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Pietro Corvaja, Julian Lawrence Demeio, Ariyan Javanpeykar, Davide Lombardo
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On the distribution of rational points on ramified covers of abelian varieties

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ABSTRACT

We prove new results on the distribution of rational points on ramified covers of abelian varieties over finitely generated fields k of characteristic zero. For example, given a ramified cover $\pi : X \rightarrow A$, where A is an abelian variety over k with a dense set of k -rational points, we prove that there is a finite-index coset $C \subset A(k)$ such that $\pi(X(k))$ is disjoint from C . Our results do not seem to be in the range of other methods available at present; they confirm predictions coming from Lang’s conjectures on rational points, and also go in the direction of an issue raised by Serre regarding possible applications to the inverse Galois problem. Finally, the conclusions of our work may be seen as a sharp version of Hilbert’s irreducibility theorem for abelian varieties.

1. Introduction

Let X be a ramified cover of an abelian variety over a number field K , i.e. there is a finite surjective non-étale morphism $X \rightarrow A$ with X normal and irreducible. The aim of this paper is to prove novel results on the distribution of rational points on X .

In this work we are guided by Lang’s conjectures on varieties of general type, and by a question of Serre on covers of abelian varieties. By Kawamata’s structure result for ramified covers of abelian varieties [Kaw81], for every ramified cover $X \rightarrow A$ of an abelian variety A over a number field K , there is a finite field extension L/K and finite étale cover $X' \rightarrow X_L$ such that X' dominates a positive-dimensional variety of general type over L . Assuming Lang’s conjecture [Jav20, Lan86], it follows that the K -rational points $X(K)$ are *not* Zariski-dense in X . Despite Faltings’ deep finiteness theorems for integral (and rational) points on subvarieties of abelian varieties [Fal94], we are far from proving such desired non-density results for rational points on ramified covers of abelian varieties (even in dimension two) over number fields.

Lang’s aforementioned conjecture predicts that, for any abelian variety A over a number field K with $A(K)$ dense and any ramified cover $\pi : X \rightarrow A$, the set $A(K) \setminus \pi(X(K))$ is again dense. Motivated by the inverse Galois problem, Serre also raised a related question [Ser08, §5.4, Problem], a positive answer to which would lead to the same conclusion. Our main results stated in the following (see Theorems 1.3 and 1.4) confirm the density of $A(K) \setminus \pi(X(K))$

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unconditionally for all abelian varieties, and thereby provide novel evidence for Lang’s conjectures on rational points and answer, in part, Serre’s question.

We stress that there are no prior results concerning the distribution of rational points on ramified covers of abelian varieties, except for some very special cases, and the same goes for integral points on ramified covers of \mathbb{G}_m^n . Our current work provides the *first* step towards understanding the Diophantine properties of ramified covers of abelian varieties.

The conclusion of our main theorem that $A(K) \setminus \pi(X(K))$ is dense bears a strong resemblance to the geometric formulation of Hilbert’s irreducibility theorem. In fact, this point of view is quite fruitful and key to many of the results of this paper.

In its simplest form, Hilbert’s irreducibility theorem states that, given a number field K and an irreducible polynomial $p(t, x_1, \dots, x_s)$ with coefficients in K , there exist infinitely many $a \in K$ such that the specialization $p(a, x_1, \dots, x_s)$ is an irreducible polynomial in the variables x_1, \dots, x_s . A similar statement applies to the simultaneous specialization of finitely many irreducible polynomials.

Following the work of several authors, Hilbert’s irreducibility theorem has been recast in geometric language as follows. An irreducible polynomial $p(t, x)$ defines in a natural way an irreducible cover (that is, a finite surjective morphism) of an open subvariety of \mathbb{P}_1 over K : this is obtained simply by projecting the zero locus of p on the t -line. Hilbert’s irreducibility theorem can then be interpreted as stating that ‘many’ fibres of this cover are irreducible over K and hence, in particular, that they have no K -rational points; in turn, this is also equivalent to ‘many’ rational points of \mathbb{P}_1 not being the image of rational points in the zero locus of p . Similar constructions can be made in the case of several variables, and this point of view is encapsulated in the following definition of Serre (for this and much more about the Hilbert property we refer the reader to the books of Serre [Ser08, Chapter 3] and Fried and Jarden [FJ08]).

DEFINITION 1.1. An integral, quasi-projective variety X over a field k satisfies the *Hilbert property over k* if, for every finite collection of finite surjective morphisms $(\pi_i : Y_i \rightarrow X)_{i=1}^n$ with Y_i a normal (integral) variety over k and $\deg \pi_i \geq 2$, the set $X(k) \setminus \bigcup_{i=1}^n \pi_i(Y_i(k))$ is dense in X .

In this language, Hilbert’s irreducibility theorem states that every projective space over a number field K has the Hilbert property. As the Hilbert property is a birational invariant, it follows that every rational variety over K has the Hilbert property.

The arithmetic nature of the ground field is of course essential for the truth of Hilbert’s irreducibility theorem, and a field k is called *Hilbertian* if \mathbb{P}_1 has the Hilbert property over k (or equivalently, if \mathbb{P}_n has the Hilbert property over k for every n). In this paper we focus on the case of finitely generated fields of characteristic zero, which are known to be Hilbertian.

Clearly not every variety has the Hilbert property: for example, in [CZ17] it is shown that, as a consequence of the Chevalley–Weil theorem, a smooth proper variety over a number field with the Hilbert property is geometrically simply connected. Nonetheless, the literature is by now very rich with examples of varieties with the Hilbert property. Several proofs exist for the case of projective space (see, for example, [BG06, §9.6], [FJ08, Chapter 11], [Sch00, §4.4], [Ser97, Chapter 9], [Ser08, Chapter 3] and [Völ96]). Note that connected reductive algebraic groups have been shown in [CTS87] to have the Hilbert property over Hilbertian fields; note that these varieties are rational over \bar{k} , but not necessarily over k). Some sporadic examples of varieties with the Hilbert property are also known (such as the K3 surface $x^4 + y^4 = z^4 + w^4 \subset \mathbb{P}_{3,\mathbb{Q}}$, which is not even geometrically unirational [CZ17, Theorem 1.4]; see also [Dem19] and [Dem20]).

Hilbert’s irreducibility theorem and its extensions have led to a number of applications (see, for example, [DZ07, Zan00]), typically motivated by the following observation: if a certain desired

object can be realized *geometrically*, and if the base space of the geometric construction has the Hilbert property, then the situation can be specialized to an *arithmetic* one. The point is of course that various tools are available on the geometric side that might make certain constructions more natural than they would be if they were carried out directly on the arithmetic side. Concretely, this has been implemented for example by Néron [Nér52], who used this idea to construct elliptic curves defined over \mathbb{Q} of large rank. By far the most important such application is to the inverse Galois problem (this was also Hilbert’s original motivation): if a group G can be realized as the Galois group of a cover $Y \rightarrow X$ defined over \mathbb{Q} , with X having the Hilbert property, then G is a Galois group over \mathbb{Q} . Noether famously used this idea to prove that S_n is a Galois group over \mathbb{Q} for all n .

Given the significance of the Hilbert property for number-theoretical questions, one would wish to extend Hilbert’s theorem to other classes of varieties in addition to the rational varieties. However, there are serious restrictions to possible generalizations of the classical results, starting with the aforementioned fact that the Hilbert property can only hold for geometrically simply connected varieties [CZ17].

Nevertheless, it is conjectured that, for a variety X over a finitely generated field k of characteristic zero with $X(k)$ Zariski-dense, the only obstruction to the Hilbert property should indeed come from unramified covers (see Conjecture 1.7 for a precise statement). Further taking into account that the original version of Hilbert’s irreducibility theorem does not need to contend with any unramified covers at all (since \mathbb{P}_n is geometrically simply connected), it seems natural to rule out such covers at the outset. This leads to the following notion.

DEFINITION 1.2 (Corvaja–Zannier). A smooth proper variety X over a field k has the *weak-Hilbert property over k* if, for all finite collections of finite surjective *ramified* morphisms $(\pi_i : Y_i \rightarrow X)_{i=1}^n$ with each Y_i an integral normal variety over k , the set $X(k) \setminus \bigcup_{i=1}^n \pi_i(Y_i(k))$ is Zariski-dense in X .

Note that we restrict ourselves to finite surjective *morphisms*, as opposed to Corvaja and Zannier who considered dominant *rational maps* of finite degree in their definition of the weak-Hilbert property (see [CZ17, §2.2]). By a standard argument using Stein factorization, one can easily show that these two definitions are equivalent (see, for example, the proof of Theorem 7.11). In addition, we stress that the weak-Hilbert property is a birational invariant amongst smooth proper varieties (see Proposition 3.1). As a consequence, all our results concerning this property automatically propagate between smooth proper varieties within the same birational equivalence class.

The weak-Hilbert property can be applied to the inverse Galois problem similarly as the classical Hilbert property; see Remark 1.5. It is studied (using some of the results of this paper) in [BSFP22, Jav22, GCM21].

Our main (though not only) focus in this paper is the weak-Hilbert property for abelian varieties over finitely generated fields of characteristic zero.

THEOREM 1.3. *Let k be a finitely generated field of characteristic zero, let A be an abelian variety over k , let $\Omega \subset A(k)$ be a Zariski-dense subgroup, and let $(\pi_i : Y_i \rightarrow A)_{i=1}^n$ be a finite collection of ramified covers with each Y_i a normal integral variety. Then, there is a finite index coset $C \subset \Omega$ such that, for every $c \in C$ and every $i = 1, \dots, n$, the scheme $Y_{i,c}$ has no k -points. In particular, if $A(k)$ is dense, then A has the weak-Hilbert property over k .*

We prove a stronger conclusion on the scheme-theoretic fibres $Y_{i,c}$, assuming however in addition that the ramified covers $\pi_i : Y_i \rightarrow A$ do not have any non-trivial étale subcovers.

Here, we say that $Y_i \rightarrow A$ has no non-trivial étale subcovers if, for every cover $X_i \rightarrow A$ of degree greater than one such that $Y_i \rightarrow A$ factors over $X_i \rightarrow A$, we have that $X_i \rightarrow A$ is ramified. Note that such a covering $Y_i \rightarrow A$ is ramified if it is of degree greater than one. Indeed, otherwise $Y_i \rightarrow A$ would be an étale subcover of itself.

THEOREM 1.4. *Let k be a finitely generated field of characteristic 0, let A be an abelian variety over k , let $\Omega \subset A(k)$ be a Zariski-dense subgroup and, for $i = 1, \dots, n$, let Y_i be a normal integral variety, and let $\pi_i : Y_i \rightarrow A$ be a finite surjective morphism with no non-trivial étale subcovers. Then, there is a finite index coset $C \subset \Omega$ such that, for every c in C and every $i = 1, \dots, n$, the k -scheme $Y_{i,c}$ is integral.*

Remark 1.5 (Inverse Galois problem). Let G be a finite group, let X be a normal geometrically integral variety over \mathbb{Q} , and let $X \rightarrow A$ be a G -Galois cover of an abelian variety A over \mathbb{Q} with no non-trivial étale subcovers. Then, if $A(\mathbb{Q})$ is dense, by our version of Hilbert’s irreducibility theorem for abelian varieties (Theorem 1.4), there is a Galois number field K with Galois group G .

Note that Theorems 1.3 and 1.4 represent the usual parallelism between irreducible fibres and fibres with no rational points. In addition, as alluded to previously, one can easily extend these results to obtain similar conclusions for finite collections of dominant *rational maps* of finite degree (see Theorem 7.11). Furthermore, we stress that the assumption that the subgroup Ω is Zariski-dense in Theorems 1.3 and 1.4 can not be weakened to the assumption that Ω is merely infinite (see Remark 7.9).

The two results directly generalize [Zan10, Theorem 2] in which the case of an abelian variety isomorphic to a power of a non-CM elliptic curve was handled. In addition, we stress that in Theorem 1.3 one cannot expect the integrality of the fibres $Y_{i,c}$. Indeed, the presence of non-trivial étale subcovers obstructs the desired integrality; see Remark 7.8 for a precise statement.

Remark 1.6 (About our proofs). Working with pairs (A, Ω) in Theorems 1.3 and 1.4 has several technical advantages. For example, an induction argument allows one to reduce from a finite collection of covers to just a single cover (Lemma 4.14). Moreover, by a specialization argument, we can easily pass from number fields to arbitrary finitely generated fields of characteristic zero (see § 7.1). Furthermore, using local–global arguments, it suffices to construct a *single* point in Ω over which the fibre is integral (Corollary 5.4). Finally, using structure results for vertically ramified covers (Lemma 2.17), we establish a product theorem for pairs (A, Ω) which allows us to consider only pairs (A, Ω) with Ω cyclic; see § 5.2 for precise statements.

As already hinted at, the weak-Hilbert property is conjectured to hold as soon as the ‘obvious’ necessary conditions are met; more precisely, one has the following (see [Cam11, § 4] for the definition of a ‘special’ smooth proper connected variety).

CONJECTURE 1.7 (Campana, Corvaja–Zannier). Let X be a smooth proper geometrically connected variety over a finitely generated field k of characteristic zero. Then the following are equivalent.

1. There is a finite field extension L/k such that X_L has the weak-Hilbert property over L .
2. There is a finite field extension M/k such that $X(M)$ is dense in X .
3. The smooth proper connected variety $X_{\bar{k}}$ is special (in the sense of Campana [Cam11, § 4]).

Frey and Jarden [FJ74] proved that an abelian variety A over a finitely generated field k of characteristic zero admits a finite extension L/k such that $A(L)$ is Zariski-dense in A (see also [Jav21a, § 3] and [HT00]). As abelian varieties are special [Cam11, § 4], by combining Frey and

Jarden's result with Theorem 1.3 we obtain a proof of Conjecture 1.7 for any variety which is birational to a smooth projective connected variety with trivial tangent bundle.

COROLLARY 1.8. *Let k be a finitely generated field of characteristic zero. Let X be a smooth proper geometrically connected variety over k which is birational to a smooth proper variety with trivial tangent bundle. Then $X_{\bar{k}}$ is special [Cam11, § 4], and there is a finite field extension L/k such that X_L has the weak-Hilbert property over L .*

Note that in view of Campana's perspective on special varieties, it is natural to study the influence of some positivity condition on the tangent bundle; the varieties under consideration in this work being precisely those with trivial tangent bundle. According to a classical theorem of Mori, a smooth projective variety with ample tangent bundle is geometrically isomorphic to a projective space, so that the Hilbert property for such varieties follows from Hilbert's original irreducibility theorem (after a possible field extension).

Motivated by Conjecture 1.7 and inspired by fundamental properties of special varieties [Cam11] and of varieties with a dense set of rational points, we establish several basic facts about the class of varieties with the weak-Hilbert property in § 3, first and foremost among them the following product theorem.

THEOREM 1.9. *Let X and Y be smooth proper varieties over a finitely generated field k of characteristic zero with the weak-Hilbert property over k . Then $X \times Y$ has the weak-Hilbert property over k .*

Theorem 1.9 was obtained jointly by the third-named author and Olivier Wittenberg, and we are grateful to Olivier Wittenberg for allowing us to include this result here.

It seems worthwhile pointing out that Theorem 1.9 is the 'weak-Hilbert' analogue of Bary-Soroker, Fehm and Petersen's product theorem for the (usual) Hilbert property [BSFP14]. This product theorem was stated by Serre as a problem in [Ser08, § 3.1].

Note that Theorem 1.9 actually reproves the product theorem of [BSFP14] when the base field is finitely generated and of characteristic zero. Indeed, if X and Y satisfy the Hilbert property (hence, weak-Hilbert property) over a finitely generated field k of characteristic zero, then $X \times Y$ has the weak-Hilbert property over k by Theorem 1.9. However, because X and Y are geometrically simply connected (by Corvaja and Zannier's theorem [CZ17, Theorem 1.6]), it follows that $X \times Y$ is geometrically simply connected (by the product property for étale fundamental groups). Thus, the smooth proper variety $X \times Y$ is geometrically simply connected and has the weak-Hilbert property over k . Therefore, by definition, it has the Hilbert property over k .

In addition to proving Theorem 1.9, in § 3 we also show that the weak-Hilbert property of a variety X is inherited by both its étale covers and its surjective images under some natural assumptions (in particular, including smooth surjective images), thus building a toolkit that simplifies proving new instances of the weak-Hilbert property.

To conclude this introduction we describe the ingredients that go into the proof of Theorem 1.3, which is the main result of this paper. Our argument follows the same broad lines as the proof of [Zan10, Theorem 2], which handled the special case of $A = E^n$ being the power of a non-CM elliptic curve with a *non-degenerate* point P , that is, a rational point P generating a Zariski-dense subgroup. The method is based on the following idea, which we simplify slightly for ease of exposition. Suppose that A is defined over a *number field* K . Given a cover $\pi : Y \rightarrow A$, one proves that there is a prime \mathfrak{p} of \mathcal{O}_K and a torsion point $\zeta \in A(\mathcal{O}_K/\mathfrak{p})$ that does not lift to $Y(\mathcal{O}_K/\mathfrak{p})$. Any rational point $Q \in A(K)$ reducing to ζ modulo \mathfrak{p} then also does not

lift to $Y(K)$, and, because this condition is adelicly open and using the group structure on A , we get the desired Zariski-dense subset of points that do not lift to Y . Naturally this requires that there is at least one $Q \in A(K)$ reducing to ζ : the last task remaining is then to prove the existence of such a point Q .

Despite the basic strategy being the same, however, we need to introduce several new ingredients with respect to [Zan10], and to reinterpret various parts of Zannier's approach in a more general context. We now briefly describe where the main novelties lie.

Some technical reductions, carried out in §4, show that multiple variants of the weak-Hilbert property for abelian varieties are essentially all equivalent. These variants describe how the weak-Hilbert property interacts with the group structure of A , and allow us to pass from statements about fibres of covers having no k -rational points to statements about fibres being irreducible over k . The core argument of our proof is contained in §6, where we study the case of abelian varieties possessing a non-degenerate point. Unlike in the case of $A = E^n$, which can essentially be reduced to the analysis of a single elliptic curve, we do not have at our disposal the full strength of Serre's open image theorem [Ser72]. In addition, some explicit computations with torsion points that are accessible in dimension one would become extremely cumbersome in general. We bypass these problems by giving a more streamlined construction of the torsion point ζ (see §6.1) and by replacing the open image theorem by an appeal to several deep results in the Kummer theory of abelian varieties (see §6.2).

All that is left to do is then to extend the result to all abelian varieties. Up to k -isogeny, any abelian variety A over k is a direct product of k -simple abelian varieties A_i , and if $A(k)$ is Zariski-dense, then $A_i(k)$ is Zariski-dense for all i . As any point of infinite order on a *simple* abelian variety is non-degenerate, defining $\Omega_i = \langle P_i \rangle$ for P_i a point of infinite order in $A_i(k)$, we may conclude at this point that, for every i , the pair (A_i, Ω_i) satisfies the conclusion of Theorem 1.3. The results of §3 (in particular, Theorem 1.9) can then be used to show that A has the weak-Hilbert property over k . However, the more precise version given by Theorem 1.3 (or Theorem 1.4) does not follow as easily. This is why in §5.2 we extend the techniques developed in §3 to prove a more specific version of Theorem 1.9, in which the factors are abelian varieties and we also take into account a (Zariski-dense) subgroup Ω . A key observation in this section is that, given a cover $Z \rightarrow A$, the existence of a *single* point $P \in \Omega$ for which the fibre Z_P is integral implies that there is a finite index coset C of Ω such that, for every c in C , the fibre over c has no K -points; see Corollary 5.4. Using the invariance of the weak-Hilbert property under isogeny, it is then a fairly straightforward matter to deduce from this the general case of our result for k a number field. Finally, we use a specialization argument to reduce from finitely generated fields of characteristic zero to number fields.

Outline of paper

In §2 we gather some preliminaries. Notably, we provide a structure result for vertically ramified covers of products (Lemma 2.17). In the following section we prove that the class of varieties over a finitely generated field of characteristic zero with the weak-Hilbert property is closed under products, finite étale covers, and smooth images. In §4 we introduce the class of (PB)-covers, i.e. ramified covers of abelian varieties with no non-trivial étale subcovers. We prove several basic properties of (PB)-covers, and provide links between variants of the Hilbert property for abelian varieties. In §5 we prove a product theorem, analogous to that obtained in §3, that applies to a *variant* of the weak-Hilbert property specific to abelian varieties. Then, in §6, we prove that this property holds for abelian varieties over number fields endowed with a non-degenerate point. Finally, in §7 we prove the theorems stated in the introduction: the results of the previous

sections suffice to handle the case of the ground field being a finite extension of \mathbb{Q} , and the general case is then proven by reduction to the number field case.

2. Preliminaries

2.1 Notation

Throughout the paper we let K denote a number field, whereas we write k for a general field (unless otherwise specified). For a number field K , we denote by M_K the set of its places, and by M_K^{fin} the subset of finite places.

A *variety* over a field k is an integral separated scheme of finite type over k . If X is a variety over k and $A \subset k$ is a subring, we define a *model* for X over A to be a pair (\mathcal{X}, ϕ) , where \mathcal{X} is a separated scheme of finite type over A and $\phi : \mathcal{X} \times_A k \rightarrow X$ is an isomorphism; we often omit ϕ from the notation.

A morphism $\pi : Y \rightarrow X$ of normal varieties over k is a *cover of X (over k)* if π is finite and surjective.

For a morphism $f : Y \rightarrow X$ of schemes, and a point $c \in X$, we denote the scheme-theoretic fibre of f over c by Y_c , or by $f^{-1}(c)$ when we need to specify the morphism to avoid ambiguity.

Let A be an abelian variety over a field k . For a prime ℓ different from the characteristic of k , we let $T_\ell A := \varprojlim_{n \rightarrow \infty} A[\ell^n]$ denote the ℓ -adic Tate module of A , where $A[\ell^n]$ is, by convention, the full geometric torsion subgroup $A[\ell^n](\bar{k})$. We similarly denote by $A[\ell^\infty]$ the union of all $A[\ell^n](\bar{k})$ for $n \geq 1$. We denote by $\text{Gal}(k'/k)$ the Galois group of a (possibly infinite) Galois field extension k'/k , and simply by Γ_k the absolute Galois group of k , namely $\Gamma_k = \text{Gal}(\bar{k}/k)$. For a rational (respectively, ℓ -adic) number $a \neq 0$, we define $v_\ell(a)$ to be the unique integer such that $a = a_1/a_2 \cdot \ell^{v_\ell(a)}$, with $a_1, a_2 \in \mathbb{Z}$ (respectively, $a_1, a_2 \in \mathbb{Z}_\ell$) and $\ell \nmid a_i, i = 1, 2$. If $a = 0$, we let $v_\ell(a) := \infty$ by convention.

2.2 Unramified morphisms

As regards unramified morphisms, we follow the conventions of the Stacks project [Sta20, Tag 02G3]: in particular, a morphism of schemes $Y \rightarrow X$ is unramified if and only if it is locally of finite type and its diagonal is an open immersion [Sta20, Tag 02GE]. We say that a morphism locally of finite type (e.g. a cover $Y \rightarrow X$) is *ramified* if it is not unramified. We need the following lemma.

