{\mathsf{CFOL}} U[\Lambda \Rightarrow^n \Omega], \text{ then } \vDash_{\mathcal{K}} \Lambda \Rightarrow^n \Omega.$$

As before, we will say that such a logic is classical on a level if it is both sub- and superclassical, incomparable to classical logic if it is neither, and strictly sub-/superclassical if it is one but not the other.

### 3.5 Taking Stock

A lot of schema types have been introduced this section. Table 1 sums them up for convenience. Recall that within both rows, the entry in the third column is a subset of that in the second column. Note also that we can largely ignore the middle column in the remainder; within each row, unilateralization is a function from the third column into the first. So it is those two columns that really matter where the classicality of a given multilateral logic is concerned.

Moreover, let us point out that the useful observation made about individual inferences at the beginning of this section also applies to all of these schemas.

**Observation 3.18** Let $\Lambda \Rightarrow^n \Omega$ be a schema of any of the types in Table 1. Let $\mathcal{K}$ be a logic on a corresponding language.

$$\vDash_{\mathcal{K}} \Lambda \Rightarrow^n \Omega \;\text{ iff }\; \vDash_{\mathcal{K}} \varnothing \Rightarrow^{n+1} (\Lambda \Rightarrow^n \Omega).$$

This ensures that disagreement on all rules of a given level, like disagreement on individual inferences, propagates upwards through the inferential hierarchy. Thus, for multilateral logics as for unilateral logics, we can speak of (super/sub)classicality *up to* a given level. This completes the full toolkit needed, not just to assess the classicality of SML, SML$^-$ and QSML$^-$, but also to investigate the possibility of improving classicality, and in particular whether this can be done without betraying the original purposes of the logics in question.

Before all of that is undertaken in the next section, a remark about the scope of the preceding developments is in order. The present section has been entirely formulated in relation to supervaluationist multilateral logics, and the issue of their comparison to classical logic. However, the core of both the comparison problem as well as its proposed solution did not in any way depend on the specifics of the logics involved, nor on the specific speech acts of the particular multilateral framework at play, nor

**Table 1** Sets of level $n$ schemas

| | Unilateral | Multilateral | Classically expressible |
|---|---|---|---|
| Propositional (modal) | $UBS^n$ | $MMS^n$ | $CMS^n$ |
| First-order | $UQS^n$ | $MQS^n$ | $CQS^n$ |

even on the fact that it is precisely *classical* logic to which we aim to compare them. Hence we take it that much if not all of what has been said applies in parallel to any bi- or multilateral logic, and its comparison to any unilateral one.[14] Additional force-markers will pose the discussed problem regarding the application of unilateral schemas, no matter which speech acts they signify (as long as they are not all strong assertion). And something like the multilateral schemas presented here—appropriately adjusted to accommodate potentially different force-markers and/or any difference in the syntax of the object language—combined with the operation of unilateralizing them, will generally work to sidestep it.

The flexibility of the methodology has already been seen in action. When we sought to adapt the accumulated methods from the propositional and modal setting to the first-order case, lifting the notion of a unilateral schema required some thought. However, subsequently constructing the comparison techniques—that is, the corresponding multilateral schemas, the method of unilateralization, and the resulting definition of (super/sub)classicality—required only trivial adjustments to those in the previous setting.

More broadly, the core ideas behind our approach might prove useful even beyond bi- and multilateral logic, to the extent that they apply to *any* formal language such that the set of formulae, for whatever reason, fails to be closed under the classical connectives.[15] Thus we expect that the overall approach advanced here can find applicability well beyond the logics under discussion in the present paper.

## 4 Results

It is finally time to put the methods developed over the previous section to work, to answer the comparison question for multilateral supervaluationism. For each of $\mathsf{SML}$, $\mathsf{SML}^-$ and $\mathsf{QSML}^-$, we will walk through the following three steps: first, determine the system's classicality status at each level as it is. Secondly, given this outcome, investigate the most natural and promising way to modify the logic so as to increase classicality. Finally, assess whether this modified version is acceptable by supervalua-

[14] That is, to any bi- or multilateral logic of the signed formula variety. Other styles of bi- or multilateralism, such as [25]'s out-of-bounds reading of consequence or the proof and refutation systems of [33], are of course a different matter.

[15] For an example of such a language from outside the bilateral tradition, see [5], where the authors use a type of signed formulae to define hierarchies of computationally tractable approximations of $\mathsf{FDE}$, $\mathsf{LP}$ and $\mathsf{K}_3$.

tionist lights and, if not, establish whether any more suitable alternatives are possible. Let us start with the first step, applied to the simpler setting.

### 4.1 The Classicality of SML and SML$^{-}$

We will make use of the following two results.

**Theorem 4.1** [15] *Take arbitrary* $\Gamma \Rightarrow^1 B \in \mathcal{L}^1_{PL}$. *Then* $\vDash_{\mathsf{CPL}} \Gamma \Rightarrow^1 B$ *iff* $\vDash_{\mathsf{SML}} \{+\gamma \mid \gamma \in \Gamma\} \Rightarrow^1 +B$.

**Theorem 4.2** [15] *Take arbitrary* $\Gamma \Rightarrow^1 B \in \mathcal{L}^1_{PL}$ *and* $\eta : Prop \to \mathcal{L}^0_{\Delta}$. *If* $\vDash_{\mathsf{CPL}} \Gamma \Rightarrow^1 B$ *then* $\vDash_{\mathsf{SML}} \{+\eta[\gamma] \mid \gamma \in \Gamma\} \Rightarrow^1 +\eta[B]$.

With these in hand, we can determine the precise strength of SML relative to classical logic, for each inferential level. To start, we show that on any level, it is at most as strong as classical logic.

**Theorem 4.3** SML *is subclassical at every level* $n$.

***Proof*** We take arbitrary $\Lambda \Rightarrow^n \Omega \in CMS^n$ such that $\vDash_{\mathsf{SML}} \Lambda \Rightarrow^n \Omega$, and show that $\vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^n \Omega]$. We make a case distinction based on whether any signed formula variables from $\Phi$ appear in $\Lambda \Rightarrow^n \Omega$.

(i) Suppose none do. Take arbitrary $\sigma : \mathcal{A} \to \mathcal{L}^0_{PL}$ to show $\vDash_{\mathsf{CPL}} \sigma[U[\Lambda \Rightarrow^n \Omega]]$. Although $\sigma$ is merely a propositional substitution, it defines an instance of $\Lambda \Rightarrow^n \Omega$ as well, because there are no signed formula variables in the latter. Thus $\vDash_{\mathsf{SML}} \Lambda \Rightarrow^n \Omega$ means $\vDash_{\mathsf{SML}} \sigma[\Lambda \Rightarrow^n \Omega]$.
We claim that this is equivalent to $\vDash_{\mathsf{CPL}} \sigma[U[\Lambda \Rightarrow^n \Omega]]$. This can be shown by an induction on $n$. For the base case, observe that $\sigma[\Lambda \Rightarrow^1 \Omega] = \{+\gamma \mid \gamma \in \sigma[U[\Lambda]]\} \Rightarrow^1 +\sigma[U[\Omega]]$. Thus the claim follows by Theorem 4.1. Induction step is trivial by the recursive definition of validity.
Hence $\vDash_{\mathsf{CPL}} \sigma[U[\Lambda \Rightarrow^n \Omega]]$, and therefore $\vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^n \Omega]$ since $\sigma$ was arbitrary.

(ii) Suppose some do. Pick fresh and distinct $A_\varphi \in \mathcal{A}$ for every $\varphi \in \Phi$ appearing in $\Lambda \Rightarrow^n \Omega$. Let $\Lambda' \Rightarrow^n \Omega'$ be the schema obtained from $\Lambda \Rightarrow^n \Omega$ by replacing every occurrence of every $\varphi \in \Phi$ with $+A_\varphi$. Every instance of $\Lambda' \Rightarrow^n \Omega'$ is an instance of $\Lambda \Rightarrow^n \Omega$, so $\vDash_{\mathsf{SML}} \Lambda \Rightarrow^n \Omega$ entails $\vDash_{\mathsf{SML}} \Lambda' \Rightarrow^n \Omega'$. Therefore $\vDash_{\mathsf{CPL}} U[\Lambda' \Rightarrow^n \Omega']$ by case (i). But $U[\Lambda' \Rightarrow^n \Omega'] = U[\Lambda \Rightarrow^n \Omega]$, so $\vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^n \Omega]$. □

Moreover, it is at least as strong as classical logic on level 1.

**Theorem 4.4** SML *is superclassical at level 1.*

***Proof*** Take arbitrary $\Lambda \Rightarrow^1 \Omega \in CMS^1$ such that $\vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^1 \Omega]$. We take arbitrary $\mu = \mu_{\mathcal{A}} \cup \mu_\Phi$ with $\mu_{\mathcal{A}} : \mathcal{A} \to \mathcal{L}^0_{\Delta}$ and $\mu_\Phi : \Phi \to \mathcal{L}^0_{\Delta,S}$, and show that $\vDash_{\mathsf{SML}} \mu[\Lambda \Rightarrow^1 \Omega]$. Again, we make a case distinction based on whether any signed formula variables from $\Phi$ appear in $\Lambda \Rightarrow^1 \Omega$.

(i) Suppose none do. Since $\vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^1 \Omega]$, then in particular $\vDash_{\mathsf{CPL}} \sigma[U[\Lambda \Rightarrow^1 \Omega]]$ where $\sigma$ maps every $A_i \in \mathcal{A}$ to a fresh and distinct $p_i \in Prop$. Now define $\eta : Prop \to \mathcal{L}^0_\Delta$ so that $\eta(p_i) = \mu(A_i)$. Theorem 4.2 ensures that $\vDash_{\mathsf{SML}} \{+\eta[\sigma[U[\lambda]]] \mid \lambda \in \Lambda\} \Rightarrow^1 +\eta[\sigma[U[\Omega]]]$. But this is just to say that $\vDash_{\mathsf{SML}} \mu[\Lambda \Rightarrow^1 \Omega]$, as desired. For applying $\eta$ after $\sigma$ is by construction the same thing as only applying $\mu$, and applying $U$ merely removes the $+$'s which are prefixed again in the end.

(ii) Suppose some $\varphi \in \Phi$ appear in $\Lambda$, but $\Omega \notin \Phi$. From $\vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^1 \Omega]$ it follows that in particular any instance where all the signed formula variables get mapped to tautologies is valid. But these variables are only in the premise set, so then the schema must already be valid without them: $\vDash_{\mathsf{CPL}} U[\Lambda - \Phi \Rightarrow^1 \Omega]$. Then by case (i), $\vDash_{\mathsf{SML}} \Lambda - \Phi \Rightarrow^1 \Omega$. By weakening, $\vDash_{\mathsf{SML}} \Lambda \Rightarrow^1 \Omega$.

