Construction and Mathematical Schematism
Kant on the Exhibition of a Concept in Intuition[1]

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in memory of David R. Lachterman

In this paper I will discuss the relevance of mathematics for our understanding of some of the central tenets of the *Critique of Pure Reason*. My purpose is not to repeat once again that Kant tried to give a transcendental foundation of the sciences on which the Newtonian world is built, nor that Kant's discussion of mathematics was bound to the historical premises and situation of eighteenth-century development of algebra, calculus and analytic geometry; nor am I concerned with the long debated problems of the relationship between visual and absolute space and the incompatibility of Kant's Transcendental Aesthetic with non-Euclidean geometries. I want rather to show why and in what sense the problem of mathematical construction, of schematism and of the apriori exhibition (*Darstellung*) of a concept in intuition are interrelated and cannot be approached independently of one another, and to what extent all interpretations of the chapter on schematism that do not pay attention to the different role of *intuition*, of *form* and of *imagination* in mathematics and philosophy are doomed to failure. I will try to analyze and discuss the plausibility of Kant's long misunderstood claim that mathematics is an apriori synthesis in time.

§ 1 *Introductory Remarks*

The elementary examples of mathematical construction that Kant gives in the *Critique of Pure Reason* have often led interpreters to ascribe a rather simplistic view of mathematics to him. This, in turn, has been in most cases the first step towards a widespread criticism of the supposed failure to include intuition in the pure development of arithmetic, geometry and algebra. I want to show that Kant's

[1] This article is a thorough reelaboration of a paper I wrote for Lachterman's seminar on Philosophy of Mathematics at Penn State in Fall 1990. I wish to acknowledge my debt to his helpful comments and to his suggestions for further bibliographical readings. It is my hope that this essay could have meant something to Lachterman. I also want to thank Professor Pierre Kerszberg and Alessandra Fussi for their very careful reading of the manuscript and valuable suggestions to improve it.

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mature theory of mathematics is, to the contrary, quite articulate, detailed and consistent. Although in order to become aware of the complexity of Kant's interest in mathematics[2] it is not necessary to move to other texts, recourse will be made to some of them. In particular I find of noteworthy importance Kant's reply to objections about the roles both of intuition in geometry and of time in the construction of irrational numbers (*Entdeckung*, correspondance with Reinhold, Schultz and Rehberg); the discussion of differential calculus in Newton's fluxions (*Reflexionen zur Mathematik*[3]) and its applicability to mechanics (*Metaphysische Anfangsgründe*); the theme of algebra as "ampliative science"[4]; and the Euclidean problem of parallel lines and the Leibnizian *analysis situs* (*Reflexionen*). The pivotal relevance of mathematics for Kant is not diminished or altered by the remark that in both these last problems Kant's knowledge of the sources was indirect[5] or mediated by Wolff, Lambert and especially by Johann Schultz. Schultz, a mathematician who was *Hofprediger* in Königsberg, author of, among other things, one of the very few reviews of the *Critique* that Kant ever praised[6], also defended Kant's views from the attacks of Maaß, Eberhard and Kästner, and collaborated with Kant in a reply to Kästner's objections to Kant's theory of geometrical space (Ak. XX, 419–22), to which I shall return later.

Many scholars, starting with Beth and Hintikka, have stressed the continuity in the evolution of Kant's position towards mathematics in order to find in his precriti-

[2] Martin (*Arithmetik und Kombinatorik bei Kant*, Freiburg 1934, Berlin–New York 1972) provides the most authoritative confirmation of Kant's deep concern with and thorough knowledge of the mathematics of his time. See also Moretto, *Sul concetto matematico di «grandezza» secondo Kant. L'Analitica del sublime della Critica del Giudizio e la grandezza infinita*, in: Verifiche XIX, 1–2, 1990, 51–125, especially about Kant's standpoint on the mathematics of the infinite.

[3] The *Critique* is cited, as usual, with the number of pages from both editions in the Akademie-Ausgabe. I use Norman Kemp Smith's English translation (New York 1929, 1965). Other translations: *Prolegomena* (Ak. Bd. IV), Engl. trans. by Carus, revised by L. W. Beck, New York 1950; *Über eine Entdeckung nach der alle neue Kritik der reinen Vernunft durch eine ältere entbehrlich gemacht werden soll*, in: Ak. VIII, Engl. trans. in: H. E. Allison, *The Kant–Eberhard Controversy*, Baltimore 1973. See also: *Vom inneren Sinne* (*Loses Blatt Leningrad 1*), ed. by R. Brandt, W. Stark and A. Gulyga, in: Voprosy Filosofii IV 1986, 128–36.

[4] Letter to J. Schultz, Nov. 25, 1788, *Briefwechsel*, Ak. Bd. X, 554–58.

[5] Lorenz' translation of Euclid's *Elements* appeared in Halle in 1773. Kant's knowledge of Greek was elementary, and the disappointing and banal remarks about the ancients found in his works come mostly from scholastic handbooks, in particular from Brucker's *Historia critica philosophiae*.

[6] See *Erläuterungen über des Herrn Professor Kant Critik der reinen Vernunft*, Königsberg 1784, and *Prüfung der kantischen Critik der reinen Vernunft*, Königsberg 1789–1792. For the relationship between Kant and Schultz, and for the latter's importance with respect to Kant's philosophy of mathematics, cf. Adickes' notes in Ak. XIV (27 ff.), as well as Martin, *Arithmetik und Kombinatorik bei Kant*, op. cit. On Schultz's reading of Kant see now Bonelli Munegato's monography (*Johann Schultz e la prima recezione del criticismo kantiano*, Trento 1992).

cal works the basis for their interpretation of intuition as evidence. What matters here is only that in the text to which they most often refer, the *Untersuchung über die Deutlichkeit der Grundsätze der natürlichen Theologie und der Moral*[7] of 1764, where Kant focuses on the difference between philosophy and mathematics, he makes no mention of construction in what he calls the "sensible signs" adopted by mathematics: mathematical judgments are here analytic. It is true that the arbitrary character of the signs adopted in mathematics and the synthetic origin of its concepts can be seen as the germ of the future notion of construction in intuition. Yet, the distinctive evidence that makes mathematics an exact science depends only on the univocity, immediate verifiability and visibility of its *signs*, as opposed to the indeterminacy of the words that metaphysics must use but cannot analyze into elementary constituents. What is missing in this essay is the mature conception of a pure intuition in which to construct the mathematical object: the very meaning of synthesis is therefore slightly different from that in the *Critique*[8]. Although in the *Dissertatio* of 1770 we already find the example of drawing a line that would occur in the chapter on transcendental deduction, in the Axioms of Intuition and in the *Methodenlehre* later in the *Critique*, the intellectual concept of a number is distinguished from its "actuation in the concrete" – a position that Kant revises in the *Critique*[9].

As reported by Cassirer, it is in the notes from the early 1770s that Kant defines his mature philosophy of mathematics. A significant passage is contained in the famous letter to Marcus Herz of 1772. Kant writes there that the key problem he had not solved in the *Dissertatio* is to account for the ground of the relation between representation and object. Human cognition has to be explained as something intermediate between the passive receptivity of the *intellectus ectypus*, where the subject is merely affected by the sensible objects, and the active creation of the *intellectus archetypus*, which produces its object as "when we conceive of divine cognitions as the models (*Urbilder*) of things". Kant adds that in mathematics objects can be represented as magnitudes because we *produce* (*erzeugen*) their repre-

[7] See Ak. Bd. II, esp. 291. Cp. E. W. Beth, *Über Lockes Allgemeines Dreieck*, in: Kant-Studien 47–8, 1955/56, 361–81 (377) and M. Capozzi Cellucci, *J. Hintikka e il metodo della matematica in Kant*, in: Il Pensiero 18, 1973, 232–67 (240 ff.).

[8] For all this see E. Cassirer, *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit*, Berlin 1911², Bk. 8, chapt. 1, § 3; F. Barone, *Logica formale e logica trascendentale*, vol. I, *Da Leibniz a Kant*, Torino 1957, 143–49; G. Tonelli, *Der Streit über die mathematische Methode in der Philosophie in der ersten Hälfte des 18. Jahrhundert und die Entstehung von Kants Schrift über die Deutlichkeit*, in: Archiv für Philosophie 9, 1959, 37–66, (61 and 65); and Capozzi (*J. Hintikka e il metodo ..., op. cit.*, 259). Tonelli shows the similarity of these points to those of Mendelssohn's essay (*Über die Evidenz in den Metaphysischen Wissenschaften*) of the same year (Mendelssohn won the competition offered by the Berlin Academy, which ranked Kant second).

[9] For the relevant passage in the *Dissertatio*, see Ak. II, 397. For a commentary on this point see H. J. De Vleeschauwer, *La déduction transcendentale dans l'œuvre de Kant*, 3 vols., Paris 1937, vol. III, 210, and Capozzi, *J. Hintikka e il metodo ..., op. cit*, 261.

sentation by taking the unity as many times as we need (*Briefwechsel*, Ak. X, 130–31).

Commenting on this passage in his beautiful book, Lachterman advances two very provocative and intriguing theses: 1. mathematics "gives our best approximation to the latter extreme (i. e., the *intellectus archetypus*), that is, to the thoroughgoing freedom to *produce* objects answering to intellectual representations". 2. Kant's philosophical enterprise takes its bearings from this conception of mathematical construction, which "remains the name for the *telos* that philosophy, in the theoretical, practical and aesthetic domains, continues to propose to itself without any hope of requital" [10]. Reason's desire to objectify itself and to exhibit its products in an alien medium, sensibility, is for Lachterman at the core both of the theory of mathematical construction and of schematism, as the infinite drive to realize – literally, give reality to – its concepts and ideas. He contents that we find the most adequate formulation of the modern project after Descartes in Kant's dictum that in mathematics "sic volo, sic jubeo": by producing a world that stands for nature we need not rely on its givenness, and, as in Maimon's phrase, we become ourselves like gods.

I think that if we took both these points without further qualification as a hermeneutical guide for the reading of the *Critique*, some conclusions we might legitimately reach could be proven wrong on the basis of a closer reading of Kant's texts. I believe one must be careful to draw a precise distinction between philosophy and mathematics (and empirical discourse as well, as we shall see) without at the same time violating their unity. In a sense, then, we should find a way between the poles indicated by Lachterman, and this essay takes its bearings from this attempt. In other words, it is crucial to find a balance between, on the one hand, the sharp separation of mathematical construction and philosophical discursive explanation, and, on the other hand, their joint inspiration of Kant's transcendental enterprise, if we do not want to give up both the unity of our theoretical experience of nature (*natura formaliter spectata*) and the different meanings of apriori synthesis. Thus in mathematics, although the finitude of our understanding is not the same as in philosophy, there cannot be a *creatio ex nihilo*, nor can the inspiration be that of a demiurgic paradigm: the understanding does not create things in themselves or a world independent of appearances in space and time. What it does is determine inner sense according to the category of quantity; but what is thus constituted is the form of appearances, not the archetypes of things. To claim the latter would be to lose sight of the unity of the understanding's spontaneous activity in the mathematical and philosophical realms, and make it aporetic, not to say contradictory, to speak of an application of mathematics to nature. On the other hand, although

[10] *The Ethics of Geometry*, New York–London, 1989, 10–12; see also his *Kant; the Faculty of Desire*, in: The Graduate Faculty Philosophy Journal 13, 2, 1990, 192–93. I give a detailed analysis of *The Ethics of Geometry* in my article *Mathesis e costruzione tra geometria antica e moderna*, in: Teoria XI/2, 1991, 87–104.

mathematics is a "splendid example" of a science which extends its knowledge apriori, and metaphysics should also try to move on the firm ground of synthetic apriori judgments (*Kr. d. r. V.*, Introduction, § V. 3), philosophy cannot imitate mathematics. There is a subtle but undeniable and crucial difference between the meaning of synthesis apriori – and of form of appearances – in transcendental philosophy and in mathematics. I hope to give a persuasive account of what I mean by this in the main body of the paper.

Kant's characterization of mathematics is indeed marked by a quasi-idealistic overtone. The autonomous *Selbstaffektion* of our spontaneity in mathematics produces its objects, which do not have existence prior to our definition of them. "While philosophical definitions are never more than expositions of given concepts, mathematical definitions are constructions of concepts, originally framed by the mind itself (...), produced synthetically (...) Mathematical definitions *make* (*machen*) their concepts" (Kant's italics, A 730/B 758). This idea, present already in the *Deutlichkeit* (Ak. II, 276), returns in the letter to Herz, where the concepts of magnitude are said to be active in their own right (*selbstthätig*, Ak. X, 131). It is here that the later distinction in the Doctrine of Method between discursive knowledge from concepts and knowledge from the construction of concepts (A 713/B 741) first appears thematically. This key notion is also at the root of the central passage in the Preface to the second edition (see also Introduction, § V) of the *Critique*, where we read that mathematics determines quite purely apriori its object (B X), and that

> a new light flashed upon the mind of the first man (be it Thales or some other) who demonstrated the properties of the isosceles triangle. The true method, so he found, was not to inspect what he discerned either in the figure, or in the bare concept of it (...) but to bring out what was necessarily implied in the concepts that he had himself formed apriori, and had put into the figure in the construction by which he presented it to himself. If he is to know anything with apriori certainty he must not ascribe to the figure anything save what necessarily follows from what he himself has set into it in accordance with his concept (B XI–XII).

Mathematics gives a "shining example"[11] of an objectifying production, in which the mere representation of the object constructed both contains the actuality of the object and shows its concept. In this sense Kant would have agreed with the spirit of Schultz' reply to Kästner, according to which "in mathematics possibility and actuality are one, and the geometer says *there are* conic sections, as soon as he has shown their possibility apriori" (Ak. XX, 386). If mathematics considers the concept *in concreto*, and not empirically but apriori, the sense of existence in mathematics differs from aposteriori empirical existence (A 719/B 747), and is made equal to constructibility[12]. The mathematician determines intuition objectively and

[11] This expression recurs at least three times in crucial passages of the *Critique*: A 4/B 8, A 39/B 55, A 712/B 740.

[12] Kant knew that this would restrict the scope of mathematical objects: for example, a regular heptagon or an imaginary number cannot be constructed. I will have more to say about this delicate point in § 4. But we certainly cannot follow Martin (*Immanuel Kant. Ontologie*

is able to bridge purely apriori the gap between discursive reason and the exhibition of an intuition that shows the concept *in concreto*, whereas all other realms of knowledge need to support their apriori cognitions *via* empirical verification. Therefore it would seem that, since the mathematician constructs or exhibits his object in an apriori intuition, thinking and knowing are not separate in mathematics.

