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ABSTRACT

A new noise, based on vortex structures in 2D (point vortices) and 3D (vortex filaments), is introduced. It is defined as the scaling limit of a jump process, which explores vortex structures, and it can be defined in any domain, also with boundary. The link with fractional Gaussian fields and Kraichnan noise is discussed. The vortex noise is finally shown to be suitable for the investigation of the eddy dissipation produced by small scale turbulence.

I. INTRODUCTION

The theory of Stochastic Partial Differential Equations (SPDEs) is nowadays very well developed (see, for instance, Refs. 8, 20, 29, and 31, with many contributions on fluid dynamics models, like).[3,7,11,12,24,33] However, with the exception of the literature making use of Kraichnan noise, which is motivated in fluid dynamics by its invariance and scaling properties, in most cases, there is no discussion about the origin of noise and its form, in connection with the fact that it is part of a fluid dynamic model. The purpose of this work is to introduce an example of noise based on vortex structures, both in 2D (point vortices) and 3D (vortex filaments). We discuss its motivations and interest for the understanding of fluid properties.

Some preliminary forms in 2D have been introduced in Refs. 13 and 19, but the noise defined here is different and goes much beyond, in particular, because we treat the 3D case on the basis of the theory of random vortex filaments (see Sec. III B).

Usually, in general or theoretical works on SPDEs, the noise is either specified by means of its covariance operator or by means of a finite or countable sum of space-functions multiplied by independent Brownian motions. Here, we start from a different viewpoint. Motivated by the emergence of vortex structures in turbulent fluids, we idealize their production/emergence process by means of a sequence of vortex impulses, mathematically structured using a jump process taking values in a set of vortex structures. This is described in Sec. II. A suitable scaling limit of this jump process gives rise to a Gaussian noise in a suitable Hilbert space. Different examples of such noise depend on different choices of the vortex structures and their statistics, at the level of the jump process. A heuristic picture then emerges of a process that fluctuates very rapidly between the elements of a family of vortex structures. The realizations of this noise are made of vortex structures, which idealize those observed in turbulent fluids—point vortices in 2D and vortex filaments in 3D.

This noise is motivated by turbulent fluids. In the physical literature, the most common noises related to turbulence are the Fractional Gaussian Field (FGF) and Kraichnan noise (see, for instance, Refs. 1, 5, 9, 10, 21–23, and 26). In Sec. IV, we show that on a torus in two and three dimensions, the vortex noise covers FGF and Kraichnan noise by a special choice of the statistical properties of the regularization parameter and the vortex intensity. The vortex noise is thus a flexible ensemble—it may cover also multifractal formalisms (see also Ref. 14)—and its realizations are the limit, as described in Secs. II and III, of localized-in-space vortex structures similar to those observed in turbulent fluids.

Finally, another main motivation for this investigation has been the recent results on eddy dissipation, showing that a transport type noise depending in a suitable way on a scaling parameter, in a transport-diffusion equation, in the scaling limit gives rise to an additional diffusion operator.[13,17] These results require that the covariance function of the noise, computed along the diagonal, $Q(x, x)$, is large, but the operator norm of the covariance is small. We check when the vortex noise satisfies these conditions. Heuristically speaking, they are satisfied when, in the scaling limit, the vortex structures defining the noise are more and more concentrated at *small scales*. This confirms the belief that eddy diffusion is a consequence of turbulence but only when it is suitably small scale.

II. JUMP NOISE AND ITS GAUSSIAN LIMIT

A. Why jump vortex noise in fluid modeling

When a fluid moves through the small obstacles of a boundary (hills, trees, and houses for the lower surface wind, mountains for the lower atmospheric layer, coast irregularities for the sea, and vegetation for a river) or it moves through small obstacles in the middle of the domain (like islands in the sea), vortices are created by these obstacles, sometimes with a regular rhythm (von Kármán vortices) or sometimes more irregularly. In principle, these vortices are the deterministic consequence of the dynamical interaction between the fluid and structure, but in very many applications, we never write the details of those obstacles when a larger scale investigation is done. Hence, it is reasonable to re-introduce the appearance of these vortices, so important for turbulence, in the form of an external perturbation of the equations of motion.

Assume that the velocity field at time t is $u(t,x)$. We may idealize the modification of $u(t,x)$ due to the emergence of a new vortex near an obstacle as an event occuring in a very short time around time t so that we have a jump,

$$u\left(t^{+},x\right) = u\left(t^{-},x\right) + \sigma(x),$$

where $\sigma(x)$ is presumably localized in space and corresponds to a vortex structure. Continuum mechanics does not make jumps; we idealize a fast change due to an instability as a jump for a cleaner mathematical description.

We may develop the previous idea in two directions. The simplest one is suitable for investigations, such as the effect of turbulence on passive scalars,[5] where a simple model of random velocity field is chosen: we consider a stepwise constant velocity field with jumps such as those described above; later on, we shall take a suitable scaling limit and get a Gaussian velocity field, delta correlated in time, with space correlation of very flexible form. A more elaborate proposal is to consider the Navier–Stokes equations with an impulsive force given by a process with jumps,

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \sum_{k\in K}\sum_i \delta\left(t - t_i^k\right)\sigma_k.$$

Here, K is an index set, and for each $k\in K$, we denote by $t_1^k < t_2^k < \cdots$ the sequence of jump times of class k and by σ_k the vortex structure (described at the level of velocity field) arisen at time t_i^k. This way the fluid moves according to the free Navier–Stokes equations between two consecutive jumps times. In Sec. II B, we formalize the noise $\sum_{k\in K}\sum_i \delta\left(t - t_i^k\right)\sigma_k$, or more precisely, similarly to what it is done for white noise and Brownian motion, we formalize the time integral of this distributional process,

$$W_t^0(x) = \sum_{k\in K}\sum_i 1\left\{t \ge t_i^k\right\}\sigma_k. \tag{1}$$

In this first heuristic formulation, it is natural to introduce an index set K, but below, we shall avoid this.

B. Jump vortex noise

Given an open domain $\mathbb{D}\subset\mathbb{R}^d$, $d = 2,3$, denote by $C^\infty_{\mathrm{c,sol}}\left(\mathbb{D},\mathbb{R}^d\right)$ the space of smooth solenoidal vector fields with compact support in $\mathbb{D}$ and denote by H the closure of $C^\infty_{\mathrm{c,sol}}\left(\mathbb{D},\mathbb{R}^d\right)$ in $L^2\left(\mathbb{D},\mathbb{R}^d\right)$. One can prove, under some regularity of the boundary, that $u\in H$ is an $L^2\left(\mathbb{D},\mathbb{R}^d\right)$-vector field, with distributional divergence equal to zero, tangent to the boundary.[32] The norm $\|u\|_H$ is given by $\|u\|_H^2 = \int_{\mathbb{D}}|u(x)|^2dx$.

The following scheme is taken from th work of Métivier,[28] first three chapters. The main tightness and convergence results for martingales, as described in Ref. 28, are due to Rebolledo.[30]

Let P be a Borel probability measure on H. Assume that

$$\int_H \varphi(\|h\|_H)P(dh) < \infty \tag{2}$$

for some nondecreasing $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ that grows faster than quadratic, i.e.,

$$\lim_{n\to\infty}\frac{\varphi(n)}{n^2} = \infty. \tag{3}$$

Denote by Q_P the trace class covariance operator defined as

$$Q_P = \int_H h\otimes hP(dh).$$

Assume that P has zero average,

$$m_P = \int_H hP(dh) = 0. \tag{4}$$

We may also define, a.s. in $x, y \in \mathbb{D}$, the covariance (matrix-valued) function,

$$Q_P(x,y) = \int_H h(x) \otimes h(y) P(dh).$$

Indeed, $\int_H |h(x)|^2 P(dh) < \infty$ for almost every $x \in \mathbb{D}$, thanks to Fubini–Tonelli theorem, since $\int_H \big(\int_{\mathbb{D}} |h(x)|^2 dx\big) P(dh) < \infty$.

Consider the continuous time jump Markov process in H with law of jumps,

$$p(v, v + A) = \frac{1}{\tau} P(A)$$

$[v \in H, A \in \mathcal{B}(H)]$, namely, with the infinitesimal generator,

$$(\mathcal{L}F)(v) = \frac{1}{\tau} \int_H (F(v+h) - F(v)) P(dh),$$

for all bounded continuous functions $F : H \to \mathbb{R}$. Here, $\tau > 0$ is the average interarrival between jumps. Denote by W_t^0 the corresponding Markov process with the initial condition $W_0^0 = 0$. The Dynkin formula

$$F\big(W_t^0\big) - F(0) = \int_0^t (\mathcal{L}F)\big(W_s^0\big) ds + M_t^F$$

gives us the decomposition in a finite variation plus a martingale term. Consider first the case when $F_1(v) = v$ (here we do not write down classical details, namely, that the computation should be done for a continuous bounded cutoff of each component $\langle v, e_i \rangle$, where (e_i) is a complete orthonormal system; see Ref. 28, p. 14). One has

$$(\mathcal{L}F_1)(v) = \frac{1}{\tau} \int_H (v + h - v) P(dh) = 0$$

because $m_P = 0$. Hence, $W_t^0 = M_t^{F_1}$, namely, the process W_t^0 is a martingale. Let us compute its Hilbert-space-valued Meyer process $\langle\langle W^0 \rangle\rangle_t$. We use the function $F_2(v) = v \otimes v$ (again one has to do the computation first for a cutoff of the functions $\langle v, e_i \rangle \langle v, e_j \rangle$),

$$\begin{aligned}(\mathcal{L}F_2)(v) &= \frac{1}{\tau} \int_H ((v+h) \otimes (v+h) - v \otimes v) P(dh)\\ &= \frac{1}{\tau} \int_H (v \otimes h + h \otimes v + h \otimes h) P(dh)\\ &= \frac{1}{\tau} Q_P.\end{aligned}$$

Therefore, $W_t^0 \otimes W_t^0 = \frac{t}{\tau} Q_P + M_t^{F_2}$. The Meyer process $\langle\langle W^0 \rangle\rangle_t$ is thus (see the definition in Ref. 28, pp. 8–12)

$$\langle\langle W \rangle\rangle_t = \frac{t}{\tau} Q_P.$$

C. Convergence of the rescaled process to a Brownian motion

Let us now parameterize and rescale the previous process. We take average interarrival between jumps given by

$$\tau_N = \frac{1}{N^2},$$

and we reduce by $\frac{1}{N}$ the size of jumps by considering a probability measure P_N on H with zero average $m_P = \int_H h P_N(dh) = 0$ and covariance Q_{P_N} given by

$$Q_{P_N} = \frac{1}{N^2} Q_P.$$

Consider the associated process W_t^N, a martingale with the Meyer process

$$\langle\langle W^N \rangle\rangle_t = \frac{t}{\tau_N} Q_{P_N} = t Q_P.$$

Definition 1. Given Q_P, denote by $(W_t)_{t\geq 0}$ a Brownian motion on H with incremental covariance Q_P.