(iii) Suppose $\Omega \in \Phi$. If $\Omega \in \Lambda$ the conclusion follows by reflexivity of SML. Suppose $\Omega \notin \Lambda$. Since $\vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^1 \Omega]$, in particular any instance where $\Omega$ gets mapped to a contradiction is valid. Hence $\vDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^1 \bot]$. Thus $\vDash_{\mathsf{SML}} \Lambda \Rightarrow^1 \bot$ by case (i) or (ii). Then $\vDash_{\mathsf{SML}} \Lambda \Rightarrow^1 \Omega$ since SML validates Explosion. □

However, as anticipated this is not the case on any higher level.

**Theorem 4.5** SML *is not superclassical at any level $n > 1$.*

***Proof*** At $n = 2$, we've seen that Multilateral Reductio, $(\varphi, +\neg A \Rightarrow^1 \bot) \Rightarrow^2 (\varphi \Rightarrow^1 +A)$, is not SML-valid due to instances such as $(\oplus p, +\neg p \Rightarrow^1 \bot) \Rightarrow^2 (\oplus p \Rightarrow^1 +p)$. But its unilateralization $(A_1, \neg A_2 \Rightarrow^1 \bot) \Rightarrow^2 (A_1 \Rightarrow^1 A_2)$ is CPL-valid.

The higher levels follow from this via Observation 3.18. □

Putting these pieces together, the full story on SML's classicality is therefore as follows.

**Corollary 4.6** SML *is classical at level 1 and strictly subclassical at every level $n > 1$.*

What about SML$^-$ then? It turns out that the removal of the $(\oplus\Delta)$ rules has no effect on classicality, and so the outcomes will be precisely the same as for SML. Moreover, they can be shown in the same way. Thus to start we need analogues of Theorems 4.1 and 4.2.

**Theorem 4.7** *Take arbitrary $\Gamma \Rightarrow^1 A \in \mathcal{L}^1_{PL}$. Then $\vDash_{CPL} \Gamma \Rightarrow^1 A$ iff $\vDash_{SML^-} \{+\gamma \mid \gamma \in \Gamma\} \Rightarrow^1 +A$.*

***Proof*** Note first that, by the fact that $\tau$ embeds SML into S5 (Theorem 2.2), SML is sound and complete with respect to the class of S5 Kripke models with signed formulae interpreted on them according to $\tau$. Now recall the model theory of SML$^-$ (Definition 2.3). It is obvious that as far as $\Delta$-free sentences are concerned, SML$^-$ models can equivalently be thought of as just such S5 models. Hence the result to be proven is equivalent to Theorem 4.1. □

**Theorem 4.8** *Take arbitrary $\Gamma \Rightarrow^1 B \in \mathcal{L}^1_{PL}$ and $\eta : Prop \to \mathcal{L}^0_\Delta$. If $\vDash_{\mathsf{CPL}} \Gamma \Rightarrow^1 B$ then $\vDash_{\mathsf{SML}^-} \{+\eta[\gamma] \mid \gamma \in \Gamma\} \Rightarrow^1 +\eta[B]$.*

***Proof*** By Theorem 4.7, $\vDash_{\mathsf{CPL}} \Gamma \Rightarrow^1 B$ entails $\vDash_{\mathsf{SML}^-} \{+\gamma \mid \gamma \in \Gamma\} \Rightarrow^1 +B$. It is clear upon inspection of the proof rules that by taking a derivation of $\{+\gamma \mid \gamma \in \Gamma\} \vdash_{\mathsf{SML}^-} +B$ and simultaneously substituting every $p \in Prop$ in it for $\eta(p)$, one obtains a derivation for $\{+\eta[\gamma] \mid \gamma \in \Gamma\} \vdash_{\mathsf{SML}^-} +\eta[B]$. □

From here we can simply copy our steps from before.

**Corollary 4.9** $\mathsf{SML}^-$ *is classical at level 1 and strictly subclassical at every level* $n > 1$.

***Proof***

- Subclassicality at every level follows in precisely the same way as it did for SML (Theorem 4.3), though by referring to Theorem 4.7 in place of Theorem 4.1.
- Superclassicality at level 1 is precisely as for SML (Theorem 4.4), though by referring to Theorem 4.8 in place of Theorem 4.2.
- Failure of superclassicality at all $n > 1$ is as it was for SML (Theorem 4.5). □

### 4.2 Increasing Classicality

Corollaries 4.6 and 4.9 are not especially surprising outcomes, as it is common for supervalautionist logics to not license the full strength of classical logic at the metalevel. Yet it remains to be seen whether this is actually *necessary* in our setting. Would it not be possible to bolster our logics, so as to match the strength of classical logic not just at level 1 but beyond, without betraying the tenets of the supervaluationist approach?

We proved the non-classicality of SML and $\mathsf{SML}^-$ via their failure to validate Multilateral Reductio: $(\varphi, +\neg A \Rightarrow^1 \bot) \Rightarrow^2 (\varphi \Rightarrow^1 +A)$. Observe that the (in)validity of Multilateral Reductio corresponds precisely to the (in)admissibility of the following negation elimination rule.[16]

$$
\begin{array}{c}
[+\neg A] \\
\vdots \\
(+\neg\mathrm{E}.^*)\ \dfrac{\bot}{+A}
\end{array}
$$

By admissibility of a rule, we mean the same thing as one would in a sequent calculus: that adding the rule to the system would not affect which (level 1) inferences can be derived in it. Thus in case of $(+\neg\mathrm{E}.^*)$, admissibility would mean that already for every $\Gamma$ and $A$ such that $\bot$ is provable from $\Gamma \cup \{+\neg A\}$, $+A$ is provable from $\Gamma$. This is of course just to say that Multilateral Reductio is valid—or rather, strictly speaking, that the schema $(\Lambda, +\neg A \Rightarrow^1 \bot) \Rightarrow^2 (\Lambda \Rightarrow^1 +A)$ is valid for each $\Lambda \subseteq \Phi$, though this is a distinction without a difference, since all our logics of interest allow for weakening.

The correspondence between schema validity and rule admissibility is important for two reasons. First, because it finally establishes an exact relation between the classical metarule of reductio, and the $(+\neg\mathrm{E}.)$ rule which [16] derive to address the concerns of [11] (recall Section 2). Namely, the relation is as follows: $(+\neg\mathrm{E}.)$ is a restricted version

[16] For more on the relations between different definitions of metainferential validity on the one hand, and proof-theoretic notions like admissibility or derivability on the other, see [6, 12].

of (+¬E.*), the admissibility of which corresponds to the validity of Multilateral Reductio, which unilateralizes to Reductio, the schema that expresses reductio for classical logic. It is in this sense that deriving (+¬E.) qualifies as showing that SML, SML$^-$, and QSML$^-$ license a weakened version of reductio. We will return to this point in Section 5. Regarding our immediate objective, however, the correspondence matters because it suggests a natural way to increase classicality: simply add (+¬E.*) as primitive. Thus we let $\vdash_{\mathsf{SML}^*}$ be the system obtained by adding (+¬E.*) to SML.

SML* is indeed more classical than SML. In fact, it is fully classical all the way up. Yet it achieves this only at the cost of a complete multilateral and modal collapse: $\ominus A$ and $+\neg\Delta A$ are SML*-equivalent to simply $+\neg A$, and $\oplus A$ to $+A$ (these facts can all be directly read off from the model theory presented in the next paragraph). In this sense, it is a rather uninteresting logic. But it is highly relevant to our present concerns, since it is the *only* extension of SML that increases classicality.[17]

We start by providing a simple model theory. We merely extend Boolean valuations, first to $\mathcal{L}_\Delta$ by stipulating that $v(\Delta A) = v(A)$, and then to $\mathcal{L}_{\Delta,S}$ by $v(+A) = v(\oplus A) = v(A)$ and $v(\ominus A) = v(\neg A)$. Let $\vDash_{\mathsf{SML}^*}$ be the $\mathcal{L}_{\Delta,S}$ consequence relation defined over such models in the usual way.

**Theorem 4.10** (SML* Soundness) *Take arbitrary* $\Gamma \Rightarrow^1 \Psi \in \mathcal{L}^1_{\Delta,S}$. *If* $\Gamma \vdash_{\mathsf{SML}^*} \Psi$ *then* $\Gamma \vDash_{\mathsf{SML}^*} \Psi$.

***Proof*** By induction on the length of derivations. The steps for the rules of SML follow from SML's embedding into S5. The case of (+¬E.*) is trivial. □

**Theorem 4.11** (SML* Completeness) *Take arbitrary* $\Gamma \Rightarrow^1 \Psi \in \mathcal{L}^1_{\Delta,S}$. *If* $\Gamma \vDash_{\mathsf{SML}^*} \Psi$ *then* $\Gamma \vdash_{\mathsf{SML}^*} \Psi$.

***Proof*** Let $\Gamma$ be an $\vdash_{\mathsf{SML}^*}$-consistent set, and construct a maximally consistent superset $\Gamma'$ by the usual procedure. Consider the valuation $v$ such that, for propositional $p$, $v(p) = 1$ iff $+p \in \Gamma'$. We first show that $v(+A) = 1$ iff $+A \in \Gamma'$ for all $A \in \mathcal{L}^0_\Delta$, by induction over the complexity of $A$.

- Base case, $\wedge$ and $\Delta$ are trivial.
- $\neg$ is also trivial once we establish that $+\neg A \in \Gamma'$ iff $+A \notin \Gamma'$. Left-to-right is obvious because $\Gamma'$ is maximally consistent and $+A, +\neg A \vdash_{\mathsf{SML}^*} \bot$ (which was already derivable for SML). Right-to-left follows from the fact that $\Gamma', +A \vdash_{\mathsf{SML}^*} \bot$ entails $\Gamma' \vdash_{\mathsf{SML}^*} +\neg A$, by the following derivation.

$$\cfrac{\cfrac{\cfrac{[+\neg A]^1 \qquad [\ominus\neg A]^2}{\bot}\,(\text{Rejection})}{+A}\,(+\neg\text{E.}^*)^1 \\ \vdots \\ \bot}{+\neg A}\,(\text{SR}_2)^2$$

[17] Yet there are other ways to understand SML*. In particular, an anonymous reviewer points out that in the equivalent presentation where strong rejection is treated as a fourth primitive speech act, (+¬E.*) is not an operational rule but a coordination principle. In similar vein, we would add that in the presence of SML's rules, the addition of (+¬E.*) is interderivable with the removal of the restrictions on (Weak Inference). There are thus various equivalent ways to think about what is 'missing' from SML to get level 2 classicality.