LEMMA 2.1. *Let $f : X \rightarrow S$ be a morphism of normal proper varieties over a field k and let $\pi : Z \rightarrow X$ be a finite surjective ramified morphism. Assume that the branch locus D of $\pi : Z \rightarrow X$ dominates S (i.e. $f(D) = S$). Then, for every point s in S , the morphism $Z_s \rightarrow X_s$ is finite surjective ramified.*

Proof. A morphism of varieties $V \rightarrow W$ over k is unramified if and only if, for every w in W , the morphism $V_w \rightarrow \text{Spec } k(w)$ is unramified (i.e. étale); see [Sta20, Tag 00UV]. Now, let s be a point of S . To show that the finite surjective morphism $Z_s \rightarrow X_s$ is ramified, let $d \in D$ be a point lying over s . Then, by the definition of the branch locus, $Z_d \rightarrow \text{Spec } k(d)$ is ramified. Note that $Z_d = Z_s \times_{X_s} d$ as schemes over $d = \text{Spec } k(d)$. As the fibre of $Z_s \rightarrow X_s$ over d is ramified, it follows that $Z_s \rightarrow X_s$ is ramified. \square

An unramified cover $X \rightarrow Y$ of varieties might not be étale [Cut18, Exercise 21.89], but this holds whenever the target is geometrically unibranch [Sta20, Tag 0BQ2], as we now show. (Recall that a scheme Y is *equidimensional* if every irreducible component has the same dimension.)

LEMMA 2.2. *Let X be a geometrically unibranch integral scheme and let $\pi : Y \rightarrow X$ be a finite surjective unramified morphism of schemes with Y equidimensional. Then π is étale.*

Proof. By [Sta20, Tag 04HJ], there is a surjective étale morphism $f : U \rightarrow X$ such that $Y_U := Y \times_{\pi, X, f} U$ has a finite disjoint union decomposition

$$Y_U = \bigsqcup_j V_j$$

such that each $V_j \rightarrow U$ is a closed immersion. Refining this decomposition if necessary, we may assume that each V_j is connected. As X is a geometrically unibranch integral scheme and $U \rightarrow X$ is a surjective étale morphism, it follows that U is a disjoint union of integral schemes. Indeed, each connected component of U is reduced because $U \rightarrow X$ is étale and X is integral, hence reduced. Furthermore, each connected component of U is irreducible, for two irreducible components would meet at some point u , and the local ring at u would have two minimal primes, contradicting the fact that X is geometrically unibranch [Sta20, Tag 06DM]. Let U_i be the connected component containing the image of $V_j \rightarrow U$. Then the restriction of the closed immersion $V_j \rightarrow U$ to the integral scheme U_i is a dominant closed immersion, as $V_j \hookrightarrow Y_U$ is an open and closed immersion, $Y_U \rightarrow U$ is finite surjective and Y_U is equidimensional (because Y is equidimensional). It follows that each non-trivial $V_j \rightarrow U_i$ is an isomorphism. This implies that $Y_U \rightarrow U$ is étale, so that $Y \rightarrow X$ is étale by étale descent. \square

The following consequence is well known and used repeatedly throughout the paper.

LEMMA 2.3. *Let X be an integral normal noetherian scheme, and let $\pi : Y \rightarrow X$ be a finite surjective morphism of integral schemes. Then π is either ramified or étale.*

Proof. Assume that π is unramified. Then, because X is a normal scheme, it is geometrically unibranch [Sta20, Tag 0BQ3]. Therefore, it follows from Lemma 2.2 that π is étale, as required. \square

2.3 Galois closures

Let $\pi : Y \rightarrow X$ be a cover of normal varieties over a field k of characteristic zero, and let $G(Y/X)$ be the automorphism group of Y over X . The arguments in the proof of [Cut18, Proposition 21.67] show the following.

PROPOSITION 2.4. *The canonical homomorphism $G(Y/X) \rightarrow \text{Aut}(k(Y)/k(X))^{\text{opp}}$ is an isomorphism.*

The cover $\pi : Y \rightarrow X$ is called *Galois* if $\#G(Y/X) = \deg \pi$. In this case we also write $\text{Gal}(Y/X)$ for $G(Y/X)$. We let $\widehat{Y} \rightarrow X$ be the normalization of X in the Galois closure of $k(Y)$ over $k(X)$; we note that the composed cover $\widehat{Y} \rightarrow X$ (which we commonly refer to as the *Galois closure of $\pi : Y \rightarrow X$*) is Galois. By Proposition 2.4, the morphism $\pi : Y \rightarrow X$ is Galois if and only if the field extension $k(Y)/k(X)$ is. Moreover, if $\pi : Y \rightarrow X$ is étale, then π is Galois if and only if it is Galois in the sense of [Sta20, Tag 03SF].

DEFINITION 2.5. If $\pi : Y \rightarrow X$ is Galois with Galois group $G = \text{Gal}(Y/X)$ and H is a subgroup of G , we let Y/H be the normalization of X in $k(Y)^H$, where $k(Y)^H$ is the subfield of $k(Y)$ fixed by H . Note that Y/H is a normal (integral) variety over k , and that one could equivalently describe Y/H as the (geometric) quotient of Y by H .

By Galois theory for $k(X) \subset k(Y)$ and Zariski’s main theorem that a birational cover of a normal variety is an isomorphism, we have the following geometric version of Galois correspondence.

PROPOSITION 2.6. *Assume that $\pi : Y \rightarrow X$ is a Galois cover with group G . There is a bijection between subgroups H of G and intermediate covers $Y \rightarrow Z \rightarrow X$ with Z normal and integral. The correspondence is given by $H \mapsto [Y \rightarrow Y/H \rightarrow X]$.*

Remark 2.7. (i) Let $k \subset F \subset F'$ be field extensions such that $\bar{k} \cap F' = k$ (i.e. \bar{k} and F' are linearly disjoint over k) and F'/F is finite. Let $\widehat{F'}/F$ be the Galois closure of F'/F , let $M/F\bar{k}$ be the Galois closure of $F'\bar{k}/F\bar{k}$ and write $L = \widehat{F'} \cap \bar{k}$. Note that L is a finite extension of k . Then $\widehat{F'}$ is also the Galois closure of $F'L$ over FL ; hence, in particular, we have $\widehat{F'} \otimes_L \bar{k} \cong \widehat{F'}\bar{k} = M$.

(ii) Proposition 2.4 and remark (i) have the following immediate consequence. If $W \xrightarrow{\phi} V$ is a cover of varieties over k , $\widehat{W} \rightarrow V$ is the Galois closure of ϕ , and $\text{Spec } L/\text{Spec } k$ is the normalization of $\text{Spec } k$ in \widehat{W} , then $\widehat{W} \times_{\text{Spec } L} \text{Spec } \bar{k}$ is the Galois closure of $W \times_k \bar{k} \xrightarrow{\phi} V \times_k \bar{k}$.

Remark 2.8. Let E'/E be a finite separable extension, and $\widehat{E'}/E$ be its Galois closure, of Galois group G , and let H be such that $\widehat{E'}^H = E'$. Then $\widehat{E'} \otimes_E E' \cong \bigoplus_{r \in G/H} \widehat{E'}^r$ as $\widehat{E'}$ -algebras.

The following proposition and its corollary tell us how Galois closure of covers behaves under smooth base change.

PROPOSITION 2.9. *Let F/E be a finite separable field extension, let E'/E be a field extension and let \widehat{F}/F be the Galois closure of F over E . Assume that $F_{E'} := E' \otimes_E F$ is a field, and let $\widehat{F}_{E'}$ be the Galois closure of $E' \otimes_E F$ over E' . There exists a surjective morphism $E' \otimes_E \widehat{F} \rightarrow \widehat{F}_{E'}$ that restricts to the identity on $E' \otimes_E F$.*

Proof. There is a canonical embedding $\iota : F \hookrightarrow E' \otimes_E F \hookrightarrow \widehat{F}_{E'}$ that is the identity on E . The field $\widehat{F}_{E'}$ is normal over E' , and contains F (through the embedding ι). Hence, there exists an embedding $\widehat{F} \hookrightarrow \widehat{F}_{E'}$ which restricts to the identity on E . Consider the morphism $\varphi : E' \otimes_E \widehat{F} \rightarrow \widehat{F}_{E'}$ which, by the universal property of the tensor product, is induced by the embedding $E' \hookrightarrow E' \otimes_E F \hookrightarrow \widehat{F}_{E'}$ and the natural embedding $\widehat{F} \hookrightarrow \widehat{F}_{E'}$. The field \widehat{F} is generated over E by the roots $\alpha_1, \dots, \alpha_n$ of a separable polynomial with coefficients in E , and these same roots also generate $\widehat{F}_{E'}$ over E' . As both E' and $\alpha_1, \dots, \alpha_n$ are contained in the image of φ , this shows that φ is surjective as desired. □

COROLLARY 2.10. *Let $Z \rightarrow X$ be a cover of normal varieties over k , let $X' \rightarrow X$ be a smooth morphism of varieties over k , and let $\widehat{Z} \rightarrow Z \rightarrow X$ be the Galois closure of $Z \rightarrow X$. Then $Z' := Z \times_X X'$ and $\widehat{Z} \times_X X'$ are normal. Assume that $Z' := Z \times_X X'$ is connected. Then Z' is an integral normal scheme and, if $Z'' \rightarrow Z' \rightarrow X'$ is the Galois closure of $Z' \rightarrow X'$, then there is an open and closed embedding $Z'' \rightarrow \widehat{Z} \times_X X'$ that commutes with projection to Z' .*

Proof. By [Sta20, Tag 034F], the schemes Z' and $\widehat{Z} \times_X X'$ are normal. In particular, if Z' is connected, then it is integral (as it is connected and normal). Define

$$E := k(X), \quad E' := k(X'), \quad F := k(Z).$$

Note that

$$k(\widehat{Z}) = \widehat{F}, \quad k(Z') = E' \otimes_E F, \quad k(Z'') = \widehat{F}_{E'}.$$

The result now follows from Proposition 2.9 and the universal property of normalization. □

2.4 Action on the fibres

Let X be a normal variety over a field k . Let G be a finite group and let $\phi : Y \rightarrow X$ be an étale (right) G -torsor over X , so that Y is also normal. In this section we discuss various properties of the action of G on the fibres of ϕ . Let $\overline{y_0} : \text{Spec } \overline{k} \rightarrow Y$ be a geometric point of Y and let $\overline{x_0} = \phi \circ \overline{y_0}$ be the corresponding geometric point of X .

The left G -action. There is a left G -action on $Y_{\overline{x_0}}$, defined as follows:

$$\begin{aligned} G \times Y_{\overline{x_0}} &\rightarrow Y_{\overline{x_0}} \\ (g, \overline{y_0} \cdot g') &\mapsto \overline{y_0} \cdot g \cdot g' \end{aligned} \tag{1}$$

This induces a morphism $\iota_{\overline{y_0}} : G \rightarrow \text{Aut}(Y_{\overline{x_0}})$. It is straightforward to check that $\iota_{\overline{y_0}}(G)$ consists of the group $\text{Aut}_G(Y_{\overline{x_0}})$ of all automorphisms of $Y_{\overline{x_0}}$ that commute with the right G -action.

Decomposition group. There is a natural left $\Gamma_{k(x_0)}$ -action on $Y_{\overline{x_0}}$ given by

$$\begin{aligned} \Gamma_{k(x_0)} \times Y_{\overline{x_0}} &\rightarrow Y_{\overline{x_0}} \\ (\gamma, \overline{y'} &\mapsto \overline{\gamma y'}, \end{aligned} \tag{2}$$

where $\overline{\gamma y'}$ denotes the composition $\text{Spec } \overline{k(x_0)} \xrightarrow{\gamma} \text{Spec } \overline{k(x_0)} \xrightarrow{\overline{y'}} Y$. As the G -action is defined over k , the action (2) commutes with the right G -action, hence yields a morphism $\Gamma_{k(x_0)} \rightarrow \text{Aut}_G(Y_{\overline{x_0}})$. Composing with the inverse of the isomorphism $\iota_{\overline{y_0}} : G \rightarrow \text{Aut}_G(Y_{\overline{x_0}})$ we get a morphism

$$\mathfrak{D}_{\overline{y_0}} : \Gamma_{k(x_0)} \rightarrow G$$

called the *decomposition morphism* of $\overline{y_0}$ under ϕ . The image of $\mathfrak{D}_{\overline{y_0}}$ is called the *decomposition group* of $\overline{y_0}$ and denoted by $D_{\overline{y_0}}$. Finally, if P is an L -rational point of X for some field $k \subseteq L \subseteq \overline{k}$, a decomposition group of P under ϕ is any subgroup of the form $D_{\overline{y_0}}$ for some geometric point $\overline{y_0}$ of Y whose image in $X(\overline{k})$ is the geometric point corresponding to P .

Remark 2.11. Note that the morphism $\mathfrak{D}_{\overline{y_0}}$ is the unique morphism that sends $\gamma \in \Gamma_{k(x_0)}$ to the unique element $g \in G$ such that $\overline{\gamma y_0} = \overline{y_0} \cdot g$.

Compatibility with subcovers. Let $Y_1 \xrightarrow{\phi_1} Y_2 \xrightarrow{\phi_2} X$ be finite étale morphisms of k -schemes of finite type such that the composition $\phi := \phi_2 \circ \phi_1$ is an étale G -torsor. Suppose that $Y_2 = Y_1/H$ for some subgroup $H \subset G$. (By Proposition 2.6, the latter is automatically satisfied if Y_1 and Y_2 are integral normal schemes.) Again let $\overline{y_0}$ be a geometric point of Y_1 and $\overline{x_0}$ be its image in X . By our discussion of the left G -action, we have a commutative diagram:

$$\begin{CD} G \times \phi^{-1}(\overline{x_0}) @>>> \phi^{-1}(\overline{x_0}) \\ @V(\text{id}, \phi_1)V V @VV\phi_1V \\ G \times \phi_2^{-1}(\overline{x_0}) @>>> \phi_2^{-1}(\overline{x_0}) \end{CD} \tag{3}$$

where the map in the upper row is the left G -action (1).

Observe that $\phi_2^{-1}(\overline{x_0})$ is isomorphic to G/H , with the isomorphism preserving the left G -action. The commutativity of (3) implies the following lemma.

LEMMA 2.12. *The fibre $\phi_2^{-1}(\overline{x_0})$ contains no k -rational points if and only if the decomposition group $D_{\overline{y_0}}$ acts with no fixed points on $\phi_2^{-1}(\overline{x_0})$. Moreover, $D_{\overline{y_0}}$ acts with no fixed points on $\phi_2^{-1}(\overline{x_0})$ if and only if $D_{\overline{y_0}} \subset G$ acts with no fixed points on G/H . \square*

The irreducibility of a fibre of a Galois cover is equivalent to the absence of rational points on the fibres of certain subcovers, as is shown implicitly in [Ser08, Proposition 3.3.1].

PROPOSITION 2.13. Let $\pi : Y \rightarrow X$ be a Galois cover with group G , and let $x \in X(k)$ be a rational point. Suppose that π is étale at x . The scheme Y_x is reducible over k if and only if there is a subgroup $H \subsetneq G$ such that the fibre of $Y/H \rightarrow X$ over x has a k -rational point.

Specialization. From now on, we assume that k is a number field K . Given an étale G -torsor $\phi : Y \rightarrow X$ as before and a place $v \in M_K^{\text{fin}}$, we say that a point $x_0 \in X(K_v)$ is of *good reduction* for ϕ if there exists an étale G -torsor $\psi : \mathcal{Y} \rightarrow \mathcal{X}$ over $\text{Spec } \mathcal{O}_{K_v}$ extending ϕ such that x_0 extends to a morphism $\text{Spec } \mathcal{O}_{K_v} \rightarrow \mathcal{X}$, where \mathcal{O}_{K_v} is the ring of integers of K_v . We fix a geometric point $\bar{y}_0 \in Y$ lying over x_0 , and have a commutative diagram:

$$\begin{array}{ccc}
 \text{Spec } \overline{\mathbb{F}_v} & \begin{array}{c} \searrow^{\bar{y}_{0v}} \\ \searrow \end{array} & \\
 & \text{Spec } \mathcal{O}_{K_v^{\text{ur}}} & \longrightarrow \mathcal{Y} \\
 \text{Spec } \overline{K_v} & \begin{array}{c} \nearrow \\ \nearrow_{\bar{y}_0} \end{array} &
 \end{array} \tag{4}$$

where $\mathcal{O}_{K_v^{\text{ur}}}$ denotes the ring of integers of the maximal unramified extension $K_v^{\text{ur}} \subset \overline{K_v}$ of K_v . Diagram (4) induces morphisms:

$$\begin{array}{ccc}
 \widehat{\mathbb{Z}} \cong \Gamma_{\mathbb{F}_v} & \searrow & \\
 & \text{Gal}(K_v^{\text{ur}}/K_v) & \\
 \Gamma_{K_v} & \nearrow &
 \end{array}$$

where the isomorphism $\widehat{\mathbb{Z}} \cong \Gamma_{\mathbb{F}_v}$ sends the topological generator $1 \in \widehat{\mathbb{Z}}$ to the Frobenius $x \mapsto x^{\#\mathbb{F}_v}$.

PROPOSITION 2.14. There exists a morphism $\text{Gal}(K_v^{\text{ur}}/K_v) \rightarrow G$ such that the following diagram commutes.

$$\begin{array}{ccc}
 \widehat{\mathbb{Z}} \cong \Gamma_{\mathbb{F}_v} & \begin{array}{c} \searrow^{\mathfrak{D}_{\bar{y}_{0v}}} \\ \searrow \end{array} & \\
 & \text{Gal}(K_v^{\text{ur}}/K_v) & \longrightarrow G \\
 \Gamma_{K_v} & \begin{array}{c} \nearrow \\ \nearrow_{\mathfrak{D}_{\bar{y}_0}} \end{array} &
 \end{array} \tag{5}$$

Proof. The fibres of $\psi : \mathcal{Y} \rightarrow \mathcal{X}$ over $\psi(\bar{y}_0)$ and over $\psi(\bar{y}_{0v})$ are both identified with G , hence with each other, in a G -equivariant way. □

The image of $1 \in \widehat{\mathbb{Z}}$ under $\mathfrak{D}_{\bar{y}_{0v}}$ is known as the Frobenius element of \bar{y}_{0v} , and is denoted by $\text{Fr}_{\phi, \bar{y}_{0v}}$. We also use the notation $\text{Fr}_{\phi, \bar{y}_0} := \text{Fr}_{\phi, \bar{y}_{0v}}$; note that this is well defined. If \bar{y}_v^1 and \bar{y}_v^2 lie above the same point $x_v \in \mathcal{X}(\mathbb{F}_v)$, and $\bar{y}_v^2 = \bar{y}_v^1 \cdot g$, then we have that $\text{Fr}_{\phi, \bar{y}_v^2} = g^{-1} \cdot \text{Fr}_{\phi, \bar{y}_v^1} \cdot g$. In particular, the conjugacy class of $\text{Fr}_{\phi, \bar{y}_{0v}}$ depends only on the base point x_v . Thus, when there is no risk of confusion, we also use the notation Fr_{x_v} to indicate the Frobenius element $\text{Fr}_{\phi, \bar{y}_{0v}}$ of any geometric point \bar{y}_{0v} above x_v . If $x_0 \in X(K_v)$ reduces to x_v in $\mathcal{X}(\mathbb{F}_v)$, we also use the notation $\text{Fr}_{x_0} := \text{Fr}_{x_v}$.

PROPOSITION 2.15. *Let $\phi : Y \rightarrow X$ be an étale cover of (normal) varieties over K , whose Galois closure $\widehat{Y} \rightarrow X$ has Galois group G , and let $H \subset G$ be such that $Y \cong \widehat{Y}/H$ as X -covers. Let v be a finite place of \mathcal{O}_K and $x_0 \in X(K_v)$ be a point with good reduction for ϕ . Then, $\phi^{-1}(x_0)(K_v) \neq \emptyset$ if and only if Fr_{x_0} acts on G/H with at least one fixed point (note that, for $g \in G$, the condition that g acts with at least one fixed point on G/H depends only on the conjugacy class of g).*

Proof. Let $\overline{x_0} : \text{Spec } \overline{K_v} \rightarrow x_0 \rightarrow X$ be a geometric point lying over x_0 . We have that $\phi^{-1}(x_0)(K_v) \neq \emptyset$ if and only if there exists a point in $\phi^{-1}(\overline{x_0})$ fixed by Fr_{x_0} . By Lemma 2.12 and Proposition 2.14, such a point exists if and only if Fr_{x_0} acts with at least one fixed point on G/H . □

2.5 Vertically ramified covers

The following structure result shows that, roughly speaking, a cover of $X \times Y$ which is ‘vertically ramified’ splits as a product, up to a finite étale cover. The fact that such a structure result might be true was first observed after many fruitful discussions between the third-named author and Olivier Wittenberg.

The following structure result is used twice in this paper. First, we use it to prove the product property for varieties with the weak-Hilbert property (Theorem 1.9). We then use it to prove a similar product property of a variant of the Hilbert property for abelian varieties (see Proposition 5.6).

DEFINITION 2.16. Let X, Y be proper smooth varieties over k and $\pi : Z \rightarrow X \times Y$ be a ramified cover. We say that π is *vertically ramified* over X if there exists a dense open subscheme $U \subset X$ such that π is unramified (hence, étale; see Lemma 2.3) over $U \times Y$.

LEMMA 2.17. *Let X, Y be proper smooth varieties over k and $\pi : Z \rightarrow X \times Y$ be a ramified cover. Let $U \subset X$ be a dense open subscheme such that π is unramified over $U \times Y$ (so that π is vertically ramified over X). Assume, furthermore, that the geometric fibres of the composition $Z \rightarrow X \times Y \xrightarrow{p_1} X$ are connected and $U(k) \neq \emptyset$. Then there exists a commutative diagram*

$$\begin{array}{ccccc}
 & & Z' \cong X' \times Y' & \longrightarrow & Z \\
 & \swarrow & \downarrow & & \downarrow \pi \\
 X' & & X \times Y' & \longrightarrow & X \times Y \\
 & \searrow & \downarrow & & \downarrow p_1 \\
 & & X & \xrightarrow{=} & X
 \end{array}$$

where:

- (1) X', Y', Z' are normal varieties over k ;
- (2) $X' \rightarrow X$ is a ramified cover;
- (3) $Z' \rightarrow Z$ and $Y' \rightarrow Y$ are finite étale;
- (4) Z' is a connected component of the fibred product $Z \times_Y Y'$; in particular, if $Z \times_Y Y'$ is connected, the upper square is Cartesian;
- (5) $Z' \rightarrow X' \rightarrow X$ is the Stein factorization of $Z' \rightarrow X$.

Proof. Let $x \in U(k)$ be a k -rational point. Then $Z_x \rightarrow \{x\} \times Y$ is a finite étale morphism. Let $Y' \rightarrow Z_x \rightarrow Y$ be the Galois closure of $Z_x \rightarrow Y$, and observe, in particular, that $Y' \rightarrow Y$ is finite étale. Let $Z' \subset Z \times_Y Y'$ be a connected component of the pull-back of $Z \rightarrow Y$ along $Y' \rightarrow Y$.

There are natural maps $Z' \rightarrow Z \rightarrow X \times Y \rightarrow X$ and $Z' \rightarrow Y'$ which induce a morphism $Z' \rightarrow X \times Y'$. Let $Z' \rightarrow X' \rightarrow X$ be the Stein factorization of the composed morphism $Z' \rightarrow X \times Y' \rightarrow X$, which (together with the obvious map $Z' \rightarrow Y'$) gives a natural morphism $Z' \rightarrow X' \times Y'$. We claim that this map is an isomorphism.

