On the rectifiability of defect measures arising in a micromagnetics model *

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1 Introduction

Given a bounded domain Ω of $\mathbf{R}^2$, we consider the space of maps $u : \Omega \to \mathbf{C}$ satisfying

$$\begin{aligned} &|u| = 1 \qquad \text{a.e. in } \Omega \\ &\operatorname{div} u = 0 \qquad \text{in } \mathcal{D}'(\Omega). \end{aligned} \tag{1.1}$$

Equivalentely, taking $u = \nabla^\perp g := (-\partial_{x_2} g, \partial_{x_1} g)$, this space coincides with the space of all functions $g : \Omega \to \mathbf{R}$ solving

$$|\nabla g|^2 = 1 \quad \text{a.e. in } \Omega.$$

Inside this large space we will restrict our attention to the following class of vector fields:

$$\mathcal{M}_{\rm div}(\Omega) := \left\{ \begin{array}{l} u : \Omega \to \mathbf{C} \ \text{ s.t. } \operatorname{div} u = 0 \text{ and } \exists \phi \in L^\infty(\Omega) \text{ satisfying } u = e^{i\,\phi} \\ \\ \text{and} \quad U_\phi := \operatorname{div}\left(e^{i\,\phi\wedge a}\right) \ \text{ is a finite Radon measure in } \Omega \times \mathbf{R} \end{array} \right\} \tag{1.2}$$

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where $\phi(x) \wedge a$ denotes the minimum between $\phi(x)$ and a. Notice that the condition on the lifting in (1.2) is nonlinear, unlike the divergence-free constraint.

The space $\mathcal{M}_{\rm div}(\Omega)$ was introduced in [RS2] and is the natural limit space of the two dimensional variational problem modelising micromagnetism without vortices (see [RS1] and [ARS] for a detailed presentation of this problem). In brief, we consider the energy

$$E_\epsilon(u) := \int_\Omega \epsilon |\nabla u|^2 + \frac{1}{\epsilon}\int_{\mathbf{R}^2} |H_u|^2 \, dx,$$

where H_u (the so-called demagnetizing field) is the curl-free vectorfield related to u by the PDE $\operatorname{div}(\tilde{u} + H_u) = 0$, $\tilde{u}$ being the extension of u to $\mathbf{R}^2 \setminus \Omega$ with the value 0. Assuming that $E_\epsilon(u_\epsilon) \leq C$ and $u_\epsilon = e^{i\,\phi_\epsilon}$ with $\phi_\epsilon \in H^1$ uniformly bounded in L^∞, in Theorem 1 of [RS2] it is shown that the family ϕ_ϵ has limit points (in the L^1 topology) as $\epsilon \to 0^+$ and that any limit point fulfils (1.2). Moreover, we have the $\Gamma \liminf$ inequality

$$\liminf_{k\to\infty} E_{\epsilon_k}(e^{i\,\phi_{\varepsilon_k}}) \geq 2|U_{\phi_\infty}|(\Omega \times \mathbf{R})$$

whenever $\phi_{\varepsilon_k} \to \phi_\infty$. In [Le] this compactness result has been extended to the $\mathcal{M}_{\rm div}$ space, see Theorem 3.6.

The proof of these facts is based, among other things, on some methods developed in [ADM] and in [DKMO1] in the very close context of the Aviles-Giga problem (see [AG1], [AG2]). In this setting one considers the energy functionals

$$F_\epsilon(v) := \int_\Omega \epsilon|\nabla^2 v|^2 + \frac{(1-|\nabla v|^2)^2}{\epsilon}\, dx,$$

so that the vector fields ∇v, up to a rotation, are exactly divergence-free but take their values on $\mathbf{S}^1$ only asymptotically.

At this stage a full Γ-convergence theorem in the micromagnetics case (and in the Aviles-Giga problem as well) is still missing, although as we said the $\Gamma \liminf$ inequality is known to hold in general and the $\Gamma \limsup$ inequality has been proved in some particular situations. Besides, the results in [RS2] and [ARS] lead to a characterization of energy minimizing configurations.

The completeness of the Γ-limit analysis of this variational problem requires a deeper understanding of the space $\mathcal{M}_{\rm div}(\Omega)$. In particular, a more precise description of the singular sets of arbitrary maps in $\mathcal{M}_{\rm div}(\Omega)$ is a very natural question.

As explained in [RS2], the measure $\operatorname{div}(e^{i\,\phi\wedge a})$ "detects" the singular set of ϕ: for instance, it is proved in [LR] that ϕ is locally Lipschitz in Ω if and only if $\operatorname{div}(e^{i\,\phi\wedge a}) = 0$ in $\mathcal{D}'(\Omega \times \mathbf{R})$. In the particular case where the lifting ϕ is a function of bounded variation it is established in [RS1], [RS2] (using the Vol'pert chain rule in BV) that the measure $\operatorname{div} e^{i\,\phi\wedge a}$ is carried by S_ϕ, where S_ϕ is the countably $\mathcal{H}^1$-rectifiable set where ϕ has a discontinuity of jump type, in an approximate sense (see Section 2). Precisely, for any $\phi\in BV(\Omega)$ such that $\operatorname{div} e^{i\,\phi}=0$ one has

$$\operatorname{div}\left(e^{i\,\phi\wedge a}\right)=\chi_{\{\phi^-<a<\phi^+\}}\,\left(e^{i\,a}-e^{i\,\phi^-}\right)\cdot\nu_\phi\ \ \mathcal{H}^1\llcorner J_\phi, \tag{1.3}$$

where $\phi^\pm$ are the approximate limits of ϕ on both sides of S_ϕ and ν_ϕ is chosen in such a way that $\phi^-<\phi^+$, and $\chi_{\{\phi^-<a<\phi^+\}}$ is the characteristic function of the interval (ϕ^-,ϕ^+) in $\mathbf{R}$. Finally $\mathcal{H}^1\llcorner J_\phi$ denotes the 1-dimensional Hausdorff measure restricted to J_ϕ.

Our main motivation in this work is to extend such a description of the jump set to liftings ϕ of vectorfields in $\mathcal{M}_{\rm div}(\Omega)$. In [ADM] an example of a vectorfield in $\mathcal{M}_{\rm div}(\Omega)$ which is not in $BV(\Omega,\mathbf{S}^1)$ is given. Precisely, the authors give an example of a map in the so-called Aviles-Giga space AG_e (see [AG1], [AG2], we follow the terminology of [ADM]) which is not in $BV(\Omega)$. We recall that $AG_e(\Omega)$ is made by all solutions u of the eikonal equation such that

$$\operatorname{div}\left(\left(\frac{\partial u}{\partial\xi}\right)^3,-\left(\frac{\partial u}{\partial\eta}\right)^3\right)\qquad\text{is a finite Radon measure in }\Omega$$

for any orthonomal basis (ξ,η) of $\mathbf{R}^2$. Because of the similarities between the two spaces it happens that this map can be made also in $\mathcal{M}_{\rm div}$ (the technical reasons is that small jumps are penalized with a power faster than 1, see (3.5) and [RS1]). Therefore the BV space is too small for our analysis and there is no hope to achieve our goal by using the classical results of the BV theory.

It is proved in [RS2] that a lifting ϕ of a vectorfield in $\mathcal{M}_{\rm div}$ solves the following kinetic equation :

$$ie^{i\,a}\cdot\nabla_x[\chi(\phi(x)-a)]=\partial_a\left(\operatorname{div} e^{i\,\phi\wedge a}\right)\qquad\text{in }\mathcal{D}'(\Omega\times\mathbf{R}), \tag{1.4}$$

where χ denotes the characteristic function of $\mathbf{R}_+$. By applying now classical results of regularity of velocity averaging of solutions to kinetic equations (see [DLM]), one gets that solutions to (1.4) for which the jump distribution $\operatorname{div}(e^{i\,\phi\wedge a})$ is a finite Radon measure are in $W^{\sigma,p}(\Omega)$ for any $\sigma<\frac{1}{5}$ and $p<\frac{5}{3}$. Taking advantage of the specificity of the solution $f=[\chi(\phi(x)-a)]$ solving the general equation $ie^{ia}\cdot\nabla_x f=\partial_a g$, where $g=\operatorname{div}(e^{i\,\phi\wedge a})$, P.E. Jabin and B. Perthame in [JP] improved the Sobolev exponents and showed that

$$\phi\in W^{\sigma,p}(\Omega)\qquad\forall\sigma<\frac{1}{3}\text{ and }p<\frac{3}{2}. \tag{1.5}$$

Still being a nice improvement, this is far from being enough to tell us something on the structure of the singular set of ϕ (one would like for instance to get as close as possible to the situation where $\sigma p=1$).

Leaving aside the classical linear Functional Analysis approach, which is perhaps not the most appropriate one to explore our non linear space $\mathcal{M}_{\rm div}(\Omega)$, we adopt here a more direct approach working directly on the singular set ϕ through a blow-up analysis of the measure $\mu_\phi(B) := |U_\phi|(B \times \mathbf{R})$.

Our main result is the following structure theorem.

Theorem 1.1. *Let ϕ be a lifting of $u \in \mathcal{M}_{\rm div}(\Omega)$ as in (1.2). Then*

(i) The jump set J_ϕ is countably $\mathcal{H}^1$-rectifiable and coincides, up to $\mathcal{H}^1$-negligible sets, with

$$\Sigma := \left\{ x \in \Omega : \limsup_{r \to 0^+} \frac{\mu_\phi(B_r(x))}{r} > 0 \right\}.$$

In addition

$$\operatorname{div}(e^{i\,\phi\wedge a}) \llcorner J_\phi = \chi_{\{\phi^- < a < \phi^+\}}\,(e^{i\,a} - e^{i\,\phi_-}) \cdot \nu_\phi\, \mathcal{H}^1 \llcorner J_\phi \qquad \forall a \in \mathbf{R}. \tag{1.6}$$

(ii) For $\mathcal{H}^1$-a.e. $x \in \Omega \setminus J_\phi$ we have the following VMO property:

$$\lim_{r \to 0^+} \frac{1}{\pi r^2} \int_{B_r(x)} |\phi - \overline{\phi}| = 0,$$

where $\overline{\phi}$ is the average of ϕ on $B_r(x)$.

(iii) The measure $\delta := \mu_\phi \llcorner (\Omega \setminus J_\phi)$ is orthogonal to $\mathcal{H}^1$, i.e.

$$B \quad \textit{Borel with } \mathcal{H}^1(B) < +\infty \quad \Longrightarrow \quad \delta(B) = 0.$$

Comparing this result with the BV theory, we expect that (ii) could be improved, showing also convergence of the mean values as $r \to 0^+$ (and thus existence of an approximate limit at $\mathcal{H}^1$-a.e. $x \in \Omega \setminus J_\phi$). Moreover, by (1.3) and the VMO condition out of J_ϕ we expect also that the measures $\operatorname{div} T^a u$ are concentrated on J_ϕ. If this is the case, by the formula (see Theorem 3.2(ii))

$$\mu_\phi = \int_{\mathbf{R}} |\operatorname{div} e^{i\,\phi\wedge a}|\, da \tag{1.7}$$

one would get that the measure δ in (iii) is identically 0 and full rectifiability of the measure μ_ϕ. All these problems are basically open, and it would be interesting even to show that δ is singular with respect to the 2-dimensional Lebesgue measure, thus showing that δ is a Cantor-type measure (according to the terminology introduced in [DeGA], [A] for BV functions). We prove that δ is identically 0 by making an additional mild regularity assumption on Σ, namely $\mathcal{H}^1(\overline{\Sigma} \cap \Omega \setminus \Sigma) = 0$, see Theorem 6.4 whose proof is based on the results in [ALR].

As explained in the paper the uniqueness of the tangent jump measure while dilating at a point where the 1-upper density of the jump measure is nonzerois strongly related to the uniqueness result established in [ALR].

It is likey that this analysis can be extended to scalar first order conservation laws with strictly convex non-linearities, where the classical Oleinik uniqueness result plays the role of our uniqueness result in [ALR]. Precisely, given a solution ϕ on $\mathbf{R} \times \mathbf{R}^+$ of

$$\frac{\partial \phi}{\partial t} + \frac{\partial (A \circ \phi)}{\partial x} = 0$$

for $A'' > 0$ and assuming that, for any $S \in \mathrm{Lip}(\mathbf{R})$, one has that

$$m = \frac{\partial (S \circ \phi)}{\partial t} + \frac{\partial (Q \circ \phi)}{\partial x} \in \mathcal{M}_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+),$$

where $S'A' = Q'$ and where $\mathcal{M}_{loc}(\mathbf{R} \times \mathbf{R}^+)$ denotes the the distributions which are Radon measures in $\mathbf{R} \times \mathbf{R}^+$, then we expect a similar structure theorem to be true for the measure m.

Now we briefly describe the contents and the techniques used in this paper. Section 2 contains some basic material about BV functions, approximate continuity, approximate jumps. The main result is Proposition 2.3, where we find a necessary and sufficient for a lifting ϕ to be a function of bounded variation.

Section 3 contains the main basic properties of the space $\mathcal{M}_{\mathrm{div}}$. In particular we show the identity (1.7) and, as a consequence, the absolute continuity of μ_ϕ with respect to $\mathcal{H}^1$.

In Section 4 we study some properties of concave functions whose gradient satisfies the eikonal equations. These properties are used in the last section of the paper for the classification of blow-ups.

