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# ON THE GAP BETWEEN THE GAMMA-LIMIT AND THE POINTWISE LIMIT FOR A NONLOCAL APPROXIMATION OF THE TOTAL VARIATION

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We consider the approximation of the total variation of a function by the family of nonlocal and nonconvex functionals introduced by H. Brezis and H.-M. Nguyen in a recent paper. The approximating functionals are defined through double integrals in which every pair of points contributes according to some interaction law.

We answer two open questions concerning the dependence of the Gamma-limit on the interaction law. In the first result, we show that the Gamma-limit depends on the full shape of the interaction law and not only on the values in a neighborhood of the origin. In the second result, we show that there do exist interaction laws for which the Gamma-limit coincides with the pointwise limit on smooth functions.

The key argument is that for some special classes of interaction laws the computation of the Gamma-limit can be reduced to studying the asymptotic behavior of suitable multivariable minimum problems.

## 1. Introduction

In a recent paper H. Brezis and H.-M. Nguyen [2018] introduced the family of nonlocal functionals

$$\Lambda_\delta(\varphi, u, \Omega) := \iint_{\Omega^2} \varphi\left(\frac{|u(y) - u(x)|}{\delta}\right) \frac{\delta}{|y - x|^{d+1}} dx dy, \quad (1-1)$$

where  $d$  is a positive integer,  $\Omega \subseteq \mathbb{R}^d$  is an open set,  $\delta > 0$  is a real parameter,  $u : \Omega \rightarrow \mathbb{R}$  is a measurable function, and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a measurable function satisfying suitable properties; see also the note [Brezis and Nguyen 2017] or the conference [Brezis 2016] for a nice presentation of the topic. The function  $\varphi$ , whose presence is motivated by problems in image processing, see [Brezis 2015; Brezis and Nguyen 2018], describes the extent to which a pair  $(x, y) \in \Omega^2$  contributes to the double integral (1-1). For this reason, in the sequel we call  $\varphi$  the “interaction law”.

Following [Brezis and Nguyen 2018], we restrict ourselves to a special set of functions.

**Definition 1.1** (admissible interaction laws). Let  $\mathcal{A}$  denote the set of all functions  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  not identically equal to 0 such that

- (i)  $\varphi$  is nondecreasing and lower semicontinuous on  $[0, +\infty)$  and is continuous except at a finite number of points in  $(0, +\infty)$ ,
- (ii) there exist a nonnegative real number  $a$  and a positive real number  $b$  such that

$$\varphi(t) \leq \min\{at^2, b\} \quad \text{for all } t \in [0, +\infty).$$



These conditions guarantee that when  $\varphi \in \mathcal{A}$ , the right-hand side of (1-1) is finite at least for every  $u$  of class  $C^1$  with compact support, and the resulting functional  $\Lambda_\delta$  is lower semicontinuous with respect to the convergence in  $L^1(\Omega)$ .

The basic example is the case where  $\varphi(t)$  coincides with

$$\varphi_1(t) := \begin{cases} 0 & \text{if } t \in [0, 1], \\ 1 & \text{if } t > 1. \end{cases} \tag{1-2}$$

In [Brezis and Nguyen 2018] also a normalization condition is included in the definition of  $\mathcal{A}$ . In this paper we do not impose any normalization condition, but instead we set

$$N(\varphi) := \int_0^{+\infty} \frac{\varphi(t)}{t^2} dt \quad \text{for all } \varphi \in \mathcal{A}, \tag{1-3}$$

and we exploit this constant as a scale factor when computing limits and Gamma-limits.

**Previous literature.** The asymptotic behavior of the family  $\Lambda_\delta$  was investigated in a series of papers, starting with the model case  $\varphi = \varphi_1$ ; see [Bourgain and Nguyen 2006; Nguyen 2006; 2007; 2008; 2011]. The idea is that  $\Lambda_\delta(\varphi, u, \Omega)$  is asymptotically proportional to the functional

$$\Lambda_0(u, \Omega) := \begin{cases} \text{total variation of } u \text{ in } \Omega & \text{if } u \in \text{BV}(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus \text{BV}(\Omega). \end{cases}$$

In order to state the precise results, let  $\mathbb{S}^{d-1} := \{\sigma \in \mathbb{R}^d : |\sigma| = 1\}$  denote the unit sphere in  $\mathbb{R}^d$ , and let us consider the geometric constant

$$G_d := \int_{\mathbb{S}^{d-1}} |\langle v, \sigma \rangle| d\sigma, \tag{1-4}$$

where  $v$  is any element of  $\mathbb{S}^{d-1}$  (of course the value of  $G_d$  does not depend on the choice of  $v$ ), and the integration is intended with respect to the  $(d-1)$ -dimensional Hausdorff measure.

The main convergence results obtained in [Brezis and Nguyen 2018] can be summed up as follows.

• *Pointwise convergence.* For every  $\varphi \in \mathcal{A}$  it turns out that

$$\lim_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u, \mathbb{R}^d) = G_d \cdot N(\varphi) \cdot \Lambda_0(u, \mathbb{R}^d) \quad \text{for all } u \in C_c^1(\mathbb{R}^d),$$

where  $G_d$  is the geometric constant defined in (1-4), and  $N(\varphi)$  is the scale factor defined in (1-3).

On the other hand, there do exist functions  $u \in W^{1,1}(\mathbb{R}^d)$  for which the left-hand side is infinite (while of course the right-hand side is finite). A precise characterization of equality cases is still unknown.

• *Gamma-convergence.* For every  $\varphi \in \mathcal{A}$ , there exists a constant  $K_d(\varphi)$ , depending a priori also on the space dimension, such that

$$\Gamma\text{-}\lim_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u, \mathbb{R}^d) = G_d \cdot N(\varphi) \cdot K_d(\varphi) \cdot \Lambda_0(u, \mathbb{R}^d) \quad \text{for all } u \in L^1(\mathbb{R}^d), \tag{1-5}$$

where the Gamma-limit is intended with respect to the usual metric of  $L^1(\mathbb{R}^d)$  (but the result would be the same with respect to the convergence in measure).

Assuming that  $K_d(\varphi)$  does not depend on  $d$ , as one reasonably expects, we could interpret (1-5) by saying that the Gamma-limit depends on the space dimension through the geometric constant  $G_d$ , on the size of  $\varphi$  through the scale factor  $N(\varphi)$ , and on the shape of  $\varphi$  through the “shape factor”  $K_d(\varphi)$ .

The behavior under rescaling clarifies the different nature of the scale and the shape factors. If we replace  $\varphi(t)$  by  $\hat{\varphi}(t) := \alpha\varphi(\beta t)$  for some positive constants  $\alpha$  and  $\beta$ , a change of variables shows that  $N(\hat{\varphi}) = \alpha\beta N(\varphi)$ , and the same scaling affects the left-hand side of (1-5). It follows that  $K_d(\varphi) = K_d(\hat{\varphi})$ , namely the shape factor is invariant by both horizontal and vertical rescaling.

Very little was known about  $K_d(\varphi)$ . In [Nguyen 2007; 2011] it was proved that  $K_d(\varphi_1) \leq \log 2$ , where  $\varphi_1$  is the model interaction law defined in (1-2). In [Brezis and Nguyen 2018] it was proved that

$$K_d(\varphi_1) \leq K_d(\varphi) \leq 1 \quad \text{for all } \varphi \in \mathcal{A}.$$

In order to shed some light on  $K_d(\varphi)$ , whose appearance in the Gamma-limit was defined in [Brezis and Nguyen 2018] as “mysterious and somewhat counterintuitive”, some open questions were explicitly stated. The first one addresses the dependence of  $K_d(\varphi)$  on the full shape of  $\varphi$ .

**Question 1** (see [Brezis and Nguyen 2018, Open Problem 4]). Assume that two functions  $\varphi$  and  $\psi$  in  $\mathcal{A}$  satisfy  $N(\varphi) = N(\psi)$ , and

$$\varphi \geq \psi \text{ near } 0 \quad (\text{resp. } \varphi = \psi \text{ near } 0).$$

Is it true that

$$K_d(\varphi) \geq K_d(\psi) \quad (\text{resp. } K_d(\varphi) = K_d(\psi))?$$

A positive answer to this question would imply that, once that the scale factor has been fixed, only the behavior of the interaction law in a neighborhood of the origin is relevant to the Gamma-limit. This intuition is supported by the observation that the kernel in (1-1) is divergent on the diagonal  $y = x$ , and therefore short-range interactions should be more relevant in the computation of  $K_d(\varphi)$ .

The second question addresses the necessity and the width of the gap between the pointwise limit and the Gamma-limit.

**Question 2** (see [Brezis and Nguyen 2018, Open Problem 2]). Is it true that  $K_d(\varphi) < 1$  for every  $\varphi \in \mathcal{A}$  (and every space dimension  $d$ )? Or even better: is it true that

$$\sup\{K_d(\varphi) : \varphi \in \mathcal{A}\} < 1? \tag{1-6}$$

A positive answer to this question, especially in the stronger form (1-6), would imply that the counter-intuitive gap is structural.

**Our results.** In this paper we show that the answer to both questions is negative. To this end, for every positive integer  $k$  we consider the interaction law  $\varphi_k : [0, +\infty) \rightarrow [0, +\infty)$  defined as

$$\varphi_k(t) := \varphi_1\left(\frac{t}{k}\right) = \begin{cases} 0 & \text{if } t \in [0, k], \\ 1 & \text{if } t > k, \end{cases} \tag{1-7}$$

and then we introduce the following special classes of interaction laws.

**Definition 1.2** (special interaction laws). Let  $\mathcal{A}$  denote the set of all admissible interaction laws defined in Definition 1.1. We consider the following special subclasses.

