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ORIGINAL RESEARCH

Every Logic is Unique: Transfinite Metainferences in Classical Set Theory

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Abstract

What is the proper way to transfinitely extend the usual hierarchy of finite metainferential levels? McAllister (*Journal of Philosophical Logic, 51*, 1345–1365, 2022; *Belief Revision About Logic*, PhD Thesis, University of Auckland, 2024) has proven that classical logic and numerous other logics are non-unique in classical set theory. On the basis of these results, she argues for a range of philosophical consequences, including problems for logical monism, classical set theory, and the identification of logics. This paper demonstrates that McAllister's key results are merely artifacts of the level labels which she adds to the transfinite hierarchy. It is proven that, in the absence of labels, every logic is unique in classical set theory. Moreover, I argue that if labels are not innocent additions, but instead influence the results in substantial ways, then they must be left out of the hierarchy. Therefore, in the final analysis, every logic is unique. The various problems which McAllister extracts from non-uniqueness accordingly dissolve.

1 Introduction

Logicians have recently begun exploring hierarchies of metainferential levels, and defining logics in terms of them. Such logics have in turn found philosophical applications e.g. in dealing with the semantic paradoxes (Pailos, 2020; Golan, 2022), and raised questions about such things as the appropriate identity criterion for logics (Barrio et al., 2020; Scambler, 2020a). The focus is usually on the countably infinite hierarchy of finite levels, consisting of first the familiar inferences, then inferences between inferences, then inferences between *those*, and so on. The present paper concerns the following question: what is the proper way to extend the inferential hierarchy into the transfinite? As it turns out, certain details about the way we do so

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have repercussions for the individuation of logics, as well as for debates such as that between logical monists and pluralists, or that between classical and paraconsistent set theories. In particular, because such details determine whether a certain type of result due to McAllister (2022, 2024) goes through or not.

McAllister (2022) first presents a proof that classical propositional logic is *non-unique* in classical set theory. This is essentially a further extension of the so-called ST-phenomenon, whereby logics can agree up to any given point in the inferential hierarchy without agreeing beyond there (Barrio et al., 2020). Moreover, McAllister considers non-uniqueness a major problem for the classically minded, especially in combination with another property she dubs *uncharacterizability*. McAllister (2024) subsequently provides similar results for a host of normal modal logics, as well as intuitionistic propositional logic and various belief revision frameworks, and expands on the consequences we might draw from such results. She argues among other things that the results rule out logical monism, at least in the context of classical set theory, and place limits on the prospects for modeling belief revision.

The main purpose of the present article is to demonstrate how the non-uniqueness results which give rise to these concerns are mere artifacts of a peculiarity in McAllister's definition of the transfinite hierarchy. Unlike e.g. the hierarchies used by Scambler (2020b) or Porter (2022), McAllister's includes what we might call *level labels*. We will see that without them, *every* logic is unique in classical set theory. Moreover, I will argue that if labels are not innocuous additions, but actually affect the results in such substantial ways, then they should be excised from the hierarchy. Therefore, in the final analysis, every logic is unique. The various problems and other consequences which McAllister extracts from non-uniqueness accordingly dissolve.

The article is structured as follows. Sect. 2 briefly introduces the necessary background on metainferences and metainferential logics. Sect. 3 then presents the relevant aspects of McAllister's work on non-uniqueness, clarifying in particular the two different types of (non-)uniqueness at play, and the significance of each. Sect. 4 will uncover the crucial role that labels play in that work, and prove that in a hierarchy without labels, every logic is unique in the sense that matters. Sect. 5 argues that if the choice is not inconsequential, then we should work with an unlabeled hierarchy. Sect. 6 addresses a potential objection to the assumptions of this note, and Sect. 7 concludes.

2 Hierarchies and Logics

Barrio et al. (2020) first considered a countably infinite hierarchy of metainferences, constructed as follows: starting with the usual inferences of a given language at level 1, an inference at level $n+1$ consists of a set of premises and a set of conclusions, each themselves inferences of level n.[1] Formally: given a logical language $\mathcal{L}$, let $\mathcal{L}_0$ be the set of its well-formed formulae.

[1] Though Barrio et al. (2020) did not originally include inferences with multiple conclusions. Moreover, the level numbering I adopt here matches that of e.g. McAllister, but differs by one from that of Barrio et al. : what we call level *n*, the latter call $n-1$.

Definition 2.1 (*BPS Hierarchy*) The *BPS hierarchy* $\mathbb{B}$ of inferential levels for a language $\mathcal{L}$ is as follows.

$$\begin{aligned}\mathcal{L}_0^{\mathbb{B}} &:= \mathcal{L}_0,\\ \mathcal{L}_{n+1}^{\mathbb{B}} &:= \{\langle\Gamma,\Delta\rangle \mid \Gamma\cup\Delta\subseteq\mathcal{L}_n^{\mathbb{B}}\}.\end{aligned}$$

Where *n* is any natural number.

Note that each language generates its own hierarchy, with its own set of inferences at each level. Although we will work with different languages in the remainder, we will denote them all by $\mathcal{L}$, since context will make clear which language this signifies at each point. We usually write $\Gamma \Rightarrow_n \Delta$ to denote the level n inference $\langle\Gamma,\Delta\rangle$, and similarly for transfinite-level inferences in the remainder.
The first transfinite generalization of the hierarchy was constructed by Scambler (2020b).

Definition 2.2 (*Scambler Hierarchy*) The *Scambler hierarchy* $\mathbb{S}$ of inferential levels for a language $\mathcal{L}$ is as follows.

$$\begin{aligned}\mathcal{L}_0^{\mathbb{S}} &:= \mathcal{L}_0,\\ \mathcal{L}_{\alpha+1}^{\mathbb{S}} &:= \{\langle\Gamma,\Delta\rangle \mid \Gamma\cup\Delta\subseteq\textstyle\bigcup_{\beta\leq\alpha}\mathcal{L}_\beta^{\mathbb{S}}\},\\ \mathcal{L}_{\lambda}^{\mathbb{S}} &:= \{\langle\Gamma,\Delta\rangle \mid \Gamma\cup\Delta\subseteq\textstyle\bigcup_{\beta<\lambda}\mathcal{L}_\beta^{\mathbb{S}}\}.\end{aligned}$$

Where λ is a limit and α is any ordinal.

But this is not the only way to set things up. McAllister (2022, 2024) opts to work with the following alternative.

Definition 2.3 (*McAllister Hierarchy*) The *McAllister hierarchy* $\mathbb{M}$ of inferential levels for a language $\mathcal{L}$ is as follows.

$$\begin{aligned}\mathcal{L}_0^{\mathbb{M}} &:= \mathcal{L}_0,\\ \mathcal{L}_{\alpha+1}^{\mathbb{M}} &:= \{\langle\Gamma,\Delta,\alpha+1\rangle \mid \Gamma\cup\Delta\subseteq\mathcal{L}_\alpha^{\mathbb{M}}\},\\ \mathcal{L}_{\lambda}^{\mathbb{M}} &:= \{\langle\Gamma,\Delta,\lambda\rangle \mid \Gamma\cup\Delta\subseteq\textstyle\bigcup_{\beta<\lambda}\mathcal{L}_\beta^{\mathbb{M}}\},\\ \mathcal{L}_{On}^{\mathbb{M}} &:= \{\langle\Gamma,\Delta,*\rangle \mid \Gamma\cup\Delta\subseteq\textstyle\bigcup_{\beta\in On}\mathcal{L}_\beta^{\mathbb{M}}\}.\end{aligned}$$

Where λ is a limit and α is any ordinal, *On* is the class of all ordinals and $*\notin On$.