Section 5 is devoted to some abstract criteria for the rectifiability of sets and measures in the plane. We use a classical blow-up technique (see [Pr] for much more on the subject), studying the asymptotic behaviour of the rescaled and renormalized measures around a point. The renormalization factor we use is simply the radius of the ball (see Definition 5.1). The new observation here is that very weak informations about the structure of blow-ups allow to show that points where the upper 1-dimensional spherical density is positive are indeed points where the *lower* 1-dimensional spherical density is positive, see Theorem 5.2. In our problem, this information is used to show that $\mu_\phi \llcorner (\Omega \setminus S_\phi)$ has zero 1-dimensional density, and therefore is orthogonal with respect to $\mathcal{H}^1$.

Section 6 is devoted to the classification of blow-ups. Here we use the idea that any vector-valued measure becomes, after blow-up, a constant multiple of a positive measure at a.e. blow-up point. This idea was first used by E. De Giorgi to classify blow-ups of sets of finite perimeter (which turn out to be halfspaces) in his fundamental work [DeG] on the rectifiability of the reduced boundary of sets of finite perimeter. Here this idea is pushed further, considering the measures

$$\int_{\mathbf{R}} e^{i\,a} \mathrm{div}\, e^{i\,\phi\wedge a}\, da, \qquad \int_{\mathbf{R}} g(a) \mathrm{div}\, e^{i\,\phi\wedge a}\, da,$$

all absolutely continuous with respect to μ_ϕ, and blowing up at Lebesgue points of all the respective densities. We show in this way that any blow-up is either constant, or jumps on a line, or jumps on a halfline, with a uniform (i.e. independent of the chosen subsequence) lower bound on the width of the jump. This suffices to apply the results of the previous sections, and to infer rectifiability.

While completing this work we learned that C. De Lellis and F. Otto independently established in [DO] a structure theorem similar to Theorem 1.1 for the Aviles-Giga space. Their proof, still based on a blow-up argument, is more elaborate, since in the case of the Aviles-Giga space the class of blow-ups is a priori richer. It is also interesting to notice that no connection with the theory of viscosity solutions is used in their paper.

We close this introduction with the following table, summarizing the notation used without further explaination in the paper.

Ω	A bounded open set in $\mathbf{R}^2$
$a \wedge b$	The minimum of a and b
$a \vee b$	The maximum of a and b
$v \cdot w$	The scalar product of v and w
$\widehat{(v,w)}$	The angle $\theta \in [0,\pi]$ such that $v \cdot w = \vert v\vert\vert w\vert \cos\theta$
$v^\perp$	The anti-clockwise $\pi/2$ rotation of v, $(-v_2, v_1)$
$e^{i\,a}$	The vector $(\cos a, \sin a)$
$B_r(x)$	The ball with centre x and radius r ($x = 0$ can be omitted)
$\mathcal{H}^1$	Hausdorff 1-dimensional measure in $\mathbf{R}^2$
$\mathbf{S}^1$	Unit sphere in $\mathbf{R}^2$
$\mathcal{M}(X)$	Finite Radon measures in X
$\mathcal{M}_+(X)$	Positive and finite Radon measures in X
$\mu \llcorner B$	Restriction of μ to B, defined by $\chi_B \mu$.

2 Continuity points, jump points, BV functions

Let us introduce some weak notions of continuity and jump, well studied in the context of BV functions. All of them have a local nature and, to fix the ideas, we give the definitions for some function $f \in L^1_{\mathrm{loc}}(\mathbf{R}^2, \mathbf{R}^m)$.

- (**Approximate limit**) We say that f has an approximate limit at x if there exists $a \in \mathbf{R}^m$ such that

$$\lim_{r\to 0^+} \frac{1}{\pi r^2} \int_{B_r(x)} |f(y) - a| dy = 0.$$

The vector a whenever exists is unique and is called the approximate limit of f at x. We denote by S_f the set of points where f has no approximate limit.

- (**Approximate jump points**) We say that x is a jump point of f if there exist a^+, $a^- \in \mathbf{R}^m$ and $\nu_x \in \mathbf{S}^1$ such that $a^+ \neq a^-$ and

$$\lim_{r\to 0^+} \frac{1}{\pi r^2} \int_{B_r^\pm(x)} |f(y) - a^\pm| \, dy = 0,$$

where $B_r^\pm(x) = \{y \in B_r(x) : \pm(y - x) \cdot \nu_x > 0\}$ are the two half balls determined by ν_x. The triple (a^+, a^-, ν_x) is uniquely determined up to a change of orientation of ν_x and a permutation of (a^+, a^-). We denote by J_f the set of jump points of f.

It is not hard to show (see [AFP]) that S_f, J_f are Borel sets, that $J_f \subset S_f$, and that S_f is Lebesgue negligible.

The following Lemma has been proved in [A1] in a more general context. For the sake of completeness we include the proof.

Lemma 2.1. *Let (χ_l) be a family of continuous functions defined on $\mathbf{R}$ which separates points. Let $\phi \in L^\infty(\mathbf{R}^2)$ and set $\phi_l := \chi_l \circ \phi$. Then the following implications hold:*

(i) ϕ has an approximate limit at x if and only if all functions ϕ_l have an approximate limit at x;

(ii) If x is either an approximate continuity point or a jump point for all functions ϕ_l, with the same normal to the jump, then the same is true for ϕ.

Proof. (i) We prove only the nontrivial implication, the "if" one. Let us set $X := [-\|\phi\|_\infty, \|\phi\|_\infty]$. By the Stone-Weierstrass theorem the algebra $\mathcal{A}$ generated by the family $(\chi_l)_{l\in\mathbb{N}}$ is dense in the set of continuous function of X, $C(X)$, endowed with the sup norm. If $\chi_l \circ \phi$ has an approximate limit at x for any l we infer that $f \circ \phi$ has an approximate limit at x for any $f \in \mathcal{A}$. Since $\mathcal{A}$ is dense in $C(X)$, the identity function is the uniform limit of a sequence of functions of $\mathcal{A}$, so that ϕ has an approximate limit at x.

(ii) The proof is similar, working in the two halfspaces determined by the common normal to the jumps. $\square$

Remark 2.2. Concerning statement (ii), notice that if we assume in addition that x is a jump point for at least one of the functions ϕ_l, then x must be a jump point of ϕ, by (i).

We are going to apply this result with $\phi_l(x) = (x \vee b_l) \wedge c_l$, where (b_l, c_l) is a family of open intervals. It is easy to check that the family (ϕ_l) separates points if and only if the closed set $\mathbf{R} \setminus \cup_l (b_l, c_l)$ has an empty interior.

We recall also some basic facts about BV functions which will be used throughout the paper. We say that $u \in L^1(\Omega, \mathbf{R}^m)$ is a BV (bounded variation) function, and we write $u \in BV(\Omega, \mathbf{R}^m)$ ($\mathbf{R}^1$ can be omitted), if its distributional derivatives $D_i u$, i.e.

$$\langle D_i u; \psi \rangle := -\int_\Omega \frac{\partial \psi}{\partial x_i} u \, dx \qquad \psi \in C_c^\infty(\Omega), \ i = 1, 2$$

are representable by finite $\mathbf{R}^m$-valued Radon measures in Ω. We denote by $|Du|(\Omega)$ the total variation of the $\mathbf{R}^{2m}$-valued measure $Du = (D_1 u, D_2 u)$. When $u \in W^{1,1}(\Omega; \mathbf{R}^m)$ we have $Du = \nabla u \mathcal{L}^2$ and therefore

$$|Du|(\Omega) = \int_\Omega |\nabla u| \, dx.$$

We recall that the jump set of a BV function u is countably $\mathcal{H}^1$-rectifiable and that

$$\int_{J_u} |u^+ - u^-| \, d\mathcal{H}^1 \leq |Du|(\Omega). \tag{2.1}$$

Moreover, $\mathcal{H}^1$-a.e. any approximate discontinuity point is a jump point, i.e.

$$\mathcal{H}^1(S_u \setminus J_u) = 0. \tag{2.2}$$

Now we investigate under which conditions a lifting of a function $u \in BV(\Omega, \mathbf{S}^1)$ is itself a BV function.

Proposition 2.3. *Let $\phi \in L^\infty(\Omega)$ be such that*

(i) $u := e^{i\phi} \in BV(\Omega, \mathbf{S}^1)$*;*

(ii) $U_\phi := \operatorname{div} e^{i\phi \wedge a} \in \mathcal{M}(\Omega \times \mathbf{R})$*.*

Then $\phi \in BV(\Omega)$ and

$$|D\phi|(\Omega) \leq C \left[|U_\phi|(\Omega \times \mathbf{R}) + |Du|(\Omega) \right]$$

for some constant C.

Proof. Let $\phi_0 \in BV(\Omega)$ be given by Lemma 2.4 below, satisfying $e^{i\,\phi_0} = e^{i\,\phi}$. Then there exists a unique $k \in L^\infty(\Omega, \mathbf{Z})$ such that $\phi = \phi_0 + 2\pi k$. The goal is to show that $k \in BV(\Omega, \mathbf{Z})$.

It is clear, since $\phi_0 \in BV(\Omega) \cap L^\infty$, that $\operatorname{div}(e^{i\,a\wedge\phi_0}) \in \mathcal{M}(\Omega \times \mathbf{R})$. Therefore we can deduce that

$$\left| \int_\Omega \int_{\mathbf{R}} \cos a \;\left(e^{i\,a\wedge\phi} - e^{i\,a\wedge\phi_0} \right) \cdot \nabla\psi \; da\, dx \right| \le C \|\psi\|_\infty \tag{2.3}$$

for any $\psi \in C_c^\infty(\Omega)$, with $C = |U_{\phi_0}|(\Omega \times \mathbf{R}) + |U_\phi|(\Omega \times \mathbf{R})$. Notice that $|U_{\phi_0}|(\Omega \times \mathbf{R})$ can be estimated (see (3.5)) with $|D\phi_0|(\Omega)$ and this, in turn, can be estimated with $|Du|(\Omega)$.

We observe that $e^{i\,a\wedge\phi} = e^{i\,a\wedge(\phi_0+2\pi k)} = e^{i\,(a-2\pi k)\wedge\phi_0}$. Fixing $x \in \Omega$ and assuming $k(x) > 0$ to fix the ideas, we deduce from the remark above that

$$\begin{aligned}
&\int_{\mathbf{R}} \cos\, a \;\left(e^{i\,a\wedge\phi(x)} - e^{i\,a\wedge\phi_0(x)} \right)\; da = \int_{\phi_0(x)}^{\phi_0(x)+2\pi k(x)} \cos a (e^{i\,a} - e^{i\,\phi_0(x)})\, da \\
&= e^{i\,\phi_0(x)} \int_0^{2\pi k(x)} \cos(b + \phi_0(x)) \;\left(e^{i\,b} - 1 \right)\; db \\
&= e^{i\,\phi_0(x)} \;(\pi k(x)\; \cos\phi_0(x) - i\pi k(x)\; \sin\phi_0(x)) \\
&= \pi k(x) \;\begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Combining this fact with (2.3) we have proved that

$$\left| \int_\Omega k \;\frac{\partial\psi}{\partial x_1} \right| \le C \;\|\psi\|_\infty \qquad \forall\psi \in C_c^\infty(\Omega).$$

This shows that $D_1 k$ is a finite Radon measure in Ω. A similar argument (replacing $\cos a$ by $\sin a$ in (2.3)) works for $D_2 k$. $\square$

In the proof above we used the following lemma, which ensures the existence of a BV lifting.

Lemma 2.4 (BV lifting). *Let $u \in BV(\Omega, \mathbf{R}^2)$ such that $|u| = 1$ almost everywhere in Ω. Then there exists $\phi_0 \in BV(\Omega, [-2\pi, 2\pi])$ verifying*

(i) $u = e^{i\,\phi_0}$ *a.e. in* Ω;

(ii) $|D\phi_0|(\Omega) \le C_0 |Du|(\Omega)$, *where* C_0 *is an absolute constant.*

Proof. Let ξ_0 be a smooth function from $\mathbf{R}^2$ into $[-\frac{\pi}{2}, +\frac{\pi}{2}]$ such that for any $z = (x_1, x_2)$ in $\mathbf{S}^1$ verifying $x_1 \geq 0$, $\xi_0(z)$ is the angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $e^{i\,\xi_0(z)} = z$. Similarly we introduce ξ_π to be a smooth map from $\mathbf{R}^2$ into $[0, 2\pi]$ such that for any $z = (x_1, x_2) \in \mathbf{S}^1$ verifying $x_1 \leq \frac{3}{4}$, $\xi_\pi(z)$ is the angle in $[0, 2\pi]$ such that $e^{i\,\xi_\pi(z)} = z$.

Since $u = (u_1, u_2)$ is in $BV(\Omega, \mathbf{R}^2)$, by the mean value theorem and the coarea formula in BV we may find $\alpha \in [\frac{1}{4}, \frac{1}{2}]$ such that

$$|D\chi_{\{x:\ u_1(x) \geq \alpha\}}|(\Omega) \leq 4|Du_1|(\Omega),$$

thus $E = \{x \in \Omega :\ u_1(x) \geq \alpha\}$ is a finite perimeter set. By virtue of the Volpert's chain rule (see for instance [AFP], Theorem 3.96), we have that both $\xi_0 \circ u$ and $\xi_\pi \circ u$ are in $L^\infty \cap BV(\Omega)$ and their total variations can be estimated with $|Du|(\Omega)$. Using now the decomposability theorem ([AFP], Theorem 3.84), we have that

$$\phi_0 := \chi_E\, \xi_0 \circ u + \chi_{\Omega \setminus E}\, \xi_\pi \circ u$$

is in $BV(\Omega)$ and

$$|D\phi_0|(\Omega) \leq [||\xi_0||_\infty + ||\xi_1||_\infty]\, |D\chi_E|(\Omega) + |D(\xi_0 \circ u)|(\Omega) + |D(\xi_1 \circ u)|(\Omega).$$

By construction we have $e^{i\,\phi} = u$ a.e. in Ω and ϕ is a solution of our problem. $\square$

3 The space $\mathcal{M}_{\mathrm{div}}(\Omega)$

In this section we introduce the main object of study of the present paper.

