Work Extraction Processes from Noisy Quantum Batteries: The Role of Nonlocal Resources

Salvatore Tirone,[1,*] Raffaele Salvia,[2,†] Stefano Chessa,[2,3,‡] and Vittorio Giovannetti[2]
[1]*Scuola Normale Superiore, I-56126 Pisa, Italy*
[2]*NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56126 Pisa, Italy*
[3]*Electrical and Computer Engineering, University of Illinois Urbana-Champaign, Urbana, Illinois 61801, USA*

(Received 5 April 2023; accepted 29 June 2023; published 8 August 2023)

We demonstrate an asymmetry between the beneficial effects one can obtain using nonlocal operations and nonlocal states to mitigate the detrimental effects of environmental noise in the work extraction process from quantum battery models. Specifically, we show that using nonlocal recovery operations after the noise action can, in general, increase the amount of work one can recover from the battery even with separable (i.e., nonentangled) input states. On the contrary, employing entangled input states with local recovery operations will generally not improve the battery performance.

DOI: 10.1103/PhysRevLett.131.060402

Introduction.—Quantum thermodynamics is a rapidly growing field that seeks to understand the behavior of small quantum systems at the nanoscale [1]. Of particular interest is the use of quantum effects to improve the charging processes of batteries [2–8], which could potentially lead to technological advancements in a variety of sectors. A crucial aspect of the problem is to assess the stability of these models when in contact with environmental noise, as this represents a more realistic scenario and it may have a significant impact on the efficiency of the energy recovery or storage. In noiseless regimes, capacities for quantum battery models have been proposed in Refs. [8–10] while, from a resource theoretical point of view, the thermodynamic capacity (in the sense of simulability) of quantum channels has been defined [11]. Concerning energy manipulation in more realistic scenarios where environmental noise is acting on the system, a few results have been obtained in specialized settings (see, e.g., Refs. [12–20]), and various schemes have been proposed to stabilize quantum batteries in the presence of specific types of perturbations [1,21–29]. In this Letter, we tackle this problem using the work capacitance functionals introduced in Ref. [30]. These quantities gauge the efficiency of the work extraction process from quantum battery models formed by large collections of identical and independent noisy elements (quantum cells, or q-cells in brief), targeting optimal state preparation schemes (encoding operations performed *before* the action of noise) and optimal recovery transformations (decoding operations performed *after* the noise). Formally, they are defined as the asymptotic limit of the ratio between the work extracted from the system and the initial energy stored in the quantum battery, and for a given quantum battery model, their values strongly depend on the type of constraints one enforces on the transformations allowed on the system—see Fig. 1. Our main finding is to provide evidence that, irrespective of the noise model, the mere use of nonlocal resources at the level of the encoding (i.e., employing quantum correlated states to store the initial energy of the battery) does not improve the efficiency of the model. On the contrary, we show that employing nonlocal transformations at the recovering stage will, in general, increase the work extraction performance of the battery. The key ingredient to attain such results is the derivation of a single-letter formula that allows us to simplify the evaluation of the local-ergotropy capacitances [30] for arbitrary noise models.

Preliminaries.—We model noisy quantum batteries (QBs) as collections of n identical and independent (noninteracting) elements (quantum cells, or q-cells in