- Let  $\mathcal{A}_0$  denote the set of interaction laws that vanish in  $[0, 1]$ , namely

$$\mathcal{A}_0 := \{\varphi \in \mathcal{A} : \varphi(t) = 0 \text{ for all } t \in [0, 1]\}. \tag{1-8}$$

- Let  $\mathcal{PCA}$  denote the set of interaction laws that can be written in the form

$$\varphi(t) = \sum_{k=1}^m \lambda_k \varphi_k(t) \quad \text{for all } t \geq 0 \tag{1-9}$$

for some positive integer  $m$  and some nonnegative real numbers  $\lambda_1, \dots, \lambda_m$  (not all equal to 0).

- Let  $\mathcal{PCA}_2$  denote the set of interaction laws of the form (1-9) whose coefficients are equal in packages of powers of 2, namely

$$\lambda_2 = \lambda_3, \quad \lambda_4 = \dots = \lambda_7, \quad \lambda_8 = \dots = \lambda_{15},$$

and so on. More precisely, every  $\varphi \in \mathcal{PCA}_2$  can be written in the form

$$\varphi(t) := \sum_{j=1}^m \left( a_j \sum_{k=2^{j-1}}^{2^j-1} \varphi_k(t) \right) \quad \text{for all } t \geq 0 \tag{1-10}$$

for some positive integer  $m$ , and some nonnegative real numbers  $a_1, \dots, a_m$  (not all equal to 0).

We observe that

$$\mathcal{A} \supseteq \mathcal{A}_0 \supseteq \mathcal{PCA} \supseteq \mathcal{PCA}_2,$$

and all inclusions are strict.

Our first result is the following.

**Theorem 1.3** (piecewise constant interaction laws). *Let  $K_d(\varphi)$  be the shape factor of an interaction law as defined by (1-5).*

*Then in every space dimension  $d$  it turns out that*

$$\sup\{K_d(\varphi) : \varphi \in \mathcal{PCA}_2\} = 1.$$

A closer look at the proof reveals that the supremum is realized for example by the interaction laws of the form (1-10) with  $a_1 = \dots = a_m = 1$ , in the limit as  $m \rightarrow +\infty$ . We point out that these achieving interaction laws are also of the form (1-9) with  $\lambda_k = 1$  and  $m$  a power of 2.

Theorem 1.3 above provides a negative answer to Question 1, as well as a negative answer to Question 2 in the stronger form (1-6). In particular, this means that the full shape of the interaction law comes into play in the computation of the Gamma-limit, which therefore takes into account both short-range and long-range interactions. At the beginning of Section 6 we present also an example with strict inequalities, namely with  $\varphi > \psi$  near the origin, but  $K_d(\varphi) < K_d(\psi)$ .

Our second result gives a stronger negative answer to Question 2, even when restricted to the smaller class  $\mathcal{A}_0$ .

**Theorem 1.4** (piecewise affine dyadic interaction laws). *Let  $K_d(\varphi)$  be the shape factor of an interaction law as defined by (1-5).*

*Then the following statements hold true:*

(1) *In every space dimension  $d$  it turns out that*

$$\max\{K_d(\varphi) : \varphi \in \mathcal{A}_0\} = 1.$$

(2) *More precisely, let  $f : \mathbb{Z} \rightarrow [0, +\infty)$  be any nondecreasing and bounded function (not identically equal to 0) such that*

$$\limsup_{n \rightarrow +\infty} f(-n) \cdot 4^n < +\infty. \tag{1-11}$$

*Let us consider the function  $\zeta : [0, +\infty) \rightarrow [0, +\infty)$  such that*

- $\zeta(0) = 0,$
- $\zeta(2^z) = f(z)$  for every  $z \in \mathbb{Z},$
- $\zeta$  is affine in the interval  $[2^z, 2^{z+1}]$  for every  $z \in \mathbb{Z}.$

*Then it turns out that  $\zeta \in \mathcal{A},$  and  $K_d(\zeta) = 1$  in every space dimension  $d.$*

The second statement of **Theorem 1.4** above shows in particular that there are large classes of interaction laws for which the Gamma-limit of (1-1) coincides with the pointwise limit for smooth functions.

**Overview of the technique.** The proof of these results follows the same strategy that in [Antonucci et al. 2020] led us to show that actually  $K_d(\varphi_1) = \log 2$  (see also [Antonucci et al. 2018] for an informal summary of our approach). Since we only need estimates from below for the shape factor, we can limit ourselves to estimating from below the Gamma-liminf. The main steps are the following.

*From local to global bounds:* In **Theorem B** we reduce the problem in any dimension to intervals of the real line, namely to showing that for  $\delta$  small enough we can estimate from below  $\Lambda_\delta(\varphi, u, (a, b))$  in terms of the oscillation of  $u$  in  $(a, b).$

*Reduction to multivariable minimum problems:* In **Proposition 4.3** we show that, when  $\varphi \in \mathcal{PCA},$  we can assume that  $u$  is a nondecreasing step function with finite image contained in  $\delta\mathbb{Z}.$  Up to vertical translations, any such function depends only on the lengths  $\ell_1, \dots, \ell_n$  of the steps, where  $n \sim \delta^{-1}.$  In this way we reduce ourselves to studying the minimum of a  $\delta$ -dependent multivariable function.

*Telescopic effect:* The minimum problems found in **Proposition 4.3** can be very complicated. Nevertheless, in the special case where  $\varphi \in \mathcal{PCA}_2,$  a telescopic effect implies that the configurations with  $\ell_1 = \dots = \ell_n$  are asymptotically optimal (see **Proposition 3.3**). At this point, we have explicit estimates from below for the Gamma-liminf, which in turn yield explicit estimates from below for shape factors, thus leading to the proof of **Theorem 1.3.**

*Approximation from below of special interaction laws:* The more general interaction laws of **Theorem 1.4** can be approximated from below by sequences of interaction laws that, up to horizontal rescaling, are in  $\mathcal{PCA}_2.$  Again this provides estimates from below for the Gamma-liminf, and hence for shape factors, that lead to the proof of **Theorem 1.4.**

**Structure of the paper.** This paper is organized as follows. In [Section 2](#) we recall the technical results from [\[Antonucci et al. 2020\]](#) that are needed in the sequel. In [Section 3](#) we investigate the asymptotic behavior of suitable sequences of minimum problems for functions of a finite number of variables. In [Section 4](#) we prove [Proposition 4.3](#), which establishes the connection between the Gamma-limit of (1-1) and the minimum problems of the previous section. In [Section 5](#) we prove [Theorems 1.3](#) and [1.4](#). In [Section 6](#) we speculate about some possible generalizations and future directions.

## 2. Preliminary results

In this section we collect, for the convenience of the reader, the results from [\[Antonucci et al. 2020\]](#) that are crucial to this paper. To begin with, we recall the definitions of truncation, vertical  $\delta$ -segmentation and nondecreasing rearrangement.

**Definition 2.1** (truncation). Let  $\mathbb{X}$  be any set, let  $w : \mathbb{X} \rightarrow \mathbb{R}$  be any function, and let  $A < B$  be two real numbers. The truncation of  $w$  between  $A$  and  $B$  is the function  $T_{A,B}w : \mathbb{X} \rightarrow \mathbb{R}$  defined by

$$T_{A,B}w(x) := \begin{cases} A & \text{if } w(x) < A, \\ w(x) & \text{if } A \leq w(x) \leq B, \\ B & \text{if } w(x) > B. \end{cases}$$

**Definition 2.2** (vertical  $\delta$ -segmentation). Let  $\mathbb{X}$  be any set, let  $w : \mathbb{X} \rightarrow \mathbb{R}$  be any function, and let  $\delta$  be a positive real number. The vertical  $\delta$ -segmentation of  $w$  is the function  $S_\delta w : \mathbb{X} \rightarrow \mathbb{R}$  defined by

$$S_\delta w(x) := \delta \left\lfloor \frac{w(x)}{\delta} \right\rfloor \quad \text{for all } x \in \mathbb{X}.$$

The function  $S_\delta w$  takes its values in  $\delta\mathbb{Z}$ , and it is uniquely characterized by the fact that  $S_\delta w(x) = k\delta$  for some  $k \in \mathbb{Z}$  if and only if  $k\delta \leq w(x) < (k+1)\delta$ .

**Definition 2.3** (nondecreasing rearrangement). Let  $(a, b) \subseteq \mathbb{R}$  be an interval, and let  $w : (a, b) \rightarrow \mathbb{R}$  be a function whose image is a finite set. The nondecreasing rearrangement of  $w$  is the function  $Mw : (a, b) \rightarrow \mathbb{R}$  defined by

$$Mw(x) := \min\{y \in \mathbb{R} : \text{meas}\{z \in (a, b) : w(z) \leq y\} \geq x - a\} \quad \text{for all } x \in \mathbb{R}. \quad (2-1)$$

As expected, the function  $Mw$  is nondecreasing and satisfies

$$\text{meas}\{x \in (a, b) : Mw(x) = y\} = \text{meas}\{x \in (a, b) : w(x) = y\} \quad \text{for all } y \in \mathbb{R}.$$

**Semidiscrete aggregation problem.** Let us recall the main result of [\[Antonucci et al. 2020, Section 2\]](#), where a class of discrete and semidiscrete aggregation problems was considered. Following the terminology introduced therein, the basic ingredients are an interval  $(a, b) \subseteq \mathbb{R}$ , an integer number  $k \geq 1$ , and a nonincreasing function  $c : (0, b - a) \rightarrow \mathbb{R}$ , possibly unbounded in a neighborhood of the origin, called the hostility function. A semidiscrete arrangement is any measurable function  $u : (a, b) \rightarrow \mathbb{Z}$  with finite image.

For any such function  $u$ , we define the total  $k$ -hostility as

$$\mathcal{F}_k(c, u) = \iint_{(a,b)^2} \varphi_k(|u(y) - u(x)|) \cdot c(|y - x|) dx dy, \tag{2-2}$$

where  $\varphi_k(t)$  is the interaction law defined by (1-7).

The key result proved in [Antonucci et al. 2020, Theorem 2.4] (and equivalent to some rearrangement inequalities found independently in [Taylor 1973; Garsia and Rodemich 1974] in a different context) is that the nondecreasing rearrangement does not increase the total hostility.