Moreover, every $\oplus A \in \mathcal{L}^0_{\Delta,S}$ is $\vdash_{\mathsf{SML}^*}$-equivalent to $+A$, by (Assertion) and the following derivation.

$$\dfrac{\dfrac{\oplus A}{\ominus\neg A}\,(\ominus\neg\mathrm{I.}) \qquad [+\neg A]^1}{\dfrac{\bot}{+\,\mathrm{A}}\,(+\neg\mathrm{E.}^*)^1}\,(\text{Rejection})$$

Therefore $v(\oplus A) = 1$ iff $\oplus A \in \Gamma'$. Lastly, each $\ominus A$ is $\vdash_{\mathsf{SML}^*}$-equivalent to $\oplus\neg A$, by ($\oplus\neg$I.) and ($\oplus\neg$E.), and thus to $+\neg A$. Therefore $v(\ominus A) = 1$ iff $\ominus A \in \Gamma'$. So $v$ is a model for $\Gamma$. □

From this model theory, full classicality easily follows (though the details would be tedious to spell out completely).

**Theorem 4.12** SML$^*$ *is classical at every inferential level.*

***Proof (sketch)*** Take any $\Lambda \Rightarrow^n \Omega \in CMS^n$. If some $\mathcal{L}_{PL}$-instance is a counterexample to the CPL-validity of $U[\Lambda \Rightarrow^n \Omega]$, then the result of prefixing every sentence with $+$ is a counterexample to the SML$^*$-validity of $\Lambda \Rightarrow^n \Omega$. Conversely, given some counterexample of the SML$^*$-validity of $\Lambda \Rightarrow^n \Omega$, removing every $+$, $\oplus$ and $\Delta$, and replacing every $\ominus$ with $\neg$, results in a counterexample to the CPL-validity of $U[\Lambda \Rightarrow^n \Omega]$. □

Yet adopting SML$^*$ is not an option for the supervaluationist. The equivalences of $\oplus A$ to $+A$, and of $\ominus A$ and $+\neg\Delta A$ to $+\neg A$, are thoroughly unacceptable to her. While the first two of these undermine the multilateral notions of weak assertion and rejection, the main concern for the supervaluationist is that all three of them are incompatible with her theory of vagueness because they rule out the existence of borderline cases. Recall that in the supervaluationist language, '$A$ is borderline' is expressed by the sentence '$\neg\Delta A \wedge \neg\Delta\neg A$' (i.e. '$A$ is neither definitely true nor definitely false'). Yet according to SML$^*$, asserting this is incoherent: $+\neg\Delta A \wedge \neg\Delta\neg A \vdash_{\mathsf{SML}^*} \bot$. Other non-definite positions, such as e.g. the combination of $\oplus A$ and $\ominus A$, fare the same; $\oplus A, \ominus A \vdash_{\mathsf{SML}^*} \bot$. Borderline cases simply cannot be entertained in SML$^*$, because to express that something is *not definitely* the case, either by $+\neg\Delta A$, $\ominus A$, or $\oplus\neg A$, is just the same thing as expressing that it is *definitely not* the case: $+\Delta\neg A$. Yet such cases are central to the supervaluationist account of vagueness.

We might think, then, that we've simply overshot the mark. Perhaps, to improve classicality, the supervaluationist ought to search for something a bit stronger than SML, but not quite as strong as SML$^*$. Yet such a search would be futile: SML$^*$ is the weakest extension of SML which is more classical than it.

**Theorem 4.13** *Let $\mathcal{K}$ be a logic on $\mathcal{L}_{\Delta,S}$, such that $\mathcal{K}$ is (i) at least as strong as* SML *on level 1, and (ii) classical on level 2. Then $\mathcal{K}$ is at least as strong as* SML$^*$ *on level 1.*

***Proof*** We assume $\Gamma \vdash_{\mathsf{SML}^*} \Psi$ by some derivation $\mathcal{D}$ and show $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \Psi$ by induction on $\mathcal{D}$. We can assume without loss of generality that $\Gamma$ is finite, because SML$^*$ is compact, and (ii) entails that $\mathcal{K}$ allows weakening.

- Suppose the last rule applied was ($\ominus\neg$I.). By (ii), $\mathcal{K}$ must be transitive, in the sense that it validates the schema $\{(\varphi_1, ..., \varphi_n \Rightarrow^1 \varphi_{n+1}), (\varphi_{n+1} \Rightarrow^1 \varphi_{n+2})\} \Rightarrow^2 (\varphi_1, ..., \varphi_n \Rightarrow^1 \varphi_{n+2})$ for every $n$. Moreover, $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \oplus A$ by induction hypothesis, and $\vDash_{\mathcal{K}} \oplus A \Rightarrow^1 \ominus\neg A$ by (i). Therefore $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \ominus\neg A$.
  The other operational rules, as well as (Rejection) and (Assertion), are analogous.
- If it was ($SR_1$), we have by induction hypothesis that $\vDash_{\mathcal{K}} \Gamma, +A \Rightarrow^1 \bot$. By (ii), we must have $\vDash_{\mathcal{K}} (\varphi_1, ..., \varphi_n, +A \Rightarrow^1 \bot) \Rightarrow^2 (\varphi_1, ..., \varphi_n \Rightarrow^1 +\neg A)$ for each $n$. Hence $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 +\neg A$. By (i), we have $\vDash_{\mathcal{K}} +\neg A \Rightarrow^1 \ominus A$, so again by transitivity we get $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \ominus A$.
  ($SR_2$) is analogous.
- If it was (Weak Inference), the induction hypothesis means that (a) $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \oplus A$ and (b) $\vDash_{\mathcal{K}} \Gamma, +A \Rightarrow^1 +B$. By (i) we have $\vDash_{\mathcal{K}} \oplus A, +\neg A \Rightarrow^1 \bot$. By (ii), $\mathcal{K}$ validates Multilateral Reductio, so this means $\vDash_{\mathcal{K}} \oplus A \Rightarrow^1 +A$. So by (a) and transitivity, $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 +A$. By (b) and transitivity, $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 +B$. Finally, by (i) we must have $\vDash_{\mathcal{K}} +B \Rightarrow^1 \oplus B$, so applying transitivity once more yields $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \oplus B$.
- If it was ($+\neg$E.*), note that $\mathcal{K}$ must validate Multilateral Reductio by (ii), from which one application of transitivity suffices. □

It's worth noting the generality of this and the next three results. For recall (Subsection 3.1) that by a logic on $\mathcal{L}_{\Delta,S}$ we mean any consequence relation whatsoever on it; that is, any subset of $\mathcal{L}^1_{\Delta,S}$, whether defined proof-theoretically or otherwise.

In fact, $\mathsf{SML}^*$ is not just the minimal such logic. Suppose that we only consider systems which are closed under uniform substitutions. To be precise, in the case of logics on $\mathcal{L}_{\Delta,S}$, this means that given any valid level 1 inference, the result of uniformly substituting its propositional letters with sentences is also valid. The constraint is plausible, since such closure is generally considered a necessary requirement in order for logics to qualify as a logical at all [24, p. 49-51]. Logic, after all, should be *formal*, and failure to be closed under uniform substitution implies that validity is determined by something other than logical form alone.

Under this assumption, $\mathsf{SML}^*$ is the *only* more classical extension of $\mathsf{SML}$.

**Theorem 4.14** *Let $\mathcal{K}$ be a logic on $\mathcal{L}_{\Delta,S}$, such that $\mathcal{K}$ is (i) at least as strong as* $\mathsf{SML}$ *on level 1, (ii) classical on level 2, and (iii) closed under uniform substitution. Then $\mathcal{K}$ is precisely as strong as* $\mathsf{SML}^*$ *on level 1.*

***Proof*** $\mathcal{K}$ is at least as strong by the previous result. Suppose for contradiction that it is strictly stronger. Then there exists $\Gamma \Rightarrow^1 \Psi \in \mathcal{L}^1_{\Delta,S}$ such that $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \Psi$ but $\nvDash_{\mathsf{SML}^*} \Gamma \Rightarrow^1 \Psi$. Any element of $\mathcal{L}^0_{\Delta,S}$ is $\mathsf{SML}^*$- and hence $\mathcal{K}$-interderivable with a strongly asserted propositional sentence (in particular, that obtained by removing all $\Delta$'s, and turning $\oplus$ into $+$ and $\ominus$ into $+\neg$). So we can assume without loss of generality that $\Gamma \cup \{\Psi\} \subseteq \{+A | A \in \mathcal{L}^0_{PL}\}$.

Now let $\Lambda \Rightarrow^1 \Omega \in CMS^1$ be the result of uniformly substituting every propositional letter in $\Gamma \Rightarrow^1 \Psi$ with a distinct $A \in \mathcal{A}$. Then $\vDash_{\mathcal{K}} \Lambda \Rightarrow^1 \Omega$ by property (iii), because $\vDash_{\mathcal{K}} \Gamma \Rightarrow^1 \Psi$ and every instance of $\Lambda \Rightarrow^1 \Omega$ is a uniform substitution instance of $\Gamma \Rightarrow^1 \Psi$. However, $\nvDash_{\mathsf{SML}^*} \Gamma \Rightarrow^1 \Psi$ and therefore $\nvDash_{\mathsf{SML}^*} \Lambda \Rightarrow^1 \Omega$. Hence $\nvDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^1 \Omega]$ by Theorem 4.12. But then we have both $\vDash_{\mathcal{K}} \Lambda \Rightarrow^1 \Omega$ and

$\nvDash_{\mathsf{CPL}} U[\Lambda \Rightarrow^1 \Omega]$, so $\mathcal{K}$ is not subclassical at level 1, which contradicts property (ii) via Observation 3.18. □

As far as bolstering SML to improve classicality is concerned, $\mathsf{SML}^*$ is really the only option.[18] And as discussed above, it is very poor one at that, especially for supervaluationist purposes.

What if we start from something weaker than SML, in particular, from $\mathsf{SML}^-$? Again, the outcomes will be precisely the same, in particular because adding $(+\neg\text{E.}^*)$ to $\mathsf{SML}^-$ allows it to recover the removed $(\oplus\Delta)$ rules, and so we get exactly $\mathsf{SML}^*$ again. This then implies the following strengthenings of our last two results.