Theorem 2. *The process $(W_t^N)_{t\geq 0}$ converges in law to $(W_t)_{t\geq 0}$, uniformly on every compact set of time, as processes with values in H.*

Proof. Using the classical theorem of tightness for martingales [cf. Ref. 28 (Chap. 2) and Ref. 30], we have that the family of laws of the processes $(W^N)_N$ is tight in the Skorohod space (because the family of laws of $\langle\langle W^N\rangle\rangle$ is tight), and every convergent subsequence has limit given by the law of a martingale W_t with $W_0 = 0$ and Meyer process,

$$\langle\langle W\rangle\rangle_t = tQ_P.$$

If we establish that W has continuous paths, then it is a Brownian motion with incremental covariance Q_P. One can prove that

$$\lim_{N\to\infty}\mathbb{P}\left(\sup_{s\in[0,T]}\left\|\Delta_s W^N\right\|_H > \epsilon\right) = 0, \tag{5}$$

where $\left\|\Delta_s W^N\right\|_H$ is the size of the jump (if any) at time s (W^N is càdlàg). Since the set $\left\{\sup_{s\in[0,T]}\left\|\Delta_s w^N\right\|_H > \epsilon\right\}$ is open in the Skorohod topology, from the Portmanteau theorem, we get

$$\mathbb{P}\left(\sup_{s\in[0,T]}\|\Delta_s W\|_H > \epsilon\right) = 0$$

for every $\epsilon > 0$, and hence, W is continuous. To show (5), denote by $\{s_i\}_{i=0}^{N_T} \subset [0,T]$ the Poisson (τ_N^{-1}) arrival times, and then, we have that

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in[0,T]}\left\|\Delta_s W^N\right\|_H > \epsilon\right) &= 1 - \mathbb{P}\left(\bigcap_{\{s_i\}\subset[0,T]}\left\{\left\|\Delta_{s_i} W^N\right\|_H \leq \epsilon\right\}\right)\\ &= 1 - \mathbb{E}\left[\prod_{\{s_i\}\subset[0,T]}\mathbb{P}\left(\left\|\Delta_{s_i} W^N\right\|_H \leq \epsilon \;\middle|\; \{s_i\}_{i=0}^{N_T}\right)\right]\\ &= 1 - \mathbb{E}\left[\left[1 - \mathbb{P}\left(\left\|\Delta W^N\right\|_H > \epsilon\right)\right]^{N_T}\right],\end{aligned}$$

where we used that given the Poisson arrival times, the laws of each jump size $\left\|\Delta_{s_i} W^N\right\|_H$ is independent of it, and identically distributed as what we simply denote by $\left\|\Delta W^N\right\|_H$. By the elementary inequality $(1-y)^n \geq 1 - ny$, for any $y \in [0,1]$ and $n \in \mathbb{N}$ and Markov's inequality, we have that

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in[0,T]}\left\|\Delta_s W^N\right\|_H > \epsilon\right) &\leq \mathbb{E}[N_T]\mathbb{P}\left(\left\|\Delta W^N\right\|_H > \epsilon\right)\\ &\leq \frac{TN^2}{\varphi(N\epsilon)}\mathbb{E}\left[\varphi\left(\|\Delta W\|_H\right)\right]\\ &= \frac{TN^2}{\varphi(N\epsilon)}\int_H \varphi\left(\|h\|_H\right)P(dh),\end{aligned}$$

which is finite by (2) and converges to zero as $N \to \infty$ by (3). ■

D. Reformulation as a PPP

This is a side section, which, however, may help the intuition [see also (1)]: we reformulate the jump process W_t^0 as a Poisson Point Process (PPP). On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{P}$ be a PPP on $[0,\infty) \times H$ with intensity measure $\lambda Leb \otimes P$, where λLeb is Lebesgue measure scaled by $\lambda > 0$ and P is the probability measure introduced in Subsections II A–II C. Heuristically,

$$\mathcal{P}(dt, du) = \sum_i \delta_{(t_i,\sigma_i)}(dt, du)$$

where (t_i, σ_i) is an i.i.d. sequence with t_i "uniformly distributed on $[0,\infty)$," σ_i distributed according to P, and t_i and σ_i independent of each other. Define the vector valued random field, defined on $(\Omega, \mathcal{F}, \mathbb{P})$,

$$W_t^0(x) = \sum_{t_i\leq t}\sigma_i(x) = \sum_i \sigma_i(x)1\{t_i \leq t\}.$$

Compared to (2), we may think that K in that formula was a finite set, and we have simply reordered the jump times (t_i^k) in a single sequence (t_i) and we have renamed the jump velocity fields. This definition is slightly heuristic because it makes use of the representation as infinite sum, which is true only in a suitable limit sense; a rigorous definition of $W(t,x)$ is

$$W_t^0(x) = \int_{[0,\infty)\times H} u(x) 1\{t' \leq t\} \mathcal{P}(dt', du).$$

However, in the sequel, for the sake of interpretability, we shall always use the heuristic expressions.

The intuition is that eddies $\sigma_i(x)$ are chosen at random with distribution P, with exponential inter-arrival times of rate λ. Condition (4) asks, heuristically speaking, that both an eddy and its opposite are equally likely to be chosen.

Rescale $W_t^0(x)$ as

$$W_t^N(x) = \frac{1}{N}\sum_i \sigma_i(x) 1\{t_i \leq N^2 t\}.$$

Let us compute the expectation and the covariance function of this process. One has [$\mathbb{E}$ denotes the Mathematical expectation on $(\Omega, \mathcal{F}, \mathbb{P})$]

$$\mathbb{E}\big[W_t^N(x)\big] = 0$$

from the independences and condition (4). Moreover,

$$\mathbb{E}\big[W_t^N(x) \otimes W_t^N(y)\big] = \frac{1}{N^2}\sum_i \mathbb{E}\big[\sigma_i(x) \otimes \sigma_i(y) 1\{t_i \leq N^2 t\}\big],$$

having used the independence when $i \neq j$ and property (4) again; hence,

$$= \frac{Q_P(x,y)}{N^2}\sum_i \mathbb{P}\big(t_i \leq N^2 t\big).$$

Proposition 3.

$$\sum_i \mathbb{P}\big(t_i \leq N^2 t\big) = N^2 \lambda t.$$

Hence,

$$\mathbb{E}\big[W_t^N(x) \otimes W_t^N(y)\big] = \lambda t Q_P(x,y).$$

Proof. We note that

$$\sum_i \mathbb{P}\big(t_i \leq N^2 t\big) = \mathbb{E}\Bigg[\sum_i 1\{t_i \leq N^2 t\}\Bigg] = \mathbb{E}\big[\eta_\lambda(N^2 t)\big] = N^2 \lambda t,$$

where $\eta_\lambda(\cdot)$ denotes a Poisson process on $\mathbb{R}_+$ with intensity λ.

This is another way of seeing the link between the noise with jumps and the covariance of the limit Brownian motion. ■

III. EXAMPLES IN 2D AND 3D

The mathematical object discussed in Sec. II B and C, although initially motivated by vortex structures, was completely general: given any probability measure P on H with covariance Q_P, the previous construction and results apply and defines a Brownian motion W_t in H with covariance operator Q_P. Note that P is not necessarily Gaussian: P and W_1 have both covariance Q_P, but only W_1 needs to be Gaussian. In a sense, we "realize" approximately samples of the Brownian motion W_t by means of samples of a possibly "nonlinear" (non-Gaussian) process W_t^N.

In this section, we give our two main examples of the measure P, highly non Gaussian. It is inspired by vortex structures.

Common to both descriptions are a few objects. First, given $\delta > 0$, we define

$$\mathbb{D}_\delta := \{x \in \mathbb{D} : \ \mathrm{dist}(x, \mathbb{D}^c) > \delta\}.$$

Second, we have a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and several $\mathcal{F}_0$-measurable r.v.'s: (a) X_0 with law $p_0(dx)$ supported on $\mathbb{D}_\delta$, which will play the role of the center of the vortex in 2D and the initial position of the vortex filament in 3D; (b) Γ, real valued, with the physical meaning of circulation, with

$$\mathbb{E}[\Gamma] = 0, \qquad \mathbb{E}\big[|\Gamma|^p\big] < \infty \text{ for some } p > 2,$$
$$\sigma^2 := \mathbb{E}\big[\Gamma^2\big];$$

(c) L, positive valued, randomizing the size of the mollification, with the property

$$\mathbb{P}(L \in (0, \delta/2)) = 1;$$

(d) U, positive valued, randomizing the length of the vortex filament. Moreover, in 3D, we also have (e) a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with values in $\mathbb{R}^3$. In the 2D case, we just take $\mathcal{F} = \mathcal{F}_0$ and do not need the filtration.

For sake of simplicity of exposition, we shall always assume that X_0, Γ, L, and U are independent, but most of the results can be extended to more general cases.

The last common element of the theory is a smooth symmetric probability density θ supported in the ball $B(0,1)$ and its rescaled mollifiers,

$$\theta_\ell(x) = \ell^{-d}\theta\big(\ell^{-1}x\big), \tag{6}$$

with support in $B(0,\ell)$.

A. Point vortices and definition of *P* in the 2D case

In 2D, by a point vortex, we mean a vorticity field of delta Dirac type, δ_{x_0}; its use in 2D fluid mechanics is manifold (see, for instance, Ref. 27). If the vorticity is assumed distributional and equal to δ_{x_0}, with x_0 in the interior of $\mathbb{D}$, then the so-called stream function $\psi_{\mathbb{D},x_0}$ is given by the solution of

$$-\Delta\psi_{\mathbb{D},x_0} = \delta_{x_0} \text{ in } \mathbb{D},$$
$$\psi_{\mathbb{D},x_0}|_{\partial\mathbb{D}} = 0,$$

and the associated velocity vector field is given by

$$u_{\mathbb{D},x_0}(x) = \nabla^\perp\psi_{\mathbb{D},x_0}(x),$$

where $\nabla^\perp f = (\partial_2 f, -\partial_1 f)$. One has

$$\psi_{\mathbb{D},x_0}(x) = \frac{1}{2\pi}\log\frac{1}{|x - x_0|} + h_{\mathbb{D},x_0}(x),$$

where $h_{\mathbb{D},x_0}$ is a smooth function, solution of the problem

$$-\Delta_x h_{\mathbb{D},x_0} = 0 \text{ in } \mathbb{D},$$
$$h_{\mathbb{D},x_0}(x) = \frac{1}{2\pi}\log|x - x_0| \text{ for } x \in \partial\mathbb{D}.$$

In the sequel, as it is customary, we shall denote $u_{\mathbb{D},x_0}(x)$ simply by $K(x,x_0)$. Hence,

$$K(x,x_0) = -\frac{1}{2\pi}\frac{(x - x_0)^\perp}{|x - x_0|^2} + \nabla^\perp h_{\mathbb{D},x_0}(x), \tag{7}$$

where $x^\perp = (x_2, -x_1)$.

Recall that θ_ℓ (6), as $\ell \to 0$ is an approximation of the Dirac delta function. Expressions of the form $\theta_\ell(x - x_0)$ are idealized smoothed point vortices, at the vorticity level, and the associated velocity field is

$$K_\ell(x,x_0) := \int_D K(x,y)\theta_\ell(y - x_0)dy.$$

With these preliminaries, let us define P.