To prove this, note that Z' is normal, as $Z' \subset Z \times_Y Y' \rightarrow Z$ is finite étale and Z is normal. Moreover, the morphism $Z' \rightarrow X' \times Y'$ is finite and surjective. By [Gro71, Proposition X.1.2], the Stein factorization of $Z' \rightarrow X$ is étale over $U \subset X$. By [Gro63, Corollaire 7.8.7], over the étale locus the Stein factorization commutes with taking fibres, so the Stein factorization of $Z'_x \rightarrow \text{Spec } k(x)$ is given by $\text{Spec } \Gamma(Z'_x, \mathcal{O}_{Z'_x})$. As $Z'_x = Z_x \times_Y Y'$ is a disjoint union of copies of Y' (as it follows from Remark 2.8 and a standard normalization argument, noting that $Y' \rightarrow Z_x \rightarrow Y$ is the Galois closure of $Z_x \rightarrow Y$), we see that $Z'_x \rightarrow X'_x \times Y'$ is an isomorphism (as it is a finite surjective morphism between the same number of copies of Y'). It follows that $Z' \rightarrow X' \times Y'$ is an isomorphism over a dense open subset, hence it is a birational morphism. As Z' and $X' \times Y'$ are integral normal varieties over k , it follows as claimed that $Z' \rightarrow X' \times Y'$ is an isomorphism by Zariski’s main theorem (see [Gro61, Corollaire 4.4.9]).

We have thus constructed the desired diagram and shown parts (1) and (3). Parts (4) and (5) are true by construction. As for part (2), we already know that $X' \rightarrow X$ is finite (it arises as the finite part of the Stein factorization of $Z' \rightarrow X$) and surjective because $Z' \rightarrow X$ is. It remains to show that $X' \rightarrow X$ is ramified; if it were not, $Z' \rightarrow X \times Y' \rightarrow X \times Y$ would be étale, hence also $Z' \rightarrow Z \rightarrow X \times Y$ would be étale. As we already know that $Z' \rightarrow Z$ is surjective and étale, by the cancellation property for étale morphisms we would get that $\pi : Z \rightarrow X \times Y$ is also étale, a contradiction. \square

3. The weak-Hilbert property

Throughout this section, we let k be a field of characteristic zero, unless otherwise specified. The goal of this section is to prove that the class of varieties with the (potential) weak-Hilbert property (Definition 1.2) has several features in common with Campana’s class of special varieties [Cam11].

We begin by showing that the weak-Hilbert property is a birational invariant among smooth proper geometrically connected varieties.

PROPOSITION 3.1 (Birational invariance). *Let X and X' be smooth proper geometrically connected varieties over k . Suppose that X and X' are birational over k . Then X has the weak-Hilbert property over k if and only if X' has the weak-Hilbert property over k .*

Proof. We denote by $\text{Cov}(X)$ (respectively, $\text{Cov}(X')$) the category of covers of X (respectively, X').

As X and X' are birational over k , we may choose:

- (i) a dense open subscheme U of X with $\text{codim}_X(X \setminus U) \geq 2$;
- (ii) a dense open subscheme U' of X' with $\text{codim}_{X'}(X' \setminus U') \geq 2$; and
- (iii) an isomorphism $\sigma : U' \rightarrow U$ with inverse $\sigma' : U \rightarrow U'$.

We let $\eta_X, \eta_{X'}$ be the generic points of X, X' , and we denote the isomorphism $\sigma'|_{\eta_X} : \eta_X \rightarrow \eta_{X'}$ as ι .

We define the functor $N\sigma^* : \text{Cov}(X) \rightarrow \text{Cov}(X')$ (respectively, $N(\sigma')^* : \text{Cov}(X') \rightarrow \text{Cov}(X)$) as sending a cover $Y \rightarrow X$ to the relative normalization $Y' \rightarrow X'$ of X' in the cover $\iota_*(Y|_{\eta_X}) \rightarrow \eta_{X'}$ (respectively, in the cover $\iota^*(Y|_{\eta_X}) \rightarrow \eta_{X'}$).

Clearly, $N(\sigma')^*$ and $N\sigma^*$ are inverse natural equivalences. We claim that these functors send étale covers to étale covers.

Let $Y \rightarrow X$ be an étale cover. We then have that $\sigma^*Y \rightarrow U'$ is étale as well. As U' is normal, it follows that σ^*Y is normal as well. Hence, $((N\sigma^*)Y)|_{U'} \cong \sigma^*Y$ as U' -schemes. In particular, $(N\sigma^*)Y \rightarrow X'$ is finite and étale over the complement of a codimension-two closed subscheme of the base. As X' is smooth and $(N\sigma^*)Y$ is normal, by Zariski–Nagata purity [Gro71, Théorème X.3.1], $(N\sigma^*)Y \rightarrow X'$ is étale, which concludes the proof of the claim (the proof for $N(\sigma')^*$ being analogous).

Therefore, because $N(\sigma')^*$ and $N\sigma^*$ are natural inverses, Lemma 2.3 implies that ramified covers $\pi : Y \rightarrow X$ give rise to *ramified* covers of X' (through $N\sigma^*$), and conversely (through $N(\sigma')^*$), which immediately implies that X has the weak-Hilbert property if and only if X' does. □

Remark 3.2. Let X be a smooth proper variety over k with the weak-Hilbert property over k , let U be a dense open of X . Then, for every finite collection of ramified covers $(Z_i \xrightarrow{\pi_i} U)_i$, we have that $U(k) \setminus \bigcup_i \pi_i(Z_i(k))$ is Zariski-dense in U . Indeed, letting Z'_i be the normalization of X in Z_i , this becomes an immediate consequence of applying the definition of the weak-Hilbert property to X and the family of ramified covers $(Z'_i \xrightarrow{\pi'_i} X)_i$.

Remark 3.3. If $Y \xrightarrow{\pi} X$ is a cover of varieties over k and Y is not geometrically connected, then $Y(k) = \emptyset$, so that $\pi(Y(k)) = \emptyset$. In particular, when studying the weak-Hilbert property for X , one may always restrict to covers $Y \rightarrow X$ with Y geometrically connected (hence, geometrically integral).

3.1 Images of varieties with the weak-Hilbert property

In this subsection we show that, under suitable assumptions on a map of varieties $X \rightarrow Y$, if the variety X has the weak-Hilbert property then so does Y .

PROPOSITION 3.4 (Finite étale images). *Let $X \xrightarrow{\phi} Y$ be a finite étale morphism of smooth proper varieties over k . If X has the weak-Hilbert property over k , then Y has the weak-Hilbert property over k .*

Proof. As X and Y are integral, it follows that ϕ is surjective. Let $(\pi_i : Z_i \rightarrow Y)_{i=1,\dots,n}$ be a finite collection of ramified covers over k . By base change, we obtain finite surjective morphisms $\varphi_i : Z_i \times_Y X \rightarrow X$.

Let Z' be a connected component of $Z_i \times_Y X$. As $Z' \subset Z_i \times_Y X$ is open and closed, the morphism $Z' \rightarrow Z_i$ is finite étale, hence surjective (by connectivity of Z' and Z_i). It follows that $\dim Z' = \dim Z_i = \dim Y = \dim X$. In particular, because $\varphi_i|_{Z'} : Z' \rightarrow X$ is finite, it is also surjective. Furthermore, the morphism $\varphi_i|_{Z'} : Z' \rightarrow X$ is ramified. Indeed, because X is smooth, if $Z' \rightarrow X$ is unramified, then it is étale (Lemma 2.3). It follows that, as $Z' \rightarrow X$ is étale and $X \rightarrow Y$ is étale, the composition $Z' \rightarrow X \rightarrow Y$ is étale, which by the surjectivity of $Z' \rightarrow Z_i$ implies that $Z_i \rightarrow Y$ is étale, contradicting our assumption that it is ramified.

Let Z'_{ij} be the connected components of $Z_i \times_Y X$. We may apply the weak-Hilbert property of X to the ramified covers $\varphi_i|_{Z'_{ij}} : Z'_{ij} \rightarrow X$ to see that $X(k) \setminus \bigcup_{i,j} \varphi_i(Z'_{ij}(k))$ is dense in X . Applying the surjective morphism ϕ to this dense set of rational points we get a dense set of rational points on Y . We claim that this dense set is contained in $Y(k) \setminus \bigcup_i \pi_i(Z_i(k))$. Indeed, suppose by contradiction that there is a point $Q \in X(k) \setminus \bigcup_{i,j} \varphi_i(Z'_{ij}(k))$ such that $\phi(Q) \in \pi_i(Z_i(k))$ for some i . Then, by the universal property of the product $Z_i \times_Y X$, the point Q belongs to $\varphi_i((Z_i \times_Y X)(k)) \subset \bigcup_{i,j} \varphi_i(Z'_{ij}(k))$, contradicting our assumption. □

Remark 3.5. The converse to Proposition 3.4 is false. Indeed, let E be an elliptic curve over \mathbb{Q} with $E(\mathbb{Q})$ dense, and let $E' \rightarrow E$ be a finite étale morphism with $E'(\mathbb{Q}) = \emptyset$; such data are easily seen to exist. Then E has the weak-Hilbert property over \mathbb{Q} by Faltings' theorem, whereas E' has no \mathbb{Q} -points and, thus, does not have the weak-Hilbert property over \mathbb{Q} . However, there is a number field K such that E'_K has the weak-Hilbert property over K . More generally, assuming k is a finitely generated field of characteristic zero, we show a converse to Proposition 3.4 in which we allow an extension of the base field; see Theorem 3.16 for a precise statement.

A surjective morphism $f : X \rightarrow Y$ of varieties over k is said to have *no multiple fibres in codimension one* if, for every point y in Y of codimension one, the scheme-theoretic fibre X_y has an irreducible component which is reduced.

PROPOSITION 3.6 (Fibrations with no multiple fibres). *Let $X \xrightarrow{\phi} Y$ be a surjective morphism of smooth proper varieties over k with geometrically connected generic fibre and no multiple fibres in codimension one. If X has the weak-Hilbert property over k , then Y has the weak-Hilbert property over k .*

Proof. Let $(\pi_i : Z_i \rightarrow Y)_{i=1, \dots, n}$ be a finite collection of ramified covers of Y over k . To prove the statement, we have to show that $Y(k) \setminus \bigcup_i \pi_i(Z_i(k))$ is Zariski-dense in Y . Let $V \subset X$ be the smooth locus of ϕ , and let U be the image $\phi(V)$. Note that V is a dense open subscheme of X , so that $U \subset Y$ is also an open subscheme. Let $\psi : V \rightarrow U$ denote the induced (smooth) morphism. As ϕ has no multiple fibres in codimension one, the complement of U in Y is of codimension at least two. As $V \xrightarrow{\phi|_V} Y$ is smooth, the morphism $Z_i \times_Y V \rightarrow Z_i$ is smooth. In particular, by [Sta20, Tag 034F] and the normality of Z_i , the scheme $W_i := Z_i \times_Y V$ is normal.

We claim that W_i is integral. Since k is of characteristic zero and X is smooth, the generic fibre $X_{k(Y)}$ of $\phi : X \rightarrow Y$ is smooth. As $X_{k(Y)}$ is geometrically connected (by assumption), $X_{k(Y)}$ is geometrically integral. It follows that the same is true for the generic fibre of $V \rightarrow Y$. In particular, the generic fibre of $W_i = Z_i \times_Y V \rightarrow Z_i$, being the base change of the generic fibre of $V \rightarrow Y$, is geometrically integral as well. Applying [Liu06, Proposition 3.8] to $(W_i)_{\bar{k}} \rightarrow (Z_i)_{\bar{k}}$, we deduce that W_i is integral.

Define $\pi'_i : W_i \rightarrow V$ to be the natural projection. We claim that π'_i is ramified. In fact, assume by contradiction that it is not. We have the following commutative diagram, which is Cartesian in the upper left corner.

$$\begin{array}{ccc}
 W_i & \longrightarrow & Z_i|_U \\
 \downarrow \pi'_i & \searrow & \downarrow \pi_i|_U \\
 V & \xrightarrow{\psi} & U
 \end{array} \tag{6}$$

As π'_i and ψ are both smooth, so is the morphism $W_i \rightarrow U$. Moreover, $W_i \rightarrow Z_i|_U$ is smooth and surjective (being a base change of the smooth surjective morphism ψ). Hence, applying [Sta20, Tag 02K5] to $(f, q, p) = (W_i \rightarrow Z_i|_U, \pi_i|_U, W_i \rightarrow U)$, we obtain that $\pi_i|_U$ is smooth, hence étale, which by the Zariski–Nagata theorem implies that π_i is étale: contradiction.

As X has the weak-Hilbert property, by Remark 3.2, the set $S := V(k) \setminus \bigcup_{i=1}^n \pi'_i(W_i(k))$ is dense in X . By the universal property of the fibred product $W_i = Z_i \times_Y V$, following the same argument used at the end of the proof of Proposition 3.4, we conclude that $\psi(S) \subset Y(k) \setminus \bigcup_i \pi_i(Z_i(k))$ is Zariski-dense in Y , as required. \square

THEOREM 3.7 (Smooth images). *Let $X \xrightarrow{\phi} Y$ be a smooth proper morphism of smooth proper varieties over k . If X has the weak-Hilbert property over k , then Y has the weak-Hilbert property over k .*

Proof. Let $X \rightarrow X' \rightarrow Y$ be the Stein factorization of $\phi : X \rightarrow Y$. Note that $X \rightarrow X'$ is a smooth proper morphism and that $X' \rightarrow Y$ is finite étale. In particular, because $X \rightarrow X'$ has no multiple fibres in codimension one, it follows that X' has the weak-Hilbert property over k by Proposition 3.6. Then, as X' has the weak-Hilbert property over k and $X' \rightarrow Y$ is a finite étale morphism of smooth proper varieties over k , we conclude that Y has the weak-Hilbert property over k (Proposition 3.4), as required. \square

3.2 Arithmetic refinements

An *arithmetic refinement of a variety Z over k* is the data of a finite index set J and, for every j in J , a cover $\psi_j : W_j \rightarrow Z$ with W_j normal and geometrically integral, with the property that $Z(k) \subset \bigcup_{j \in J} \psi_j(W_j(k))$. When ‘testing’ the weak-Hilbert property for a variety X , one can replace a given cover by arithmetic refinements. Let us be more precise.

Let $n \geq 1$ be an integer and, for $i = 1, \dots, n$, let $\pi_i : Z_i \rightarrow X$ be a ramified cover of normal projective geometrically integral varieties over k . For every i , let J_i be a finite set. For $i = 1, \dots, n$ and j in J_i , let $\psi_{i,j} : W_{i,j} \rightarrow Z_i$ be a cover. Assume that, for $i = 1, \dots, n$, the collection $\{\psi_{i,j} : W_{i,j} \rightarrow Z_i\}_{j \in J_i}$ is an arithmetic refinement of Z_i . Then

$$X(k) \setminus \bigcup_{i=1}^n \bigcup_{j \in J_i} \pi_i(\psi_{i,j}(W_{i,j}(k))) = X(k) \setminus \bigcup_{i=1}^n \pi_i(Z_i(k)).$$

Thus, when ‘testing’ the weak-Hilbert property for a variety (Definition 1.2), one may replace each cover $Z_i \rightarrow X$ in a given finite collection of covers $(Z_i \rightarrow X)_i$ by an arithmetic refinement of Z_i .

3.3 Chevalley–Weil for finitely generated fields of characteristic zero

Let $f : X \rightarrow Y$ be a finite étale surjective morphism of proper schemes over a field k . When k is a number field, a classical result (which goes back to Chevalley and Weil [CW32]) shows that there exists a finite extension L of k such that every k -rational point of Y lifts to an L -rational point of X . The same statement holds when k is a finitely generated field of characteristic zero; because we were unable to find this precise statement in the literature, we include a short proof.

THEOREM 3.8. *If $f : X \rightarrow Y$ is a finite étale surjective morphism of proper schemes over a finitely generated field k of characteristic zero, then there is a finite field extension L/k such that $Y(k) \subset f(X(L))$.*

Proof. By standard spreading out arguments, we may choose a regular \mathbb{Z} -finitely generated integral domain $A \subset k$, a proper model \mathcal{X} for X over A , a proper model \mathcal{Y} for Y over A , and a finite étale surjective morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ extending f . By properness of \mathcal{Y} over A , for every k -point $y : \text{Spec } k \rightarrow Y$, there exist a dense open subscheme $U_y \subset \text{Spec } A$ whose complement in $\text{Spec } A$ is of codimension at least two and a morphism $U_y \rightarrow \mathcal{Y}$ extending the morphism $y : \text{Spec } k \rightarrow Y$. Pulling back $U_y \rightarrow \mathcal{Y}$ along $F : \mathcal{X} \rightarrow \mathcal{Y}$, we obtain a finite étale surjective morphism $V_y := U_y \times_{\mathcal{Y}} \mathcal{X} \rightarrow U_y$ of degree $\deg(f)$. By purity of the branch locus [Gro71, Théorème X.3.1], the finite étale morphism $V_y \rightarrow U_y$ extends to a finite étale morphism $\overline{V}_y \rightarrow \text{Spec } A$. Since the set of isomorphism classes of finite étale covers of $\text{Spec } A$ with bounded degree is finite by the Hermite–Minkowski theorem for arithmetic schemes [HH09], the set of isomorphism classes of the \overline{V}_y (and, thus, V_y) appearing above (with $y \in Y(k)$) is finite. In particular, we may choose a finite field extension L/k such that, for all k -points $y : \text{Spec } k \rightarrow Y$ and every connected (hence, irreducible) component V' of V_y , the function field $K(V')$ of V' is contained in L . This readily implies that $Y(k) \subset f(X(L))$, as required. \square

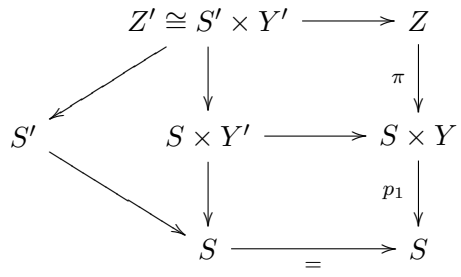
3.4 Applying the Chevalley–Weil theorem

Let X and Y be smooth projective geometrically connected varieties over k . We stress that projectivity is only assumed for technical reasons, as it is used to ensure the existence of certain Weil restrictions in the proof of Theorem 3.9.

Let $\pi : Z \rightarrow X \times Y$ be a cover which is vertically ramified (see Definition 2.16) over X . We apply the Chevalley–Weil theorem (Theorem 3.8) and use Weil’s restriction of scalars to construct suitable arithmetic refinements of the vertically ramified morphism $Z \rightarrow X \times Y$. The precise statement we prove reads as follows.

THEOREM 3.9 (Arithmetic refinement). *Let k be a finitely generated field of characteristic zero. Let X and Y be smooth projective geometrically connected varieties over k . Let $\pi : Z \rightarrow X \times Y$ be a cover which is vertically ramified over X . Assume that $Z(k)$ is dense in Z , so that, in particular, $X(k)$ and $Y(k)$ are dense in X and Y , respectively. There exists an arithmetic refinement $\{f_j : W_j \rightarrow Z\}_{j \in J}$ of Z such that, for every j in J , the Stein factorization of $W_j \rightarrow X \times Y \rightarrow X$ is ramified over X .*

Proof. Let $Z \rightarrow S \rightarrow X$ be the Stein factorization of the composed map $Z \rightarrow X \times Y \rightarrow X$ and note that S is a normal geometrically integral variety over k . If $S \rightarrow X$ is ramified, there is no need to replace $Z \rightarrow X \times Y$ by an arithmetic refinement, and we are done. Thus, we may and do assume that $S \rightarrow X$ is unramified, hence étale (Lemma 2.3). Note that the finite surjective morphism $Z \rightarrow X \times Y$ factors over a finite surjective morphism $Z \rightarrow S \times Y$ which is vertically ramified with respect to $S \times Y \rightarrow S$ (note that $S \rightarrow X$ is étale). As the geometric fibres of the composed morphism $Z \rightarrow S \times Y \rightarrow S$ are connected and $S(k)$ is dense, it follows from Lemma 2.17 that there is the following commutative diagram.



Here $S' \rightarrow S$ is a ramified cover, $Y' \rightarrow Y$ is finite étale and $\psi : Z' \cong S' \times Y' \rightarrow Z$ is finite étale. In particular, because k is a finitely generated field of characteristic zero, it follows from the Chevalley–Weil theorem (Theorem 3.8) that there is a finite field extension L/k such that $Z(k) \subset \psi(Z'(L))$.

Consider the induced morphism of Weil restriction of scalars $R_{L/k}(\psi_L) : R_{L/k}Z'_L \rightarrow R_{L/k}Z_L$; see [BLR90, Chapter 7.6]. Let $\Delta : Z \rightarrow R_{L/k}Z_L$ be the diagonal morphism. Let $(W_j)_{j \in J}$ be the connected components of $Z \times_{\Delta, R_{L/k}Z_L} R_{L/k}Z'_L$ such that $W_j(k) \neq \emptyset$ and let $f_j : W_j \rightarrow Z$ be the natural morphism. As W_j is normal connected and with a k -rational point, it is geometrically integral. Moreover, note that, by the defining properties of the Weil restriction of scalars, the finite set of covers $\{f_j : W_j \rightarrow Z\}_{j \in J}$ is an arithmetic refinement of Z .

After base change to \bar{k} , the fibre product $Z \times_{\Delta, R_{L/k}Z_L} R_{L/k}Z'_L$ is given by $Z'_{\bar{k}} \times_{Z_{\bar{k}}} \dots \times_{Z_{\bar{k}}} Z'_{\bar{k}}$. Hence, the morphism $(W_j)_{\bar{k}} \rightarrow X_{\bar{k}}$ factors as $(W_j)_{\bar{k}} \rightarrow Z'_{\bar{k}} \rightarrow Z_{\bar{k}} \rightarrow X_{\bar{k}}$, with the morphism $(W_j)_{\bar{k}} \rightarrow Z'_{\bar{k}}$ being finite étale. From this it follows that the Stein factorization of $(W_j)_{\bar{k}} \rightarrow X_{\bar{k}}$ factors over the Stein factorization of $Z'_{\bar{k}} = S'_{\bar{k}} \times Y'_{\bar{k}} \rightarrow X_{\bar{k}}$ which implies that the Stein factorization of $(W_j)_{\bar{k}} \rightarrow X_{\bar{k}}$ is ramified over $X_{\bar{k}}$. As Stein factorization commutes with flat base change

(in particular, base change to \bar{k} in our case), we deduce that the Stein factorization of $W_j \rightarrow X$ is ramified over X , as required. \square

Example 3.10 (Wittenberg). It can happen that the geometric fibres of $Z \rightarrow X$ are connected. Indeed, let $E' \rightarrow E$ be a degree-two isogeny between elliptic curves over k , and let G be its kernel. Let G act on E' by translation and on $\mathbb{P}_{1,k}$ by the involution $x \mapsto -x$. Define

$$Y = E'/G = E, \quad X' = \mathbb{P}_{1,k}, \quad X = X'/G \cong \mathbb{P}_{1,k}, \quad Z = (E' \times \mathbb{P}_{1,k})/G.$$

The natural morphism $Z \rightarrow X \times Y$ is a finite surjective ramified morphism of degree two which is vertically ramified with respect to $X \times Y \rightarrow X$. In fact, the branch locus of $Z \rightarrow X \times Y$ lies over precisely two points of X . Note that the geometric fibres of $Z \rightarrow X \times Y \rightarrow X$ are connected, so that the Stein factorization of $Z \rightarrow X$ is étale (even an isomorphism) over X . Define $Z' = E' \times \mathbb{P}_{1,k}$. Then $Z' \rightarrow Z$ is a finite étale morphism (as the action of G on $E' \times \mathbb{P}_{1,k}$ is free), and the Stein factorization of the composed morphism $Z' \rightarrow Z \rightarrow X \times Y \rightarrow X$ is given by $Z \rightarrow X' \rightarrow X$. Note that $X' \rightarrow X$ is ramified (as the action of G on X' is not free). This example shows that it is necessary to pass to a finite étale cover of Z to guarantee that the Stein factorization is ramified over the base.