There are three differences between the Scambler and McAllister hierarchies. Firstly, McAllister's includes an additional final level, corresponding to the class of ordinals. Note that $\mathcal{L}^{\mathbb{M}}_{On}$ thus defined is a proper class. So too is $\bigcup_{\beta \in On} \mathcal{L}^{\mathbb{M}}_{\beta}$, therefore the '$\subseteq$' appearing in the definition of $\mathcal{L}^{\mathbb{M}}_{On}$ denotes the subclass relation. Secondly, for Scambler all levels are cumulative, in the sense that the inferences can have premises and conclusions of *any* lower lever. For McAllister only the limit stages are cumulative; at successor stages, inferences only take premises and conclusions of the predecessor level. Thirdly, inferences of $\mathbb{M}$ are ordered *triples*, rather than pairs, with the third coordinate acting as a label that tells us which level the inference belongs to. In the case of $\mathcal{L}^{\mathbb{M}}_{On}$, we use some special object $*$ because *On* itself, being a proper class, cannot feature in ordered tuples, and we require that $* \notin On$ to make sure that the label is different from that of any other level.[2]
The first of these three is the least fundamental; we could simply extend $\mathbb{S}$ with the extra level. The second is substantial in that it changes at which level many inferences will make their first appearance. This might be quite important for some purposes, but it won't be for ours.[3] The seemingly harmless third variation, the presence or absence of labels, will turn out to make all the difference.

So much for inferences, what about logics? In the context of metainferential hierarchies such as these, it won't do to think of logics simply as consequence relations that judge every regular inference as valid or invalid. Rather, we need them to rule every inference of any level in the hierarchy as valid or invalid. Thus a logic on language $\mathcal{L}$ and hierarchy $\mathbb{H}$ is formally understood as a subclass of $\bigcup_{i \in On \vee i = On} \mathcal{L}^{\mathbb{H}}_{i}$. We therefore think of logics as full consequence relations in the sense of Ripley (2022), though extended to transfinite levels. In the remainder, we will often talk simply about "logics on $\mathbb{H}$" (or simply "logics"), if the language (and the hierarchy) are clear from context. Where $\mathbf{L}$ is a logic, we write $\mathbf{L} \vDash \Gamma \Rightarrow_{\alpha} \Delta$ to mean that the level α inference $\Gamma \Rightarrow_{\alpha} \Delta$ is valid according to $\mathbf{L}$ (formally: that $\Gamma \Rightarrow_{\alpha} \Delta \in \mathbf{L}$).

This definition of 'logics' is somewhat more abstract and liberal than what is common. We omit any reference to the usual counterexample or satisfaction relations, or to local or global (or other) validity. We don't assume that our logics are determined by sequences of standards or validity notions (Scambler, 2020a), let alone by single standards which generate the rest of the sequence (Barrio et al. 2020; Pailos & Da Ré 2023). I adopt this abstracted perspective because it is simpler, while sufficing for the current aims. There is no need to go into the underlying constructions of logics, because the observations of this note apply to any full consequence relation whatsoever. Nonetheless, at certain points it will be useful to keep in mind that the literature often assumes a more restrictive notion of logic than the one I use here. This explains why certain results from the literature, which I will reference below, are not trivial.

[2] McAllister instead lets inferences in $\mathcal{L}^{\mathbb{M}}_{On}$ be ordered pairs, leaving out the label only for them. Thus they are implicitly labeled by the very absence of an explicit label. This is essentially equivalent to our solution.

[3] Ferguson and Ramírez-Cámara (2022) advocate for cumulative successor levels on the basis of independent philosophical reasons. But their arguments are to the effect that mixed inferences, with premises and conclusions of different levels, merit inclusion in the hierarchy. In a setting such as theirs, with finite levels only, including them requires cumulative successor stages. In a transfinite hierarchy, the cumulative limit stages ensure that all mixed inferences show up anyway (Lemma 4.10 below).

Moreover, we won't need to define any particular full logics. All we have to know is that there is a certain customary way to extend familiar 'level 1' logics, such as the propositional and modal logics that concern McAllister, to full consequence relations. See e.g. Scambler (2020a) or Ripley (2022) for details on the basic construction in case of classical propositional logic, and Scambler (2020b) for its transfinite generalization. We will henceforth use the acronyms '**CL**' and '**IL**' to refer to the full logic obtained by applying this customary construction to respectively classical and intuitionistic propositional logic.[4] Technically, logics like **CL** and **IL** are different classes depending on which hierarchy we have extended them to, so it would be more proper to write e.g. '$\mathbf{CL}^{\mathbb{H}}$' for classical propositional logic extended by the standard construction to the hierarchy $\mathbb{H}$. But I will omit this notation and let context disambiguate. With that, we are set to go.

3 Non-Uniqueness and its Consequences

Most of McAllister's (2022; 2024) work on uniqueness is concerned with demonstrating non-uniqueness results, and coming to grips with their consequences, within the context of classical set theory.[5] Another significant part reconsiders the situation assuming a paraconsistent set theory as the metatheory instead. The present note, however, only deals with the state of affairs in classical set theory, leaving the paraconsistent setting aside.

What does (non-)uniqueness mean? Say that two logics agree on an inferential level $\mathcal{L}_i^{\mathbb{H}}$ if they agree on the validity of every inference $\Gamma \Rightarrow_i \Delta \in \mathcal{L}_i^{\mathbb{H}}$. Furthermore, say that they agree on a collection of levels if they agree on every level in the collection.

Definition 3.1 (*Uniqueness*) A logic $\mathbf{L}$ on $\mathbb{H}$ is *unique* on a collection of levels $\mathcal{C}$ from $\mathbb{H}$ when, for all logics $\mathbf{M}$ on $\mathbb{H}$: if $\mathbf{L}$ and $\mathbf{M}$ agree on $\mathcal{C}$, then $\mathbf{L}$ and $\mathbf{M}$ agree on all levels of $\mathbb{H}$.

In other words, $\mathbf{L}$ is non-unique on $\mathcal{C}$ just in case there exists some logic $\mathbf{M}$ which agrees with $\mathbf{L}$ completely on $\mathcal{C}$, yet disagrees on some inference $\Gamma \Rightarrow_i \Delta \in \mathcal{L}_i^{\mathbb{H}} \notin \mathcal{C}$. Intuitively, the idea is that $\mathbf{L}$ is unique on $\mathcal{C}$ only when complete information about the (in)validities in $\mathcal{C}$ suffices to uniquely single out $\mathbf{L}$ (at least up to full equivalence regarding validities). Put differently: when it is the only logic on $\mathbb{H}$ (at least up to full equivalence regarding validities) with precisely its $\mathcal{C}$ validities.[6] Observe that non-uniqueness on a class of levels entails non-uniqueness on any subclass.

[4] Yet this isn't meant to imply that the customary way of generalizing logics to higher levels is also the *canonical* one. For instance, Ripley (2022) raises questions about the relevant sequence of standards being canonical, whereas Kortenbach (2025) takes issue with the use of local validity.

[5] McAllister explains that we may take 'classical set theory' to mean ZFC, or "possibly NBG [...] if object-language discussion of proper classes is desired" (2024, p. 19).

[6] The qualification between parentheses in the last two sentences is needed, because as we will see in a moment, when two logics can rightfully be identified is an open question. However, it is at least clear that

For the moment, we are only interested in uniqueness on the McAllister hierarchy. So from now on until otherwise specified, let it be understood that we are talking about logics on $\mathbb{M}$.