**Theorem A** (total hostility minimization). *Let  $(a, b) \subseteq \mathbb{R}$  be an interval, let  $k \geq 1$  be an integer, and let  $c : (0, b - a) \rightarrow \mathbb{R}$  be a nonincreasing function. Let  $u : (a, b) \rightarrow \mathbb{Z}$  be a measurable function with finite image, let  $Mu : (a, b) \rightarrow \mathbb{Z}$  be its nondecreasing rearrangement defined by (2-1), and let  $\mathcal{F}_k(c, u)$  be the functional defined by (2-2).*

*Then it turns out that*

$$\mathcal{F}_k(c, u) \geq \mathcal{F}_k(c, Mu).$$

**Localization technique.** One of the main points in [Antonucci et al. 2020] was obtaining a localized version of the Gamma-liminf inequality, namely an asymptotic estimate from below for  $\Lambda_\delta(\varphi, u_\delta, (a, b))$  in terms of the oscillation of  $u_\delta$  in  $(a, b)$ . After such an estimate has been established, a quite classical path, see for example [Gobbino 1998; Gobbino and Mora 2001], independent of the presence of the interaction law  $\varphi$ , leads to an estimate from below for the Gamma-liminf of  $\Lambda_\delta$  in any space dimension, and hence to an estimate from below for the shape factor of  $\varphi$ . The precise statement is the following.

**Theorem B** (from local to global bounds). *Let  $\varphi \in \mathcal{A}$  be an interaction law, with scale factor  $N(\varphi)$  defined by (1-3) and shape factor  $K_d(\varphi)$  defined by (1-5).*

*Let us assume that there exists a constant  $K_0$  such that, for every interval  $(a, b) \subseteq \mathbb{R}$  and every family  $\{u_\delta\}_{\delta>0} \subseteq L^1((a, b))$ , it happens that*

$$\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u_\delta, (a, b)) \geq K_0 \cdot \liminf_{\delta \rightarrow 0^+} \text{osc}(u_\delta, (a, b)), \tag{2-3}$$

where  $\text{osc}(u_\delta, (a, b))$  denotes the essential oscillation of  $u_\delta$  in  $(a, b)$ .

*Then for every positive integer  $d$  it turns out that*

$$\Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u, \mathbb{R}^d) \geq G_d \cdot \frac{1}{2} K_0 \cdot \Lambda_0(u, \mathbb{R}^d) \quad \text{for all } u \in L^1(\mathbb{R}^d). \tag{2-4}$$

The proof of **Theorem B** has two distinct steps.

The first one, for which we refer to [Antonucci et al. 2020, Section 3.2], exploits a piecewise affine approximation in order to deduce (2-4) in dimension 1 from the local estimate (2-3). We remark that  $G_1 = 2$ , and therefore in dimension 1 the geometric constant cancels the denominator, so that (2-3) is exactly the localized version of (2-4).

The second step, for which we refer to [Antonucci et al. 2020, Section 4], relies on an integral-geometric representation of both the total variation and the double integral (1-1). This representation leads from (2-4) in dimension 1 to the analogous inequality in any space dimension.



### 3. A family of multivariable minimum problems

In this section we introduce some notation, and we prove asymptotic estimates for a family of inequalities involving multivariable functions. Roughly speaking, these multivariable functions represent the functional  $\Lambda_\delta$  computed on a piecewise constant function with a finite number of steps of lengths  $\ell_1, \dots, \ell_n$ . The main result of this section, namely estimate (3-10), is going to play a crucial role when combined with the result of Proposition 4.3, where we estimate from below the asymptotic cost of oscillations.

Let  $n$  be a positive integer, and let  $(\ell_1, \dots, \ell_n)$  be an  $n$ -tuple of nonnegative real numbers. For every positive integer  $k \leq n$  we consider all possible sums of  $k$  consecutive terms

$$S_{i,k}(\ell_1, \dots, \ell_n) := \sum_{h=0}^{k-1} \ell_{i+h} \quad \text{for all } i \in \{1, \dots, n - k + 1\}, \tag{3-1}$$

and we define the set

$$D_{n,k} := \{(\ell_1, \dots, \ell_n) \in [0, +\infty)^n : S_{i,k}(\ell_1, \dots, \ell_n) > 0 \text{ for all } i \in \{1, \dots, n - k + 1\}\} \tag{3-2}$$

of all  $n$ -tuples of nonnegative real numbers without  $k$  consecutive components equal to 0. When  $n \geq k + 1$ , for every  $(\ell_1, \dots, \ell_n) \in D_{n,k}$  we can set

$$L_k(\ell_1, \dots, \ell_n) := \sum_{i=1}^{n-k} \log \frac{[S_{i,k+1}(\ell_1, \dots, \ell_n)]^2}{S_{i,k}(\ell_1, \dots, \ell_n) \cdot S_{i+1,k}(\ell_1, \dots, \ell_n)}. \tag{3-3}$$

Given any interaction law  $\varphi \in \mathcal{PCA}$ , we call  $\mu(\varphi)$  the smallest integer  $k$  such that  $\lambda_k \neq 0$  in the representation (1-9), and for every  $n \geq m + 1$  we consider the homogeneous function  $P_{n,\varphi} : D_{n,\mu(\varphi)} \rightarrow \mathbb{R}$  defined by

$$P_{n,\varphi}(\ell_1, \dots, \ell_n) := \sum_{k=\mu(\varphi)}^m \lambda_k L_k(\ell_1, \dots, \ell_n) \quad \text{for all } (\ell_1, \dots, \ell_n) \in D_{n,\mu(\varphi)}, \tag{3-4}$$

and its infimum

$$\begin{aligned} I_n(\varphi) &:= \inf\{P_{n,\varphi}(\ell_1, \dots, \ell_n) : (\ell_1, \dots, \ell_n) \in D_{n,\mu(\varphi)}\} \\ &= \inf\{P_{n,\varphi}(\ell_1, \dots, \ell_n) : (\ell_1, \dots, \ell_n) \in (0, +\infty)^n\}. \end{aligned} \tag{3-5}$$

Let us consider for example the interaction law  $\varphi_1$  defined in (1-2). In this case  $\mu(\varphi_1) = 1$ , the set  $D_{n,1}$  is just  $(0, +\infty)^n$ , and the function in (3-4) has the form

$$P_{n,\varphi_1}(\ell_1, \dots, \ell_n) = L_1(\ell_1, \dots, \ell_n) = \log \frac{(\ell_1 + \ell_2)^2}{\ell_1 \ell_2} + \log \frac{(\ell_2 + \ell_3)^2}{\ell_2 \ell_3} + \dots + \log \frac{(\ell_{n-1} + \ell_n)^2}{\ell_{n-1} \ell_n}.$$

All the fractions inside the logarithms are greater than or equal to 4, and hence  $I_n(\varphi_1) = (n - 1) \log 4$ , with the minimum realized when all the variables are equal. This computation was the final step in the proof of the crucial estimate in [Antonucci et al. 2020]. In Section 4 of the present paper we show that the asymptotic behavior of  $I_n(\varphi)$  plays a fundamental role in estimating from below the shape factor of any interaction law  $\varphi \in \mathcal{PCA}$ .

Unfortunately, things are not that simple for larger values of  $k$ . For example, when the interaction law is  $\varphi_3$ , the function  $L_3(\ell_1, \dots, \ell_n)$  is given by

$$\log \frac{(\ell_1 + \ell_2 + \ell_3 + \ell_4)^2}{(\ell_1 + \ell_2 + \ell_3)(\ell_2 + \ell_3 + \ell_4)} + \log \frac{(\ell_2 + \ell_3 + \ell_4 + \ell_5)^2}{(\ell_2 + \ell_3 + \ell_4)(\ell_3 + \ell_4 + \ell_5)} + \dots,$$

and now the minimum is not realized when all the variables are equal (for example the periodic pattern  $1, 0, 0, 1, 0, 0, \dots$  is better).

Of course any interaction law of the form  $\varphi_k$  can be dealt with as a rescaling of  $\varphi_1$ , but nevertheless the appearance of different patterns in the minimization process seems to suggest that things get worse and worse when we take linear combinations of the form (3-4). Fortunately this is not always the case. Indeed, when we expand linear combinations of this form, the numerators of the terms of  $L_k$  can partially cancel with the denominators of the terms of  $L_{k+1}$ , leading to the following result.

**Lemma 3.1** (telescopic effect). *Let  $a, b$ , and  $n$  be positive integers such that  $a \leq b \leq n - 1$ . Let  $S_{i,k}$  and  $L_k$  be the functions of  $n$  variables defined in (3-1) and (3-3).*

*Then for every  $(\ell_1, \dots, \ell_n) \in D_{n,a}$  it turns out that*

$$\sum_{j=a}^b L_j(\ell_1, \dots, \ell_n) \geq \sum_{i=1}^{n-b} \log \frac{[S_{i,b+1}(\ell_1, \dots, \ell_n)]^2}{S_{i,a}(\ell_1, \dots, \ell_n) \cdot S_{i+(b-a)+1,a}(\ell_1, \dots, \ell_n)}. \tag{3-6}$$

*Proof.* To begin with, we observe that (3-6) is an equality when  $b = a$ . Therefore, in the sequel we assume that  $b \geq a + 1$ . For the sake of shortness, throughout this proof we omit the explicit dependence on the variables  $\ell_1, \dots, \ell_n$ . The left-hand side of (3-6) can be written in the form

$$\sum_{j=a}^b L_j = 2\Sigma_1 - \Sigma_2 - \Sigma_3, \tag{3-7}$$

where

$$\Sigma_1 := \sum_{j=a}^b \sum_{i=1}^{n-j} \log S_{i,j+1}, \quad \Sigma_2 := \sum_{j=a}^b \sum_{i=1}^{n-j} \log S_{i,j}, \quad \Sigma_3 := \sum_{j=a}^b \sum_{i=1}^{n-j} \log S_{i+1,j}.$$