Definition 4. In the 2D case, the probability measure P on the space H is the law of the H-valued r.v.,

$$\Gamma K_L(x,X_0) = \Gamma\int_{\mathbb{D}} K(x,y)\theta_L(y - X_0)dy. \tag{8}$$

For future reference, the spatial covariance matrix of the vortex noise in 2D is given by

$$Q_{\text{vortex}}(x,x') = \mathbb{E}\big[\Gamma K_L(x,X_0) \otimes \Gamma K_L(x',X_0)\big], \quad x,x' \in \mathbb{D}. \tag{9}$$

Proposition 5. The random vector field of Definition 4 takes values in H. If

$$\mathbb{E}\big(|\Gamma|^p L^{-p}\big) < \infty,$$

for some $p > 2$, then it satisfies (2) and (3). Moreover, it satisfies (4).

Proof. Fixing any $p > 2$, we compute by independence between X_0, Γ, and Hölder's inequality [with $p_1 = p/2, p_2 = p/(p-2)$ such that $1/p_1 + 1/p_2 = 1$]

$$\begin{aligned}\int_H \|h\|_H^p P(dh) &= \mathbb{E}\bigg[\bigg(\int_{\mathbb{D}} |\Gamma K_L(x, X_0)|^2 dx\bigg)^{p/2}\bigg]\\ &= \mathbb{E}\bigg(|\Gamma|^2 \int_{\mathbb{D}}\bigg|\int_{\mathbb{D}} K_L(x,y)\theta_L(y - X_0)dy\bigg|^2 dx\bigg)^{p/2}\\ &\le \mathbb{E}\big[|\Gamma|^p\big]|\mathbb{D}|^{\frac{p}{2}-1}\mathbb{E}\int_{\mathbb{D}}\bigg|\int_{\mathbb{D}} K_L(x,y)\theta_L(y - X_0)dy\bigg|^p dx,\end{aligned}$$

where recall that

$$K(x,y) = \frac{1}{2\pi}\frac{(x-y)^\perp}{|x-y|^2} + h_{\mathbb{D},y}(x).$$

Per fixed $x \in \mathbb{D}$, we perform the following analysis. Since $X_0 \in \mathbb{D}_\delta$ and $|y - X_0| \le L < \delta/2$ for any y contributing to the above integral, we have $y \in \mathbb{D}_{\delta/2}$. Therefore, the part $\nabla_x^\perp h_{\mathbb{D},y}(x)$ of the kernel $K(x,y)$ is smooth as a function of $x \in \mathbb{D}$ for every $y \in \mathbb{D}_{\delta/2}$. Due to continuous dependence of $h_{\mathbb{D},y}(x)$ on boundary conditions, hence on the variable y, the following constant is finite:

$$C(\mathbb{D},\delta) := \sup_{y\in\overline{\mathbb{D}}_{\delta/2}}\sup_{x\in\overline{\mathbb{D}}}|\nabla_x^\perp h_{\mathbb{D},y}(x)|.$$

The contribution of $\nabla_x^\perp h_{\mathbb{D},y}(x)$ to the above integral hence is finite, i.e.,

$$\begin{aligned}&\mathbb{E}\bigg|\int_{\mathbb{D}}\nabla_x^\perp h_{\mathbb{D},y}(x)\theta_L(y - X_0)dy\bigg|^p dx\\ &\quad\le \mathbb{E}\bigg(\int_{\mathbb{D}}|\nabla_x^\perp h_{\mathbb{D},y}(x)|\theta_L(y - X_0)dy\bigg)^p\\ &\quad\le C(\mathbb{D},\delta)^p\mathbb{E}\bigg(\int_{\mathbb{D}}\theta_L(y - X_0)dy\bigg)^p \le C(\mathbb{D},\delta)^p,\end{aligned}$$

where we used that $\int_{\mathbb{D}}\theta_L(y - X_0)dy = 1$ for any realization of X_0. It suffices now to focus on the other part of the kernel $(2\pi)^{-1}\frac{(x-y)^\perp}{|x-y|^2}$. We have that

$$\begin{aligned}&\mathbb{E}\bigg|\int_{\mathbb{D}}\frac{(x-y)^\perp}{|x-y|^2}\theta_L(y - X_0)dy\bigg|^p\\ &\quad\le \mathbb{E}\bigg(\int_{\mathbb{D}}\frac{1}{|x-y|}\theta_L(y - X_0)dy\bigg)^p\\ &\quad\overset{y'=L^{-1}y}{=}\quad \mathbb{E}\bigg(\int_{\mathbb{D}}\frac{L^{-1}}{|L^{-1}x - y'|}\theta(y' - L^{-1}X_0)dy'\bigg)^p\\ &\quad\le \|\theta\|_\infty^p\mathbb{E}\bigg(L^{-1}\int_{B(L^{-1}X_0,1)}\frac{1}{|L^{-1}x - y'|}dy'\bigg)^p\\ &\quad\le \|\theta\|_\infty^p\mathbb{E}\bigg(L^{-1}\int_{B(L^{-1}(x-X_0),1)}\frac{1}{|y''|}dy''\bigg)^p\\ &\quad\le \|\theta\|_\infty^p\mathbb{E}\bigg(L^{-1}\int_{B(0,1)}\frac{1}{|y''|}dy''\bigg)^p\\ &\quad\le C_{p,\theta}\mathbb{E}[L^{-p}],\end{aligned}$$

where we use the fact that the integral of $|y''|^{-1}$ over a unit ball centered anywhere in $\mathbb{R}^2$ is maximized when the center is the origin, and nonrandom constant $C_{p,\theta}$ is independent of x. Hence, we get that

$$\int_H \|h\|_H^p P(dh) \le C_{p,\theta} |\mathbb{D}|^{\frac{p}{2}} \mathbb{E}\big(|\Gamma|^p L^{-p}\big).$$

Finally, it satisfies (4),

$$\mathbb{E}\bigg[\Gamma \int_{\mathbb{D}} K(\cdot,y)\theta_L(y-X_0)dy\bigg] = \mathbb{E}[\Gamma]\mathbb{E}\bigg[\int_{\mathbb{D}} K(\cdot,y)\theta_L(y-X_0)dy\bigg] = 0$$

because the second expectation is finite and the first one is equal to zero, by assumption. ■

The case when $L = 0$ is outside the previous definition and result. The velocity field $K(x,x_0)$ is not of class H. Nevertheless, it is of class $L^p(\mathbb{D},\mathbb{R}^2)$ for $p < 2$ or of class $H^{-s}(\mathbb{D},\mathbb{R}^2)$ for $s > 0$. Therefore, we may consider the random field,

$$\Gamma K(x, X_0),$$

taking values in these spaces and call P its law. We shall see below that it satisfies certain special properties.

B. Vortex filaments and the definition of *P* in the 3D case

In 3D, by vortex filament we mean a distributional vector valued field (a "current," in the language of Calculus of Variations[18]), given by

$$\int_0^{U\wedge\tau} \delta_{X_t} dX_t,$$

where X_t is a function or a process such that the previous expression is well defined. We have already introduced a possibly relevant stopping time τ because it may help to cope with the presence of a boundary. Stochastic currents have been introduced and investigated in some works.[2,4,15,16] We do not need, strictly speaking, that theory here since we shall always deal with mollified objects, except in one section where we explain what is necessary. In this work, we shall always assume that (X_t) has the law of a Brownian motion, but it is interesting to investigate also other processes, for instance, directed polymers, such as in Ref. 25.

The following construction of a vortex filament in 3D is due to Ref. 14 (which we slightly modify). Let $(\Gamma, U, \ell) \in \mathbb{R}_+^3$ be a triple whose joint distribution is given by some probability measure $\nu(d\gamma, du, d\ell)$ (assumed to be a product measure for simplicity). Let $(X_t)_{t\ge 0}$ denote a 3D Brownian motion starting with X_0 distributed with a probability density $p_0(x)$ supported in $\mathbb{D}_\delta$, where $p_0(x) \in [p_{\min}, p_{\max}] \subset (0,\infty)$. We call $\mathcal{W}$ its law, which we assume to be independent of $\nu(\cdot)$. Define the first exit time from $\mathbb{D}_\delta$ of (X_t) by

$$\tau = \tau^{\mathbb{D}_\delta} := \inf\{t \ge 0 : \ X_t \in \mathbb{D}_\delta^c\} \in [0,\infty).$$

We consider random vorticity fields defined as

$$\int_0^{U\wedge\tau} (\theta * \delta_{X_t})(x) dX_t = \int_0^{U\wedge\tau} \theta(x - X_t) dX_t.$$

Let $A(x)$ be the vector potential defined path by path by the solution of the equation

$$-\Delta A(x) = \int_0^{U\wedge\tau} \theta(x - X_t) dX_t \ \text{ in } \mathbb{D},$$
$$A|_{\partial\mathbb{D}} = 0$$

and extend $A = 0$ outside of $\mathbb{D}$, when necessary. Then, the associated velocity is given by

$$u(x) = \operatorname{curl} A(x).$$

Concerning the Biot–Savart kernel, here we have

$$\psi_{\mathbb{D},x_0}(x) = \frac{1}{4\pi}\frac{1}{|x - x_0|} + h_{\mathbb{D},x_0}(x),$$

where $h_{\mathbb{D},x_0}$ is a smooth function, solution of the problem

$$-\Delta_x h_{\mathbb{D},x_0} = 0 \ \text{ in } \mathbb{D},$$
$$h_{\mathbb{D},x_0}(x) = -\frac{1}{4\pi}\frac{1}{|x - x_0|} \ \text{ for } x \in \partial\mathbb{D}.$$

As usual, we shall denote curl $\psi_{\mathbb{D},x_0}(x)$ simply by $K(x,x_0)$, which now is vector valued and its action on a generic vector v is given by

$$K(x,x_0)\times v := -\frac{1}{4\pi}\frac{(x-x_0)\times v}{|x-x_0|^3} + \nabla_x h_{\mathbb{D},x_0}(x)\times v. \tag{10}$$

Definition 6. In the 3D case, the probability measure P on the space H is the law of the H-valued r.v.,

$$\Gamma K_L(x,X_\cdot) := \Gamma\int_{\mathbb{D}} K(x,y)\times\left(\int_0^{U\wedge\tau}\theta_L(y-X_t)dX_t\right)dy. \tag{11}$$

Remark 7. We use the killed BM, not the normally reflected BM, in the definition of the filament because the latter is not a local martingale, only a semimartingale due to the boundary push term, which leads to difficulties in integration against dX_t.

For future reference, the spatial covariance matrix of the vortex noise in 3D is given by

$$Q_{\text{vortex}}(x,x') = \mathbb{E}\left[\Gamma K_L(x,X_\cdot)\otimes\Gamma K_L(x',X_\cdot)\right],\quad x,x'\in\mathbb{D}. \tag{12}$$

Proposition 8. The random vector field of Definition 4 takes values in H. If

$$\mathbb{E}\left(|\Gamma|^p U^{\frac{p}{2}}L^{-2p}\right)<\infty,$$

for some $p>2$, then it satisfies (2) and (3). Moreover, it satisfies (4).