Definition 3.1. We denote by $\mathcal{M}_{\mathrm{div}}(\Omega)$ the space of two-dimensional vector fields u in $L^1(\Omega, \mathbf{S}^1)$ satisfying

(P1) $\operatorname{div} u = 0$ in $\mathcal{D}'(\Omega)$;

(P2) there exists a lifting $\phi \in L^\infty(\Omega)$, i.e. a map ϕ satisfying $u = e^{i\,\phi}$, such that the distribution U_ϕ in $\mathcal{D}'(\Omega \times \mathbf{R})$ defined by

$$\langle U_\phi; \psi(x, a)\rangle := -\int_{\mathbf{R}} \int_\Omega e^{i\,\phi(x) \wedge a} \cdot \nabla_x \psi(x, a)\, dx da$$

is a finite Radon measure in $\Omega \times \mathbf{R}$.

For $a \in \mathbf{R}$ we set $T^a u := e^{i\phi \wedge a} \in L^1(\Omega, \mathbf{S}^1)$ (this is a slight abuse of notation, since $T^a u$ depends on the lifting and not only on u, but it is justified by the fact that in the following the lifting of u will be kept fixed), so that

$$\langle U_\phi; \psi\rangle := \int_{\mathbf{R}} \langle \operatorname{div} T^a u; \psi(\cdot, a)\rangle\, da. \qquad \forall \psi \in C_c^\infty(\Omega \times \mathbf{R}).$$

Notice that, since $\phi \in L^\infty(\Omega)$, then (P1) implies that $\operatorname{div} T^a u = 0$ for all $a \in \mathbf{R}$ such that $|a| > \|\phi\|_\infty$. Finally, we denote by μ_ϕ the projection of $|U_\phi|$ on the first variable, i.e.

$$\mu_\phi(B) := |U_\phi|(B \times \mathbf{R}) \qquad \text{for any } B \subset \Omega \text{ Borel.}$$

In the following theorem we state some basic properties of the truncated vector fields $T^a u$ and a useful representation formula for U_ϕ.

Theorem 3.2. *Let $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$. Then, the following properties hold:*

(i) The map $a \mapsto \operatorname{div} T^a u$ satisfies the Lipschitz condition

$$\left|\langle \operatorname{div} T^a u; \psi\rangle - \langle \operatorname{div} T^b u; \psi\rangle\right| \le \mathcal{L}^n(\Omega)\|\nabla\psi\|_\infty |b-a| \qquad \forall \psi \in C_c^\infty(\Omega). \quad (3.1)$$

(ii) $\mu_\phi(B) = \int_{\mathbf{R}} |\operatorname{div} T^a u|(B)\, da$ for any Borel set $B \subset \Omega$. In particular $\operatorname{div} T^a u$ is a finite Radon measure in Ω for a.e. $a \in \mathbf{R}$.

(iii) For a.e. $a \in \mathbf{R}$ we have

$$\frac{1}{2\delta}\int_{a-\delta}^{a+\delta} \operatorname{div} T^b u\, db \underset{\delta \to 0^+}{\longrightarrow} \operatorname{div} T^a u \quad \textit{in } \mathcal{M}'(\Omega).$$

Proof. (i) Follows by the elementary inequality $|T^a u - T^b u| \le |b-a|$.

(ii) For any $\psi(x,a) = f(x)g(a)$, with $f \in C_c^\infty(\Omega)$ and $g \in C_c^\infty(\mathbf{R})$ we have

$$\langle U_\phi; \psi(x,a)\rangle = -\int_{\mathbf{R}} g(a) \int_\Omega T^a u \cdot \nabla f(x)\, dx da.$$

By approximation, the same identity holds if g is a bounded Borel function with compact support. Now, choosing an open set $A \subset \Omega$ and $f \in C_c^\infty(A)$ with $\|f\|_\infty \le 1$ and $g = \chi_{(a,a+\delta)}$ we get

$$|U_\phi|\,(A \times (a, a+\delta]) \ge -\int_a^{a+\delta} \int_\Omega T^b u \cdot \nabla f(x)\, dx db,$$

so that

$$\frac{d}{da}|U_\phi|\,(A \times (-\infty, a]) \ge -\int_\Omega T^a u \cdot \nabla f(x)\, dx \quad \forall a \in \mathbf{R}.$$

Being f arbitrary, this gives that $\operatorname{div} T^a u$ is a finite Radon measure in A and

$$\frac{d}{da}|U_\phi|\,(A \times (-\infty, a]) \ge |\operatorname{div} T^a u|(A)$$

for a.e. $a \in \mathbf{R}$. By integration it follows that

$$|U_\phi|(A \times \mathbf{R}) \geq \int_{\mathbf{R}} |\mathrm{div}\, T^a u|(A)\, da \tag{3.2}$$

for any open set $A \subset \Omega$. On the other hand, the inequality

$$|U_\phi|(\Omega \times \mathbf{R}) \leq \int_{\mathbf{R}} |\mathrm{div}\, T^a u|(\Omega)\, da. \tag{3.3}$$

is easy to prove, using the definition of U_ϕ. From (3.2) and (3.3) we obtain the coincidence of the measures μ_ϕ and $\int |\mathrm{div}\, T^a u|\, da$.

The property (iii) is an easy consequences of (ii) and of the Lipschitz property (3.1): it suffices to choose Lebesgue points of the integrable function $a \mapsto |\mathrm{div}\, T^a u|(\Omega)$. □

The following covering technical lemma will be used to show the absolute continuity of μ_ϕ with respect to $\mathcal{H}^1$.

Lemma 3.3. *Let K be a compact set of Ω. Then, there exists a sequence $(\psi_n) \subset C_c^\infty(\Omega, [0,1])$ such that:*

(i) $\psi_n = 1$ on K and spt $\psi_n \to K$ as $n \to \infty$;

(ii) $\displaystyle\limsup_{n\to+\infty} \int_\Omega |\nabla \psi_n|\, dx \leq \pi \mathcal{H}^1(K)$.

Proof. Let $L = \mathcal{H}^1(K)$. By the definition of Hausdorff measure, for any $n \geq 1$ we can find a finite number of balls $B_i = B(x_i, r_i)$ whose union covers K and such that $r_i < 1/n$ and $\sum_i 2r_i < L + 1/n$. By the subadditivity of perimeter, the open set $A_n := \cup_i B_i$ has perimeter less than $\pi L + \pi/n$. Then, we set $\psi_n = \chi_{A_n} * \rho_{\epsilon_n}$, where $\epsilon_n < 1/n$ is chosen so small that still $\psi_n = 1$ on K (it suffices that $\epsilon_n < \mathrm{dist}(K, \partial A_n)$) and the support of ψ_n is compact. Since the total variation does not increase under convolution (see for instance Proposition 3.2(c) of [AFP]) we have

$$\int_\Omega |\nabla \psi_n|\, dx = |D\psi_n|(\Omega) \leq |D\chi_{A_n}|(\Omega) \leq \pi L + \frac{\pi}{n}$$

and therefore ψ_n has all the stated properties. □

Theorem 3.4 (Absolute continuity). *The measure μ_ϕ is absolutely continuous with respect to $\mathcal{H}^1$, i.e. $\mu(B) = 0$ whenever B is a Borel $\mathcal{H}^1$-negligible set.*

Proof. By the inner regularity of μ_ϕ it suffices to show that there exists $C > 0$ such that, for all compact sets $K \subset \Omega$, $\mu_\phi(K) \leq C\mathcal{H}^1(K)$. We will prove that, for all $a \in \mathbf{R}$ such that $\operatorname{div} T^a u$ is a Radon measure on Ω, the inequality $|\operatorname{div} T^a u|(K) \leq 2\pi\mathcal{H}^1(K)$ holds for any compact set $K \subset \Omega$. Then, since $\operatorname{div} T^a u = 0$ as soon as $|a| > \|\phi\|_\infty$, by Theorem 3.2(iii) we obtain

$$\mu_\phi(K) = \int_{\mathbf{R}} |\operatorname{div} T^a u|(K)\, da \leq 2\pi\|\phi\|_\infty \mathcal{H}^1(K).$$

Let $a \in \mathbf{R}$ be such that $\nu := \operatorname{div} T^a u$ is a finite Radon measure on Ω. By the Hahn decomposition theorem, there exists two disjoint Borel sets A^+, A^- such that, if ν^+ and ν^- denote respectively the positive and negative parts of ν, then $\nu^\pm = \pm\nu \llcorner A^\pm$. Since $|\nu| = \nu^+ + \nu^-$, it suffices to prove that $\nu^+(K) \leq \pi\mathcal{H}^1(K)$ for any $K \subset A^+$ compact and $\nu^-(K) \leq \pi\mathcal{H}^1(K)$ for any $K \subset A^-$ compact.

Let $K \subset A^+$ be compact and let $(\psi_n) \subset C_c^\infty(\Omega, [0,1])$ be given by Lemma 3.3. We have

$$\begin{aligned}
\nu^+(K) &= \nu(K) \leq \lim_{n\to\infty} \int_\Omega \psi_n \, d\nu = -\lim_{n\to\infty} \int_\Omega T^a u \cdot \nabla\psi_n \, dx \\
&\leq \limsup_{n\to\infty} \int_\Omega |\nabla\psi_n| \, dx \leq \pi\mathcal{H}^1(K).
\end{aligned}$$

A similar argument works for compact sets $K \subset A^-$. $\square$

In the case when $\phi \in BV_{\mathrm{loc}}(\Omega)$ one can use Volpert's chain rule in BV to obtain an explicit formula for $\operatorname{div} T^a u$, see [RS2]: it turns out that

$$\operatorname{div} T^a u = \chi(a, \phi^+, \phi^-)(e^{i\,a} - e^{i\,\phi^- \wedge \phi^+}) \cdot \nu_\phi \, \mathcal{H}^1 \llcorner J_\phi, \tag{3.4}$$

where

$$\chi(a, \phi^+, \phi^-) := \begin{cases} \;\;1 & \text{if } \phi^- < a < \phi^+ \\ -1 & \text{if } \phi^+ < a < \phi^- \\ \;\;0 & \text{else.} \end{cases}$$

Moreover, the divergence free condition gives $e^{i\,\phi^+} \cdot \nu_\phi = e^{i\,\phi^-} \cdot \nu_\phi$ at any point in J_ϕ. In particular, choosing ν_ϕ in such a way that $\phi^+ > \phi^-$, Fubini theorem and (2.1) give

$$\begin{aligned}
|U_\phi|(\Omega \times \mathbf{R}) &= \int_{\mathbf{R}} |\operatorname{div} T^a u|(\Omega)\, da = \int_{\mathbf{R}} \int_{J_\phi} \chi_{\{\phi^- \leq a \leq \phi^+\}} |e^{i\,a} - e^{i\,\phi^-}| \, d\mathcal{H}^1 da \\
&\leq \int_{J_\phi} |\phi^+ - \phi^-| (2 \wedge \frac{1}{2}|\phi^+ - \phi^-|) \, d\mathcal{H}^1 \leq 2|D\phi|(\Omega). \qquad (3.5)
\end{aligned}$$

The following lemma provides an integral representation of the divergence, assuming rectifiability of the measure and existence of jumps.

Lemma 3.5. *Let $u \in L^\infty(\mathbf{R}^2, \mathbf{R}^2)$ and let $K \subset \mathbf{R}^2$ be countably $\mathcal{H}^1$-rectifiable. If* $\operatorname{div} u$ *is a Radon measure in $\mathbf{R}^2$ and $\mathcal{H}^1(K \cap S_u \setminus J_u) = 0$, then*

$$\operatorname{div} u \llcorner K = (u^+ - u^-) \cdot \nu \mathcal{H}^1 \llcorner K \cap J_u.$$

Proof. Arguing as in Theorem 3.4 and using Lemma 3.3 one can easily show that $\operatorname{div} u << \mathcal{H}^1$, hence $\operatorname{div} u \llcorner K$ is representable by $\theta \mathcal{H}^1 \llcorner K$ for some density function θ. The function θ can be characterized by a blow-up argument, using the fact that K becomes a line (here the rectifiability of K plays a role) after blow-up and u becomes a jump function or a constant function at $\mathcal{H}^1$-a.e. blow-up point of K. □

The following compactness result has been proved in [Le] adapting the truncation argument of [RS2].

Theorem 3.6 (Compactness). *For any constant $M \geq 0$ the set*

$$\{\phi \in L^\infty(\Omega) : \ \|\phi\|_\infty + |U_\phi|(\Omega \times \mathbf{R}) \leq M\}$$

is compact in $L^1(\Omega)$ with respect to the strong topology.

4 Some properties of concave functions

In this section we study some properties of concave functions g whose gradient satisfies the eikonal equation. We recall that the superdifferential $\partial g(x)$ of g at x is the closed convex set defined by

$$\partial g(x) := \left\{ p \in \mathbf{R}^2 : \ g(y) \leq g(x) + p \cdot (y - x) \ \ \forall y \in \mathbf{R}^2 \right\}.$$

It follows immediately from the definition that the graph of ∂g, i.e. $\{(x, p) : \ p \in \partial g(x)\}$ is a closed subset of $\mathbf{R}^2 \times \mathbf{R}^2$. Moreover, the Lipschitz assumption on g gives $\partial g(x) \subset \overline{B}_1$ for any x. Finally, $\partial g(x) = \{\nabla g(x)\}$ at any differentiability point of g.