With some algebra (shift of indices and separation of the terms corresponding to the first or last value of some index) we can rewrite the three sums as

$$\begin{aligned} \Sigma_1 &= \sum_{i=1}^{n-b} \log S_{i,b+1} + \sum_{j=a+1}^b \log S_{1,j} + \sum_{j=a+1}^b \log S_{n-j+1,j} + \sum_{j=a+1}^b \sum_{i=2}^{n-j} \log S_{i,j}, \\ \Sigma_2 &= \sum_{i=1}^{n-a} \log S_{i,a} + \sum_{j=a+1}^b \log S_{1,j} + \sum_{j=a+1}^b \sum_{i=2}^{n-j} \log S_{i,j}, \\ \Sigma_3 &= \sum_{i=2}^{n-a+1} \log S_{i,a} + \sum_{j=a+1}^b \log S_{n-j+1,j} + \sum_{j=a+1}^b \sum_{i=2}^{n-j} \log S_{i,j}. \end{aligned}$$

When we plug these three equalities into (3-7), all double sums cancel, and also the second sums in  $\Sigma_2$  and  $\Sigma_3$  cancel with a part of the second and third terms in  $\Sigma_1$ . We end up with

$$2\Sigma_1 - \Sigma_2 - \Sigma_3 = 2 \sum_{i=1}^{n-b} \log S_{i,b+1} + \sum_{j=a+1}^b \log S_{1,j} + \sum_{j=a+1}^b \log S_{n-j+1,j} - \sum_{i=1}^{n-a} \log S_{i,a} - \sum_{i=2}^{n-a+1} \log S_{i,a}. \quad (3-8)$$

Let us reorganize these terms. In the second and third sums we change the indices and we rewrite them as

$$\begin{aligned} \sum_{j=a+1}^b \log S_{1,j} &= \sum_{k=1}^{b-a} \log S_{1,k+a}, \\ \sum_{j=a+1}^b \log S_{n-j+1,j} &= \sum_{k=n-b+1}^{n-a} \log S_{k,n+1-k}. \end{aligned}$$

In the fourth sum we split the terms as

$$\sum_{i=1}^{n-a} \log S_{i,a} = \sum_{i=1}^{n-b} \log S_{i,a} + \sum_{k=n-b+1}^{n-a} \log S_{k,a}.$$

In the fifth sum we split the terms, and then we shift one index in order to rewrite the sum as

$$\sum_{i=2}^{n-a+1} \log S_{i,a} = \sum_{i=2}^{b-a+1} \log S_{i,a} + \sum_{i=b-a+2}^{n-a+1} \log S_{i,a} = \sum_{k=1}^{b-a} \log S_{k+1,a} + \sum_{i=1}^{n-b} \log S_{i+(b-a)+1,a}.$$

Plugging all these equalities into (3-8) we find that

$$2\Sigma_1 - \Sigma_2 - \Sigma_3 = \sum_{i=1}^{n-b} \log \frac{[S_{i,b+1}]^2}{S_{i,a} \cdot S_{i+(b-a)+1,a}} + \sum_{k=1}^{b-a} \log \frac{S_{1,k+a}}{S_{k+1,a}} + \sum_{k=n-b+1}^{n-a} \log \frac{S_{k,n+1-k}}{S_{k,a}}.$$

In the sums of the last line, all terms are nonnegative because in all the fractions the numerators are greater than or equal to the corresponding denominators. Recalling (3-7), it follows that

$$\sum_{j=a}^b L_j = 2\Sigma_1 - \Sigma_2 - \Sigma_3 \geq \sum_{i=1}^{n-b} \log \frac{[S_{i,b+1}]^2}{S_{i,a} \cdot S_{i+(b-a)+1,a}},$$

which completes the proof of (3-6). □

**Corollary 3.2.** *Let us consider the situation described in Lemma 3.1 in the special case where  $a = 2^{m-1}$  and  $b = 2^m - 1$  for some positive integer  $m$ .*

*Then for every  $n \geq 2^m$  it turns out that*

$$\sum_{k=2^{m-1}}^{2^m-1} L_k(\ell_1, \dots, \ell_n) \geq (n - 2^m + 1) \cdot 2 \log 2 \quad \text{for all } (\ell_1, \dots, \ell_n) \in D_{n,a}. \quad (3-9)$$

*Proof.* In this special case it turns out that

$$S_{i,b+1} = S_{i,a} + S_{i+(b-a)+1,a} \quad \text{for all } i \leq n - 2^m + 1.$$

Therefore, from the inequality between arithmetic mean and geometric mean, we deduce that all the fractions in the right-hand side of (3-6) are greater than or equal to 4, and this is enough to establish (3-9).  $\square$

From Corollary 3.2 we deduce a lower bound for the asymptotic behavior of  $I_n(\varphi)$  for interaction laws  $\varphi \in \mathcal{PCA}_2$ .

**Proposition 3.3** (interaction laws with package structure). *Let  $m$  be a positive integer, let  $a_1, \dots, a_m$  be nonnegative real numbers (not all equal to 0), and let  $\varphi \in \mathcal{PCA}_2$  be defined as in (1-10).*

*For every integer  $n \geq 2^m$ , let  $P_{n,\varphi}$  be the homogeneous function defined by (3-4), and let  $I_n(\varphi)$  be its infimum as in (3-5).*

*Then it turns out that*

$$\liminf_{n \rightarrow +\infty} \frac{I_n(\varphi)}{n} \geq 2 \log 2 \cdot \sum_{j=1}^m a_j. \tag{3-10}$$

*Proof.* Let  $m_0(\varphi)$  denote the smallest integer  $k$  such that  $a_k > 0$ . From Corollary 3.2 we know

$$\sum_{k=2^{j-1}}^{2^j-1} L_k(\ell_1, \dots, \ell_n) \geq (n - 2^j + 1) \cdot 2 \log 2$$

for every  $j \in \{m_0(\varphi), \dots, m\}$ , and therefore

$$P_{n,\varphi}(\ell_1, \dots, \ell_n) \geq \sum_{j=m_0(\varphi)}^m a_j \cdot (n - 2^j + 1) \cdot 2 \log 2 \geq (n - 2^m) \cdot 2 \log 2 \cdot \sum_{j=1}^m a_j$$

for every admissible choice of  $\ell_1, \dots, \ell_n$ .

Dividing by  $n$ , and letting  $n \rightarrow +\infty$ , we obtain (3-10).  $\square$

### 4. Asymptotic cost of oscillations

In this section we clarify the connection between the Gamma-limit of the family (1-1) and the multivariable functions of Section 3. In analogy with [Antonucci et al. 2020], the question we address is the following. Let us assume that a function  $u_\delta(x)$  oscillates between two values  $A$  and  $B$  in some interval  $(a, b)$ . Does this provide an estimate from below for  $\Lambda_\delta(\varphi, u_\delta, (a, b))$ , at least when  $\delta$  is small enough? A quantitative answer is provided by Proposition 4.3 and Corollary 4.4, and this answer is connected to the Gamma-limit of the family (1-1) by Theorem B, as we clarify in Proposition 4.5.

To begin with, we show that three simplifying operations can be performed on  $u_\delta$  without changing its oscillation between  $A$  and  $B$  or increasing its energy.

**Lemma 4.1** (truncation, segmentation, rearrangement). *Let  $a < b$  and  $A < B$  be real numbers, let  $u : (a, b) \rightarrow \mathbb{R}$  be a measurable function, and let  $\varphi \in \mathcal{PCA}$ .*

*Then for every  $\delta > 0$  it turns out that*

$$\Lambda_\delta(\varphi, u, (a, b)) \geq \Lambda_\delta(\varphi, MS_\delta T_{A,B} u, (a, b)), \tag{4-1}$$

where  $T_{A,B}$ ,  $S_\delta$ , and  $M$  are the operators of truncation, vertical  $\delta$ -segmentation, and nondecreasing rearrangement defined at the beginning of Section 2.



*Proof.* Since  $\Lambda_\delta$  is linear with respect to  $\varphi$ , it is enough to show inequality (4-1) when  $\varphi = \varphi_k$  for some positive integer  $k$ , in which case

$$\Lambda_\delta(\varphi_k, u, (a, b)) = \iint_{I_k(\delta, u, (a, b))} \frac{\delta}{(y-x)^2} dx dy,$$

where

$$I_k(\delta, u, (a, b)) := \{(x, y) \in (a, b)^2 : |u(y) - u(x)| > k\delta\}.$$

Let us examine the effects on  $\Lambda_\delta$  of the three operations performed on  $u$ . The arguments are the same as in the first part of the proof of [Antonucci et al. 2020, Proposition 3.4], where however only the case of  $\varphi_1$  was considered.

*Truncation:* For every  $x$  and  $y$  in  $(a, b)$  it turns out that

$$|T_{A,B}u(y) - T_{A,B}u(x)| > k\delta \implies |u(y) - u(x)| > k\delta.$$

This implies  $I_k(\delta, T_{A,B}u, (a, b)) \subseteq I_k(\delta, u, (a, b))$ , and therefore

$$\Lambda_\delta(\varphi_k, u, (a, b)) \geq \Lambda_\delta(\varphi_k, T_{A,B}u, (a, b)). \tag{4-2}$$

*Vertical  $\delta$ -segmentation:* For every  $x$  and  $y$  in  $(a, b)$  it turns out that

$$|S_\delta u(y) - S_\delta u(x)| > k\delta \implies |S_\delta u(y) - S_\delta u(x)| \geq (k+1)\delta \implies |u(y) - u(x)| > k\delta.$$

As before this implies

$$\Lambda_\delta(\varphi_k, T_{A,B}u, (a, b)) \geq \Lambda_\delta(\varphi_k, S_\delta T_{A,B}u, (a, b)). \tag{4-3}$$

*Nondecreasing rearrangement:* We claim that

$$\Lambda_\delta(\varphi_k, S_\delta T_{A,B}u, (a, b)) \geq \Lambda_\delta(\varphi_k, MS_\delta T_{A,B}u, (a, b)). \tag{4-4}$$

This inequality, together with (4-2) and (4-3), completes the proof of (4-1).