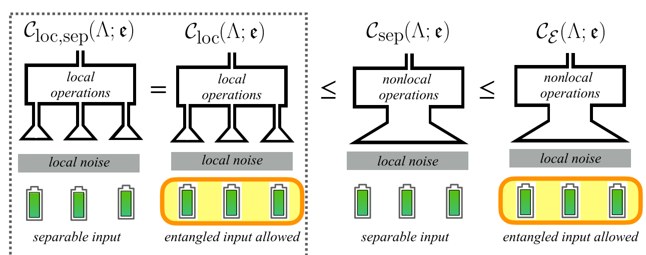

FIG. 1. Resource accounting for work extraction in noisy QB models composed of n q-cells (green elements of the figure) affected by local noise (gray). From left to right: separable-input, local ergotropy capacitance $C_{\rm loc,sep}(\Lambda;\mathfrak{e})$ (maximum work extractable per unit cell, when both the state preparation of the QB and the recovery operations applied after the noise action are restricted to local resources); local ergotropy capacitance $C_{\rm loc}(\Lambda;\mathfrak{e})$ (here, locality is enforced only at the recovery level); separable-input capacitance $C_{\rm sep}(\Lambda;\mathfrak{e})$ (locality is enforced only at the state preparation level); ergotropy capacitance $C_{\mathcal{E}}(\Lambda;\mathfrak{e})$ (no local restrictions imposed). As shown in Theorem 1, irrespective of the noise model, $C_{\rm loc}(\Lambda;\mathfrak{e})$ and $C_{\rm loc,sep}(\Lambda;\mathfrak{e})$ always coincide.

brief), each capable of storing energy in the internal degrees of freedom associated with their local Hamiltonians $\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_n$, all of which are locally identical to a single q-cell Hamiltonian $\hat{h}$, and perturbed by the same local noise source, which we describe in terms of a linear, completely positive and trace-preserving (LCTP) super-operator Λ [31–33]. The work extraction efficiency in these systems can be measured using as a figure of merit the work capacitances introduced in Ref. [30]. A first example is provided by the ergotropic capacitance $C_{\mathcal{E}}(\Lambda; \mathfrak{e})$. Ergotropy is a well-established measure of the maximum work one can extract from a quantum state $\hat{\rho}$ by means of unitary operations that preserve the total energy of the system [9,34,35]. For a d-dimensional system characterized by a Hamiltonian $\hat{H}$, it can be expressed as

$$\mathcal{E}(\hat{\rho}; \hat{H}) \coloneqq \max_{\hat{U}\in\mathbb{U}(d)} \{\mathfrak{E}(\hat{\rho}; \hat{H}) - \mathfrak{E}(\hat{U}\,\hat{\rho}\,\hat{U}^\dagger; \hat{H})\}, \tag{1}$$

where $\mathfrak{E}(\hat{\rho}; \hat{H}) \coloneqq \mathrm{Tr}[\hat{\rho}\,\hat{H}]$ is the average energy of a quantum state $\hat{\rho}$ and $\mathbb{U}(d)$ is the d-dimensional representation of the unitary group. In view of this definition, a reasonable way to gauge the maximum work one can retrieve from the QB after the action of the noise Λ is obtained by considering

$$\mathcal{E}^{(n)}(\Lambda; E) \coloneqq \max_{\hat{\rho}^{(n)}\in\mathfrak{S}_E^{(n)}} \mathcal{E}(\Lambda^{\otimes n}(\hat{\rho}^{(n)}); \hat{H}^{(n)}), \tag{2}$$

where $\hat{H}^{(n)} \coloneqq \hat{h}_1 + \cdots + \hat{h}_n$ represents the battery Hamiltonian and where the maximization is performed on the set $\mathfrak{S}_E^{(n)}$, which is the set of all the n q-cell states $\hat{\rho}^{(n)}$ with average energy $\mathfrak{E}(\hat{\rho}^{(n)}) \leq E$. The ergotropic capacitance $C_{\mathcal{E}}(\Lambda; \mathfrak{e})$ is now defined as the proper regularization of $\mathcal{E}^{(n)}(\Lambda; E)$ for $n \to \infty$, under the assumption that (on average) each of the n q-cells stores no more than a fraction $\mathfrak{e} \in [0, \|\hat{h}\|_\infty]$ of the total input energy [30], i.e.,

$$C_{\mathcal{E}}(\Lambda; \mathfrak{e}) \coloneqq \lim_{n\to\infty} \frac{\mathcal{E}^{(n)}(\Lambda; E = n\mathfrak{e})}{n}, \tag{3}$$