One may be willing to concede this point, but still find the relationship between concept and intuition controversial or obscure. A certain number of works have been devoted to illustrating Kant's philosophy of mathematics. My contention in what follows is that by and large – obviously with a few exceptions – the works in the secondary literature cannot successfully make sense of the syntheticity of mathematics suggested by Kant for the simple reason that his assertions about mathematics are not related, as seems necessary to me, to the notion of inner sense in the transcendental deduction, and to the schematism. This is the source of many misunderstandings. For instance, Hintikka not only incorrectly reconstructs Kant's evolution in mathematics within the precritical period and wrongly posits a continuity between Kant's notion of construction and the Euclidean εκθεσις and κατασκευη[13], he also interprets intuition as a singularity having no relation to space and time as forms of our sensibility. The shortcoming of this approach is that time, which is for Kant essential to mathematics, does not have any more meaning than the necessity of the schema, because Hintikka's nominalistic semantics, which operates with variables and individuals, makes the intuition already homogeneous to the concept[14]. It is hard to say what is left of Kant, especially since the syntheticity of mathematics is lowered to the mere question of the ampliative character of demonstrations. Succession in time is involved in every mathematical concept for Kant. What this means is not easy to show, but certainly this rules out the possibility that mathematical construction as exhibition of an apriori intuition amounts to a test or a process of verification of an intellectual concept, and also that construction in geometry is dispensable, as suggested by Winterbourne[15]. Beth, Parsons and Young

und Wissenschaftstheorie, Berlin–New York 1969, 25), who argues that since a chiliagon cannot form an image, it is not a geometrical figure. Kant says very clearly in the *Entdeckung* that we cannot deny it the right of citizenship in mathematics. The distinction between schematic and technical construction – which belongs to empirical art, not to science – serves the purpose of distinguishing schema and image and their respective tasks in mathematics (Ak. VIII, 191–92, n.; Engl. tr. Allison 110 n.).

13 See for ex. *Kant on the Mathematical Method*, in: The Monist 51, 1967, 352–75 (361–68). For a different, more accurate understanding of these terms in Euclid see Lachterman, *The Ethics of Geometry, op. cit.*, § 2 (57 and 118); for a criticism of Hintikka's evaluation of Euclid with respect to Kant, cf. Capozzi, *J. Hintikka e il metodo ..., op. cit.*, 250–53.

14 *Kant on the Mathematical Method, op. cit.*, 358. For a very insightful and attentive review of Hintikka's works, see Capozzi, *J. Hintikka e il metodo ..., op. cit.*

15 See T. Winterbourne, *Construction and the Role of Schematism in Kant's Philosophy of Mathematics*, in: Studies in History and Philosophy of Science 12, 1981, 33–46 (36–38 and 43–4).

interpret intuition as the empirical proxy for concepts given non-intuitively, so that they cannot explain the relation between concept and intuition – or, for that matter, the *origin* of mathematical concepts. We need to take seriously Kant's notion that mathematical construction is the understanding's determination of sense: the intuition in which we construct mathematical objects is not just a means, an auxiliary ladder to throw away after using it, because it exhibits the objective validity of mathematical definitions in space and time. And the question of syntheticity in mathematics cannot be reduced to a discussion of its method or of its demonstrative procedure: intuition accounts first of all for the synthetic genesis of concepts and judgments.

That this is immediately related by Kant to their objective reality and validity is apparent. "Mathematical definitions can never be in error. For since the concept is first given through the definition, it includes nothing except precisely what the definition intends should be understood by it" (A 731/B 759). Unlike philosophical and empirical types of definition, mathematical definitions are not logical or nominal, but real definitions, insofar as they show the objective validity of their concepts by providing the corresponding intuition[16]. In a letter to Reinhold, where he defends his position against Eberhard's criticism, Kant writes:

> The definition which Apollonius gives, e. g., of a parabola, is itself the exhibition of a concept in intuition, namely, the intersection of a cone under certain conditions, and in establishing the objective reality of the concept, that the definition here, as always in geometry, is at the same time the construction of the concept (...) If a circle is defined as a curve line on which all points are equidistant from a center, is not that concept given in intuition? (*Briefwechsel*, Ak. XI, 43–4, Engl. tr. Allison 167).

In other words, the synthesis here involves the necessity to go beyond the concept and show its pure, apriori determination of a spatio-temporal intuition: the guidance for the construction of the object. And a synthetic judgment is not a formal, discursive relation between the subject and its predicate, but the activity of exhibiting in intuition the real belonging of a property to its object (A 718–19/B 746–47, and *Entdeckung*, Ak. VIII, 242, Allison 153).

§ 2 *Intuition and Concept in Mathematical Synthesis*

Of course this is not, by any means, a sufficient account of Kant's view of mathematics, which will have to be fully articulated in this and the next two sections. In order to understand what Kant means by the autonomous determination of inner sense in mathematics, and to see the relation between concept, schema, image and intuition in mathematics and philosophy, we have to proceed carefully through an analysis of the problem of space and time in the transcendental deduction. As I

[16] *Kr. d. r. V.* A 713/B 741; *Entdeckung*, Ak. VIII, 191 (Engl. tr. Allison 110); *Kr. d. r. V.* B XXVI, n. and A 241–2, n.; see also *Deutlichkeit*, Ak. II, 276, Kr. d. U. § 62, *Logik*, § 106.

already anticipated, I find it misleading to accept a view such as Cohen's that construction as found in the Discipline of the Doctrine of Method "becomes now a notion of method that takes the place of the pure intuition of the 'Doctrine of Elements'"[17]. Moreover, the idea that the "productivity of imagination relates itself simply to the understanding and its unities, the categories, certainly not to the manifold of intuition"[18] also needs a closer examination. This latter remark does not recognize that the *Selbstaffektion* of the productive imagination does in fact determine the manifold of intuition in the inner sense. It also assumes that there is no shift in meaning between the mere receptivity of space and time in the Aesthetic and their more "active" role in the Analytic, the latter being, so I think, the premise for an understanding of both the schematism and the *Methodenlehre*.

For one thing, the relationship between time and space in the Aesthetic is not thematic, while in the Analytic of Principles Kant shows their interconnection in the act of drawing a line, in the principle of the permanence of substance and in general in the Analogies of Experience, and in the refutation of subjective idealism. But what is most important is that in the Aesthetic time and space are only pure, empty forms of intuition that precede all intuition of objects in space and time, while in the Analytic – in the second edition – they become themselves positive objects of intuition in the context of a discussion of geometrical representations and selfconsciousness (§§ 24–26).

According to Kant, that the inner sense is modified by us, i. e., apriori without any appeal to an external givenness or an empirical *Affektion*, can be shown by every act of attention (B 156 n.). We cannot think of a triangle, or of a line, without drawing it in thought. Drawing a line is a *synthesis speciosa* (B 151), i. e., a transcendental synthesis of imagination applied to the manifold of intuition. In the act of drawing a line we successively determine our inner sense, so that the pure description of a space is a synthetic activity, a motion on the part of the subject that originally produces the concept of succession as the progressive composition of homogeneous parts (B 154–55). The understanding does not therefore *find* in inner sense a ready-made unification of the manifold, but rather, with the aid of productive imagination, *produces* it by modifying time. Kant returns to this in § 26, where he writes that, in this context, space and time are no longer merely forms of sensible intuitions, but rather apriori intuitions themselves, containing a manifold.

In the footnote at B 160–61, Kant specifies what he has in mind: space, represented as an object ("as we are required to do in geometry"), contains more than the mere form of intuition, namely the combination (*Zusammenfassung*) of the given manifold in an intuitive representation. If the form of intuition only gives the manifold, the *formale Anschauung* gives the unity of the representation of the manifold (*ibid.*). If the form of intuition is something all-embracing, given, one, underlying all intuitions, formal intuitions are limitations on that one intuition. The general

[17] *Commentar zur Kritik der reinen Vernunft*, 2. Aufl., Leipzig 1917, 192.
[18] *Kants Theorie der Erfahrung*, 2. Aufl., Berlin 1885, 310.

concepts of diverse times or spaces, the manifold in the form of intuition, are contained within the pure form of space and time as an infinite number of possible representations (B 39–40; see also B 136, n.). This idea is taken up again as the leading thread of Kant's and Schultz' reply to Kästner (Ak. XX, 410–23: 419, Engl. tr. Allison 175–77: 176): if metaphysics shows how one *has* the representation of space, geometry teaches how to *describe* a space. Whereas in the former space is considered "as given, before all determinations (...) in the latter it is considered as it is generated (*gemacht*). In the former it is *original* and only one (single) *space*. In the latter it is derived and there are (many) *spaces*". But the geometrician admits that these spaces, the singular representations or intuitions generated as limitations of the one space, are thought as parts of the one original space, which he too then represents as original, infinite and given.

This distinction seems to me a crucial shift in Kant's argument[19]. It seems to compel us to think of formal intuitions as in themselves positively active in the determination of the content of our knowledge, as *actual* representations opposed to mere forms negatively restricting the use of our concepts (a distinction which is reminiscent of the Aristotelian εξις–κτησις difference). It would seem that the separation between understanding and sensibility cannot be maintained in the terms in which we find it in the *Dissertatio* or even in the Aesthetic. In fact we must realize that this shift takes us to a deeper level than Kant's distinction between receptivity and activity. In light of this, I think we should treat Kant's recurrent characterization of sensibility as pure receptivity as opposed to the understanding's pure spontaneity as a systematic problem, rather than resort to a description of those views as a residue of the precritical stage[20].

The ambiguity concerns the autonomous role of formal intuitions, and it is a tension to be found in different passages in the text, not just in Kant's development. It can be stated as follows. Both the Aesthetic and the Analytic claim the sufficiency of intuitions for laying the foundations of mathematics: the principles of mathematics are derived from intuition, not from concepts[21]. The footnote at B 160 says that the unity of the formal intuition belongs merely to sensibility and precedes any concept, and recalls that this was already stated in the Aesthetic, where we read that it is a property of the form of spatial intuition that in a triangle two sides together are greater than the third (A 25/B 39)[22]. This seems to imply that it is a

19 Many commentators are aware of the distinction between form of intuition and formal intuition, or render it thematic in different ways (see Gram, Graubner, Allison, Pippin, Palumbo, Blasche, Moretto among others).

20 As do for ex. Cassirer, *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit, op. cit.*, Bk. 8, chapt. 1, § 3, and S. Vanni Rovighi, *Introduzione allo studio di Kant*, Brescia 1968, 144.

21 A 149, B 188; A 159–60, B 198–99: note that Kant is here speaking of the axioms of mathematics as those immediately evident and certain synthetic apriori principles of space and time, such as 'a straight line is the shortest distance between two points'.

22 For the commonly held view that the aesthetic is the sufficient ground for mathematics, see N. Rotenstreich, *The Schematism in its Context*, in: Id., *Experience and its Systematiza-*

property of space as such, before all concepts, to allow for our three-dimensional geometry. And this could make sense in view of Kant's notion that it is the transcendental exposition of space as pure juxtaposition (*Nebeneinandersein*) that lays the ground for its geometrical three-dimensionality[23]. But a passage from the *Prolegomena* clearly conflicts with this autonomous role of formal intuitions and their independence of the activity of the understanding: intuitions presuppose the subsumption "under the concept of magnitude, which is certainly no mere intuition, but has its seat in the understanding alone and serves to determine the intuition (of the line) with regard to the judgments which may be made about it" (§ 20, Ak. IV, 302, Engl. tr. 49; see also the passage from B 137–38 quoted below). As a consequence, if it is clear that formal intuitions are the starting point presupposed by mathematics as the objects it represents *per se*, and that formal intuitions, as actual representations, are already a composition of the manifold (therefore they cannot be pure receptivity), it is arguable to what extent they are independent, as mathematical objects, of the schematized concepts of quantity. In this respect I find it significant that in his later writings Kant leans more and more unambiguously toward the view of the *Prolegomena* and toward what appears to be the standard view of the second edition of the *Critique*, according to which the figurative synthesis of productive imagination, the unity of apperception, and the pure concepts of the understanding are the objective conditions for mathematical knowledge. In the *Fortschritte der Metaphysik*, formal intuitions require a composition or synthesis (*Zusammensetzung*) of the manifold performed by the categories of the understanding (Ak. XX, 271 and 276).

The notion of formal intuition and the centrality of inner sense established in the second edition of the *Critique* are the decisive elements for the dramatic redefi-

tion, The Hague, 26–43, 1972², 29. H. Graubner (*Form und Wesen*, Kant-Studien Ergänzungshefte 104, Bonn 1972, 147–65) gives the best and most articulate analysis of Kant's footnote: he argues that the absence of concepts in the unity of formal sensibility points to the precategorial influence of the apperception on the pure intuiting ("als Bewußtmachen des reinen Anschauens", *ibid.*, 151). For him a formal intuition differs from a formal intuition taken as object (through the category of quantity in mathematics). I base my disagreement with Graubner regarding the plausibility of his oxymoric talk of a "precategorial operation of the understanding" (*ibid.*, 156) on Kant's statement of the inseparability of – if not virtual identity between – apperception and understanding *qua* objective conditions of unity in theoretical knowledge (e.g., B 137–38; 152; 154, and 169).

[23] From A 24 and B 41 it would appear that axioms of geometry are synthetic, formal intuitions: the principles of geometry and the necessary three-dimensionality of space are themselves derived from the possibility of representing objects "alongside and outside one another" (A 23/B 38), they are not implicit in the form of intuition *per se*. However, this is not clear enough. Also, it is not clear to what extent the axioms of time differ from the axioms of space: Kant speaks as though the principles of time (such as its being one-dimensional, successive, etc.: A 31/B 47) had to belong to time as a form of intuition, not as a formal intuition. See A 31/B 46, where Kant writes that only if we presuppose time we can represent something in a relation of simultaneity or succession with something else. While I believe that Kant is right in ascribing succession to the form of intuition, and not

nition of the role of imagination. But this transition from the first to the second edition of the *Critique* does not reduce the importance of imagination, as is commonly thought (and not only by Heidegger). Although, formally, imagination is no more a third faculty beyond understanding and sensibility, substantially its new definition as spontaneous *Selbstaffektion* makes it the tacit key element at work in the constitution of formal intuitions and of inner sense. As the regulated function of figurative synthesis, an as ancillary to the understanding, imagination works progressively more openly as the concrete determination of sensibility according to the categories. However, formal intuitions arise only where pure space and time are constructed apriori as objects, and this is possible only in the category of quantity. Thus, the extension of the understanding of schematism as production of formal intuitions to the scope of other categories is more problematic than it appears, as we shall see.

But before I anticipate my conclusions, let me summarize what we have seen so far. What the issue of formal intuitions shows is that the very genesis of mathematical objects is synthetic, that the concept is constructed as the production of its corresponding intuition. The judgment *qua* discursive formulation of a cognition is synthetic in a secondary, derivative but, as we shall see, closely related sense: as an extension of the meaning of the subject to a predicate (in the case of mathematics, an intuitive property) that was not included in its representation.