In order to prove (4-4), we rely on the theory of semidiscrete arrangements. To this end, we consider the semidiscrete arrangement  $v_\delta : (a, b) \rightarrow \mathbb{Z}$  defined by

$$v_\delta(x) := \frac{1}{\delta} S_\delta T_{A,B}u(x) \quad \text{for all } x \in (a, b) \tag{4-5}$$

(we recall that  $S_\delta T_{A,B}u$  takes its values in  $\delta\mathbb{Z}$ , and hence  $v_\delta(x)$  is integer-valued) and the hostility function  $c : (0, b-a) \rightarrow \mathbb{R}$  defined by  $c(\sigma) := \delta\sigma^{-2}$ . We observe that

$$MS_\delta T_{A,B}u(x) = \delta Mv_\delta(x) \quad \text{for all } x \in (a, b),$$

where  $Mv_\delta$  is the nondecreasing rearrangement of  $v_\delta$ . From (4-5) and (2-2) it turns out that

$$\Lambda_\delta(\varphi_k, S_\delta T_{A,B}u, (a, b)) = \delta \Lambda_1(\varphi_k, v_\delta, (a, b)) = \mathcal{F}_k(c, v_\delta),$$

and similarly

$$\Lambda_\delta(\varphi_k, MS_\delta T_{A,B}u, (a, b)) = \delta \Lambda_1(\varphi_k, Mv_\delta, (a, b)) = \mathcal{F}_k(c, Mv_\delta),$$

so that now (4-4) follows from Theorem A. □

As a second simplifying step, we show that the double integral over  $(a, b)^2$  can be replaced, in the computation of the liminf, by a double integral over an infinite strip, which is easier to handle. To this end, we introduce the family of functionals

$$\hat{\Lambda}_\delta(\varphi, u, (c, d)) := \int_c^d dx \int_{-\infty}^{+\infty} \varphi\left(\frac{|u(y) - u(x)|}{\delta}\right) \frac{\delta}{(y-x)^2} dy,$$

and we prove the following result.

**Lemma 4.2** (extension to a vertical strip). *Let  $a < c < d < b$  be real numbers, and let  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  be a bounded measurable function. For every  $\delta > 0$ , let  $u_\delta : (a, b) \rightarrow \mathbb{R}$  be a measurable function. Let us extend  $u_\delta$  to the whole real line by setting  $u_\delta(x) = 0$  for every  $x \notin (a, b)$ .*

*Then it turns out that*

$$\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u_\delta, (a, b)) \geq \liminf_{\delta \rightarrow 0^+} \hat{\Lambda}_\delta(\varphi, u_\delta, (c, d)). \tag{4-6}$$

*Proof.* Let us set for shortness

$$f_\delta(x, y) := \varphi\left(\frac{|u_\delta(y) - u_\delta(x)|}{\delta}\right) \frac{\delta}{(y-x)^2} \quad \text{for all } (x, y) \in (a, b) \times \mathbb{R}.$$

Since  $\varphi$  is nonnegative and  $(c, d) \subseteq (a, b)$ , for every  $\delta > 0$  it turns out that

$$\begin{aligned} \Lambda_\delta(\varphi, u_\delta, (a, b)) &\geq \int_c^d dx \int_a^b f_\delta(x, y) dy \\ &= \hat{\Lambda}_\delta(\varphi, u_\delta, (c, d)) - \int_c^d dx \int_{\mathbb{R} \setminus [a, b]} f_\delta(x, y) dy. \end{aligned} \tag{4-7}$$

From the boundedness of  $\varphi$  it follows that

$$\int_c^d dx \int_b^{+\infty} f_\delta(x, y) dy \leq \delta \|\varphi\|_\infty \int_c^d dx \int_b^{+\infty} \frac{1}{(y-x)^2} dy.$$

Since  $d < b$ , the double integral in the right-hand side is convergent, and hence

$$\lim_{\delta \rightarrow 0^+} \int_c^d dx \int_b^{+\infty} f_\delta(x, y) dy = 0. \tag{4-8}$$

In an analogous way we obtain

$$\lim_{\delta \rightarrow 0^+} \int_c^d dx \int_{-\infty}^a f_\delta(x, y) dy = 0. \tag{4-9}$$

At this point, (4-6) follows from (4-7), (4-8), and (4-9). □

We are now ready to state and prove the main result of this section.

**Proposition 4.3** (limit cost of vertical oscillations). *Let  $(a, b) \subseteq \mathbb{R}$  be an interval, let  $\{u_\delta\}_{\delta>0} \subseteq L^1((a, b))$  be a family of functions, let  $\varphi \in \mathcal{PCA}$  be a piecewise constant interaction law, let  $P_{n,\varphi}$  be the multivariable function defined by (3-4), and let  $I_n(\varphi)$  be its infimum as in (3-5).*

Let us assume that there exist two real numbers  $A \leq B$  such that

$$\liminf_{\delta \rightarrow 0^+} \text{meas}\{x \in (a, b) : u_\delta(x) \leq A + \varepsilon\} > 0 \quad \text{for all } \varepsilon > 0, \tag{4-10}$$

$$\liminf_{\delta \rightarrow 0^+} \text{meas}\{x \in (a, b) : u_\delta(x) \geq B - \varepsilon\} > 0 \quad \text{for all } \varepsilon > 0. \tag{4-11}$$

Then it turns out that

$$\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u_\delta, (a, b)) \geq (B - A) \cdot \liminf_{n \rightarrow +\infty} \frac{I_n(\varphi)}{n}. \tag{4-12}$$

*Proof.* To begin with, we observe that (4-12) is trivial if  $A = B$ , or if the left-hand side is infinite. Up to restricting ourselves to a sequence  $\delta_k \rightarrow 0^+$ , we can also assume that the liminf is actually a limit. Therefore, in the sequel we assume that the left-hand side of (4-12) is uniformly bounded from above and that  $A < B$ .

Let us fix  $\varepsilon > 0$  such that  $4\varepsilon < B - A$ . Due to assumptions (4-10) and (4-11), there exist  $\eta > 0$  and  $\delta_0 > 0$  such that

$$\text{meas}\{x \in (a, b) : u_\delta(x) \leq A + \varepsilon\} \geq \eta \quad \text{for all } \delta \in (0, \delta_0), \tag{4-13}$$

$$\text{meas}\{x \in (a, b) : u_\delta(x) \geq B - \varepsilon\} \geq \eta \quad \text{for all } \delta \in (0, \delta_0). \tag{4-14}$$

Let us consider the modified family  $\hat{u}_\delta := MS_\delta T_{A,B} u_\delta$  as in Lemma 4.1. From (4-13) and (4-14) it follows that the nondecreasing function  $\hat{u}_\delta$  satisfies

$$\hat{u}_\delta(x) \leq A + 2\varepsilon \quad \text{for all } x \in (a, a + \eta), \text{ for all } \delta \in (0, \delta_1), \tag{4-15}$$

$$\hat{u}_\delta(x) \geq B - 2\varepsilon \quad \text{for all } x \in (b - \eta, b), \text{ for all } \delta \in (0, \delta_1), \tag{4-16}$$

where  $\delta_1 := \min\{\varepsilon, \delta_0\}$ . Moreover, from Lemmas 4.1 and 4.2 it follows that

$$\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u_\delta, (a, b)) \geq \liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, \hat{u}_\delta, (a, b)) \geq \liminf_{\delta \rightarrow 0^+} \hat{\Lambda}_\delta(\varphi, \hat{u}_\delta, (a + \eta, b - \eta)),$$

where in the computation of the latter we imagine that  $\hat{u}_\delta$  has been extended to the whole real line by setting it equal to 0 (or any other value) outside  $(a, b)$ .

In order to compute the last liminf, we need a deeper description of the structure of  $\hat{u}_\delta$ . We know that  $\hat{u}_\delta$  is nondecreasing and that its image is contained in  $\delta\mathbb{Z}$ . Let  $\delta m_0$  denote the value of  $\hat{u}_\delta$  in a right neighborhood of  $a$ , let us set  $x_0 = a$ , and for every positive integer  $i$  let us define

$$x_i := \sup\{x \in (a, b) : \hat{u}_\delta(x) < (m_0 + i)\delta\}.$$

The sequence  $x_i$  is nondecreasing, and  $x_i = b$  for every large enough index  $i$ . If  $x_{i+1} > x_i$  for some index  $i$ , then it turns out that

$$\hat{u}_\delta(x) = (m_0 + i)\delta \quad \text{for all } x \in (x_i, x_{i+1}).$$

If  $x_{i+1} = x_i$  for some index  $i$ , this means that

$$\text{meas}\{x \in (a, b) : \hat{u}_\delta(x) = (m_0 + i)\delta\} = 0.$$

Let  $\alpha$  and  $\beta$  be the two indices (which of course do depend on  $\delta$ ) such that

$$\alpha := \min\{i \in \mathbb{N} : x_i \geq a + \eta\} \quad \text{and} \quad \beta := \max\{i \in \mathbb{N} : x_i \leq b - \eta\}.$$

Let us consider now the interaction law  $\varphi$ , which we assumed of the form (1-9), and let  $\mu(\varphi)$  denote the smallest index  $k \leq m$  such that  $\lambda_k > 0$ . To begin with, we show that  $x_{i+\mu(\varphi)} > x_i$  for every index  $i$  with  $\alpha \leq i \leq \beta$ . Indeed, if this is not the case, then it turns out that

$$\hat{u}_\delta(y) - \hat{u}_\delta(x) \geq (\mu(\varphi) + 1)\delta \quad \text{for all } (x, y) \in (a, x_i) \times (x_i, b),$$

and in particular

$$\Lambda_\delta(\varphi, \hat{u}_\delta, (a, b)) \geq \lambda_{\mu(\varphi)} \Lambda_\delta(\varphi_{\mu(\varphi)}, \hat{u}_\delta, (a, b)) \geq \lambda_{\mu(\varphi)} \int_a^{x_i} dx \int_{x_i}^b \frac{\delta}{(y-x)^2} dy,$$

which is absurd because the left-hand side is uniformly bounded from above, while the double integral in the right-hand side is divergent.