3.5 Products

We are now ready to show that a product of varieties satisfying the weak-Hilbert property over a finitely generated field k of characteristic zero has the weak-Hilbert property over k .

Proof of Theorem 1.9. Let X and Y be smooth proper varieties over k with the weak-Hilbert property over k . We may assume, without loss of generality, by Chow’s lemma and Proposition 3.1, that X and Y are projective (so that we may appeal to Theorem 3.9). We aim to show that $X \times Y$ has the weak-Hilbert property over k . For $i = 1, \dots, n$, let $\pi_i : Z_i \rightarrow X \times Y$ be a ramified cover with Z_i a normal proper geometrically integral variety over k (we remind the reader of Remark 3.3). Let $\psi_i : S_i \rightarrow X$ denote the Stein factorization of $Z_i \rightarrow X \times Y \rightarrow X$, so that $Z_i \rightarrow S_i$ has geometrically connected fibres. By Theorem 3.9 we may replace each vertically ramified cover $Z_i \rightarrow X \times Y$ with an arithmetic refinement $W_{i,j} \rightarrow Z_i \rightarrow X \times Y$. Therefore, reindexing if necessary, we may assume that there exists $1 \leq m \leq n$ such that:

- (i) ψ_i is ramified for $i = 1, \dots, m$;
- (ii) ψ_i is étale (see Lemma 2.3) and the branch locus of $Z_i \rightarrow X \times Y$ dominates X for $i = m + 1, \dots, n$.

We define

$$\Sigma := X(k) \setminus \bigcup_{i=1}^m \psi_i(S_i(k)),$$

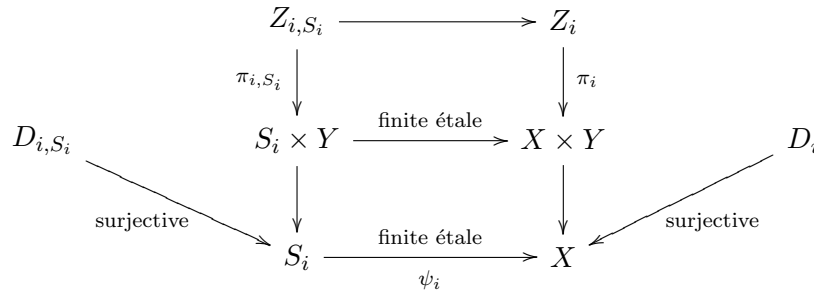
which is Zariski-dense in X because X has the weak-Hilbert property over k . Moreover, we define

$$\begin{aligned} \Psi &:= \bigcup_{x \in \Sigma} \left(\{x\} \times \left(Y(k) \setminus \bigcup_{i=m+1}^n \pi_{i,x}(Z_{i,x}(k)) \right) \right) = (\Sigma \times Y(k)) \setminus \bigcup_{i=m+1}^n \pi_i(Z_i(k)) \\ &\subseteq (X(k) \times Y(k)) \setminus \bigcup_{i=1}^n \pi_i(Z_i(k)). \end{aligned}$$

To conclude the proof it suffices to show that Ψ is dense in $X \times Y$. By [Gro66, Theorem 12.2.4 (iv)] and [Gro65, Theorem 6.9.1], there is a dense open subscheme $U \subset X$ such that, for every x in $U(k)$, the fibre Z_x of $Z \rightarrow X \times Y \rightarrow X$ over x is a normal (not necessarily connected) scheme.

As $\Sigma \cap U$ is dense in U , to show the density of Ψ it suffices to prove that, for every x in $U(k)$, the set $Y(k) \setminus \bigcup_{i=m+1}^n \pi_{i,x}(Z_{i,x}(k))$ is dense in Y . To do so, it suffices to show that, for $m + 1 \leq i \leq n$ and x in $U(k)$, and for every connected component Z' of $Z_{i,x}$, the restriction of the morphism $\pi_{i,x} : Z_{i,x} \rightarrow \{x\} \times Y \cong Y$ to Z' is a ramified cover.

Let $i \in \{m + 1, \dots, n\}$ and fix x in $U(k)$. Let $\psi_i^{-1}(x) = s_1 \sqcup \dots \sqcup s_r$ (where the s_i are the points in the fibre). Then the normal scheme $Z_{i,x}$ decomposes as a disjoint union $Z_{i,x} = Z_{i,s_1} \sqcup \dots \sqcup Z_{i,s_r}$ of normal connected varieties. Recall that the branch locus D_i of $\pi_i : Z_i \rightarrow X \times Y$ dominates X . We have the following commutative diagram of morphisms.



As the branch locus D_i of π_i dominates X , it follows that the branch locus D_{i,S_i} of $\pi_{i,S_i} : Z_{i,S_i} \rightarrow S_i \times Y$ dominates S_i . This implies that, for all s in S_i , the morphism $Z_{i,s} \rightarrow \{s\} \times Y$ is ramified (Lemma 2.1), hence so is the composition $Z_{i,s} \rightarrow \{s\} \times Y \rightarrow \{\psi_i(s)\} \times Y$, as required. \square

Remark 3.11. If X and Y are special varieties over k , then $X \times Y$ is special [Cam11]. Thus, the product theorem for the weak-Hilbert property is in accordance with the conjecture of Campana and Corvaja and Zannier (Conjecture 1.7) that a variety is special if and only if it has the weak-Hilbert property over some finite field extension of the base.

Remark 3.12. Note that Theorem 1.9 improves on the main result of [Jav21b] over finitely generated fields of characteristic zero. Indeed, in [Jav21b] it is proven that $X \times Y$ has the ‘very-weak-Hilbert property’ (see [Jav21b, Definition 1.1]), albeit without any assumption on the base field. The main novelty of our current approach is Theorem 3.9, in which we deal with ‘vertically ramified’ covers.

Example 3.13. New examples of varieties with the weak-Hilbert property can be given by taking products. For example, let X be the K3 surface defined by $x^4 + y^4 = z^4 + w^4$ in $\mathbb{P}_{3,\mathbb{Q}}$, and note that X has the weak-Hilbert property over \mathbb{Q} (see [CZ17, Theorem 1.4]). Let A be an abelian variety over \mathbb{Q} with $A(\mathbb{Q})$ dense, so that A has the weak-Hilbert property (Theorem 1.3). Then, for every positive integer m , because $\mathbb{P}_{n,\mathbb{Q}}$ has the (weak-)Hilbert property over \mathbb{Q} , the variety $A \times \mathbb{P}_{n,\mathbb{Q}} \times X^m$ has the weak-Hilbert property over \mathbb{Q} by Theorem 1.9.

3.6 Finite étale covers and extending the base field

We now adapt the arguments in the proof of Theorem 3.9 to prove that the (potential) weak-Hilbert property of a variety X/k is inherited by every base change X_L with L/k finite, as well as its étale covers when k is finitely generated over \mathbb{Q} . We use the following technical lemma in both situations.

LEMMA 3.14. *Let L/k be a finite extension of fields of characteristic zero, let X be a smooth proper geometrically connected variety over L , let Y be a smooth proper geometrically connected variety over k and let $\pi : X \rightarrow Y$ be a finite étale morphism. Assume that the variety Y has the*

weak-Hilbert property over k , and that $Y(k) \subset \pi(X(L))$. Then X has the weak-Hilbert property over L .

Proof. As X is geometrically connected and $X \rightarrow Y$ is surjective, it follows that Y is geometrically connected over k .

We now explain how to reduce to the case that Y (hence, X) is projective over k . To do so, let $Y' \rightarrow Y$ be a proper birational surjective morphism with Y' a smooth projective geometrically connected variety over k . Define $X' := X \times_Y Y'$ and note that the natural projection $\pi' : X' \rightarrow Y'$ is finite étale (surjective), as it is the base change of $\pi : X \rightarrow Y$ along $Y' \rightarrow Y$. In particular, because X' is finite étale over the smooth variety Y' over k , the scheme X' is a smooth projective geometrically connected variety over L . As $X' \rightarrow X$ is a proper birational surjective morphism, to show that X has the weak-Hilbert property over L , by Proposition 3.1, it suffices to show that X' has the weak-Hilbert property over L . Finally, because $Y(k) \subset \pi(X(L))$, it readily follows from the universal property of fibre products that $Y'(k) \subset \pi'(X'(L))$. Thus, replacing X by X' and Y by Y' if necessary, we may and do assume that X and Y are projective over k .

Now, let Z be a normal proper variety over L and let $Z \rightarrow X$ be a ramified cover. As X and Y_L are projective over L , we may consider the Weil restrictions $R_{L/k}(Z) \rightarrow R_{L/k}(X) \rightarrow R_{L/k}(Y_L)$ and the diagonal map $Y \xrightarrow{\Delta} R_{L/k}(Y_L)$. Let $Z' := R_{L/k}(Z) \times_{R_{L/k}(Y_L)} Y$ and write $Z' = \bigcup_i Z'_i$ for its decomposition in irreducible components. We note that, for each i , the natural morphism $Z'_i \rightarrow Y$ is finite.

We claim that, if the morphism $Z'_i \rightarrow Y$ is surjective, then the composed morphism $\tilde{Z}'_i \rightarrow Y$ is ramified, where \tilde{Z}'_i is the normalization of Z'_i . To prove this, it suffices to show that $\tilde{Z}'_{iL} \rightarrow Y_L$ is ramified. We have the following commutative diagram:

$$\begin{array}{ccccccc}
 Z & \longleftarrow & (R_{L/k}Z)_L & \longleftarrow & (Z'_i)_L & \xleftarrow{\text{normalization}} & \tilde{Z}'_i \\
 \downarrow & & \downarrow & & \downarrow & & \\
 Y_L & \longleftarrow & (R_{L/k}Y_L)_L & \xleftarrow{\Delta_L} & Y_L & &
 \end{array}$$

where the composition of the maps in the lower row is the identity. As the composition $(Z'_i)_L \rightarrow Z \rightarrow Y_L$ is finite surjective and Z is connected, we have that $(Z'_i)_L \rightarrow Z$ is surjective. In particular, because $Z \rightarrow Y_L$ (which is equal to the composition $Z \rightarrow X \rightarrow Y_L$) is ramified, it follows that the composition $\tilde{Z}'_i \rightarrow Z \rightarrow Y_L$ is ramified. (Indeed, if it were unramified, then it would be étale by Lemma 2.3 which would contradict the fact that $Z \rightarrow Y_L$ is ramified, as Z is normal.)

Now, by the weak-Hilbert property for Y over k , there is a Zariski-dense subset $C \subset Y(k)$ of points not lifting to k -points in Z' . Let $\Omega \subset X(L)$ be the inverse image of C in $X(L)$ via $\pi : X(L) \rightarrow Y(L)$. Note that the assumption $Y(k) \subseteq \pi(X(L))$ ensures that $\pi(\Omega) = C$, hence Ω is a dense subset of X . We claim that, for every c in Ω , the fibre Z_c has no L -point. Indeed, such an L -point would induce a k -point of $R_{L/k}(Z)$ and a k -point of Y and, thus, a k -point of Z' mapping to $\pi(c) \in Y(k)$: contradiction. \square

As a first direct consequence, we deduce that the weak-Hilbert property persists along finite extensions of the field of definition.

PROPOSITION 3.15 (Base change). *Let L/k be a finite extension of fields of characteristic zero and let V be a smooth proper (geometrically) connected variety over k with the weak-Hilbert property over k . Then V_L has the weak-Hilbert property over L .*

Proof. Define $X := V_L$ and $Y := V$. As the morphism $\pi : X = V_L \rightarrow V = Y$ is finite étale and $Y(k) = V(k) \subset V(L) = \pi(X(L))$, the proposition follows from Lemma 3.14. \square

THEOREM 3.16. *Let k be a finitely generated field of characteristic zero. Let $f : V \rightarrow U$ be a finite étale morphism of smooth proper geometrically connected varieties over k . If U has the weak-Hilbert property over k , then there is a finite field extension L/k such that V_L has the weak-Hilbert property over L .*

Proof. As $f : V \rightarrow U$ is a finite étale morphism and k is a finitely generated field of characteristic zero, by the Chevalley–Weil theorem (Theorem 3.8), there is a finite extension L/k such that $U(k) \subset f(V(L))$. Define $X := V_L$ and $Y = U$, and let $\pi : X \rightarrow Y$ be the composed morphism $V_L \rightarrow V \rightarrow U$. Note that $Y(k) = U(k) \subset f(V(L)) = \pi(X(L))$. Thus, by Lemma 3.14, the weak-Hilbert property holds for $X = V_L$ over L . \square

4. Variants of the Hilbert property for abelian varieties

With a view towards proving that the weak-Hilbert property holds for abelian varieties with a dense set of rational points (Theorem 1.3) we study properties and relations between variants of the Hilbert property for abelian varieties over an arbitrary field k of characteristic zero.

DEFINITION 4.1. Let $\pi : Y \rightarrow A$ be a cover of an abelian variety A over k . The cover π satisfies:

- (PB) for *pullback*, if, for every positive integer m , the scheme $[m]^*Y = Y \times_{\pi, A, [m]} A$ is geometrically integral;
- (GPB) for *Galois pullback*, if the Galois closure of π is a (PB)-cover; and
- (Ram) for *ramified*, if π is ramified.

As we show later in Lemma 4.4, a cover $\pi : Y \rightarrow A$ of an abelian variety A over k satisfies (PB) if and only if it has no non-trivial étale subcovers.

Note that, if A is an abelian variety over k and $\pi : Y \rightarrow A$ is a cover of A which satisfies the property (PB), then the scheme $[m]^*Y$ is a geometrically integral normal projective variety over k for every positive integer m . We stress that the properties (PB) and (Ram) are central to our paper, whereas the notion (GPB) is introduced only for technical purposes.

DEFINITION 4.2. Let A be an abelian variety over k , let $\pi : Y \rightarrow A$ be a cover of A over k , and let $\Omega \subseteq A(k)$ be a finitely generated Zariski-dense subgroup. We say that the pair (π, Ω) satisfies:

- (PF) for *pointless (reduced) fibres*, if either $\deg \pi = 1$ or there is a finite-index coset $C \subseteq \Omega$ such that, for every c in C , the scheme $\pi^{-1}(c)$ is reduced and has no k -rational points;
- (IF) for *integral fibres*, if there is a finite-index coset $C \subseteq \Omega$ such that, for every c in C , the scheme $\pi^{-1}(c)$ is integral.

In addition, we say that property (PF), respectively property (IF), holds for π (without specifying Ω) if property (PF), respectively property (IF), holds for all pairs (π, Ω) with Ω a finitely generated Zariski-dense subgroup of $A(k)$. Let r be a positive integer. We say that property (PF), respectively property (IF), holds for π up to rank r if property (PF), respectively property (IF), holds for all pairs (π, Ω) with Ω a finitely generated Zariski-dense subgroup of $A(k)$ of rank at most r .

We say that a property (\star) in the list of Definition 4.1 implies property (PF), respectively property (IF), for a pair (A, Ω) if the following holds: for every cover $\pi : Y \rightarrow A$ such that π satisfies property (\star) , the pair (π, Ω) also satisfies property (PF), respectively property (IF).

We record the following obvious observation.

Remark 4.3. Let $Z \rightarrow A$ be a cover that factors through a subcover $Z \rightarrow Z' \xrightarrow{\psi} A$ with $\deg \psi \geq 2$. If $Z' \rightarrow A$ satisfies property (PF), then so does $Z \rightarrow A$.

As we show in the following lemma, a cover of an abelian variety has property (PB) if and only if it has no étale subcovers; this generalizes [Zan10, Proposition 2.1] to non-algebraically closed fields.

LEMMA 4.4. Let A be an abelian variety over k . Let $\pi : Z \rightarrow A$ be a cover with Z geometrically integral over k . Then there is a factorization

$$\pi : Z \xrightarrow{\phi} B \xrightarrow{\lambda} A,$$

where $\phi_{\bar{k}}$ has property (PB) and λ is unramified. In particular, π is a (PB)-cover if and only if it has no étale subcovers.

Proof. The maximal étale subcover of $Z \rightarrow A$ is unique up to unique isomorphism (as a subcover of $Z \rightarrow A$), so that, by Galois descent, the morphism $Z \rightarrow A$ factors through a non-trivial étale subcover if and only if $Z_{\bar{k}} \rightarrow A_{\bar{k}}$ does. The lemma now follows from [Zan10, Proposition 2.1] (where the base field is assumed to be algebraically closed). \square

In fact, [Zan10, Proposition 2.1] also proves that a cover satisfies property (PB) if its total space remains geometrically integral after pull-back along multiplication by the degree of the cover.

LEMMA 4.5. Let A be an abelian variety over k , and $\phi : X \rightarrow A$ be a cover of degree d with X geometrically integral variety over k . Then ϕ has property (PB) if and only if the fibre product $X \times_{\phi, A, [d]} A$ is geometrically integral.

4.1 Avoiding the branch locus

We now show that, given a Zariski-dense subgroup Ω of $A(k)$, one may always find a finite-index coset C of Ω that avoids any given (proper) closed subscheme of A . In particular, given a cover $\pi : Y \rightarrow A$, there exists a finite-index coset of Ω that is disjoint from the branch locus of π . This gives us an ample supply of points of Ω over which the fibre of π is unramified; we use this fact in several of our subsequent proofs.

LEMMA 4.6. Let A be an abelian variety over k , let $\Omega \subset A(k)$ be a finitely generated Zariski-dense subgroup, and let $Z \subsetneq A$ be a closed subscheme. Then there exists a finite index coset C of Ω such that $C \cap Z = \emptyset$.

Proof. As Ω is Zariski-dense, there exists $P \in \Omega$ such that $P \notin Z$. Now, by standard spreading out arguments, we may choose the following data:

- (1) a \mathbb{Z} -finitely generated subring $R \subset k$, an abelian scheme \mathcal{A} over R and an isomorphism $\mathcal{A} \times_R k \cong A$ such that $\Omega \subset A(k)$ lies in the subgroup $\mathcal{A}(R) \subset A(k)$;
- (2) a closed subscheme $\mathcal{Z} \subset \mathcal{A}$ defining $Z \subset A$ over R ;
- (3) a section $\sigma_P : \text{Spec } R \rightarrow \mathcal{A}$ which coincides with P in $A(k)$.

As $P \notin Z$, we have that $\sigma_P^* \mathcal{Z} \subsetneq \text{Spec } R$ is a proper closed subset. In particular, there exists a closed point $v \in \text{Spec } R$ such that $P \bmod v = \sigma_P(\text{Spec } \mathbb{F}_v) \notin \mathcal{Z}$. As $\mathcal{A}(\text{Spec } \mathbb{F}_v)$ is a finite group, the kernel Ω' of the specialization map $\mathcal{A}(R) \rightarrow \mathcal{A}(\mathbb{F}_v)$ is of finite index. In particular, for each c in the finite index coset $C := P + \Omega'$ of Ω , we have that $P \equiv c \bmod v$. Finally, for each $c \in C$, the element $c \bmod v$ does not lie in the closed subscheme $\mathcal{Z} \subset \mathcal{A}$, hence $c \notin Z = \mathcal{Z} \cap A$. \square

COROLLARY 4.7. *Let A be an abelian variety over k , let $\Omega \subset A(k)$ be a finitely generated Zariski-dense subgroup, and let $\phi : Y \rightarrow A$ be a ramified cover. Then there exists a finite index coset C of Ω such that, for each $c \in C$, the scheme $\phi^{-1}(c)$ is étale over k .*

Proof. Let $B \subset A$ be the branch locus of ϕ . As A is normal, by Lemma 2.3, the morphism ϕ is étale over $U := A \setminus B$. Moreover, by Lemma 4.6, there is a finite index coset C of Ω such that $C \cap B = \emptyset$. In particular, for every $c \in C$, we have that $c \in U$ and ϕ is étale over c . \square

4.2 Base change of (PB)-covers

The first basic observation is that the total space of a cover $Y \rightarrow A$ which has property (PB) remains connected after pull-back along étale covers of A .

LEMMA 4.8. *Let A be an abelian variety over k and let $\pi : Y \rightarrow A$ be a cover of A . Then π satisfies property (PB) if and only if, for every isogeny $\phi : A' \rightarrow A_{\bar{k}}$ of abelian varieties over \bar{k} , the fibre product $Y_{\bar{k}} \times_{\pi, A_{\bar{k}}, \phi} A'$ is connected.*

Proof. We note that $Y_{\bar{k}} \times_{\pi, A_{\bar{k}}, \phi} A'$, being étale over $Y_{\bar{k}}$, which is normal, is itself normal. The ‘if’ direction follows because $[N]$ is, in particular, an isogeny and normal connected schemes are integral. For the ‘only if’ direction, observe that given any isogeny $\phi : A' \rightarrow A_{\bar{k}}$ we may find another isogeny $\psi : A_{\bar{k}} \rightarrow A'$ such that $\phi \circ \psi = [\text{deg } \phi]$. As π satisfies property (PB), the fibre product $Y_{\bar{k}} \times_{\pi, A_{\bar{k}}, [\text{deg } \phi]} A_{\bar{k}}$ is integral. The claim follows from the commutative diagram

$$\begin{array}{ccccc}
 Y_{\bar{k}} \times_{\pi, A_{\bar{k}}, [\text{deg } \phi]} A_{\bar{k}} & \longrightarrow & Y_{\bar{k}} \times_{\pi, A_{\bar{k}}, \phi} A' & \longrightarrow & Y_{\bar{k}} \\
 \downarrow & & \downarrow & & \downarrow \pi \\
 A_{\bar{k}} & \xrightarrow{\psi} & A' & \xrightarrow{\phi} & A_{\bar{k}} \\
 & \searrow & \text{[deg } \phi] & \nearrow & \\
 & & & &
 \end{array}$$

where both squares are Cartesian, the horizontal morphisms are surjective, and $Y_{\bar{k}} \times_{\pi, A_{\bar{k}}, [\text{deg } \phi]} A_{\bar{k}}$ is irreducible, hence so is $Y_{\bar{k}} \times_{\pi, A_{\bar{k}}, \phi} A'$ as desired. \square

Remark 4.9. Let $f : X \rightarrow Y$ be a continuous, surjective and open morphism of topological spaces. Assume that Y is connected and that, for all points $y \in Y$, the fibre X_y is connected. Then X is connected. Indeed, assume by contradiction that $X = U_1 \sqcup U_2$, with U_1 and U_2 non-empty open subsets of X . Then $Y = f(U_1) \cup f(U_2)$. As f is open, $f(U_1)$ and $f(U_2)$ are open, and, because Y is connected, $f(U_1) \cap f(U_2) \neq \emptyset$. Let $y \in f(U_1) \cap f(U_2)$. Then $X_y = (X_y \cap U_1) \sqcup (X_y \cap U_2)$, and both $X_y \cap U_1$ and $X_y \cap U_2$ are non-empty: contradiction.