**Theorem 4.15** *Let $\mathcal{K}$ be a logic on $\mathcal{L}_{\Delta,S}$, such that $\mathcal{K}$ is (i) at least as strong as* $\mathsf{SML}^-$ *on level 1, and (ii) classical on level 2. Then $\mathcal{K}$ is at least as strong as* $\mathsf{SML}^*$ *on level 1.*

***Proof*** First note that $\mathcal{K}$ must be at least as strong as $\mathsf{SML}^{-*}$, the result of adding $(+\neg\text{E.}^*)$ to $\mathsf{SML}^-$, by the very same argument as the proof of Theorem 4.13.

Moreover, $\mathsf{SML}^{-*}$ is precisely as strong as $\mathsf{SML}^*$. For the only difference is that the former lacks the $(\oplus\Delta\text{I.})$ and $(\oplus\Delta\text{E.})$ rules as primitive, but they are derivable in it nonetheless, via the following respective proofs.[19]

$$
\dfrac{\dfrac{+A}{+\Delta A}\,(+\Delta\text{I.})}{\oplus\Delta A}\,(\text{Assertion})
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\oplus\Delta A}{\ominus\neg\Delta A}\,(\ominus\neg\text{I.}) \qquad [+\neg\Delta A]^1}{\bot}\,(\text{Rejection})}{+\Delta A}\,(+\neg\text{E.}^*)^1}{+A}\,(+\Delta\text{E.})
$$

□

**Theorem 4.16** *Let $\mathcal{K}$ be a logic on $\mathcal{L}_{\Delta,S}$, such that $\mathcal{K}$ is (i) at least as strong as* $\mathsf{SML}^-$ *on level 1, (ii) classical on level 2, and (iii) closed under uniform substitution. Then $\mathcal{K}$ is precisely as strong as* $\mathsf{SML}^*$ *on level 1.*

***Proof*** As that of Theorem 4.14. □

Should we then take something even weaker than $\mathsf{SML}^-$ as our starting point, from which to work towards level 2 classicality? This is unlikely. For it seems to us that, as per [16], each of the remaining rules of the calculus is soundly justified from the supervaluationists perspective. $\mathsf{SML}^-$ is already minimal, in the sense that none of its operational rules codify anything beyond the core meaning of the relevant connective, and similarly each of the coordination principles establishes no more than the basic supervaluationist interpretation of the force-markers. In light of said interpretation, there does not appear to be any room for further weakening of the system. Moreover,

[18] We have only shown that any logic meeting conditions (i) - (iii) of the last result must agree with $\mathsf{SML}^*$ *on level 1*. However, on our Definition 3.2 of validity, all the higher level validities are fully determined by those at level 1. So there cannot be two logics on the same language which agree on some higher level despite agreeing completely on level 1.

[19] As [16] also notes, this derivation for $(\oplus\Delta\text{I.})$ is already available in $\mathsf{SML}^-$ itself. It is the recovery of $(\oplus\Delta\text{E.})$ that requires $(+\neg\text{E.}^*)$.

such weakening wouldn't get at the heart of the problem, which in its simplest form boils down to this: level 2 classicality requires licensing (+¬E.*), but (+¬E.*) encapsulates the principle that if something cannot be superfalse then it must be supertrue, which means there are no borderline cases.

Thus we conclude that, as far as the basic propositional setting goes, multilateral supervaluationism leads to supervaluationist logic being classical at level 1, but strictly subclassical beyond.

### 4.3 QSML$^-$

We now turn our attention back over to the quantified setting, and investigate the classicality status of QSML$^-$. We start by establishing its subclassicality at all levels, the path to which is the same as it was in the cases of SML and SML$^-$.

**Theorem 4.17** *Take arbitrary* $\Gamma \Rightarrow^1 A \in \mathcal{L}^1_=$. *Then* $\vDash_{\mathsf{CFOL}} \Gamma \Rightarrow^1 A$ *iff* $\vDash_{\mathsf{QSML}^-} \{+\gamma \mid \gamma \in \Gamma\} \Rightarrow^1 +A$.

***Proof*** Recall the model theory of QSML$^-$ (Definition 2.4). As far as $\Delta$-free sentences are concerned, a QSML$^-$ model is essentially just a set $\mathcal{W}$ of classical first-order models, and a strong assertion is satisfied just in case the sentence is classically satisfied by every model in $\mathcal{W}$.

Thus if $\nvDash_{\mathsf{CFOL}} \Gamma \Rightarrow^1 A$ by some countermodel $M$, then $\nvDash_{\mathsf{QSML}^-} \{+\gamma \mid \gamma \in \Gamma\} \Rightarrow^1 +A$ by countermodel $\{M\}$. Conversely, if $\nvDash_{\mathsf{QSML}^-} \{+\gamma \mid \gamma \in \Gamma\} \Rightarrow^1 +A$ by some countermodel $\mathcal{W}$, then there must be a particular $w \in \mathcal{W}$ where all of $\Gamma$ but not $A$ is satisfied, which means $w$ is our countermodel showing $\nvDash_{\mathsf{CFOL}} \Gamma \Rightarrow^1 A$. □

**Theorem 4.18** QSML$^-$ *is subclassical at every level.*

***Proof*** From Theorem 4.17, as Theorem 4.3 from 4.1. □

Yet at this point, the similarities between the two situations end. SML and SML$^-$ turned out to be metainferentially weaker than classical logic, as is within the realm of expectations for a supveraluationist logic. Yet supervaluationism is also known to produce classicality on the inferential level. And indeed, SML and SML$^-$ matched classical logic's strength perfectly on level 1. Not so for QSML$^-$: it is strictly weaker than classical logic *already on level 1*. This is due to the failure of the following schema, expressing the substitution of identicals.[20]

$$\varnothing \Rightarrow^1 +[\exists x(x = a \wedge A)] \rightarrow [\forall x(x = a \rightarrow A)] \quad \text{(SI)}$$

**Theorem 4.19** QSML$^-$ *invalidates* (SI).

[20] A more direct and perhaps more common way of expressing the substitution of identicals would be via something like $\varnothing \Rightarrow^1 +\forall x, y((x = y \wedge A(x)) \rightarrow A(y))$. Yet this is not a schema under our definitions, for the machinery specifying that both occurrences of $A$ be applied to different open variables is not available to them (cf. our discussion of expressive limitations in Subsection 3.4). This is no real disadvantage, however, since our (SI) does the same job.

***Proof*** Consider the instance where $\mu(A) = \Delta Fx$. There are two types of $\mathsf{QSML}^-$ countermodels to it. We will discuss them both, since they each offer useful insights.

First, consider $\mathcal{M}_1$ such that: $\mathcal{W}_1 = \{w, v\} = R_1(Fa, v) = R_1(Fb, v)$, $a^{\mathcal{I}_1(w)} \neq b^{\mathcal{I}_1(w)}$ with $a^{\mathcal{I}_1(w)} \in F^{\mathcal{I}_1(w)}$ but $b^{\mathcal{I}_1(w)} \notin F^{\mathcal{I}_1(w)}$, whilst $a^{\mathcal{I}(v)} = b^{\mathcal{I}_1(v)} \in F^{\mathcal{I}_1(v)}$.

Then $v \vDash a = a \wedge \Delta Fa$, hence $v \vDash \exists x(x = a \wedge \Delta Fx)$. However, we also have $v \vDash b = a$, yet since $w \in R_1(Fb, v)$ and $w \nvDash Fb$, we have $v \nvDash \Delta Fb$. So $v \nvDash b = a \to \Delta Fb$ and hence $v \nvDash \forall x(x = a \to \Delta Fx)$. Therefore $v \nvDash [\exists x(x = a \wedge \Delta Fx)] \to [\forall x(x = a \to \Delta Fx)]$, and thus $\mathcal{M}_1 \not\Vdash +[\exists x(x = a \wedge \Delta Fx)] \to [\forall x(x = a \to \Delta Fx)]$.

Secondly, define $\mathcal{M}_2$ such that $\mathcal{W}_2 = \{w, v\}$, $a^{\mathcal{I}_2(w)} = b^{\mathcal{I}_2(w)} \notin F^{\mathcal{I}_2(w)}$ and $a^{\mathcal{I}_2(v)} = b^{\mathcal{I}_2(v)} \in F^{\mathcal{I}_2(v)}$. Suppose furthermore that $R_2(Fa, v) = \{v\}$, but $R_2(Fb, v) = \{w, v\}$.

Then $v \vDash \Delta Fa$ and so $v \vDash \exists x(x = a \wedge \Delta Fx)$. Yet $v \nvDash \Delta Fb$ even though $v \vDash b = a$, therefore $v \nvDash \forall x(x = a \to \Delta Fx)$. Thus we again have $v \nvDash [\exists x(x = a \wedge \Delta Fx)] \to [\forall x(x = a \to \Delta Fx)]$, and so $\mathcal{M}_2 \not\Vdash +[\exists x(x = a \wedge \Delta Fx)] \to [\forall x(x = a \to \Delta Fx)]$. □

In Section 5 we will consider the failure of (SI) from a philosophical angle, in particular its implications for the underlying supervaluationism. For now, we proceed with a technical investigation of its causes and consequences. First of all, since the unilateralization of (SI) is clearly classically valid, we get the following (with the usual help from Observation 3.18).

**Corollary 4.20** $\mathsf{QSML}^-$ *is not superclassical at any inferential level.*

Thus, combined with the earlier result:

**Corollary 4.21** $\mathsf{QSML}^-$ *is strictly subclassical at every inferential level.*

We did not explicitly include an inferential level 0, corresponding to the logical theorems, in our definitions. But it is worth noting $\mathsf{QSML}^-$ would fail to be classical even there, since (SI) has empty premises.

As in the case of $\mathsf{SML}$ and $\mathsf{SML}^-$, the question arises whether we can do any better than Corollary 4.21. It is all the more pressing now, because failure of classicality at the *first* inferential level is not a known or expected product of supervaluationism. Moreover, the source of this strict subclassicality is ostensibly very different from that at level 2: it does not seem to have much to do with vagueness or borderline cases at all. Instead, it apparently arises only because of some idiosyncrasies in the way $\mathsf{QSML}^-$ treats identity, regarding which the supervaluationist might have room for modifying maneuvers. Thus, whilst it is easy to see that reaching for level 2 classicality will end in $\mathsf{SML}^*$-like collapse also in the first-order case, there might still be a natural and appealing way to obtain it at level 1.