Proof. Fix any $p>2$, then we compute

$$\int_H \|h\|_H^p P(dh) = \mathbb{E}\left[\left(\int_{\mathbb{D}}|\Gamma u(x)|^2dx\right)^{p/2}\right].$$

Fixing any realization of (Γ,U,L) according to measure ν, we take expectation with respect to the Wiener measure $\mathcal{W}$ first. By Hölder's inequality and $p/2>1$ and Burkholder–Davis–Gundy inequality, we compute

$$\begin{aligned}
&\mathcal{W}\left[\left(\int_{\mathbb{D}}|u(x)|^2dx\right)^{p/2}\right]\\
&\quad=\mathcal{W}\left[\left(\int_{\mathbb{D}}\left|\int_0^{U\wedge\tau}\int_{\mathbb{D}_\delta}K(x,y)\theta_L(y-X_t)dy\times dX_t\right|^2dx\right)^{p/2}\right]\\
&\quad\le|\mathbb{D}|^{\frac{p}{2}-1}\mathcal{W}\left[\int_{\mathbb{D}}\left|\int_0^{U\wedge\tau}\int_{\mathbb{D}_\delta}K(x,y)\theta_L(y-X_t)dy\times dX_t\right|^p dx\right]\\
&\quad=|\mathbb{D}|^{\frac{p}{2}-1}\int_{\mathbb{D}}dx\,\mathcal{W}\left[\left|\int_0^{U\wedge\tau}\int_{\mathbb{D}_\delta}K(x,y)\theta_L(y-X_t)dy\times dX_t\right|^p\right]\\
&\quad\le|\mathbb{D}|^{\frac{p}{2}-1}\int_{\mathbb{D}}dx\,\mathcal{W}\left[\left|\int_0^{U\wedge\tau}2\left|\int_{\mathbb{D}_\delta}K(x,y)\theta_L(y-X_t)dy\right|^2dt\right|^{p/2}\right].
\end{aligned}$$

Since $X_{t\wedge\tau}\in\mathbb{D}_\delta$, we have that any y that contributes to the above integral is supported in $y\in\mathbb{D}_{\delta/2}$; hence, $\nabla_x h_{\mathbb{D},y}(x)$ part of the kernel $K(x,y)$ is uniformly bounded, i.e.,

$$\sup_{y\in\overline{\mathbb{D}}_{\delta/2}}\sup_{x\in\overline{\mathbb{D}}}|\nabla_x h_{\mathbb{D},y}(x)|\le C(\mathbb{D},\delta).$$

Hence, its contribution in the above integral can be computed, as for any $x\in\mathbb{D}$,

$$\begin{aligned}
&\mathcal{W}\left[\left|\int_0^{U\wedge\tau}\left|\int_{\mathbb{D}}\nabla_x h_{\mathbb{D},y}(x)\theta_L(y-X_t)dy\right|^2dt\right|^{p/2}\right]\\
&\quad\le U^{\frac{p}{2}-1}\int_0^U dt\,\mathcal{W}\left[\left|\int_{\mathbb{D}}\nabla_x h_{\mathbb{D},y}(x)\theta_L(y-X_t)dy\right|^p 1_{\{t\le\tau\}}\right]
\end{aligned}$$

$$\le U^{\frac{p}{2}-1} C(\mathbb{D},\delta)^p \int_0^U dt\ \mathcal{W}\left[\left|\int_{\mathbb{D}} \theta_L(y-X_t)dy\right|^p 1_{\{t\le\tau\}}\right]$$
$$\le U^{\frac{p}{2}} C(\mathbb{D},\delta)^p$$

using that $\int \theta_L(y-X_t)dy = 1$ for every possible realization of $X_{t\wedge\tau} \in \mathbb{D}_\delta$.

It suffices to focus on the other part of the kernel $(4\pi)^{-1}\frac{x-y}{|x-y|^3}$. We can do an explicit calculation: by Hölder's inequality and then a change of variables, we have that for any $x \in \mathbb{D}$,

$$\mathcal{W}\left[\left|\int_0^{U\wedge\tau}\left|\int_{\mathbb{D}}\frac{x-y}{|x-y|^3}\theta_L(y-X_t)dy\right|^2 dt\right|^{p/2}\right]$$
$$\le U^{\frac{p}{2}-1}\int_0^U dt\ \mathcal{W}\left[\left|\int_{\mathbb{D}}\frac{x-y}{|x-y|^3}\theta_L(y-X_t)dy\right|^p 1_{\{t\le\tau\}}\right]$$
$$\overset{y'=L^{-1}y}{=} \quad U^{\frac{p}{2}-1}\int_0^U dt\ \mathcal{W}\left[\left|\int_{L^{-1}\mathbb{D}}\frac{L^{-2}(L^{-1}x-y')}{|L^{-1}x-y'|^3}\theta(y'-L^{-1}X_t)dy'\right|^p 1_{\{t\le\tau\}}\right]$$
$$\le U^{\frac{p}{2}-1}L^{-2p}\|\theta\|_\infty^p\int_0^U dt\ \mathcal{W}\left[\left|\int_{B(L^{-1}X_t,1)}\frac{1}{|L^{-1}x-y'|^2}dy'\right|^p 1_{\{t\le\tau\}}\right]$$
$$= U^{\frac{p}{2}-1}L^{-2p}\|\theta\|_\infty^p\int_0^U dt\ \mathcal{W}\left[\left|\int_{B(L^{-1}(x-X_t),1)}\frac{1}{|y''|^2}dy''\right|^p\right]$$
$$\le U^{\frac{p}{2}}L^{-2p}\|\theta\|_\infty^p\ \mathcal{W}\left[\left|\int_{B(0,1)}\frac{1}{|y''|^2}dy''\right|^p\right]$$
$$\le C_{p,\theta}U^{\frac{p}{2}}L^{-2p},$$

where $C_{p,\theta}$ is a non-random constant independent of x. Indeed, we used the geometric fact that the integral of the function $|y''|^{-2}$ over a unit ball centered at anywhere in $\mathbb{R}^3$ is maximized when the center is the origin.

Thus, we can conclude that

$$\mathbb{E}\left[\left(\int_{\mathbb{D}}|\Gamma u(x)|^2dx\right)^{p/2}\right]\le C_{p,\theta}|\mathbb{D}|^{\frac{p}{2}}\mathbb{E}\left(|\Gamma|^pU^{\frac{p}{2}}L^{-2p}\right)$$

with the finiteness of the RHS providing a sufficient condition.

Finally, it satisfies (4),

$$\mathbb{E}\left[\Gamma\int_{\mathbb{D}}K(\cdot,y)\times\left(\int_0^{U\wedge\tau}\theta_L(y-X_t)dX_t\right)dy\right]$$
$$=\mathbb{E}[\Gamma]\mathbb{E}\left[\int_{\mathbb{D}}K(\cdot,y)\times\left(\int_0^{U\wedge\tau}\theta_L(y-X_t)dX_t\right)dy\right]=0$$

because the second expectation is finite and the first one is equal to zero, by assumption. ■

IV. VORTEX NOISES REPRODUCE FRACTIONAL GAUSSIAN FIELDS AND KRAICHNAN NOISE

In this section, we analyze the covariance operators of our vortex noises constructed above in 2D and 3D and show that our vortex noises are instances of Fractional Gaussian Fields,[26] which is a broad class of Gaussian generalized random fields that includes Gaussian Free Field (GFF) and Kraichnan noise. We show that by choosing the statistical parameters of our model suitably, we can reproduce a large class of FGF. It may also reproduce multifractal vector fields, which was the main motivation of study in Ref. 14.

For simplicity, our fields are defined on the torus $\mathbb{T}^d$, $d = 2, 3$.

In the scalar case and on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, the classical d-dimensional FGF of index $s \in \mathbb{R}$ is the Gaussian field with covariance $(-\Delta)^{-s}$, where Δ is the Laplacian in on $\mathbb{T}^d$ (see Ref. 26). The case $s = 1$ is called Gaussian Free Field (GFF). Similarly, let us introduce a Gaussian measure on solenoidal vector fields. Let H be the space of mean zero periodic L^2 solenoidal vector fields. The Stokes operator is defined as

$$A : D(A) \subset H \to H,$$
$$D(A) = H^2\left(\mathbb{T}^d, \mathbb{R}^d\right),$$
$$Av = \Delta v$$

(no projection of $L^2(\mathbb{T}^d, \mathbb{R}^d)$ to H is needed here, opposite to the case of a bounded domain with Dirichlet boundary conditions). The Laplacian Δv is computed componentwise. The operator A is invertible in H (see Ref. 32). With these definitions at hand, we call Solenoidal Fractional Gaussian Field (SFGF) of index $s \in \mathbb{R}$ the Gaussian measure with covariance $(-A)^{-s}$. The case $s = 1$ will be called Solenoidal Gaussian Free Field (SGFF).

A. Covariance of 2D vortex noise

Let us first consider the 2D case, and recall the definition of the noise based on point vortices (8).

The covariance operator of our noise is given by

$$\langle \mathbb{Q}v, w\rangle = \mathbb{E}\left[\Gamma^2 \int_{\mathbb{T}^2} K_L(x, X_0)\cdot v(x)dx \int_{\mathbb{T}^2} K_L(x', X_0)\cdot w(x')dx'\right].$$

Call $Q_{\text{vortex}}(x, x')$ its covariance function (matrix-valued) such that

$$\langle \mathbb{Q}v, w\rangle = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} v(x)^T Q_{\text{vortex}}(x, x') w(x') dx dx'.$$

It is clear (and proved below) that it is homogeneous,

$$Q_{\text{vortex}}(x, x') = Q_{\text{vortex}}(x - x')$$

for a matrix function $Q_{\text{vortex}}(x)$. In the sequel, we denote by $\mathbb{Z}_0^d$ the set $\mathbb{Z}^d \backslash \{0\}$.