For any $\omega \in \mathbf{S}^1$ and any $x \in \mathbf{R}^2$, the left and right directional derivative along ω of g at x are defined by

$$\nabla^\pm_\omega g(x) := \lim_{r \to 0^\pm} \frac{g(x + r\omega) - g(x)}{r}.$$

For any $x \in J_{\nabla g}$ we denote in the following by $(\nabla g^+, \nabla g^-, \nu_x)$ the triple defined in Section 2.

Proposition 4.1. *Let $g : \mathbf{R}^2 \to \mathbf{R}$ be concave and satisfying $|\nabla g| = 1$ a.e. in $\mathbf{R}^2$. Then, g satisfies the following properties:*

(i) If ∇g has an approximate limit at x, then g is differentiable at x. Moreover, setting $D_x := \{x + t\nabla g(x) : \ t < 0\}$, for $\mathcal{H}^1$-a.e. $y \in D_x$, ∇g has an approximate limit at y equal to $\nabla g(x)$.

(ii) Let J be the set of approximate jump points of ∇g and let $x \in J$. For any $\omega \in \mathbf{S}^1$ such that $\omega \cdot \nu_x > 0$, the partial derivatives $\nabla^{\pm}_{\omega} g(x)$ exist and

$$\nabla^{-}_{\omega} g(x) = \omega \cdot \nabla g^{-}(x) \geq \omega \cdot \nabla g^{+}(x) = \nabla^{+}_{\omega} g(x). \tag{4.1}$$

Moreover, setting $D^{\pm}_x := \{x + t\nabla g^{\pm}(x) : \ t < 0\}$, for $\mathcal{H}^1$-a.e. $y \in D^{\pm}_x$, ∇g has an approximate limit equal to $\nabla g^{\pm}(x)$.

(iii) For all $\alpha > 0$, we define the following sets

$$J_\alpha := \{x \in J : \ |\nabla g^{+}(x) - \nabla g^{-}(x)| \geq \alpha\}$$

$$\Sigma_\alpha := \{x \in \mathbb{R}^2 : \ \mathrm{diam}\,(\partial g(x)) \geq \alpha\}.$$

Then, $J_\alpha \subset \Sigma_\alpha$ and Σ_α is closed.

Proof. The first two statements can be proved in the same way and we prove only the second. By the definition of J there exist $\nu_x \in \mathbf{S}^1$ and $\nabla g^{+}(x)$, $\nabla g^{-}(x) \in \mathbf{S}^1$ such that

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}^2(B^{\pm}_r(x))} \int_{B^{\pm}_r(x)} |\nabla g(y) - \nabla g^{\pm}(x)| dy = 0. \tag{4.2}$$

For $r > 0$, let us define

$$g_r(y) := \frac{g(x + ry) - g(x)}{r}, \qquad y \in \overline{B}_1.$$

Then, $\nabla g_r(y) = \nabla g(x + ry)$. By (4.2), ∇g_r converge in $L^1(B_1)$ when $r \to 0^+$ to the function

$$G_0(y) := \begin{cases} \nabla g^{+}(x) & \text{if } y \cdot \nu_x > 0 \\ \nabla g^{-}(x) & \text{if } y \cdot \nu_x < 0. \end{cases}$$

By Sobolev embedding, this implies that (g_r) uniformly converges in $\overline{B}_1$ to a 1-Lipschitz function g_0 satisfying $\nabla g_0 = G_0$. Since $g_r(0) = 0$ we have that $g_0(0) = 0$ and therefore g_0 is uniquely determined:

$$g_0(y) := \begin{cases} y \cdot \nabla g^{+}(x) & \text{if } y \cdot \nu_x \geq 0 \\ y \cdot \nabla g^{-}(x) & \text{if } y \cdot \nu_x \leq 0 \end{cases}$$

But, for any $\omega \in \mathbf{S}^1$, we have

$$\nabla^{+}_{\omega} g(x) = \lim_{r \to 0^+} \frac{g(x + r\omega) - g(x)}{r} = \lim_{r \to 0^+} g_r(\omega) = g_0(\omega)$$

and

$$\nabla^-_\omega g(x) = \lim_{r\to 0^+} \frac{g(x-r\omega)-g(x)}{-r} = \lim_{r\to 0^+} -g_r(-\omega) = -g_0(-\omega).$$

Therefore, if we assume that $\omega \cdot \nu_x > 0$, then

$$\nabla^-_\omega g(x) = \omega \cdot \nabla g^-(x) \quad \text{and} \quad \nabla^+_\omega g(x) = \omega \cdot \nabla g^+(x).$$

Moreover, we have $\nabla^-_\omega g(x) \geq \nabla^+_\omega g(x)$ since the restriction of g to $\mathbf{R}\omega$ is concave.

Let us now prove the second part. Let $x \in J$ and let $y \in D^-_x$. Since the restriction of g to D^-_x is concave, we have

$$\nabla^-_{\nabla g^-(x)} g(y) \geq \nabla^+_{\nabla g^-(x)} g(y) \geq \nabla^-_{\nabla g^-(x)} g(x) = \nabla g^-(x) \cdot \nabla g^-(x) = 1.$$

Since g is 1-Lipschitz we obtain that

$$\nabla^-_{\nabla g^-(x)} g(y) = \nabla^+_{\nabla g^-(x)} g(y) = 1. \tag{4.3}$$

By (2.2), for $\mathcal{H}^1$-a.e. $y \in \mathbf{R}^2$ either ∇g has an approximate limit at y, or y is an approximate jump point of ∇g. If $y \in D^-_x$, y can't be a jump point of ∇g. Indeed, assuming that $y \in J$ and applying (4.1) with $\omega = \nabla g^-(x)$, we have

$$\nabla^-_{\nabla g^-(x)} g(y) = \nabla g^-(y) \cdot \nabla g^-(x) \geq \nabla g^+(y) \cdot \nabla g^-(x) = \nabla^+_{\nabla g^-(x)} g(y).$$

By (4.3), $\nabla g^-(y) \cdot \nabla g^-(x) = \nabla^+ g(y) \cdot \nabla g^-(x) = 1$. Thus, $\nabla g^-(y) = \nabla g^+(y) = \nabla g^-(x)$, which contradicts the assumption $y \in J$. Therefore, ∇g has an approximate limit equal to $\nabla g^-(x)$ at $\mathcal{H}^1$-a.e. $y \in D^-_x$. The same argument can be used for D^+_x and (ii) is proved.

(iii) First, let us show that $J_\alpha \subset \Sigma_\alpha$. Indeed, since g is differentiable a.e., for any $x \in J_\alpha$ we can find differentiability points $x^\pm_h$ converging to x such that $\nabla g(x^\pm_h)$ converge to $\nabla g^\pm$, hence the closednedd os the graph of ∂g gives that $\nabla g^+(x)$ and $\nabla g^-(x)$ are in $\partial g(x)$. Thus, $\operatorname{diam} \partial g(x) \geq \alpha$ and $x \in \Sigma_\alpha$.

The closedness of Σ_α is an immediate consequence of a compactness argument based on the closedness of the graph of ∂g and on the fact that $\partial g(x) \subset \overline{B}_1$ for any x. □

5 Rectifiability of 1-dimensional measures in the plane

In this section we consider a measure $\mu \in \mathcal{M}_+(\Omega)$ absolutely continuous with respect to $\mathcal{H}^1$, i.e. vanishing on any $\mathcal{H}^1$-negligible set. We define

$$\Theta_*(\mu, x) := \liminf_{r\to 0^+} \frac{\mu(B_r(x))}{r}, \qquad \Theta^*(\mu, x) := \limsup_{r\to 0^+} \frac{\mu(B_r(x))}{r}. \tag{5.1}$$

A general property is that $\Theta^*(\mu,x)$ is finite for $\mathcal{H}^1$-a.e. x (see for instance [AFP]) hence the absolute continuity assumption gives that $\Theta^*(\mu,x)$ is finite for μ-a.e. x. We define also

$$\Sigma_\mu^+ := \{x\in\Omega:\ \Theta^*(\mu,x)>0\}\,, \qquad \Sigma_\mu^- := \{x\in\Omega:\ \Theta_*(\mu,x)>0\} \tag{5.2}$$

and notice that $\Sigma_\mu^\pm$ are Borel sets and $\Sigma_\mu^-\subset\Sigma_\mu^+$. Notice also that Σ_μ^+ is σ-finite with respect to $\mathcal{H}^1$, as all the sets

$$\Sigma_\alpha := \{x\in\Omega:\ \Theta^*(\mu,x)\geq\alpha\} \tag{5.3}$$

satisfy $\mathcal{H}^1(\Sigma_\alpha)\leq 2\mu(\Omega)/\alpha$ (see [AFP], Theorem 2.56). Therefore, by the Radon–Nikodým theorem, we can represent

$$\mu = \mu\llcorner\Sigma_\mu^+ + \mu\llcorner(\Omega\setminus\Sigma_\mu^+) = f\mathcal{H}^1\llcorner\Sigma_\mu^+ + \mu\llcorner(\Omega\setminus\Sigma_\mu^+) \tag{5.4}$$

for some $f\in L^1(\mathcal{H}^1\llcorner\Sigma_\mu^+)$. Notice that the residual part $\mu^r := \mu\llcorner(\Omega\setminus\Sigma_\mu^+)$ is "orthogonal" to $\mathcal{H}^1$ in the following sense:

$$\mathcal{H}^1(B)<+\infty \quad\Longrightarrow\quad \mu^r(B)=0.$$

This is a consequence of the fact that $\Theta^*(\mu^r,x)$ is 0 for μ^r-a.e. x.

The following definition is a particular case of the general one given in the fundamental paper [Pr].

Definition 5.1 (Tangent space to μ). Given $x\in\Omega$ and $r>0$, we define the rescaled measures $\mu_{x,r}\in\mathcal{M}\left((\Omega-x)/r\right)$ by

$$\mu_{x,r}(B) := \frac{\mu(x+rB)}{r}$$

for any Borel set $B\subset(\Omega-x)/r$, so that

$$\int\phi(y)\,d\mu_{x,r}(y) = \frac{1}{r}\int\phi(\frac{y-x}{r})\,d\mu(y) \qquad \forall\phi\in C_c\left((\Omega-x)/r\right).$$

We denote by $\mathrm{Tan}(\mu,x)$ the collection of all limit points as $r\to 0^+$ of $\mu_{x,r}$, in the duality with $C_c(\mathbf{R}^2)$.

Notice that the definition above makes sense because the sets $(\Omega-x)/r$ invade $\mathbf{R}^2$ as $r\to 0^+$. Moreover, as

$$\mu_{x,r}(B_R) = \frac{\mu(B_{Rr}(x))}{r} \leq 1+R\Theta^*(\mu,x) \qquad \forall R>0$$

for r sufficiently small (depending on R), a simple diagonal argument shows that $\mathrm{Tan}(\mu,x)$ is not empty whenever $\Theta^*(\mu,x)$ is finite (and thus μ-a.e.).

Theorem 5.2 (Positive upper density implies positive lower density). *Assume that for some $x \in \Sigma_\mu^+$ the following properties hold*

(i) The density function $f(r) := \mu(B_r(x))/r$ is continuous in $(0, \delta)$ for some $\delta \in (0, \operatorname{dist}(x, \partial\Omega))$;

(ii) $\Theta^*(\mu, x)$ is finite;

(iii) There exists $c_x > 0$ such that any nonzero measure $\nu \in \operatorname{Tan}(\mu, x)$ is representable by $c\mathcal{H}^1 \llcorner L$, where $c \geq c_x$ and L is either a line or a halfline (not necessarily passing through the origin).

Then $x \in \Sigma_\mu^-$.

Proof. We introduce first some notation. Given a line or a half line L intersecting the open ball B_1, we denote by $\hat{L}$ the line containing it and by ξ its direction (if $L = \hat{L}$ the orientation does not matter). We denote by $h_L \in [0, 1)$ the distance of $\hat{L}$ from the origin. Finally we define $d_L \in [-1, 1]$ so that

$$y \in L \cap B_1 \qquad \Longleftrightarrow \qquad y \in \hat{L} \cap B_1 \text{ and } y \cdot \xi > -d_L.$$

An elementary geometric argument shows that, if $d_L \geq 0$ and $\mathcal{H}^1(L \cap B_1) \leq 1/2$, then $h_L \geq \sqrt{3}/2$.

We assume by contradiction that $x \notin \Sigma_\mu^-$, i.e. $\Theta_*(\mu, x) = 0$. Henceforth, we fix a positive number $q < \min\{c_x/2, \Theta^*(\mu, x)\}$ and find a decreasing sequence (R_i) with $f(R_i) < q/4$ and then $r_i < R_i$ such that $f(r_i) = q$ and $f(t) < q$ for $t \in (r_i, R_i]$ (r_i is the first r below R_i at which f hits q). Notice that necessarily $R_i/r_i \geq 4$.

Possibly extracting a subsequence, by assumptions (ii), (iii) we can assume that the rescaled measures $\mu_i = \mu_{x,r_i}$ weakly converge, in the duality with $C_c(\mathbf{R}^2)$, to a Radon measure $\nu = c\mathcal{H}^1 \llcorner L$, where L is either a line or a halfline and $c \geq c_x$.