McAllister herself only defines uniqueness of a logic on an individual level, rather than on a class of levels, but this doesn't quite suffice. It is sometimes important whether the validities at a given class of levels suffice to uniquely single out a logic, and uniqueness on a class cannot always be reduced to uniqueness on any one level. This is so even if we assume some restriction on logics that makes them downward determining, in the sense that the validities at any given level fix those at each lower level.[7] In that case, if $\mathcal{C}$ contains a highest level $\mathcal{L}_i^{\mathbb{M}}$, uniqueness on $\mathcal{C}$ reduces to uniqueness on $\mathcal{L}_i^{\mathbb{M}}$. But even so, we are occasionally interested in classes which lack a highest level, such as those of all finite or all ordinal levels, in which case such a restriction doesn't help. Hence the generalized definition of uniqueness is needed.

We can recast the familiar ST-phenomenon in this terminology. The well-known central results of Barrio et al. (2020) can be understood as showing that **CL** is non-unique on any class of levels of the form $\{\mathcal{L}_n^{\mathbb{M}} \mid n < m\}$ with $m \in \omega$. Scambler (2020b) generalizes this, establishing that **CL** is not unique on any class $\{\mathcal{L}_\alpha^{\mathbb{M}} \mid \alpha < \beta\}$ with $\beta \in On$.[8] Here is one reason these results matter. Most logics can be presented via a host of different formal systems. Think, for instance, of the various model theories or proof systems one might use to characterize classical propositional logic. We normally don't think of these systems as each a different logic, but rather as different ways of presenting and understanding one and the same logic. What justifies this identification? The initially tempting answer is that those systems all define the same logic because they produce exactly the same level 1 validities. But Barrio et al.'s result shows that this is not a sufficient condition for identity, because e.g. classical logic is not uniquely singled out by its validities at level 1. Nor, more generally, by its set of validities up to any finite level, so we cannot identify logics by their validities up to any finite level either. For this reason Barrio et al. proposed a different identity criterion for logics, which takes into account all the levels in $\{\mathcal{L}_n^{\mathbb{M}} \mid n < \omega\}$. Scambler's result, however, shows that this too is insufficient, as is any other proper initial segment of the ordinals. This, in turn, suggests that we might identify logics just in case they agree on the entirety of $\{\mathcal{L}_\alpha^{\mathbb{M}} \mid \alpha \in On\}$.

Yet this is where McAllister comes in. Her initial non-uniqueness result was as follows.

Theorem 3.2 *(McAllister, 2022)* ***CL*** *is non-unique on the class of levels* $\{\mathcal{L}_\alpha^{\mathbb{M}} \mid \alpha \in On\}$.

She has subsequently added the following.

agreement on all levels in $\mathbb{H}$ is a necessary requirement for identity between logics in $\mathbb{H}$. For the sake of readability we will drop the qualification in the remainder.

[7] For instance by some restriction along the lines of Scambler's (2020a, p. 359) Eqs. 2–3 equivalence or Ripley's (2022, p. 1242) downward coherence, suitably adapted to our setting.

[8] Of course, Barrio et al. and Scambler actually proved the analogous facts for the BPS and Scambler hierarchies, respectively, but the proofs equally apply to McAllister's. Note, moreover, that these results would be trivial under the fully liberal definition of 'logic' that I gave in the previous section. Their interest hinges on the fact that they also hold under the more restrictive notions of logic which were mentioned above.

Theorem 3.3 *(McAllister, 2024, Ch. 3)* ***IL** is non-unique on each class of levels* $\{\mathcal{L}_\alpha^{\mathbb{M}}\}$ *with* $\alpha \in On$.

Theorem 3.4 *(McAllister, 2024, Ch. 3) Every classical modal logic characterized by a class of Kripke frames is non-unique on each class of levels* $\{\mathcal{L}_\alpha^{\mathbb{M}}\}$ *with* $\alpha \in On$.

Lastly, McAllister (2024, Ch. 9) proves analogues to Theorems 3.3 and 3.4 for various belief revision systems.

Two observations stand out. First: Theorem 3.2 is of a much stronger sort than the others. It essentially says that all the ordinal levels *together* don't provide enough information to uniquely single out **CL**, whereas Theorems 3.3 and 3.4 only tell us that each ordinal level *individually* falls short of uniquely picking out the intuitionistic and modal logics. Let us call a logic *collectively non-unique* if it is non-unique on $\{\mathcal{L}_\alpha^{\mathbb{M}} \mid \alpha \in On\}$, and *individually non-unique* if it is non-unique on $\{\mathcal{L}_\alpha^{\mathbb{M}}\}$ for each $\alpha \in On$. Otherwise we call it, respectively, *collectively unique* and *individually unique*. Note that collective non-uniqueness concerns a class without a highest level, and hence cannot be expressed as a claim about non-uniqueness on single levels, even under the assumption of downward determination. If we do assume downward determination, individual non-uniqueness means that no proper initial segment of the ordinals suffices for uniqueness, like Scambler's result for **CL**. But only a collective non-uniqueness result tells us that also the *entire* sequence of ordinal levels is insufficient. McAllister does not remark on the difference, or on why she stops short of proving collective non-uniqueness for the logics other than **CL**. Yet I think the difference is crucial. In particular, I will argue in a moment that only collective non-uniqueness results can support the kinds of consequences she proceeds to draw.

The second observation is that the non-uniqueness results are only about the ordinal levels; none of them mention $\mathcal{L}_{On}^{\mathbb{M}}$. In line with e.g. Barrio et al.'s takeaway from the initial ST-phenomenon, our first response might therefore be to think that we still just haven't included enough levels. Apparently, when identifying logics, we really need to take the *whole* inferential hierarchy into account, including $\mathcal{L}_{On}^{\mathbb{M}}$. Yet according to McAllister, $\mathcal{L}_{On}^{\mathbb{M}}$ cannot help us with the individuation of logics, because of uncharacterizability. In brief, McAllister has it that the logics of concern are not characterizable at $\mathcal{L}_{On}^{\mathbb{M}}$, because not all of their validities at this order are formally definable. She argues that inferences like e.g. $\{p \wedge \neg p\} \Rightarrow_{On} \bigcup_{\beta\in On} \mathcal{L}_\beta^{\mathbb{M}}$ are (or should be) classically and intuitonistically valid, because the premise is a contradiction and each of the conclusions is a well-formed formula/inference. Yet the ordered tuple $\langle \{p \wedge \neg p\}, \bigcup_{\beta\in On} \mathcal{L}_\beta^{\mathbb{M}}, *\rangle$ cannot be formed, because $\bigcup_{\beta\in On} \mathcal{L}_\beta^{\mathbb{M}}$ is a proper class. For McAllister, uncharacterizability is quite literal, in that it disqualifies $\mathcal{L}_{On}^{\mathbb{M}}$ as a tool for the characterization of logics, which is what makes the non-uniqueness results on the levels below $\mathcal{L}_{On}^{\mathbb{M}}$ so consequential.

To clarify: McAllister has it that we cannot *build on* $\mathcal{L}_{On}^{\mathbb{M}}$ when identifying or characterizing logics, but that does not mean that we do not need to *account for* $\mathcal{L}_{On}^{\mathbb{M}}$. In particular, when we universally quantify over the levels of $\mathbb{M}$ in the definition of uniqueness, that still includes $\mathcal{L}_{On}^{\mathbb{M}}$. Indeed, if it didn't, all logics would trivially be collectively unique, for then only agreement on the levels that are already in the class $\{\mathcal{L}_\alpha^{\mathbb{M}} \mid \alpha \in On\}$ would be required. As it stands, a logic is collectively unique just in case it can be uniquely singled out up to *and including* its validities at $\mathcal{L}^{\mathbb{M}}_{On}$, without appealing to its validities at $\mathcal{L}^{\mathbb{M}}_{On}$ in the process.