As a consequence of Lemma 4.8 and Remark 4.9, we show that a (PB)-cover remains a (PB)-cover after pull-back along any dominant morphism of abelian varieties. In other words, the property of being a (PB)-cover of an abelian variety commutes with dominant base change.

LEMMA 4.10. *Let $\phi : Z \rightarrow A$ be a (PB)-cover of an abelian variety A over k , and let $\lambda : A' \rightarrow A$ be a dominant morphism of abelian varieties. Then $\lambda^* Z := A' \times_{\lambda, A, \phi} Z$ is geometrically integral and $\lambda^* Z \rightarrow A'$ is a (PB)-cover.*

Proof. Let $n \geq 1$ be an integer and let $\mu := [n]_{A'}$ be multiplication by n on A' . By using properties of Stein factorization, we may factor the morphism $\lambda \circ \mu$ as

$$\lambda \circ \mu = \lambda_1 \circ \lambda_2 : A' \xrightarrow{\lambda_2} A'' \xrightarrow{\lambda_1} A,$$

where λ_2 has geometrically integral fibres and λ_1 is an isogeny. Note that $(\lambda \circ \mu)^*Z \cong \lambda_2^*\lambda_1^*Z$. Moreover, because ϕ satisfies property (PB), by Lemma 4.8, the morphism $\lambda_1^*Z \rightarrow A''$ is a (PB)-cover, so that λ_1^*Z is geometrically connected. As λ_2 is smooth and has geometrically connected fibres, the morphism $\lambda_2^*(\lambda_1^*Z) \rightarrow \lambda_1^*Z$ is smooth and has geometrically connected fibres. As λ_1^*Z is geometrically connected and smooth morphisms are flat, hence open, we have that $\lambda_2^*(\lambda_1^*Z)$ is geometrically connected by Remark 4.9. As λ_1^*Z is normal (hence, geometrically normal), $\lambda_2^*(\lambda_1^*Z)$ is geometrically normal by [Sta20, Tag 034F]. As $\lambda_2^*(\lambda_1^*Z)$ is geometrically normal and connected, it is geometrically integral. Therefore, $(\lambda \circ \mu)^*Z \cong \lambda_2^*\lambda_1^*Z$ is as well. As this holds for every integer $n \geq 1$, this proves that λ^*Z is geometrically integral and that $\lambda^*Z \rightarrow A'$ satisfies the property (PB). \square

4.3 Spreading out (PB)-covers

We now show that a cover of an abelian variety over a function field which satisfies property (PB) can be spread out to a cover of an abelian scheme which fibrewise satisfies property (PB).

LEMMA 4.11. *Let S be an integral scheme of characteristic zero with function field $k(S)$ and let $\mathcal{A} \rightarrow S$ be an abelian scheme. Let $Z \rightarrow \mathcal{A}$ be a finite surjective morphism such that $Z_{k(S)} \rightarrow \mathcal{A}_{k(S)}$ is a (PB)-cover of $\mathcal{A}_{k(S)}$ over $k(S)$. Then, there exists a dense open subscheme $U \subset S$ such that, for every $u \in U$, the morphism $Z_u \rightarrow \mathcal{A}_u$ is a (PB)-cover of \mathcal{A}_u over $k(u)$.*

Proof. Let d be the degree of Z over \mathcal{A} . The isomorphism $[d]^*Z_{k(S)} \cong ([d]^*Z) \times_S k(S)$ and the assumption that $Z_{k(S)} \rightarrow \mathcal{A}_{k(S)}$ is a (PB)-cover show that the generic fibre of $[d]^*Z \rightarrow \mathcal{A}$ is geometrically irreducible. Since Z is normal and S is of characteristic zero, the scheme $Z_{k(S)}$ is normal. By spreading out (see [Gro65, Théorème 6.9.1] and [Gro66, Théorème 9.7.7(iv) and Théorème 12.2.4(iv)]), there is a dense open subscheme U of S such that, for all $u \in U$, the scheme $[d]^*Z_u$ is normal and geometrically irreducible over $k(u)$. In particular, by Lemma 4.5, for every $u \in U$, the cover $Z_u \rightarrow \mathcal{A}_u$ of degree d is a (PB)-cover, as required. \square

4.4 Invariance under translation

Since the class of (PB)-covers (respectively, ramified covers) is invariant by translation on A , we obtain the following proposition.

PROPOSITION 4.12. *Let A be an abelian variety over k , and let r be a positive integer. Then the following two statements are equivalent.*

- (1) *For every non-trivial (PB)-cover $Z \rightarrow A$ and every finitely generated Zariski-dense subgroup $\Omega \subset A(k)$ of rank at most r , there exists a finite index coset C of Ω such that, for every $c \in C$, the scheme Z_c is étale over k and has no k -points (respectively, is integral).*
- (2) *For every non-trivial (PB)-cover $Z \rightarrow A$, for every $b \in A(k)$, and every finitely generated Zariski-dense subgroup $\Omega \subset A(k)$ of rank at most r , there exists a finite index coset C of Ω such that, for every $c \in C$, the scheme Z_{b+c} is étale over k and has no k -points (respectively, is integral).*

The same equivalence holds with (PB) replaced by (Ram).

Remark 4.13. In the case of (Ram) we actually only care about the equivalence of statements (1) and (2) for property (PF) because, by Remark 7.8, the fibres $\phi^{-1}(c)$ will rarely be integral.

4.5 From one cover to many covers

LEMMA 4.14. *Let A be an abelian variety over k , and let r be a positive integer. Then the following statements hold.*

- (1) Suppose that every (PB)-cover of A satisfies property (IF) (respectively, property (PF)) up to rank r . Then, for any finite collection $(\pi_i : Y_i \rightarrow A)_{i=1, \dots, n}$ of non-trivial (PB)-covers and for every finitely generated Zariski-dense subgroup $\Omega \subset A(k)$ of rank at most r , there is a finite-index coset C of Ω such that, for every $c \in C$ and $i \in \{1, \dots, n\}$, the scheme $\pi_i^{-1}(c)$ is integral (respectively, reduced with no k -rational points).
- (2) Suppose that every (Ram)-cover of A satisfies property (PF) up to rank r . Then, for any finite collection $(\pi_i : Y_i \rightarrow A)_{i=1, \dots, n}$ of ramified covers and for every finitely generated Zariski-dense subgroup $\Omega \subset A(k)$ of rank at most r , there is a finite-index coset C of Ω such that, for every $c \in C$ and $i \in \{1, \dots, n\}$, the scheme $\pi_i^{-1}(c)$ is reduced and has no k -rational points.

Proof. We argue by induction on n ; the case $n = 1$ being true by assumption. Suppose now the claim is true for $n - 1$; we prove it for n . Let π_1, \dots, π_n be non-trivial covers of A that satisfy property (PB) (respectively, property (Ram)) and let Ω be a Zariski-dense subgroup of $A(k)$ of rank at most r . By the inductive hypothesis, there exists a finite-index coset $C_{n-1} \subseteq \Omega$ such that, for every c in C_{n-1} and $i \in \{1, \dots, n - 1\}$, the scheme $\pi_i^{-1}(c)$ is integral (respectively, is reduced and has no k -rational points). Let $\Omega' \subset \Omega$ be a finite index (Zariski-dense) subgroup (of rank at most r) and let $c_{n-1} \in A$ be such that $C_{n-1} = c_{n-1} + \Omega'$. By Proposition 4.12, there is a finite-index coset C of Ω' such that, for every c in C , the fibre $\pi_n^{-1}(c)$ is integral (respectively, is reduced and has no k -rational points). It is immediate to check that C is a finite-index coset of Ω with the desired properties. □

4.6 Invariance under isogeny

The analogue of Proposition 3.4 in the context of the Hilbert property for pairs (A, Ω) reads as follows.

PROPOSITION 4.15. *Let $\phi : A \rightarrow B$ be an isogeny of abelian varieties over k . If every (PB)-cover of A has property (PF), then every (PB)-cover of B has property (PF).*

Proof. Let $\pi : Z \rightarrow B$ be a non-trivial (PB)-cover and let $\Omega \subset B(k)$ be a finitely generated Zariski-dense subgroup. To show that there is a finite index coset $C \subset \Omega$ such that $\pi^{-1}(c)$ has no k -rational points for every $c \in C$, replacing Ω by a finite-index subgroup if necessary, we may and do assume that Ω is in the image of $A(k) \xrightarrow{\phi} B(k)$. Then $\Omega' := \phi^{-1}\Omega \cap A(k)$ is a finitely generated Zariski-dense subgroup of A . Moreover, let $\pi' : Z' \rightarrow A$ be the pull-back of $\pi : Z \rightarrow B$ along $\phi : A \rightarrow B$. As $Z \rightarrow B$ is a (PB)-cover, it follows that Z' is integral (Lemma 4.8). Moreover, because $Z' \rightarrow Z$ is finite étale, we see that Z' is normal. In particular, again by Lemma 4.8, it follows that the cover $Z' \rightarrow A$ is a non-trivial (PB)-cover. As every (PB)-cover of A has property (PF), the (non-trivial) cover $Z' \rightarrow A$ satisfies property (PF). In particular, we can choose a finite-index coset C' of Ω' such that, for all c in C' , the fibre $(\pi')^{-1}(c)$ has no k -rational points and is reduced. The universal property of fibre products then implies that $(\pi')^{-1}(c) \cong \pi^{-1}(\phi(c))$, so that for every c in the finite-index coset $C := \phi(C') \subseteq \Omega$ the fibre $\pi^{-1}(c)$ has no k -rational points and is reduced. This proves that $\pi : Z \rightarrow B$ satisfies property (PF) as desired. □

4.7 A special class of products

LEMMA 4.16. *Assume that for all positive integers r , for all abelian varieties A_1, \dots, A_r over k , and for all cyclic, Zariski-dense subgroups $\Omega_i \subset A_i(k)$ (for $i = 1, \dots, r$) property (PB) implies property (IF) for the pair $(\prod_{i=1}^r A_i, \prod_{i=1}^r \Omega_i)$. Then property (PB) implies property (IF) for all pairs (A, Ω) , where A is an abelian variety A over k and Ω is a finitely generated Zariski-dense subgroup of $A(k)$.*

Proof. Let A be an abelian variety over k , $\Omega \subset A(k)$ be a finitely generated Zariski-dense subgroup, and $\phi : Z \rightarrow A$ a (PB)-cover. We want to prove that there exists a finite index coset $C \subset \Omega$ such that, for every c in C , the scheme $\phi^{-1}(c)$ is integral. As every finitely generated abelian group contains a free abelian subgroup of finite index, we can assume, up to replacing Ω with a finite-index subgroup, that $\Omega = \bigoplus_{i=1}^r \Omega_i$, with $\Omega_i = \langle \omega_i \rangle \subset \Omega$ cyclic of infinite order. Moreover, we can assume without loss of generality (up to replacing Ω with a finite-index subgroup again) that the Zariski closure A_i of Ω_i is irreducible and, hence, an abelian subvariety of A . Let $\lambda : \bigoplus_{i=1}^r A_i \rightarrow A$ be the natural surjective morphism (defined by the colimit property of the direct sum). It follows from Lemma 4.10 that λ^*Z is a (PB)-cover of $\bigoplus_{i=1}^r A_i$. Moreover, $\lambda|_{\bigoplus_i \Omega_i} : \bigoplus_{i=1}^r \Omega_i \rightarrow \Omega$ is an isomorphism.

Let C' be a finite index coset of $\bigoplus_{i=1}^r \Omega_i \subset \bigoplus_{i=1}^r A_i$ such that, for each c in C' , the scheme $(\lambda^*\phi)^{-1}(c)$ is integral. Then $C := \lambda(C')$ is a finite index coset of Ω . Moreover, because $(\lambda^*\phi)^{-1}(c) \cong \phi^{-1}(\lambda(c))$ for every c in C' , it follows that, for every c in C , the scheme $\phi^{-1}(c)$ is integral, as required. \square

4.8 From (PB)-covers to ramified covers

The following proposition shows that, when studying the weak-Hilbert property for abelian varieties, it is natural to only consider (PB)-covers.

PROPOSITION 4.17. *Suppose that, for every abelian variety B over k , every (PB)-cover $\pi : Y \rightarrow B$ has property (PF). Then, for every abelian variety A over k , every (Ram)-cover $\psi : Z \rightarrow A$ has property (PF).*

Proof. Let A be an abelian variety over k , let $\psi : Z \rightarrow A$ be a ramified cover and let $\Omega \subset A(k)$ be a finitely generated Zariski-dense subgroup. To prove the proposition, by Corollary 4.7, translating the origin on A and replacing Ω by a finite index subgroup if necessary, we may and do assume that $\psi : Z \rightarrow A$ is étale over every c in Ω .

By Lemma 4.4, there exists a factorization

$$\psi : Z \xrightarrow{\psi_0} A' \xrightarrow{\lambda} A,$$

where λ is étale and ψ_0 is a non-trivial (PB)-cover. If $A'(k) = \emptyset$, then the claim trivially holds, so that we can assume that $A'(k) \neq \emptyset$ and that λ is an isogeny. We define $\Omega' := \lambda^{-1}(\Omega) \cap A'(k)$ and $N := A'(k) \cap \text{Ker } \lambda$, and note that N is a finite group. We have a natural exact sequence

$$N \hookrightarrow \Omega' \rightarrow \Omega. \tag{7}$$

Note that the cokernel of the last morphism is finite, as every isogeny between abelian varieties is a left and right factor of multiplication by some integer $m \in \mathbb{N}_{>0}$.

For each n in the finite group N , we define ψ_n to be the cover of A' given by the composition $Z \xrightarrow{\psi_0} A' \xrightarrow{+n} A'$. By Lemma 4.14, as every ψ_n is a (PB)-cover, there exists a finite index coset $C \subset \Omega'$ such that, for every $n \in N$ and every $c \in C$, the scheme $\psi_n^{-1}(c)$ has no k -points. Let $c \in \lambda(C) \subset \Omega$, and choose a representative $c_0 \in C$ such that $\lambda(c_0) = c$. We have

$$\psi^{-1}(c) = \psi_0^{-1}(\lambda^{-1}(c) \setminus A'(k)) \cup \bigcup_{n \in N} \psi_0^{-1}(c_0 + n) = \psi_0^{-1}(\lambda^{-1}(c) \setminus A'(k)) \cup \bigcup_{n \in N} \psi_n^{-1}(c_0),$$

and, by construction, this is étale and has no k -points. As $\lambda(C) \subset \Omega$ is a coset of finite index, this concludes the proof. \square

4.9 Galois closures and (PB)-covers

The next proposition is very close in spirit to the first part of the proof of [Zan10, Theorem 1].

PROPOSITION 4.18. *Let A be an abelian variety over k and let r be a positive integer.*

- (1) *Suppose that, for every finite field extension L/k , every (GPB)-cover $\pi : Y \rightarrow A_L$ of A_L has property (IF) up to rank r over L . Then, for every finite field extension M/k , every (PB)-cover $\pi : Y \rightarrow A_M$ of A_M has property (IF) up to rank r over M .*
- (2) *Suppose that, for every finite field extension L/k , every (GPB)-cover $\pi : Y \rightarrow A_L$ of A_L has property (PF) up to rank r over L . Then, for every finite field extension M/k , every (PB)-cover $\pi : Y \rightarrow A_M$ of A_M has property (PF) up to rank r over M .*

Proof. We only prove part (1), as the proof of part (2) is essentially identical. First, we may and do assume that $M = k$. Thus, let $\pi : Y \rightarrow A$ be a (PB)-cover of A , and let $\Omega \subset A(k)$ be a finitely generated Zariski-dense subgroup of rank at most r . Define $B := (\deg \pi)!$ and consider $[B]^*\pi : [B]^*Y \rightarrow A$. To prove that (A, Ω) has property (IF), replacing Ω with its finite-index subgroup $[B]\Omega = \{[B]\omega : \omega \in \Omega\}$, we may assume that the image of $A(k) \xrightarrow{[B]} A(k)$ contains Ω . Let $[B]^*\Omega = \{x \in A(k) : Bx \in \Omega\}$.

We now choose a finite field extension L/k such that, if $\widehat{Y}_L \rightarrow Y_L \rightarrow A_L$ is the Galois closure of $Y_L \rightarrow A_L$, then \widehat{Y}_L is geometrically integral over L . Replacing L by a finite field extension if necessary, let $\psi : A' \rightarrow A_L$ be an isogeny of abelian varieties such that $\widehat{Y}_L \rightarrow A' \rightarrow A_L$ is the (PB)-factorization of $\widehat{Y}_L \rightarrow A_L$ (see Lemma 4.4). Replacing L by a finite field extension if necessary, we may suppose that $\ker \psi$ is L -rational. We observe that $\deg \psi$ divides the degree of $\widehat{Y}_L \rightarrow A_L$, which divides $(\deg \pi)! = B$. Consider the following diagram in which every square is Cartesian.

$$\begin{array}{ccccc}
 V & \longrightarrow & \widehat{Y}_L \times_{A_L, [B]} A_L & \longrightarrow & \widehat{Y}_L \\
 \downarrow & & \downarrow & & \downarrow \\
 A_L \times \{0\} & \longrightarrow & A_L \times \ker \psi = A_L \times_{[B], A_L, \psi} A' & \longrightarrow & A' \\
 & & \downarrow & & \downarrow \psi \\
 & & A_L & \xrightarrow{[B]} & A_L
 \end{array}$$

As we observed previously, $[B]$ kills $\text{Ker } \psi$. It follows that ψ is an isogeny factor of $[B]$, which gives the identity at the center of the diagram. Note that the composed morphism $A_L \cong A_L \times \{0\} \rightarrow A_L \times \ker \psi \rightarrow A'$ is an isogeny. Therefore, because $\widehat{Y}_L \rightarrow A'$ is a (PB)-cover, it follows from Lemma 4.10 that $V \rightarrow A_L$ is a (PB)-cover, so that, in particular, V is geometrically integral over L .

We now apply Corollary 2.10 to see that the composed morphism $V \rightarrow [B]^*Y_L \rightarrow A_L$ is a Galois closure of the morphism $[B]^*Y_L \rightarrow A_L$. Explicitly, applying Corollary 2.10 with

$$X := A_L, X' := A_L, Z := Y_L,$$

and with $X' \rightarrow X$ given by multiplication with B , we deduce that the Galois closure $\widehat{[B]^*Y_L}$ of $[B]^*Y_L \rightarrow A_L$ embeds as a connected component of $\widehat{Y}_L \times_{A_L, [B]} A_L$, and the embedding commutes with projection to A_L . As $\text{Ker } \psi(\overline{K}) = \text{Ker } \psi(K)$, all connected components of $\widehat{Y}_L \times_{A_L, [B]} A_L$ are isomorphic, over A_L , to V . Hence, we deduce that there exists an isomorphism $\widehat{[B]^*Y_L} \cong V$ which commutes with projection to A_L . As $V \rightarrow A_L$ is a (PB)-cover, we conclude that $[B]^*Y_L \rightarrow A_L$ is a (GPB)-cover of A_L .

For $x \in A_L(L)$ we have the following Cartesian diagram:

$$\begin{array}{ccccc}
 ([B]^*\pi_L)^{-1}(x) & \longrightarrow & [B]^*Y_L & \longrightarrow & Y_L \\
 \downarrow & & [B]^*\pi_L \downarrow & & \pi_L \downarrow \\
 x & \longrightarrow & A_L & \xrightarrow{[B]} & A_L
 \end{array} \tag{8}$$

and, hence, a canonical isomorphism of schemes (with Galois action),

$$([B]^*\pi_L)^{-1}(x) \cong \pi_L^{-1}([B]x).$$

As every (GPB)-cover of A_L has property (IF) up to rank r over L , there is a finite-index coset C of $[B]^*\Omega$ such that the fibre of $[B]^*\pi_L$ over every $c \in C$ is integral. The previous isomorphism shows that $[B]C$ is a finite-index coset of Ω such that the fibre of $\pi : Y_L \rightarrow A_L$ over any $[B]c \in [B]C$ is integral, hence $(\pi_L : Y_L \rightarrow A_L, \Omega)$ has property (IF) over L . This implies (readily) that $(\pi : Y \rightarrow A, \Omega)$ has property (IF) over k , as required. \square

PROPOSITION 4.19. *Let A be an abelian variety over k , and let r be a positive integer. Suppose that, for every finite field extension L/k , every (GPB)-cover $\pi : Y \rightarrow A_L$ of A_L has property (PF) up to rank r over L . Then, for every finite field extension M/k , every (PB)-cover $\pi : Y \rightarrow A_M$ of A_M has property (IF) up to rank r over M .*

Proof. By Proposition 4.18(1), it suffices to prove that, for every finite field extension L/k , every (GPB)-cover $\pi : Y \rightarrow A_L$ has property (IF) up to rank r over L . To do so, let $\Omega \subset A(k)$ be a finitely generated Zariski-dense subgroup of rank at most r , let $\pi : Y \rightarrow A_L$ be a (GPB)-cover of A_L , and let $\widehat{\pi} : \widehat{Y} \rightarrow A_L$ be its Galois closure with group G .

Let $H_1 := \{e\}, \dots, H_s$ be an enumeration of the proper subgroups of G . By our definition of (GPB)-cover, the cover $\widehat{\pi}$ satisfies property (PB), hence for each $i = 1, \dots, s$ so does its subcover $\widehat{Y}_{H_i} \rightarrow A_L$. By Proposition 4.18(2), the hypothesis implies that every (PB)-cover $\pi : Y \rightarrow A_L$ has property (PF) up to rank r , so that by Lemma 4.14 there is a finite-index coset C of Ω such that, for each $c \in C$ and each $i = 1, \dots, s$, the fibre of $\widehat{Y}_{H_i} \rightarrow A_L$ over c has no L -rational points and is reduced. As $\widehat{Y} = \widehat{Y}_{H_1} \rightarrow A_L$ is étale over the points of C , by Proposition 2.13, we conclude that the fibre $\widehat{\pi}^{-1}(c)$ is integral, as required. \square

5. A product theorem for abelian varieties

Throughout this section we let K be a number field.

5.1 Local considerations

LEMMA 5.1. *If E/K is a non-trivial finite extension of number fields, then there exist infinitely many places v of K such that $(\text{Spec } E)(K_v) = \emptyset$.*

Proof. We may and do assume that E is Galois over K . By Chebotarev’s density theorem, the set of finite places v of K which are unramified and not totally split in E is infinite. As E over K is Galois, for any of the (infinitely many) finite places v of K unramified and not totally split in E , the K_v -algebra $E \otimes_K K_v$ is isomorphic to the power M^ℓ of a non-trivial extension M of K_v (with $\ell \geq 1$ an integer). For such a v , because M is non-trivial over K_v , we have that

$$(\text{Spec } E)(K_v) = (\text{Spec } E \otimes_K K_v)(K_v) = ((\text{Spec } M)(K_v))^\ell = \emptyset.$$

This concludes the proof. \square

For the next two propositions we let $\mathbb{A}_K^{\text{fin}}$ be the ring of finite adèles of K . We also implicitly consider K as embedded diagonally in $\mathbb{A}_K^{\text{fin}}$.