According to Kant this is what Eberhard and Kästner misunderstood. Eberhard, who intended to defend Leibniz's view about space and time and treated them as images, had tried to undermine Kant's notion of mathematical syntheticity by writing that "the mathematicians themselves completed the delineation of entire sciences without saying a single word about the reality of their object"[24]. Kant replied to this point in the aforementioned passage on Apollonius. He argued one proved the objective reality of the concept of conic sections by providing the corresponding intuition in accordance with the pure concept. In a letter to Reinhold, complaining that Eberhard "seeks in vain for Kant's principle of synthetic judgment", Kant writes: "But this principle is completely unambiguously presented in the whole *Critique*, from the chapter on the schematism on (...) It is: *All synthetic judgments of theoretical knowledge are only possible through the relation of a given concept to an intuition*" (Ak. XI, 39–40, Engl. tr. Allison 164, Kant's ital.).

It is Kant himself, then, who stresses the pivotal role of this notoriously enigmatic chapter for an understanding of synthesis in general, and of mathematical synthesis

to formal intuitions, I think that this will give rise to a problem about Kant's use of the phrase 'generation of time', which I will discuss in section 4.

[24] Quoted by Kant in: *Entdeckung*, Ak. VIII, 190 (Engl. tr. Allison 110). Analogously Strawson (*The Bounds of Sense*, London 1966, Part V), in order to rescue Kant from the threat of inadequacy represented by non-Euclidean spaces, interprets geometry as a purely visual knowledge without any bearing on what he calls the "physical interpretation" of space: he suggests Kant would have done better to keep geometry and physics separate. However, as I will argue later, for Kant because space is one, it must allow for both mathematics

in particular. And what the schematism does is to exhibit apriori the pure intuitions that realize the concepts of the understanding. The peculiarity – and privilege – of mathematics and transcendental philosophy is that they can specify, besides the rule, the instance to which the rule applies: this is how their concepts relate to objects necessarily and apriori (A 135/B 174–75). Without the activity unifying the manifold into formal intuitions performed *via* schematized concepts, I cannot know an object or generate a mathematical concept. It emerges in the Analytic with all desirable clarity that concepts of magnitude become actual through the understanding's synthetic activity of construction and exhibition in an intuition, which, of course, by itself would be blind. An important passage in this regard reads as follows:

> The mere form of outer sensible intuition, space, is not yet knowledge; it supplies only the manifold of apriori intuition for a possible knowledge. To know anything in space (for instance, a line), I must *draw* it, and thus synthetically bring into being a determinate combination of the given manifold, so that the unity of this act is at the same time the unity of consciousness (as in the concept of a line); and it is through this unity of consciousness that an object (a determinate space) is first known (B 137–38).

But exactly what elements are involved in a synthesis, and how do they relate to one another? The synthesis or composition (*Verbindung*) is the only representation which I cannot possibly acquire from experience[25]: the spontaneous activity of the understanding generates it. But by itself the understanding is just the function of thought, its synthesis is merely logical (B 152). It would remain empty if it could not relate apriori to intuitions of possible objects. Thus, "synthesis in general is the mere result of the power of imagination, a blind but indispensable function of the soul, without which we should have no knowledge whatsoever, but of which we are scarcely ever conscious" (A 78/B 103)[26]. The characterization of imagination as "blind" immediately reminds us of intuition. In fact, imagination lets us represent objects in intuition, and therefore belongs to sensibility. But it is not mere receptivity. Rather than being determined by appearances, imagination determines sense spontaneously and apriori in accordance with the unity of apperception. Therefore it belongs to the understanding as well. It constitutes the understanding's effect (*Wirkung*) on sensibility and the ground of its "application to the objects of our

and its application to the science of nature, for both the production of the mathematical form and its apprehension or recognition in appearances.

25 Note that Kant presupposes that, without the unifying activity of the understanding, a mere scattered multiplicity of material elements would appear in experience. For him something like the Platonic-Aristotelian νοησις, the apprehension of a structured form in nature, could not make sense because a unity does not exist by nature (save as a living organism for teleological judgment). I will return to this later.

26 This seems to conflict with B 152, which allows at least the possibility of a purely intellectual synthesis as opposed to the figurative synthesis of the imagination. Generally Kant refers to the activity of imagination as synthesis, while he reserves the name of unity for the activity of the understanding.

possible intuition" (B 152). Through productive imagination or figurative synthesis (*synthesis speciosa*), then, the understanding determines inner sense: it unifies the manifold of intuition by bringing it under apperception, i. e., the consciousness of the *Wirkung* of imagination on inner sense. It is this conscious activity that makes the determination of sensibility in formal intuitions possible. The consciousness of our generation, in space and time, of one whole representation through the progressive combination of their parts is the condition of possibility of our representing an object as a *quantum* or extensive magnitude (A 162–63/B 203). "We cannot think a line without *drawing* it in thought, or a circle without *describing* it, or represent the three dimensions of space save by *setting* three lines at right angles to one another from the same point" (Kant's ital., B 154). But by so synthesizing, or progressively producing, a manifold in space – whereby we abstract from the spatial manifold to which we give rise and focus only on our activity – we generate the very notion of temporal succession. Since this is a delicate point (to which I will have to return in section 4), let me restate Kant's argument as follows. Time is given, as the indeterminate form of our intuition (as the possibility of a serial order): but the order of the succession (its sense) is the result of our positing a relation among representations. This relation, the order thus produced, is itself the unity of a representation of a *quantum*, the whole that combines the parts given in the succession. Inner sense *per se* does not contain any determinate (formal) intuition. It is the apperceptive activity of the understanding, its "motion" (B 154–55 and n.), that connects intuitions in time and *produces* the manifold of time as the representation of before and after. All our representations of objects in sensible intuition are subject to the order of inner sense determined by our spontaneity. *Prima facie*, Kant seems to accept the traditional figurative representation of time as a straight line which dates back to Aristotle's *Physics* (and is later found in Hegel's *Nacheinandersein*): a line is the symbolic image of time as the succession of nows divided into past and future by the continuously flowing instant of the present (A 33/B 50; B 156; B 292)[27].

The task of the productive imagination is to provide the understanding with schemata. Kant's defensive tone bears witness to the growing intricacy of the issue[28]. The function of the schematism is to bridge the gap between category and

[27] Aristotle, however, does not admit of any motion of νους in its intellection of ειδη.

[28] Imagination is, as we saw, a blind but indispensable function of which we are scarcely ever conscious; it is also called "an art concealed in the depths of the human soul, whose real modes of activity nature is hardly likely to allow us to discover, and to have open to our gaze" (A 141–42/B 180–81). Unfortunately Kant does not find it necessary at B 182 to "be further delayed by a dry and tedious analysis" of transcendental schemata. In the *Prolegomena* he refers to this chapter of the *Critique* as an "important and even indispensable, though very dry, investigation" (§ 34, Ak. IV 315–16, Engl. tr. 63). This section is, significantly enough, one that posterity has traditionally acknowledged as among the most obscure in the *Critique* (see C. La Rocca, *Schematismo e linguaggio*, in: *Strutture kantiane*, Pisa 1990, 21–73: cf. 21–3). It is one of the passages that Kant's second edition does not alter in the least, while at the same time the role and collocation of imagination between

intuition, thus allowing the subsumption of the one under the other and the application of categories to appearances. There must be *ein Drittes*, as Kant says, a source of homogeneity between the otherwise heterogeneous understanding and sensibility. Time, as the condition of all representations in our inner sense, contains the formal possibility of an apriori manifold in pure intuition, and is determined successively by the understanding according to an apriori rule. As homogeneous to the appearance, it is contained in every empirical representation of the manifold (A 138–9/B 177–8). The pure and yet sensible representation of time is the transcendental schema: the "phenomenon, or sensible concept, of an object in agreement with the category" (A 146/B 186). The transcendental schema is a product of imagination, which does not aim at a singular intuition but only at the unity in the determination of sensibility. This is what distinguishes the particular, sensible image from the schema, which can be defined as a monogram of pure apriori imagination[29]. It is

> rather the representation of a method whereby a multiplicity, for instance a thousand, may be represented in an image in conformity with a certain concept, than the image itself (...) This representation of a universal procedure of imagination in providing an image for a concept I entitle the schema of this concept (A 140/B 179–80).

A schema then is a general method of giving our categories a meaning, an intuitive representation. As to our "pure sensible" (A 141/B 180 – i. e., mathematical) concepts, it is schemata, not images of objects, that underlie them. Kant tries to convey this idea with the example of a triangle. No image would ever be adequate to the concept of a triangle in general because it would be right-angled or obtuse-angled, that is always determinate, so that it would never attain the universality of the concept. Therefore it cannot exist except in thought, where it has the property of universality. The schema of the triangle is then "a rule of synthesis of the imagination, in respect to pure figures in space" (A 141/B 180). Kant's other examples include the schema of 'dog' (as a rule to delineate the figure of a four-footed animal), and of number ("the successive addition of homogeneous units ... the unity of the synthesis of the manifold of a homogeneous intuition in general, a unity due to my generating time itself in the apprehension of the intuition", A 143/B 182).

Let me emphasize two points here. Kant sometimes calls a schema a rule, sometimes a procedure or method. Since concepts are also rules, how do they differ from

understanding and sensibility undergo a radical change. As I try to show in my essay *Kant's Productive Imagination and its Alleged Antecedents*, in: The Graduate Faculty Philosophy Journal, 18, 1, 1995, pp. 1–27, Kant's attitude towards imagination can be explained by what he felt was its novelty, a novelty of which he was aware and which he stresses with a pride that contrasts with his usual moderate and unpretentious tone (see A 120 n.). What is definitely new about Kant's productive imagination is the combination of two elements: the fact that its activity is ruled apriori by the understanding, and that it exhibits figuratively the understanding's constructions and representations in intuition.

29 Lachterman (*Vico, Doria e la geometria sintetica*, in: Bollettino del centro di studi vichiani, X, 1980, 10–35, 18–9, n.) finds in Vico a previous occurrence of "monogram" in an analogous sense.

schemata[30]? Does not the distinction between concepts and schemata appear blurred again? A provisional answer on Kant's behalf, which will prove less than satisfactory, is the following: rules are logical functions of unity, while schemata provide them with an intuitive meaning. Schematism allows us to apply concepts to intuitions, i. e., to give them a sensible figurative representation, *and* to subsume intuitions under concepts, i. e., to find in an intuition the instance of a concept or to interpret it as a sensibilized rule. Thus, whereas concepts as such are discursive functions that unify *linguistic* marks (*Merkmale*), transcendental schemata are the synthetic procedures to *subsume* possible objects of intuition under the condition of time, mathematical schemata are rules to *construct intuitive* objects, and empirical schemata are ways to describe spatial figures as images of natural objects (or artefacts: e. g., a house) in general. But then we are forced to acknowledge that there is something essentially ambiguous in the examples chosen by Kant. Therefore, they all deserve more attention than Kant scholars are usually willing to pay to them[31]. To say the least, putting the schema of a dog and that of a triangle on the same level is misleading because, although both need to be imagined according to a concept, there is nothing common to their exhibitions *in concreto*. As to the pure concepts of the understanding, they cannot be brought into any image whatsoever.

Kant is somewhat clearer about this in the Doctrine of Method. In the Discipline Kant discusses the example of the triangle, and so provides a valuable illustration of the difference in the method a mathematician and a philosopher follow with regard to this figure. "Suppose a philosopher be given the concept of a triangle and

[30] M. S. Gram (*Ontology and the Apriori*, Evanston 1968, 91–129) points out the circularity of the notion of schemata as rules: this would presuppose that the intuitive manifold we have is already constructed according to a rule, i. e., it presupposes that we already know what it would contain as an image of a pure concept. But because rules as such cannot be intuited, Gram interprets schemata as intuitions. Thus, he sees a logical relation between schema and image, between a semantic particular and its instance, thereby disregarding their transcendental relation and so the basic difference between pure and empirical intuition. Furthermore, his thesis of the centrality of the issue of verifiability in the synthetic judgments of metaphysics seems to me to miss the point. No verification is at stake in metaphysics for Kant, and rightly so: either metaphysics is reduced to philosophy of nature, which Kant certainly could not accept, or its concepts as discursive functions can only be explained, not verified.

[31] Among the interpreters sensitive to this issue is La Rocca, *Schematismo e linguaggio, op. cit.*, who however does not deal specifically with mathematical schematism. In his classic essay on schematism, Kaulbach (*Schema, Bild und Modell nach den Voraussetzungen des kantischen Denkens*, in: Studium Generale, XVIII, 1965, 464–79: 470–71) too finds the example of the dog out of place, but for the reason that the dog is a lived body (a *Leib*, not a *Körper*), thus the expression of something internal (the first emergence of subjectivity), and not merely an object. I will show later why I think that Kaulbach's reading would better fit Hegel's self-objectifying reason than Kant's schematism. Here it is sufficient to say that his reading misses the point because he uniformly interprets schemata as something constructible that accounts for the transition from internal to external (from subject to substance, one could say), regardless of how the transition takes places – how objectivity arises in the different domains.

he be left to find out, in his own way, what relation the sum of its angles bears to its right angle" (A 716/B 744). However long he meditates, he will never produce anything new, because all he will do is clarify the concepts given, without arriving at anything which is not already contained in the concepts. For his part, the geometrician

> at once begins by constructing a triangle. Since he *knows* that the sum of two right angles is exactly equal to the sum of all the adjacent angles which can be constructed from a single point on a straight line, he *prolongs* one side and *obtains* two adjacent angles (...) then *divides* the external angle by drawing a line parallel to the opposite side of the triangle, and *observes* that he has thus *obtained* an external adjacent angle (...). In this fashion, through a chain of inferences *guided throughout by intuition*, he arrives at a fully evident and universally valid solution of the problem (*ibid.*).

I italicized Kant's verbs in order to highlight a problem that still must be addressed. It seems from this passage that geometrical construction proceedes from previous knowledge in order to produce a final vision of a property of the figure. This would seem to entail that geometrical concepts can neither be *sic et simpliciter* equated with nor reduced to construction, as a reading of the chapter on schematism could suggest. Drawing any such conclusion, though, would be hasty. Once again it is necessary to keep in mind a difference in the meaning of synthesis. Mathematical concepts are in their own right procedures to operate on limitations of space and time, or to exhibit in formal intuitions (to construct) objects by providing the corresponding intuition. The definitions of their objects thus guarantee their actuality. Their concepts have no meaning or existence prior to the objective reality bestowed upon them by our arbitrary construction. In the case of mathematics then there seems to be no difference between a discursive rule and a schema – recall that we cannot think a rule (triangle, locus) without constructing it. But are the properties we demonstrate or learn of triangles properties of the objects or of our understanding? Kant gives a straightforward answer to this question in the *Prolegomena*. This answer is worth quoting in full.

> If we consider the properties of a circle by which this figure combines in itself so many arbitrary determinations of space in a universal rule, we cannot avoid attributing a nature to this geometrical thing. Two straight lines, for example, which intersect each other and the circle, howsoever they may be drawn, are always divided so that the rectangle constructed with the segments of the one is equal to that constructed with the segments of the other. The question now is: Does this law lie in the circle or in the understanding? (...) When we follow the proofs of this law, we soon perceive that it can only be derived from *the conditions on which the understanding founds the construction of this figure*, namely, the concept of the equality of the radii (...) Space is something so uniform and as to all its properties so indeterminate (...) that we should not seek a store of laws in it. Whereas *that which determines space to assume the form of a circle, or the figures of a cone and a sphere, is the understanding*, so far as it contains the ground of the unity of their construction (§ 38, AA 321–22, Engl. 67–8, my ital.; see also *KU* § 62).