Let us consider now an integer  $k \in \{\mu(\varphi), \dots, m\}$ , and for every  $x \in (a, b)$  let us set

$$H_{k,+}(x) := \int_x^{+\infty} \varphi_k \left( \frac{|\hat{u}_\delta(y) - \hat{u}_\delta(x)|}{\delta} \right) \frac{\delta}{(y-x)^2} dy,$$

$$H_{k,-}(x) := \int_{-\infty}^x \varphi_k \left( \frac{|\hat{u}_\delta(y) - \hat{u}_\delta(x)|}{\delta} \right) \frac{\delta}{(y-x)^2} dy.$$

With this notation it turns out that

$$\begin{aligned} \hat{\Lambda}_\delta(\varphi_k, \hat{u}_\delta, (a + \eta, b - \eta)) &= \int_{a+\eta}^{b-\eta} dx \int_{-\infty}^{+\infty} \varphi_k \left( \frac{|\hat{u}_\delta(y) - \hat{u}_\delta(x)|}{\delta} \right) \frac{\delta}{(y-x)^2} dy \\ &= \int_{a+\eta}^{b-\eta} (H_{k,+}(x) + H_{k,-}(x)) dx \\ &\geq \int_{x_\alpha}^{x_{\beta-k}} H_{k,+}(x) dx + \int_{x_{\alpha+k}}^{x_\beta} H_{k,-}(x) dx. \end{aligned} \tag{4-17}$$

Let us compute the last two integrals separately. For every index  $i \in \{\alpha + 1, \dots, \beta - k\}$  it turns out that

$$H_{k,+}(x) = \int_{x_{i+k}}^{+\infty} \frac{\delta}{(y-x)^2} dy = \frac{\delta}{x_{i+k} - x} \quad \text{for all } x \in (x_{i-1}, x_i).$$

The previous equality assumes that  $x_{i-1} < x_i$ , but actually it is true for trivial reasons also if  $x_{i-1} = x_i$ . It follows that

$$\int_{x_{i-1}}^{x_i} H_{k,+}(x) dx = \delta \log \frac{x_{i+k} - x_{i-1}}{x_{i+k} - x_i}$$

for every  $i \in \{\alpha + 1, \dots, \beta - k\}$ , and therefore

$$\int_{x_\alpha}^{x_{\beta-k}} H_{k,+}(x) dx = \sum_{i=\alpha+1}^{\beta-k} \int_{x_{i-1}}^{x_i} H_{k,+}(x) dx = \delta \sum_{i=\alpha+1}^{\beta-k} \log \frac{x_{i+k} - x_{i-1}}{x_{i+k} - x_i}. \tag{4-18}$$



In an analogous way, for every index  $i \in \{\alpha + k + 1, \dots, \beta\}$  it turns out that

$$H_{k,-}(x) = \int_{-\infty}^{x_{i-k-1}} \frac{\delta}{(y-x)^2} dy = \frac{\delta}{x-x_{i-k-1}} \quad \text{for all } x \in (x_{i-1}, x_i),$$

so that, with a shift of indices, we obtain

$$\int_{x_{i+k-1}}^{x_{i+k}} H_{k,-}(x) dx = \delta \log \frac{x_{i+k} - x_{i-1}}{x_{i+k-1} - x_{i-1}}$$

for every  $i \in \{\alpha + 1, \dots, \beta - k\}$ , and therefore

$$\int_{x_{\alpha+k}}^{x_{\beta}} H_{k,-}(x) dx = \sum_{i=\alpha+1}^{\beta-k} \int_{x_{i+k-1}}^{x_{i+k}} H_{k,-}(x) dx = \delta \sum_{i=\alpha+1}^{\beta-k} \log \frac{x_{i+k} - x_{i-1}}{x_{i+k-1} - x_{i-1}}. \tag{4-19}$$

Plugging (4-18) and (4-19) into (4-17), we find that

$$\hat{\Lambda}_{\delta}(\varphi_k, \hat{u}_{\delta}, (a + \eta, b - \eta)) \geq \delta \sum_{i=\alpha+1}^{\beta-k} \log \frac{(x_{i+k} - x_{i-1})^2}{(x_{i+k-1} - x_{i-1})(x_{i+k} - x_i)}.$$

Setting  $\ell_i := x_{\alpha+i} - x_{\alpha+i-1}$  for every  $i \in \{1, \dots, \beta - \alpha\}$ , we can write the last inequality in the form

$$\begin{aligned} \hat{\Lambda}_{\delta}(\varphi_k, \hat{u}_{\delta}, (a + \eta, b - \eta)) &\geq \delta \sum_{i=1}^{\beta-\alpha-k} \log \frac{(\ell_i + \dots + \ell_{i+k})^2}{(\ell_i + \dots + \ell_{i+k-1})(\ell_{i+1} + \dots + \ell_{i+k})} \\ &= \delta L_k(\ell_1, \dots, \ell_{\beta-\alpha}), \end{aligned}$$

where  $L_k$  is the multivariable function defined in (3-3). We observe that the denominators do not vanish because  $k \geq \mu(\varphi)$ , and we have already proved that  $x_{i+\mu(\varphi)} > x_i$ , which is equivalent to saying that the list  $(\ell_1, \dots, \ell_{\beta-\alpha})$  contains no  $k$  consecutive terms that vanish. Since  $\hat{\Lambda}_{\delta}$  is linear with respect to  $\varphi$ , we deduce

$$\begin{aligned} \hat{\Lambda}_{\delta}(\varphi, \hat{u}_{\delta}, (a + \eta, b - \eta)) &\geq \delta \sum_{k=\mu(\varphi)}^m \lambda_k L_k(\ell_1, \dots, \ell_{\beta-\alpha}) \\ &= \delta P_{\beta-\alpha, \varphi}(\ell_1, \dots, \ell_{\beta-\alpha}) \geq \delta I_{\beta-\alpha}(\varphi). \end{aligned}$$

Letting  $\delta \rightarrow 0^+$ , and observing that  $\beta - \alpha \rightarrow +\infty$ , we conclude that

$$\begin{aligned} \liminf_{\delta \rightarrow 0^+} \hat{\Lambda}_{\delta}(\varphi, \hat{u}_{\delta}, (a + \eta, b - \eta)) &\geq \liminf_{\delta \rightarrow 0^+} \delta I_{\beta-\alpha}(\varphi) \\ &\geq \liminf_{\delta \rightarrow 0^+} \delta(\beta - \alpha) \cdot \liminf_{\delta \rightarrow 0^+} \frac{I_{\beta-\alpha}(\varphi)}{\beta - \alpha} \\ &\geq \liminf_{\delta \rightarrow 0^+} \delta(\beta - \alpha) \cdot \liminf_{n \rightarrow +\infty} \frac{I_n(\varphi)}{n}. \end{aligned} \tag{4-20}$$

It remains to compute the liminf of  $\delta(\beta - \alpha)$ . To this end, from (4-15) and the minimality of  $\alpha$  we deduce

$$A + 2\varepsilon \geq \hat{u}_{\delta}(x) = (m_0 + \alpha - 1)\delta \quad \text{for all } x \in (x_{\alpha-1}, x_{\alpha}).$$

Similarly, from (4-16) and the maximality of  $\beta$  we deduce

$$B - 2\varepsilon \leq \hat{u}_\delta(x) = (m_0 + \beta)\delta \quad \text{for all } x \in (x_\beta, x_{\beta+1}).$$

It follows that  $(\beta - \alpha)\delta \geq B - A - 4\varepsilon - \delta$ , and therefore from (4-20) we conclude that

$$\liminf_{\delta \rightarrow 0^+} \hat{\Lambda}_\delta(\varphi, \hat{u}_\delta, (a + \eta, b - \eta)) \geq (B - A - 4\varepsilon) \cdot \liminf_{n \rightarrow +\infty} \frac{I_n(\varphi)}{n}.$$

Letting  $\varepsilon \rightarrow 0^+$ , we finally deduce (4-12). □

As observed in [Antonucci et al. 2020], we can rewrite Proposition 4.3 as a relation between the liminf of the energy and the liminf of oscillations as follows.

**Corollary 4.4.** *Let  $(a, b)$ ,  $u_\delta$ ,  $\varphi$ , and  $I_n(\varphi)$  be as in Proposition 4.3. For every  $\delta > 0$ , let  $\text{osc}(u_\delta, (a, b))$  denote the essential oscillation of  $u_\delta$  in  $(a, b)$ .*

*Then it turns out that*

$$\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u_\delta, (a, b)) \geq \left( \liminf_{\delta \rightarrow 0^+} \text{osc}(u_\delta, (a, b)) \right) \cdot \liminf_{n \rightarrow +\infty} \frac{I_n(\varphi)}{n}.$$

*Proof.* Let  $i_\delta$  and  $s_\delta$  denote the essential infimum and the essential supremum of  $u_\delta(x)$  in  $(a, b)$ , respectively. Let us assume that  $i_\delta$  and  $s_\delta$  are real numbers (otherwise an analogous argument works with standard minor changes). Let us set  $w_\delta(x) := u_\delta(x) - i_\delta$ , and let us observe that

$$\Lambda_\delta(\varphi, u_\delta, (a, b)) = \Lambda_\delta(\varphi, w_\delta, (a, b)) \quad \text{for all } \delta > 0$$

and that  $w_\delta$  satisfies (4-10) and (4-11) with  $A := 0$  and

$$B := \liminf_{\delta \rightarrow 0^+} (s_\delta - i_\delta) = \liminf_{\delta \rightarrow 0^+} \text{osc}(u_\delta, (a, b)).$$

At this point the conclusion follows from Proposition 4.3. □

Combining Theorem B and Corollary 4.4, we obtain the following result, which connects the Gamma-liminf of the family (1-1) to the multivariable minimum problems of Section 3.