As $\mu_i(\overline{B}_1) = q$ we obtain that $\nu(B_1) = \nu(\overline{B}_1) \geq q$. On the other hand, as $\mu_i(B_r) \leq qr$ for any $r \in (1, 4)$ we obtain

$$\nu(B_1) = q \qquad \text{and} \qquad \nu(B_r) \leq qr \quad \forall r \in (1, 4).$$

In particular the right derivative of $g(r) := \nu(B_r)/r$ at $r = 1$ is nonpositive.

On the other hand, we have

$$\nu(B_r) = c\left(d_L + \sqrt{r^2 - h_L^2}\right) \qquad \forall r \geq 1,$$

so that

$$\frac{d}{dr^+} g(r)\bigg|_{r=1} = c\frac{d}{dr^+}\frac{d_L + \sqrt{r^2 - h_L^2}}{r}\bigg|_{r=1} = c\frac{h_L^2 - d_L\sqrt{1 - h_L^2}}{\sqrt{1 - h_L^2}}.$$

This derivative is strictly positive if $d_L < 0$. If $d_L \geq 0$ we notice that

$$\mathcal{H}^1(L \cap B_1) = \frac{q}{c} \leq \frac{q}{c_x} < \frac{1}{2},$$

hence $h_L \geq \sqrt{3}/2$ and $h_L^2 > \sqrt{1-h_L^2}$. Therefore the derivative above is strictly positive in any case. This contradiction proves the theorem. $\square$

The following rectifiability result is part of the folklore on the subject, but we include a proof for convenience of the reader.

Theorem 5.3 (Rectifiability criterion). *Assume that for μ-a.e. $x \in \Sigma_\mu^-$ there exists a unit vector $\xi = \xi(x)$ such that any measure $\nu \in \mathrm{Tan}(\mu, x)$ is concentrated on a line parallel to ξ. Then Σ_μ^- is countably $\mathcal{H}^1$-rectifiable.*

Proof. For $n \geq 1$, let S_n be defined by

$$S_n := \left\{ x \in \Omega : \ \Theta_*(\mu, x) \geq \frac{1}{n} \right\}$$

As $\mathcal{H}^1 \llcorner S_n \leq 2n\mu$ it follows that $\mathcal{H}^1(S_n) < +\infty$, therefore by the decomposition theorem (see Corollary 2.10 in [F]) we can write $S_n = S_n^r \cup S_n^u$, where $S_n^r \cap S_n^u = \emptyset$, S_n^r is countably $\mathcal{H}^1$-rectifiable and S_n^u is purely unrectifiable, i.e. its intersection with any rectifiable curve is $\mathcal{H}^1$-negligible. Let us show that $\mathcal{H}^1(S_n^u) = 0$. Then, Σ_μ^- will be contained in a countable union of rectifiable curves and Theorem 5.3 will be proved.

Let us define, for any direction $\omega \in \mathbf{S}^1$, for any angle $\theta \in (0, \frac{\pi}{2})$, $x \in \mathbf{R}^2$ and $r > 0$, $S_r(x, \omega, \theta)$ as the intersection of $\overline{B}_r(x)$ with the cone

$$\left\{ y \in \mathbf{R}^2 \setminus \{x\} : \ |\cos(\widehat{y - x, \omega})| > |\cos\theta| \right\}.$$

having $x + \mathbf{R}\nu$ as axis. Since S_n^u is purely unrectifiable, by Theorem 3.29 in [F], for $\mathcal{H}^1$-a.e. $x \in S_n^u$ we have

$$\limsup_{r \to 0^+} \frac{\mathcal{H}^1(S_n^u \cap S_r(x, \omega, \theta))}{r} \geq \frac{1}{6}\sin\theta \qquad \forall \omega \in \mathbf{S}^1, \ \forall \theta \in (0, \frac{\pi}{2}).$$

In particular, fixing θ, we have

$$\limsup_{r \to 0^+} \frac{\mu\left(S_r(x, \xi^\perp(x), \theta)\right)}{r} \geq \frac{1}{12n}\sin\theta \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in S_n^u.$$

Assuming by contradiction that $\mathcal{H}^1(S_n^u) > 0$, choose $x \in S_n^u$ where the above density property holds and a sequence $r_i \downarrow 0$ such that $\mu_{x,r_i} \to \nu$ locally weakly in $\mathbf{R}^2$ and

$$\lim_{i\to\infty} \frac{\mu\left(S_{r_i}(x, \xi^\perp(x), \theta)\right)}{r_i} \geq \frac{1}{12n} \sin\theta. \tag{5.5}$$

By assumption we know that ν is concentrated on a line L parallel to ξ, and (5.5) gives

$$\nu\left(S_1(0, \xi^\perp(x), \theta)\right) \geq \frac{1}{12n} \sin\theta > 0.$$

We will obtain a contradiction by showing that the line L passes through the origin. If not, there is $c > 0$ such that $\nu(\overline{B}_c) = 0$, so that $\mu(B_{cr_i}(x))/r_i$ is infinitesimal as $i \to \infty$. This is not possible because $x \in \Sigma_\mu^-$. □

6 Classification of blow-ups and rectifiability

In this section we analyze the asymptotic behaviour of good liftings ϕ of vector fields $u \in \mathcal{M}_{\rm div}(\Omega)$. In Proposition 6.1 and Theorem 6.2 we show that generically a blow-up produces a lifting ϕ_∞ with special features, i.e. either approximately continuous or jumping on a line or on a halfline. Moreover, there is a rich family of truncations which turns ϕ_∞ into a $BV_{\rm loc}$ vector field.

Then, in Theorem 6.3 we prove rectifiability of the 1-dimensional part of μ_ϕ by showing that the normal to the jump is independent of the sequence of radii chosen for the blow-up, and a lower bound on the width of the jump of ϕ_∞. The first information comes choosing a Lebesgue point for the density function $\vec{H}$ characterized by

$$\int_{\mathbf{R}} e^{i\,a} {\rm div}\, T^a u \, da = \vec{H}\mu_\phi.$$

The second information comes choosing Lebesgue point for the density functions $\vec{H}_k$ characterized by

$$\int_{\mathbf{R}} e^{i\,ka} {\rm div}\, T^a u \, da = \vec{H}_k\mu_\phi, \qquad k \in (1,2) \cap \mathbf{Q}.$$

This aspect of the proof is quite delicate, since a priori the jump can be arbitrarily small and no universal constant in the lower bound can be expected, unlike in the theory of minimal surfaces. A linearization around $k = 1$ shows that small jumps are uniquely determined by all vectors $\vec{H}_k$.

Proposition 6.1. *Let $u \in \mathcal{M}_{\rm div}(\Omega)$ and let $\phi \in L^\infty(\Omega)$ be a lifting satisfying (P2) in Definition 3.1. For μ_ϕ-almost every $x_0 \in \Omega$, from any sequence $r_n \to 0^+$ one can* *extract a subsequence r_i such that the functions $\phi_{r_i}(x) := \phi(x_0 + r_i x)$ converge to ϕ_∞ in $L^1_{\mathrm{loc}}(\mathbf{R}^2)$.*

Moreover setting $u_\infty := e^{i\,\phi_\infty}$, the following properties hold:

(i) There exist a nonnegative Radon measure ν on $\mathbf{R}^2$ and a Lipschitz map $h : \mathbf{R} \to \mathbf{R}$ such that

$$\mathrm{div}\, T^a u_\infty = h(a)\nu \qquad \forall a \in \mathbf{R}.$$

(ii) There exists a finite or countable family of open segments (possibly unbounded) $I_l = (b_l, c_l)$ such that

(a) $\mathbf{R} \setminus \cup_l I_l$ has an empty interior;

(b) for all l, $\mathrm{div}\, T^{c_l}_{b_l} u_\infty = 0$;

(c) for all l, either $\mathrm{div}\, T^a_{b_l} u_\infty$ is a nonnegative measure for all $a \in I_l$ or $\mathrm{div}\, T^a_{b_l} u_\infty$ is a non-positive measure for all $a \in I_l$.

Proof. By Theorem 3.4 we know that μ_ϕ is absolutely continuous with respect to $\mathcal{H}^1$, hence (see Section 5) the upper density $\Theta^*(\mu_\phi, x)$ is finite for μ_ϕ-a.e. x. Henceforth, we choose x_0 with this property. Since $\mu_{\phi_r}(B_R) = \mu_\phi(B_{Rr}(x_0))/r$ is equibounded with respect to r for any fixed R, the compactness Theorem 3.6 and a diagonal argument ensure the first part of the statement. We can also assume that the rescaled measures $(\mu_\phi)_{x_0, r_i}$ as in Definition 5.1 weakly converge, in the duality with $C_c(\mathbf{R}^2)$, to some Radon measure ν.

In order to obtain the property stated in (i) we impose additional (but generic) conditions on x_0. By Theorem 3.2(ii) we have that, for all $g \in C_c(R)$, the Radon measure $\int_{\mathbf{R}} g(a)\mathrm{div}\, T^a u\, da$ is absolutely continuous with respect to μ_ϕ. Let D be a countable set dense in $C_c(\mathbf{R})$ and set

$$\nu_g := \int_{\mathbf{R}} g(a)\mathrm{div}\, T^a u\, da \qquad \forall g \in D.$$

Then, by the Radon-Nikodým theorem there exist functions $h_g \in L^1(\Omega, \mu_\phi)$ such that $\nu_g = h_g \mu_\phi$. By Proposition 3.2(ii) again we obtain

$$\int_\Omega (h_g - h_{g'})\psi\, dx = \int_{\mathbf{R}} (g(a) - g'(a))\langle \mathrm{div}\, T^a u; \psi\rangle\, da \le \sup|g - g'| \int_\Omega |\psi|\, d\mu_\phi$$

for any $\psi \in C^\infty_c(\Omega)$ and any g, $g' \in C_c(\mathbf{R})$, hence $\|h_g - h_{g'}\|_\infty \le \sup|g - g'|$ (the L^∞ norm is computed using μ_ϕ as reference measure).

Let us consider the Borel set $\Omega' = \Omega \setminus \cup_{g \in D} S_{h_g}$ of approximate continuity points of all maps h_g, for $g \in D$. Let $\mathcal{B}^\infty(\Omega')$ be the space of bounded Borel functions on Ω', endowed with the sup norm. By the previous estimate, the map R which associates to $g \in D$ the function

$$R_g(x) := \text{ap}-\lim_{y \to x} h_g(y), \qquad x \in \Omega'$$

is 1-Lipschitz between D and $\mathcal{B}^\infty(\Omega')$. By a density argument R extends to a 1-Lipschitz map defined on the whole of $C_c(\mathbf{R})$ and each point x of Ω' is an approximate continuity point of all functions h_g, $g \in C_c(\mathbf{R})$, with approximate limit $R_g(x)$.

We fix $x_0 \in \Omega'$. Rescaling ν_g as in Definition 5.1 we obtain

$$(\nu_g)_{x_0,r} = h_g(x_0 + r\cdot)(\mu_\phi)_{x_0,r}$$

and the approximate continuity of h_g at x_0, together with the fact that the upper density is finite, ensures that $(\nu_g)_{x_0,r_i}$ weakly converge, in the duality with $C_c(\mathbf{R}^2)$, to $R_g(x_0)\nu$. On the other hand, the identity

$$(\nu_g)_{x_0,r_i} = \int_{\mathbf{R}} g(a) \mathrm{div}\, T^a u_{r_i}\, da$$

and the convergence in the sense of distributions of $\mathrm{div}\, T^a u_{r_i}$ to $\mathrm{div}\, T^a u_\infty$ give

$$\int_{\mathbf{R}} g(a) \langle \mathrm{div}\, T^a u_\infty; \xi \rangle \, da = R_g(x_0) \int_{\mathbf{R}^2} \xi d\nu \qquad \forall g \in C_c(\mathbf{R}^2),\ \xi \in C_c^\infty(\mathbf{R}^2).$$

Now we fix $\xi_0 \in C_c^\infty(\mathbf{R}^2)$ such that $\int_{\mathbf{R}^2} \xi_0 \, d\nu = 1$ (assuming with no loss of generality that $\nu(\mathbf{R}^2) > 0$) and notice that consequently

$$|R_g(x_0)| \leq \|\nabla \xi_0\|_\infty \int_{\mathbf{R}} |g(a)| \, da.$$

If particular, if g_k weakly converge to the Dirac mass at a, then $R_{g_k}(x_0)$ is bounded, and any limit point h satisfies

$$\langle \mathrm{div}\, T^a u_\infty; \xi \rangle = h \int_{\mathbf{R}^2} \xi \, d\nu \qquad \forall \xi \in C_c^\infty(\mathbf{R}^2).$$

This implies that h does not depend on the approximating sequence, but only on a. The Lipschitz property of h follows directly by Proposition 3.2(i), using ξ_0 as test function.