The difference in the meaning of synthesis amounts then to this: because mathematical objects have a synthetic genesis and can be constructed in intuition, so that our real definition originates them and gives them objective validity, then mathematics can have its synthetic proof-structure. Synthetic demonstration is the second, derivative sense of synthesis: the mathematical object, thanks simply to the rule for its construction, contains many properties that, from a consideration of the figure produced according to a universal rule, allow for a synthetic increase of theoretical knowledge. The understanding can *learn* what it has put into the figure, it can *be taught* properties of the circle it had not thought when it constructed it according to the rule of the equality of the radii. In this sense the subject of a judgment is synthetically supplemented with a new predicate, the demonstrated property (and this sense is implied in the famous passage from the Introduction about the distinction between synthetic and analytic judgments: A 7/B 10). But this also shows that intuition *here* has to stand for both the determination of sensibility, say into a triangular space, *and* the ostensive evidence or visible verifiability of the properties we see when we prove a theorem of the space we have described. This is particularly clear in the passage quoted above about the "chain of inferences guided throughout by intuition". The ampliative character of mathematics, namely its synthetic apriori judgments, is grounded on the synthetic, intuitive origin of its objects. It is synthesis in the first sense that makes possible the transition from the representational system of discursive *Charakterismen* or linguistic signs, to use the terminology of the *Critique of Judgment* (§ 59), to the system of ostensive hypotyposes. Because schematized concepts in mathematics provide the corresponding objects, so that their actuality is thereby completed, we can have an apriori growth in knowledge without having to rely on experience. Here the influence of the understanding on sensibility makes it possible to dispense with the necessity for appearances to affect it. This *Selbstaffektion* is sufficient to determine purely and apriori its objects. This twofold synthesis, this twofold apriori intuition, is something unique to mathematics. Neither transcendental philosophy nor any science can have a demonstrative development *and* an ostensive production of their objects in intuition. But this *Selbstaffektion* influences sensibility, and it means to construct or produce – not create – the objects in space and time; it does not stand for reason's manifestation of itself, for a world of forms through whose constitution the understanding reflects its imaginative power.

§ 3 *Philosophy, Empirical Discourse and Mathematics. What do We Produce and What do We Find?*

It is indeed tempting to find in mathematics the paradigm which philosophy would like to emulate. The centrality of mathematics in the *Critique* gives us reasons to pursue this point. At crucial passages mathematics is mentioned as the clear example of a solution that philosophy cannot otherwise find. If what the *Critique* shows is the possibility of synthetic apriori judgments, it is mathematics that takes advantage of this ampliative principle with greatest confidence and success, a principle that philosophy has not yet made its guide but needs to follow. Thereby, however, the suspicion arises that the *Critique* is not just an explanation of concepts, as philosophy is described, but involves at least the principle of their application to nature and the effort not to rest within the limits of concepts, but to go beyond them (B XV–XVIII; B 18). The schematism would be the key point allowing for philosophy's synthetic judgments, insofar as it shows the sensible meaning of our categories and their use under the condition of time. It is after all one and the same influence of the understanding on sensibility that accounts for the representation of a house and of a circle.

But the very difference in nature between the schematism at work in mathematics and in philosophy pushes us in a different direction. Now we need to stress this point through an analysis of the Doctrine of Method and the Analytic of Principles, in order to explain why philosophy cannot have such a twofold synthesis.

The purpose of the Discipline of Pure Reason in the *Methodenlehre* is to illustrate "the twofold employment of reason", in accordance with concepts and through construction of concepts (A 723/B 751). It purports to distinguish between mathematical and philosophical method, and, if I can express myself thus, to give philosophy its *memento mori*, an admonition not to rise above finitude and pretend to mimic the dogmatism of mathematics with its "splendid example of the successful extension of pure reason without the help of experience" (A 712/B 740). While mathematical knowledge considers the universal in the singular, and yet apriori and by way of reason, philosophy considers the particular in the universal. Philosophy cannot start from definitions. It must rest content to expound given concepts, an *in abstracto* enterprise for which the German language has *Erklärung* as a family name, as Kant reminds us. Definitions in philosophy are always incomplete, and the best they can do is to approximate the given concept by an enumeration of its most essential marks; so they should rather come at the end of the inquiry, as the crowning of our efforts (A 731/B 759).

Let us go back to the examples of schemata. The schema of a triangle, contrary to the empirical one of a dog or the pure transcendental one of cause, yields a sensible *singularity* as the apriori exhibition of a concept, which is capable nonetheless of being regarded as universally valid for all possible intuitions of the same concept. When, thanks to my productive imagination, I construct a triangle, I pay attention solely to the operation of the construction of a concept according to universal rules. This is the privilege of mathematics: its concepts are universally determinate, they have a concrete apriori *Darstellung* in space and time, as formal intuitions synthesizing the temporal manifold in number and the spatial manifold in figures. The schema of a dog cannot be analogous to that of a triangle for the reason that an intuition of a dog which exhibits its concept *in concreto* has to be provided by experience. A consequence of the greatest importance, which Kant never states, is that in mathematics schematism has no residual matter. There is no radical split between aposteriori content and apriori form, between *spectare materiale et formale*. The matter is the concrete image of the triangle drawn on this piece of paper, not a jumping dog in flesh and bones – something I have produced, and not something I find. In other words, if in philosophical and empirical schemata there is a three-term relationship among *Schema*, *Bild* and a given phenomenon, in mathematical schematism the relationship contains only two terms, schema and image, and the intuition is not of an appearance but of the image itself as representative of the method of its construction. This is what justifies its apriori character. Of course we cannot say that we intuit a method. We always see only images. But Kant's emphasis on the mediating activity of imagination means that the sensible image points to something else, and has to be – is – regarded as the *image of* something (in the case of the triangle, of its apriori universal, in the case of the dog, of its being a member of a class). (As we shall see at the end of this section, there are problems in this doctrine). In mathematical construction we cannot say that we have generated autonomous things in themselves or produced an archetype of the form of appearances. What we can and must say is that only mathematics can determine its objects apriori, and that any talk of construction about what is not the quantitative form of appearances, figures and numbers, is misleading[32].

This indicates, it seems to me, the peculiar role of the category of quantity (and to some extent of quality as well) with respect to exhibition, and consequently to the difference among schemata. There is a different relation between concept, schema and intuition in the case of mathematical schemata and, say, the case of the schema of a dog. Further, the meaning of existence changes. In both cases what we can determine apriori is the form of the appearance. To be sure, the dog has a magnitude which is apprehended through progressive combination of parts. But in the case of the dog there is a separation between its quantitative form and its matter which does not hold for mathematical objects. These are completed apriori through the quantitative determinations in space and time in a way in which other objects cannot be.

32 Kaulbach thinks that reason behaves with respect to space and time as the writer does with respect to paper: we *produce signs*, and thereby give objective meaning to concepts – as the example of drawing a line supposedly shows (*Schema, Bild und Modell ..., op. cit.*, 464 f.). As I tried to point out, the example of the line cannot be generalized so as to support his conclusion that being is equal to being constructed (ibid., 468). This is what I referred to in the first page of this paper as the necessity of approaching the schematism with a careful attention to the differences between mathematics, philosophy and empirical discourse; and this is what marks the difference between Kant and the Hegelian spirit of Kaulbauch's interpretation. He takes the schematism as the self-externalization of reason in objectivity, such that its activity extends to realms where it does not belong – i. e., even the idea of right would be realized in institutions thanks to the action of schematism in the metaphysics of morals. For Kant construction concerns only quantity, and is only theoretical, not practical; it is the apriori exhibition of an intuition corresponding to the concept. Only mathematics can construct its objects (A 713–716/B 741–744). This is not stressed enough in the literature. For example, even M. J. Young (*Construction, Schematism and Imagination*, in: Topoi 3, 1984, 123–31: 129) extends construction to the "construction of a maple tree". A maple tree is apprehended as extensive magnitude, i. e., through progressive combination of its spatial parts; it is not constructed.

The only time Kant says something illuminating about this difference he writes explicitly that only the category of quantity can be constructed, but then mentions that while we can form in intuition "the shape of a cone" apriori, its colour must be given in experience (A 714–5/B 742–3). It would seem that the difference lies in the totally secondary and theoretically insignificant matter of the aposteriori perception of the cone, as opposed to the living dog, whose lived presence cannot be given save in experience. But even the livelihood of the dog is irrelevant to our apriori knowledge of it (provided this phrase makes sense). The difference lies then in the apriori actuality of the object. Mathematical objects have a sort of definite presence or ideal existence that no philosophical or empirical concepts can attain[33]. But this existence differs from the ordinary notion of existence that can be given only in experience. Neither a dog nor a transcendental concept can be arbitrarily constructed and thus exhibited completely in intuition. Here the intuition of the form, produced, not found by us, is sufficient to exhaustively determine the object, and existence is in this case what gives us the relation – or figurative application – of the formal intuition to a given appearance.

This shows that quantity has a closer relation to space and time than the other categories. But we should not take this to entail that quantity is not a pure concept but rather ought to belong to the Aesthetic[34]. What distinguishes the concepts constructed according to the category of quantity is that we can anticipate the *content* of objects in intuition, their spatio-temporal form, while the *matter* of their sensation, their existence in reality or what we perceive in appearances, is only given aposteriori. The categories of quantity, and of quality, refer to what can be known apriori in intuition, not to the existence of their objects, as dynamical categories do (A 719/B 747; B 110; Kr. d. U. § 62 n.).

A passage in the *Metaphysische Anfangsgründe* (Ak. IV, 469) says that existence, in the sense of empirical *Dasein*, can never "be exhibited in intuition". This entails a further difference: while in the laws of natural science the understanding applies its principles to particular instances, which however have to be furnished by experi-

[33] De Vleeschauwer, just to take an authoritative representative of a widespread conception among Kant scholars, misses the peculiar actuality germane to mathematics when he writes that the triangle has to be "réalisé dans une chose triangulaire, dont la matière nous est donnée par la perception", in order to be objectively known; otherwise (i. e., without the empirical existence of the mathematical object), "nous n'avons que des schémas d'objets dans l'imagination productive" (*La déduction transcendentale ..., op. cit.*, vol. III, 170). On the contrary Kant, as we saw, could not be more explicit than when he writes, distinguishing between nominal and real definitions, that real definitions do not only clarify a concept, but also provide "its *objective reality.* Mathematical definitions, exhibiting the object in intuition according to its concept, are of this kind" (Kant's italics, A 242 n./B 300 n.). But De Vleeschauwer is right to say that "Kant revient constamment sur cette idée" (*ibid.*); Kant has a somewhat complicated position which, if not carefully developed, leads to this kind of misinterpretation, as we shall see in a moment.

[34] This is how Vanni Rovighi (*Introduzione ..., op. cit.*, 159) interprets the Axioms of Intuition.

ence, in mathematics the individual instance is not aposteriori in the same sense. The singularity of the constructed intuition is the sensible and yet universal and produced apriori instance. Here, paradoxically, it is the arbitrariness of the construction that constitutes its universality.

This issue becomes immediately crucial when we turn to a consideration of pure and empirical construction and to the application of mathematics to nature. At A 720/B 748 Kant reminds the reader of his basic notion that the only intuition given apriori is the form of appearances, space and time, which by itself cannot yield "an apriori intuition of the real object", since this must be empirical. From what I have said so far it would seem, on the contrary, that mathematics does yield an apriori intuition of its objects, as Kant literally says three pages later: objects as *quanta* are created, determined apriori in intuition through a homogeneous synthesis. This alternation runs throughout the *Critique*, as we shall see.

Emphasis on the uniqueness of pure construction and on the apriori determination of sensibility should not mislead us into claiming its sharp separation from the world of appearances. I mean that there is an important problem regarding the application of mathematics to appearances and the role of construction in the constitution of artefacts. We construct an ellipse in intuition, but we also find it in a planetary orbit. We draw a certain rectangular shape, and then transpose it in the architectural project for a house. Or, as in the example from the *Critique of Judgment* (§ 62), the order of my design and of the garden to which it has been applied, though different, must be related.

It would seem at first that Kant's distinction between productive and reproductive imagination could help us define the roles of the understanding in producing and finding forms in intuition. Kant writes at B 152 that it is the business of productive imagination to perform schematic construction, whereas reproductive imagination is "entirely subject to empirical laws of association", and therefore contributes nothing to the explanation of the possibility of apriori knowledge. Thus, it "falls within the domain, not of transcendental philosophy, but of psychology". The point is made with even greater clarity in the *Anthropologie in pragmatischer Hinsicht* (§ 28; see also the *Metaphysik L*): the productive imagination is the *exhibitio originaria* of pure intuitions of space and time, while reproductive imagination arises from experience and presupposes a connection of repeated sensations with the corresponding objects, which produces a uniform experience[35]. It would seem then possible to ascribe to productive imagination the function of producing intuitions apriori, while all activities based on association, as language for example is according to Kant, should be reduced to the empirical formation of discursive signs. If so, contemporary attempts at connecting schematism or productive imagination to language (Aschenberg, Hogrebe, La Rocca, etc.) would be substantially misguided.

[35] Note that here productive imagination is not creative absolutely, it needs sooner or later some sensible representation to substantiate its images.

It is soon apparent, however, that nothing could be more misleading than to take our bearings by this distinction. The separation between productive and reproductive functions is very shaky, as for example an examination of the difficulties in keeping separate *ab-, nach-, vorbildende Kraft* in the lectures known as *Metaphysik L* would show[36]. But above all an absolute production would somehow have to come to terms with our recognition in experience of the very forms we produce; otherwise we would actually be in what I referred to earlier as a created world irremediably lost to experience. Kant is, or understands himself to be, one decisive step beyond the Cartesian project to simply substitute his *ordo et mensura* for a given nature, where the motion of things is resolved into the figurative reconstruction of the steps of the mind's motion. Kant's appearances are not just the stage of the understanding's self-realization as master of nature *via* geometricized physics, but the spatio-temporal world given to us and to which our productive imagination must somehow be essentially, not just externally, related. Thus, we should not forget that a more fundamental claim, compared to which all these distinctions among eventually overlapping functions of imagination become virtually worthless or, at least, are to be subordinated, is contained in different passages in the *Critique*. At A 224/B 271 we read that the "formative (*bildende*; also: *figurative*) synthesis through which we construct a triangle in imagination is *precisely the same* as that which we exercise in the apprehension of an appearance" (my ital.). From this we must draw the important conclusion that imagination is involved in the synthetic construction as well as in the synthesis of apprehension, namely the "combination of the manifold in an intuition, whereby perception, that is, empirical consciousness of the intuition (as appearance), is possible" (B 160). As is well-known, in the first edition, the function of the imagination was more articulated and ample. In B the threefold synthesis of imagination (apprehension, reproduction, recognition) is revised and substantially altered. But these passages show that Kant does not drop the idea of imagination as an active ingredient in perception. In the words of A 120, "since imagination has to bring the manifold of intuition into the form of an image, it must previously have taken the impressions up into its activity, that is, have apprehended them".