Proposition 9. Assume θ symmetric and X_0 independent of (Γ, L) and uniformly distributed. Then,

$$Q_{vortex}(x) = \sum_{\mathbf{k}\in\mathbb{Z}_0^2} \mathbb{E}\left[\Gamma^2 \left|\widehat{\theta}(L\mathbf{k})\right|^2\right] \frac{1}{|\mathbf{k}|^2} P_{\mathbf{k}} e^{i\mathbf{k}\cdot x}. \tag{13}$$

Proof. We may rewrite

$$\begin{aligned}\int_{\mathbb{T}^2} K_L(x, X_0)\cdot v(x)dx &= \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K(x, y)\cdot v(x)\theta_L(y - X_0)dydx \\ &= (\theta_L * K * v)(X_0).\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \mathbb{Q}v, w\rangle &= \mathbb{E}\left[\Gamma^2 (\theta_L * K * v)(X_0)(\theta_L * K * w)(X_0)\right] \\ &= \mathbb{E}\left[\Gamma^2 \int_{\mathbb{T}^2} (\theta_L * K * v)(x)(\theta_L * K * w)(x)dx\right].\end{aligned}$$

By the Parseval theorem,

$$\begin{aligned}\langle \mathbb{Q}v, w\rangle &= \mathbb{E}\left[\Gamma^2 \sum_{\mathbf{k}} \widehat{\theta_L * K * v}(\mathbf{k})\overline{\widehat{\theta_L * K * w}(\mathbf{k})}\right] \\ &= \sum_{\mathbf{k}\in\mathbb{Z}_0^2} \mathbb{E}\left[\Gamma^2 \left|\widehat{\theta_L^T}(\mathbf{k})\right|^2\right] \frac{1}{|\mathbf{k}|^2} \left\langle P_{\mathbf{k}}\widehat{v}(\mathbf{k}), \overline{\widehat{w}(\mathbf{k})}\right\rangle,\end{aligned}$$

recalling that

$$\widehat{K}(\mathbf{k}) = i\frac{\mathbf{k}^{\perp}}{|\mathbf{k}|^2}$$

and calling $P_{\mathbf{k}} = I - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}$ is the projection on the orthogonal to $\mathbf{k}$. Therefore,

$$Q_{\text{vortex}}(x) = \sum_{\mathbf{k}\in\mathbb{Z}_0^2} \mathbb{E}\left[\Gamma^2 \left|\widehat{\theta_L}(\mathbf{k})\right|^2\right] \frac{1}{|\mathbf{k}|^2} P_{\mathbf{k}} e^{i\mathbf{k}\cdot x}.$$

Since $\widehat{\theta_\ell}(\mathbf{k}) = \widehat{\theta}(\ell\mathbf{k})$, we get the result. ∎

Corollary 10. In addition, assume θ is a smooth function with $\widehat{\theta}(\mathbf{k}) = \widehat{\theta}(|\mathbf{k}|)$, let f_L be the probability density of L, and assume Γ is a function of L: $\Gamma = \gamma(L)$. Assume

$$\gamma^2(r) f_L(r) = Cr^\alpha$$

for some $C > 0$ and

$$\alpha > -1.$$

Call

$$D := \int_0^\infty \left|\widehat{\theta}(r)\right|^2 \gamma^2(r) f_L(r) dr,$$

which is a finite constant. Then,

$$Q_{vortex}(x) = D \sum_{\mathbf{k} \in \mathbb{Z}_0^2} \frac{1}{|\mathbf{k}|^{3+\alpha}} P_{\mathbf{k}} e^{i\mathbf{k}\cdot x}.$$

This is the covariance function of a SFGF of index,

$$s = \frac{3+\alpha}{2}.$$

Proof. Since θ is smooth, $\widehat{\theta}(r)$ has a fast decay, which makes $\widehat{\theta}(r) r^\alpha$ integrable at infinity for every α; it is also integrable at zero because $\alpha > -1$. From the assumptions,

$$\begin{aligned}
\mathbb{E}\left[\Gamma^2 \left|\widehat{\theta}(L\mathbf{k})\right|^2\right] &= \mathbb{E}\left[\gamma^2(L) \left|\widehat{\theta}(|L\mathbf{k}|)\right|^2\right] \\
&= \int_0^\infty \gamma^2(\ell) \left|\widehat{\theta}(\ell|\mathbf{k}|)\right|^2 f_L(\ell) d\ell \\
&= |\mathbf{k}|^{-1} \int_0^\infty \left|\widehat{\theta}(r)\right|^2 \gamma^2\left(|\mathbf{k}|^{-1} r\right) f_L\left(|\mathbf{k}|^{-1} r\right) dr \\
&= |\mathbf{k}|^{-1-\alpha} D.
\end{aligned}$$

∎

Note that $\alpha > -1$ corresponds to

$$s > 1,$$

so the SGFF ($s = 1$) is a (just excluded) limit case.

Recall that the solenoidal Kraichnan model with scaling parameter ζ is defined, on the torus $\mathbb{T}^d$, by the covariance function

$$Q_{\text{Kraichnan}}(x) = D \sum_{\mathbf{k} \in \mathbb{Z}_0^d} \frac{1}{|\mathbf{k}|^{d+\zeta}} P_{\mathbf{k}} e^{i\mathbf{k}\cdot x}.$$

We see thus that the vortex noise, in dimension $d = 2$ (see Sec. IV B for $d = 3$), covers the Kraichnan model with scaling parameter,

$$\zeta = 1 + \alpha > 0$$

(any positive ζ is covered).

The space-scale ℓ of the vortices is free in the previous results. If we restrict ourselves to small vortices, namely, we take $f_L(r) = 0$ for $r > k_0^{-1}$, we get the following corollary:

Corollary 11. Under the same assumptions of the previous corollary except for

$$\gamma^2(r) f_L(r) = Cr^\alpha 1_{\{r \le k_0^{-1}\}}$$

for some $C, k_0 > 0$ and $\alpha > -1$, we get

$$Q_{vortex}(x) = \frac{1}{k_0^{3+\alpha}} \sum_{|\mathbf{k}|>k_0} \frac{D(|\mathbf{k}|/k_0)}{(|\mathbf{k}|/k_0)^{3+\alpha}} P_{\mathbf{k}} e^{i\mathbf{k}\cdot x} + R_{k_0}(x),$$

where

$$\lim_{\kappa\to\infty} D(\kappa) = D,$$

$$\|R_{k_0}(x)\| \le \frac{C'}{\alpha+1} \frac{\log k_0}{k_0^{1+\alpha}}$$

for some constant $C' > 0$.

Proof. As above,

$$\mathbb{E}\Big[\Gamma^2\big|\widehat{\theta}(L\mathbf{k})\big|^2\Big] = |\mathbf{k}|^{-1-\alpha}D(|\mathbf{k}|/k_0).$$

The first limit property is obvious. Moreover (using also $\|\widehat{\theta}\|_\infty \le 1$),

$$D(\kappa) \le C\frac{\kappa^{\alpha+1}}{\alpha+1},$$

and hence,

$$\frac{1}{k_0^{3+\alpha}}\sum_{|\mathbf{k}|\le k_0}\frac{D(|\mathbf{k}|/k_0)}{(|\mathbf{k}|/k_0)^{3+\alpha}} \le \frac{C}{\alpha+1}\frac{1}{k_0^{1+\alpha}}\sum_{|\mathbf{k}|\le k_0}|\mathbf{k}|^{-2} \le \frac{C'}{\alpha+1}\frac{1}{k_0^{1+\alpha}}\log k_0.$$

■

We thus see that, up to lower order terms, the vortex model with cutoff corresponds to the Kraichnan model with infrared cutoff k_0 [cf. Ref. 10, Eq. (2.3)].

Finally, we remark that the model has the flexibility of multifractality. To explain it in the simplest possible case, assume

$$\gamma^2(r)f(r) = \sum_{i=1}^{N} C_i r^{\alpha_i}$$

$$D_i := \int_0^\infty \big|\widehat{\theta}(r)\big|^2 C_i r^{\alpha_i}dr.$$

Then, we get

$$Q_{\text{vortex}}(x) = \sum_{i=1}^{N} D_i\sum_{\mathbf{k}}\frac{1}{|\mathbf{k}|^{3+\alpha_i}}P_{\mathbf{k}}e^{i\mathbf{k}\cdot x}.$$

Clearly, one can do the same with a continuously distributed multifractality in place of the finite sum (we void to introduce additional notations to explain this point).

Remark 12. An intriguing but extremely difficult question (we thank an anonymous referee for it) is whether we may infer the value of the scaling exponent ζ of the Kraichnan model, or a multifractal version of it, from the similarity with the vortex noise. It was the main aim of the outstanding book,[6] which—as admitted by the author—remained open at the time of the book and it is still open now. Two examples of attempts in this direction have been Refs. 14 and 25; in the latter work, a multifractal formalism based on vortex filaments was developed. However, it must be stressed that no one of these works deduced K41 or other scalings from vortex models; they could only reproduce scalings chosen a priori.

B. Covariance of 3D vortex noise

Next, we turn to the 3D case, and recall the definition of the noise based on vortex filaments (11). The covariance of the noise is given by

$$\langle\mathbb{Q}v, w\rangle = \mathbb{E}\Big[\Gamma^2\int_{\mathbb{T}^3}K_L(x, X.)\cdot v(x)dx\int_{\mathbb{T}^3}K_L(x', X.)\cdot w(x')dx'\Big],$$

where

$$\int_{\mathbb{T}^3}K_L(x, X.)\cdot v(x)dx = \int_{\mathbb{T}^3}\int_{\mathbb{T}^3}v(x)\cdot K(x,y)\times\int_0^U\theta_L(y-X_t)\,dX_t\,dxdy.$$

For simplicity, we set from now on the time-horizon $U = 1$, and assume that the 3D Brownian motion (X_t) starts from uniform distribution on $\mathbb{T}^3$, and hence, for any time $t > 0$, the distribution of X_t remains uniform. (X_t) is also independent of (Γ, L). Using vector identity, we may rewrite

$$\begin{aligned}\int_{\mathbb{T}^3}K_L(x, X.)\cdot v(x)dx &= \int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\int_0^1\theta_L(y-X_t)v(x)\times K(x,y)\cdot dX_t dxdy\\ &= \int_0^1\Big[\theta_L^T * \Big(\int_{\mathbb{T}^3}v(x)\times K(x-\cdot)dx\Big)\Big](X_t)\cdot dX_t.\end{aligned}$$

For the 3D kernel K (10), we still have the property that $K(x,a) = K(x-a) = -K(a-x)$.

Our first result is that in 3D, the vortex noise has the same covariance structure as in the 2D case.

Proposition 13. Assume θ symmetric and (X_t) independent of (Γ, L) and starts from uniform distribution on $\mathbb{T}^3$. Then,

$$Q_{vortex}(x) = \sum_{\mathbf{k}\in\mathbb{Z}_0^3} \mathbb{E}\Big[\Gamma^2\big|\widehat{\theta}(L\mathbf{k})\big|^2\Big]\frac{1}{|\mathbf{k}|^2}P_{\mathbf{k}}e^{i\mathbf{k}\cdot x}.$$

Proof.

$$\begin{aligned}\langle\mathbb{Q}v,w\rangle &= \mathbb{E}\Big[\Gamma^2\int_0^1\Big[\theta_L*\Big(\int_{\mathbb{T}^3}v(x)\times K(x-\cdot)dx\Big)\Big](X_t)\cdot dX_t\\ &\qquad\int_0^1\Big[\theta_L*\Big(\int_{\mathbb{T}^3}w(x')\times K(x'-\cdot)dx'\Big)\Big](X_t)\cdot dX_t\Big]\\ &= \mathbb{E}\Big[\Gamma^2\int_0^1\Big[\theta_L*\Big(\int_{\mathbb{T}^3}v(x)\times K(x-\cdot)dx\Big)\Big](X_t)\cdot\Big[\theta_L*\Big(\int_{\mathbb{T}^3}w(x')\times K(x'-\cdot)dx'\Big)\Big](X_t)\,dt\Big]\\ &= \mathbb{E}\Big[\Gamma^2\int_{\mathbb{T}^3}\Big[\theta_L*\Big(\int_{\mathbb{T}^3}v(x)\times K(x-\cdot)dx\Big)\Big](z)\cdot\Big[\theta_L*\Big(\int_{\mathbb{T}^3}w(x')\times K(x'-\cdot)dx'\Big)\Big](z)\,dz\Big],\end{aligned}$$

where we take conditional expectation with respect to (X_t) first using its time-stationarity and uniform distribution, whereas the randomness of (Γ, L) remains.