Let us now prove that (i) implies (ii), assuming with no loss of generality that ν is a nonzero measure. Then it suffices to take as intervals the connected components of $\{h \neq 0\}$ and the connected components of the interior of $\{h = 0\}$. By construction the complement of the union of these intervals has an empty interior. $\square$

Theorem 6.2. *Let $u \in \mathcal{M}_{\mathrm{div}}(\Omega)$ and let $\phi \in L^\infty(\Omega)$ be a lifting satisfying (P2) in Definition 3.1. For μ_ϕ-almost every $x_0 \in \Omega$, from any sequence $r_n \to 0^+$ one can extract a subsequence r_i such that the functions $\phi_{r_i}(x) := \phi(x_0 + r_i x)$ converge to in $L^1_{\mathrm{loc}}(\mathbf{R}^2)$ to ϕ_∞. Moreover the jump set J_{ϕ_∞} of ϕ_∞ coincides, up to $\mathcal{H}^1$-negligible sets, either with the empty set, or with a line or with a halfline K, not necessarily passing through the origin.*

If K is a line and $\omega_K \in \mathbf{S}^1$, $A \in \mathbf{R}^2$ are such that $K = A + \mathbf{R}\omega_K^\perp$ (see Figure 1), then ϕ_∞ is constant in the halfspaces $\Gamma^\pm$ defined by

$$\Gamma^+ := \{y \in \mathbf{R}^2 :\ (y - A)\cdot \omega_K > 0\}, \qquad \Gamma^- := \{y \in \mathbf{R}^2 :\ (y - A)\cdot \omega_K < 0, \}. \tag{6.1}$$

If K is a halfline and $\omega_K \in \mathbf{S}^1$, $A \in \mathbf{R}^2$ are such that $K = \{A + t\omega_K^\perp :\ t > 0\}$ (see Figure 1), then the approximate limits ϕ_∞^+ and ϕ_∞^- are constant $\mathcal{H}^1$ a.e. on K. Moreover, ϕ_∞ is equal to $\phi_\infty^\pm$ a.e. in $\Gamma_A^\pm$, where

$$\Gamma_A^\pm := \Gamma^\pm \cap \{y \in \mathbf{R}^2 :\ \frac{y-A}{|y-A|}\cdot \omega_K^\perp \geq -u_l^\pm \cdot \omega_K)\}.$$

Proof. Keeping the notation of Proposition 6.1, in the following we denote by L^0 the set of all l such that I_l is not a connected component of the interior of $\{h = 0\}$. Then, if $l \notin L^0$, $\operatorname{div} T^a u_\infty = 0$ for any $a \in I_l$. If $l \in L^0$, either $\operatorname{div} T^a u_\infty$ is nonnegative and nonzero for any $a \in I_l$ or $\operatorname{div} T^a u_\infty$ is nonpositive and nonzero for any $a \in I_l$.

Let us set $u_l := e^{i\,(\phi_\infty \vee b_l)\wedge c_l}$. Then u_l is divergence free, because $\operatorname{div} T^{b_l} u_\infty = \operatorname{div} T^{c_l} u_\infty = 0$ and

$$e^{i\,(\phi_\infty \vee b_l)\wedge c_l} + e^{i\,\phi_\infty \wedge b_l} = e^{i\,b_l} + e^{i\,\phi_\infty \wedge c_l}.$$

Since

$$e^{i\,(\phi_\infty \vee b_l)\wedge a} + e^{i\,\phi_\infty \wedge b_l} = e^{i\,b_l} + e^{i\,\phi_\infty \wedge a}, \tag{6.2}$$

we obtain that $\operatorname{div} T^a u_l = \operatorname{div} T^a u_\infty$ for $a \in I_l$, therefore

$$\operatorname{div} T^a u_l = \begin{cases} h(a)\nu & \text{if } a \in I_l \\ 0 & \text{else.} \end{cases} \tag{6.3}$$

In particular $u_l \in \mathcal{M}_{\mathrm{div}}(\Omega)$. Moreover, either $\operatorname{div} T^a u_l$ is nonnegative for any $a \in \mathbf{R}$ or $\operatorname{div} T^a u_l$ is non-positive for any $a \in \mathbf{R}$. If we are in the first situation, by Theorem I.1 of [ALR], any function $g_l \in W^{1,\infty}(\mathbf{R}^2)$ such that $u_l = -\nabla^\perp g_l$ is a viscosity solution of the eikonal equation $|\nabla g|^2 - 1 = 0$ on $\mathbf{R}^2$. Therefore, g_l is concave and $u_l \in BV_{\mathrm{loc}}(\mathbf{R}^2)$ (see [AD]). If we are in the second situation, for any function $g_l \in W^{1,\infty}(\mathbf{R}^2)$ such that $u_l = \nabla^\perp g_l$ we have the same statement. In both cases, by applying Proposition 2.3, we obtain that $\phi_l \in BV_{\mathrm{loc}}(\mathbf{R}^2)$.

In order to study the jump set of ϕ_∞ we first study the behaviour of the functions ϕ_l. If $l \notin L^0$, by the previous discussion we obtain that any function g_l satisfying $u_l = -\nabla^\perp g_l$ is affine (being concave and convex) and therefore u_l is constant. As $U_{\phi_l} = 0$, from Proposition 2.3 we obtain that ϕ_l is constant as well.

In the following we consider $l \in L_0$ and, to fix the ideas (since the argument is similar for both cases), we assume that $\operatorname{div} T^a u_l$ is a nonzero and nonnegative measure for any $a \in I_l$.

We denote by J_l the set of approximate jump points of ϕ_l, by ω_l a unit normal of J_l and by ϕ_l^+, ϕ_l^- the corresponding approximate limits of ϕ_l on each side of J_l. Since $\operatorname{div} u_l = 0$, then $\omega_l \cdot e^{i\,\phi_l^+} = \omega_l \cdot e^{i\,\phi_l^-}$ on J_l. Thus, $\omega_l = \pm e^{\frac{i}{2}(\phi_l^+ + \phi_l^-)}$ and we choose $\omega_l = e^{\frac{i}{2}(\phi_l^+ + \phi_l^-)}$. Then, the explicit formula (3.4) given in Section 3 gives

$$\operatorname{div} T^a u_l = \chi(a, \phi_l^+, \phi_l^-)(e^{i\,a} - e^{i\,\phi_l^-}) \cdot \omega_l \, \mathcal{H}^1 \llcorner J_l,$$

where

$$\chi(a, \phi_l^+, \phi_l^-) := \begin{cases} 1 & \text{if } \phi_l^- < a < \phi_l^+ \\ -1 & \text{if } \phi_l^+ < a < \phi_l^- \\ 0 & \text{else.} \end{cases}$$

But, $\operatorname{div} T^a u_l$ is nonnegative for all $a \in \mathbf{R}$. Then, $|\phi_l^+ - \phi_l^-| < 2\pi$, since, otherwise, there would exist $a \in \mathbf{R}$ such that $\chi(a, \phi_l^+, \phi_l^-)(e^{i\,a} - e^{i\,\phi_l^-}) \cdot \omega_l < 0$. In particular J_l is also the set of approximate jump points of u_l. If $\phi_l^- > \phi_l^+$, then $(e^{i\,a} - e^{i\,\phi_l^-}) \cdot \omega_l \geq 0$ for any $a \in [\phi_l^+, \phi_l^-]$. Therefore, we must have $\phi_l^+ > \phi_l^-$ and $|\phi_l^+ - \phi_l^-| < 2\pi$ $\mathcal{H}^1$-a.e. on J_l and

$$\operatorname{div} T^a u_l = \chi_{(\phi_l^-, \phi_l^+)}(a)(e^{i\,a} - e^{i\,\phi_l^-}) \cdot \omega_l \, \mathcal{H}^1 \llcorner J_l. \tag{6.4}$$

Claim 1. $\phi_l^+ = c_l$ and $\phi_l^- = b_l$ $\mathcal{H}^1$-a.e. on J_l.
First of all, we notice that $c_l \geq \phi_l^+ > \phi_l^- \geq b_l$ $\mathcal{H}^1$-a.e. on J_l. Assuming by contradiction that $\{\phi_l^+ < c_l\}$ has positive $\mathcal{H}^1$-measure, we can find $\epsilon > 0$ such that $\{\phi_l^+ < c_l\} \cap \{\phi_\epsilon^+ - \phi_l^- > \epsilon\}$ has positive $\mathcal{H}^1$-measure, and then an interval $(\beta, \beta') \subset (b_l, c_l)$ with length less than $\epsilon/2$ such that

$$E := \{\phi_l^+ \in (\beta, \beta')\} \cap \{\phi_l^+ - \phi_l^- > \epsilon\}$$

has positive $\mathcal{H}^1$-measure. From (6.4) we infer that $\operatorname{div} T^a u_l \llcorner E = 0$ for $a \in (\beta', c_l)$, while $\operatorname{div} T^a u_l(E) > 0$ for $a \in (\beta - \epsilon/2, \beta)$. Since $h > 0$ on (b_l, c_l), this contradicts (6.3). The argument for ϕ_l^- is similar.

Claim 2. For any choice of l, $m \in L^0$ we have $\mathcal{H}^1(J_l \setminus J_m) = 0$.
Suppose that there exist l, $m \in L^0$ and $A \subset J_l \setminus J_m$ such that $\mathcal{H}^1(A) > 0$. Since

$A \cap J_m = \emptyset$, (6.4) yields $\operatorname{div} T^a u_m \llcorner A = 0$ for any $a \in \mathbf{R}$ and (6.3) yields $h(a)\nu(A) = 0$, so that $\nu(A) = 0$. On the other hand, the function $\chi_{(\phi_l^-, \phi_l^+)}(a)(e^{i\,a} - e^{i\,\phi_l^-}) \cdot \omega_l$ is constant $\mathcal{H}^1$-a.e. on J_l by Claim 1. Moreover, this constant is not 0 for any $a \in I_l$. Since $\nu(A) = 0$, then $|\operatorname{div} T^a u_l|(A) = 0$ and therefore $\mathcal{H}^1(J_l \cap A) = 0$. Since $A \subset J_l$, then $\mathcal{H}^1(A) = 0$ which contradicts the hypothesis and proves the claim.

Claim 3. For any $l \in L^0$, J_l is contained in one line.
Let us recall that the normal unit vector ω_l to J_l is given by $e^{\frac{i}{2}(\phi_l^+ + \phi_l^-)}$ and is constant $\mathcal{H}^1$-a.e. on J_l. Let us assume that there exist x_1, $x_2 \in J_l$ such that $(x_2 - x_1) \cdot \omega_l \neq 0$ and assume (up to a permutation of x_1 and x_2) that the scalar product is positive. We set $\omega := \frac{x_2 - x_1}{|x_2 - x_1|}$, so that $\omega \cdot \omega_l > 0$. Since the restriction of g_l to the line $\mathbf{R}\,\omega$ is concave, we must have

$$\nabla^+_\omega g_l(x_1) \geq \nabla^-_\omega g_l(x_2).$$

By Proposition 4.1 we get

$$\nabla^+_\omega g_l(x_1) = \omega \cdot \nabla g_l^+(x_1) = \omega \cdot (e^{i\phi_l^+})^\perp$$

and

$$\nabla^-_\omega g_l(x_2) = \omega \cdot \nabla g_l^-(x_2) = \omega \cdot (e^{i\phi_l^-})^\perp,$$

so that $\omega \cdot (e^{i\phi_l^+})^\perp \geq \omega \cdot (e^{i\phi_l^+})^\perp$. On the other hand, since $\omega \cdot \omega_l > 0$ and $\phi_l^+ > \phi_l^-$, then $\omega \cdot (e^{i\phi_l^+})^\perp < \omega \cdot (e^{i\phi_l^-})^\perp$, a contradiction (this inequality can be easily checked in a frame where $\phi_l^+ + \phi_l^- = 0$, so that $\omega_1 > 0$). Therefore J_l must be contained in one line. By Claim 2, all sets J_l with strictly positive $\mathcal{H}^1$-measure (i.e. those corresponding to $l \in L^0$) are contained in the same line. Let us denote this line by R.

Claim 4. There exists a closed set $K_l \subset R$ such that $\mathcal{H}^1(K_l \Delta J_l) = 0$.
Let us recall that J_l coincides with the set $J_{\nabla g_l}$ of approximate jump points of $\nabla g_l = (e^{i\,\phi_l})^\perp$, where g_l is concave and satisfies $|\nabla g_l| = 1$. Since $\phi_l^+ = c_l$ and $\phi_l^- = b_l$ $\mathcal{H}^1$-a.e. on J_l, taking $\alpha = |e^{i\,c_l} - e^{i\,b_l}|$, it is clear that the closure K_l of $J^\alpha := \{x \in J_{g_l} : \ |\nabla g_l^+(x) - \nabla g_l^-(x)| \geq \alpha\}$ contains $\mathcal{H}^1$-almost all of J_l. By Proposition 4.1, $J^\alpha \subset \Sigma_\alpha$, where $\Sigma_\alpha := \{x \in \mathbf{R}^2 : \ \operatorname{diam}(\partial g_l(x)) \geq \alpha\}$ is a closed set. Therefore $K_l \subset \Sigma_\alpha$. But, $\Sigma_\alpha \subset S_{\nabla g_l}$, where $S_{\nabla g_l}$ is the set of points where ∇g_l doesn't have an approximate limit. Indeed, by Proposition 4.1 , at any point x where ∇g_l has an approximate limit the function g_l is differentiable, hence $\partial g_l(x)$ is a singleton. By (2.2) we infer

$$\mathcal{H}^1(K_l \setminus J_l) \leq \mathcal{H}^1(\Sigma_\alpha \setminus J_l) \leq \mathcal{H}^1(S_{g_l} \setminus J_{g_l}) = 0.$$

For any $l \in L^0$, for $\mathcal{H}^1$-almost every $x \in K_l$, ∇g_l has an approximate limit at $\mathcal{H}^1$-almost every $y \in D_x^-$ and the approximate limit at a.e. point in the strip $\bigcup\limits_{x \in K_l} D_x^-$

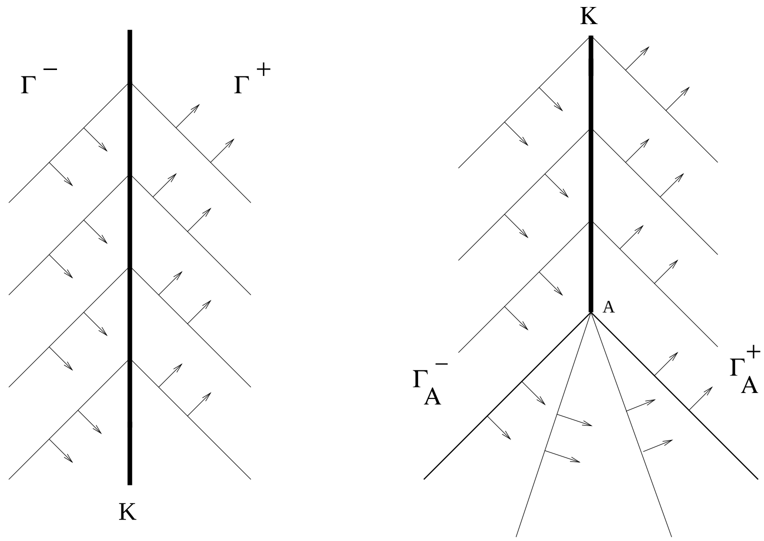

Figure 1: Behaviour of $e^{i\,\phi_\infty}$ when K is a line or a halfline

is equal to $\nabla g_l^-(x) = (e^{i\,b_l})^\perp$, since ϕ_l^- is constant equal to b_l $\mathcal{H}^1$-a.e. on K. In the same way, one can show that ∇g_l has an approximate limit at a.e. point in $\bigcup_{x\in K_l} D_x^+$ equal to $(e^{i\,c_l})^\perp$. If K_l is the whole line, then $\bigcup_{x\in K_l} D_x^\pm = \Gamma^\pm$, where $\Gamma^\pm$ are the sets defined in (6.1). Therefore u_l is constant a.e. in $\Gamma^\pm$ and equal to $e^{i\phi_l^\pm}$. By Proposition 2.3 we obtain that ϕ_l is constant in the two halfspaces as well.

Now, let us assume that K_l is not the whole line and let us show that K_l must be a halfline. Assume that K_l is not connected. There exists a bounded open interval S contained in $R\setminus K_l$, whose endpoints s_1, s_2 belong to K_l. We will denote by K_1, K_2 the components of K_l containing s_1 and s_2 respectively. Set $R_i^\pm := \bigcup_{x\in K_i} D_x^\pm$, $i = 1, 2$. The region $\mathbf{R}^2\setminus(R_1^-\cup R_1^+\cup R_2^-\cup R_2^+)$ can be divided into three parts A^+, A^-, C (see Figure 2). If $y \in A^+$ is a point of approximate continuity of ∇g_l, then $\nabla g_l(y)$ must be equal to $(e^{i\,c_l})^\perp$, otherwise the halfline D_y^+ would cross R_1^- or R_2^- and this would contradict the result of Proposition 4.1 (ii). If $y \in A^-$ is a point of approximate continuity of ∇g_l, by the same argument, $\nabla g_l(y) = (e^{i\,b_l})^\perp$. If $y \in C$ is a point of approximate continuity of ∇g_l, then $\nabla g_l(y)$ can only be equal to $(e^{i\,c_l})^\perp$ or $(e^{i\,b_l})^\perp$ (see Figure 2). Then, C contains a set of approximate jump points of ∇g_l. But, by hypothesis, $K_l \cap C = \emptyset$, hence $\mathcal{H}^1(J_l \cap C) = 0$. Therefore, K_l must be connected.

If K_l is not the whole line, then K_l has one or two endpoints. Let A be one endpoint of K_l and let ω_K be the unit normal to K_l such that $K_l \subset \{A + t\omega_K^\perp : t \geq 0\}$.

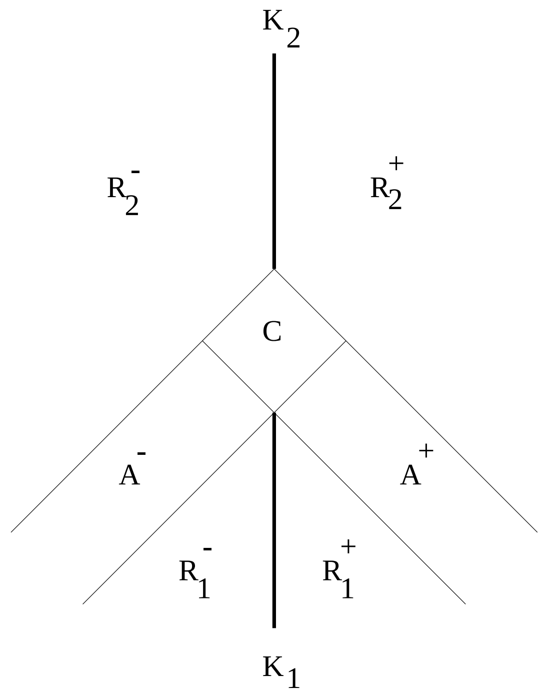

Figure 2: K must be connected

Let us define the cone $\mathcal{C}$ by

$$\mathcal{C} := \big\{ y \in \mathbf{R}^2 \setminus \{A\} : \ \frac{y-A}{|y-A|} \cdot \omega_K^\perp \le -e^{i\,\phi_l^-} \cdot \omega_K \big\}.$$

Let $\mathcal{C}'$ be any open set containing $\mathcal{C}$ such that $\overline{\mathcal{C}'} \cap K = \{A\}$ and $\mathcal{C}' \cap K = \emptyset$. Then, $\operatorname{div} T^a u_l = 0$ in $\mathcal{D}'(\mathcal{C}')$ for any $a \in \mathbf{R}$. Using the result of [LR], ϕ_l is locally Lipschitz in $\mathcal{C}'$. Therefore, for a.e. x in $\mathcal{C}'$, $\nabla \phi_l(x)$ exists and $\operatorname{div} e^{i\,\phi_l}(x) = (e^{i\,\phi_l(x)})^\perp \cdot \nabla \phi_l(x) = 0$. Then, $\nabla \phi_l(x)$ is parallel to $e^{i\,\phi_l(x)}$ for a.e. $x \in \mathcal{C}'$. Therefore for any $a \in \mathbf{R}$ the tangent to the level set $\{\phi_l = a\}$ at x is orthogonal to $e^{i\,\phi_l(x)}$ which is equal to $e^{i\,a}$ on $\{\phi_l = a\}$. Hence, the level sets $\{\phi_l = a\}$ are straight lines oriented by $(e^{i\,a})^\perp$ and the only possible configuration in $\mathcal{C}$ is the one described in Figure 1.

Finally, we can exclude the case of K_l is a segment or a single point (Figure 3). Indeed, choose $R > 0$ such that $K_l \subset B_R$. Since $\mathcal{H}^1(J_l \setminus B_R) = 0$, the slicing theory of BV functions (see [AFP], Theorem 3.108) shows that for a.e. $r \in (R, R+1)$ the restriction of ϕ_l to ∂B_r is (equivalent to) a continuous BV function. Therefore u_l has a continuous lifting in ∂B_r and its topological degree is 0. This is in contradiction with the fact that there are vortices which have the same orientation α_l at the two endpoints of K_l. Therefore, K_l is a halfline.

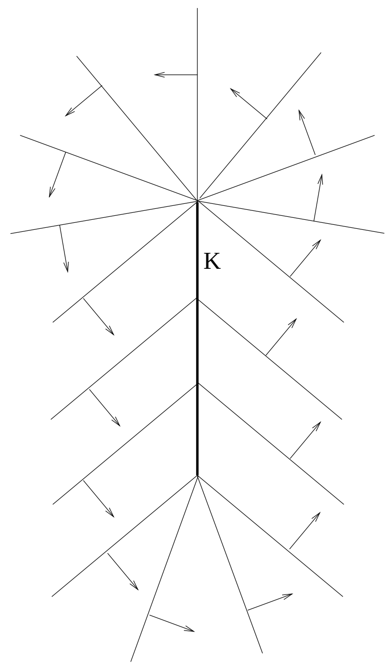

Figure 3: K can't be a segment

By Claims 2 and 4 we obtain that all lines (or halflines) K_l, $l \in L^0$, coincide. Henceforth we set $K = K_l$. By Lemma 2.1 and Remark 2.2 we obtain that ϕ_∞ has an approximate limit $\mathcal{H}^1$-a.e. in $\mathbf{R}^2 \setminus K$ and $\mathcal{H}^1$-a.e. point of K is a jump point of ϕ_∞. Moreover, as all limits $\phi_l^\pm$ are constant on J_{ϕ_∞}, the same is true for $\phi_\infty^\pm$. $\square$

Theorem 6.3 (Main rectifiability theorem). *Let $u \in \mathcal{M}_{\rm div}(\Omega)$ and let $\phi \in L^\infty(\Omega)$ be a lifting satisfying (P2) in Definition 3.1. Then the set*

$$\Sigma := \{x \in \Omega : \ \Theta^*(\mu_\phi, x) > 0\} \tag{6.5}$$

is countably $\mathcal{H}^1$-rectifiable and coincides, up to $\mathcal{H}^1$-negligible sets, with J_ϕ. Moreover, for $\mathcal{H}^1$-a.e. $x \in \Omega \setminus J_\phi$ we have

$$\lim_{r\to 0^+} \frac{1}{\pi r^2} \min_{c\in\mathbf{R}} \int_{B_r(x)} |\phi(y) - c| \, dy = 0. \tag{6.6}$$

Proof. **Step 1.** We show that $\Sigma' := \{\Theta_*(\mu_\phi, \cdot) > 0\}$ is countably rectifiable, using Theorem 5.3. To this aim we show that for μ-a.e. x any $\sigma = \lim_i (\mu_\phi)_{x,r_i} \in \mathrm{Tan}(\mu_\phi, x)$ is supported on a line whose direction depends on x only.

We proved in Theorem 6.2 that (possibly passing to a subsequence) we can assume that $\phi_{r_i} = \phi(x + r_i y) \to \phi_\infty$ in $L^1_{\rm loc}(\mathbf{R}^2)$. Moreover, there exists a closed set

K, the empty set, a line or a halfline, such that $\mathcal{H}^1(K\Delta J_{\phi_\infty})=0$. Denoting by ω_K the orientation of K such that $e^{i\,(\phi^+_\infty+\phi^-_\infty)/2}=\omega_K$, now we show that

$$\operatorname{div} T^a u_\infty = (T^a u^+_\infty - T^a u^-_\infty)\cdot\omega_K \mathcal{H}^1 \llcorner K \qquad \forall a\in\mathbf{R} \tag{6.7}$$

using Lemma 3.5. To this aim, we need only to check that $T^a u_\infty$ is divergence-free in $\Omega\setminus K$. If a belongs to some interval (b_l,c_l), this follows by the identity $\operatorname{div} T^a u_\infty=\operatorname{div} T^a u_l$ (see (6.2) and by (3.4), because $J_{\phi_l}\subset K$ up to $\mathcal{H}^1$-negligible sets. In the general case one can argue by approximation, using the fact that the complement of $\cup_l(b_l,c_l)$ has an empty interior.

One can show, by a direct computation based on (6.7), that the vector-valued measure $\int_{\mathbf{R}} e^{i\,a}\operatorname{div} T^a u_\infty\,da$ is oriented by ω_K, and precisely

$$\int_{\mathbf{R}} e^{i\,a}\operatorname{div} T^a u_\infty\,da=\frac{1}{2}\left(\phi^+_\infty-\phi^-_\infty-\sin(\phi^+_\infty-\phi^-_\infty)\right)\omega_K\mathcal{H}^1\llcorner K \tag{6.8}$$

(this computation is easily done in a frame where $\omega_K=(1,0)$, so that $\phi^+_\infty=-\phi^-_\infty+4k\pi$ for some $k\in\mathbf{Z}$ and the periodicity and the odness of the integrand show that the integral of the second component is 0). Moreover, the vector-valued measure $\lambda_1:=\int_{\mathbf{R}} e^{i\,a}\operatorname{div} T^a u\,da$ satisfies, by Theorem 3.2(ii), the inequality $|\lambda_1|\le\mu_\phi$. Thus, there exists a vector-valued function $\vec{H}\in L^1(\Omega,\mu_\phi)$ such that $\lambda_1=\vec{H}\mu_\phi$ and $|\vec{H}|\le 1$. In addition to the previous generic conditions imposed on x_0, assume also that x_0 is a Lebesgue point of $\vec{H}$, relative to the measure μ_ϕ. Then

$$(\lambda_1)_{x_0,r_i}\to\vec{H}(x_0)\sigma \quad \text{ in } \mathcal{M}'(\mathbf{R}^2).$$

On the other hand, the convergence of ϕ_{r_i} to ϕ_∞ implies

$$(\lambda_1)_{x_0,r_i}=\int_{\mathbf{R}} e^{i\,a}\operatorname{div} T^a u_{r_i}\,da\to\int_{\mathbf{R}} e^{i\,a}\operatorname{div} T^a u_\infty\,da \quad \text{ in } \mathcal{D}'(\mathbf{R}^2).$$

Therefore

$$\int_{\mathbf{R}} e^{i\,a}\operatorname{div} T^a u_\infty\,da=\vec{H}(x_0)\sigma.$$

Comparing this expression with (6.8) we obtain

$$\frac{1}{2}\left(\phi^+_\infty-\phi^-_\infty-\sin(\phi^+_\infty-\phi^-_\infty)\right)\omega_K\mathcal{H}^1\llcorner K=\vec{H}(x_0)\sigma. \tag{6.9}$$

Therefore ω_K does not depend on the sequence chosen, but only on x_0.