This is the only way to explain the opening passage of the chapter on schematism (and, incidentally, to see why Vaihinger's emendation, accepted by Kemp Smith, misses the point): "the empirical concept of a *plate* is homogeneous with the pure geometrical concept of a *circle*. The roundness which is thought in the former can be intuited in the latter" (A 137/B 176). If we recall that a schema has to allow both for the application of a concept to an intuition *and* the *subsumption* of an intuition under a concept, we have to interpret the figurative synthesis of the circle

[36] See now, besides H. Mörchen (*Die Einbildungskraft bei Kant*, in: Jahrbuch für Philosophie und phänomenologische Forschung, hrsg. v. E. Husserl, Bd. 11, 1930, 311–495), R. A. Makkreel (*Imagination and Interpretation in Kant*, Chicago and London 1991, 14–9) for a commentary on this point.

as active both in its apprehension in space and in its production in intuition. Subsumption is the understanding's activity of finding in intuition the concept according to which we constructed it (with artefacts: one finds the triangle in the side of a pyramid), or according to which we formally represent the given object (with nature: in the sun the circle)[37].

This is crucial for the success of Kant's explanation of the possibility of our concepts relating apriori to objects. As I wrote earlier, there is an alternation in Kant's tone when he speaks of the sufficiency of mathematics for knowing the form of appearances. It should now become more evident why this is a central point. In both mathematics and ποιησις do our productions acquire an independent actuality before us. But mathematics cannot be equated to ποιησις. In the world of ποιησις we shape a matter and make it assume the form we want. Although Kant does not say much about technical artefacts, this much is clear: here the form designed by the understanding is in principle undefinable and indeterminate. The definition of a ship's clock does not "assure me of the existence or of the possibility of its object (...), and my explanation may be better described as a declaration of my project than as a definition of an object" (A 729/B 757). In mathematics we have real definitions which complete the objective actuality of the form of appearances we might or might not encounter, but which are essentially related to, and indeed defined by, the possibility of being considered as the forms of actual appearances. So we read that through the determination of pure intuition we have

> apriori knowledge of objects, as in mathematics, but only in regard to their form, as appearances; whether there can be things which must be intuited in this form, is still left undecided. Mathematical concepts are not, therefore, by themselves *knowledge*, except on the supposition that there are things which allow of being presented to us only in accordance with the form of that pure sensible intuition (B 147, my ital.).

In this passage knowledge *must* mean knowledge of forms *as forms of* existing appearances; and this is understandable in the context of a discussion about the application of categories to objects of experience. But Kant cannot create a gap between mathematical and experiential knowledge. Although he does not have an articulate 'ontological' theory about the relation between the form and matter of appearances, he cannot allow a discrepancy between forms as such and as forms of the appearances (the traditional Platonic-Aristotelian problem). However, a striking passage in the chapter on phenomena and noumena says that, since all apriori concepts relate to empirical intuitions, "apart from this relation they have no objective validity, and in respect of their representations are a mere play of imagination or of understanding" (A 239/B 298). Kant's example is of mathematical concepts, that would mean nothing if they were not made sensible. The difficulty with this passage is that Kant seems to contemplate at least the possibility that mathematical

[37] La Rocca explains the subsuming as the judgment's interpretative apprehension of a *casus datae legis* (see *Schematismo e linguaggio, op. cit.*, 32–3 and 47–51); cf. also Palumbo (*Immaginazione e matematica in Kant*, Bari 1984, 18, 27, 30, 38 and 72).

concepts do *not* relate apriori to possible intuitions. However it may be, I think that an interpretation of the sort given by De Vleeschauwer is one-sided. Kant does mean here an empirical intuition, not a pure one. But the empirical and the pure are not related here as they would be in experience, because they are *both* generated apriori. It would be catastrophic for Kant to abandon this point. In the passage we are reading he continues:

> The mathematician meets this demand by the construction of a figure, which, although produced apriori, is an appearance present to the senses. In the same science the concept of magnitude seeks its support and sensible meaning in number, and this in turn in the fingers, in the beads of the abacus, or in strokes and points which can be placed before the eyes. The concept itself is always apriori in origin, and so likewise are the synthetic principles or formulas derived from such concepts; but their employment (*Gebrauch*) and their relation to their professed objects can in the end be sought nowhere but in experience, of whose possibility they contain the formal conditions.
>
> That this is also the case with all categories and the principles derived from them, appears from the following consideration. We cannot define any one of them in any real fashion (...) without at once descending to the conditions of sensibility (A 240/B 299).

This passage is of considerable importance for more than one reason. For one thing, the meaning of "employment" here helps us explain a passage we could not otherwise understand (A 160/B 199), in which Kant writes that mathematics derives its principles from pure intuition, but that only the understanding can apply them to experience. If application (*Anwendung*) here stands for the exhibition in sensibility of the conditions under which our concepts relate to objects of experience, and here understanding has the broad sense of synthesis, i. e., inclusive of the figurative synthesis of schematism, then it is no longer so obscure that the understanding applies, namely relates, necessarily its forms to the forms of appearances. This is the meaning of real definitions, as we saw in the previous section. Without its schematization, mathematics would be a logical science, just as the very idea of a two-angled figure formed by two lines in a non-Euclidean locus (A 221/B 268)[38] is only a logical possibility, deprived of significance for our synthetic geometry of a real space – the space to which Kant has to be committed. Second, this very "application" is said to be the same for the pure concepts of the understanding. We will return to that in a moment, in the context of an account of the different syntheses in philosophy and mathematics. A third reason is that pure and empirical intuition are explicitly related by Kant. Their relation, unlike the one that philosophy articulates, is a relation of immediacy. I mean that the sensible image of the triangle is, as such, disregarded. In geometry we take it for granted that we must regard it as an instance of the concept. The image gives way necessarily to the schema. As universally determinate, it is immediately *subsumed* under the universal procedures

[38] In the Amphiboly, however (A 291/B 348), Kant denies even the logical consistency of such a figure, not just the impossibility of constructing it in intuition. On this flat contradiction see G. Martin, *Das geradlinige Zweieck. Ein offener Widerspruch in der Kritik der reinen Vernunft*, in: *Tradition und Kritik*. Festschrift für R. Zocher, Stuttgart 1967, 229–35.

according to which we constructed it. At A 713/B 742 Kant emphasizes that the constructions "by imagination alone, in pure intuition, or in accordance therewith also on paper, in empirical intuition" both occur completely apriori.

The reason for the privilege granted mathematics is that its objects are constructed in accordance with the form of sensibility. This means that we can apply mathematics to appearances insofar as appearances and mathematics both refer to the same space and time as the form of all possible intuitions. Because space is a pure intuition to which all appearances are subject, we can have a geometry of real space as the form of apprehension of external appearances. Only if mathematics bases its procedure on the form of intuition which "has its seat in the subject only" (B 41), only if the formal intuitions which constitute its objects are limitations of the one, pure form of intuition, can its synthetic knowledge be knowledge of the same space and time in which we encounter objects in our experience.

As Kant writes in the Axioms of Intuition, "empirical intuition is possible only by means of the pure intuition of space and of time. What geometry asserts of pure intuition is therefore undeniably valid of empirical intuition" (A 165/B 206). If a triangle bore no relation to the form of our intuition, to the space in which we intuit or construct it, we could never conclude anything about the properties of the triangle. We would have a concept of three lines, from which we could never infer anything new, e. g., the figure, as "something which must necessarily be met with in the object" (A 48/B 65–66). If a triangle were a thing in itself, if it were given outside a necessary relation to the form of my intuition, it "would be given antecedently to your knowledge, and not by means of it" (*ibid.*).

This has two very important consequences. If it is true that "properly speaking there is as much science in the doctrine of nature as mathematics is contained in it" (*M. A. d. N.*, 470), natural science presupposes mathematical apriori construction in intuition. The concept of a combined motion in phoronomy, for example, is equated with its apriori construction in intuition (*ivi*, 486). A science of nature, in direct contrast to Newton's thesis of their identity[39], presupposes a metaphysics of nature capable of exhibiting its concepts in intuition (*ivi*, 469–70)[40].

The second consequence is that the mathematical principles of the mathematization of nature themselves are justified transcendentally by a non-mathematical critique that shows their possibility. The possibility of mathematics, physics and all the theoretical sciences of nature is not shown in and by those sciences, but only in the critique of reason. However tempting it may be, then, to stress the centrality

[39] See K. Cramer, *Nicht-reine synthetische Urteile apriori*, Heidelberg 1985, 119 n., and G. Büchel, *Geometrie und Philosophie*, Berlin–New York 1987.

[40] Again this does not mean that mathematics has no objective meaning beyond its application to nature, as suggested by Martin (*Immanuel Kant ..., op. cit.*, 40). As Kant writes in the *Critique of Judgment* (§ 62), the ancients undertook to inspect the properties of geometrical figures completely apriori, without worrying about the possible application of ellipses or parabolas. They unknowingly worked for the benefit of posterity: mathematics would not contain synthetic knowledge if its advances had to rely on an application to experience.

of mathematics in the *Critique*, we must not forget that Kant writes a "propaedeutic" to the system of pure reason (A 11/B 25) and to metaphysics (A 841/B 869; the sketch of the Architectonic is taken up again in *M. A. d. N.*, 468–70). It is transcendental philosophy, then, that shows why and how we have synthetic sciences by showing the possibility – the transcendental origin – of our scientific cognitions in connection with the forms of knowledge we have, intuition and concept. Incidentally, this also means that the outline of space and time as forms of our intuition given in the Aesthetic is independent of – or at least it does not take its bearings from – any consideration of geometry or mechanics[41].

The question of the nature of philosophical synthesis emerges now. If philosophy can only expound concepts, if it cannot go beyond given concepts to pure or empirical intuitions, it would seem that it can only contain analytic propositions. We saw that in the first instance the question of synthesis is that of the transcendental origin of our knowledge. The ampliative and, even more decidedly, the logical senses of synthesis are derivative. Even in transcendental philosophy synthesis refers first and foremost to the intuitive origin of our knowledge[42]. But in this case the synthesis is only indirect, and unlike mathematical synthesis it does not determine exhaustively any actual object, indeed, it cannot even give us any object in intuition. What it does give us is the principle on which all possible intuitions rest, even those of mathematical objects (A 149/B 188–89). An apriori concept can either contain in itself a pure intuition, and can therefore be constructed, or contain only the synthesis of possible intuitions. Thus, for instance, if I have the concept of causality or of substance, I only have the principle of the synthesis of possible empirical intuitions under the condition of time. There is no way that I can pass to the intuition that would represent *in concreto* the concept of a cause. Therefore philosophy is mere discursive knowledge from concepts, it can never construct or exhibit its concepts in apriori intuitions. But if it must not pretend to the apriori syntheticity of mathematics, what it can achieve is an explanation, namely in the schematism and in the Analytic of Principles[43], of how its pure concepts can refer apriori to possible intuitions. Showing that our apriori knowledge is possible in experience, i. e., that it is knowledge of possible appearances, bestows upon it objective reality (A 156/B 195). However, unlike mathematical definitions, as I said, what here acquires objective reality is the synthesis of possible objects, never the intuition of actual ones

[41] In the last page of his book (*The Bounds of Sense, op. cit.*, 292), Strawson asserts that Kant's failure to distinguish between his exposition of geometry and its physical interpretation makes the thesis of the transcendental subjectivity of space inadequate. I do not see a connection between the two issues. See R. P. Horstmann, *Space as Intuition and Geometry*, in: Ratio XVIII, 1, 1976, 17–30, for a criticism of Russell and Strawson on this point.

[42] Kant does say, however, that "metaphysics consists, at least *in intention*, entirely of apriori synthetic propositions" (B 18), such as "the world must have a first beginning". But it seems to me that this must fall under the blows of the Dialectic.

[43] I think Cohen is perfectly right to find in this key part the touchstone of the positive teaching of the *Critique* (*Kants Theorie der Erfahrung*, 2. Aufl., Berlin 1885, 261 ff.).

(A 720–22/B 748–50). Again in contrast with mathematics, philosophy does not have axioms. The Axioms of Intuition in the Analytic of Principles are the principle of the possibility of axioms in general, and of mathematical axioms in particular (A 734/B 762). Philosophy is also not an apodictic science. Only directly intuitive proofs, not the discursive arguments of philosophy, are immediately certain.

Schematic synthesis in philosophy, then, only gives us the rule for possible intuitions in space and time. As an immediate consequence, it would seem that transcendental philosophy has nothing to say about empirical concepts. Its concern is solely with the apriori conditions of experience. Indeed Kant writes that the pure concepts of the understanding are the sole object of inquiry in the Analytic (A 64/B 89). This, however, besides making the example of the dog all the more troubling, points to the crucial problem of the relation between pure concepts, pure sensible – i. e., mathematical – concepts, and empirical concepts. It must be the case, although Kant does not deal with this at length, that the pure concepts of the understanding are the condition of possibility of the other two types of concepts. But how exactly are things supposed to go here?

When Kant says that his concern is with the origin of our apriori knowledge of the form of appearances, he assumes the traditional distinction between primary and secondary qualities[44]. While the form is what we can know objectively, properties such as colours or taste are "only changes in the subject, changes which may, indeed, be different for different men" (B 45; see also *Fortschritte der Metaphysik*, Ak. XX, 268–69). The distinction goes back, as is well-known, to the Greek atomists and to Plato's *Theaetetus*, and is developed by Aristotle under the heading of the κοινα as a specific object of sensation. But in modernity it is transformed into the Galilean grounding principle for the mathematization of nature, and it is in this form that Kant takes it up. The decisive step taken by Kant here is that of separating this notion from any relation to a given eidetic structure or to inner representation. Concepts are for Kant, unlike their predecessors in Descartes, who still depicted them as mental figures, the purely discursive product of the unifying activity of the understanding. Thinking is essentially judging. That is, concepts do not represent the ideal structure of the world or intrinsically refer to objects, but are the elements of judgment, of an objective relation among given cognitions brought under the unity of apperception (B 141). This logical form in its turn depends on the fact that the understanding is not intuitive, but yields knowledge only by means of concepts: "whereas all intuitions, as sensible, rest on affections, concepts rest on functions. By function I mean the unity of bringing various representations under one common representation" (A 68/B 93). While sensible intuitions are based on the receptivity of impressions, concepts are spontaneous, and the only use the understanding can make of such concepts is

[44] See M. Baum, *Erkennen und Machen in der 'Kritik der reinen Vernunft'*, in: *Probleme der 'Kritik der reinen Vernunft'*, Kant-Tagung Marburg 1981, hrsg. v. B. Tuschling, Berlin–New York 1984, 161–77. For the relevance of this distinction for schematism, see my *Kant's Productive Imagination and Its Alleged Antecedents, op. cit.*

to judge by means of them. Since no representation, save when it is an intuition, is in immediate relation to an object, no concept is ever related to an object immediately, but to some other representation of it, be that other representation an intuition, or itself a concept. Judgment is therefore the mediate knowledge of an object, that is, the representation of a representation of it (A 68/B 93).