By Parseval theorem and vector identities, we may rewrite

$$\begin{aligned}\langle\mathbb{Q}v,w\rangle &= \mathbb{E}\Bigg[\Gamma^2\sum_{\mathbf{k}\in\mathbb{Z}_0^3}\widehat{\theta}_L(\mathbf{k})\Big(\int_{\mathbb{T}^3}v(x)\times K(x-\cdot)dx\Big)^{\wedge}(\mathbf{k})\cdot\overline{\widehat{\theta}_L(\mathbf{k})\Big(\int_{\mathbb{T}^3}w(x)\times K(x-\cdot)dx\Big)^{\wedge}(\mathbf{k})}\Bigg]\\ &= \sum_{\mathbf{k}\in\mathbb{Z}_0^3}\mathbb{E}\Big[\Gamma^2\big|\widehat{\theta}_L(\mathbf{k})\big|^2\big(\widehat{v}(\mathbf{k})\times\widehat{K}(\mathbf{k})\big)\cdot\big(\overline{\widehat{w}(\mathbf{k})}\times\overline{\widehat{K}(\mathbf{k})}\big)\Big]\\ &= \sum_{\mathbf{k}\in\mathbb{Z}_0^3}\mathbb{E}\Big[\Gamma^2\big|\widehat{\theta}_L(\mathbf{k})\big|^2\overline{\widehat{w}(\mathbf{k})}\cdot\Big(\overline{\widehat{K}(\mathbf{k})}\times\big(\widehat{v}(\mathbf{k})\times\widehat{K}(\mathbf{k})\big)\Big)\Big].\end{aligned}$$

By properties of the triple cross product, we have that

$$\overline{\widehat{K}(\mathbf{k})}\times\big(\widehat{v}(\mathbf{k})\times\widehat{K}(\mathbf{k})\big) = \widehat{v}(\mathbf{k})\Big(\widehat{K}(\mathbf{k})\cdot\overline{\widehat{K}(\mathbf{k})}\Big) - \widehat{K}(\mathbf{k})\Big(\overline{\widehat{K}(\mathbf{k})}\cdot\widehat{v}(\mathbf{k})\Big),$$

and hence,

$$\begin{aligned}&\overline{\widehat{w}(\mathbf{k})}\cdot\Big(\overline{\widehat{K}(\mathbf{k})}\times\big(\widehat{v}(\mathbf{k})\times\widehat{K}(\mathbf{k})\big)\Big)\\ &\quad= \big|\widehat{K}(\mathbf{k})\big|^2\Big(\widehat{v}(\mathbf{k})\cdot\overline{\widehat{w}(\mathbf{k})}\Big) - \Big(\overline{\widehat{w}(\mathbf{k})}\cdot\widehat{K}(\mathbf{k})\Big)\Big(\overline{\widehat{K}(\mathbf{k})}\cdot\widehat{v}(\mathbf{k})\Big)\\ &\quad= \big|\widehat{K}(\mathbf{k})\big|^2\Big(\widehat{v}(\mathbf{k})\cdot\overline{\widehat{w}(\mathbf{k})}\Big) - \overline{\widehat{w}(\mathbf{k})}^T\Big(\widehat{K}(\mathbf{k})\otimes\overline{\widehat{K}(\mathbf{k})}\Big)\widehat{v}(\mathbf{k})\\ &\quad= \frac{1}{|\mathbf{k}|^2}\Big(\widehat{v}(\mathbf{k})\cdot\overline{\widehat{w}(\mathbf{k})}\Big) - \overline{\widehat{w}(\mathbf{k})}^T\Big(\frac{\mathbf{k}}{|\mathbf{k}|^2}\otimes\frac{\mathbf{k}}{|\mathbf{k}|^2}\Big)\widehat{v}(\mathbf{k})\\ &\quad= \frac{1}{|\mathbf{k}|^2}\Big\langle\Big(I-\frac{\mathbf{k}}{|\mathbf{k}|}\otimes\frac{\mathbf{k}}{|\mathbf{k}|}\Big)\widehat{v}(\mathbf{k}),\overline{\widehat{w}(\mathbf{k})}\Big\rangle,\end{aligned}$$

recalling that in 3D,

$$\widehat{K}(\mathbf{k}) = i\frac{\mathbf{k}}{|\mathbf{k}|^2}.$$

Thus, we may conclude that

$$\langle\mathbb{Q}v,w\rangle = \sum_{\mathbf{k}\in\mathbb{Z}_0^3}\mathbb{E}\Big[\Gamma^2\big|\widehat{\theta}_L(\mathbf{k})\big|^2\Big]\frac{1}{|\mathbf{k}|^2}\Big\langle P_{\mathbf{k}}\widehat{v}(\mathbf{k}),\overline{\widehat{w}(\mathbf{k})}\Big\rangle,$$

where $P_{\mathbf{k}} = I - \frac{\mathbf{k}}{|\mathbf{k}|}\otimes\frac{\mathbf{k}}{|\mathbf{k}|}$ is the projector on the orthogonal to $\mathbf{k}$. This yields, in turn, that the covariance matrix of the noise is given by ■

$$Q_{\text{vortex}}(x,x') = \sum_{\mathbf{k}\in\mathbb{Z}_0^3}\mathbb{E}\Big[\Gamma^2\big|\widehat{\theta}_L(\mathbf{k})\big|^2\Big]\frac{1}{|\mathbf{k}|^2}P_{\mathbf{k}}e^{i\mathbf{k}\cdot(x-x')}.$$

This formula agrees with formula (13) obtained for 2D; hence, Corollary 10 applies in 3D without change (except for summation over $\mathbf{k} \in \mathbb{Z}_0^3$).

Our result in 3D covers Kraichnan noise with parameter

$$\zeta = \alpha > -1.$$

We can also restrict the vortices to small scales by introducing a cutoff k_0, as in Corollary 11. Here, we need to restrict to $\alpha > 0$ in its statement so that the remainder $R_{k_0}(x)$ is of lower order,

$$\|R_{k_0}(x)\| \leq \frac{C'}{\alpha+1}\frac{1}{k_0^{\alpha}}.$$

V. THE EFFECT OF VORTEX STRUCTURE NOISE ON PASSIVE SCALARS

A. Introduction

Regarding eddy diffusion enhancement in domains with boundary, we recall the following theorem proved in Ref. 13 (Theorems 1.1 and 1.3). Here, we have a passive scalar θ driven by the white-in-time, correlated-in-space noise $\partial_t W$ produced by our vortex structures, where $W(t, x)$ is the limit Gaussian process obtained via the invariance principle in Theorem 2,

$$\partial_t \theta + \partial_t W \circ \nabla\theta = \kappa\Delta\theta,$$

$\circ$ denotes Stratonovich integration, and scalar $\kappa > 0$. We denote the smallest eigenvalue of the matrix $Q(x, x)$ by

$$q(x,x) := \min_{0\neq\xi\in\mathbb{R}^d} \frac{\xi^T Q(x,x)\xi}{\xi^T\xi}$$

and the squared operator norm $\|Q^{1/2}\|^2_{L^2(\mathbb{D})\to L^2(\mathbb{D})}$ by

$$\epsilon_Q := \sup_{0\neq v\in H} \frac{\int_{\mathbb{D}}\int_{\mathbb{D}} v^T(x)Q(x,y)v(y)dxdy}{\int_{\mathbb{D}} v(x)^T v(x)dx}.$$

Theorem 14 *(Ref. 13, Theorems 1.1 and 1.3).*

(a) *For any $\theta_0 \in H$ measurable and any $t \geq 0$, we have that*

$$\mathbb{E}\left[\left(\int_{\mathbb{D}}|\theta(t,x)|dx\right)^2\right] \leq \left(\frac{\epsilon_Q}{\kappa} + 2|\mathbb{D}|e^{-2t\lambda_{\mathbb{D},\kappa,Q}}\right)\mathbb{E}\left[\|\theta_0\|^2_{L^2}\right],$$

where $\lambda_{\mathbb{D},\kappa,Q}$ is the first eigenvalue of the elliptic operator $-A_Q$ for

$$A_Q := \kappa\Delta + \frac{1}{2}\,div(Q(x,x)\nabla\cdot).$$

(b) *There exists a constant $C_{D,d} > 0$ such that*

$$\lambda_{\mathbb{D},\kappa,Q} \geq C_{D,d}\min\left(\sigma^2, \kappa/\delta\right)$$

for every Q such that

$$\inf_{x\in D_\delta} q(x,x) \geq \sigma^2.$$

In view of this theorem, our aim is to show that the noises based on vortex structures in 2D and 3D that we constructed in Sec. III, for small L, enjoy the property that they have small ϵ_Q and large $q(x, x)$, simultaneously, once the other parameters of the model are tuned properly. Here, we assume that $\Gamma, U, L, X.$ are independent.

For technical reasons, we demonstrate this only for the torus $\mathbb{D} = \mathbb{T}^d$, $d = 2, 3$, in this section. The same conclusions should be true for any regular domains D, but the corrector part of the Green function is difficult to handle; hence, we prefer to state in the simple case of torus. Note in this case, we do not have a boundary, and hence, $\mathbb{D}_\delta = \mathbb{D}$, $\delta = 0$, and we can put the stopping time $\tau = \infty$ in the 3D case.

B. The 2D case

The following theorem applies to any realization ℓ of L. For fixed $\ell > 0$, we shall use [recall (8)]

$$Q_\ell(x,y) = \mathbb{E}\big[\Gamma^2 K_\ell(x,X_0) \otimes K_\ell(y,X_0)\big].$$

Therefore, for $\xi \in \mathbb{R}^2$, we have

$$\xi^T Q_\ell(x,x)\xi = \mathbb{E}\big[\Gamma^2 |K_\ell(x,X_0)\cdot\xi|^2\big],$$

while for $v \in H$,

$$\langle \mathbb{Q}_\ell v, v\rangle = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} v(x)^T Q_\ell(x,y)v(y)dxdy = \mathbb{E}\left[\Gamma^2\left(\int_{\mathbb{T}^2} v(x)\cdot K_\ell(x,X_0)dx\right)^2\right].$$

In the next statement, we set $\sigma^2 = \mathbb{E}(\Gamma^2)$.

Theorem 15. *(i) There exists a finite constant C such that for every $v \in H$ and $\ell \in (0,1)$,*

$$\frac{\langle \mathbb{Q}_\ell v, v\rangle}{\|v\|_H^2} \le C\sigma^2.$$

(ii) For every $x \in \mathbb{T}^2$, let $q_\ell(x) \ge 0$ be the largest number such that for any $v \in \mathbb{R}^2$ and $\ell \in (0,1)$,

$$\frac{v^T Q_\ell(x,x)v}{|v|^2} \ge q_\ell(x).$$

Then, there exists some positive constant c such that

$$\inf_{x\in\mathbb{T}^2} q_\ell(x) \ge c\,\sigma^2|\log\ell|.$$

Remark 16. We can choose $\sigma^2 = \mathbb{E}(\Gamma^2)$ to be small and then choose ℓ small enough such that $\sigma^2|\log\ell|$ is large to fulfill the conditions in Theorem 14.

Proof. Since $\mathbb{D} = \mathbb{T}^2$, the function $\nabla_x^\perp h_{\mathbb{D}}(x,y)$ is bounded above uniformly and does not affect the computations on $K(x,y)$, which will be based only on the term $\frac{1}{2\pi}\frac{(x-y)^\perp}{|x-y|^2}$. Thus, we use the approximation for all $x \in \mathbb{T}^2$, a.s.,

$$|K_\ell(x,X_0)| \le \int_{\mathbb{T}^2} |K(x,y)|\theta_\ell(y-X_0)dy \sim \frac{1}{2\pi}\int_{\mathbb{T}^2}\frac{1}{|x-y|}\theta_\ell(y-X_0)dy.$$

Let C_{K_ℓ} be the random variable defined as

$$C_{K_\ell} := \int_{\mathbb{T}^2}|K_\ell(x,X_0)|dx.$$

Under our approximation, we have

$$\begin{aligned}\int_{\mathbb{T}^2}|K_\ell(x,X_0)|dx &\lesssim \frac{1}{2\pi}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\frac{1}{|x-y|}\theta_\ell(y-X_0)dydx\\ &= \frac{1}{2\pi}\int_{\mathbb{T}^2}\left(\int_{\mathbb{T}^2}\frac{1}{|x-y|}dx\right)\theta_\ell(y-X_0)dy\\ &\le C\int_{\mathbb{T}^2}\theta_\ell(y-X_0)dy = C,\end{aligned}$$

and hence, C_{K_ℓ} is finite a.s. and even uniformly bounded above. Then,

$$\begin{aligned}\langle \mathbb{Q}_\ell v, v\rangle &\le \mathbb{E}\left[\Gamma^2\left(\int_{\mathbb{T}^2}|v(x)||K_\ell(x,X_0)|dx\right)^2\right]\\ &\le \mathbb{E}\left[\Gamma^2 C_{K_\ell}^2\left(\int_{\mathbb{T}^2}|v(x)|\frac{|K_\ell(x,X_0)|}{C_{K_\ell}}dx\right)^2\right]\\ &\le \mathbb{E}\left[\Gamma^2 C_{K_\ell}^2\int_{\mathbb{T}^2}|v(x)|^2\frac{|K_\ell(x,X_0)|}{C_{K_\ell}}dx\right]\\ &= \mathbb{E}\left[\Gamma^2 C_{K_\ell}\int_{\mathbb{T}^2}|v(x)|^2|K_\ell(x,X_0)|dx\right].\end{aligned}$$

Let $\widetilde{C}_{K_\ell}$ be the deterministic constant defined as

$$\widetilde{C}_{K_\ell} := \sup_{x\in\mathbb{T}^2}\mathbb{E}\left[\Gamma^2 C_{K_\ell}|K_\ell(x,X_0)|\right] < \infty.$$

We have proved

$$\langle \mathbb{Q}_\ell v, v\rangle \le \widetilde{C}_{K_\ell}\|v\|_H^2.$$

Concerning the size of $\widetilde{C}_{K_\ell}$, under the assumptions that p_0 has a bounded density, we have

$$\begin{aligned}\widetilde{C}_{K_\ell} &\le C\sup_{x\in\mathbb{T}^2}\mathbb{E}\left[\Gamma^2|K_\ell(x,X_0)|\right]\\ &\sim \frac{C}{2\pi}\sup_{x\in\mathbb{T}^2}\mathbb{E}\left[\Gamma^2\int_{\mathbb{T}^2}\frac{1}{|x-y|}\theta_\ell(y-X_0)dy\right]\\ &= \frac{C}{2\pi}\sup_{x\in\mathbb{T}^2}\mathbb{E}\left[\Gamma^2\int_{\mathbb{T}^2}\int\frac{1}{|x-y|}\theta_\ell(y-x_0)p_0(x_0)dx_0dy\right]\\ &\le \frac{Cp_{max}}{2\pi}\sup_{x\in\mathbb{T}^2}\mathbb{E}\left[\Gamma^2\int_{\mathbb{T}^2}\frac{1}{|x-y|}dy\right]\\ &\le C(p_{max},\mathbb{T}^2)\mathbb{E}(\Gamma^2)\end{aligned}$$

since

$$\sup_{x\in\mathbb{T}^2}\int_{\mathbb{T}^2}\frac{1}{|x-y|}dy \le C_{\mathbb{T}^2}.$$

Therefore,

$$\langle \mathbb{Q}_\ell v, v\rangle \le C\mathbb{E}(\Gamma^2)\|v\|_H^2.$$

This quantity is small if $\mathbb{E}(\Gamma^2)$ is small.

Concerning $v^T Q(x,x)v$, $v\in\mathbb{R}^2$, using again the simplified asymptotics, we have

$$\begin{aligned}v^T Q(x,x)v &= \mathbb{E}\left(\Gamma^2\int_{\mathbb{T}^2}|K_\ell(x,x_0)\cdot v|^2 p_0(dx_0)\right)\\ &\sim \mathbb{E}\left(\frac{\Gamma^2}{(2\pi)^2}\int_{\mathbb{T}^2}\left|\int_{\mathbb{T}^2}\frac{(x-y)^\perp\cdot v}{|x-y|^2}\theta_\ell(y-x_0)dy\right|^2 p_0(dx_0)\right).\end{aligned}$$

Given any $x\in\mathbb{T}^2$ and unit vector $v\in\mathbb{R}^2$, there is a cone $C(x,v)\subset\mathbb{T}^2$ (a set of the form $x+rw$, $r\in[0,r_0]$, $|w|=1$, $w\cdot e\ge\alpha$ for some $|e|=1$ and $\alpha\in(0,1)$) such that

$$(x-x_0)^\perp\cdot v \ge \frac{1}{2}|x-x_0||v| \text{ for every } x_0\in C(x,v)$$

and

$$|C(x,v)|\ge\eta>0.$$

Moreover, assume $p_0(dx_0)$ is bounded below by $p_{\min} Leb$ for some constant $p_{\min} > 0$. We then have

$$v^T Q_\ell(x,x)v \geq \mathbb{E}\left(\frac{\Gamma^2 p_{\min}}{(2\pi)^2}\int_{C(x,v)}\left|\int_{B(x_0,\ell)}\frac{(x-y)^\perp\cdot v}{|x-y|^2}\theta_\ell(y-x_0)dy\right|^2 dx_0\right).$$

Taking $\ell > 0$ very small, reduce the cone $C(x,v)$ to the set

$$C_\ell(x,v)\subset C(x,v)$$

of points x_0 such that

$$\mathrm{dist}(x_0,\partial C(x,v))\geq 2\ell.$$

We then have

$$y\in C(x,v)\ \text{ if } y\in B(x_0,\ell)\ \text{ with } x_0\in C_\ell(x,v),$$

and thus,

$$\begin{aligned}
v^T Q_\ell(x,x)v &\geq \mathbb{E}\left(\frac{\Gamma^2 p_{\min}}{(2\pi)^2}\int_{C_\ell(x,v)}\left|\int_{B(x_0,\ell)}\frac{\frac{1}{2}|x-y||v|}{|x-y|^2}\theta_\ell(y-x_0)dy\right|^2 dx_0\right)\\
&= \mathbb{E}\left(\frac{\Gamma^2 p_{\min}|v|^2}{4(2\pi)^2}\int_{C_\ell(x,v)}\left|\int_{B(x_0,\ell)}\frac{1}{|x-y|}\theta_\ell(y-x_0)dy\right|^2 dx_0\right)\\
&= \mathbb{E}\left(\frac{\Gamma^2 p_{\min}|v|^2}{4(2\pi)^2}\int_{C_\ell(x,v)}\left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x-x_0)dx_0\right)\\
&= \mathbb{E}\left(\frac{\Gamma^2 p_{\min}|v|^2}{4(2\pi)^2}\int_{C_\ell(0,v)}\left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x_0)dx_0\right)\\
&\geq c(p_{\min},\eta)|v|^2\mathbb{E}(\Gamma^2)\int\left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x_0)dx_0.
\end{aligned}$$

The last inequality is because the quantity $x_0 \mapsto \left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x_0)$ is rotationally invariant; hence, the integral $\int_{C_\ell(0,v)}\left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x_0)dx_0$ does not depend on v. Since $|C(0,v)| \geq \eta$, we have that

$$\int_{C_\ell(0,v)}\left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x_0)dx_0 \geq c\eta^{-1}\int\left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x_0)dx_0.$$

Let us investigate the problem of the scaling in ℓ of the quantity $\int\left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x)dx$. Given the mollifier $\theta_\ell(x) = \ell^{-2}\theta(\ell^{-1}x)$ that we assume the best possible one (non-negative, smooth, symmetric), let us introduce the smooth symmetric pdf, compactly supported in $B(0,2)$,

$$\theta^{(2)}(z) := \int \theta(z-z')\theta(z')dz'.$$

Then,

$$\begin{aligned}
\theta_\ell^{(2)}(z) = \ell^{-2}\theta^{(2)}(\ell^{-1}x) &= \int \ell^{-2}\theta(\ell^{-1}z - z')\theta(z')dz'\\
&\overset{z'=\ell^{-1}w}{=} \int \ell^{-2}\theta(\ell^{-1}(z-w))\theta(\ell^{-1}w)\ell^{-2}dw\\
&= \int \theta_\ell(z-w)\theta_\ell(w)dw\\
&= (\theta_\ell * \theta_\ell)(z).
\end{aligned}$$

Below we shall use the formula,

$$\theta_\ell^{(2)}(y-y') = \int \theta_\ell(x-y)\theta_\ell(x-y')dx$$

true because

$$\theta_\ell^{(2)}(y-y') = \int \theta_\ell(y-y'-w)\theta_\ell(w)dw \overset{w=x-y'}{=} \int \theta_\ell(y-x)\theta_\ell(x-y')dx$$

(recall θ is symmetric). After these preliminaries, we have

$$\begin{aligned}\int \left(\theta_\ell * \frac{1}{|\cdot|}\right)^2(x)dx &= \int \left(\int \theta_\ell(x-y)\frac{1}{|y|}dy\right)^2 dx \\ &= \iiint \theta_\ell(x-y)\theta_\ell(x-y')\frac{1}{|y|}\frac{1}{|y'|}dydy'dx \\ &= \iint \theta_\ell^{(2)}(y-y')\frac{1}{|y|}\frac{1}{|y'|}dydy' \\ &= \iint \theta_\ell^{(2)}(z)\frac{1}{|y|}\frac{1}{|y-z|}dydz \\ &= \int \left(\int \frac{1}{|y|}\frac{1}{|y-z|}dy\right)\theta_\ell^{(2)}(z)dz.\end{aligned}$$

Now, we have to understand first the behavior of

$$z \mapsto \int \frac{1}{|y|}\frac{1}{|y-z|}dy.$$

We can prove that for $|z| \le 1$,

$$\int_{\mathbb{R}^2} \frac{1}{|y|}\frac{1}{|y-z|}dy \ge |\log|z||.$$

Indeed, since $|y-z| \le |y| + |z|$,

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{1}{|y|}\frac{1}{|y-z|}dy &\ge \int \frac{1}{|y|}\frac{1}{|y|+|z|}dy \\ &\ge \int_0^1 \frac{1}{\rho}\frac{1}{\rho+|z|}\rho\, d\rho = \log(1+|z|) - \log|z| \\ &\ge -\log|z| = |\log|z||.\end{aligned}$$

Then, $\int \left(\int \frac{1}{|y|}\frac{1}{|y-z|}dy\right)\theta_\ell^{(2)}(z)dz$ can be bounded below by

$$\begin{aligned}&\int \theta_\ell^{(2)}(z)|\log|z||dz \\ &\ge \ell^{-2}\int_{|z|\le\ell} \theta^{(2)}(\ell^{-1}z)|\log|z||dz \\ &\ge -c_\theta \ell^{-2}\int_0^\ell r\log r dr \\ &= c_\theta\left(-\ell^{-2}\left[\frac{r^2}{2}\log r\right]_{r=0}^{r=\ell} + \ell^{-2}\int_0^\ell \frac{r^2}{2}\frac{1}{r}dr\right) \\ &= c_\theta\left(-\ell^{-2}\frac{\ell^2}{2}\log \ell + \ell^{-2}\frac{\ell^2}{2}\right) \\ &= c_\theta\left(|\log \ell| + \frac{1}{2}\right),\end{aligned}$$

where without loss of generality

$$c_\theta := \inf_{z\in B(0,1)} \theta^{(2)}(z) > 0.$$

This yields that

$$\frac{v^T Q(x,x)v}{|v|^2} \geq c\,\mathbb{E}(\Gamma^2)|\log \ell|$$

for some $c > 0$ and any $\ell \in (0,1)$.

C. The 3D case

Recall that we take $\mathbb{D} = \mathbb{T}^3$; hence, the computation below can be based solely on the $\frac{1}{4\pi}\frac{x-y}{|x-y|^3}\times$ part of the kernel $K(x,y)$ (10), with the other part from $\nabla_x h_{\mathbb{D},x_0}(x)\times$ uniformly bounded. We also set $\tau = \infty$. The following theorem applies to any realization ℓ of L. We shall use the notation $Q_\ell(x,y)$ and $\mathbb{Q}_\ell$ for fixed ℓ, similarly to what is done in the 2D case, while recalling (11).

In the next statement, we set $\sigma^2 = \mathbb{E}(\Gamma^2)$.

Theorem 17. *(i) There exists a constant $C < \infty$ such that for every $v \in H$ and $\ell \in (0,1)$,*

$$\frac{\langle \mathbb{Q}_\ell v, v\rangle}{\|v\|_H^2} \leq C\,\mathbb{E}(U)\sigma^2.$$

(ii) There exists a constant $c > 0$ such that for all $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$, and $\ell \in (0,1)$,

$$\frac{v^T Q_\ell(x,x)v}{|v|^2} \geq c\,\mathbb{E}(U)\sigma^2\ell^{-1}.$$

Remark 18. We can choose the distribution of (Γ, U) such that $\mathbb{E}(U)\sigma^2$ is small and then choose ℓ small enough such that $\mathbb{E}(U)\sigma^2\ell^{-1}$ is large to fulfill the conditions in Theorem 14.

Proof. Taking any $v \in H$, we consider

$$\begin{aligned}\langle \mathbb{Q}_\ell v, v\rangle &= \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v(x)^T Q_\ell(x,y)v(y)\,dxdy\\ &= \mathbb{E}\left[\Gamma^2\left(\int_{\mathbb{T}^3} v(x)\cdot\int_{\mathbb{T}^3} K(x,y)\times\left(\int_0^{U\wedge\tau}\theta_\ell(y-X_t)dX_t\right)dydx\right)^2\right].\end{aligned}$$

For any fixed realization of (Γ, U), we take expectation over $\mathcal{W}$ first

$$\begin{aligned}\langle \mathbb{Q}_\ell v, v\rangle &\sim \Gamma^2\mathcal{W}\left[\left(\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\int_0^U \theta_\ell(y-X_t)v(x)\cdot\frac{1}{4\pi}\frac{x-y}{|x-y|^3}\times dX_t dydx\right)^2\right]\\ &= \Gamma^2\mathcal{W}\left[\left(\int_0^U\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\theta_\ell(y-X_t)v(x)\times\frac{1}{4\pi}\frac{x-y}{|x-y|^3}dydx\cdot dX_t\right)^2\right]\\ &= \Gamma^2\mathcal{W}\left[\int_0^U\left|\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\theta_\ell(y-X_t)v(x)\times\frac{1}{4\pi}\frac{x-y}{|x-y|^3}dydx\right|^2 dt\right],\end{aligned}$$

where the last step is due to Itô isometry. We further bound it above by moving the norm inside the integral,

$$\begin{aligned}&\Gamma^2\mathcal{W}\left[\int_0^U\left(\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\theta_\ell(y-X_t)|v(x)|\frac{1}{4\pi}\frac{1}{|x-y|^2}dydx\right)^2 dt\right]\\ &= \Gamma^2\mathcal{W}\left[\int_0^U C_u^2\left(\int_{\mathbb{T}^3}|v(x)|\frac{\int_{\mathbb{T}^3}\theta_\ell(y-X_t)\frac{1}{4\pi}\frac{1}{|x-y|^2}dy}{C_u}dx\right)^2 dt\right]\\ &\leq \Gamma^2\mathcal{W}\left[\int_0^U C_u\int_{\mathbb{T}^3}|v(x)|^2\int_{\mathbb{T}^3}\theta_\ell(y-X_t)\frac{1}{4\pi}\frac{1}{|x-y|^2}dydxdt\right],\end{aligned}$$

where the random constant C_u is

$$C_u := \int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\theta_\ell(y-X_t)\frac{1}{4\pi}\frac{1}{|x-y|^2}dydx \le C_{\mathbb{T}^3}$$

for some deterministic finite constant $C_{\mathbb{T}^3}$ (integrate first dx then dy). Set

$$C'_u := \sup_{x\in\mathbb{T}^3}\mathcal{W}\left[\int_0^U\int_{\mathbb{T}^3}\theta_\ell(y-X_t)\frac{1}{4\pi}\frac{1}{|x-y|^2}dydt\right].$$

Recall that X_0 has density $p_0(x)$, which is bounded above uniformly by $p_{\max}$. Since the heat semigroup is an L^∞-contraction, the density of X_t at any later time t is bounded above by $p_{\max}$, and thus, we have

$$C'_u \le Up_{\max}\sup_{x\in\mathbb{T}^3}\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\theta_\ell(y-z)\frac{1}{4\pi}\frac{1}{|x-y|^2}dydz \le UC'_{\mathbb{T}^3}$$

(integrating first dz then dy) for some deterministic finite constant $C'_{\mathbb{T}^3}$. We conclude with

$$\langle\mathbb{Q}v,v\rangle \le \mathbb{E}\left(\Gamma^2C'_u\right)\|v\|_H^2 \le C'_{\mathbb{T}^3}\sigma^2\mathbb{E}(U)\|v\|_H^2.$$

Taking now any unit vector $v\in\mathbb{R}^3$, for any $x\in\mathbb{T}^3$, we consider the quantity

$$v^TQ(x,x)v = \mathbb{E}\left[\Gamma^2|v\cdot u(x)|^2\right].$$

We again fix any realization of (Γ,U,ℓ) and take expectation over $\mathcal{W}$ first,

$$\begin{aligned}
&\mathcal{W}\left[\Gamma^2|v\cdot u(x)|^2\right]\\
&\quad=\Gamma^2\mathcal{W}\left[\left(\int_{\mathbb{T}^3}\int_0^U\theta_\ell(y-X_t)v\cdot\frac{1}{4\pi}\frac{x-y}{|x-y|^3}\times dX_tdy\right)^2\right]\\
&\quad=\Gamma^2\mathcal{W}\left[\left(\int_0^U\int_{\mathbb{T}^3}\theta_\ell(y-X_t)v\times\frac{1}{4\pi}\frac{x-y}{|x-y|^3}dy\cdot dX_t\right)^2\right]\\
&\quad=\Gamma^2\mathcal{W}\left[\int_0^U\left|\int_{\mathbb{T}^3}\theta_\ell(y-X_t)v\times\frac{1}{4\pi}\frac{x-y}{|x-y|^3}dy\right|^2dt\right],
\end{aligned}$$

where the last step is due to Itô isometry.

Since $\mathbb{D}=\mathbb{T}^3$ is compact, the density of X_t, denoted $p_t(z)$, converges to the uniform distribution, and hence, it is not hard to see that there exists some $p_{\min}>0$ independent of t such that

$$p_t(z)\ge p_{\min},\quad z\in\mathbb{T}^3,\ t\in[0,U].$$

Then, we can continue to bound below $\mathcal{W}\left[\Gamma^2|v\cdot u(x)|^2\right]$ by

$$\begin{aligned}
&\Gamma^2\int_0^U\int_{\mathbb{T}^3}\left|\int_{\mathbb{T}^3}\theta_\ell(y-z)v\times\frac{1}{4\pi}\frac{x-y}{|x-y|^3}dy\right|^2p_t(z)dz\\
&\qquad\ge\Gamma^2p_{\min}U\int_{\mathbb{T}^3}\left|\int_{\mathbb{T}^3}\theta_\ell(y-z)v\times\frac{1}{4\pi}\frac{x-y}{|x-y|^3}dy\right|^2dz.
\end{aligned}$$

For any $x\in\mathbb{T}^3$, there exist a cone $C(x,v)$ and a ball $B=B(x^*,\ell/2)\subset C(x,v)$ of radius $\ell/2$ with center x^* with $|x-x^*|=2\ell$ such that provided $z\in B$, we have all the y that contribute to the above integral be contained in $B(x^*,3\ell/2)$ and $\ell/2\le|x-y|\le7\ell/2$, and on the other hand, the orientation of the cone is chosen such that $v\times(x-y)$ are roughly in the same direction for all the y. This implies that for some absolute constant $c>0$ and any $z\in B$,

$$\left|\int_{\mathbb{T}^3}\theta_\ell(y-z)v\times\frac{1}{4\pi}\frac{x-y}{|x-y|^3}dy\right|\ge c|v|\int_{\mathbb{T}^3}\theta_\ell(y-z)\ell^{-2}dy=c\ell^{-2}.$$

Thus, we have that upon squaring and using $|B| \asymp \ell^3$,

$$v^T Q_\ell(x,x)v \geq c p_{\min} \mathbb{E}\left(\Gamma^2 U \int_B \ell^{-4} dz\right) = c p_{\min} \mathbb{E}\left(\Gamma^2 U\right)\ell^{-1}.$$

This completes the proof. ∎

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Franco Flandoli: Formal analysis (equal); Investigation (equal); Writing – original draft (equal). **Ruojun Huang**: Formal analysis (equal); Investigation (equal); Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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