The initial consequence of non-uniqueness and uncharacterizability, suggested by McAllister (2022, p. 1353), concerns the identity criterion for logics. As mentioned above, we'd want a criterion for the identification of logics, in terms of their validities. But if the criterion doesn't include agreement on $\mathcal{L}^{\mathbb{M}}_{On}$, non-uniqueness means it won't provide sufficient conditions. Yet it can't include $\mathcal{L}^{\mathbb{M}}_{On}$, due to uncharacterizability. Related to this is a broader problem which McAllister (2024, Ch. 4) calls *logical underdetermination*: "when a logician expresses the belief that some collection of inferences is valid [...] this belief alone cannot uniquely determine which logic they are committed to" (2024, p. 97). The idea here is that if collections of logical principles cannot fully determine unique logics, then *commitment to* such a collection cannot entail commitment to any one specific logic either. From underdetermination McAllister draws a range of more specific consequences, including the impossibility of logical monism. For how can there be *One* True Logic if no one logic can be determined even by the One True collection of validities? She similarly rules out specific belief revision about logic: how could one rationally change their belief to a specific new logic, if belief in a given collection of validities cannot commit one to such?

I won't analyze these arguments much, except to make one observation: insofar as they succeed, their success depends on *collective* non-uniqueness. Even if we cannot rely on $\mathcal{L}^{\mathbb{M}}_{On}$ when characterizing logics, there is no reason to think that we have to limit ourselves to a single level below it. If we have individual but not collective non-uniqueness, the premise that our identity criterion won't provide sufficient conditions unless it includes $\mathcal{L}^{\mathbb{M}}_{On}$ is simply false. Similarly, logical underdetermination and its further consequences wouldn't follow from mere individual non-uniqueness either. As long as logics are still collectively unique, the collection of their validities across all ordinal levels suffices to fully determine a unique logic. Hence commitment to such a collection entails commitment to a specific logic. Such a collection could thereby also single out a One True Logic, and so on and so on. There might be other noteworthy consequences of individual non-uniqueness. But the arguments which McAllister herself has thus far based on non-uniqueness all require the stronger, collective variant to get of the ground.

Of course this would hardly be a noteworthy point by itself. We know from Theorem 3.2 that at least one logic is collectively non-unique, and one is all it takes for the identification and underdetermination worries to arise. Moreover, adapting McAllister's techniques to prove it of other logics looks to be straightforward. The trouble is that, as we will see in the next section, that theorem and those techniques depend crucially on the level labels. In the absence of labels, collective non-uniqueness results are impossible.

I first briefly mention that McAllister (2022, Sect. 4; 2024, Sect. 4.4) outlines another group of takeaways from her work on the transfinite hierarchy. Namely, she objects to classical set theory—at least *qua* metatheory for the construction of logics—taking issue with the fact that it cannot treat the class of all classical validities as a set, and moreover cannot allow the class to include those 'proper class-sized' inferences. The only thing I want to point out about this second family of arguments is that they don't appear to require non-uniqueness of either type. They are partly based on uncharacterizability, but independent of uniqueness. The present note is focused on uniqueness, so we can leave the arguments in this second category largely untouched. Insofar as one finds them convincing, I won't say much to the contrary (though see Sect. 6 below).

4 Every Logic is Unique

How does McAllister herself view the purpose of the labels? Consider first her reason for defining successor stages non-cumulatively, in contrast to Scambler (2020b):

> [Scambler's] construction entails that when $\alpha < \beta$, every $\mathcal{L}_\alpha$ formula is also a $\mathcal{L}_\beta$ formula. This cumulative approach works fine, but it complicates matters slightly. The non-cumulative construction I use ensures that the lowest level language that a formula belongs to is the *only* language that a formula belongs to. Since the lowest level language is what really matters [...] defining languages non-cumulatively streamlines things. (McAllister, 2022, fn. 4, emphasis in original)

Note that the third sentence here is not quite correct. McAllister's limit levels are still cumulative, which means that every inference still reappears infinitely many times, namely at each limit above its initial level. Nonetheless, it is strictly speaking true that all the levels of $\mathbb{M}$ are pairwise disjoint. But the real reason for this is unrelated to the non-cumulative successor levels. It is instead the labels. As McAllister recognizes elsewhere:

> Treating formulas [...] as triples rather than as pairs ensures that every formula belongs to exactly one language. Given two formulas $\Gamma \Rightarrow_\alpha \Delta$ and $\Gamma \Rightarrow_\lambda \Delta$, for which $\alpha < \lambda$, these formulas are distinct since $\langle \Gamma, \Delta, \alpha \rangle \neq \langle \Gamma, \Delta, \lambda \rangle$. (McAllister, 2022, p. 1347)

Piecing together these quotations, the motivation for the labels becomes clear: having the levels be pairwise disjoint simplifies subsequent definitions and proofs, and the labels ensure that they are. However, we should keep in mind that this disjointness is despite the fact that there *are* infinitely many 'copies' of any given inference—in fact, as many as there are ordinals—spread across equally many levels. It is just that they are made to be formally distinct by their labels.

Now consider McAllister's (2022, Sect. 2; 2024, Sect. 2.2) proof of Theorem 3.2. To show **CL** non-unique on $\{\mathcal{L}^{\mathbb{M}}_\alpha \mid \alpha \in On\}$, she constructs a logic $\mathbf{T}^{limit}_{On}$ and proves that

(i) $\mathbf{T}^{limit}_{On}$ agrees with **CL** on every $\mathcal{L}^{\mathbb{M}}_\alpha$ with $\alpha \in On$, yet
(ii) $\mathbf{T}^{limit}_{On}$ disagrees with **CL** on $\mathcal{L}^{\mathbb{M}}_{On}$.

In particular, she shows (ii) by demonstrating that $\mathbf{T}^{limit}_{On} \nvDash p \wedge \neg p \Rightarrow_{On} q$, whereas of course we have $\mathbf{CL} \vDash p \wedge \neg p \Rightarrow_{On} q$.

However, observe that from (i), it follows that $\mathbf{T}_{On}^{limit} \vDash p \wedge \neg p \Rightarrow_1 q$. So we have both $\mathbf{T}_{On}^{limit} \vDash p \wedge \neg p \Rightarrow_1 q$ and $\mathbf{T}_{On}^{limit} \nvDash p \wedge \neg p \Rightarrow_{On} q$. How can this be? Well, it is made possible precisely by the fact that $\langle \{p \wedge \neg p\}, \{q\}, 1\rangle \neq \langle \{p \wedge \neg p\}, \{q\}, *\rangle$. But this means that the proof relies fundamentally on the labels. In a label-free hierarchy, '$p \wedge \neg p \Rightarrow_1 q$' and '$p \wedge \neg p \Rightarrow_{On} q$' are just two different ways of denoting the same inference—formally, the pair $\langle \{p \wedge \neg p\}, \{q\}\rangle$—therefore it is trivial that $\mathbf{L} \vDash p \wedge \neg p \Rightarrow_1 q$ iff $\mathbf{L} \vDash p \wedge \neg p \Rightarrow_{On} q$ for all logics $\mathbf{L}$. Hence in the absence of labels, no logic can agree with $\mathbf{CL}$ on level 1 while invalidating $p \wedge \neg p \Rightarrow_{On} q$, which means that McAllister's proof strategy would have no hope of succeeding.