The production of a formal unity is then the exclusive function of the understanding. Kant's version of the question of primary and secondary qualities, therefore, is his distinction of apriori form and aposteriori matter, which immediately makes it necessary to restate in different terms the *crux* of the teaching of the post-Aristotelian tradition on the origin and nature of the universal.

We have seen that in mathematics what we produce is the form and content of the object. Experience in this case shows that they belong to an appearance, that they are the *form of* an appearance. In philosophy we do not produce any intuition, but merely show the possibility of our pure concepts referring to objects under the condition of sensibility. Now, what do we make in the case of empirical concepts? We spontaneously produce the unity of the concept, the discursive form under which we relate impressions. Empirical concepts "cannot be defined at all" (A 727/ B 755). What we produce, according to the *Logik* (§ 6), is the unity of their characteristics through comparison, reflection and abstraction. Here the form is the logical form; but matter, what we acquire from experience, is, paradoxically enough, the various formal elements or characteristics that experience offers to the understanding for its unification. The understanding brings under the unity of apperception the sensible marks of the object. In Kant's words, the activity of the understanding is an act of unification of what affects the receptivity of the senses. However, the problem with this separation between activity and receptivity is: how can the apriori formally constitutive activity take place if I have not already unified the manifold of empirical intuition? A spontaneous synthetic activity – but in this case an unconscious and forever subjective, associative *synthesis speciosa*, not as in the construction of formal intuitions – of subsuming the empirical intuition under an image would have to be presupposed here as having already taken place[45]. And if, generally speaking, the application of the understanding to an intuition and the subsumption of the intuition under a concept are the activities of the same imagination, what we should conclude that we produce in the case of empirical concepts is the figurative synthesis, the apprehension of the unity of the matter – the formal elements or characteristics – in an image. Here, however, productive imagination does not produce apriori the schema for the apprehension of the object in an image, but apprehends the particular dog as an intuitive image of a member of a class. This radically differs from the mathematical schematism. Formal intuitions are limitations of the given unitary form of intuition. They do not *fall under* a concept,

[45] Kant himself would actually seem to imply this in a passage from the first edition (A 120 and n.).

while the dog is the individual instance of a concept, that, furthermore, I have not produced apriori and arbitrarily.

But the more important problem – in a *Gestaltist* or Wittgensteinian spirit, if you will – is the one I mentioned a moment ago under the rubric of the unconscious synthesis: how can I see the manifold of the dog and ascribe it to one object unless I have already unified the manifold (through reproductive imagination?) in the representation of one object? In other words, my sight needs to have selectively isolated from their background the characteristics of the dog *as belonging to the dog* for me to be able to formally unify the marks in a discursive concept. Here the difference between schema and concept is very obscure. The reason is that no pure apriori concept or schema of the dog makes sense. Above all, therefore, while with formal intuitions the identity of productive and reproductive imagination, or, better, of application and subsumption, is rooted in the same schema, in the same method for constructing an object – e.g., the circle according to the equality of the radii –, with empirical concepts the task of productive imagination is limited to the transformation of the sensible object into its figurative image without any apriori productive counterpart. A schema here is not a method, but rather works as an admonition to find the one in and for the many, to interpret and picture the 'this' as an essence. In such a way we would save all the characteristics of the schematism – figurative synthesis, the irreducibility of the pure methodical nature of the schema to the image, the homogeneity between sensibility and spontaneity, the hermeneutics of judgment – which together make the case of the dog similar to that of the triangle, but we cannot speak of an arbitrary apriori production of the monogram. And this is the main difference between progressive apprehension and schematic construction of extensive magnitudes.

Let me restate the difficulty in Aristotelian terms. I must have a representation of the τοδε τι as a τοιονδε, of this thing as a dog, in order to compare, reflect on and abstract from – the functions performed by the understanding in the formation of empirical concepts – the characteristics in question. Once again, this seems to speak against the characterization of sensibility as pure receptivity and of the passivity of our impressions in the very origin of empirical concepts. Kant's emphasis on the sameness of imagination in the apprehension that transforms the appearance into an image and in the apriori determination of forms of appearances runs against the sharp separation between apriori form and aposteriori matter. But what it really calls into question is the role of matter in the formation of concepts[46]. If the alleged matter which affects our senses is something our spontaneity has already formed, we must stress that this in its turn is possible only insofar as the dog is seen as

[46] It is as if Kant were eventually forced, *malgré lui*, to return to something like the noetic intuition of Plato and Aristotle. This idea is at the root, it seems to me, of Pippin's pages (*Kant's Theory of Form*, New Haven and London 1982, 106–22). The idea that Kant, and philosophy in general, cannot help but take its bearings from a noetic intuition is central in S. Rosen's essay (*Is Metaphysics possible?*, in: The Review of Metaphysics, XLV, 2, 1991, 235–57; see also his *Plato's Sophist*, New Haven–London 1983, *passim*).

itself one, it is itself the unity of its characteristics. Then we do not *produce* the unity in the discursive concept at all, we "re-produce" it by looking to the appearance itself, or to our image of it. It is the vision of the dog – our sensibility, not our understanding – that is the ultimate ground of our unitary image of the dog and thereby of our discursive concept. But then the distinction between a form and a matter cannot stand. Thus, the production of the image can very well be called arbitrary and even free, in the Hegelian sense of being dependent on the inner nature and idealizing work of imagination, but it can hardly be called apriori in the same sense in which a mathematical arbitrary construction is. This is less clear in the Kantian examples of water or gold, in which we combine marks which could well belong to other things, and in which – to stick to an Aristotelian terminology – we do not have independent ουσιαι. But even there, as in the case of the dog or of the different trees (*Logik*, § 6), it should have been evident to Kant that his claims about the origin of empirical concepts needed further qualification: in particular, a discussion of what it means for imagination to be productive in the case of empirical concepts, and a redefinition of the meaning of the schemata of empirical concepts as generalities paradigmatic for – but at the same time irreducible to – particular images.

The question of the relation between matter and form is a delicate point as well in Kant's discussion of the construction of motion in natural science, although there Kant's answers are, predictably, more persuasive. Physics seems to be on the borderline between what we can determine thoroughly apriori and what needs determination from experience. To put it more appropriately, motion always presupposes *ein Bewegliches*, a portion of matter of which we assume only the property of mobility. The construction of a combined motion is its exhibition in intuition (*M. A. d. N.*, 486). But the description of a space in phoronomy differs from the merely geometric construction because it includes the consideration of velocity, which presupposes something empirical and its relation to time, not only to space. Velocity, motion, force, mass and other basic concepts of natural science are apriori, yet not pure concepts. What is their transcendental origin? They are not empirical concepts in the sense just outlined, although in the end they must be *concepts of*, laws *referring to*, something empirical: they are the application of mathematics to experience. However, so far it has seemed that, in the application of mathematical concepts to experience, all that was needed was the apriori construction of mathematical objects in pure intuition, whose belonging to appearances could be verified by experience. The necessity of matter, of given (however ideal) bodies in space and time in the science of nature seems now to show that the mathematical concepts in the physical sciences are not just the external application of our geometrical and arithmetical construction to given appearances, but are of a different nature. Whereas mathematical concepts are apriori *concepts of pure intuitions*, having limitations of space and time as their object, physical concepts are universal laws ruling the motion of appearances *in* space and time (see A 41/B 58). This is what prevents physics from extending its knowledge in completely apriori synthetic construction.

No pure apriori *Selbstaffektion* is sufficient to determine appearances in physical laws. Phoronomy does construct motion apriori, but to the simple description of space it must add "a cause of the alteration of motion which cannot be simply space" (*M. A. d. N*, 495). Reference to something more than the mere form of intuition, i. e., to matter, is necessary for it.

If the mathematical configuration of our laws is the pure apriori product of our understanding, which does not look to anything in experience to extend its knowledge, the source of the particular laws of nature which are transcribed in that configuration cannot itself be the understanding in its mathematical use. In physical experiments the activity of the understanding is to compel nature to answer "questions of reason's own determining" (BXIII), because "reason has insight only into that which it produces after a plan of its own". What we determine apriori here is, however, just a fruitful way of asking questions and of turning to nature, looking to it, "in order to be taught by it" (ibid.). This is Kant's version of the radical dissociation of geometry from physics, which is never determinable entirely apriori, in Descartes' *mathesis universalis*[47]. But should we not conclude then that the mathematical status of universal physical laws does not determine its objects in the same sense in which mathematics determines its? If the being of appearances is always divided into form and matter, and hence is irreducible to its intelligibility, is the different intelligibility of appearances in mathematics and physics not itself ultimately rooted in the very givenness of matter?

The world of apriori forms and that of aposteriori matter are, then, not so clearly and univocally distinguishable by Kant's own standards. It is the same form of a possible matter which is both mathematically and physically intelligible, and spontaneously unified in empirical concepts. Thus there seems to be a multiplication of

[47] I think I can say now that my disagreement with Lachterman boils down to just this. He is persuaded that the Cartesian project of a *mathesis universalis* defines modernity, and its final stage is Kant's philosophy with its emphasis on construction. He finds a continuity stretching from the Cartesian reduction of the world to geometry through Leibniz' *ars characteristica*, to Lambert's intermediacy of symbolic cognition (*The Ethics of Geometry, op. cit.*, 52), and finally to mathematical construction in Kant, whose immediate antecedent would be Wolff's construction of an algebraic equation (*op. cit.*, 11). Therefore algebra, and symbolic cognition, would play in Kant the same role that geometry is assigned in Descartes' 'absolute idealism'. The same problem, then, which according to Lachterman haunts Descartes' imagination would also be a radical objection to Kant. Cartesian imagination's double function "as both the instrument by which and the medium in which the prescribed courses of technical genesis can be both carried out and appreciated for what they produce" (*op. cit.*, 181) shows its necessity of relying on the condition "that the appearances it sets out to master continue to appear" (*op. cit.*, 203; on this point see already S. Rosen, *A Central Ambiguity in Descartes*, in: *Cartesian Essays*, ed. by B. Magnus, The Hague 1969). However, as I said, Descartes substitutes his *ordo et mensura* for given nature, while Kant stresses that the apriori quantitative determination of nature is not the whole story about experience: an empirical manifold has to be given, and the mathematical apriori determination of forms is directly related to the possibility of exhibiting those very forms as the forms of appearances.

layers not amenable to unity: the pure mathematical determination of forms, the universal laws in which physics states its knowledge of the world of appearances, the empirical experience of that very world (its form *and* matter) in our everyday life, and the philosophical synthesis represented by the transcendental foundation of our apriori knowledge of forms. In general, I believe there is something obtuse about frequent appeals to Occam's razor in philosophy. However, especially if we consider also the reflective judgment, this multiplication of layers tallies with a proliferation of the senses of form and its relation to matter, so that not only does nothing eventually unify the theoretical, practical, aesthetic, telelogical and religious uses of reason, but even the various respects in which we consider appearances theoretically are a dispersed hierarchy of essentially different, although communicating, levels.

The notion of exhibition *in concreto* and, consequently, the role and sufficiency of imagination, then, differ according to the relation between form and matter. If the possibility of dispensing with experience distinguishes mathematics and its apriori development, we must, however, go beyond Kant's presentation of the Analytic and the Doctrine of Method as irrelevant and unrelated to one another, and focus on the relation between schematism and method. In other words, the discussion about the privilege of mathematics belongs in the context of the philosophical analysis of our determination of form. Kant is right in pointing out that it is transcendental philosophy which must answer these questions, not science of nature. But transcendental philosophy cannot leave the problem of the relation between apriori determination and empirical knowledge unresolved on any of its levels. To rest content with a declaration of indifference with regard to this issue would mean to give up the possibility for an adequate and comprehensive philosophical account of the distinction between form and matter.

§ 4 *Time or Space-time? Geometry, Arithmetic and Algebra*

The relation between space and time is not only central to natural science, but also to our concern with mathematical schematism. It would be pointless to deny that the Kantian examples I have mentioned so far always seemed more plausible and appropriate in connection with geometry than when arithmetic was the subject-matter. If it is clear that geometry is a determination of outer sense, the ambiguity about inner sense is that it is difficult to understand the difference between determination in time and determination *of* time. But it is Kant's standpoint that arithmetic and algebra are sciences based on numbering, which is defined as the "production of time" (A 145/B 184).

We have two possibilities. We can take seriously Kant's repeated characterization of number as the generation of time, and try to reconstruct its sense. Alternatively, we can minimize the difficulty, and think that Kant's terminology was imprecise and that it should not lead us astray. Time would then be merely the medium in which we synthesize the manifold; it would not itself be the object of synthesis in the same sense as space is the object of geometric construction. The second alternative banalizes Kant's thought. It assumes that numbers are abstract entities involving no intuition other than the vision of their symbolic configuration. According to this reading, temporal succession is irrelevant to the essence of number and has to do only with the operations we perform in time. I believe that the interpreter has no choice: Kant's letter cannot be rejected until it is shown to conflict with the spirit of his philosophy. This, however, can only be done on the basis of a global analysis of the texts, not by concentrating on some passages at the discredit of others. So we must now try to see what time has to do with mathematics, and how time and space are related in construction.

Admittedly, Kant does not say much about it. In the chapter on schematism there is a definite priority of time over space, whereas in the Refutation of Idealism the relation seems inverted. To be sure, in the *Critique* there are numerous asymmetries in the relative priority of space and time. In the schematism the greater universality of time has to do only derivatively with the fact that all appearances are subject to the condition of time, but not all are spatial or external appearances (A 142/B 182). The main reason for our understanding's privileged relation to time is that time is the medium in which every manifold is apprehended and successively unified in our consciousness. Only time, and not space, mediates between concept and sensibility in schematism. We generate and apprehend appearances as quantities by successive temporal synthesis. This is the fundamental idea at the root of the principle of extensive magnitudes, which is presupposed by any mathematical consideration of formal intuitions. I mean that I can construct a triangle because I can generate a magnitude through the progressive synthesis of productive imagination in time. Only the production of unity, plurality and totality in consciousness is crucial for this generation, while the generation of the image is the immediate spatial outcome of a temporal succession.