where, without loss of generality, we set to zero the ground-state energy of the local Hamiltonians $\hat{h}$. In the absence of noise (i.e., Λ is the identity channel), $C_{\mathcal{E}}(\Lambda; \mathfrak{e})$ is equal to $\mathfrak{e}$, signaling that, by properly preparing the input state of the q-cells, we can retrieve all the energy we have initially stored in the battery. Dissipation and decoherence will instead tend to produce smaller values of $C_{\mathcal{E}}(\Lambda, \mathfrak{e})$, indicating that the performance of the model gets degraded irrespective of the choice we make at the level of state preparation of the QB. Setting restrictions on the allowed operations one can perform on the battery will also reduce the value of $C_{\mathcal{E}}(\Lambda, \mathfrak{e})$. For instance, assuming the maximization in Eq. (2) to run only on separable input states of the q-cells will lead us to replace $\mathcal{E}^{(n)}(\Lambda; n\mathfrak{e})$ with $\mathcal{E}_{\mathrm{sep}}^{(n)}(\Lambda; n\mathfrak{e})(\leq \mathcal{E}^{(n)}(\Lambda; n\mathfrak{e}))$, which, regularized as in Eq. (3), gives the separable-input capacitance $C_{\mathrm{sep}}(\Lambda, \mathfrak{e})$ of the model. This, in turn, expresses the asymptotic work we can extract per q-cell in the absence of initial entanglement between these elements [30]. Similarly, by restricting the optimization in Eq. (1) to include only unitary operations acting locally on the q-cells [i.e., replacing $\mathcal{E}(\hat{\rho}; \hat{H})$ with the local ergotropy [36]] will lead us to identify $\mathcal{E}_{\mathrm{loc}}^{(n)}(\Lambda; n\mathfrak{e})$ with its regularized limit $C_{\mathrm{loc}}(\Lambda; \mathfrak{e})$. Finally, assuming the optimizations to be restricted to separable states and to local unitary operations, one can define $\mathcal{E}_{\mathrm{loc,sep}}^{(n)}(\Lambda; n\mathfrak{e})$ and $C_{\mathrm{loc,sep}}(\Lambda; \mathfrak{e})$. See Ref. [37] for the formal definitions. Simple resource counting arguments can be used to show that a natural partial ordering exists among these functionals [30] which identifies $C_{\mathcal{E}}(\Lambda; \mathfrak{e})$ and $C_{\mathrm{loc,sep}}(\Lambda; \mathfrak{e})$ as the largest and smallest terms, respectively, leaving the role of intermediate quantities to $C_{\mathrm{sep}}(\Lambda; \mathfrak{e})$ and $C_{\mathrm{loc}}(\Lambda; \mathfrak{e})$, i.e.,

$$C_{\mathcal{E}}(\Lambda; \mathfrak{e}) \geq C_{\mathrm{sep}}(\Lambda; \mathfrak{e}), \quad C_{\mathrm{loc}}(\Lambda; \mathfrak{e}) \geq C_{\mathrm{loc,sep}}(\Lambda; \mathfrak{e}). \tag{4}$$

As we shall see, one of the main goals of the present work is to refine Eq. (4), showing that, for all noise models, no gap exists between $C_{\mathrm{loc,sep}}(\Lambda; \mathfrak{e})$ and $C_{\mathrm{loc}}(\Lambda; \mathfrak{e})$, and that $C_{\mathrm{sep}}(\Lambda; \mathfrak{e}) \geq C_{\mathrm{loc}}(\Lambda; \mathfrak{e})$.

Closed formulas and bounds.—We now show that, regardless of the LCPT map Λ, the separable-input, local ergotropy capacitance $C_{\mathrm{loc,sep}}(\Lambda; \mathfrak{e})$ and the local ergotropy capacitance $C_{\mathrm{loc}}(\Lambda, \mathfrak{e})$ coincide and admit a simple single-letter expression in terms of the single-shot ($n = 1$) maximal output ergotropy functional (2).

Theorem 1: For any LCPT map Λ, and for any $\mathfrak{e} \in [0, \|\hat{h}\|_\infty]$, we can write

$$C_{\mathrm{loc}}(\Lambda; \mathfrak{e}) = C_{\mathrm{loc,sep}}(\Lambda; \mathfrak{e}) = \chi(\Lambda; \mathfrak{e}), \tag{5}$$

$$\chi(\Lambda; \mathfrak{e}) \coloneqq \sup_{\{p_j, \mathfrak{e}_j\}} \sum_j p_j \mathcal{E}^{(1)}(\Lambda; \mathfrak{e}_j), \tag{6}$$

where the supremum is taken over all the distributions $\{p_j, \mathfrak{e}_j\}$ of input q-cell energy $\mathfrak{e}_j \in [0, \|\hat{h}\|_\infty]$ that fulfill the constraint

$$\sum_j p_j \mathfrak{e}_j \leq \mathfrak{e}. \tag{7}$$

Proof.—In view of Eq. (4), to derive Eq. (5), it is sufficient to show that $\chi(\Lambda; \mathfrak{e})$ is (i) a lower bound for $C_{\mathrm{loc,sep}}(\Lambda; \mathfrak{e})$ and (ii) an upper bound for $C_{\mathrm{loc}}(\Lambda; \mathfrak{e})$.