McAllister finds non-identities of the form $\Gamma \Rightarrow_\alpha \Delta \neq \Gamma \Rightarrow_\lambda \Delta$ helpful because they streamline things. But that is far from all they do. They also open the door to logics validating some-but-not-all of the differently labeled copies of a given inference. Let's give this phenomenon a name. When there exist Γ, Δ, i and j such that $\mathbf{L} \vDash \Gamma \Rightarrow_i \Delta$ but $\mathbf{L} \nvDash \Gamma \Rightarrow_j \Delta$, we will say that $\mathbf{L}$ is *label sensitive* (about $\Gamma \Rightarrow \Delta$). As we've just seen, label sensitivity is not merely an abstract possibility; it occurs in the logics McAllister studies, and plays a crucial role in the proof of Theorem 3.2. Of course, that doesn't yet mean that it is also indispensable to the result itself. The question now is: could another example logic and/or inference witness the collective non-uniqueness of $\mathbf{CL}$, without exploiting the labels? It will be straightforward to check that the answer is no.

To see why, first reconsider the notion of agreement between logics. We say that two logics agree on a level $\mathcal{L}_i^{\mathbb{M}}$ when they agree on the validity of every inference at $\mathcal{L}_i^{\mathbb{M}}$. The point to emphasize is that in the case of $\mathcal{L}_{On}^{\mathbb{M}}$, this means every formally definable inference in $\mathcal{L}_{On}^{\mathbb{M}}$. We might agree with McAllister that there "exist", in some sense, inferences such as $\{p \wedge \neg p\} \Rightarrow_{On} \bigcup_{\beta \in On} \mathcal{L}_\beta^{\mathbb{M}}$ which cannot formally be defined. We might furthermore agree that they are in some sense "of" level $\mathcal{L}_{On}^{\mathbb{M}}$ –even though, being undefinable, they are not actually elements of the class $\mathcal{L}_{On}^{\mathbb{M}}$. We might even agree with her that their undefinability is undesirable, and represents a notable cost of classical set theory. Nevertheless, the fact is that as things stand in classical set theory, they cannot be formally defined, which means that they are not elements of $\mathcal{L}_{On}^{\mathbb{M}}$ and cannot play a role in our definitions of agreement on $\mathcal{L}_{On}^{\mathbb{M}}$.

With that in mind, we need a piece of notation, a definition, and two simple lemmas. Let π_3 be the function mapping ordered triples to their third coordinate.

Definition 4.1 An inference $\Gamma \Rightarrow_{On} \Delta \in \mathcal{L}_{On}^{\mathbb{M}}$ is *bounded* when $\pi_3[\Gamma \cup \Delta] = \{\beta \mid \langle \Gamma', \Delta', \beta\rangle \in \Gamma \cup \Delta \text{ for some } \Gamma', \Delta'\}$ has an upper bound in the ordinals.

Lemma 4.2 *Every $\Gamma \Rightarrow_{On} \Delta \in \mathcal{L}_{On}^{\mathbb{M}}$ is bounded.*

Proof If $\Gamma \Rightarrow_{On} \Delta = \langle \Gamma, \Delta, *\rangle \in \mathcal{L}_{On}^{\mathbb{M}}$, then $\langle \Gamma, \Delta, *\rangle$ must be a set, which means $\Gamma \cup \Delta$ is a set. Then so is $\pi_3[\Gamma \cup \Delta]$, by the axiom schema of replacement. Then $\bigcup \pi_3[\Gamma \cup \Delta]$ is an upper bound of $\pi_3[\Gamma \cup \Delta]$ in the ordinals. $\square$

Lemma 4.3 *If $\Gamma \Rightarrow_{On} \Delta$ is bounded, then there exists $\alpha \in On$ such that $\Gamma \Rightarrow_{\alpha} \Delta \in \mathcal{L}^{\mathbb{M}}_{\alpha}$.*

Proof Let α be any limit ordinal bigger than the upper bound of $\pi_3[\Gamma \cup \Delta]$. Then $\Gamma \Rightarrow_{\alpha} \Delta \in \mathcal{L}^{\mathbb{M}}_{\alpha}$ by the limit clause of the definition of $\mathbb{M}$. ☐

Putting these lemmas together produces the following.

Corollary 4.4 *For every $\Gamma \Rightarrow_{On} \Delta \in \mathcal{L}^{\mathbb{M}}_{On}$, there exists $\alpha \in On$ such that $\Gamma \Rightarrow_{\alpha} \Delta \in \mathcal{L}^{\mathbb{M}}_{\alpha}$.*

In other words, $\mathcal{L}^{\mathbb{M}}_{On}$ doesn't include any genuinely new inferences at all. It consists entirely of relabeled copies of inferences that already appeared below it in the hierarchy. This means that if two logics agree with each other on every level below $\mathcal{L}^{\mathbb{M}}_{On}$, yet disagree on $\mathcal{L}^{\mathbb{M}}_{On}$ itself, this must be due to label sensitivity.

Theorem 4.5 *The collective non-uniqueness of any logic can only be witnessed in virtue of label sensitivity.*

Proof Take any two logics $\mathbf{L}$ and $\mathbf{M}$ such that (i) $\mathbf{L}$ and $\mathbf{M}$ agree on $\{\mathcal{L}^{\mathbb{M}}_{\alpha} \mid \alpha \in On\}$ but (ii) $\mathbf{L}$ and $\mathbf{M}$ don't agree on $\mathcal{L}^{\mathbb{M}}_{On}$. Point (ii) must be witnessed by some $\Gamma \Rightarrow_{On} \Delta \in \mathcal{L}^{\mathbb{M}}_{On}$ such that $\mathbf{L}$ and $\mathbf{M}$ disagree on its validity. By Corollary 4.4, $\Gamma \Rightarrow_{\alpha} \Delta \in \mathcal{L}^{\mathbb{M}}_{\alpha}$ for some $\alpha \in On$. By (i), $\mathbf{L}$ and $\mathbf{M}$ agree on the validity of $\Gamma \Rightarrow_{\alpha} \Delta$. Hence either $\mathbf{L}$ or $\mathbf{M}$ must be label sensitive about $\Gamma \Rightarrow \Delta$. ☐

Thus Theorem 3.2 in particular only holds true thanks to the labels.

We can make this point even more explicit by turning our attention over to a hierarchy without labels. We could use Scambler's, but for a more direct comparison we can simply leave out the labels from McAllister's. This gives us essentially the hierarchy used by Porter (2022), except extended with the final level for *On*.

Definition 4.6 (*Extended Porter Hierarchy*) The *extended Porter hierarchy* $\mathbb{P}$ of inferential levels for a language $\mathcal{L}$ is as follows.

$$\begin{aligned} \mathcal{L}^{\mathbb{P}}_{0} &:= \mathcal{L}_0, \\ \mathcal{L}^{\mathbb{P}}_{\alpha+1} &:= \{\langle \Gamma, \Delta \rangle \mid \Gamma \cup \Delta \subseteq \mathcal{L}^{\mathbb{P}}_{\alpha}\}, \\ \mathcal{L}^{\mathbb{P}}_{\lambda} &:= \{\langle \Gamma, \Delta \rangle \mid \Gamma \cup \Delta \subseteq \bigcup\nolimits_{\beta<\lambda} \mathcal{L}^{\mathbb{P}}_{\beta}\}, \\ \mathcal{L}^{\mathbb{P}}_{On} &:= \{\langle \Gamma, \Delta \rangle \mid \Gamma \cup \Delta \subseteq \bigcup\nolimits_{\beta\in On} \mathcal{L}^{\mathbb{P}}_{\beta}\}. \end{aligned}$$

Where λ is a limit and α is any ordinal.

Now we can easily obtain the analogue of Corollary 4.4.

Proposition 4.7 $\mathcal{L}^{\mathbb{P}}_{On} \subseteq \bigcup_{\beta \in On} \mathcal{L}^{\mathbb{P}}_{\beta}$.