But space is more essentially related to time, it seems to me, than Kant argues in the schematism section. The production of the image must have an essential relation to the temporal production, and figurative synthesis cannot be just a side-product of imagination[48]. This has an immediate bearing both on the geometrical construction and on the spatial character of the signs used by algebra and arithmetic. When he transforms time from a form of intuition to the principle of schematism, Kant may have an eye on the condition of applicability of the concepts of physical science to the objects of sensibility, as Cohen argues. But what is more important is that the very concept of change in inner sense is made comprehensible, in the Analogies of Experience and the Refutation of Idealism, by an appeal to the permanence of objects in external sense. Time itself can be externally represented only in a spatial

[48] I develop these points in my *Mathematical Synthesis, Intuition and Productive Imagination in Kant*, in: *The Sovereignty of Construction*. Essays in Memory of David R. Lachterman, ed. by P. Kerszberg and D. Conway, Amsterdam, forthcoming, § 2.

intuition. The symbolic representation of time as a straight line makes succession depend on the imagination's motion of generating the manifold of space. Imagination thereby determines inner sense, or produces the temporal manifold as the representation of before and after, as we saw in section 2. But this determination takes place as the description of a space. Or, better, it is itself the organization of *Vorstellungen* in an ordered series, hence the positing of figurative relations in a succession. The intuition of the movement of a point in space and its presence at different locations on the line is what alone yields an intuition of alteration, and makes comprehensible the successive existence of our selves in different states (A 33/B 50, B 292, and *Reflexionen zur Mathematik*, Ak. XIV, 55).

As Schultz rightly remarked, there is not a particular science of time – as geometry can be called a science of space – other than this "geometry of the straight line"[49]. Schultz' predictable conclusion was that it is not worth studying independently. Kant no more agreed on this latter point than on most of Schultz' remarks about arithmetic, as we shall see. But it is true that arithmetic is not a form of time in the same sense in which geometry is a form of space. In the Aesthetic, after writing that geometry is the science of space, Kant quite surprisingly writes that our concept of time explains the possibility not of arithmetic, which had until then accompanied geometry along with the respective examples of spatial figures and of '5 + 7 = 12', but of "the general doctrine of motion" (A 32/B 49). While the figure of the line comes up very often in the Aesthetic and in the first part of the Analytic, we have to wait – with the notable exception of his comment on the synthetic nature of '5 + 7 = 12' in the fifth section of the Introduction – until A 78/B 104 for a simple mention of counting.

This, however, rather than pointing to a subordinate view of arithmetic, shows the more fundamental role of *numbering* as such. It is number that is the schema of the category of quantity. Number is the most fundamental form of producing a plurality and the most universal means of determining objects as magnitudes in intuition, so much so that the relation of number to time and space is now strikingly equated to that of the schema to its pure images (A 142/B 182). What should strike us here is the definition of space and time as pure images. But, thereby, the fundamental relation between schema and intuition is not altered, because space and time are considered themselves as magnitudes to be determined numerically. It is the homogeneity of number that makes magnitudes comparable and reducible to one another for all subsequent mathematical operations. As the *Critique of Judgment* (§ 25) says, *that* something is a *quantum* can be learnt absolutely; but to determine *how* great it is we need a standard, a unity of measure, that allows us to compare one magnitude to another. Even the geometrical construction of a triangle presupposes the employment of number.

Number is defined as "the representation which comprises the successive addition of homogeneous units"; the outcome is "a unity due to my generating time itself in

[49] Quoted by Martin (*Arithmetik und Kombinatorik bei Kant, op. cit.*, 101).

the apprehension of the intuition" (A 142–43/B 182). Number is not just a means to compare magnitudes. With regard to its transcendental constitution or ideal genesis, number is rather the production of time as the counting of a manifold[50]. Temporal succession, far from being irrelevant to number, is its condition of possibility. This explanation of the genesis of number makes possible all the particular numbers we use in counting and in arithmetical operations.

All this characterization requires considerable attention. A great deal is implicit in, and many points must be developed out of, this intricate knot. To begin with, what we have here is the principle for the definition of natural numbers, the basis for all higher mathematics, which in its turn rests on the succession: 0, 1, 2, 3, etc. But in this very succession it is noteworthy that '0' is not a magnitude and therefore in its own right it has nothing to do with the succession of natural numbers[51]. Even the One ('1') is heterogeneous to natural numbers, but for the different reason that it is the condition of possibility of the generation of numbers, as the unity of measure by whose repetition we produce integers. The One cannot be generated by the same synthesis that generates integers as, as it were, unities of units. In fact, it cannot be generated at all. We can call it a monad. The successive addition of units presupposes the given unit of which we make use and the unity of consciousness, for which only counting is possible. Martin[52] cites a passage in the first edition of the *Critique* (A 103), where Kant writes that the unity of consciousness in memory is presupposed in counting. There, in the synthesis of recognition in the concept, Kant mentions the three elements necessary in all our knowledge: the inspection of the manifold of intuition, its reproduction in a synthesis and its combination in a concept. Obviously, none of these operations can be accomplished with respect to the One. Since no manifold is given, no synthetic unity in a concept is possible.

From very similar considerations Pythagoras, Plato and Aristotle positively determined the One as the αρχη of numbering. Why is Kant silent about the peculiarity of the One? The first thing that comes to mind in the attempt to answer this question is that for the Pythagoreans number was the principle of the ordered arrangement of the world[53]. Numbers had a nature, so that for example the Ten was an intrinsic element of the order of things and of counting, while for Kant the decimal basis of our numbering is the result of an arbitrary choice[54]. Kant never says any-

50 I would rather say that number is the quantification of time. In Kant's words there is an inescapable circularity between time *qua* the possibility of number and time *qua* produced by my activity of numbering. I discuss this point in my *Mathematical Synthesis, Intuition and Productive Imagination ..., op. cit*, § 2.

51 '0' is a *nihil privativum*, as Moretto puts it (*Sul concetto matematico di grandezza ..., op. cit.*, 60).

52 *Arithmetik und Kombinatorik, op. cit.*, 106–09.

53 See J. Klein, *Die griechische Logistik und die Entstehung der Algebra*, 1934–36, Engl. tr. by E. Brann, *Greek Mathematical Thought and the Origin of Algebra*, Cambridge M.I.T. 1968, 63 ff.

54 See *Kr. d. U.* § 26. See Kant's enlightened mockery of the "childish" nature of "the mystical importance" ascribed to the number 7 in the Appendix to § 39 of the *Anthropologie*.

thing about the kind of problems which were at the center of Plato's and Aristotle's philosophy. For him the One does not present an ontological question, as for instance it does later for idealism, and unity is thematic only as the synthetic unification of the manifold performed by the understanding or as the first category of quantity. What Kant seems to imply is that units are nows, time intervals which are posited as discrete by the activity of differentiating and articulating the *continuum* of time. In the construction of number the production of time is a repeated positing of the unity of measure ('Setzung', *Reflexionen zur Mathematik*, Ak. XIV, 54). This now takes on, it seems to me, the function of the indivisible unit for the construction of numbers *via* combination and successive generation of the manifold in the unity of apperception[55]. The One, the temporal unit, is then the quantitative expression of the original synthetic unity of apperception. *Selbstaffektion* in mathematics, i. e., imagination's determination of inner sense into an actual intuition, is the positing of the unity of measure resulting in the autonomous determination of the temporal succession.

If this interpretation is correct, however, it is ambiguous and even improper to say that we generate time[56]. Time must be given, as an indeterminate form of our intuition, as the possibility of succession. We can *modify* time. What we produce or generate thereby is not time, but actual temporal relations: the order – the meaning – of the succession. We do produce synthetically the unification of the manifold *of* time (not just *in* time). Empirically, this occurs in our inner experience when we posit the order of our representations in the unity of our memory or consciousness of them. With respect to mathematics and to the category of quantity, this occurs when we constitute the order of a plurality in a number. However, we do not "*produce* the concept of succession" (B 154–55). We establish a determinate succession as the unitary composition of homogeneous parts. Taking up Lachterman's metaphor again, a metronome makes time assume the shape it wants, it determines its length, its cadence. But it can do so only insofar as it disciplines a given one-dimensional flux, a homogeneous *continuum*.

Going back to the One and to the ontological questions concerning it mentioned above, a further problem is this. Aristotle's chief criticism of Plato – that no reason was given for the unity of the many – seems to be potentially destructive for Kant's unity of units as well: I want to say that there is a certain circularity in his definition of number. On the one hand, a number is supposed to be a totality (B 111), the result of the union of unity and plurality. On the other hand, "the concept of a magnitude in general (...) is that determination of a thing whereby we are enabled to say how many times a unit is posited in it. But this how-many-times is based on successive repetition, and therefore on time and the synthesis of the homogeneous

[55] In the restrictive context of the category of quantity, we can apply Lachterman's ingenious (but broader) suggestion and regard numbering as the rhythm of a metronome in thought (*Kant: the Faculty of Desire, op. cit.*, 198).

[56] See footnote 23 above.

in time" (A 242/B 300). In light of this, it is difficult to understand how you can have a method for constructing the number 64 in an image, as the successive addition of units, without having to use that very number in determining when to stop once the units have been posited in the number 64, no less and no more than that number of times. The same problem emerges in the very notion of extensive magnitudes. If the manifold is an aggregate, what directs my construction to stop? In geometry it is the *quality* of the magnitude, i. e., its figure. In the construction of a triangle I stop drawing the line when it meets another line to form a given angle, but it is not clear how I generate the number 64 by simply adding units without a direction telling me when to stop adding units.

It would be very dangerous here to argue that it is *sic et simpliciter* the concept to direct me. It is true that Kant seems to be primarily interested in the constructive origin of our mathematical synthesis, and he says little about, say, the properties of arithmetical numbers. So, for example, it is difficult to see how Kant's theory of construction can account for the number 17's property of being a prime number. This should cause no surprise. As I pointed out earlier, an articulate view of mathematics is not the central concern of the *Critique*, which only has to show how and why mathematics, as one kind of apriori knowledge, is possible as a pure apriori science. Construction in intuition is the necessary and sufficient answer, from Kant's standpoint, to his query. Now, though, the question concerns what time has to do with numbers, and this can hardly be peripheral to our concern. Is time determinative only for arithmetic operations or is it also necessarily involved in the construction of numbers themselves?

In an attempt to make sense of Kant's philosophy of arithmetic, Parsons argues that intuition in arithmetic covers the role of the verification of abstract structures[57]. For Young, schematism provides the rules of the procedures for identifying perceptible collections with numbers[58], thereby making possible the application of arithmetical concepts to intuited things[59]. Their views rest on an interpretation of Kant's distinction between number and magnitude which misleads them into thinking that Kant took intuition to be the sensible token representative of abstract concepts – in this case, number would be the perceptible instance of a *quantitas*. Since they do not focus on the modification of time produced by us dealt with in the transcendental deduction and schematism, they cannot account for the plausibility of treating number as generated by the successive addition of homogeneous units in time. I think that their interpretation relies heavily on Frege's concept of number, and that if the modification of inner sense is not conceived of as already ruled in itself by productive imagination, one ought to draw a conclusion that they actually do not state, namely that time has nothing to do with the construction of

[57] *Kant's Philosophy of Arithmetic*, in: *Philosophy, Science and Method*, Essays in Honor of E. Nagel, Ed. S. Morgenbesser *et al.*, 1969, 568–94 (588–90).
[58] *Construction, Schematism and Imagination*, op. cit., 127.
[59] *Kant on the Construction of Arithmetical Concepts*, in: Kant-Studien 73, 1982, 17–46: 38.

numbers and is involved only in arithmetical operations. However, if one accepts their view, saying that 7 + 5 = 12 involves time is just a form of psychologism which does not affect the timeless entities on which one operates. Parsons recognizes that number and arithmetic involve succession, but to this claim from the Schematism he immediately opposes an analysis of Kant's letter to Schultz[60]. According to his interpretation, the letter marks Kant's return to the precritical consideration (in the *Dissertatio*) of numbers as a pure intellectual synthesis referring to a concept of a thing in general, not as a schematic or *figürliche Synthesis* of intuition[61]. He concludes that "Kant did not reach a stable position on the place of the concept of number in relation to the categories and the forms of intuition"[62].

Kant's letter to Schultz is indeed central. Schultz had written that while he agreed that geometry was synthetic, he still found arithmetic analytic. Kant replies that general arithmetic is an ampliative science, and therefore cannot ground its synthesis on analytic judgment. In fact, time has no influence on the properties of numbers, and the science of number is a purely intellectual synthesis in thought, "unless one has to attend to the succession required in the construction of the magnitude. But insofar as magnitudes have to be determined, they have to be given so that we can represent their intuition successively in time" (*Briefwechsel*, Ak. X, 557). Numbers then are subject to the condition of time whenever we construct them or exhibit them in intuition. This does not mean that they have a separate existence. For Kant their mode of being is peculiar, and different from the mode of being of geometrical figures. Although we can employ numbers generally, numbers as such are forever particular objects, unlike geometrical universal objects. This is why Kant calls arithmetical propositions "numerical formulas" (A 165/B 206). The function of construction shows, again, to be closer to that of a constitutional monarch, rather than to the arbitrary *determinatio ex nihilo* of an absolute sovereign: it has to take into account the given form of intuition and the different nature of the means through and in which it constructs its objects. The means at the disposal of geometry, and of physics, is the spatio-temporal intuition of a *quantum*. Algebra and arithmetic, on the other hand, are not directly related to intuition but to number, which is the representation of the "how-many-times", of *quantitas*.

As already pointed out, all that matters for Kant is the origin of numbers as the expression of the construction of magnitude, and all that Kant wants to prove is that they are subject to the condition of time in their construction in intuition. Another way to put it is to say that Kant wants to explain the synthetic nature of arithmetic by grounding here too what I have called the synthesis as ampliative extension of knowledge on the synthesis as the determination of sensibility, the exhibition and production of the object in temporal intuition.

[60] *Kant's Philosophy of Arithmetic, op. cit.*, 585. For Kant's letter (November 25, 1788), see *Briefwechsel*, Ak. X, 554–58.

[61] *Kant's Philosophy of Arithmetic, op. cit.*, 586–87.

[62] *Arithmetic and the Categories*, in: Topoi 3, 1984, 109–21: 118.

The ampliative synthesis is connected to the presence in arithmetic of what Kant strangely calls primitive, indemonstrable "postulates", as opposed to the intuitive axioms grounding the construction of objects as *quanta* in geometry (A 234/B 287). What they represent is the practical task (*Aufgabe*) of constructing an object synthetically, of performing an operation such as addition. As Kant writes in the letter to Schultz, 7 is not the analysis of the concept of the task of thinking 3 and 4 together in a number, but the synthetic generation of the result of the addition. "That 3 and 4 yield the concept of one magnitude (...) is a mere thought; but the number 7 is the exhibition of this concept in a 'counting together'" (*Zusammenzählung*, Ak. X, 556)[63]. In 16, I do not think 2^4, 4^*4, 6 + 10, 64,000/4,000, etc. All these are the result of different synthetic operations (*ivi*, 555).