The obvious starting point lies with the rules for identity. How should we amend them? In the proof of Theorem 4.19, countermodel $\mathcal{M}_1$ exploits the fact that, in $\mathsf{QSML}^-$ models, identity need not be rigid across precisifications. Thus, forcing rigidity through the proof rules would avoid at least one of the counterexamples to (SI), and thereby go some way towards raised classicality.[21] Fortunately, an elegant way to do this is already known. Schlöder [30] sets up the following rules for rigid identity, denoted by '≡', in multilateral natural deduction calculi.[22]

$$(+\equiv\text{I.})\ \frac{\begin{array}{cc}[+Fa] & [+\neg Fa]\\ \vdots & \vdots \\ \oplus Fb & \ominus Fb\end{array}}{+a \equiv b}$$ where $F$ is a predicate symbol not occurring in undischarged leafs

$$(+\equiv\text{E.}_1)\ \frac{+a \equiv b \quad +Fa}{\oplus Fb}$$ where $F$ is a predicate symbol $$(+\equiv\text{E.}_2)\ \frac{+a \equiv b \quad +\neg Fa}{\ominus Fb}$$ where $F$ is a predicate symbol

Schlöder shows that, given the same coordination principles as those of $\mathsf{QSML}^-$, identity as defined by these rules is rigid, in the sense that e.g. $\oplus a \equiv b \vdash +a \equiv b$. This would certainly be enough to rule out $\mathcal{M}_1$. However, plugging in these rules as they stand would not suffice to bring about classicality. For note that $\mathcal{M}_2$, the other countermodel to (SI), does not rely on non-rigidity across precisifications. Unlike in $\mathcal{M}_1$, all points in $\mathcal{M}_2$ agree on the identity between $a$ and $b$. The same instance of (SI) nevertheless fails to hold, but this time it is because $R_2(Fa, v) \neq R_2(Fb, v)$. Conversely, this possibility was not utilized in $\mathcal{M}_1$, where $R_1(Fa, v) = R_1(Fb, v)$.[23] Thus there are (at least) two distinct and independent causes underlying (SI)'s invalidity. Moving to Schlöder's ≡-rules would only solve one of them, so we need to do something else to avoid the other.

In particular, the problem with $\mathcal{M}_2$ seems to be that $R_2(Fa, v) \neq R_2(Fb, v)$, despite the fact that a and b are identical at $v$. On the proof-theoretic side, this corresponds to the fact that $(+=\text{E.}_1)$ and $(+=\text{E.}_2)$ can only be directly applied to sentences of the form $Fa$, where $F$ is some predicate. Generalized versions—which apply, in particular, to sentences of the form $\Delta Fa$—would be derivable from there, but only with the help of the $(\oplus\Delta\text{I.})$ and $(\oplus\Delta\text{E.})$ rules which have been given up in $\mathsf{QSML}^-$. The same point applies to $(+\equiv\text{E.}_1)$ and $(+\equiv\text{E.}_2)$. Abandoning the $\oplus\Delta$ rules was required to accommodate higher-order vagueness, and so they should not be reintroduced. But

21 Though mind that forcing complete rigidity is strictly speaking overkill, as far as ruling out countermodels like $\mathcal{M}_1$ is concerned. For upon closer inspection, the problem with $\mathcal{M}_1$ is not so much that $w$ and $v$ disagree on whether $a = b$, but more specifically that they do so whilst being connected by $R_1$ in a particular way. Thus avoiding $\mathcal{M}_1$ does not require total rigidity, in the sense that $a^{I(v)} = b^{I(v)}$ for some $v$ entails $a^{I(w)} = b^{I(w)}$ for *all* $w$. A weaker condition suffices; e.g. such that $a^{I(v)} = b^{I(v)}$ for some $v$ entails $a^{I(w)} = b^{I(w)}$ for all those $w$ such that $w \in R(v, A)$ for some $A$ which features $a$. Imposing such a condition might even be enough to rule out all countermodels to level 1 classicality which exploit non-rigidity. But simply enforcing full rigidity looks to be the most natural way to ensure this more complex condition.

22 The second subderivation in the (+≡I.) rule is superfluous and could be removed, but then the ruleset is no longer obviously harmonious. To achieve harmony was one of Schlöder's goals. Although we are not concerned with harmony here, we repeat Schlöder's versions of the rules.

23 Another interesting difference between the two is that $\mathcal{M}_1$ but not $\mathcal{M}_2$ also functions as a countermodel to the rule version of (SI): $+\exists x(x = a \wedge A) \Rightarrow^1 +\forall x(x = a \rightarrow A)$. For in $\mathcal{M}_1$, if we further specify that $R_1(Fa, w) = \{w\}$, it would follow that all points in the model satisfy $\exists x(x = a \wedge \Delta Fx)$. In $\mathcal{M}_2$ this would not work, since the point there is precisely that $a^{\mathcal{I}_2(w)} = b^{\mathcal{I}_2(w)} \notin F^{\mathcal{I}_2(w)}$. Proof-theoretically, the rule version being slightly weaker than (SI) itself is explained by the restrictions on the derivable $(+\rightarrow\text{I.})$ rule (see Section 2).

what we might do is simply make general versions of the E-rules primitive. Combined with Schlöder's approach to rigidity, which we retain to address the phenomenon of models like $\mathcal{M}_1$, this gives us the following set.[24]

$$(+=\mathrm{I}.^{\#})\ \frac{\begin{matrix}[+Fa] \\ \vdots \\ \oplus Fb\end{matrix} \qquad \begin{matrix}[+\neg Fa] \\ \vdots \\ \ominus Fb\end{matrix}}{+a=b}$$ where $F$ is a predicate symbol not occurring in undischarged leafs

$$(+=\mathrm{E}.^{\#}_1)\ \frac{+a_1=b_1,\ ...,\ +a_n=b_n \qquad +A[a_i/x_i]_{i\leqslant n}}{\oplus A[b_i/x_i]_{i\leqslant n}}$$

$$(+=\mathrm{E}.^{\#}_2)\ \frac{+a_1=b_1,\ ...,\ +a_n=b_n \qquad +\neg A[a_i/x_i]_{i\leqslant n}}{\ominus A[b_i/x_i]_{i\leqslant n}}$$

Let $\vdash_{\mathsf{QSML}^{\#}}$ be the calculus obtained by adding these to QSML$^{-}$, and removing (+ =I.), (+ =E.$_1$) and (+ =E.$_2$). The first thing to check is that such adjustments correspond precisely to ruling out the two types of countermodel to (SI). So let a QSML$^{\#}$ model be a QSML$^{-}$ model such that:

- (=-Rig) For all terms $a$ and $b$, and for all points $w, v$: $a^{\mathcal{I}(w)} = b^{\mathcal{I}(w)}$ iff $a^{\mathcal{I}(v)} = b^{\mathcal{I}(v)}$.
- ($\Delta$-Sub) For all points $v$, sentences $A$ and terms $a_1, ..., a_n$ and $b_1, ..., b_n$: if $a_i^{\mathcal{I}(v)} = b_i^{\mathcal{I}(v)}$ for all $i \leqslant n$, then $R(v, A[a_i/x_i]_{i\leqslant n}) = R(v, A[b_i/x_i]_{i\leqslant n})$.

We write $\vDash_{\mathsf{QSML}^{\#}}$ for the consequence relation on these models. We will make repeated use of the following technical lemma.

**Lemma 4.22** *Take arbitrary QSML$^{\#}$ model $\mathcal{M}$, and terms $a_1, ..., a_n$ and $b_1, ..., b_n$. If $a_i^{\mathcal{I}(w)} = b_i^{\mathcal{I}(w)}$ for some point $w$, then for any (potentially different) point $v$: $\mathcal{M}, v \vDash A[a_i/x_i]_{i\leqslant n}$ iff $\mathcal{M}, v \vDash A[b_i/x_i]_{i\leqslant n}$ for all $A$.*

***Proof*** By induction on $A$. Base cases follow immediately from =-Rig. The step for $\Delta$ follows from $\Delta$-Sub. The other induction steps are trivial as well. □

We proceed with soundness and completeness.

**Theorem 4.23** (QSML$^{\#}$ Soundness) *Take arbitrary $\Gamma \Rightarrow^1 \Psi \in \mathcal{L}^1_{\Delta,=,S}$. If $\Gamma \vdash_{\mathsf{QSML}^{\#}} \Psi$ then $\Gamma \vDash_{\mathsf{QSML}^{\#}} \Psi$.*

[24] Again, this may technically be slight overkill. As mentioned in footnote 21, we don't seem to need complete rigidity on the model-theoretic side. Hence we probably don't have to enforce complete rigidity in the proof theory, as these Schlöder-style rules do, in order to obtain the desired level 1 classicality. In particular, one might suspect that a system just like QSML$^{-}$ except with the (+ =E.) rules generalized (as we have generalized Schlöder's rules) would be sound and complete for a model theory with something like the quasi-rigidity constraint mentioned in footnote 21, as well as being level 1 classical. If so, such a system could play essentially the same role that we will afford to QSML$^{\#}$ within the remainder of the paper. Be that as it may, we have opted for the QSML$^{\#}$ route in order to work with the simpler, full rigidity condition, which ensures the desired behaviour.

***Proof*** By induction on the derivation. The steps for the rules of QSML$^{-}$ follow from QSML$^{-}$'s soundness.

- For ($+ =$I.$^{\#}$), assume $\Gamma, +Fa \vDash_{\mathsf{QSML}^{\#}} \oplus Fb$ and $\Gamma', +\neg Fa \vDash_{\mathsf{QSML}^{\#}} \ominus Fb$, with $F$ not occurring in $\Gamma$. Suppose for *reductio* that $\Gamma \nvDash_{\mathsf{QSML}^{\#}} +a = b$. So there exists $\mathcal{M} = \langle \mathcal{W}, \mathcal{D}, \mathcal{I}, R\rangle$ such that $\mathcal{M} \Vdash \Gamma$ but $\mathcal{M} \nVdash +a = b$. By =-Rig, this means $a^{\mathcal{I}(v)} \neq b^{\mathcal{I}(v)}$ on all $v \in \mathcal{W}$. Then there is a model $\mathcal{M}' = \langle \mathcal{W}, \mathcal{D}, \mathcal{I}', R\rangle$, where $\mathcal{I}'$ is exactly like $\mathcal{I}$ except $F^{\mathcal{I}'(v)} = (F^{\mathcal{I}(v)} \cup \{a^{\mathcal{I}(v)}\}) - \{b^{\mathcal{I}(v)}\}$ for all $v$. Then $\mathcal{M}' \Vdash \Gamma$ and $\mathcal{M}' \Vdash +Fa$ but $\mathcal{M}' \nVdash \oplus Fb$, contradicting the induction hypothesis.
- ($+ =$E.$^{\#}_{1}$) and ($+ =$E.$^{\#}_{2}$) follow from the fact that if $a_i^{\mathcal{I}(v)} = b_i^{\mathcal{I}(v)}$, then $v \vDash A[a_i/x_i]_{i \leqslant n}$ iff $v \vDash A[b_i/x_i]_{i \leqslant n}$. This is in turn a special case of Lemma 4.22 where both points are identical. $\square$

**Theorem 4.24** (QSML$^{\#}$ Completeness) *Take arbitrary* $\Gamma \Rightarrow^{1} \Psi \in \mathcal{L}^{1}_{\Delta,=,S}$. *If* $\Gamma \vDash_{\mathsf{QSML}^{\#}} \Psi$ *then* $\Gamma \vdash_{\mathsf{QSML}^{\#}} \Psi$.