Proof Consider an operation F on inferences from $\mathbb{M}$ which deletes all labels. Recursively: $F(\varphi) := \varphi$ for $\varphi \in \mathcal{L}_0$, and $F(\langle \Gamma, \Delta, i \rangle) := \langle F[\Gamma], F[\Delta] \rangle$ for $i \in On$ or $i = *$. Two facts are immediate.

(i) If $\langle \Gamma, \Delta, j \rangle \in \mathcal{L}^{\mathbb{M}}_i$, then $F(\langle \Gamma, \Delta, j \rangle) \in \mathcal{L}^{\mathbb{P}}_i$.
(ii) For every $\langle \Gamma, \Delta \rangle \in \mathcal{L}^{\mathbb{P}}_i$, there is some $\langle \Gamma', \Delta', j \rangle \in \mathcal{L}^{\mathbb{M}}_i$ such that $F(\langle \Gamma', \Delta', j \rangle) = \langle \Gamma, \Delta \rangle$.

Now take any $\langle \Gamma, \Delta \rangle \in \mathcal{L}^{\mathbb{P}}_{On}$. By (ii), there exists some $\langle \Gamma', \Delta', * \rangle \in \mathcal{L}^{\mathbb{M}}_{On}$ such that $\langle \Gamma, \Delta \rangle = F(\langle \Gamma', \Delta', * \rangle)$. By Corollary 4.4, there exists $\alpha \in On$ such that $\langle \Gamma', \Delta', \alpha \rangle \in \mathcal{L}^{\mathbb{M}}_{\alpha}$. By (i), $F(\langle \Gamma', \Delta', \alpha \rangle) \in \mathcal{L}^{\mathbb{P}}_{\alpha}$. But $F(\langle \Gamma', \Delta', \alpha \rangle) = F(\langle \Gamma', \Delta', * \rangle) = \langle \Gamma, \Delta \rangle$. Therefore $\langle \Gamma, \Delta \rangle \in \mathcal{L}^{\mathbb{P}}_{\alpha}$. Thus $\langle \Gamma, \Delta \rangle \in \bigcup_{\beta \in On} \mathcal{L}^{\mathbb{P}}_{\beta}$. Since $\langle \Gamma, \Delta \rangle$ was arbitrary, we conclude that $\mathcal{L}^{\mathbb{P}}_{On} \subseteq \bigcup_{\beta \in On} \mathcal{L}^{\mathbb{P}}_{\beta}$.[9] □

This gives us our titular result.

Theorem 4.8 *Every logic on* $\mathbb{P}$ *is collectively unique.*

Proof Take any logic $\mathbf{L}$ on $\mathbb{P}$, and $\mathbf{M}$ on $\mathbb{P}$ which agrees with $\mathbf{L}$ on $\{\mathcal{L}^{\mathbb{P}}_{\alpha} \mid \alpha \in On\}$. By Proposition 4.7, this means that $\mathbf{L}$ and $\mathbf{M}$ must also agree on $\mathcal{L}^{\mathbb{P}}_{On}$. Therefore they agree everywhere. □

To be thorough, let's make sure that the same outcome holds if we extend Scambler's hierarchy in the obvious way.

Definition 4.9 (*Extended Scambler Hierarchy*) The *extended Scambler hierarchy* $\mathbb{S}$ of inferential levels for a language $\mathcal{L}$ is as follows.

$$
\begin{aligned}
\mathcal{L}^{\mathbb{S}}_0 &:= \mathcal{L}_0, \\
\mathcal{L}^{\mathbb{S}}_{\alpha+1} &:= \{\langle \Gamma, \Delta \rangle \mid \Gamma \cup \Delta \subseteq \bigcup\nolimits_{\beta \leq \alpha} \mathcal{L}^{\mathbb{S}}_{\beta}\}, \\
\mathcal{L}^{\mathbb{S}}_{\lambda} &:= \{\langle \Gamma, \Delta \rangle \mid \Gamma \cup \Delta \subseteq \bigcup\nolimits_{\beta < \lambda} \mathcal{L}^{\mathbb{S}}_{\beta}\}, \\
\mathcal{L}^{\mathbb{S}}_{On} &:= \{\langle \Gamma, \Delta \rangle \mid \Gamma \cup \Delta \subseteq \bigcup\nolimits_{\beta \in On} \mathcal{L}^{\mathbb{S}}_{\beta}\}.
\end{aligned}
$$

Where λ is a limit and α is any ordinal.

[9] Of course, instead of piggybacking on Corollary 4.4, we could also have proven this claim directly. In particular, we could have done so by mirroring the steps we took towards Corollary 4.4. Though it would be slightly more cumbersome this time around, precisely because as McAllister has pointed out, having the levels overlap does complicate some proofs a bit.

We first show that while the Porter and Scambler hierarchies differ about *when* certain inferences first appear, they agree on which ones eventually appear.

Lemma 4.10 $\bigcup_{\beta\in On}\mathcal{L}^{\mathbb{P}}_{\beta}=\bigcup_{\beta\in On}\mathcal{L}^{\mathbb{S}}_{\beta}$.

Proof For the left to right inclusion: it is immediate from the constructions that $\mathcal{L}^{\mathbb{P}}_{\beta}\subseteq\mathcal{L}^{\mathbb{S}}_{\beta}$ for each $\beta\in On$.

For the other direction, we show that if $X\in\mathcal{L}^{\mathbb{S}}_{\beta}$, then there exists $\alpha\in On$ such that $X\in\mathcal{L}^{\mathbb{P}}_{\alpha}$, by induction on β. If $\beta=0$ and X is a formula, this is trivial. If β is a successor or a limit ordinal and $X=\langle\Gamma,\Delta\rangle$, we suppose for the induction hypothesis that the claim holds at all $\beta'<\beta$. We know by definition that every $\gamma\in\Gamma\cup\Delta$ is in $\mathcal{L}^{\mathbb{S}}_{\beta'}$ for some $\beta'<\beta$, so by the induction hypothesis, we can let α_γ be the smallest ordinal such that $\gamma\in\mathcal{L}^{\mathbb{P}}_{\alpha_\gamma}$. Since $\Gamma\cup\Delta$ is a set, it follows by the axiom schema of replacement that $\{\alpha_\gamma\mid\gamma\in\Gamma\cup\Delta\}$ is too. Thus we let α be the first limit ordinal bigger than $\bigcup\{\alpha_\gamma\mid\gamma\in\Gamma\cup\Delta\}$. Then $\langle\Gamma,\Delta\rangle\in\mathcal{L}^{\mathbb{P}}_{\alpha}$ by the limit clause of Definition 4.6. $\square$

The other results then carry over from the Porter hierarchy.

Proposition 4.11 $\mathcal{L}^{\mathbb{S}}_{On}\subseteq\bigcup_{\beta\in On}\mathcal{L}^{\mathbb{S}}_{\beta}$.

Proof $\mathcal{L}^{\mathbb{S}}_{On}=\mathcal{L}^{\mathbb{P}}_{On}\subseteq\bigcup_{\beta\in On}\mathcal{L}^{\mathbb{P}}_{\beta}=\bigcup_{\beta\in On}\mathcal{L}^{\mathbb{S}}_{\beta}$, where the subclass relation in the middle is Proposition 4.7, and the two identities on either side are by Lemma 4.10. $\square$

Theorem 4.12 *Every logic on $\mathbb{S}$ is collectively unique.*

Proof Analogous to Theorem 4.8. $\square$

So given a hierarchy without labels, whether it be with or without cumulative successors, every logic is collectively unique.