If this is clear, what is not is the meaning of intuition here – in number as such, not in addition –, and how time and space are related to numerical construction. Furthermore, this thesis seems to deny the possibility of conceiving of certain classes of objects in everyday use in mathematics, such as, for example, irrational numbers. It was Rehberg who raised a simple but apparently fatal objection: in order to see the truth of arithmetical propositions no intuition of time is required, because proofs are evident from the mere concepts of numbers. The form of our sensibility notwithstanding, we can think of irrational numbers that do not require intuition, and, in fact, cannot be synthesized in time either (*Briefwechsel*, Ak. XI, 205–6). Kant's reply is that the *possibility* of algebra and arithmetic is not subject to the condition of time, but their actuality is. The construction of the magnitude is its representation in the figurative synthesis of imagination, without which no mathematical object can be given. Thus, to be sure, $\sqrt{2}$ can be thought, but it does not remain an empty concept, because the geometrical representation of the diagonal of a square represents it in intuition. $\sqrt{2}$ designates a root, but in order to find its value we have to construct it in time and in space. This shows that space and time are necessarily interrelated in determining the objects of our intuition, and, again, that without space time could not be represented as a magnitude (*Briefwechsel*, Ak. XI, 209).

In the *Reflexionen zur Mathematik* Kant reiterates his position on the actuality of mathematical objects. If we recall the question of the objective validity of mathematical definitions, we note that here Kant's rejection of the idea that mathematics could produce fictional objects in imagination, which thus have no reality as the possible form of appearances, tallies with the close tie now found between space and time, and consequently between arithmetic and geometry. Geometry *proves* the actuality of $\sqrt{2}$.

If we did not have any concept of space, the magnitude of $\sqrt{2}$ would have no meaning for us, because we would then be able to represent each number as a set of indivisible units. Instead

[63] See also B 16: "That 5 should be added to 7, I have indeed already *thought* in the concept of a sum = 7 + 5, but not that this sum is equivalent to the number 12. Arithmetical

we represent to ourselves a line in fluxion, namely as produced in time; in it we do not represent anything simple, and can think 1/10, 1/100, etc. of the given unity (Ak. XIV, 53).

What is crucial is the mention of fluxions in this context (see also A 169–70/ B 211–12). As in Newtonian analysis, numerical magnitudes express a continuous motion, and continuous motion presupposes variations in time and their figuration in space[64]. However, Kant has silently moved here to intensive magnitudes, which, as functions or relations among magnitudes, apply to calculus, analysis and physics. We are no longer dealing with extensive, geometric magnitudes. It is the principle of intensive magnitudes that the construction of a magnitude – in this case, the intuitive representation of an irrational number – is not an aggregation of indivisible units because its construction is continuous and cannot stop at anything simple. Although it is always possible to exhibit a proportional magnitude between two homogeneous numbers, here we cannot give "that middle proportional magnitude *in one integer*" (Ak. XIV, 57, Kant's ital.). At this level, geometry and arithmetic are not separate, rather geometrical construction is the apriori exhibition of numerical magnitudes in intuition. However, in light of this, and of the interpretation sketched above according to which numbering is the positing of nows that articulates the *continuum* of inner sense in discrete units, we must conclude that for Kant natural numbers, although primitives for our construction of all other numbers, are not fundamental elements in their own right. What is actually primitive in number as the schema of quantity, therefore, is only the determination of the temporal succession: "the determination of magnitudes through a rule of counting" in time (*Reflexionen zur Mathematik*, Ak. XIV, 57). In other words, the transcendental origin of number is an activity, rather than its products, whereby nothing is given or presupposed other than the seriality of nows: the one-dimensional form of time.

Going back to the notion of the spatial actuality of numerical magnitudes, I think that this shows the limits of construction. Throughout the letter to Rehberg Kant stresses that thinking would be empty if it did not make (*machen*) what it thinks in productive imagination: we must *find* a positive determination for all magnitudes, including $\sqrt{2}$. What cannot be constructed, though, is merely imaginary (and not in the sense of the result of productive imagination, which in mathematics always constitutes the actuality of its objects, but in the sense of something vacuous). Not only does it have no objective validity, it is even contradictory, as imaginary numbers are (*Briefwechsel*, Ak. XI, 209, n.)[65].

propositions are therefore always synthetic". With the aid of intuition I see "the number 12 *come into being*" (my ital.).

64 Hamilton's later project of "Algebra as the science of pure time" (see his *Mathematical Papers*, vol. III, Cambridge 1967) is Kantian in at least this respect: magnitudes are not regarded as formed and fixed, "but rather as nascent, or in process of generation". Newton's theory of fluxions "involves the notion of time" (p. 5).

65 Schultz writes that their meaning is comparable to that of a square circle (quoted by Martin, *Arithmetik und Kombinatorik ...*, *op. cit.*, 71).

A second issue is this: it was a well-known Cartesian, Leibnizian and Wolffian thesis that arithmetic and geometry could collaborate in the solution of problems. Kästner in fact shares this position. It is noteworthy that Schultz replies to him by arguing that mathematics has to keep separate irrational and infinitesimal numbers from any geometrical consideration. Schultz tacitly recalls Aristotle's *Posterior Analytics* when he writes that it would be a μεταβασις εις αλλο γενος to use a proof from trigonometry in calculus[66].

Let me mention that we have here two interesting alternatives. On the one hand, let us assume that Kant, who knew Schultz' *Anfangsgründe* so well that he actively intervened in the composition of the text, agreed on this point. He could do so only by separating proofs from exhibition in intuition. This would imply, however, that Kant would not allow a *mathesis universalis* in the Cartesian or in the Leibnizian sense. This would seem to be in conflict with the characterization of the *Metaphysische Anfangsgründe* as the "*pure mathesis*" (Ak. IV, 489) of magnitudes – that is of the continuous, intensive magnitudes of calculus in the first place. Furthermore, Kant would have had to separate construction from synthesis: the visual intuitive exhibition would then have had no bearing on the ampliative character of arithmetic, and this contrasts with that close relation of the two senses of synthesis I have pointed out all along. On the other hand, if Kant did not agree, he would have sustained Leibniz' claim. This might have unpalatable consequences, because this claim can itself be shown to follow from the relational view of space and time which Kant sets out to criticize in the first place.

Kant's texts do not allow us to develop any further the details of or his response to these alternatives. What we can say is that there is a further sense in which space and time are related, a sense which sheds light on Kant's distinction between *quantitas* and *quantum* and on the separation of the realms of geometry, arithmetic and algebra. This returns us to the aforementioned role of intuition in arithmetic.

Beginning with the *Deutlichkeit* and the *Metaphysik L* on, Kant insists repeatedly that arithmetic deals with signs substituting for things. In the Doctrine of Method Kant writes that algebra constructs magnitude as such.

> In this it abstracts completely from the properties of the object, (...) then chooses a certain notation for all constructions of magnitude as such (numbers), that is, for addition, subtraction, extraction of roots, etc. Once it has adopted a notation for the general concept of magnitudes so far as their different relations are concerned, it exhibits in intuition (...) all the various operations through which the magnitudes are produced and modified. When, for instance, one magnitude is to be divided by another, their symbols are placed together, in accordance with the sign for division, and similarly in the other processes; and thus in algebra by means of a symbolic construction, just as in geometry by means of an ostensive construction (...), we succeed in arriving at results which discursive knowledge could never have reached by means of mere concepts (A 717/B 745).

We have to note here two ambiguities. First, algebra as the science of quantity is what Kant calls elsewhere, as for example in the Axioms of Intuition, arithmetic. Second, the two differ first of all in their degree of abstraction, and consequently

[66] Quoted by Martin (*Arithmetik und Kombinatorik ..., op. cit.*, 111).

in the different modes of exhibition of their objects, which may be either schematic or characteristic. So there is also an ambiguity in Kant's use of the term 'symbol'. Here it stands for the marks used by algebra, while in the *Critique of Judgment* (§ 59) or in the *Fortschritte der Metaphysik* (Ak. XX, 279–80) a symbol is one sort of hypotyposis, namely the analogical or indirect exhibition (for example of the ideas of reason). As such it is opposed to the *Charakterismen* of language or algebra, which are the sensible signs used as the means for the associative reproduction of concepts.

But the more important point has to do with the recourse to intuition in arithmetic (and algebra). It has often been pointed out that Kant's quotations from Segner (A 240/B 299; *Reflexionen zur Mathematik*, Ak. XIV, 55), and his appeal to signs, fingers, etc. – Eberhard was the first to find it laughable – seems to apply only to the most elementary operations with very small numbers. There is a serious misunderstanding behind this criticism. The confusion is between pure and empirical intuition. The notion that numbers need "pure sensible images" (*Reflexionen zur Mathematik*, Ak. XIV, 55) does mean that also the number has an intuitive visible referent. The objective spatial representation of numbers in signs makes all arithmetic operations, both the addition of up to ten items empirically representable by fingers and all more complex apriori operations on signs, an exhibition *in concreto*. When Kant refers, in the passage quoted above, to the "placing together" of symbols according to the sign for division, he implies that the spatial configuration, the ordered placing of the elements of the operation, is essential for its success. It allows us to check the result easily, with a single glance. Even in this case, then, as in geometry, it would seem that the productive imagination's pure determination of time as the successive addition of units finds its exhibition in empirical intuition, and that synthetic judgments are grounded on the intuitive origin of our mathematical concepts.

However, in arithmetic and algebra, although intuition (of time) is still the starting-point, the exhibition has primarily an instrumental or auxiliary function. In this respect a geometric schema differs as substantially from number, the schema of quantity, as space differs from time. In contradistinction to geometric schemata, number as the schema of quantity shares the priority of time over space typical of the schematism, as well as the higher generality with regard to its exhibition. The serial temporal character of the number finds in the sensible mark standing for it only a contingently connected image, for a sign does not contain anything belonging to the intuition of the object, as the *Critique of Judgment* (§ 59) reads. The relation between schema and sensible intuition is in this case indirect, whereas for spatial figures it is immanent. If the schema of the triangle is a procedure to construct the image of a figure, in the image or application of the schema of number we must pass from time to a representation in space, and this passage is marked by a gap which does not affect something *per se* spatial. While there is nothing like a geometric schema in general, but only particular monograms or schemata of geometric figures as *quanta* – triangle, circle, etc. –, number remains irreducible to all particular numbers. Thus we have a curious inversion of what earlier was referred to as the particularity of numbers as opposed to the universality of geometric figures. But clearly that was the relation of particular numbers – Kant's "numerical formulas" – to the activity of numbering, whereas here the question is that of a plurality of geometric schemata as opposed to an alleged schema of a spatial figure in general, which is an empty representation. If in the case of geometry the universality of figures is contrasted with images, however apriori the production of these concrete instances may be; in the case of arithmetic, particularity refers to determinate numbers standing in a relation of heterogeneity with the schematic activity of generating them in time.

This explains why the schema of quantity, the most fundamental determination of magnitudes, is number – or, as I said, numbering. Numbering makes the particular numbers the most adequate designations of pluralities in appearances. If geometry is based on schematic exhibition in intuition, and arithmetic and algebra operate with the intuition of symbols – or characters, or signs –, which are not the exhibition of something essential, but arbitrary expressions which have a conventional relationship with the concepts they stand for, then the superior generality of arithmetic and algebra over geometry explains why the first two operate on pure *quantitas*, while geometry constructs extensive magnitudes as *quanta*.

In sum, intuition in arithmetic and algebra points primarily to the genesis of our schemata as a production of time, and secondarily to the visible signs, the empirical images, used in our constructions. Synthesis, in turn, is this relation of concepts with the pure intuition of time, and in a second sense it refers to the ampliative character of the proof-structure of arithmetic and algebra.

Finally, there is a further difference between arithmetic and algebra: the latter must also interpret its signs, which do not have a meaning by themselves. But this does not entail that algebraic propositions would be in themselves analytic. They too use an empirical intuition to represent their symbols, and are ampliative. So we cannot read the passage where analytic propositions are first introduced as a refutation of the universal synthetic nature of mathematical propositions.

Some few fundamental propositions, presupposed by the geometrician, are, indeed, really analytic (...). But, as identical propositions, they serve only as links in the chain of method and not as principles; for instance, $a = a$; the whole is equal to itself; or $(a + b) > a$, that is, the whole is greater than its part. And even these propositions, though they are valid according to pure concepts, are only admitted in mathematics because they can be exhibited in intuition (B 16–7).

Kant refers here to analytic principles of quantity, not to mathematical or algebraic propositions. The question is, Kant continues, not what we have to think in addition to a given concept, but, as in the letters to Schultz and to Rehberg, what we *really think* in it, albeit only obscurely. So it is clear that the predicate does necessarily adhere to those concepts by way of an intuition which goes beyond the concept.

In conclusion, if schemata are what gives categories meaning, mathematical concepts in geometry, arithmetic and algebra are all schematizable in pure and empirical intuitions, which exhibit apriori their content.

§ 5 *Conclusion*

We have seen why mathematical synthesis has a twofold meaning. Construction as exhibition in intuition occurs purely apriori; but, as I have pointed out all along, the relation between universal and instance, between matter and form, and the very notion of existence change in the context of mathematical schematism. The category of quantity seems to have a peculiar role with respect to the other categories. To say that synthesis and *Darstellung* in mathematics and in philosophy differ only in method cannot suffice unless one is aware of the impossibility of regarding method as extrinsic to the content. I have spelled out the different meaning that synthesis, schematism and exhibition in intuition have in mathematics, in transcendental philosophy and in the formation of empirical concepts, and given a comprehensive account of the relation between space and time in mathematics. I think I have shown the plausibility of Kant's basic concept of an apriori determination of sensibility, without passing over what I believe still remains problematic in it.

If we do not proceed through an accurate analysis of the intricate context in which these notions are embedded, we may very easily lose sight of Kant's original intentions. I think that only a global interpretation of these knots in the *Critique* can explain their relation. In particular I mean to say that contemporary attempts to rescue Kant's thought on mathematics, such as Hintikka's or Parsons', fail precisely because they take their bearings from the context and meaning of *form, intuition* and *signs* which are germane to the logical thought stemming from Frege (and, of course, from Leibniz). I hope I have shown why this tradition, which remains the guiding thread for most contemporary confrontations with Kant's philosophy of mathematics, cannot be the background for a genuine interpretation of it.

On the other hand, we must not underestimate the novelty of Kant's productive imagination, and of its relation with exhibition and mathematical construction, in comparison with its ancient and modern antecedents. It is necessary to be clear about what is specific and distinctive about schematism in its relation to the different modes of exhibition in intuition. In Heidegger's (and Mörchen's) interpretation imagination's *synthesis speciosa* remains productive only in that it spontaneously pictures the given figuratively. I think that their hermeneutical reconstruction cannot adequately make sense of the notion of construction, of the methodical nature of the mathematical schemata and of imagination's apriori production of signs and sensible intuitions. Since on the basis of the Kantian text it is not possible to establish universally the priority of imagination over the understanding, I think that we cannot approach the doctrine of schematism and of the determination of time homogeneously, i. e., treat it as a univocal and undifferentiated datum[67].

[67] I comment in detail on Kant's productive imagination and on these two exegetic traditions, the Fregean and the Heideggerian, in my *Mathematical Synthesis and Productive Imagination ..., op. cit.*