**Proposition 4.5.** *For every positive integer  $d$  and every interaction law  $\varphi \in \mathcal{PCA}$  it turns out that*

$$\Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u, \mathbb{R}^d) \geq G_d \cdot \frac{1}{2} \liminf_{n \rightarrow +\infty} \frac{I_\varphi(n)}{n} \cdot \Lambda_0(u, \mathbb{R}^d) \quad \text{for all } u \in L^1(\mathbb{R}^d).$$

### 5. Proofs of our main results

*Proof of Theorem 1.3.* Let us consider, for every positive integer  $m$ , the interaction law defined as

$$\psi_m(t) := \sum_{k=1}^{2^m-1} \varphi_k(t) \quad \text{for all } t \geq 0. \tag{5-1}$$

This interaction law can be written in the form (1-10) with  $a_1 = \dots = a_m = 1$ , and hence  $\psi_m \in \mathcal{PCA}_2$ . As a consequence, from Proposition 3.3 we deduce

$$\liminf_{n \rightarrow +\infty} \frac{I_n(\psi_m)}{n} \geq m \cdot 2 \log 2,$$

and therefore from [Proposition 4.5](#) we obtain

$$\Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\psi_m, u, \mathbb{R}^d) \geq G_d \cdot m \log 2 \cdot \Lambda_0(u, \mathbb{R}^d). \tag{5-2}$$

On the other hand, from [\(1-5\)](#) we know

$$\Gamma\text{-}\lim_{\delta \rightarrow 0^+} \Lambda_\delta(\psi_m, u, \mathbb{R}^d) = G_d \cdot N(\psi_m) \cdot K_d(\psi_m) \cdot \Lambda_0(u, \mathbb{R}^d), \tag{5-3}$$

and with some simple calculus we find that

$$N(\psi_m) = \int_0^{+\infty} \frac{\psi_m(t)}{t^2} dt = \sum_{k=1}^{2^m-1} \int_0^{+\infty} \frac{\varphi_k(t)}{t^2} dt = \sum_{k=1}^{2^m-1} \int_k^{+\infty} \frac{1}{t^2} dt = \sum_{k=1}^{2^m-1} \frac{1}{k}. \tag{5-4}$$

Comparing [\(5-2\)](#) and [\(5-3\)](#) we obtain

$$K_d(\psi_m) \geq \frac{m \log 2}{N(\psi_m)} \quad \text{for all } m \geq 1. \tag{5-5}$$

Now from [\(5-4\)](#) we know that  $N(\psi_m) \sim m \log 2$  as  $m \rightarrow +\infty$ , and therefore we conclude that  $K_d(\psi_m) \rightarrow 1$  as  $m \rightarrow +\infty$ , independently of the space dimension.  $\square$

*Proof of [Theorem 1.4](#)(1).* Let us consider the interaction law  $\theta \in \mathcal{A}_0$  defined by

$$\theta(t) := \begin{cases} 0 & \text{if } t \in [0, 1], \\ t - 1 & \text{if } t \in [1, 2], \\ 1 & \text{if } t \geq 2. \end{cases} \tag{5-6}$$

We claim that the shape factor of  $\theta$  is 1 in any space dimension.

For every positive integer  $m$  we consider the interaction law

$$\theta_m(t) := \sum_{k=2^{m-1}}^{2^m-1} \varphi_k(t),$$

and the rescaled function

$$\hat{\theta}_m(t) := \frac{1}{2^{m-1}} \theta_m((2^{m-1} - 1)t).$$

To begin with, we show that

$$\theta(t) \geq \hat{\theta}_m(t) \quad \text{for all } t \geq 0, \text{ for all } m \geq 1. \tag{5-7}$$

To this end, we distinguish three cases.

Case 1: If  $t \in [0, 1]$ , then  $(2^{m-1} - 1)t \leq 2^{m-1} - 1$ , and hence  $\varphi_k((2^{m-1} - 1)t) = 0$  for every  $k \geq 2^{m-1}$ . It follows that  $\hat{\theta}_m(t) = 0$ , and hence [\(5-7\)](#) is trivial.

Case 2: If  $t \geq 2$ , then

$$\hat{\theta}_m(t) \leq \frac{1}{2^{m-1}} \sum_{k=2^{m-1}}^{2^m-1} 1 = 1 = \theta(t),$$

and therefore [\(5-7\)](#) is again satisfied.

Case 3: If  $t \in (1, 2)$ , let us choose  $i \in \{2^{m-1}, \dots, 2^m - 1\}$  such that

$$\frac{i}{2^{m-1}} < t \leq \frac{i+1}{2^{m-1}}.$$

Since

$$(2^{m-1} - 1)t \leq (2^{m-1} - 1) \cdot \frac{i+1}{2^{m-1}} = i - \frac{i+1-2^{m-1}}{2^{m-1}} \leq i,$$

we deduce

$$\varphi_k((2^{m-1} - 1)t) = 0 \quad \text{for all } k \geq i,$$

and therefore

$$\hat{\theta}_m(t) \leq \frac{1}{2^{m-1}} \sum_{k=2^{m-1}}^{i-1} \varphi_k(t) \leq \frac{1}{2^{m-1}}(i - 2^{m-1}) = \frac{i}{2^{m-1}} - 1 \leq t - 1 = \theta(t),$$

which proves (5-7) also in this case.

From inequality (5-7), and the rescaling properties of  $\Lambda_\delta$  with respect to the interaction law, we deduce

$$\begin{aligned} \Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\theta, u, \mathbb{R}^d) &\geq \Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\hat{\theta}_m, u, \mathbb{R}^d) \\ &= \frac{2^{m-1} - 1}{2^{m-1}} \cdot \Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\theta_m, u, \mathbb{R}^d). \end{aligned} \tag{5-8}$$

Now we observe that the interaction law  $\theta_m(t)$  can be written in the form (1-10) with  $a_1 = \dots = a_{m-1} = 0$  and  $a_m = 1$ , and hence  $\theta_m \in \mathcal{PCA}_2$ . As a consequence, from Proposition 3.3 we deduce

$$\liminf_{n \rightarrow +\infty} \frac{I_n(\theta_m)}{n} \geq 2 \log 2,$$

and therefore from Proposition 4.5 we obtain

$$\Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\theta_m, u, \mathbb{R}^d) \geq G_d \cdot \log 2 \cdot \Lambda_0(u, \mathbb{R}^d).$$

Plugging this estimate into (5-8), and letting  $m \rightarrow +\infty$ , we deduce

$$\Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\theta, u, \mathbb{R}^d) \geq G_d \cdot \log 2 \cdot \Lambda_0(u, \mathbb{R}^d). \tag{5-9}$$

On the other hand, from (1-5) we know

$$\Gamma\text{-}\lim_{\delta \rightarrow 0^+} \Lambda_\delta(\theta, u, \mathbb{R}^d) = G_d \cdot N(\theta) \cdot K_d(\theta) \cdot \Lambda_0(u, \mathbb{R}^d), \tag{5-10}$$

and with some simple calculus we find that

$$N(\theta) = \int_1^2 \frac{t-1}{t^2} dt + \int_2^{+\infty} \frac{1}{t^2} dt = \log 2.$$

Comparing (5-9) and (5-10) we conclude that  $K_d(\theta) = 1$  in any space dimension.



*Proof of Theorem 1.4(2).* The function  $\zeta$  is continuous because  $f(z) \rightarrow 0$  as  $z \rightarrow -\infty$ . It is also bounded and monotone due to the corresponding assumptions on  $f(z)$ . Finally, assumption (1-11) implies the existence of a constant  $a$  such that  $\zeta(t) \leq at^2$  for every  $t \geq 0$ . This proves that  $\zeta \in \mathcal{A}$ .

In order to compute scale and shape factor of  $\zeta$ , we observe that it can be written in the form

$$\zeta(t) = \sum_{z=-\infty}^{+\infty} (f(z+1) - f(z)) \cdot \theta(2^{-z}t) \quad \text{for all } t \geq 0,$$

where  $\theta$  is the interaction law defined in (5-6). Due to the additivity and to the rescaling properties of  $\Lambda_\delta$  with respect to the interaction law, from this representation it follows that

$$\begin{aligned} \Gamma\text{-}\lim_{\delta \rightarrow 0^+} \Lambda_\delta(\zeta, u, \mathbb{R}^d) &\geq \sum_{z=-\infty}^{+\infty} (f(z+1) - f(z))2^{-z} \cdot \Gamma\text{-}\liminf_{\delta \rightarrow 0^+} \Lambda_\delta(\theta, u, \mathbb{R}^d) \\ &= \sum_{z=-\infty}^{+\infty} (f(z+1) - f(z))2^{-z} \cdot G_d \cdot N(\theta) \cdot \Lambda_0(u, \mathbb{R}^d), \end{aligned} \tag{5-11}$$

where in the last equality we have exploited that  $K_d(\theta) = 1$ .

On the other hand, from (1-5) we know

$$\Gamma\text{-}\lim_{\delta \rightarrow 0^+} \Lambda_\delta(\zeta, u, \mathbb{R}^d) = G_d \cdot N(\zeta) \cdot K_d(\zeta) \cdot \Lambda_0(u, \mathbb{R}^d). \tag{5-12}$$

Since

$$N(\zeta) = N(\theta) \cdot \sum_{z=-\infty}^{+\infty} (f(z+1) - f(z))2^{-z},$$

comparing (5-11) and (5-12) we conclude that  $K_d(\zeta) = 1$  in any space dimension. □

### 6. Final remarks

In this section we present some variants of our main results, and we speculate about some possible future extensions of the theory developed in this paper.