A proof of inequality (i) follows by observing that given $\mathfrak{e} \in [0, \|\hat{h}\|_\infty]$ and the n integer, we have that the maximum output local ergotropy of the model $\mathcal{E}_{\mathrm{loc,sep}}^{(n)}(\Lambda; E = n\mathfrak{e})$ is certainly smaller than the output local ergotropy computed on the (factorized) state of the form $\hat{\rho}^{(n)}_{\rm fact} := \hat{\rho}_1 \otimes \hat{\rho}_2 \otimes \cdots \otimes \hat{\rho}_n$, where for $i = 1, \ldots, n$, $\hat{\rho}_i$ is a density matrix of the ith q-cell with input energy $\tilde{\mathfrak{e}}_i \in [0, \|\hat{h}\|_\infty]$ fulfilling the constraint $\sum_{i=1}^n \tilde{\mathfrak{e}}_i/n \leq \mathfrak{e}$. In other words, $\mathcal{E}^{(n)}_{\rm loc,sep}(\Lambda; E = n\mathfrak{e}) \geq \mathcal{E}_{\rm loc}(\Lambda^{\otimes n}(\hat{\rho}^{(n)}_{\rm fact}); \hat{H}^{(n)}) = \sum_{i=1}^n \mathcal{E}(\Lambda(\hat{\rho}_i); \hat{h})$, where in the second part we use the fact that the local ergotropy for noninteracting systems is additive [30,36]. In particular, selecting the $\tilde{\rho}_i$ so that they maximize the single-shot maximal output ergotropy $\mathcal{E}^{(1)}(\Lambda; \tilde{\mathfrak{e}}_i)$, we can translate the above inequality into

$$\frac{\mathcal{E}^{(n)}_{\rm loc,sep}(\Lambda; E = n\mathfrak{e})}{n} \geq \sum_{i=1}^n \frac{\mathcal{E}^{(1)}(\Lambda; \tilde{\mathfrak{e}}_i)}{n}. \tag{8}$$

In the $n \to \infty$ limit, the lhs converges toward $C_{\rm loc,sep}(\Lambda; \mathfrak{e})$. On the contrary, given an arbitrary distribution $\{p_j, \mathfrak{e}_j\}$ that fulfills the constraint (7), we can force the rhs of Eq. (8) to converge to $\sum_j p_j \mathcal{E}^{(1)}(\Lambda; \mathfrak{e}_j)$. Accordingly, we can write $C_{\rm loc,sep}(\Lambda; \mathfrak{e}) \geq \sum_j p_j \mathcal{E}^{(1)}(\Lambda; \mathfrak{e}_j)$, which, upon optimization over all choices of $\{p_j, \mathfrak{e}_j\}$, shows that the rhs of Eq. (5) is indeed a lower bound for $C_{\rm loc,sep}(\Lambda; \mathfrak{e})$.

We now prove property (ii). For this purpose, consider the optimal state $\hat{\rho}^{(n)}_*$, which allows us to saturate the maximization of $\mathcal{E}^{(n)}_{\rm loc}(\Lambda; E = n\mathfrak{e})$ for fixed n and $\mathfrak{e}$. Using the additivity of the local ergotropy, we can write

$$\begin{aligned}\frac{\mathcal{E}^{(n)}_{\rm loc}(\Lambda; E = n\mathfrak{e})}{n} &= \frac{\mathcal{E}_{\rm loc}(\Lambda^{\otimes n}(\hat{\rho}^{(n)}_*); \hat{H}^{(n)})}{n} \\ &= \sum_{i=1}^n \frac{\mathcal{E}(\Lambda(\hat{\rho}_i); \hat{h})}{n} \leq \sum_{i=1}^n \frac{\mathcal{E}^{(1)}(\Lambda; \mathfrak{e}_i)}{n},\end{aligned} \tag{9}$$

where for $i \in \{1, \ldots, n\}$, $\hat{\rho}_i$ is the reduced density matrix of $\hat{\rho}^{(n)}_*$ associated with the ith q-cell and where $\mathfrak{e}_i$ indicates its mean energy, which, by construction, must fulfill the condition $\sum_{i=1}^n \mathfrak{e}_i/n \leq \mathfrak{e}$. Next, we observe that since $\{p_i = 1/n, \mathfrak{e}_i\}$ is a special instance of energy distribution satisfying Eq. (7), the last term of Eq. (9) is certainly smaller than or equal to $\chi(\Lambda; \mathfrak{e})$. Taking the $n \to \infty$ limit, we finally arrive at the thesis. ■

Remark 1: For noise models where the single-shot ($n = 1$) maximal output ergotropy $\mathcal{E}^{(1)}(\Lambda; \mathfrak{e})$ is a concave function of the energy parameter $\mathfrak{e}$, the optimization in Eq. (5) can be explicitly performed, leading to a more compact expression:

$$C_{\rm loc}(\Lambda; \mathfrak{e}) = C_{\rm loc,sep}(\Lambda; \mathfrak{e}) = \mathcal{E}^{(1)}(\Lambda; \mathfrak{e}). \tag{10}$$

Since $C_{\rm loc,sep}(\Lambda; \mathfrak{e})$ is certainly not larger than $C_{\rm sep}(\Lambda; \mathfrak{e})$, the result of Theorem 1 allows us to introduce a definite ordering among $C_{\rm sep}(\Lambda; \mathfrak{e})$ and $C_{\rm loc}(\Lambda; \mathfrak{e})$, i.e.,

$$C_{\rm sep}(\Lambda; \mathfrak{e}) \geq C_{\rm loc}(\Lambda; \mathfrak{e}). \tag{11}$$

Physically, this implies that, at variance with what happens in other quantum information settings like those of quantum communication [38], in the case of work extraction tasks, the use of nonlocal resources at the decoding stage is, in principle, *always* preferable to their use at the encoding stage. In order to strengthen this statement, we next derive a nontrivial lower bound for $C_{\rm sep}(\Lambda; \mathfrak{e})$, which, for QB models made of q-cells of dimension larger than 2, is typically larger than $\chi(\Lambda, \mathfrak{e})$.

Corollary 1: For any LCPT map Λ, and for any $\mathfrak{e} \in [0, \|\hat{h}\|_\infty]$, we can write

$$C_{\rm sep}(\Lambda; \mathfrak{e}) \geq \chi_{\rm tot}(\Lambda; \mathfrak{e}) := \sup_{\{p_j, \mathfrak{e}_j\}} \sum_j p_j \mathcal{E}^{(1)}_{\rm tot}(\Lambda; \mathfrak{e}_j), \tag{12}$$

where $\mathcal{E}^{(1)}_{\rm tot}(\Lambda; \mathfrak{e})$ is the single-shot, energy-constrained, maximum total ergotropy [9] that one can get at the output of the channel Λ; here, as in the case of Eq. (5), the supremum is taken over all the distributions $\{p_j, \mathfrak{e}_j\}$ of input q-cell energies $\mathfrak{e}_j \in [0, \|\hat{h}\|_\infty]$ satisfying the constraint (7).

Ultimately, the inequality (12) is a consequence of the property shown in Ref. [30] that $C_{\rm sep}(\Lambda; \mathfrak{e})$ coincides with $C_{\rm sep,tot}(\Lambda; \mathfrak{e})$ [the latter being obtained by replacing the ergotropy appearing in $\mathcal{E}^{(n)}_{\rm sep}(\Lambda; E)$ with the total ergotropy]. For the sake of completeness, however, we provide an independent proof of the corollary in Ref. [37].

Under special circumstances, we can show that the rhs of Eq. (12) provides the exact value of $C_{\rm sep}(\Lambda; \mathfrak{e})$.