Step 2. We show that $\mu_\phi(\Sigma\setminus\Sigma')=0$ using Theorem 5.2. Since $\mathcal{H}^1(S\cap S')=0$ whenever $S\neq S'$ are circles, the family of all circles S such that $\mu_\phi(S)>0$ is at most countable, and the same is true for their centers. Therefore we can choose x_0 out of this set, so that the density function $f(r) := \mu_\phi(B_r(x_0))$ is continuous. In order to check condition (iii) of Theorem 5.2, for $k \in \mathbf{Q} \cap (1,2)$ we define the measures $\lambda_k := \int_{\mathbf{R}} e^{i\,ka} \mathrm{div}\, T^a u\, da$, all absolutely continuous with respect to μ_ϕ, we denote by $\vec{H}_k \in L^1(\Omega, \mu_\phi)$ their densities with respect to μ_ϕ and we choose a Lebesgue point x_0 for all functions $\vec{H}_k$ (relative to μ_ϕ).

Assuming that σ is not identically 0, we have to show that $\sigma = c\mathcal{H}^1 \llcorner K$ with $c \geq c(x_0) > 0$. By (6.9) and since $\phi_\infty^\pm$ are constant on K, we know that $\sigma = c\mathcal{H}^1 \llcorner K$, where c is constant on K. Moreover,

$$c|\vec{H}(x_0)| = \frac{1}{2} \left| (\phi_\infty^+ - \phi_\infty^-) - \sin(\phi_\infty^+ - \phi_\infty^-) \right|. \tag{6.10}$$

Therefore, if $|\phi_\infty^+ - \phi_\infty^-| \geq \pi/2$, we have $c \geq (\pi/2 - 1)/2$ because $|\vec{H}(x_0)| \leq 1$. Setting $d := |\phi_\infty^+ - \phi_\infty^-|/2 > 0$, in the following we show that d (and therefore c, by (6.10)) is uniquely determined by $\vec{H}_k(x_0)$ whenever $d \leq \pi/4$. We can assume with no loss of generality (possibly making a rotation and adding to ϕ_∞ an integer multiple of 2π) that $\omega_K = (1,0)$, $\phi_\infty^+ = \pm d$ and $\phi_\infty^- = \mp d$. Then, arguing as in Step 1 we get

$$\int_{\mathbf{R}} e^{i\,ka} \mathrm{div}\, T^a u_\infty\, da = \vec{H}_k(x_0)\sigma \qquad \forall k \in (1,2) \cap \mathbf{Q}.$$

On the other hand, computing the left side we find that its real part equals $\frac{2}{k(k^2-1)} F_d(k) \mathcal{H}^1 \llcorner K$, where

$$F_d(k) := (\sin kd \cos d - k \cos kd \sin d).$$

Then $F_d(k) \neq 0$ if and only if $\vec{H}_k(x_0) \cdot \omega_K \neq 0$ and

$$c = \frac{2}{k(k^2-1)} \frac{F_d(k)}{\vec{H}_k(x_0) \cdot \omega_K}. \tag{6.11}$$

It turns out that the ratios

$$\Phi_{k,m}(d) := \frac{F_d(k)}{F_d(m)} = \frac{k(k^2-1)}{m(m^2-1)} \frac{\vec{H}_k(x_0) \cdot \omega_K}{\vec{H}_m(x_0) \cdot \omega_K} \tag{6.12}$$

(when defined) depend on x_0, k and m but not on d, so that the functions F_d and $F_{d'}$ are proportional whenever d, d' satisfy (6.12). A Taylor expansion at $k = 1$ gives

$$F_t(k) = (k-1)(t - \sin t \cos t) + (k-1)^2 t \sin^2 t.$$

Therefore $F_t(k) \neq 0$ for $k - 1$ sufficiently small and the constant ratio between F_d and $F_{d'}$ must be equal to

$$\frac{d - \sin d \cos d}{d' - \sin d' \cos d'} \qquad \text{and} \qquad \frac{d \sin^2 d}{d' \sin^2 d'}.$$

Therefore $g(d) = g(d')$, where

$$g(t) := \frac{t - \sin t \cos t}{t \sin^2 t}.$$

A direct computation shows that g is strictly decreasing in $(0, \pi/4)$. Therefore $d = d'$.

Step 3. Now we show the last part of the statement. Since we know that $\mu_\phi \llcorner \Sigma$ is a rectifiable measure, by Theorem 2.83 of [AFP] we know that $\mathrm{Tan}(\mu_\phi \llcorner \Sigma, x)$, is a singleton for $\mathcal{H}^1$-a.e. $x \in \Omega$, therefore $\mathrm{Tan}(\mu_\phi, x)$ is a singleton for $\mathcal{H}^1$-a.e. $x \in \Sigma$. Coming back to (6.9) we obtain that the jump $\phi_\infty^+ - \phi_\infty^-$ is uniquely determined $\mathcal{H}^1$-a.e., and the same is true for $\phi_\infty^+ + \phi_\infty^-$ modulo 2π. Hence, ϕ_∞^+ is only determined modulo 2π, $\mathcal{H}^1$-a.e. on Σ and ϕ_∞^- is given by $\phi_\infty^- = \phi_\infty^+ - (\phi_\infty^+ - \phi_\infty^-)$ when ϕ_∞^+ is known.

Let us define the following measures, all absolutely continuous with respect to μ_ϕ:

$$\tau_k := \int_{2k\pi}^{2(k+1)\pi} \operatorname{div} T^a u \, da, \qquad \forall k \in \mathbf{Z}.$$

Let us denote by $t_k \in L^1(\Omega, \mu_\phi)$ their densities with respect to μ_ϕ and let us choose a Lebesgue point x_0 of all functions of t_k. As in Step 1, we have

$$\int_{2k\pi}^{2(k+1)\pi} \operatorname{div} T^a u_\infty \, da = t_k(x_0)\sigma \qquad \forall k \in \mathbf{Z}.$$

By (6.7), $\operatorname{div} T^a u_\infty = 0$ as soon as $a \notin [\phi_\infty^-, \phi_\infty^+]$. Let us define $X_0 := \{k \in \mathbf{Z} : t_k(x_0) = 0\}$. Then, $k \in X_0$ if and only if $(2k\pi, 2(k+1)\pi) \cap [\phi_\infty^-, \phi_\infty^+] = \emptyset$. Let $k_0 \in \mathbf{Z}$ be such that $\phi_\infty^+ \in [2k_0\pi, 2(k_0+1)\pi)$. Then, $\phi_\infty^- \in [2(k_0 - l_0)\pi, 2(k_0 - l_0 + 1)\pi)$, where $l_0 \in \mathbf{N}$ depends only on $\phi_\infty^+ - \phi_\infty^-$ and k_0 depends on X_0 in the following way: $\mathbf{Z} \setminus X_0 = \{k_0 - j : \ 0 \le j \le l_0\}$. Since X_0 only depends on x_0, then k_0 only depends on x_0 and ϕ_∞^+ is uniquely determined. Thus ϕ_∞^+ and ϕ_∞^- are uniquely determined μ-a.e. on Σ. Henceforth $\mathcal{H}^1$-a.e. $x_0 \in \Sigma$ is a jump point of ϕ.

Finally, (6.6) and the inclusion $J_\phi \subset \Sigma$ follow by the fact that any blow-up limit ϕ_∞ at points $x \notin \Sigma$ is constant. Indeed, $e^{i\phi_\infty} = -\nabla^\perp g_\infty$ is constant, (being g_∞ concave and affine, see [ALR]) and $U_{\phi_\infty} = 0$, so that ϕ_∞ is constant by Proposition 2.3. $\square$

In conclusion, the statements made in Theorem 1.1 of the introduction follow by Theorem 6.3 with the only exception of (1.6). The latter follows by applying Lemma 3.5 to the vectorfield $T^a u$, with $K = J_\phi$.

Theorem 6.4. *Let u, ϕ as in Theorem 6.3 and assume that*

$$\mathcal{H}^1(\overline{\Sigma} \cap \Omega \setminus \Sigma) = 0,$$

where Σ is defined by (6.5). Then μ_ϕ is concentrated on J_ϕ and therefore is a 1-dimensional rectifiable measure.

Proof. Let g be a 1-Lipschitz function such that $u = -\nabla^\perp g$ and recall that Σ coincides, up to $\mathcal{H}^1$-negligible sets, with J_ϕ. The blow-up argument in [ALR] shows that g is a viscosity solution of the eikonal equation $|\nabla g|^2 - 1 = 0$ in set $\Omega \setminus \Sigma$, since $U_{\phi_\infty} = 0$ for any blow-up function ϕ_∞ at any point $x \in \Omega \setminus \Sigma$. Therefore, g is locally semiconcave in the open set $A := \Omega \setminus \overline{\Sigma}$ and its gradient (and u as well) is a BV_{loc} function in A. By Proposition 2.3 we obtain that $\phi \in BV_{\mathrm{loc}}(A)$ and (3.4) gives $\mu_\phi \llcorner A = 0$ because $A \cap J_\phi$ is $\mathcal{H}^1$-negligible. Therefore μ_ϕ is supported on $\overline{\Sigma}$ and the absolute continuity of μ_ϕ with respect to $\mathcal{H}^1$ leads us to the conclusion. $\square$

References

[ARS] F. Alouges, T. Rivière and S. Serfaty, Néel Walls and Cross-tie Walls for micromagnetic materials having a strong planar anisotropy, COCV (2002), Volume à la mémoire de J.L.Lions.

[A] L. Ambrosio, Variational problems in SBV and image segmentation, *Acta Appl. Math.*, **17** (1989), 1-40.

[A1] L. Ambrosio, Metric space valued functions of bounded variation, *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* (4), (1990), 439-478.

[AD] L. Ambrosio and N. Dancer, Calculus of variations and partial differential equations. Topics on geometrical evolution problems and degree theory. Papers from the Summer School held in Pisa, September 1996, Edited by G.Buttazzo, A.Marino and M.K.V.Murthy, Springer, 2000.

[ADM] L. Ambrosio, C. De Lellis, and C. Mantegazza, Line energies for gradient vector fields in the plane, *Calc. Var. PDE*, **9** (1999), 327-355.

[AFP] L. Ambrosio, N. Fusco, and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, 2000.

[ALR] L. Ambrosio, M. Lecumberry and T. Riviere, A viscosity property of minimizing micromagnetic configurations, submitted to CPAM, 2002.

[AG1] P. Aviles and Y. Giga, A mathematical problem related to the physical theory of liquid crystals configurations, *Proc. Centre Math. Anal. Austral. Nat. Univ., 12,* (1987), 1-16.

[AG2] P. Aviles and Y. Giga, On lower semicontinuity of a defect obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields, *Proc. Royal Soc. Edinburgh, Sec A, 129,* (1999), 1-17.

[CEL] M. Crandall, L.C. Evans and P.L.Lions, Some Properties of Viscosity Solutions of Hamilton-Jacobi Equations, *Trans. Amer. Math. Soc.*, 282, (1984), 487-502

[DeG] E. De Giorgi, Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio a r dimensioni, *Ricerche Mat.*, **4** (1955), 95-113.

[DeGA] E.De Giorgi, L.Ambrosio, Un nuovo funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8) Mat. Appl.*, **82** (1988), 199-210.

[DKMO1] A. DeSimone, R.V. Kohn, S. Müller and F. Otto, A compactness result in the gradient theory of phase transitions, *Proc. Royal Soc. Edinburgh.*, **131** (2001), 833-844.

[DKMO2] A. DeSimone, R.V. Kohn, S. Müller and F. Otto, Magnetic microstructures, a paradigm of multiscale problems, to appear in *Proceedings of ICIAM*, (1999).

[DLM] R. DiPerna, P.L. Lions and Y. Meyer, L^p regularity of velocity averages *Annales IHP, Analyse non linéaire*, 8, (1991), 271-287

[DO] C. De Lellis, F. Otto, Structure of entropy solutions: application to variational problems. preprint.

[EG] L.C. Evans and R.F. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advances Mathematics, (1992)

[ET] I. Ekeland and R. Temam, Analyse Convexe et Problèmes Variationnels, Dunod Gauthier-Villars, 1974.

[F] K.J. Falconer, The Geometry of Fractal Sets, Cambridge University Press, 1985.

[JK] W. Jin and R. Kohn, Singular Perturbation and the Energy of Folds, *Journal of Nonlinear Science, 10, No. 3*, (2000), 355-390.

[JOP] P.E. Jabin, F. Otto and B. Perthame, Line Energy Ginzburg-Landau models: zero energy states, preprint (2001).

[JP] P.E. Jabin and B. Perthame, Compactness in Ginzburg-Landau energy by kinetic averaging, *Comm. Pure and App. Math.*, 54, (2001), 1096-1109.

[Le] M. Lecumberry, The structure of micromagnetic defects, PhD Thesis in preparation.

[LR] M. Lecumberry and T. Rivière, Regularity for micromagnetic configurations having zero jump energy, to appear in *Calc. of Var. and P.D.E.* (2002).

[Pr] D. Preiss, Geometry of measures in $\mathbf{R}^n$: distributions, rectifiability, and densities, *Ann. of Math.*, **125** (1987), 537-643.

[RS1] T. Rivière and S. Serfaty, Limiting Domain Wall Energy for a Problem Related to Micromagnetics, *Comm. Pure Appl. Math.*, 54, (2001), 294-338.

[RS2] T. Rivière and S. Serfaty, Compactness, kinetic formulation and entropies for a problem related to micromagnetics, to appear in *Comm. in P.D.E.* (2002).