***Proof*** By a modification of completeness proofs found in e.g. [15] and [30], which are in turn variations on the standard model existence approach, so we can skip some familiar details. We adopt in particular the notion of s-consistency. We write $\Gamma \vdash^{s} \varphi$ to mean $\Gamma \vdash_{\mathsf{QSML}^{\#}} \varphi$ can be proved via a derivation which only uses strongly asserted premises and does not apply the $\Delta$-introduction rules. A set of formulae $\Gamma \subseteq \mathcal{L}^{0}_{\Delta,=,S}$ is s-consistent when $\Gamma \nvdash^{s} \bot$.

Let $\Gamma \subseteq \mathcal{L}^{0}_{\Delta,=,S}$ be any QSML$^{\#}$-consistent set of formulae. We first close $\Gamma$ under $\vdash_{\mathsf{QSML}^{\#}}$, and then take a Henkin extension $\Gamma^{H}$ of this closure.[25] Now take an enumeration of all ordered pairs of closed terms from the extended signature. Define a sequence with $\Gamma_0 = \Gamma^{H}$ and, where $\langle a, b\rangle$ is the $n$th pair in the enumeration, $\Gamma_{n+1} = \Gamma_n \cup \{+a = b\}$ if this is QSML$^{\#}$-consistent, $\Gamma_{n+1} = \Gamma_n \cup \{+\neg a_n = b_n\}$ otherwise. Note that (precisely) one of these two must always be consistent, provided that $\Gamma_n$ was, in virtue of the fact that $\oplus a = b \vdash_{\mathsf{QSML}^{\#}} +a = b$. Take the limit $\bigcup_{n\in\omega} \Gamma_n$, and let $\Gamma^{=}$ be its closure under $\vdash_{\mathsf{QSML}^{\#}}$. Finally, let $\mathcal{E}$ be the set of all maximally s-consistent extensions of $\Gamma^{=}$.

Our model is $\mathcal{M} = \langle \mathcal{E}, \mathcal{D}, \mathcal{I}, R\rangle$, with $\mathcal{D}(v) = \{[a] \mid \text{a is a closed term}\}$, where $[a]$ is the equivalence class under the relation $a \sim b$ iff $+a = b \in \Gamma^{=}$. Note that $\sim$ must be an equivalence relation, because the following rules can be derived as in [30].

$$\text{(Reflexivity)}\ \frac{}{+a = a} \qquad \text{(Symmetry)}\ \frac{+a = b}{+b = a} \qquad \text{(Transitivity)}\ \frac{+a = b \qquad +b = c}{+a = c}$$

$\mathcal{I}$ is defined as usual, and we set $R(v, A) = \{w \in \mathcal{E} \mid \text{if } +\Delta A \in v \text{ then } +A \in w\}$.

For $\mathcal{M}$ to be a QSML$^{\#}$ model, we must show that it meets the constraints =-Rig and $\Delta$-Sub. The former holds because all points are extensions of $\Gamma^{=}$, which we have constructed to include a decision on every potential identity. For the latter, suppose $a_i^{\mathcal{I}(v)} = b_i^{\mathcal{I}(v)}$ for $i \leqslant n$. Then $+a_i = b_i \in \Gamma^{=} \subseteq v \cap w$ for any $w$. By ($+ =$E.$^{\#}_{1}$)

25 In the multilateral context, a set of formulae is Henkin when for every sentence $\forall x A$ there is a constant $c_A$ such that the set contains $+(\neg\forall x A) \to \neg A[c_A/x]$. This is the usual notion of being Henkin, except for the insertion of a '+'. Accordingly, the standard constructions, with +'s inserted in the obvious places, suffice to ensure that every consistent set of formulae has a consistent Henkin extension.

this means $+\Delta A[a_i/x_i]_{i\leqslant n} \in v$ iff $+\Delta A[b_i/x_i]_{i\leqslant n} \in v$, and $+A[a_i/x_i]_{i\leqslant n} \in w$ iff $+A[b_i/x_i]_{i\leqslant n} \in w$. Therefore $w \in R(v, A[a_i/x_i]_{i\leqslant n})$ iff $w \in R(v, A[b_i/x_i]_{i\leqslant n})$.

Induction on $A$ then shows our truth lemma: $+A \in v$ iff $\mathcal{M}, v \vDash A$, for all $v \in \mathcal{E}$. We only show the base case for $=$ and the induction step for $\forall$, since the rest is as in e.g. the SML$^-$ completeness proof in [16].

(=) Observe simply that $+a = b \in v$ iff $+a = b \in \Gamma^=$ iff $a^{\mathcal{I}(v)} = [a] = [b] = b^{\mathcal{I}(v)}$ iff $\mathcal{M}, v \vDash a = b$.

($\forall$) For left-to-right, suppose $\mathcal{M}, v \nvDash \forall x A$. Then by definition of satisfaction $\mathcal{M}, v \vDash \neg A[t/x]$ for some closed term $t$, hence $+\neg A[t/x] \in v$ by induction hypothesis, hence $+\forall x A \notin v$ by s-consistency of $v$.
For right-to-left, suppose $+\forall x A \notin v$. Then $+\neg\forall x A \in v$ by maximal s-consistency. Since $v$ extends $\Gamma^H$, $v$ is Henkin and hence $+(\neg\forall x A) \to \neg A[c_A/x] \in v$ for some constant $c_A$. By the derivable (+→E.) rule [15, pp. 513-514], $+\neg A[c_A/x] \in v$. By induction hypothesis, $\mathcal{M}, v \vDash \neg A[c_A/x]$. Thus $\mathcal{M}, v \nvDash \forall x A$.

This completes the truth lemma.

It remains to show that $\mathcal{M}$ is a model for $\Gamma$. For any $+A \in \Gamma$, since $\Gamma \subseteq \Gamma^= \subseteq \bigcap\mathcal{E}$, the truth lemma tells us that $\mathcal{M} \Vdash +A$. For $\oplus A \in \Gamma$, by consistency $+\neg A \notin \Gamma^=$, so there is at least one $v \in \mathcal{E}$ where $+A \in v$, and thus by the truth lemma $\mathcal{M} \Vdash \oplus A$.[26] Similarly for $\ominus A \in \Gamma$. Having covered all three force-markers, we conclude that $\mathcal{M} \Vdash \Gamma$. Since $\Gamma$ was arbitrary, it follows that every consistent set has a model, as desired. □

It remains to verify that all of this actually suffices to validate (SI), and more generally to achieve the desired classicality. To start, we must check that we have not overshot the mark and inadvertently gone beyond classical strength. Fortunately, this is easy to see upon inspection of our QSML$^-$ subclassicality proof.

**Theorem 4.25** QSML$^\#$ *is subclassical at every level.*

***Proof*** As that of Theorem 4.18. □

The hard part is showing that we have nevertheless done enough to get superclassicality at level 1.

**Theorem 4.26** QSML$^\#$ *is superclassical at level 1.*

***Proof*** Take arbitrary $\Lambda \Rightarrow^1 \Omega \in CQS^1$. We suppose for contraposition that $\nvDash_{\mathsf{QSML}^\#} \Lambda \Rightarrow^1 \Omega$, and show that $\nvDash_{\mathsf{CFOL}} U[\Lambda \Rightarrow^1 \Omega]$. We may assume without loss of generality that no signed formula variables appear in $\Lambda \Rightarrow^1 \Omega$, since other cases can be reduced to this as in the superclassicality proof of SML (Theorem 4.4). It will help to add some notation: if $\varphi$ is a signed formula of the form $+A$, let us write $\varphi^C$ for its unsigned 'content' $A$, and extend this to sets of formulae as usual.

We know by assumption that there is some $\mu : \mathcal{A} \to \mathcal{L}^0_{=,\Delta}$ and QSML$^\#$-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{D}, \mathcal{I}, R\rangle$ such that $\mathcal{M} \Vdash \mu[\Lambda]$ but $\mathcal{M} \nVdash \mu[\Omega]$. In the new notation, it follows that

[26] At this point it is crucial that $\mathcal{E}$ contains the maximally *s-consistent*, rather than simply consistent, extensions. For alongside $\oplus A$, $\Gamma$ might consistently contain formulae like $\ominus A$ or $+\neg\Delta A$, in which case there are no (maximally) consistent extensions which contain $+A$. There must be a maximally s-consistent one, however, because if $\Gamma, +A \vdash^s \bot$ then $\Gamma, \oplus A \vdash_{\mathsf{QSML}^\#} \bot$ by (Weak Inference).

there is some particular point $v$ such that $\mathcal{M}, v \vDash \mu[\Lambda]^C$, and yet $\mathcal{M}, v \nvDash \mu[\Omega]^C$. We will use $\mu$ and $v$ to construct our classical counterexample to the schema $U[\Lambda \Rightarrow^1 \Omega]$.