Corollary 2: Suppose that Λ is a LCPT map such that, for all $\mathfrak{e} \in [0, \|\hat{h}\|_\infty]$ and n integers, it admits a single-site state $\hat{\sigma}_\mathfrak{e}$, possibly dependent on n, with mean energy $\mathfrak{E}(\hat{\sigma}; \hat{h}) \leq \mathfrak{e}$, such that

$$\mathcal{E}(\Lambda^{\otimes n}(|\Psi^{(n)}_{\rm fact}\rangle\langle\Psi^{(n)}_{\rm fact}|); \hat{H}^{(n)}) \leq \mathcal{E}((\Lambda(\hat{\sigma}_\mathfrak{e}))^{\otimes n}; \hat{H}^{(n)}), \tag{13}$$

for all factorized pure states $|\Psi^{(n)}_{\rm fact}\rangle$ with mean energy $\mathfrak{E}(|\Psi^{(n)}_{\rm fact}\rangle; \hat{H}^{(n)}) \leq n\mathfrak{e}$. Then, the inequality (12) is saturated, i.e.,

$$C_{\rm sep}(\Lambda; \mathfrak{e}) = \chi_{\rm tot}(\Lambda, \mathfrak{e}). \tag{14}$$

Proof.—Let $\{P_k, |\Psi^{(n)}_{\rm fact}(k)\rangle\}$ be an ensemble of factorized states allowing us to express a given separable density matrix $\hat{\rho}^{(n)}_{\rm sep}$ of the QB, i.e., $\hat{\rho}^{(n)}_{\rm sep} = \sum_k P_k |\Psi^{(n)}_{\rm fact}(k)\rangle\langle\Psi^{(n)}_{\rm fact}(k)|$. Notice that if $\hat{\rho}^{(n)}_{\rm sep}$ has mean energy $\mathfrak{E}(\hat{\rho}^{(n)}_{\rm sep}; \hat{H}^{(n)}) \leq n\mathfrak{e}$, then we must have $\sum_k P_k \mathfrak{e}_k \leq \mathfrak{e}$, with $n\mathfrak{e}_k$ being the mean energy of $|\Psi^{(n)}_{\rm fact}(k)\rangle$, i.e., $\mathfrak{E}(|\Psi^{(n)}_{\rm fact}(k)\rangle; \hat{H}^{(n)}) = n\mathfrak{e}_k$. Thanks to the convexity of the ergotropy functional, it follows that

$$
\begin{aligned}
\frac{1}{n}\mathcal{E}(\Lambda^{\otimes n}(\hat{\rho}^{(n)}_{\mathrm{sep}});\hat{H}^{(n)}) &\le \frac{1}{n}\sum_k P_k \mathcal{E}(\Lambda^{\otimes n}(|\Psi^{(n)}_{\mathrm{fact}}(k)\rangle);\hat{H}^{(n)}) \\
&\le \frac{1}{n}\sum_k P_k \mathcal{E}((\Lambda(\hat{\sigma}_{\mathfrak{e}_k}))^{\otimes n};\hat{H}^{(n)}) \\
&\le \sum_k P_k \sup_{n'} \frac{\mathcal{E}((\Lambda(\hat{\sigma}_{\mathfrak{e}_k}))^{\otimes n'};\hat{H}^{(n')})}{n'} \\
&= \sum_k P_k \mathcal{E}_{\mathrm{tot}}(\Lambda(\hat{\sigma}_{\mathfrak{e}_k});\hat{h}) \\
&\le \sum_k P_k \mathcal{E}^{(1)}_{\mathrm{tot}}(\Lambda,\mathfrak{e}_k) \le \chi_{\mathrm{tot}}(\Lambda;\mathfrak{e}),
\end{aligned}
$$

where, in the second inequality, we used Eq. (13), while in the last three formulas, we invoked the definitions of total ergotropy and of maximum energy-constrained, output total ergotropy. Observing that $\{P_k, \mathfrak{e}_k\}$ is a special instance of the ensembles entering the supremum of the rhs of Eq. (14), we can hence conclude that $(1/n)\mathcal{E}(\Lambda^{\otimes n}(\hat{\rho}^{(n)}_{\mathrm{sep}});\hat{H}^{(n)}) \le \chi_{\mathrm{tot}}(\Lambda;\mathfrak{e})$, which holds true for all separable inputs that have mean energy smaller than or equal to $n\mathfrak{e}$. The thesis finally follows by taking the supremum with respect to all $\hat{\rho}^{(n)}_{\mathrm{sep}}$ and taking the $n \to \infty$ limit. ∎