Recall that, for a finite-type separated scheme X over K , the adelic topology is defined as follows. Fix a model \mathcal{X} (separated, of finite type) of X over $\text{Spec } \mathcal{O}_K$. The adelic topology on $X(\mathbb{A}_K^{\text{fin}}) \subseteq \prod_{v \in M_K^{\text{fin}}} X(K_v)$ is by definition the topology generated by all subsets of the form

$$\prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v),$$

where S is a finite subset of M_K^{fin} , each U_v is an open subset of $X(K_v)$ (for its natural v -adic topology) and \mathcal{O}_v is the ring of integers of K_v . One checks that this definition is independent of the choice of model \mathcal{X} . When X is proper over K , the valuative criterion for properness easily implies that $X(\mathbb{A}_K^{\text{fin}}) = \prod_{v \in M_K^{\text{fin}}} X(K_v)$ as topological spaces.

PROPOSITION 5.2. *If $f : Z' \rightarrow Z$ is a projective morphism of separated schemes of finite type over K , then the induced morphism $f : Z'(\mathbb{A}_K^{\text{fin}}) \rightarrow Z(\mathbb{A}_K^{\text{fin}})$ is closed.*

Proof. Let v be a finite place of K . As f is projective, there exists $n \in \mathbb{N}$ such that f factors as $X \hookrightarrow \mathbb{P}_{n,Y} \rightarrow Y$, where $X \hookrightarrow \mathbb{P}_{n,Y}$ is a closed embedding and $\mathbb{P}_{n,Y} \rightarrow Y$ is the standard projection. It is straightforward to see that $X(K_v) \hookrightarrow \mathbb{P}_{n,Y}(K_v)$ is a closed embedding, hence it is a proper map of topological spaces. Moreover, because $\mathbb{P}_n(K_v)$ is compact, the composed map $\mathbb{P}_{n,Y}(K_v) = \mathbb{P}_n(K_v) \times Y(K_v) \rightarrow Y(K_v)$ is proper. As composition of proper maps is proper, the map $f_{K_v} : Z'(K_v) \rightarrow Z(K_v)$ is proper, hence closed. Choose a finite set of finite places S of K and a proper morphism $\phi : Z' \rightarrow Z$ of finite-type separated $\mathcal{O}_{K,S}$ -schemes extending $f : Z' \rightarrow Z$ over $\mathcal{O}_{K,S}$. By the valuative criterion for properness, for all $v \notin S$, we have $\phi(Z'(\mathcal{O}_v)) = f(Z'(K_v)) \cap Z(\mathcal{O}_v)$. It is now a straightforward verification to see that $f : Z'(\mathbb{A}_K^{\text{fin}}) \rightarrow Z(\mathbb{A}_K^{\text{fin}})$ is closed. \square

COROLLARY 5.3. *Let $\phi : Z \rightarrow A$ be a cover of normal varieties over K , and let $P \in A(K)$ such that ϕ is étale over P . Suppose that one of the following conditions hold:*

- (1) *the set $Z_P(\mathbb{A}_K^{\text{fin}})$ is empty;*
- (2) *the morphism ϕ is Galois and $Z_P(K) = \emptyset$;*
- (3) *the scheme Z_P is integral and $\deg \phi > 1$.*

Then there is an adelic open neighbourhood $U_P \subset A(\mathbb{A}_K^{\text{fin}})$ of P such that, for every Q in $U_P \cap A(K)$, the set $Z_Q(\mathbb{A}_K^{\text{fin}})$ is empty.

Proof. We first show that condition (2) implies condition (1) and condition (3) implies condition (1), respectively. Indeed, if ϕ is Galois, because ϕ is étale over P , the scheme Z_P is a Galois étale cover of $P \cong \text{Spec } K$. Therefore, if $Z_P(K) = \emptyset$, then $Z_P \cong \text{Spec } E \sqcup \dots \sqcup \text{Spec } E$, where E is a finite non-trivial Galois extension of K . By Lemma 5.1, there is a finite place v of K such that $Z_P(K_v) = \emptyset$. In particular, the set $Z_P(\mathbb{A}_K^{\text{fin}})$ is empty. This shows that condition (2) implies condition (1).

Now, assume condition (3) holds, so that Z_P is integral. Then, because $Z_P \rightarrow \text{Spec } K$ has degree greater than one (as $\deg \phi > 1$ and Z_P is integral), by Lemma 5.1, there exists a finite place v of K such that $Z_P(K_v) = \emptyset$, so that again $Z_P(\mathbb{A}_K^{\text{fin}}) = \emptyset$. This shows that condition (3) implies condition (1).

Thus, to prove the corollary, we may assume that condition (1) holds. As ϕ is finite, hence projective [Sta20, Tag 0B3I], the subset $\phi(Z(\mathbb{A}_K^{\text{fin}})) \subset A(\mathbb{A}_K^{\text{fin}})$ is closed by Proposition 5.2. Moreover, as $\phi(Z(\mathbb{A}_K^{\text{fin}}))$ does not contain P by condition (1), the corollary follows by taking U_P to be the complement of $\phi(Z(\mathbb{A}_K^{\text{fin}}))$ in $A(\mathbb{A}_K^{\text{fin}})$. \square

COROLLARY 5.4. *Let $\phi : Z \rightarrow A$ be a cover of an abelian variety A over K with $\deg \phi > 1$, and let $\Omega \subset A(K)$ be a subgroup. If there is a point $P \in \Omega$ such that Z_P is integral, then there exists a finite index coset C of Ω such that, for each c in C , the set $Z_c(K)$ is empty.*

Proof. By Corollary 5.3, there is an open adelic neighbourhood $U_P \subset A(\mathbb{A}_K^{\text{fin}})$ of P such that the set $\phi^{-1}(U_P)(\mathbb{A}_K^{\text{fin}})$ is empty. As A is proper over K , the adelic topology on $A(\mathbb{A}_K^{\text{fin}}) = \prod_{v \in M_K^{\text{fin}}} A(K_v)$ coincides with the product topology. Therefore, we may assume that there is a finite subset of finite places $V := \{v_1, \dots, v_r\} \subset M_K^{\text{fin}}$ and, for each $i = 1, \dots, r$, an open subset $U_i \subset A(K_{v_i})$ containing P such that

$$\prod_{i=1}^r U_i \times \prod_{v \notin V} A(K_v) \subset U_P.$$

For $i = 1, \dots, r$, let $W_i := -P + U_i \subset A(K_{v_i})$, and note that W_i is an open subset of $A(K_{v_i})$ containing the identity.

Let p_i be the residue characteristic of v_i . By [Mat55], the topological group $A(K_{v_i})$ contains a finite index open subgroup isomorphic to the additive group $\mathcal{O}_{v_i}^{\dim A}$. Let $t_i := [A(K_{v_i}) : \mathcal{O}_{v_i}^{\dim A}]$. Then, for n_i sufficiently large, the subgroup $(p_i^{n_i} t_i)\Omega$ is contained in W_i . Define $N := \prod_{i=1}^r p_i^{n_i} t_i$, and $\omega := N\Omega$. As Ω is a finitely generated abelian group (by Mordell–Weil), the subgroup ω is of finite index, so that the finite index coset $C := \omega + P$ of Ω satisfies the condition in the statement. □

5.2 Products of abelian varieties

The main result of this section (Corollary 5.7) will allow us to reduce the proof of Hilbert’s irreducibility theorem for pairs (A, Ω) to the case that Ω is cyclic.

LEMMA 5.5. *Let A, B be abelian varieties over K . Let $\pi : Z \rightarrow A \times B$ be a (PB)-cover of $A \times B$, and let $Z \rightarrow S \xrightarrow{\phi} A$ be the Stein factorization of $p_1 \circ \pi$, where $p_1 : A \times B \rightarrow A$ denotes the projection. Then ϕ is a (PB)-cover of A .*

Proof. By the universal property of fibre products, if $S \rightarrow S' \rightarrow A$ is a non-trivial étale subcover of $S \rightarrow A$, the finite étale morphism $S' \times B \rightarrow A \times B$ is a non-trivial étale subcover of $Z \rightarrow A \times B$. Therefore, the result follows from Lemma 4.4. □

PROPOSITION 5.6. *Let A, B be abelian varieties over K , and $\Omega_A \subset A(K)$, $\Omega_B \subset B(K)$ be Zariski-dense subgroups. Suppose that, for every finite extension L of K , property (PB) implies property (IF) for both (A_L, Ω_A) and (B_L, Ω_B) . Then property (PB) implies property (PF) for $(A \times B, \Omega_A \times \Omega_B)$.*

Proof. Let $\pi : Z \rightarrow A \times B$ be a (PB)-cover, which we may assume to be of degree at least two. By Corollary 4.7, there exist finite index cosets $C_A \subset \Omega_A$ and $C_B \subset \Omega_B$ such that π is étale over $C_A \times C_B$. To prove the proposition, up to composing with a translation of $A \times B$, we may and do assume that C_A and C_B are actually subgroups. Therefore, replacing Ω_A (respectively, Ω_B) with C_A (respectively, C_B), we may assume that π is étale over $\Omega_A \times \Omega_B$.

Let $Z \rightarrow S \rightarrow A$ be the Stein factorization of the composition $Z \rightarrow A \times B \rightarrow A$ as follows.

$$\begin{array}{ccc}
 Z & & \\
 \pi \downarrow & \searrow \psi & \\
 A \times B & & S \\
 \downarrow & \swarrow \phi & \\
 A & &
 \end{array}$$

We note that $S \rightarrow A$ has property (PB) by Lemma 5.5. Note that $S \rightarrow A$ could be an isomorphism.

We now distinguish two cases.

(1) Assume $S \rightarrow A$ is not an isomorphism. Applying the assumption to the (PB)-cover $\phi : S \rightarrow A$ and to the (finitely generated) subgroup Ω_A , we get a finite-index coset $C \subseteq \Omega_A$ with the property that, for every $c \in C$, the fibre $\phi^{-1}(c)$ has no K -rational points. It follows immediately that $C \times \Omega_B$ is a finite-index coset of $\Omega_A \times \Omega_B$ such that, for any $(c, b) \in C \times \Omega_B$, we have $\pi^{-1}((c, b))(K) \subseteq \psi^{-1}\phi^{-1}(c)(K) = \emptyset$. As $\pi^{-1}((c, b))$ is reduced by assumption, this proves that $\pi : Z \rightarrow A \times B$ has property (PF) in this case.

(2) Assume that $S \rightarrow A$ is an isomorphism, so that $Z \rightarrow A$ has geometrically connected fibres. As K is of characteristic zero and Z is a normal geometrically integral variety over K , $Z_{\overline{K}}$ is normal by [Sta20, Tag 034F]. Hence, the generic fibre of $Z_{\overline{K}} \rightarrow A_{\overline{K}}$ is normal and integral by a simple localization argument. Therefore, by [Gro66, Theorem 12.2.4(iv)] and [Gro65, Theorem 6.9.1], there is a dense open subscheme \mathcal{U} of A over which the geometric fibres of $Z \rightarrow A$ are normal (integral) varieties. In particular, for every $a \in \mathcal{U}(K)$, the morphism $Z_a \rightarrow \{a\} \times B$ is a cover of B over K .

Let $\eta_A \in A$ be the generic point. By Lemma 4.4, there is a finite étale cover $\tilde{B} \rightarrow B_{\eta_A}$ and a cover $Z_{\eta_A} \rightarrow \tilde{B}$ such that $Z_{\eta_A} \rightarrow B_{\eta_A}$ factors as

$$Z_{\eta_A} \rightarrow \tilde{B} \rightarrow B_{\eta_A}, \tag{9}$$

and the morphism $Z_{\overline{K(A)}} \rightarrow \tilde{B}_{\overline{K(A)}}$ is a (PB)-cover of the abelian variety $\tilde{B}_{\overline{K(A)}}$. Let \tilde{Z} be the normalization of $A \times B$ in (the function field of) \tilde{B} . Note that \tilde{Z} is a normal variety over K and that the factorization 9 induces a factorization

$$Z \rightarrow \tilde{Z} \xrightarrow{\lambda} A \times B, \tag{10}$$

with $\tilde{Z}_{\eta_A} = \tilde{B}$. By construction λ is vertically ramified over A (see Definition 2.16). Moreover, λ is a (PB)-cover of $A \times B$, as it is a factor of the (PB)-cover $Z \rightarrow A \times B$.

- (a) Suppose that $\tilde{Z} = A \times B$ (i.e. $\deg \lambda = 1$), so that $Z_{\eta_A} \rightarrow B_{\eta_A}$ is a (PB)-cover. Therefore, by applying Lemma 4.11 to the abelian scheme $\mathcal{A} := A \times B \rightarrow A := S$, there is a dense open subscheme $U \subset A$ such that, for every a in $U(K)$, the morphism $Z_a \rightarrow \{a\} \times B$ is (PB)-cover of B . As Ω_A is dense in A , there is a K -rational point a of U contained in Ω_A . In particular, the assumption on (B, Ω_B) implies that there is a finite-index coset C of Ω_B such that, for every c in C , the scheme $\pi^{-1}((a, c))$ is integral. By Corollary 5.4, the existence of such a point implies that $(\pi, \Omega_A \times \Omega_B)$ satisfies property (PF) as desired.
- (b) Suppose that $\deg \lambda \geq 2$. Then, property (PF) for $(\pi, \Omega_A \times \Omega_B)$ is implied by property (PF) for $(\lambda, \Omega_A \times \Omega_B)$ (see Remark 4.3). Thus, we may and do assume that $\tilde{Z} = Z$, so that

6. Abelian varieties with a non-degenerate point

The main result of this section (Theorem 6.15) says, roughly speaking, that an abelian variety over a number field endowed with a non-degenerate point has the weak-Hilbert property. We prove this theorem using Kummer theory for abelian varieties.

6.1 Constructing a suitable torsion point

PROPOSITION 6.1. *Let A be an abelian variety over a finite field \mathbb{F}_q of cardinality q , and let ℓ be a rational prime coprime with q . Then*

$$v_\ell(\det(\rho(\text{Fr}_q) - 1)) = v_\ell(\#A[\ell^\infty](\mathbb{F}_q)),$$

where $\text{Fr}_q \in \Gamma_{\mathbb{F}_q}$ is the Frobenius of \mathbb{F}_q and $\rho : \Gamma_{\mathbb{F}_q} \rightarrow T_\ell A$ is the Galois action on the ℓ -adic Tate module.

Proof. It follows from the standard theory of abelian varieties and Tate modules that, if $\phi : A \rightarrow A$ denotes the Frobenius endomorphism, then $\det(\rho(\text{Fr}_q) - 1) = \deg(\phi - 1) = \# \text{Ker}(\phi - 1)$. As $\text{Ker}(\phi - 1) = A(\mathbb{F}_q)$, the proposition follows. □

The following elementary result in group theory goes back to Jordan [Jor72].

PROPOSITION 6.2. *Let G be a group acting transitively on a set X . If $|X| \geq 2$, then there exists $g \in G$ that acts on X with no fixed points.*

We recall that throughout this paper, we let K denote a number field. The main result of this subsection is the following lemma which gives a torsion point ζ over which the fibre has no $K(\zeta)$ -points.

LEMMA 6.3. *Let $\phi : Y \rightarrow A$ be a non-trivial (GPB)-cover of an abelian variety A over K with Galois closure $\hat{\phi} : \hat{Y} \rightarrow A$ and Galois group G . For every finite set S of prime numbers, there exists a torsion point $\zeta \in A(\overline{K})_{\text{tors}}$ such that $\text{ord}(\zeta)$ is coprime with all the finite primes in S , the fibre $\phi^{-1}(\zeta)$ is étale over $K(\zeta)$ and there exists an element g in a decomposition group at ζ for $\hat{\phi}$ such that g acts with no fixed points on the geometric fibre $\phi^{-1}(\bar{\zeta})$.*

Proof. By Remark 2.7(ii), the Galois closure of $Y_{\overline{K}} \rightarrow A_{\overline{K}}$ is the base change to \overline{K} of the Galois closure of $Y \rightarrow A$, hence the Galois closure of $Y_{\overline{K}} \rightarrow A_{\overline{K}}$ is a (PB)-cover.

Choose a finite set of places $S' \subset M_K^{\text{fin}}$ and an abelian scheme $\mathcal{A} \rightarrow \text{Spec}_{\mathcal{O}_{K,S'}}$ extending $A \rightarrow \text{Spec } K$; this is possible by spreading out. After possibly enlarging S' , we may choose a normal integral model \mathcal{Y} of Y over $\mathcal{O}_{K,S'}$ and a finite surjective morphism $\psi : \mathcal{Y} \rightarrow \mathcal{A}$ extending ϕ . We let $\mathcal{B} \subset \mathcal{A}$ denote the branch locus of ψ ; note that \mathcal{B} is a closed subscheme of \mathcal{A} of pure codimension one by [Gro71, Théorème X.3.1]. As \mathcal{A} is normal and ψ is finite surjective, the morphism ψ is étale precisely where it is unramified (Lemma 2.3), hence $\psi|_{\psi^{-1}(\mathcal{A} \setminus \mathcal{B})}$ is étale. We define $\mathcal{U} := \mathcal{A} \setminus \mathcal{B}$ and $\mathcal{V} := \psi^{-1}(\mathcal{U})$. In addition, we define $U := \mathcal{U}_K$ and $V := \mathcal{V}_K$.

Let now $\hat{\psi} : \hat{\mathcal{V}} \rightarrow \mathcal{U}$ be a Galois closure of the finite étale morphism $\mathcal{V} \rightarrow \mathcal{U}$ (with Galois group G) and let $\hat{V} := \hat{\mathcal{V}}_K$. As $\hat{\mathcal{V}}$ is regular and connected, it is integral. We let $H \subset G$ be the subgroup corresponding to \mathcal{V} by Galois theory [Sta20, Tag 0BND], so that $\hat{\mathcal{V}}/H \rightarrow \mathcal{V}$ is an isomorphism.

As G acts transitively on G/H , by Proposition 6.2 there exists an element $g \in G$ that acts with no fixed points on G/H .

If v is a finite place of K not in S' and $\zeta_v \in \mathcal{U}(\mathbb{F}_v)$ is a torsion point of order m coprime with S such that a Frobenius of ζ_v for $\hat{\psi}$ is g , then any $\zeta \in \mathcal{A}[m](K_v)$ that reduces to ζ_v modulo v is such that $\psi^{-1}(\zeta)$ has no K_v -points by Proposition 2.15. In particular, for such v, ζ_v and ζ , the

fibre $\psi^{-1}(\zeta)$ is reduced and has no $K(\zeta)$ -points. Thus, to prove the lemma it suffices to show that there exist v and ζ_v as above.

For each prime number ℓ , we denote by $\rho_\ell : \Gamma_K \rightarrow \text{Aut}(T_\ell(A))$ the natural ℓ -adic Galois representation attached to A . For every positive integer C we define the following open subset of Γ_K :

$$\square_C := \{\gamma \in \Gamma_K \mid v_{\ell_i}(\det(\rho_{\ell_i}(\gamma) - 1)) \leq v_{\ell_i}(C) \text{ for all } \ell_i \in S\}. \tag{12}$$

We claim that there exists a positive integer C such that $\square_C \neq \emptyset$. To see this, choose a finite place w_0 of K at which A has good reduction and that does not lie over any prime in S . Let now $\text{Fr}_{w_0} \in \Gamma_K$ be a Frobenius element corresponding to a place of \overline{K} lying over w_0 , and let $C := \prod_{\ell_i \in S} \{\det(\rho_{\ell_i}(\text{Fr}_{w_0}) - 1)\}_{\ell_i}$ (where, for a rational number $q \in \mathbb{Q}$ and a rational prime ℓ , $\{q\}_\ell := \ell^{v_\ell(q)}$). By Proposition 6.1, this number is non-zero, and $\text{Fr}_{w_0} \in \square_C$, so that the claim holds. We fix from now on a value of C for which $\square_C \neq \emptyset$.

Let M be a finite Galois extension of K such that $\square_C = \pi_{\overline{K}/M}^{-1}(\square'_C)$ for some subset $\square'_C \subset \text{Gal}(M/K)$, where $\pi_{\overline{K}/M} : \Gamma_K \rightarrow \text{Gal}(M/K)$ denotes the natural projection. By enlarging S' if necessary, we may assume that it contains all the places that ramify in the extension M/K . Fix an element $\gamma_0 \in \square'_C$ and consider the following Cartesian diagram.

$$\begin{array}{ccc} [C]^*\hat{\mathcal{Y}} & \longrightarrow & \hat{\mathcal{Y}} \\ [C]^*\hat{\psi} \downarrow \text{étale} & & \downarrow \text{étale} \\ [C]^{-1}\mathcal{U} & \xrightarrow{[C]} & \mathcal{U} \end{array}$$

As $\hat{\mathcal{Y}} \rightarrow A$ is a (PB)-cover, it follows that $[C]^*\hat{\mathcal{Y}}$ and, thus, $[C]^*\hat{\mathcal{V}}$, is geometrically integral over K . Consider the finite étale Galois morphism

$$r : [C]^*\hat{\mathcal{V}} \times_{\mathcal{O}_{K,S'}} \mathcal{O}_{M,S'} \rightarrow [C]^*\hat{\mathcal{V}} \rightarrow [C]^{-1}\mathcal{U},$$

where $[C]^*\hat{\mathcal{V}} \times_{\mathcal{O}_{K,S'}} \mathcal{O}_{M,S'}$ is integral. Note that the Galois group G' of r can be identified with $\text{Gal}(M/K) \times G$. Let $\gamma := (\gamma_0, g)$ in G' .

By the geometric Chebotarev theorem [Pin97, Theorem B.9] there exist a finite place v of K not in S' and a point $P_v \in [C]^{-1}\mathcal{U}(\mathbb{F}_v)$ such that a Frobenius at P_v for r is γ . As the residue field at P_v is \mathbb{F}_v , and because γ_0 is an element of \square'_C , by Proposition 6.1 we have $\{\#\mathcal{A}(\mathbb{F}_v)\}_{\ell_i} \mid C$ for every $\ell_i \in S$. In particular, for every element $P \in \mathcal{A}(\mathbb{F}_v)$ and every $\ell_i \in S$, the ℓ_i part of the order of P (which is a divisor of $\#\mathcal{A}(\mathbb{F}_v)$) divides C , hence the order of $[C]P$ is not divisible by any prime in S . The place v and the point $\zeta_v := [C]P_v$ satisfy all the conditions we need: v is not in S' , the order of ζ_v is not divisible by any prime in S , and a Frobenius at ζ_v for ψ is the projection of γ to G , which by definition is g . □

6.2 Kummer theory

Let A be an abelian variety over the number field K . Recall that a point $P \in A(K)$ is said to be *non-degenerate* if the set $\mathbb{Z}P := \{nP : n \in \mathbb{Z}\}$ is dense in A . We fix throughout a non-degenerate point $P \in A(K)$.

The aim of this section is to construct a finite place v of K and a finite index coset C of $\mathbb{Z}P$ whose elements all reduce modulo v to the torsion point ζ constructed in Lemma 6.3.