It is worth pausing briefly to observe the robustness of this outcome. Recall that we have taken a 'logic' on $\mathbb{H}$ to be *any* subclass of $\bigcup_{i\in On\vee i=On}\mathcal{L}^{\mathbb{H}}_{i}$, and this means that the results above are independent of whether we are thinking in terms of e.g. local or global validity. We can go a bit further still. It has been argued that logics consist not just of their validities, but also their so-called antivalidities (Scambler, 2020a) and/or contingencies (Barrio & Pailos, 2022) at each level. If we think that these are each independent of the validities, we could accordingly have defined a logic on $\mathbb{H}$ as consisting of *three* subclasses of $\bigcup_{i\in On\vee i=On}\mathcal{L}^{\mathbb{H}}_{i}$, representing the valid, antivalid and contingent inferences of the logic. In similar vein, McAllister (2024, Chs. 3 & 9) herself at points understands logics as quadruples of full consequence relations, since she wants to distinguish between the local and global validities and antivalidities. It is easy to see that the results of this section would still hold under any such conception.

This robustness is due to the fact that the results are ultimately about the structure of the hierarchies themselves.

5 Let's not Put a Label on It

The conclusion that I think we should draw from Sect. 4 is as follows: no logic suffers from collective non-uniqueness, hence logical underdetermination and the related difficulty with identifying logics do not occur. Therefore further consequences, such as the supposed problems for logical monism and belief revision about logic, dissolve in turn as well. However, we are not quite in a position to conclude this just yet. So far, all we have seen is that with labels, collective non-uniqueness applies to at least some logics, and without them it does not. This leads to the conclusion just described only because there are strong grounds to prefer a set-up which is (for all intents and purposes) unlabeled.

To start, let's consider what counts in favour of labels. The only justification McAllister provides for them is the one we have already seen: that their presence allows us to simplify and streamline notation, definitions and proofs. However, this is a point about mere convenience. Such considerations only come into play when nothing important hinges on the choice in question. The fact that one definitional set-up leads to shorter subsequent proofs than another cannot be a factor when the results themselves are substantially affected. And as we have seen in Sect. 4, key results about collective uniqueness turn on the choice between having labels or not. In light of this fact, the point about simplicity becomes moot. Furthermore, there does not appear to be any other, more principled way to motivate the inclusion of labels.

On the other hand, the addition of labels comes at sizable costs. First of all: the labels are to blame for collective non-uniqueness, and thereby for logical underdetermination and the like, which McAllister herself regards as deeply unwelcome phenomena. However, the more basic problem with the labels is simply that they open the door to label sensitivity. For label sensitivity is highly implausible. Recall that we defined a logic as label sensitive when it validates some inference $\Gamma \Rightarrow_i \Delta$, yet invalidates some relabeled copy $\Gamma \Rightarrow_j \Delta$. Such logics make possible the following scenario. Suppose logician A asks their colleague, logician B, whether Δ follows from Γ or not according to B's favourite logic, whereupon B responds: "Well, that depends on the level. There are as many versions of the inference from Γ to Δ as there are ordinals, you see, so you'll have to be a tad more specific. For example, at α, the answer is yes. But at λ? Certainly not. At λ', however, I'd have to check. Now at...". This scenario strikes me as bizarre. The (in)validity of an inference according to a logic can and should depend on nothing more than the logic, on the one hand, and the premises and conclusions of the inference on the other. There is nothing in our understanding of logical validity which suggests that it is relative to the specification of an ordinal. Indeed, it is difficult to see what the difference between two relabeled copies of an inference could even be, beyond the purely formal level of their settheoretic representation, let alone how this difference could explain why one is valid but the "other" is not.[10]

It is worth noting that the phenomenon of label sensitivity is actually much broader than discussed thus far. Let us say that two inferences $\langle\Gamma, \Delta, i\rangle$ and $\langle\Gamma', \Delta', j\rangle$ are *label variants* just in case $F(\langle\Gamma, \Delta, i\rangle) = F(\langle\Gamma', \Delta', j\rangle)$, where F removes all labels (see the proof of Proposition 4.7). Until now we've focused on the simplest case of label variance, where $\Gamma = \Gamma'$ and $\Delta = \Delta'$, which means that $i \neq j$ if the inferences are to be distinct. But this is just the tip of the iceberg. The inferences might instead be label variants because $F[\Gamma] = F[\Gamma']$ and $F[\Delta] = F[\Delta']$, in which case they can be distinct even when $i = j$. In other words: $\mathbb{M}$ not only distinguishes label variants which have the same relata, but also ones which have *label variants of* the same relata. To illustrate, consider a propositional version of contraposition: $\{p \Rightarrow_1 q\} \Rightarrow_2 \{\neg q \Rightarrow_1 \neg p\}$. Of course, for each limit level $\mathcal{L}^{\mathbb{M}}_{\lambda}$, we get a copy $\{p \Rightarrow_1 q\} \Rightarrow_{\lambda} \{\neg q \Rightarrow_1 \neg p\}$. But what's more, at the same level we also get distinct copies $\{p \Rightarrow_{\lambda'} q\} \Rightarrow_{\lambda} \{\neg q \Rightarrow_{\lambda''} \neg p\}$ for every $\lambda', \lambda'' < \lambda$. In fact, things go one step further still: a premise or conclusion set might contain *multiple* label variants of any given premise/conclusion. In the presence of labels, sets of inferences start behaving almost a bit like multisets. Therefore the label variants of our contraposition example, within level $\mathcal{L}^{\mathbb{M}}_{\lambda}$, include any inference $\Gamma \Rightarrow_{\lambda} \Delta$ such that $\Gamma \subseteq \{p \Rightarrow_{\lambda'} q \mid \lambda' < \lambda\}$ and $\Delta \subseteq \{\neg q \Rightarrow_{\lambda'} \neg p \mid \lambda' < \lambda\}$. The number of label variants that one inference can have—even within a single, relatively low level—quickly rises as we add more relata and nest them more deeply. And as before, the non-identity of all these variants, as formally represented in $\mathbb{M}$, means that logics on $\mathbb{M}$ can be sensitive to this kind of variation as well, which enables a larger scale version of the bizarre scenario sketched above.

One might point out that excluding label sensitive logics does not necessarily require excluding the labels altogether. While that is true, recall also that collective non-uniqueness results like Theorem 3.2 turned out to be artifacts not just of labels, but of label sensitivity in particular. That is just what Theorem 4.5 tells us. Hence excluding label sensitivity, even while keeping labels, suffices to ensure collective uniqueness. For instance: we could redefine a 'logic' on $\mathbb{M}$ as a logic (in the sense used before) which is not label sensitive. Accordingly, we now say such a logic is unique on $\mathcal{C}$ if it is the only logic (in the not-label-sensitive sense) with precisely its validities on $\mathcal{C}$, and so on. In this terminology, Theorem 4.5 says exactly that every logic on $\mathbb{M}$ is collectively unique. Another option, suggested by an anonymous referee, would be to restrict our hierarchy, by adding at each stage of the construction only those inferences which don't already have a differently labeled copy at a lower level. This way, we could keep the labels, yet there wouldn't be any relabeled copies, and hence no opportunity for label sensitivity. This is certainly possible. But Corollary 4.4 tells us that in a hierarchy built with that restriction, $\mathcal{L}^{\mathbb{H}}_{On}$ would be

[10] We may think that validity is dependent on things like contexts or points of view, as an anonymous reviewer suggests. However, this wouldn't do much to justify sensitivity to *level* labels, since contexts or points of view don't correspond to inferential levels. Perhaps one could model context-sensitivity by adding "context labels", but the ordinals wouldn't be a natural choice for this, and there would be no reason to have the inferences be distributed across the different contexts in the way that they are distributed across the inferential levels.

empty. Hence any two logics agree on $\mathcal{L}^{\mathbb{H}}_{On}$ vacuously, and therefore every logic is collectively unique. The moral is that whether we shut out label sensitive logics by removing labels altogether, or by some other means, it is all the same as far as uniqueness is concerned.

Whatever set-up we use should be label free *in effect*. And when it is label free in effect, every logic is collectively unique.

6 Formalization and Definability

Before wrapping up, we have to discuss a possible objection, striking at one of the basic assumptions made throughout this note. Namely, so far I've followed McAllister in supposing that when Γ or Δ is a proper class, the inference $\Gamma \Rightarrow_i \Delta$ is really not formally definable in classical set theory, by any means. McAllister assumes this when arguing that logics are uncharacterizable at $\mathcal{L}^{\mathbb{M}}_{On}$, and I have relied on it when proving that $\mathcal{L}^{\mathbb{H}}_{On}$ cannot feature any genuinely new inferences, which was in turn used to establish that all logics are collectively unique in the absence of label sensitivity. How justified is this supposition? I think there are two potential paths around it which deserve some attention.

Firstly: the specific technical reason why we could not formally treat $\Gamma \Rightarrow_i \Delta$ with Γ or Δ a proper class is that proper classes cannot be elements of anything, and so the relevant ordered pair $\langle \Gamma, \Delta \rangle$ or triple $\langle \Gamma, \Delta, i \rangle$ cannot be formed. But this seems to be a fact about ordered tuples, rather than about inferences themselves. Arguably, the tuples are just a tool which we use to formally represent the inferences. Moreover, we *choose* to use this particular representation method, and we do so because it is convenient. Then perhaps the lesson from these undefinable proper class-sized inferences should just be that, when extending the inferential hierarchy transfinitely, ordered tuples cease to be a convenient mode of representation, because they don't allow us to define all the inferences we want to. In that case, the question is whether some other representation scheme can do better, or whether there are general reasons why we will always run into similar limitations. Answering this lies beyond our current scope. We will see in a moment where it leaves us.

The second path is as follows. Even if we stick to ordered tuples, it might be submitted that the property of being a proper class is rather less absolute than we have made it out to be. For it is well known that classical set theory admits many models, and that what is a proper class from the perspective of one model is just another set from the perspective of a different, larger model (Zermelo, 1930). Thus one might think that the ordered tuples we need can always be found by moving to a sufficiently large model. Some care must be taken with this, however. The class of all ordinals of a given model $\mathcal{M}$ will constitute a set within a larger model $\mathcal{N}$, but only because from the perspective of $\mathcal{N}$ it won't be the class of *all* ordinals. Similarly, the entire inferential hierarchy of one model $\mathcal{M}$ will be a set within a larger model $\mathcal{N}$, but then from the perspective of $\mathcal{N}$ it isn't indexed by the entire sequence of ordinals, and hence doesn't look like a full hierarchy anymore. Lastly, what $\mathcal{M}$ thinks of as

	Non-Cumulative Successors	Cumulative Successors
No Labels	$\mathbb{P}$	$\mathbb{S}$
Labels	$\mathbb{M}$	$\mathbb{X}$

Fig. 1 Types of transfinite inferential hierarchies

e.g. $\bigcup_{\beta \in On} \mathcal{L}_\beta^{\mathbb{M}}$ will be a set to $\mathcal{N}$, and can hence feature as premise or conclusion set in an ordered tuple, which can then be added to the hierarchy. However, to $\mathcal{N}$ it doesn't look like $\bigcup_{\beta \in On} \mathcal{L}_\beta^{\mathbb{M}}$, as it doesn't include inferences indexed by every ordinal. Similarly for any other proper class we might want to feature in an inference. So it is not clear how much can be achieved by this maneuver. Moreover, the issue seems intertwined with the debate between universe and multiverse views on the different models of set theory (Antos et al., 2015). As in the previous case, exploring this path in depth would take us too far afield.

Instead, we can sidestep these complications by noting that for our current purposes, it ultimately does not matter much whether either (or both) of these paths can be followed successfully. For even if proper class-sized inferences can be formalized in some satisfactory way, that would not only undermine our results from Sect. 4, but also dispel McAllisters uncharacterizability concerns. Hence it would grant us access to $\mathcal{L}_{On}$ when characterizing logics. That means we can simply identify logics by their collection of validities from the entire hierarchy, on which, trivially, every logic is unique. In fact, it is easy to see that every logic on $\mathbb{P}$ or $\mathbb{S}$ is unique even on respectively $\mathcal{L}_{On}^{\mathbb{P}}$ or $\mathcal{L}_{On}^{\mathbb{S}}$ alone, and a logic on $\mathbb{M}$ can be non-unique on $\mathcal{L}_{On}^{\mathbb{M}}$ only in virtue of label sensitivity. Thus we get a different dissolution of the problem, but a dissolution nonetheless.

The overall argument of this note, then, might be read as a proof by cases. Either proper class-sized inferences can after all be formally represented, in which case the point is simply that every logic is unique on $\mathcal{L}_{On}^{\mathbb{H}}$—or on the entire hierarchy, if one prefers. Or they cannot, in which case the considerations of Sects. 4 and 5 kick in to ensure that every logic is unique on $\{\mathcal{L}_\alpha^{\mathbb{H}} \mid \alpha \in On\}$. Either way, at the end of the day we don't have any trouble uniquely identifying logics, nor do we run into logical underdetermination.

7 Conclusion

The introduction raised the question of how we should extend the usual inferential hierarchy into the transfinite. The main options for doing so are tabulated in Fig. 1. The $\mathbb{X}$ in the bottom right represents a breed of hierarchy which, to my knowledge, has not been proposed. Regarding the three explored alternatives, Pailos & Da Ré comment that "Brian Porter and Isabella McAllister get to the transfinite level in a slightly different way [from Scambler], that is, though, similar with respect to each other" (2023, p. 58). This refers, of course, to the choice between cumulative or non-cumulative successor stages, i.e. to the left and right columns of Fig. 1. Surprisingly, however, it instead turns out that—at least as far as uniqueness is concerned—the pivotal division is the one between the table's top and bottom row.[11]

The principal line of reasoning in this note passed through a few key steps. First, we discussed towards the end of Sect. 3 that only collective non-uniqueness leads to such problems as logical underdetermination. We then proved in Sect. 4 that collective non-uniqueness results are impossible in hierarchies which are (effectively) unlabeled. In Sect. 5, I argued that if labels are not merely innocent, simplifying additions, but actually affect the results, then we should get rid of them (or at least of their effects). And finally, in Sect. 6, we observed that if we do manage to formalize proper class-sized inferences, that means we can even rely on $\mathcal{L}^{\mathbb{H}}_{On}$ to uniquely identify logics. Putting these steps together produces two central morals. One: we can and should stick to the top row of Fig. 1, as labels have no business in our metainferential hierarchies. Two: every logic is sufficiently unique, so we can put to rest our worries about logical underdetermination.

There's still plenty more to be figured out about transfinite metainferences. For example, this note focused on the structure of the inferential hierarchies themselves, leaving aside the construction of logics on those hierarchies. On the model-theoretic side, there are open questions about how the new and improved notions of global validity are to be generalized to transfinite levels (Kortenbach, 2025). Moreover, as an anonymous referee points out, there are several known methods of building proof systems for logics defined on the initial hierarchy $\mathbb{B}$ (Cobreros et al., 2022; Da Ré & Pailos, 2022; Fjellstad, 2022; Golan, 2023), and it remains to be seen if and how these can be extended to logics defined on the different types of transfinite hierarchies. I suspect this might be one area in which the choice between $\mathbb{P}$ and $\mathbb{S}$ turns out to make a difference, despite Lemma 4.10. I leave these topics for future occasions.

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[11] While we have not covered hierarchies of type $\mathbb{X}$, it is easy enough to see that any non-uniqueness results for $\mathbb{M}$ must carry over.

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