Examples.—Since the total-ergotropy functional and the ergotropy always coincide for systems of dimension 2 (see Ref. [37]), in view of Theorem 1 and Corollary 1, the best option to identify QBs that exhibit a finite gap ΔC between $C_{\mathrm{sep}}(\Lambda;\mathfrak{e})$ and $C_{\mathrm{loc}}(\Lambda;\mathfrak{e})$ is to focus on models with q-cell elements having dimension $d \ge 3$. A first example of this kind can be found in Ref. [30], which solved the values of $C_{\mathrm{sep}}(\Lambda;\mathfrak{e})$ and $C_{\mathrm{loc}}(\Lambda;\mathfrak{e})$ for depolarizing maps of arbitrary dimension. In what follows, we present a couple of extra cases. In panel (a) of Fig. 2, for instance, we plot the gap we obtained by numerically solving the optimizations in Eqs. (5) and (12) for a three-level system subject to the action of a multilevel amplitude damping channel [39,40]. Another example is obtained by focusing on QB models formed by the collection of independent (infinite-dimensional) harmonic oscillators affected by noise models described by phase-insensitive bosonic Gaussian channels (PI-BGCs) [38,41]. Thanks to the results of Ref. [42], we know that, in these systems, the maximal output ergotropy and maximal total ergotropy are obtained using coherent states as input configurations for the q-cells. If Λ corresponds to a single-mode PI-BGC Φ_1, this implies that $C_{\mathcal{E}}(\Phi_1;\mathfrak{e}) = C_{\mathrm{sep}}(\Phi_1;\mathfrak{e})$ for all input energies $\mathfrak{e}$, leading to no gap between $C_{\mathrm{sep}}(\Phi_1;\mathfrak{e})$ and $C_{\mathrm{loc}}(\Phi_1;\mathfrak{e})$. In particular, we have $C_{\mathcal{E}}(\mathcal{L}_{\lambda,N};\mathfrak{e}) = \lambda\mathfrak{e}$, $C_{\mathcal{E}}(\mathcal{A}_{\mu,N};\mathfrak{e}) = \mu\mathfrak{e}$, and $C_{\mathcal{E}}(\mathcal{N}_N;\mathfrak{e}) = \mathfrak{e}$, where $\mathcal{L}_{\lambda,N}$ is the thermal attenuator, $\mathcal{A}_{\mu,N}$ is the thermal amplifier, and $\mathcal{N}_N$ is the additive noise channel [38,41]. The situation becomes more interesting if Λ represents a multimode PI-BGC. As a matter of fact, here we can observe cases where C_{sep} and C_{loc} exhibit a gap. For instance, in panel (b) of Fig. 2, we consider a two-mode PI-BGC $\Phi_2 = \mathcal{L}_{\lambda,N} \otimes \mathcal{L}_{0,0}$, which acts as a thermal attenuator on one of the modes and outputs the vacuum state on the other mode. For this channel, the optimal output ergotropy $\mathcal{E}^{(1)}(\Phi_2;\mathfrak{e})$ and the optimal output total ergotropy $\mathcal{E}^{(1)}_{\mathrm{tot}}(\Phi_2;\mathfrak{e})$ are both affine but with different functions of the input energy $\mathfrak{e}$, implying $C_{\mathrm{loc}}(\Phi_2;\mathfrak{e}) = \mathcal{E}^{(1)}(\Phi;\mathfrak{e})$ and $C_{\mathrm{sep}}(\Phi_2;\mathfrak{e}) = \mathcal{E}^{(1)}_{\mathrm{tot}}(\Phi;\mathfrak{e})$. As shown in Ref. [37], the resulting gap ΔC is a constant with respect to the input energy and only depends on the noise parameters of the model (i.e., the constants λ and N) via an implicit functional of λ and N.