To start, we must pick an instance of the schema to counterexemplify, by defining $\sigma : \mathcal{A} \to \mathcal{L}^0_=$. Recall that for a given $A_i \in \mathcal{A}$, $\mu(A_i)$ may feature some open variables, as long as they are all bound in each occurrence of $\mu(A_i)$ in the schema. Let $\langle x_1, ..., x_n \rangle$ be (some ordering of) the open variables in $\mu(A_i)$.

$$\sigma(A_i) := \begin{cases} \top & \text{if } \mu(A_i) \text{ is closed and } \mathcal{M}, v \vDash \mu(A_i) \\ \bot & \text{if } \mu(A_i) \text{ is closed and } \mathcal{M}, v \nvDash \mu(A_i) \\ P_i(x_1, ..., x_n) \text{ for fresh } n\text{-ary } P_i & \text{if } \mu(A_i) \text{ has } n \text{ many open variables.} \end{cases}$$

Where '$\top$' and '$\bot$' can be any classical tautology and contradiction, respectively. Next we build our classical countermodel $\mathcal{M}' = \langle \mathcal{D}', \mathcal{I}' \rangle$ to this instance. Our domain $\mathcal{D}'$ is simply $\mathcal{D}(v)$ from $\mathcal{M}$. On terms and function symbols, $\mathcal{I}'$ is also defined as $\mathcal{I}(v)$. Moreover, for predicate symbols $P$ occurring in $\Lambda \Rightarrow \Omega$, we set $P^{\mathcal{I}'} = P^{\mathcal{I}(v)}$. That leaves the fresh $P_i$, which correspond to $A_i \in \mathcal{A}$ with $\langle x_1, ..., x_n \rangle$ open in $\mu(A_i)$. For them, we set $\langle d_1, ..., d_n \rangle \in P_i^{\mathcal{I}'}$ iff $\mathcal{M}, v \vDash \mu(A_i)[a_j/x_j]_{j \leqslant n}$, where $a_1, ..., a_n$ is some sequence of closed terms such that $a_j^{I(v)} = d_j$ for all $j \leqslant n$. Note that it does not matter which such sequence we choose, thanks to Lemma 4.22.

Now we claim the following: let $B$ be any element of the closure of $\mathcal{A} \cup At^{\mathcal{L}_=}$ under $\neg$, $\wedge$ and $\forall$. Let $\langle x_1, ..., x_n \rangle$ be the open variables in $B$, should there be any, and let $\langle a_1, ..., a_n \rangle$ be any sequence of closed terms. Then $\mathcal{M}' \vDash \sigma[B][a_j/x_j]_{j \leqslant n}$ iff $\mathcal{M}, v \vDash \mu[B][a_j/x_j]_{j \leqslant n}$. This can be shown by a straightforward induction on $B$, where the base case $B = A_i \in \mathcal{A}$ relies on Lemma 4.22. The other base cases follow immediately from our constructions, and the induction steps are trivial.

Then we are nearly there. For all elements of $\Lambda^C$, and $\Omega^C$, are cases of such $B$'s. Therefore, by the above claim and our initial assumptions about $v$, it follows that $\mathcal{M}' \vDash \sigma[\Lambda^C]$ but $\mathcal{M}' \nvDash \sigma[\Omega^C]$. But this is just to say that $\mathcal{M}' \vDash \sigma[U[\Lambda]]$ and $\mathcal{M}' \nvDash \sigma[U[\Omega]]$. Hence $\mathcal{M}' \nvDash \sigma[U[\Lambda \Rightarrow^1 \Omega]]$ and so $\nvDash_{\mathsf{CFOL}} U[\Lambda \Rightarrow^1 \Omega]$, as desired. □

Observe that the points where this proof would fail in case of QSML$^-$ are nicely marked by the appeals to Lemma 4.22.

On the other hand, we have not done anything to validate Multilateral Reductio, and so we are still weaker than classical at level 2.

**Theorem 4.27** QSML$^\#$ *is not superclassical at any level* $n > 1$.

***Proof*** As that of Theorem 4.5. □

Summarizing once more:

**Corollary 4.28** QSML$^\#$ *is classical at level 1, and strictly subclassical beyond.*

Given the collapses required to achieve level 2 classicality in the mutlilateral setting, as we have already seen in the case of SML and its variants, this seems like the optimal outcome. However, in the first-order setting, there is another concern.

### 4.4 One Step is Too Much

Part of the motivation for the development of $\mathsf{QSML}^-$ in [16] is to show how multilateral supervaluationism can account for higher-order vagueness also in its first-order variants. In particular, the authors show how the system avoids an argument due to Zardini [36], the success of which would be problematic.

Let $a_1$ to $a_n$ be a Sorites series for some vague predicate $P$, say baldness, so that $a_1$ is (definitely) not bald and $a_n$ is (definitely) bald. The supervaluationist maintains that, somewhere in between, there are borderline cases, which are neither definitely bald nor definitely not bald.

$$+\,\exists m(\neg\Delta Pa_m \wedge \neg\Delta\neg Pa_m) \qquad \text{(0-Borderline)}$$

Since vagueness persists at higher orders, there must also be borderline cases of definite baldness (such that it is neither definitely the case that they are definitely bald, nor definitely the case that they are not definitely bald), of definite definite baldness, and so on. Letting $\Delta_k$ be short for $k$-many $\Delta$'s:

$$+\,\exists m(\neg\Delta\Delta_k Pa_m \wedge \neg\Delta\neg\Delta_k Pa_m) \qquad (k\text{-Borderline})$$

Zardini's proof is to the effect that supervaluationistically acceptable reasoning allows one to pass from (0-Borderline), and some basic assumptions about the structure of the Sorites sequence, to the following principle.[27]

$$+\,\forall m(\Delta Pa_m \rightarrow \neg\Delta\neg Pa_{m+1}) \qquad \text{(0-Gap)}$$

By analogous argument, we get a version of this at each higher order.

$$+\,\forall m(\Delta\Delta_k Pa_m \rightarrow \neg\Delta\neg\Delta_k Pa_{m+1}) \qquad (k\text{-Gap})$$

Yet Graff Fara [11] has shown that the Gap principles are jointly inconsistent assuming $\Delta$-strengthening; the argument from $+A$ to $+\Delta A$. Yet as Graff Fara also points out, the validity of $\Delta$-strengthening is a crucial component of the supervaluationists account. $\Delta A$ expresses in the object language the supertruth of $A$. Hence the equivalence of $+A$ and $+\Delta A$ follows from the supervaluationists identification of truth with supertruth.

So we want to (weakly) reject the Gap principles, in order to be able to maintain $\Delta$-strengthening. But we do not want to give up the Borderline principles. Hence Zardini's argument must be undermined, to allow for the rejection of the Gap principles. Incurvati & Schlöder [16] show how the proof is blocked in $\mathsf{QSML}^-$. Crucially, however, their strategy is to resist the substitution of identicals in $\Delta$-contexts. But such substitution is perfectly legitimate in $\mathsf{QSML}^\#$, as can be seen e.g. from Lemma 4.22. Thus Zardini's argument, more precisely the multilateral reconstruction of it found in [16, p. 1572], goes through for $\mathsf{QSML}^\#$. Since we still have $\Delta$-strengthening, it follows that $\mathsf{QSML}^\#$ is inconsistent with higher-order vagueness in Sorites sequences. $\mathsf{QSML}^\#$, as $\mathsf{SML}^*$, improves classicality at a price the supervaluationist can ill afford to pay.

[27] Although Zardini in [36] is working in a unilateral setting, and so his arguments treat unsigned formulations of each of these principles, they apply equally under strong assertion. The same goes for the work in [11] mentioned below.

So again the question arises whether we cannot find a system in the middle; stronger than QSML$^-$, to gain classicality, yet weaker than QSML$^\#$, so as to avoid Zardini's proof. But here we find another limitative result standing in our way.

**Theorem 4.29** *Let $\mathcal{K}$ be any natural deduction system on $\mathcal{L}_{=,\Delta,S}$ such that it (i) includes the rules of* QSML$^-$ *and (ii) is superclassical on level 1. Then $\mathcal{K}$ derives Zardini's argument.*

***Proof*** The derivation of Zardini's argument is that provided in [16, p. 1572]. All of the steps in said derivation follow by the rules of QSML$^-$, except that it involves substitutions of identicals in $\Delta$ contexts. Hence in order for the derivation to work in $\mathcal{K}$, we have to show that $\mathcal{K}$ can derive full intersubstitutivity of identicals.

By (ii), $\mathcal{K}$ must derive (SI). By (i), the following is then a derivation in $\mathcal{K}$, for any $A$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Reflexivity)}
\UnaryInfC{$+a = a$}
\AxiomC{$+A[a/x]$}
\RightLabel{$(+\wedge\text{I.})$}
\BinaryInfC{$+a = a \wedge A[a/x]$}
\RightLabel{$(+\exists\text{I.})$}
\UnaryInfC{$+\exists x(x = a \wedge A)$}
\AxiomC{(SI)}
\RightLabel{$(+\rightarrow\text{E.})$}
\BinaryInfC{$+\forall x(x = a \rightarrow A)$}
\RightLabel{$(+\forall\text{E.})$}
\UnaryInfC{$+b = a \rightarrow A[b/a]$}
\AxiomC{$+b = a$}
\RightLabel{$(+\rightarrow\text{E.})$}
\BinaryInfC{$+A[b/x]$}
\end{prooftree}
$$

Note that this derivation does not use (+ΔI.). This is crucial, because the uses of intersubstitutivity in [16]'s derivation occur in the subderivations of applications of (Weak Inferece), or rules derived from (Weak Inference), which disallow (+ΔI.).[28] □

Note that this statement is quite different from that of Theorem 4.13. A result analogous to the latter—i.e. to the effect that any logic on $\mathcal{L}_{=,\Delta,S}$ which is as strong as QSML$^-$ and classical at level 1 must be as strong as QSML$^\#$—is not available here. For one thing, as discussed in footnote 21, validating (SI) does not require full rigidity, and so something weaker than QSML$^\#$ may suffice for level 1 classicality. More importantly, however, the entire proof strategy of Theorem 4.13 is built on the assumption of level 2 classicality, and only assuming the much weaker level 1 classicality does not allow one to deduce analogous consequences. Nonetheless, the takeaways of the result are quite analogous.

It is possible to strengthen QSML$^-$ in a natural way, so as to match up to classical logic at level 1. QSML$^\#$ shows as much. However, similarly to the basic propositional modal case, though for different reasons, it is just not possible to do this in a way which the supervaluationist might find acceptable. Consequently, we might once again wonder whether we should take something even weaker than QSML$^-$ as our starting point, from which to work towards level 1 classicality. We take it the answer is again no, for reasons similar to those offered before: each of the remaining operational rules

[28] What if the derivation of the relevant instance of (SI) already used (+ΔI.)? Since (SI) has empty premises, $+\exists x(x = a \wedge A) \rightarrow \forall x(x = a \rightarrow A)$ must be derivable, for every $A$, *from no assumptions*. In such cases, the coordination principles can always be used to rewrite the overall derivation so that the application of the disallowed proof rule (+ΔI.) no longer occurs inside the subderivation of the proof rule (Weak Inference) which disallows it. See [17, Sect. 5] for details on this trick.

constitutes the very meaning of the operator involved, and the coordination principles directly reflect the supervaluationist interpretation of the force-markers.

It has been argued [21, 27] that the ST-sequence of increasingly classical logics puts pressure on proponents of ST to climb further and further up the ladder in pursuit of improved classicality. Ripley [26] counters that whilst we may have reasons to attain level 1 classicality, going any higher from there is no longer beneficial; one step is enough. Theorem 4.29 shows that, for the multilateral supervaluationist, one step is *too much*.

## 5 Discussion

In the introduction, we discussed the issue of supervaluationism's allegedly problematic revisionism of classical reasoning, by breaking it up into three components: the formalization, comparison, and acceptability questions. The first concerns how to define supervaluationist logic, the second how the result measures up to the strength of classical logic, and the third whether the differences in strength, if there are any, are worrisome for supervaluationism. We set out to answer the comparison question, with respect to Incurvati and Schlöder's [16] multilateral solution to the formalization question. The results of Section 4 provide the answer: as far as propositional logic with a 'definitely' operator goes, multilateral supervalautionism allows one to retain classical reasoning on the levels of theorems and regular inferences, but requires a forfeit of some classical metarules. Once we move to a first-order language with identity, however, even some regular inferences and theorems from the classical canon must be rejected. This is only the case for first-order logic *with identity* though: in a language with first-order quantifiers and predicates but without identity, we can get level 1 classicality without problems. This can be seen from the fact that $\mathsf{QSML}^{\#}$ is level 1 classical, but indistinguishable from $\mathsf{QSML}^{-}$ as far as the identity-free fragment of the language is concerned, and so the latter is already classical on said fragment.

Before continuing, two brief remarks about the methods we employed to get here. First, as advanced in Subsection 3.5, we take it that the comparison techniques developed may be usefully applied in a variety of settings other than the present one. Most straightforwardly, of course, to the comparison of other bi- or multilateral logics to unilateral ones, but also to the study of other unconventional syntaxes which fail to be closed under Boolean operations. Secondly, we want to point out that defining the measures necessarily involved making some choices. In particular, when defining inferential validity at levels $n > 2$, and when setting up the machinery of quantified schemas. We think our choices on these matters are well motivated, but alternative decisions might be possible, and would potentially lead one to different outcomes.

With that said, what do our results mean, in turn, for the acceptability question? We have discussed how [16] already sketches a partial answer to the acceptability concern, by deriving in $\mathsf{SML}$, $\mathsf{SML}^{-}$ and $\mathsf{QSML}^{-}$ certain metarules which appear to be restricted versions of the forfeited classical ones (see Section 2). This shows that, contra the arguments of Graff Fara [11] and Williamson [35], supervaluationists can provide a plausible account of good reasoning, even without unrestricted versions of the classical metarules that play such a central role in our deductive practices. Yet as a consequence of lacking a precise answer to the comparison question, this resolution was necessarily incomplete in some ways. We can now fill in the gaps.

For one, prior to establishing a systematic criterion for identifying rules across multi- and unilateral languages, it could not be determined which multilateral principles would need to be validated in order for the logics to *fully* license the relevant classical metarules, hence it was also not clear in what sense [16]'s derived principles are restricted versions thereof. We can now give a precise explanation, as we illustrated in Subsection 4.2 via the example of *reductio*. The derived metarules are restricted multilateral versions of their classical counterparts in the following sense: each has a restriction on when it can be applied, and the rule that would be obtained from removing this restriction is such that its admissibility corresponds to the validity of a multilateral schema, the unilateralization of which is the classical metarule in question.[29]

Let us walk through the details again with another example. Recall the typical supervaluationistic failure of conditional proof, to which Incurvati & Schlöder in [16] respond by deriving the following rule in their systems.

$$(+\to\text{I.})\ \frac{\begin{array}{c}[+A]\\ +\vdots\\ +B\end{array}}{+A\to B}\ \begin{array}{l}\text{if } (+\Delta\text{I.}) \text{ and } (\oplus\Delta\text{I.})\\ \text{were not used to derive}\\ +B\end{array}$$

What is the exact relationship between this proof rule, and conditional proof? For unilateral logics, conditional proof is expressed by the schema $(A_1 \Rightarrow^1 A_2) \Rightarrow^2 (\varnothing \Rightarrow^1 A_1 \to A_2)$—strictly speaking, the set of schemas $(\Lambda, A_1 \Rightarrow^1 A_2) \Rightarrow^2 (\Lambda \Rightarrow^1 A_1 \to A_2)$ for each $\Lambda \subseteq \mathcal{A}$, though again, this is equivalent for logics with weakening. The multilateral analogue of this schema, via unilateralization, is $(+A_1 \Rightarrow^1 +A_2) \Rightarrow^2 (\varnothing \Rightarrow^1 +A_1 \to +A_2)$—strictly speaking $(\Lambda, +A_1 \Rightarrow^1 +A_2) \Rightarrow^2 (\Lambda \Rightarrow^1 +A_1 \to +A_2)$ for each $\Lambda \subseteq \Phi$. The validity of this (set of) schema(s) corresponds precisely to the admissibility of the following natural deduction rule.

$$(+\to\text{I.}^*)\ \frac{\begin{array}{c}[+A]\\ \vdots\\ +B\end{array}}{+A\to B}$$

So to fully license conditional proof, a multilateral logic would need to have $(+\to\text{I.}^*)$ as an admissible rule. But $(+\to\text{I.})$ is the very same rule, except it comes with a restriction on when it can be used. Hence the latter's derivability and consequent admissibility demonstrate a restricted but supervaluationistically licensed version of conditional proof.

29 The case of Existential Instantiation is slightly different, due to the expressive limitations of our quantified schemas, discussed in Subsection 3.4. We don't have a single unilateral schema that expresses general Existential Instantiation, but one for every sentence. The relation between the derived $(+\exists\ \text{E.})$ rule and the classical principle of Existential Instantiation must thus also be understood 'per sentence'. The derivability of $(+\exists\ \text{E.})$ includes the derivability of a dedicated 'subrule' for each sentence, such that removing the restriction from each of them results in a set of rules which each correspond via admissibility to a particular multilateral schema, again one for each sentence. Each of these multilateral schemas then unilateralizes to the unilateral schema which expresses Existential Instantiation for that sentence.

Secondly, [16]'s response to the acceptability question was naturally focused on supervaluationisms known failure of the classical metarules. Yet it was uncertain whether they were the *only* deviations from classicality to worry about regarding the multilateral systems. Now we have seen that SML and SML$^{-}$ are perfectly classical on the inferential level, and moreover don't validate anything that classical logic invalidates on the higher levels, so the relative weakness on levels 2 and above are indeed the only departures from classicality. QSML$^{-}$, however, turned out to be weaker than classical logic already on level 1. This apparently raises a separate instance of the acceptability question, i.e. whether we can accept the failure of (SI).

Though it does not at first glance appear to constitute a failure of classicality, Incurvati and Schlöder [16] do of course note the failure of the intersubstitutivity of identicals in $\Delta$-contexts, relying on it to block Zardini's argument. They show, for instance, that $+\forall x, y((x = y \wedge \Delta Fx) \rightarrow \Delta Fy)$ is not a theorem of QSML$^{-}$. They argue that the phenomenon fits naturally with a contextualist interpretation of the logic:

> "Two sentences *a is definitely tall* and *b is definitely tall* may be precisified to different standards of definite tallness [...]. This is plausible, for instance, when a is a building and b is a person, since different standards of (definite) tallness apply. More dramatically, this can also happen when *a* and *b* co-refer. We submit that this is still plausible if vagueness is context-sensitive, since the way an object is referred to can affect the relevant standards. One may refer to, say, Jeremy Lin as *that person* or as *that basketball player*. These two ways of referring are associated with different standards of (definite) tallness."
> [16, p. 1576]

We can also apply this interpretation to explain the invalidity of (SI). Recall that the instances of it that fail are those such as $\varnothing \Rightarrow^1 +[\exists x(x = a \wedge \Delta Fx)] \rightarrow [\forall x(x = a \rightarrow \Delta Fx)]$. This amounts to the principle that if there is one way to refer to an object such that it is definitely $F$, then on all ways of referring to it, it is definitely $F$. Such a principle obviously runs counter to the contextualist interpretation cited above. Thus, insofar as one finds this interpretation agreeable, it also serves to address the newfound instance of the acceptability question. Note furthermore that, due to our results on QSML$^{\#}$, we can be sure that there are no other types of revision with respect to the classical inferences and theorems, since ruling out the countermodels to (SI) proved sufficient to achieve full classicality at level 1.

Finally, considerations on the acceptability of each of these departures from classicality would be largely irrelevant if they were merely byproducts of coincidental features of the definitions of the logics in question. However, as the results on SML$^{*}$ and Theorem 4.29 respectively show, both the invalidated metarules and the failure of (SI) are necessary consequences of the supervaluationist account, at least in its multilateral formalization. The former in order to have room for the very existence of borderline cases, and the second in order to accomodate higher-order vagueness.

To close, let us highlight a similarity, and potential relation, between our results in the propositional modal setting, and some recent work by Schechter [29]. Assuming the standard formalization of supervaluationism (i.e. superentailment as defined in Section 2), which can be understood as 'generated' by S5 in the sense that $\Gamma \vDash_{SV} A$ iff $\Delta\Gamma \vdash_{S5} \Delta A$, Schechter investigates whether we might get classicality at level 2 by starting with a different modal logic as the 'base'. Much like our upshots from Subsection 4.2, he concludes that prospects are dire, since it will inevitably lead to a collapse:

> "[...] there will be no difference between $\alpha$ and it being determinately true that $\alpha$. This seems incompatible with the motivations for a supervaluational treatment of vagueness."
> [29, Sect. 10].

This is based on his proof that Triv is the only normal modal logic which (i') meets certain structural assumptions (ii') extends KT, (iii') is itself classical at level 2, and (iv') generates a logic that is level 2 classical [29, Prop. 4.6]. Triv is what results from adding $A \leftrightarrow \Delta A$ to an axiomatization of K, and so it generates itself. Although this result is obviously rather different from our Theorems 4.14 and 4.16, there is one striking similarity in that Triv is exactly SML$^*$ without force-markers. For it is easy to see that Triv is sound and complete for Boolean valuations extended with the clause $v(\Delta A) = v(A)$.

We leave it for future work to investigate the extent to which this is a coincidence, or whether there are deeper underlying reasons why such different strategies towards increasing classicality, and starting out from such different formalizations, are bound to end in the same collapse.

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