For every positive integer m , we set $K_m := K(A[m])$ and $K_{P,m} := K(A[m], [m]^{-1}P)$, where

$$[m]^{-1}P = \{x \in A(\overline{K}) \mid [m]x = P\}.$$

The extensions $K_{P,m}/K_m$, $K_{P,m}/K$ and K_m/K are all Galois. Note that the extension of K generated by *all* m -division points of P also contains all (the coordinates of the) m -torsion points of A , because every m -torsion point is the difference of two m -division points of P . The Galois group $G_m := \text{Gal}(K_{P,m}/K)$ sits in the obvious exact sequence

$$1 \rightarrow N_m \rightarrow G_m \rightarrow H_m \rightarrow 1,$$

where $H_m = \text{Gal}(K_m/K)$ and $N_m = \text{Gal}(K_{P,m}/K_m)$. Fix once and for all a compatible system $(P_m)_{m \geq 1}$ of points of $A(\overline{K})$ that satisfy $nP_m = P_{m/n}$ for all positive integers $n \mid m$. This choice allows us to interpret G_m as a subgroup of $A[m] \rtimes \text{Aut } A[m]$ via the map

$$\begin{aligned} G_m &\hookrightarrow A[m] \rtimes \text{Aut } A[m] \\ \sigma &\mapsto (\sigma(P_m) - P_m, \sigma|_{A[m]}). \end{aligned} \tag{13}$$

In this way, H_m gets identified with a subgroup of $\text{Aut } A[m]$, and the kernel N_m with a subgroup of $A[m]$.

Restricting our attention to positive integers m of the form $m = \ell^n$, where ℓ is a fixed prime, and passing to the limit $n \rightarrow \infty$ in (13), we obtain an injection

$$G_{\ell^\infty} := \text{Gal}(K(A[\ell^\infty], \ell^{-\infty}P)/K) \hookrightarrow T_\ell(A) \rtimes \text{Aut } T_\ell(A),$$

where $\ell^{-\infty}P := \{x \in A(\overline{K}) : \ell^n x = P \text{ for some } n \geq 1\}$. We may therefore write elements of G_{ℓ^∞} as pairs (t, M) with $t \in T_\ell(A)$ and $M \in \text{Aut } T_\ell(A)$. As previously, note that $K(A[\ell^\infty])$ is a subfield of $K(\ell^{-\infty}P)$.

Finally, there are obvious variants of this construction where one works with finitely many primes ℓ_1, \dots, ℓ_r . In this case, the Galois group of $K(\ell_1^{-\infty}P, \dots, \ell_r^{-\infty}P)$ over K embeds into $\prod_{i=1}^r T_{\ell_i}(A) \rtimes \prod_{i=1}^r \text{Aut}(T_{\ell_i}(A))$ by passing to the inverse limit in (13) over positive integers m all of whose prime factors lie in $\{\ell_1, \dots, \ell_r\}$. Moreover, $K(A[\ell_1^\infty], \dots, A[\ell_r^\infty])$ is a subfield of $K(\ell_1^{-\infty}P, \dots, \ell_r^{-\infty}P)$, and the Galois group $\text{Gal}(K(\ell_1^{-\infty}P, \dots, \ell_r^{-\infty}P)/K(A[\ell_1^\infty], \dots, A[\ell_r^\infty]))$ is identified with a subgroup of $\prod_{i=1}^r T_{\ell_i}(A)$.

6.2.1 The ℓ -part of the reductions of P . Let S^b be the set of finite places of bad reduction of A , define $\mathcal{U} = \text{Spec } \mathcal{O}_{K, S^b}$ and let \mathcal{A} be the (smooth proper) Néron model of A over \mathcal{U} . The point P gives rise to a point of infinite order of $\mathcal{A}(\mathcal{U})$ that we still denote by P . For a finite place v of K at which A has good reduction we may then consider the image of P in $\mathcal{A}(\mathbb{F}_v)$, where \mathbb{F}_v is the residue field at v . For the sake of simplicity, we denote this image by $P \bmod v$, and we abuse notation and write simply $A(\mathbb{F}_v)$ instead of $\mathcal{A}(\mathbb{F}_v)$.

We need the following notion of the ℓ part of an element in an abelian group.

DEFINITION 6.4. Let B be an abelian group and $g \in B$ be an element of finite order. The subgroup of B generated by g is a finite abelian group that we may write as the (internal) direct product of its Sylow subgroups: let

$$B = S_{\ell_1} \times \dots \times S_{\ell_r},$$

where the ℓ_i are pairwise distinct prime numbers and S_{ℓ_i} is the corresponding Sylow subgroup. Given a prime ℓ , the ℓ part of g is the projection of g on S_ℓ (which may be trivial). We denote it by $[g]_\ell$.

Remark 6.5. An equivalent construction of the ℓ part of g is as follows. Let $m = \text{ord}(g)$ and let ℓ^e be the exact power of ℓ that divides m . Write $m = \ell^e q$ and choose an integer $s \equiv 1 \pmod{\ell^e}$, $s \equiv 0 \pmod{q}$. The ℓ part of g is then $s \cdot g$.

Let v be a finite place of good reduction for A . We may then consider the ℓ part of $P \bmod v$, where we (equivalently) view $P \bmod v$ as a point of finite order of either $A(\mathbb{F}_v)$ or $A(\overline{\mathbb{F}}_v)$. We now wish to describe $[P \bmod v]_\ell$ in terms of Kummer theory. Note that $[P \bmod v]_\ell$ is an element of $A(\mathbb{F}_v)[\ell^\infty] \subseteq A(\overline{\mathbb{F}}_v)[\ell^\infty]$, and that, provided that ℓ is different from the residue characteristic of v , there is a canonical identification $A(\overline{K})[\ell^\infty] \cong A(\overline{\mathbb{F}}_v)[\ell^\infty]$ given by reduction modulo (a place of \overline{K} over) v : indeed, for any positive integer n coprime with the residue characteristic of v , the n -torsion subscheme of \mathcal{A} is finite étale over \mathcal{O}_{K_v} . We may therefore also view $[P \bmod v]_\ell$ as an element of $A(\overline{K})[\ell^\infty]$. Recall, on the other hand, that we have a canonical identification

$$\pi_\ell : V_\ell(A)/T_\ell(A) \cong A(\overline{K})[\ell^\infty]. \tag{14}$$

It will be useful to make this identification more explicit.

Remark 6.6. Representing points in $T_\ell(A)$ as compatible sequences (t_0, t_1, t_2, \dots) of torsion points (with $t_i \in A[\ell^i]$ and $\ell t_i = t_{i-1}$) and elements of $V_\ell(A)$ as $\ell^{-n}(t_0, t_1, t_2, \dots)$, the torsion point corresponding to $\ell^{-n}(t_0, t_1, t_2, \dots) \bmod T_\ell(A)$ is simply t_n . The inverse map, from $A(\overline{K})[\ell^\infty]$ to $V_\ell(A)/T_\ell(A)$, may be described as follows. Let $t \in A(\overline{K})[\ell^\infty]$ be a torsion point of exact order ℓ^n and choose $t_{n+1}, t_{n+2}, \dots \in A(\overline{K})[\ell^\infty]$ with the property that $\ell^i t_{n+i} = t$ and $\ell t_{n+i+1} = t_{n+i}$ for all $i \geq 1$. The element

$$\ell^{-n}(\ell^n t, \ell^{n-1} t, \dots, \ell t, t, t_{n+1}, t_{n+2}, \dots) \in V_\ell(A)$$

is then well-defined up to addition of elements of $T_\ell(A)$, and its image via π_ℓ is t .

A general principle of Kummer theory is that all the information about the ℓ -part of the reductions of P should be captured by the Galois group G_{ℓ^∞} . In our context, using the above identification π_ℓ , this has been made concrete by Pink as follows.

THEOREM 6.7 [Pin04, Proposition 3.2]. *Let v be a finite place of K at which A has good reduction and let ℓ be a prime different from the residue characteristic of v . Let $\text{Fr}_v \in G_{\ell^\infty}$ be a Frobenius element at v for the extension $K(\ell^{-\infty}P)/K$. Writing $\text{Fr}_v = (t, M) \in T_\ell(A) \times \text{Aut } T_\ell(A)$, we have*

$$[P \bmod v]_\ell = \pi_\ell((M - \text{Id})^{-1}t).$$

Remark 6.8. Note that, as a consequence of the Weil conjectures, the eigenvalues of M (as in Theorem 6.7) are all of absolute value $\sqrt{\#\overline{\mathbb{F}}_v}$. In particular, $M - \text{Id} : V_\ell(A) \rightarrow V_\ell(A)$ is invertible. Also note that, in general, $(M - \text{Id})^{-1}t$ is an element of $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, but not necessarily of $T_\ell(A)$. Finally, to make the connection between our language and that of [Pin04], note that

$$\begin{aligned} \lambda : \mathbb{Z}[1/\ell] &\rightarrow A(\overline{K}) \\ \frac{b}{\ell^k} &\mapsto bP_{\ell^k} \end{aligned}$$

is a *special splitting* in the sense of [Pin04, § 2].

6.2.2 Adelic Kummer theory for abelian varieties. We now review the last result we need from Kummer theory. The following theorem is essentially due to Ribet [Rib79]; for the version quoted here see Hindry [Hin88, § 2, Proposition 1] or Bertrand [Ber88, Theorem 1].

THEOREM 6.9. *Let A be an abelian variety over a number field K and let $P \in A(K)$ be a non-degenerate point. There is a constant $c = c(A/K, P)$, which only depends on K, A and P , such that, for every integer m , the index of N_m in $A[m]$ (under the identification given by (13)) is bounded by c .*

Remark 6.10. Proposition 1 of [Hin88, §2] is stated under the assumption that P be indivisible in $A(K)$; the conclusion is then that the constant $c(A/K, P)$ is independent of P . One may readily check that (allowing c to also depend on P) the hypothesis that P be indivisible may be removed; this is also the statement given in [Ber88].

We use this result in the form of the following obvious corollary.

COROLLARY 6.11. *Let c be as in Theorem 6.9. Then, for any choice of primes ℓ_1, \dots, ℓ_r , all strictly larger than c , the Galois group of*

$$K(\ell_1^{-\infty}P, \dots, \ell_r^{-\infty}P)/K(A[\ell_1^\infty], \dots, A[\ell_r^\infty])$$

is isomorphic to $\prod_{i=1}^r T_{\ell_i}(A)$ under the injection induced by (13).

Proof. It suffices to show that for each $m = \ell_1^{e_1} \cdots \ell_r^{e_r}$ the image of N_m via (13) is all of $A[m] \times \{1\}$. If this were not the case, then by our assumption that $\ell_1, \dots, \ell_r > c$, the index $[A[m] : N_m]$ would be at least $\min\{\ell_i\} > c$, which contradicts Theorem 6.9. \square

6.3 The case of a cyclic dense subgroup

Let A be an abelian variety over a number field K and let $P \in A(K)$ be a non-degenerate point. Let $c = c(A/K, P)$ be as in Theorem 6.9 (and Corollary 6.11) and fix a positive integer d ; when applying the results of this section to our study of covers of abelian varieties, the integer d will represent the degree of a cover. We define

$$S := \{\ell \text{ prime number} \mid \ell \leq \max\{c, d\}\}.$$

Note that S is clearly finite; it is the set of ‘bad’ primes we will have to avoid in our arguments. Let $\zeta \in A(\overline{K})$ be a torsion point of order $m = \ell_1^{e_1} \cdots \ell_r^{e_r}$, and assume that each prime factor ℓ_i is not in S . Let L be a finite Galois extension of $K(\zeta)$ of degree dividing $d!$ and write Δ for the Galois group of L over $K(\zeta)$. Finally fix a non-empty conjugacy-stable subset $\delta \subseteq \Delta$. The following proposition and corollary show that, given such a torsion point ζ , one can find a finite place v of $K(\zeta)$ and a finite index coset C of $\langle P \rangle$ such that every c in C is congruent to ζ modulo v , whilst simultaneously fixing the Frobenius conjugacy class of v in the extension $L/K(\zeta)$.

PROPOSITION 6.12. *With the notation as before, there is a positive density set of finite places v of $K(\zeta)$ with the following properties:*

- (1) v is unramified in L , and the corresponding Frobenius element in Δ lies in δ ;
- (2) $A \times_K K(\zeta)$ has good reduction at v , and v does not divide $m = \text{ord}(\zeta)$;
- (3) the ℓ_i -part of $P \bmod v$ is equal to the ℓ_i -part of ζ for all $i = 1, \dots, r$.

Before proving the proposition we briefly pause to explain its role in our approach. Given a (GPB)-cover $\phi : Y \rightarrow A$, Lemma 6.3 will provide us with a torsion point $\zeta \in A(\overline{K})$ that does not lift to $Y(K(\zeta))$. Fix in addition a Zariski-dense cyclic subgroup $\Omega \subseteq A(K)$. We use Proposition 6.12 to show that there is a finite index coset of Ω all of whose points reduce to ζ modulo a suitable place v of $K(\zeta)$: the idea is then to show that the points in this coset do not lift to Y . In order to draw this conclusion we also need to impose some additional conditions related to the behaviour of the place v in the field over which the fibre $\phi^{-1}(\zeta)$ splits: in the present setting, such conditions are encapsulated by the field L and by the conjugacy class δ . Finally, we point out that this result has no analogue in [Zan10], and replaces a completely different argument that used Serre’s open image theorem for elliptic curves, a tool that is not available to us in the context of general abelian varieties.

Proof. Let $K_{P,\infty}$ be the extension of K generated by $\ell_i^{-\infty}P$ for all $i = 1, \dots, r$ and $K_\infty = K(A[\ell_i^\infty] : i = 1, \dots, r)$. Note that $K_\infty = K(\zeta)(A[\ell_i^\infty] : i = 1, \dots, r)$ and similarly for $K_{P,\infty}$. Letting $G = \text{Gal}(K_{P,\infty}/K(\zeta))$, $H = \text{Gal}(K_\infty/K(\zeta))$ and $N = \text{Gal}(K_{P,\infty}/K_\infty)$, we have the exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1.$$

Writing $T := \prod_{i=1}^r T_{\ell_i}(A)$, by passing to the limit in (13) we may identify G with a subgroup of $T \rtimes \text{Aut}(T)$. As every prime ℓ_i is strictly larger than c , by Corollary 6.11, the subgroup N can be identified with $T \rtimes \{1\}$. We write t_i for the projection of an element $t \in T$ to $T_{\ell_i}(A)$.

It will also be useful to fix \mathbb{Z}_{ℓ_i} -bases of the Tate modules $T_{\ell_i}(A)$ as follows. For each prime divisor ℓ_i of m , write $[\zeta]_{\ell_i}$ as $n_i\zeta$, and note that it is a torsion point of order $\ell_i^{e_i}$. For every such ℓ_i , fix a basis $v_{i,1}, \dots, v_{i,2g}$ of $T_{\ell_i}(A)$ with the property that $\pi_{\ell_i}(\ell_i^{-e_i}v_{i,1}) = n_i\zeta$, where π_{ℓ_i} is defined by (14). This is possible because $n_i\zeta$ is a torsion point of exact order $\ell_i^{e_i}$. We use these bases to identify $T \cong \prod_{i=1}^r T_{\ell_i}(A)$ with $\prod_{i=1}^r \mathbb{Z}_{\ell_i}^{2g}$ and $\text{Aut}(T)$ with $\prod_{i=1}^r \text{GL}_{2g}(\mathbb{Z}_{\ell_i})$. By the definition of π_{ℓ_i} in (14) and Remark 6.6, we see that a representative in $V_{\ell_i}(A) \cong \mathbb{Q}_{\ell_i}^{2g}$ of $n_i\zeta$ is $\ell_i^{-e_i}(1 \ 0 \ \dots \ 0)^T$.

Let F be the extension of K generated by $K_{P,\infty}$ and by L . The Galois group G_1 of F over $K(\zeta)$ injects into $\Delta \times G \subseteq \Delta \times (T \rtimes \text{Aut}(T))$. We represent elements g_1 of G_1 as triples $(\gamma, (t, M))$ with $\gamma \in \Delta$, $t \in T$ and $M \in \text{Aut}(T)$. Note that M corresponds to a collection (M_1, \dots, M_r) of matrices with $M_i \in \text{GL}_{2g}(\mathbb{Z}_{\ell_i})$ or, equivalently, M is a matrix in $\text{GL}_{2g}(\prod_{i=1}^r \mathbb{Z}_{\ell_i})$. It thus makes sense to compute the determinant of M as an element of the ring $\prod_{i=1}^r \mathbb{Z}_{\ell_i}$ (independent of the choice of basis); equivalently, $\det(M)$ is the collection $(\det(M_1), \dots, \det(M_r))$.

We claim that G_1 contains $\{1\} \times (T \rtimes \{1\})$. On the one hand, multiplication by $\#\Delta$ is invertible on the profinite abelian group T (because, by assumption, $\#\Delta$ is relatively prime to the order of any finite quotient of T), so, given $t \in T$, we may find $t' \in T$ such that $[\#\Delta]t' = t$. On the other hand, basic Galois theory shows that G_1 surjects onto G , which contains $T \rtimes \{1\}$. Therefore, given any $t \in T$, we know that G_1 contains an element of the form $g_1 = (\gamma, (t', 1))$ for some $\gamma \in \Delta$ and $t' \in T$ such that $[\#\Delta]t' = t$. It follows that G_1 contains $g_1^{\#\Delta} = (1, (t, 1))$, which proves the claim.

We now consider

$$U = \left\{ (\gamma, (t, M)) \in G_1 \left| \begin{array}{l} \det(M_i - \text{Id}) \neq 0 \text{ for all } i = 1, \dots, r \\ \gamma \in \delta \\ \pi_{\ell_i}(((M - \text{Id})^{-1}t)_i) = [\zeta]_{\ell_i}, \text{ for } i = 1, \dots, r \end{array} \right. \right\}.$$

Note that U is open in G_1 : indeed, the first condition is clearly open, whereas the condition $\gamma \in \delta$ factors via the finite quotient $G_1 \rightarrow \Delta$. Finally, $(\gamma, (t, M)) \mapsto \pi_{\ell_i}(((M - \text{Id})^{-1}t)_i) = \pi_{\ell_i}((M_i - \text{Id})^{-1}t_i)$ is a continuous function with values in the discrete set $A[\ell_i^\infty]$, hence it is locally constant. As we are imposing only finitely many of these conditions, U is the finite intersection of open subsets of G_1 and is therefore open.

To prove the proposition, it suffices to show that U is non-empty. To see this, note that the Chebotarev density theorem yields the existence of a positive density set of places unramified in L and whose corresponding Frobenius conjugacy class lies in U . Discarding the finitely many places that do not satisfy condition (2) still leaves us with a positive density set of places that satisfy all conditions in the statement (note that all places whose Frobenius belongs to U satisfy condition (3) thanks to Theorem 6.7). Thus, it suffices to show that U is non-empty.

We start by choosing an element $g_1 = (\gamma, (t, M)) \in G_1$ with $\gamma \in \delta$ and $\det(M_i - \text{Id}) \neq 0$ for every $i = 1, \dots, r$. To see that such an element exists, note first that $G_1 \rightarrow \Delta$ is surjective, and that by the Chebotarev density theorem (applied to the extension $F/K(\zeta)$, with Galois group

G_1) we may find an element g_1 whose image in Δ lies in δ and which is the Frobenius of some place w of $K(\zeta)$. As a consequence of the Weil conjectures, all the eigenvalues of M_i (which corresponds to the action of a Frobenius at w on $T_{\ell_i}(A)$) then have absolute value $\sqrt{\#\overline{\mathbb{F}}_w} \neq 1$, which ensures that $\det(M_i - \text{Id})$ is non-zero for all i .

The fact that $g_1 = (\gamma, (t, M))$ fixes ζ (recall that $g_1 \in G_1 = \text{Gal}(F/K(\zeta))$) translates into the following condition. Write as before $M = (M_1, \dots, M_r)$ with $M_i \in \text{GL}_{2g}(\mathbb{Z}_{\ell_i})$. Reducing M_i modulo $\ell_i^{e_i}$ gives the action of g_1 on $A[\ell_i^{e_i}]$, which must be trivial on the first basis element because G_1 fixes ζ (hence, also $n_i\zeta$). It follows that we have

$$M_i \equiv \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \pmod{\ell_i^{e_i}} \tag{15}$$

where each entry $*$ may in principle be arbitrary. We now construct an element of U as a product $(1, (t', 1)) \cdot g_1$ for a suitable $t' \in T$. We have already observed that all elements of the form $(1, (t', 1))$ are in G_1 , and g_1 is in G_1 by construction, so for any choice of t' we do indeed get an element of G_1 . For $t' \in T$ (to be determined later), consider the element

$$(1, (t', 1)) \cdot (\gamma, (t, M)) = (\gamma, (t' + t, M)).$$

Note that its image in Δ is $\gamma \in \delta$, and that $\det(M_i - \text{Id}) \neq 0$ for all $i = 1, \dots, r$ by our choice of M . Hence, for this element to lie in U , we only need to ensure that we can satisfy the conditions

$$\pi_{\ell_i}(((M - \text{Id})^{-1}(t' + t))_i) = [\zeta]_{\ell_i} \quad \forall i = 1, \dots, r.$$

This condition is implied by the equality (which we write in the coordinates chosen previously)

$$(M_i - \text{Id})^{-1}(t' + t)_i = \ell_i^{-e_i}(1 \ 0 \ \dots \ 0)^T \in V_{\ell_i}(A).$$

As $M_i - \text{Id}$ is invertible, the latter equality is, in turn, equivalent to

$$(t' + t)_i = (M_i - \text{Id})\ell_i^{-e_i}(1 \ 0 \ \dots \ 0)^T.$$

This equation clearly has the unique solution

$$t'_i = -t_i + (M_i - \text{Id})\ell_i^{-e_i}(1 \ 0 \ \dots \ 0)^T$$

in $V_{\ell_i}(A)$, and we claim that this element t'_i lies in $T_{\ell_i}(A)$. Indeed, the congruence in (15) ensures that the first column of $M_i - \text{Id}$ is an integral multiple of $\ell_i^{e_i}$, so that the vector $(M_i - \text{Id})\ell_i^{-e_i}(1 \ 0 \ \dots \ 0)^T$ (which is simply $\ell_i^{-e_i}$ times the first column of $M_i - \text{Id}$) has integral coordinates; equivalently, given our identifications, it corresponds to an element of $T_{\ell_i}(A)$. We may then take $t' = (t'_1, \dots, t'_r)$, where each t'_i is as constructed previously, to obtain as desired that the product $(1, (t', 1)) \cdot g_1$ lies in U . This proves that U is non-empty, which by our previous remarks finishes the proof of the proposition. \square

COROLLARY 6.13. *Let v be as in Proposition 6.12. Then there exist integers s and q such that every Q in the coset $sP + \mathbb{Z}(mqP)$ of $\mathbb{Z}P$ is congruent to ζ modulo v .*

Proof. Let $n = \text{ord}(P \bmod v)$. As the ℓ -parts of $(P \bmod v)$ and ζ agree for all prime divisors ℓ of $m = \text{ord}(\zeta)$, we see, in particular, that for all $\ell \mid m$ we have $v_\ell(m) = v_\ell(n)$. This shows that $m \mid n$ and that writing $n = mq$ we have $(m, q) = 1$. By the Chinese remainder theorem, there is an integer s such that $s \equiv 0 \pmod q$ and $s \equiv 1 \pmod m$. As the order of $sP \bmod v$ equals m and,

for every prime divisor ℓ of m ,

$$[sP \bmod v]_\ell = [\zeta]_\ell,$$

it follows that sP is congruent to ζ modulo v . As mqP reduces to the trivial element modulo v , the integers q and s have the desired property. \square

6.4 Main theorem: cyclic case

PROPOSITION 6.14. *Let A be an abelian variety over a number field K . Let $\Omega = \langle P \rangle \subset A(K)$ be a Zariski-dense cyclic subgroup of A . Then property (GPB) implies property (PF) for (A, Ω) .*

Proof. Let $\phi : Y \rightarrow A$ be a (GPB)-cover of A of degree d , $\hat{\phi} : \hat{Y} \rightarrow A$ be the Galois closure of ϕ , and G be its Galois group. We have to prove that there is a finite index coset C of Ω such that, for each $c \in C$, the fibre $\phi^{-1}(c)$ is reduced and has no K -rational points.