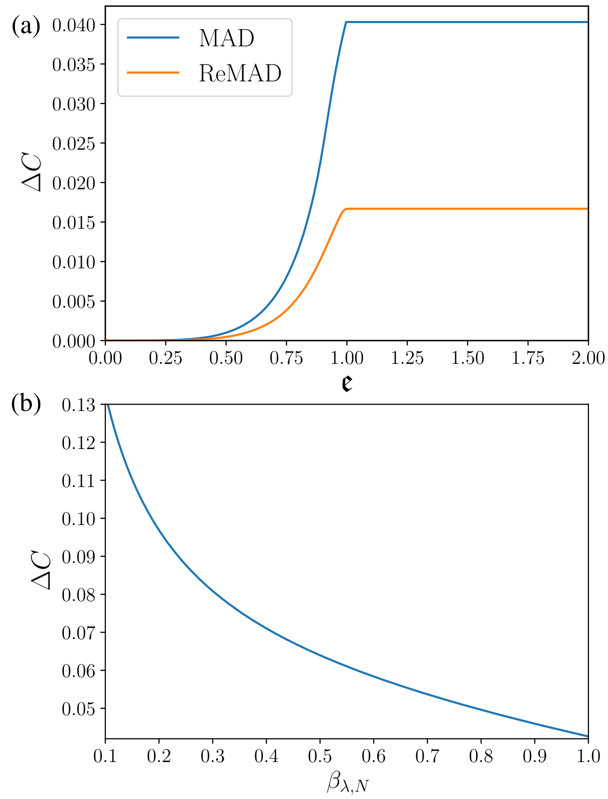

FIG. 2. Capacitance gap $\Delta C \coloneqq C_{\mathrm{sep}}(\Lambda;\mathfrak{e}) - C_{\mathrm{loc}}(\Lambda;\mathfrak{e})$. (a) Plot of the lower bound on ΔC derived from Eqs. (5) and (12) for the MAD channel $\Phi_{\gamma_1,\gamma_2,\gamma_3}$ [39] and the ReMAD channel $\Gamma_{\gamma_1,\gamma_2,\gamma_3}$ [40], both acting on a qutrit system of Hamiltonian $\hat{h} = \varepsilon_0(|1\rangle\langle 1| + 2|2\rangle\langle 2|)$. Here, the energy is rescaled by the factor ε_0, and the noise parameters have been fixed equal to $\gamma_1 = 0.3$, $\gamma_2 = 0.2$, and $\gamma_3 = 0.6$. (b) Plot of ΔC for the two-mode Gaussian attenuator $\Phi_2 = \mathcal{L}_{\lambda,N} \otimes \mathcal{L}_{0,0}$ as a function of the effective temperature $\beta_{\lambda,N}$ [see Eq. (30) in Ref. [37]]; in this case, ΔC is independent on the average input energy $\mathfrak{e}$. In the plot, the value of ΔC has been rescaled by $\hbar\omega$, with ω being the mode's frequency.

Discussion.—We have observed an asymmetry in the role of nonlocal resources in the investigation of work extraction from noisy QB models. Specifically, our findings indicate that when energy is recovered through local operations, the use of entangled input states of the q-cells does not enhance the QB's resistance to noise. Conversely, through the examination of a few examples, we demonstrate that incorporating nonlocality in the extraction operations can be advantageous. This implies that nonlocality provides a distinct benefit in energy retrieval but may not be beneficial if employed in state preparation. Moving forward, we plan to extend these results to models where the QB Hamiltonian exhibits interactions among the individual q-cells.

We acknowledge financial support by MUR (Ministero dell' Università e della Ricerca) through the following projects: PNRR (Piano Nazionale di Ripresa e Resilienza) MUR Project No. PE0000023-NQSTI, PRIN 2017 (Progetti di Rilevante Interesse Nazionale) Taming Complexity via Quantum Strategies: A Hybrid Integrated Photonic Approach (QUSHIP) Id. No. 2017SRN-BRK, and the PRO3 Quantum Pathfinder project. S. C. is also supported by a grant through the IBM-Illinois Discovery Accelerator Institute.

*salvatore.tirone@sns.it
†raffaele.salvia@sns.it
‡schessa@illinois.edu

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