**A counterexample to the short-range question with strict inequalities.** Let us consider the interaction laws

$$\varphi_\varepsilon(t) := \begin{cases} c_{1,\varepsilon} \cdot \varepsilon t^2 & \text{if } t \in [0, 1], \\ c_{1,\varepsilon} & \text{if } t > 1, \end{cases} \quad \psi(t) := c_2 \psi_2(t),$$

where  $\psi_2(t)$  is defined by (5-1) with  $m = 2$ , and the constants  $c_{1,\varepsilon}$  and  $c_2$  are chosen in such a way that  $N(\varphi_\varepsilon) = N(\psi) = 1$ .

From (5-5) and (5-4) with  $m = 2$  it follows that  $K_d(\psi) \geq (12/11) \log 2$ . On the other hand, it is possible (but not completely trivial) to show that  $K_d(\varphi_\varepsilon) \rightarrow \log 2$  as  $\varepsilon \rightarrow 0^+$ .

Therefore, when  $\varepsilon$  is small enough, this is an example of two interaction laws  $\varphi_\varepsilon$  and  $\psi$  with equal scale factor, satisfying  $\varphi_\varepsilon(t) > \psi(t)$  for every  $t \in (0, 1]$ , but nevertheless  $K_d(\varphi_\varepsilon) < K_d(\psi)$  in every space dimension. This provides a counterexample to Question 1 with strict inequalities.

**True Gamma-limits and smooth recovery families.** In this paper we limited ourselves to providing estimates from below for the Gamma-liminf, since they are enough to establish both Theorems 1.3 and 1.4. On the other hand, with little further effort we could prove that actually the lower bound coincides with the Gamma-limit.

This is evident in the case of the interaction laws with shape factor equal to 1, for example all those provided by statement (2) of Theorem 1.4, because for them the pointwise limit coincides on smooth functions with the estimate from below for the Gamma-liminf. Therefore, for all these interaction laws we now know both the Gamma-limit with exact values of the constants in any space dimension and the existence of smooth recovery families.

As for the interaction laws  $\psi_m(t)$  defined by (5-1), again we can show that the Gamma-limit coincides with the lower bound we obtained for the Gamma-liminf, namely

$$\Gamma\text{-}\lim_{\delta \rightarrow 0^+} \Lambda_\delta(\psi_m, u, \mathbb{R}^d) = G_d \cdot m \log 2 \cdot \Lambda_0(u, \mathbb{R}^d). \tag{6-1}$$

In order to prove this result, one should follow the path we pursued in [Antonucci et al. 2020]. The main idea is that in any space dimension the family  $S_{\delta u}$  of vertical  $\delta$ -segmentations of  $u$  is a recovery family when  $u$  is piecewise  $C^1$  or piecewise affine with compact support, and those classes are dense in energy for the right-hand side of (6-1). Since vertical  $\delta$ -segmentations of piecewise affine functions with compact support are step functions with level sets that are finite unions of polytopes, it is enough to further approximate them in order to produce recovery sequences made by functions of class  $C^\infty$  with compact support. We refer to [Antonucci et al. 2020] for the details. Therefore, also in the case of the interaction laws  $\psi_m(t)$ , we end up with a Gamma-convergence result with both exact values of the constants in any space dimension, and existence of smooth recovery families.

The same argument should work for all interaction laws in  $\mathcal{PCA}_2$ , and more generally for all interaction laws  $\varphi \in \mathcal{PCA}$  for which  $I_n(\varphi)$  is realized asymptotically when all the variables are equal.

**Toward a general formula for the Gamma-limit.** We suspect that the lower bound in Proposition 4.3 might be optimal for every  $\varphi \in \mathcal{PCA}$ , and that the liminf in the right-hand side of (4-12) is actually a limit. Thanks to Proposition 4.5, this would imply

$$\Gamma\text{-}\lim_{\delta \rightarrow 0^+} \Lambda_\delta(\varphi, u, \mathbb{R}^d) = G_d \cdot \frac{1}{2} \lim_{n \rightarrow +\infty} \frac{I_n(\varphi)}{n} \cdot \Lambda_0(u, \mathbb{R}^d) \quad \text{for all } u \in L^1(\mathbb{R}^d), \tag{6-2}$$

for every  $\varphi \in \mathcal{PCA}$ . In order to prove this result, the main difficulty seems to be the construction of recovery sequences, which in general can no longer be obtained simply by vertical  $\delta$ -segmentation. On the contrary, the construction should now take into account the pattern that realizes the infimum  $I_n(\varphi)$ .

A representation of the form (6-2), if true, would be important because any interaction law can be approximated from below by piecewise constant interaction laws with steps of equal horizontal length (as we did in the proof of statement (1) of Theorem 1.4), and these laws are rescalings of laws in  $\mathcal{PCA}$ .

This kind of representation would be even more important if it were true that the Gamma-limit of  $\Lambda_\delta(\varphi, u, \mathbb{R}^d)$  is the supremum of the Gamma-limits of  $\Lambda_\delta(\psi, u, \mathbb{R}^d)$  as  $\psi$  varies in the set of all piecewise constant interaction laws, with steps of equal horizontal length that are less than or equal to  $\varphi$ . A

confirmation of this conjecture would open the way for answering several questions raised in [Brezis and Nguyen 2018]: a simplified proof of the Gamma-convergence result in full generality, a less implicit formula for shape factors, and existence of smooth recovery families.

**Characterization of interaction laws without gap.** Concerning the gap between the pointwise limit and the Gamma-limit, the challenge is now characterizing all interaction laws with shape factor equal to 1. Let us summarize what we know for the time being on this specific issue.

- Continuity does not guarantee lack of the gap. Among continuous interaction laws, we have both examples without gap (all interaction laws provided by Theorem 1.4) and interaction laws with gap. Indeed, it is possible to show that the piecewise affine interaction law that is equal to 0 for  $t \in [0, 1 - \varepsilon]$  and equal to 1 for  $t \geq 1$  has a shape factor that tends to  $\log 2$  as  $\varepsilon \rightarrow 0^+$ .

Conversely, we have no example of discontinuous interaction law without gap.

- It is not a matter of vanishing in a neighborhood of the origin. Among the interactions laws in  $\mathcal{A}_0$  we have both examples without gap (the interaction law  $\theta$  defined in (5-6)) and examples with gap (the model interaction law  $\varphi_1$ ). Among the interaction laws that are positive for every  $t > 0$  we have both examples without gap (defined as in statement (2) of Theorem 1.4) and examples with gap (the interaction law  $\varphi_\varepsilon(t)$  defined at the beginning of this section).

- The shape factor is concave when restricted to interaction laws with equal scale factor. As a consequence, any convex combination of interaction laws with the same scale factor, and shape factor equal to 1, has again shape factor equal to 1. Considering that now we know many interaction laws with shape factor equal to 1, this leads us to guess that the set of interaction laws with shape factor equal to 1 might be quite large.

**More general exponents.** It should not be difficult to extend the results of this paper to the more general family of functionals

$$\Lambda_{\delta,p}(\varphi, u, \Omega) := \iint_{\Omega^2} \varphi\left(\frac{|u(y) - u(x)|}{\delta}\right) \frac{\delta}{|y - x|^{d+p}} dx dy,$$

where  $p > 1$  is a real number. This case was investigated in [Brezis and Nguyen 2020]. The Gamma-limit turns out to be a multiple of the  $L^p$ -norm of the gradient of  $u$ , and the exact constant was found in [Antonucci et al. 2020] in the case  $\varphi = \varphi_1$ . When extending the results of this paper, the presence of the general exponent  $p > 1$  requires probably only a change in definition (3-3), which now should be replaced by something like

$$\begin{aligned} L_{k,p} &:= \int_{S_{i,k}}^{S_{i,k+1}} \frac{1}{\sigma^p} d\sigma + \int_{S_{i+1,k}}^{S_{i,k+1}} \frac{1}{\sigma^p} d\sigma \\ &= \frac{1}{p-1} \sum_{i=1}^{n-k} \left( -\frac{2}{[S_{i,k+1}]^{p-1}} + \frac{1}{[S_{i,k}]^{p-1}} + \frac{1}{[S_{i+1,k}]^{p-1}} \right). \end{aligned}$$

Again, the special structure of the terms of the sum should guarantee the telescopic effect as in Lemma 3.1.

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# ANALYSIS & PDE

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On the gap between the Gamma-limit and the pointwise limit for a nonlocal approximation of the total variation	627
CLARA ANTONUCCI, MASSIMO GOBBINO and NICOLA PICENNI	
External boundary control of the motion of a rigid body immersed in a perfect two-dimensional fluid	651
OLIVIER GLASS, JÓZSEF J. KOLUMBÁN and FRANCK SUEUR	
Distance graphs and sets of positive upper density in $\mathbb{R}^d$	685
NEIL LYALL and ÁKOS MAGYAR	
Isolated singularities for semilinear elliptic systems with power-law nonlinearity	701
MARIUS GHERGU, SUNGHAN KIM and HENRIK SHAHGHOIAN	
Regularity of the free boundary for the vectorial Bernoulli problem	741
DARIO MAZZOLENI, SUSANNA TERRACINI and BOZHIDAR VELICHKOV	
On the discrete Fuglede and Pompeiu problems	765
GERGELY KISS, ROMANOS DIOGENES MALIKIOSIS, GÁBOR SOMLAI and MÁTÉ VIZER	
Energy conservation for the compressible Euler and Navier–Stokes equations with vacuum	789
IBROKHIMBEK AKRAMOV, TOMASZ DĘBIEC, JACK SKIPPER and EMIL WIEDEMANN	
A higher-dimensional Bourgain–Dyatlov fractal uncertainty principle	813
RUI HAN and WILHELM SCHLAG	
Local minimality results for the Mumford–Shah functional via monotonicity	865
DORIN BUCUR, ILARIA FRAGALÀ and ALESSANDRO GIACOMINI	
The gradient flow of the Möbius energy: $\varepsilon$ -regularity and consequences	901
SIMON BLATT	
Correction to the article The heat kernel on an asymptotically conic manifold	943
DAVID A. SHER	