First, we choose a finite set S^b of finite places of K , an abelian scheme \mathcal{A} over A extending A over \mathcal{O}_{K,S^b} , a normal proper model \mathcal{Y} for Y over \mathcal{O}_{K,S^b} and a finite surjective morphism $\psi : \mathcal{Y} \rightarrow \mathcal{A}$ extending $\phi : Y \rightarrow A$ over \mathcal{O}_{K,S^b} . Now, we let $c(A/K, P)$ be the constant of Theorem 6.9, and let S be a finite set of rational primes containing $\{\ell \mid \ell \leq \max\{c(A/K, P), d\}\}$. By Lemma 6.3, there is a torsion point $\zeta \in A(\bar{K})$ with the following properties:

- (1) the order of ζ is coprime with S ;
- (2) the fibre $\phi^{-1}(\zeta)$ is étale over $\text{Spec } K(\zeta)$;
- (3) if $\Delta \subset G$ is a decomposition group for ζ (associated with an implicit choice of a geometric point $\bar{y}_0 \in \hat{\phi}^{-1}(\zeta)(\bar{K})$ lying over ζ), there exists an element $\delta_0 \in \Delta$ such that δ_0 acts with no fixed points on $\phi^{-1}(\bar{\zeta})$.

We define $K_0 := K(\zeta)$. Let \mathcal{U} be a dense open subscheme of \mathcal{A} such that ψ is étale over \mathcal{U} and $\mathcal{U}(K_0)$ contains ζ . After possibly enlarging S^b , we may assume that the Zariski closure $\{\bar{\zeta}\} \subset \mathcal{A}$ is contained in \mathcal{U} . Write $\mathcal{V} := \psi^{-1}(\mathcal{U})$, so that $\psi|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{U}$ is finite étale. Let $\hat{\psi} : \hat{\mathcal{V}} \rightarrow \mathcal{U}$ be the Galois closure of $\psi|_{\mathcal{V}}$, and note that its Galois group is G .

Let L/K_0 be the finite Galois extension corresponding to $\text{Ker}(\mathfrak{D}_\zeta : \Gamma_{K_0} \rightarrow G)$. The canonical identification of $\text{Gal}(L/K_0)$ with Δ provided by \mathfrak{D}_ζ allows us to assume, with a slight abuse of notation, that these two groups are the same.

Note that L is the Galois closure over K_0 of the minimal field of definition of \bar{y}_0 , so that L/K_0 has degree dividing $d!$. In particular, we may apply Proposition 6.12 and Corollary 6.13 to see that there exist a finite place $v \in M_{K_0}$ not lying over any place of S^b , and a finite index coset C of Ω such that each $c \in C$ reduces to $\zeta \bmod v$ and such that a Frobenius (after implicitly fixing a place $w \in M_{\bar{K}}$ lying over v) for $v \in M_{K_0}$ in Δ is δ_0 . In particular, for each c in C , a decomposition group for c contains the Frobenius $\delta_0 = \text{Fr}_{\bar{y}_0, w, \hat{\psi}} \in \Delta$. The latter acts without fixed points on $\psi^{-1}(c) \bmod v$, so that by Lemma 2.12 the \mathcal{O}_{K_v} -scheme $\psi^{-1}(c)_{\mathcal{O}_{K_v}}$ has no \mathbb{F}_v -points. As the \mathcal{O}_{K_v} -scheme $\psi^{-1}(c)_{\mathcal{O}_{K_v}}$ is finite over \mathcal{O}_{K_v} , it has no K_v -points either. Therefore, the scheme $\phi^{-1}(c)$ has no K -points, as required. This concludes the proof. \square

THEOREM 6.15. *Let A be an abelian variety over a number field K . Let $\Omega = \langle P \rangle \subset A(K)$ be a Zariski-dense cyclic subgroup of A . Then, for every (PB)-cover $\pi : Y \rightarrow A$ the pair (π, Ω) satisfies property (IF).*

Proof. It immediately follows from Proposition 6.14 that, for every number field K , every abelian variety A over K , and every finitely generated Zariski-dense subgroup Ω of rank one, we have that property (GPB) implies property (PF) for (A, Ω) . Therefore, the result follows from Proposition 4.19 (with $r = 1$). \square

7. Proof of the main results

If k is a finitely generated field of characteristic zero and A is an abelian variety over k , then $A(k)$ is a finitely generated abelian group [Con06, Corollary 7.2]. Therefore, the results of §4 apply to every subgroup Ω of $A(k)$.

7.1 From number fields to finitely generated fields over \mathbb{Q}

We deduce the weak-Hilbert property for all abelian varieties over finitely generated fields by a specialization argument to number fields (see Proposition 7.6). For the proof we rely on a general result essentially due to Serre, for which we need some notation. Let S be a variety over a number field K , with generic point η , and let $\mathcal{A} \rightarrow S$ be an abelian scheme.

DEFINITION 7.1. For positive integers d and h , we let $S(d, h)$ be the set of closed points of S of degree at most d over K and of logarithmic height at most h (after some appropriate embedding $S \hookrightarrow \mathbb{P}_{N,K}$ has been chosen).

THEOREM 7.2. There exist an integer d_0 and constant $c > 0$ such that the set

$$S(d_0, h)_{\text{End}} := \{s \in S(d_0, h) : \text{End}_\eta \mathcal{A}_\eta \xrightarrow{\sim} \text{End}_{k(s)} \mathcal{A}_s\}$$

has cardinality at least e^{ch} for $h \gg 1$, and the set $\bigcup_{h \geq 1} S(d_0, h)_{\text{End}}$ is Zariski-dense in S .

Proof. This can be deduced from the main results of Masser [Mas96] or upon combining work of Serre [Ser00, Letter to Ken Ribet of January 1st, 1981, §1] and Noot [Noo95, Corollary 1.5], as we now explain.

Fix an auxiliary prime ℓ . Serre constructs the following data:

- (1) a dense open $X_0 \subset \mathbb{A}_{\mathbb{Q}}^{\dim S}$;
- (2) a dense open $X \subset S$ and a finite étale morphism $\pi : X \rightarrow X_0$;
- (3) an étale cover $X^* \rightarrow X$,

such that, if B_0 is the set of points $x_0 \in X_0(\mathbb{Q})$ for which the fibre of $X^* \rightarrow X_0$ is irreducible, and B is its inverse image under π , then the images of the ℓ -adic Galois representations attached to \mathcal{A}_s are equal to those attached to \mathcal{A}_η for all $s \in B$. Note that, by Hilbert’s (classical) irreducibility theorem, B_0 is the complement of a thin set.

Let $k = k(\eta)$ be the function field of S . Then, for all s in B , we have

$$\text{End}_{\text{Gal}(\bar{k}/k)}(T_\ell \mathcal{A}_\eta) \otimes \mathbb{Q}_\ell \cong \text{End}_{\text{Gal}(\bar{k}(s)/k(s))}(T_\ell \mathcal{A}_s) \otimes \mathbb{Q}_\ell.$$

By Tate’s conjecture on endomorphisms (which holds for all finitely generated fields of characteristic zero; see [Fal84]) we thus obtain $\text{End}_k(\mathcal{A}_\eta) \otimes \mathbb{Q}_\ell \cong \text{End}_{k(s)}(\mathcal{A}_s) \otimes \mathbb{Q}_\ell$. As the inclusion $\text{End}_k(\mathcal{A}_\eta) \hookrightarrow \text{End}_{k(s)}(\mathcal{A}_s)$ has torsion-free cokernel, we see that all points in B satisfy the desired property concerning endomorphisms and have bounded degree over \mathbb{Q} (we take, in particular, $d_0 = \deg \pi$). Note that B is Zariski dense, so that $\bigcup_{h \geq 1} S(d_0, h)_{\text{End}}$ is Zariski dense. To conclude the proof, it only remains to show that the cardinality of $S(d_0, h)_{\text{End}}$ grows exponentially in h . This follows from the fact that the complement of a thin set of $\mathbb{A}_{\mathbb{Q}}^{\dim S}$ contains $\gg e^{ch}$ points of logarithmic height h for some positive constant c (see [Ser08, Theorem 3.4.4]). \square

LEMMA 7.3. Let $\Omega \subset \mathcal{A}(S)$ be a Zariski-dense subgroup. There exists a Zariski-dense set of closed points $s \in S$ of bounded degree over K such that the image $\Omega_s := \text{Im}_{\mathcal{A}(S) \rightarrow \mathcal{A}(k(s))}(\Omega)$ is Zariski-dense in \mathcal{A}_s and the restriction morphism $\Omega \rightarrow \Omega_s$ is an isomorphism.

Proof. We need the following observation, which follows at once from Poincaré’s reducibility theorem for abelian varieties: if A is an abelian variety over a field k , and $B \subset A$ is an abelian

subvariety, there exists an endomorphism $\phi : A \rightarrow A$ such that $B \subset \text{Ker } \phi$, and the quotient $(\text{Ker } \phi)/B$ is finite. In particular, a finitely generated subgroup $\langle Q_1, \dots, Q_r \rangle \subset A(k)$ is Zariski-dense in A if and only if there does not exist any non-zero endomorphism $\phi \in \text{End}_k(A)$ such that $\phi(Q_i) = 0$ for all $i = 1, \dots, r$.

As Ω is finitely generated (see [Con06, Corollary 7.2]), we may write $\Omega = \langle P_1, \dots, P_r \rangle$. Let $\Gamma := \text{End}_S(\mathcal{A}) \cdot \Omega \subset \mathcal{A}(S)$. By the main theorem of [Mas89], the cardinality of the set of $s \in S(d_0, h)$ such that the reduction morphism $\Gamma \rightarrow \mathcal{A}_s(k(s))$ is not injective is bounded above by $C \cdot h^k$, for some $k \in \mathbb{N}$ and $C \in \mathbb{R}_{>0}$. Defining

$$S(d_0, h)_{\text{good}} := \{s \in S(d_0, h) \mid \text{End}_\eta \mathcal{A}_\eta \xrightarrow{\sim} \text{End}_{k(s)} \mathcal{A}_s \text{ and } \Gamma \rightarrow \mathcal{A}_s(k(s)) \text{ is injective}\},$$

we claim that $\bigcup_{h \geq 1} S(d_0, h)_{\text{good}}$ is Zariski-dense in S . Indeed, let $U \subset S$ be a dense open subset. By Theorem 7.2, there exist an integer d_0 and a constant $c > 0$ (possibly depending on U) such that, for every $h \geq 1$, the set

$$U(d_0, h)_{\text{End}} := \{s \in U(d_0, h) : \text{End}_k \mathcal{A}_\eta \xrightarrow{\sim} \text{End}_{k(s)} \mathcal{A}_s\}$$

has cardinality at least e^{ch} . By comparing cardinality bounds, because $e^{ch} > C \cdot h^k$ for h sufficiently large, there is an integer $h \geq 1$ such that the intersection of U and $S(d_0, h)_{\text{good}}$ is non-empty, so that $\bigcup_{h \geq 1} S(d_0, h)_{\text{good}}$ meets all open dense subschemes U as desired.

We claim that, for a point s in $\bigcup_{h \geq 1} S(d_0, h)_{\text{good}}$, the reduction $\Omega_s = \langle \overline{P}_1, \dots, \overline{P}_r \rangle$ is Zariski-dense in \mathcal{A}_s , where \overline{P}_i denotes the reduction of $P_i \in \mathcal{A}(S)$ in $\mathcal{A}_s(k(s))$.

Assume, by contradiction, that this does not hold. By the observation in the first paragraph, there exists a non-zero endomorphism $\bar{\phi} \in \text{End}_{k(s)}(\mathcal{A}_s)$ such that $\bar{\phi}(\overline{P}_i) = 0$ for each $i = 1, \dots, r$. As $\text{End}_\eta \mathcal{A}_\eta \xrightarrow{\sim} \text{End}_{k(s)} \mathcal{A}_s$, the endomorphism $\bar{\phi}$ is the reduction of some (necessarily non-zero) $\phi \in \text{End}_\eta(\mathcal{A}_\eta)$. As $\Gamma \rightarrow \mathcal{A}_s(k(s))$ is injective, we obtain $\phi(P_i) = 0$ for each $i = 1, \dots, r$. In particular, we see that $\Omega \subset \text{Ker } \phi$, contradicting the Zariski-density of Ω . \square

Remark 7.4. Lemma 7.3 can also be deduced from Néron’s [Nér52, Théorème 6].

Remark 7.5. A different proof of Lemma 7.3 can also be obtained by combining Silverman’s specialization theorem [Sil83], which gives boundedness of height for the points $s \in S^{(0)}$ for which the specialization map $\mathcal{A}(S) \rightarrow \mathcal{A}_s(k(s))$ is not injective, with a weaker version of Theorem 7.2, where we only require that the set in question be infinite.

PROPOSITION 7.6. *Assume that, for every number field L , every abelian variety A over L , and every Zariski-dense subgroup $\Omega \subset A(L)$, we have that property (PB) implies property (IF) for (A, Ω) . Then, for every finitely generated field k of characteristic zero, every abelian variety A' over k , and every Zariski-dense subgroup $\Omega' \subset A'(k)$, we have that property (PB) implies property (IF) for (A', Ω') .*

Proof. Let A be an abelian variety defined over a finitely generated field k of characteristic zero, and let $\Omega \subset A(k)$ be a Zariski-dense (finitely generated) subgroup. Let S be a smooth geometrically connected variety over a number field K such that $K(S) = k$, and let $\eta \in S$ be its generic point. As Ω is finitely generated, by spreading out (and possibly restricting S), we may assume that A extends to an abelian scheme $\mathcal{A} \rightarrow S$ with generic fibre $\mathcal{A}_\eta = A$, and that $\Omega \subset \mathcal{A}(S)$. Let us show that property (PB) implies property (IF) for (A, Ω) .

Let $\phi : X \rightarrow A$ be a (PB)-cover. Replacing S by a dense open subscheme if necessary, we may assume that $\phi : X \rightarrow A$ extends to a finite surjective morphism $\psi : \mathcal{X} \rightarrow \mathcal{A}$. By Lemma 4.11, replacing S by a dense open subscheme if necessary, for every s in S , the specialization $\psi_s : \mathcal{X}_s \rightarrow \mathcal{A}_s$ is a (PB)-cover of the abelian variety \mathcal{A}_s over $\kappa(s)$.

For s in S , let $\text{red}_s : \mathcal{A}(S) \rightarrow \mathcal{A}_s(\kappa(s))$ denote the reduction map. Then, by Lemma 7.3, there exists a closed point $s \in S$ such that the specialization $\Omega_s := \text{red}_s(\Omega)$ is isomorphic to Ω and Zariski-dense in \mathcal{A}_s . In particular, by our assumption on the pair $(\mathcal{A}_s, \Omega_s)$ over the number field $\kappa(s)$, there exists a finite index coset C of $\Omega_s \cong \Omega$ such that, for all $c \in C$, the scheme $\psi_s^{-1}(c) \cong \psi^{-1}(c)$ is integral.

Define $C' := \text{red}_s^{-1}(C)$ to be the inverse image of C in Ω , and note that $C' \subset \Omega$ is a finite index coset. Let c be an element of $C' \subset \mathcal{A}(S)$. The $c(S)$ -scheme $\mathcal{X}|_{c(S)}$ is connected, and finite étale over the point $c(s)$. In particular, the fibre of $\mathcal{X}|_{c(S)} \rightarrow c(S)$ over $c(\eta)$ is étale and integral. Thus, for every c in the finite index coset C' of Ω , the scheme $\phi^{-1}(c) = \psi^{-1}(\eta_c)$ is integral. We conclude that property (IF) holds for (A, Ω) . □

7.2 Main results

We now show that every (PB)-cover $Y \rightarrow A$ of an abelian variety over a finitely generated field of characteristic zero satisfies property (IF), and that every (Ram)-cover $Y \rightarrow A$ satisfies property (PF).

THEOREM 7.7. *Let k be a finitely generated field of characteristic zero, let A be an abelian variety over k , let $\Omega \subset A(k)$ be a Zariski-dense subgroup and let $(\pi_i : Y_i \rightarrow A)_{i=1}^n$ be a finite collection of (PB)-covers. Then, there is a finite index coset $C \subset \Omega$ such that, for every c in C and every $i = 1, \dots, n$, the scheme $Y_{i,c}$ is integral.*

Proof. By Lemma 4.14(1), we may and do assume that $n = 1$. By Proposition 7.6, it suffices to prove that property (PB) implies property (IF) for all covers $Y \rightarrow A$ and all $\Omega \subseteq A(K)$, where K is a number field, A is an abelian variety over K and Ω is a (finitely generated) Zariski-dense subgroup of $A(K)$. By Lemma 4.16 it then suffices to prove this statement for $(A, \Omega) = (\prod A_i, \prod \Omega_i)$, where each Ω_i is cyclic and Zariski-dense in A_i . By Theorem 6.15, for every finite field extension L/K , we know that property (PB) implies property (IF) over L for every pair $((A_i)_L, \Omega_i)$. Therefore, the statement follows from Corollary 5.7 and an immediate induction. □

Proof of Theorem 1.3. It suffices to prove the first statement. To do so, note that, by Lemma 4.14(2), we may assume that $n = 1$. The result then follows from Proposition 4.17 and Theorem 7.7. □

Proof of Theorem 1.4. This is Theorem 7.7, as (PB)-covers are precisely covers with no non-trivial étale subcovers (Lemma 4.4). □

Remark 7.8. Note that property (Ram) does not imply property (IF) in general. Indeed, if $\phi : A \rightarrow A'$ is a non-trivial isogeny of abelian varieties such that all points in $\ker \phi$ are defined over k , and $X \rightarrow A$ is any ramified cover, then the composition $\pi : X \rightarrow A'$ is ramified. However, for every P in the subgroup $\Omega := \text{Im}(A(k) \rightarrow A'(k))$, the fibre of π over P splits into at least $\#\ker \phi$ components.

Remark 7.9 (Necessity of density). In Theorems 1.3 and 1.4 it is necessary to assume that the subgroup Ω be Zariski-dense. Indeed, let E be an elliptic curve over a number field K with $E(K)$ infinite. Let $\pi : C \rightarrow E$ be a ramified double cover with C a smooth projective connected curve of genus at least two such that $0_E \in \pi(C(K))$. Let $\Omega = E(K) \times \{0\}$. Note that Ω is an infinite (non-dense) subgroup of $A := E \times E$. Define $Y := E \times C$ and note that $Y \rightarrow A$ is a ramified (PB)-cover. Note that $\Omega \setminus \pi(Y(K))$ is empty. Thus, the group Ω does not contain a finite index coset (nor even a non-empty subset) C such that Y_C has no K -points.

Proof of Theorem 1.8. As the weak-Hilbert property is a birational invariant among smooth proper varieties (Proposition 3.1), we may and do assume that X has trivial tangent bundle. In particular, because k is of characteristic zero, there is a finite field extension L/k such that X_L is an abelian variety (see [Bri12, Corollary 2.3]). Replacing L by a finite field extension if necessary, by the potential density of rational points on abelian varieties (see [Jav21a, §3]), we may assume that $X(L)$ is Zariski-dense in X . It then follows from Theorem 1.3 that X_L has the weak-Hilbert property over L . Finally, it is shown in [Cam11] that $X_{\bar{k}}$ is special. This concludes the proof. \square

We record the following consequence of Theorem 1.3 which, roughly speaking, says that abelian varieties also satisfy a version of the Hilbert property with respect to the complex-analytic topology.

Remark 7.10 (Zariski density versus complex-analytic density). Let k be a finitely generated field of characteristic zero, let A be an abelian variety over k and let $(\pi_i : Y_i \rightarrow A)_{i=1}^n$ be a finite collection of ramified covers. Choose an embedding $k \rightarrow \mathbb{C}$ and assume that $A(k)$ is dense in $A(\mathbb{C})$ with respect to the complex-analytic topology on $A(\mathbb{C})$. Then, because $A(k) \setminus \bigcup_{i=1}^n \pi_i(Y_i(k))$ contains a finite index coset of $A(k)$ (by Theorem 1.3), the subset $A(k) \setminus \bigcup_{i=1}^n \pi_i(Y_i(k))$ is complex-analytically dense in $A(\mathbb{C})$. (Indeed, a finite index coset C of a complex-analytically dense subgroup Ω of $A(\mathbb{C})$ is complex-analytically dense.)

We conclude our paper by showing that our main results concerning rational points on ramified covers of abelian varieties can be used to obtain similar conclusions for rational points on varieties which only admit a dominant generically finite rational map to an abelian variety.

THEOREM 7.11 (Improving Theorem 1.4). *Let k be a finitely generated field of characteristic zero, let A be an abelian variety over k and let $\Omega \subset A(k)$ be a Zariski-dense subgroup. For $i = 1, \dots, n$, let Z_i be a normal projective variety and let $\pi_i : Z_i \dashrightarrow A$ be a dominant generically finite rational map which does not (rationally) dominate any non-trivial étale cover of A . Then, there is a finite index coset $C \subset \Omega$ such that, for every c in C and every $i = 1, \dots, n$, the rational map π_i is defined above c and the k -scheme $Z_{i,c}$ is integral.*

Proof. For $i = 1, \dots, n$, let W_i be a normal projective variety over k and let $W_i \rightarrow Z_i$ be a proper birational surjective morphism such that the composed rational map $W_i \rightarrow Z_i \dashrightarrow A$ is a (proper surjective generically finite) morphism. Let $W_i \rightarrow Y_i \rightarrow A$ be the Stein factorization of $W_i \rightarrow A$, so that Y_i is a normal projective variety, $Y_i \rightarrow A$ is a finite morphism and $W_i \rightarrow Y_i$ is a proper birational surjective morphism. As $W_i \rightarrow Z_i$ and $W_i \rightarrow Y_i$ are proper birational surjective morphisms, we can choose a closed subset $W_i^0 \subset W_i$ such that $W_i \rightarrow Z_i$ and $W_i \rightarrow Y_i$ are isomorphisms away from W_i^0 . Let $Z \subset A$ be the image of $\bigcup_{i=1}^n W_i^0$ in A . By Lemma 4.6, replacing Ω by a finite index coset if necessary and translating the origin of A , we may and do assume that $\Omega \cap Z = \emptyset$.

As $Z_i \dashrightarrow A$ does not (rationally) dominate any non-trivial étale cover of A , it follows that $Y_i \rightarrow A$ has no non-trivial étale subcovers. In particular, by Theorem 1.4, there is a finite index coset $C \subset \Omega$ such that, for every c in C and every $i = 1, \dots, n$, the k -scheme $Y_{i,c}$ is integral. As $C \cap Z = \emptyset$, it follows that $Z_{i,c} = W_{i,c} = Y_{i,c}$ is integral, as required. \square

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Pietro Corvaja pietro.corvaja@uniud.it

Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze, 206,
33100 Udine, Italy

Julian Lawrence Demeio julian.demeio@sns.it

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

Ariyan Javanpeykar ariyanjavan@gmail.com

Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9,
55099 Mainz, Germany

Davide Lombardo davide.lombardo@unipi.it

Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

Umberto Zannier umberto.zannier@sns.it

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy