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PERTURBATIVE TECHNIQUES
FOR THE CONSTRUCTION OF SPIKE-LAYERS

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Dedicated to Wei-Ming Ni with admiration

Abstract. In this paper we survey some results concerning the construction of *spike-layers*, namely solutions to singularly perturbed equations that exhibit a concentration behaviour. Their study is motivated by the analysis of pattern formation in biological systems such as the Keller-Segel or the Gierer-Meinhardt's. We describe some general perturbative variational strategy useful to study concentration at points, and also at spheres in radially symmetric situations.

1. **Introduction.**

This paper surveys some results over the past decades concerning the study of spike-layers, on which W.M. Ni gave some of the most important contributions. Here we denote by *spike-layers* solutions of the following problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{P_ε}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $p > 1$, $\varepsilon > 0$ is a small parameter and ν stands for the unit normal to $\partial\Omega$. We will also consider the same problem under Dirichlet boundary conditions: although our equation is of specific type, in the literature more general nonlinearities were also considered.

Such a problem has different motivations, which are well described for example in [32] or [47]. One of them concerns the stationary *Keller-Segel system*, meant to describe chemotactic aggregation

$$\begin{cases} D_1 \Delta \mathcal{U} - \chi \nabla \cdot (\mathcal{U} \nabla \log \mathcal{V}) = 0 & \text{in } \Omega, \\ D_2 \Delta \mathcal{V} - a\mathcal{V} + b\mathcal{U} = 0 & \text{in } \Omega, \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{KS}$$

Here χ, a, b and D_1, D_2 are positive parameters in suitable ranges of $(0, +\infty)$, while $\mathcal{U}, \mathcal{V}$ are unknown functions in Ω. Another relater system is the *Gierer-Meinhard*'s,

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describing an activator-inhibitor system in biological pattern formation

$$\begin{cases} d_1\Delta\mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} = 0 & \text{in } \Omega, \\ d_2\Delta\mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} = 0 & \text{in } \Omega, \\ \frac{\partial\mathcal{U}}{\partial\nu} = \frac{\partial\mathcal{V}}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{GM}$$

where all parameters involved are again positive.

In both models, $\mathcal{U}$ and $\mathcal{V}$ represent densities of either some chemical substance or of a biological population, and a phenomenon that is observed is the presence of solutions that are higly concentrated near some subsets of Ω, especially when the two diffusivities of the components are very different. This is in the spirit of Turing's instability for reaction-diffusion systems, [57], while single equations may not exhibit (stable) patterns ([13], [45]).

In some asymptotic regimes for the diffusivities, one component tends to become more and more homogeneous in Ω, so the above systems in their parabolic versions reduce to *shadow systems* where an unknown function is coupled to a constant that depend on time. In the static version, the other unknown will solve (P_ε) with a good approximation. Another motivation for the study of (P_ε) (in presence of a potential and/or in unbounded domains like the whole Euclidean space) arises from the nonlinear Schrödinger equation in the semi-classical limit, where the small parameter ε plays the role of Planck's constant: some classical references will be given below.

Among the first papers analyzing rigorously the pattern formation for the above two systems we mention [33] and [48]: here it was shown via a-priori estimates that for small values of the diffusivity of $\mathcal{V}$ in (KS) (or of $\mathcal{U}$ in (GM)) only constant solutions may arise. On the other hand, in the opposite regime, there is the appearance of solutions with sharp profiles. In showing the latter property, the analysis of (P_ε) was crucial: in particular the authors analysed its variational structure and derived basic estimates on its mountain-pass energy level. This study was continued in [49], where a detailed analysis of the least-energy solutions was performed (even for non-linearities more general than those in (P_ε)). Using rather sharp estimates, where the main asymptotic of the energy was derived, it was shown that those have to converge to the boundary of the domain, and that as $\varepsilon \to 0$ they only have one global maximum.

The prototypical asymptotics for solutions u_ε to (P_ε) can be guessed making the change of variables $u_\varepsilon(x) \sim u_0\left(\frac{x-Q}{\varepsilon}\right)$, where Q is some point of $\overline{\Omega}$ (to be determined), and where u_0 solves

$$-\Delta u_0 + u_0 = u_0^p \qquad \text{in } \mathbb{R}^N \quad (\text{or in } \mathbb{R}^N_+ = \{x_1, \dots, x_N \in \mathbb{R}^N \ : \ x_N > 0\}). \tag{1}$$

The choice of the limiting domain depends on whether solutions *concentrate* in the interior of Ω or at the boundary the domain: in the latter case Neumann conditions are imposed.

When $p < \frac{N+2}{N-2}$ (in fact, only in this case, see [11]), problem (1) is well-known to have a positive radial solution U satisfying

$$\lim_{r\to+\infty} e^r r^{\frac{N-1}{2}} U(r) = \alpha_{N,p}, \tag{2}$$

where $\alpha_{N,p} > 0$ depends only on N and p, as well as

$$\lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -1; \qquad \lim_{r\to+\infty} \frac{U''(r)}{U(r)} = 1. \tag{3}$$

Problem (P_ε) has variational structure, with Euler-Lagrange functional given by

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx; \qquad u \in H^1(\Omega). \tag{4}$$

In [50] it was proved that solutions with minimal energy converge to a boundary point with maximal mean curvature. For doing this, the authors expanded the energy of the mountain-pass solution up to the second main term, showing that the correction in the expansion is proportional to that of the volume (induced by the mean curvature) of metric balls in the domain centered at points of the boundary. Rigorous estimates were obtained using the decay of the above solution U, together with the study of the linearized equation of (1) at U.

As we will explain, the characterization of the kernel of the linearized equation (both in the whole $\mathbb{R}^N$ or in a half space), together with the variational feature of the problem allows also the construction of solutions at suitable critical points of the mean curvature of the boundary. These methods, relying on finite-dimensional reductions, can be used to construct a rich family of solutions, namely with interior peaks (even with Dirichlet boundary conditions), or with multiple ones, both at the boundary and at the interior of the domain, see e.g. [14], [16], [18], [25], [26], [27], [28], [31], [32], [52], [59], [60], [61]. Related results were obtained regarding semiclassical states of nonlinear Schrödinger equations, see e.g. [1], [17], [22], [53].

As it was conjectured for some time, see e.g. [47], one might expect that (P_ε) also has solutions concentrating at k-dimensional sets, for every integer $k \in \{1, \dots, N-1\}$: the literature on this phenomeon is indeed more recent.

In [3], [4] the finite-dimensional reduction technique was used to prove existence of solutions concentrating on spheres, for both problem (P_ε), the corresponding Dirichlet problem and also for the nonlinear Schrödinger equation in the whole space. An interesting feature of this phenomenon is that the location of the concentration set is driven not only by the geometry of the domain (or the potential in case of the NLS) but also on the *volume* of spherical shells where concentration occurs.

The general case, without symmety assumptions, is more delicate since strong resonance phenomena occur (see also [37], [46] for the geometric problem of finding constant mean curvature surfaces). In fact, radially symmetric solutions concentrating on spheres have bounded Morse index within the class of radial functions, while the index among arbitrary Sobolev functions diverges as ε tends to zero. Moreover, in this limit, more and more eigenvalues approach zero.

A different strategy was then needed, relying on more sophisticated implicit function arguments. We will not discuss them in detail here (some general description can be found in [40]), limiting ourselves to mention the principal ideas of the construction and some more recent progress. First, approximate solutions with high degree of accuracy are constructed. Then, a detailed study of the linearized equation is done, for which invertibility is shown only for a suitable sequence $\varepsilon_j \to 0$. In [42], [43] existence of solutions concentrating at the whole boundary was proved (in dimension two and arbitrary, respectively), while in [39], [37] concentration at non-degenerate minimal k-dimensional submanifolds of the boundary was proved (for $(N, k) = (3, 1)$ and (N, k) arbitrary, respectively). In [6], solutions developing an increasing number of boundary spikes were found, approaching a proper subset of the boundary (see also [55] for the special case of a rectangle). In [21] instead, a supercritical problem was considered, and existence of solutions with interior profiles approaching suitable submanifolds of the boundary were found (see also [15]).

In [34] solutions with a growing number of peaks (as $\varepsilon \to 0$) were constructed. In [29] and [62] solutions concentrating at interior lines or surfaces (orthogonal to the boundary) were found. In [5] the authors built solutions forming a triple junction in the interior of the domain, relater to the entire profiles constructed in [41] (see also [54]).

The plan of the paper is the following. In Section 2 we recall a general perturbative and variational theory that allows to treat concentration at points: we will focus on both Dirichlet and Neumann conditions. In Section 3 instead we will treat concentration at spheres in radially symmetric situations, showing a competing effect between volume energy and boundary conditions, than generate solutions with spherical profiles.

2. Concentration at points and spheres.

In this section we recall a general perturbative method, variational in nature, which allows to produce solutions concentrating at points via a finite-dimensional reduction, see e.g. [2] for a general treatment on this topic.

2.1. Perturbative critical point theory.

Here we recall some general strategy to tackle variational problems involving a small parameter ε. We consider a Hilbert space $\mathcal{H}$ (possibly depending on ε) containing a finite-dimensional submanifold Z_ε satisfying the following properties

: *i)* Z_ε has dimension d and $\exists\, C, r > 0$ such that for any $z \in Z_\varepsilon$, $Z \cap B_r(z)$ is parameterized by $\xi \in B_1(0) \subseteq \mathbb{R}^d$ with C^3-derivative bounded by C.

On $\mathcal{H}$ it is defined a $C^{2,\alpha}$ functional I_ε such that

: *ii)* $\|\nabla I_\varepsilon(z)\| \leq a(\varepsilon)$ for every $z \in Z_\varepsilon$ and $\|\nabla^2 I_\varepsilon(z)[q]\| \leq b(\varepsilon)\|q\|$ for every $z \in Z_\varepsilon$ and $q \in T_z Z_\varepsilon$, where $a, b : (0, \varepsilon_0) \to \mathbb{R}$ are a smooth functions tending to zero as $\varepsilon \to 0$;

: *iii)* $\exists\, C$, $\alpha \in (0,1]$, $r_0 > 0$ such that $\|I''_\varepsilon\|_{C^\alpha} \leq C$ in $\{u \,:\, dist(u, Z_\varepsilon) < r_0\}$;

: *iv)* let P_z be the projection on the orthogonal complement of $T_z Z_\varepsilon$. Then $\exists C > 0$ such that, on $(T_z Z_\varepsilon)^\perp$, $P_z \nabla^2 I_\varepsilon(z)$ is invertible from $(T_z Z_\varepsilon)^\perp$ in itself, with inverse satisfying $\|(P_z \nabla^2 I_\varepsilon(z))^{-1}\| \leq C$.

Let W denote the orthogonal space $W = (T_z Z_\varepsilon)^\perp$: since by the above property *ii)* all points of Z_ε are approximate critical points of I_ε, it is natural to look for *true* critical points in the form $u = z + \omega$, $z \in Z_\varepsilon$ $\omega \in W$. The conditions $I'_\varepsilon(z + \omega) = 0$ then becomes the following system:

$$\begin{cases} P_z I'_\varepsilon(z + \omega) = 0 & (\textit{auxiliary equation}); \\ (Id - P_z) I'_\varepsilon(z + \omega) = 0 & (\textit{bifurcation equation}). \end{cases} \tag{5}$$

From the contraction mapping theorem one can prove the following result.

Proposition 1. *Suppose the above conditions i)-iv) hold. Then* $\exists\, \varepsilon_0 > 0$ *such that for all* $|\varepsilon| < \varepsilon_0$ *and* $z \in Z_\varepsilon$, *the auxiliary equation in (5) possesses a unique solution* $\omega = \omega_\varepsilon \in W = (T_z Z_\varepsilon)^\perp$, *of class* C^1 *in* z *and such that, for* $|\varepsilon| \to 0$, $\|\omega_\varepsilon(z)\| \leq C_1 a(\varepsilon)$ *and such that* $\|\partial_\xi \omega_\varepsilon(z)\| \leq C_1(a(\varepsilon)^\alpha + b(\varepsilon))$.

Given the equivalence $I'_\varepsilon(z+\omega)$ to the above system (5), we are left with solving the bifurcation equation. For doing this, it is possible to exploit the variational structure of the problem, considering the *reduced functional* $\mathbf{I}_\varepsilon : Z \to \mathbb{R}$ given by

$$\mathbf{I}_\varepsilon(z) = I_\varepsilon(z + \omega_\varepsilon(z)). \tag{6}$$

As stated in the next proposition, this finite-dimensional quantity determines precisely the critical points in a neighborhood of Z of fixed size.

Proposition 2. *Consider the same assumptions as in Proposition 1. If $\mathbf{I}_\varepsilon$ has a critical point z_ε then $u_\varepsilon = z_\varepsilon + \omega_\varepsilon(z_\varepsilon)$ is also critical point of I_ε. Moreover, $\exists\, \tilde{c}, \tilde{r} > 0$ small so that if u is critical for I_ε with $dist(u, Z_{\varepsilon,\tilde{c}}) < \tilde{r}$, where*

$$Z_{\varepsilon,\tilde{c}} = \{z \in Z_\varepsilon \ : \ dist(z, \partial Z_\varepsilon) > \tilde{c}\},$$

then there exists $z_\varepsilon \in Z_\varepsilon$ such that u is of the form $z_\varepsilon + \omega_\varepsilon(z_\varepsilon)$.

The proof of the first statement can be geometrically described as follows. Consider the *perturbed manifold*

$$\tilde{Z}_\varepsilon := \{z + \omega_\varepsilon(z) \ : \ z \in Z_\varepsilon\}.$$

Since also the C^1-norm of $z \mapsto \omega_\varepsilon(z)$ tends to zero as $\varepsilon \to 0$, the two tangent spaces $T_z Z_\varepsilon$ and $T_{z+\omega_\varepsilon(z)}\tilde{Z}_\varepsilon$ are nearly parallel. By Lagrange's multipliers rule, the gradient of I_ε at $z_\varepsilon + \omega_\varepsilon(z_\varepsilon)$ is orthogonal to $T_{z_\varepsilon+\omega_\varepsilon(z_\varepsilon)}\tilde{Z}_\varepsilon$. On the other hand, by the auxiliary equation in (5), this gradient must also be orthogonal to $T_{z_\varepsilon} Z_\varepsilon$, but since the two tangent spaces are nearly parallel, it must eventually vanish identically. The proof of the second statement relies instead on the uniqueness of the fixed point in the contraction mapping.

The above abstract results will be next applied to the concrete settings of singularly Neumann and Dirichlet problems, dealing with both concentration at points or spheres.

2.2. Concentration at boundary points for the Neumann problem.

Here we discuss the construction of boundary spike-layers for problem (P_ε), giving only general ideas and referring to [2] for more details. It is convenient to perform a change of variables, so that the Neumann problem (P_ε) becomes

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon; \\ u > 0 & \text{in } \Omega_\varepsilon, \end{cases} \qquad \Omega_\varepsilon = \frac{1}{\varepsilon}\Omega. \tag{7}$$

For $p \leq \frac{N+2}{N-2}$, solutions of the latter problem are critical points of the Euler-Lagrange energy

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(|\nabla u|^2 + u^2\right) - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^{p+1}, \qquad u \in H^1(\Omega_\varepsilon). \tag{8}$$

In the limit $\varepsilon \to 0$, after a proper translation and rotation, Ω_ε converges to the half-space $\mathbb{R}^N_+$. The limit problem then becomes

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^N_+; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}^N_+; \\ u > 0 & \text{in } \mathbb{R}^N_+. \end{cases} \tag{9}$$

The last problem admits as a solution the radial function U discussed in the introduction, satisfying the asymptotics in (2) and (3). It is also known that the linearization of (9) at U has minimal degeneracy, namely its kernel is formed by the infinitesimal generators of translations of U along the boundary, namely by the functions $\partial_{x_1}U, \dots, \partial_{x_{N-1}}U$. This will guarantee property $iv)$ is the abstract setting of Subsection 2.1.

We construct next the manifold Z_ε for this concrete setting: for doing this, we need to introduce a parametrization of the boundary of Ω_ε near one of its points, which we call X. We can suppose that $X = 0 \in \mathbb{R}^N$, that $\{x_N = 0\}$ is the tangent plane of $\partial\Omega_\varepsilon$ (or $\partial\Omega$) at X, and that the unit normal to Ω_ε at X is $\nu(X) = (0, \dots, 0, -1)$. Assuming the same conditions on the original domain Ω, let $x_N = \psi(x')$ be a local parametrization of $\partial\Omega$. Then for some μ_0 small there holds

$$x_N = \psi(x') := \tfrac{1}{2}\langle A_X x', x'\rangle + O(|x'|^3); \qquad |x'| < \mu_0. \tag{10}$$

Here A_X is the hessian of ψ, and the mean curvature H at X satisfies $H(X) = \frac{1}{N-1} tr A_X$. Dilating the domain, we easily see that the boundary of Ω_ε is parameterized by the function $y_N = \psi_\varepsilon(x') := \frac{1}{\varepsilon}\psi(\varepsilon x')$, and one has that

$$\psi_\varepsilon(x') = \frac{\varepsilon}{2}\langle A_X x', x'\rangle + \varepsilon^2 O(|x'|^3).$$

The outer unit normal ν to $\partial\Omega_\varepsilon$ can be expanded in these coordinates as

$$\nu = \frac{\left(\frac{\partial\psi_\varepsilon}{\partial x_1}, \dots, \frac{\partial\psi_\varepsilon}{\partial x_{N-1}}, -1\right)}{\sqrt{1 + |\nabla\psi_\varepsilon|^2}} = (\varepsilon(A_X x'), -1) + \varepsilon^2 O(|x'|^2). \tag{11}$$

Given μ_0 as in (10), we straighten the coordinates on $B_{\frac{\mu_0}{\varepsilon}}(X) \cap \Omega_\varepsilon$ as follows. Define

$$y' = x'; \qquad\qquad y_N = x_N - \psi_\varepsilon(x') : \tag{12}$$

It these coordinates the metric coefficients (g_{ij}) are given by

$$(g_{ij}) = \left(\langle \frac{\partial x}{\partial y_i}, \frac{\partial x}{\partial y_i}\rangle\right) = \begin{pmatrix} & & & \frac{\partial\psi_\varepsilon}{\partial y_1} \\ & \delta_{ij} + \frac{\partial\psi_\varepsilon}{\partial y_i}\frac{\partial\psi_\varepsilon}{\partial y_j} & & \vdots \\ & & & \frac{\partial\psi_\varepsilon}{\partial y_{N-1}} \\ \frac{\partial\psi_\varepsilon}{\partial y_1} & \dots & \frac{\partial\psi_\varepsilon}{\partial y_{N-1}} & 1 \end{pmatrix},$$

and they satisfy?

$$g_{ij} = Id + \varepsilon A + O(\varepsilon^2|y'|^2); \qquad \partial_{y_k}(g_{ij}) = \varepsilon\partial_{y_k}A + O(\varepsilon^2|y'|), \tag{13}$$

where $A = \begin{pmatrix} 0 & A_X y' \\ (A_X y')^t & 0 \end{pmatrix}$. It is also easy to check that the inverse matrix (g^{ij}) is of the form $g^{ij} = Id - \varepsilon A + O(\varepsilon^2|y'|^2)$, and that $\partial_{y_k}(g^{ij}) = -\varepsilon\partial_{y_k}A + O(\varepsilon^2|y'|)$. Since (12) preserves volume, one has also that $\det(g)_{ij} \equiv 1$. The Laplace operator with respect to a given Riemannian metric is

$$\Delta_g u = \frac{1}{\sqrt{\det g}}\partial_j\left(g^{ij}\sqrt{\det g}\right)\partial_i u + g^{ij}\partial^2_{ij}u,$$

so when the determinant of g is identically equal to 1 this simplifies to

$$\Delta_g u = g^{ij}u_{ij} + \partial_i(g^{ij})\partial_j u.$$

From (13), is u is a smooth function, we then obtain

$$\begin{aligned} \Delta_g u = \Delta u &- \varepsilon\left(2\langle A_X y', \nabla_{y'}\partial_{y_N}u\rangle + tr A_X\partial_{y_N}u\right) \\ &+ O(\varepsilon^2|y'|)|\nabla u| + O(\varepsilon^2|y'|^2)|\nabla^2 u|. \end{aligned} \tag{14}$$

The area-element of the boundary of Ω_ε can be written as

$$d\sigma = (1 + O(\varepsilon^2 |y'|^2)) dy'. \tag{15}$$

Choose a radial non-increasing cut-off function ψ_{μ_0} identically equal to 1 on $B_{\frac{\mu_0}{4}}(0)$, vanishing outside $B_{\frac{\mu_0}{2}}(0)$, and then define

$$z_{\varepsilon,X}(y) = \psi_{\mu_0}(\varepsilon y) U(y). \tag{16}$$

We next want to apply the abstract framework in Subsection 2.1 by choosing $I_\varepsilon = J_\varepsilon$ (see (8)) and as Z_ε the following manifold

$$Z_\varepsilon = \left\{ z_{\varepsilon,X} \ : \ X \in \partial\Omega_\varepsilon \right\}. \tag{17}$$

We already discussed the role of non-degeneracy of U with respect to condition $iv)$: we next aim to show here the first part of conditions $i)$ with $a(\varepsilon) = O(\varepsilon)$, the other ones being more technical. We have the following result.

Lemma 2.1. *There exists a constant $C > 0$ such that for ε small one has the inequality*

$$\|\nabla J_\varepsilon(z_{\varepsilon,X})\| \leq C\varepsilon; \qquad\qquad \textit{for all } X \in \partial\Omega_\varepsilon.$$

Proof. Consider an arbitrary function $v \in W^{1,2}(\Omega_\varepsilon)$. Since $z_{\varepsilon,X}$ is supported in $B_{\frac{\mu_0}{2\varepsilon}}(X)$, see (16), the coordinates y are globally defined in this set, and we get

$$\nabla J_\varepsilon(z_{\varepsilon,X})[v] = \int_{\partial\Omega_\varepsilon} \frac{\partial z_{\varepsilon,X}}{\partial\tilde{\nu}} v d\sigma + \int_{\Omega_\varepsilon} \left(-\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z_{\varepsilon,X}^p \right) v dy. \tag{18}$$

Concerning the normal derivative $\frac{\partial z_{\varepsilon,X}}{\partial\tilde{\nu}}$, one has

$$\frac{\partial z_{\varepsilon,X}}{\partial\tilde{\nu}} \quad = \quad U \nabla\psi_{\mu_0}(\varepsilon y) \cdot \tilde{\nu} + \psi_{\mu_0}(\varepsilon y) \nabla U \cdot \tilde{\nu}.$$

Since $\nabla\psi_{\mu_0}(\varepsilon\cdot)$ is supported in $\mathbb{R}^N \setminus B_{\frac{\mu_0}{4\varepsilon}}$, and by properties (2)-(3), we have

$$|U \nabla\psi_{\mu_0}(\varepsilon y) \cdot \tilde{\nu}| \leq C(1 + |y|^C) e^{-\frac{1}{C\varepsilon}} e^{-|y|}.$$

On the other hand, since U has zero normal derivative on hyperplanes passing through the origin and by (11) we find that

$$\frac{\partial z_{\varepsilon,X}}{\partial\tilde{\nu}} = O(\varepsilon^2 |y'| |\nabla w|) + O(\varepsilon^2 |y'|^2 |\nabla U|); \qquad |y| \leq \frac{\mu_0}{4\varepsilon};$$

$$\left| \frac{\partial z_{\varepsilon,X}}{\partial\tilde{\nu}} \right| \leq C e^{-|y|} + \overline{C}\varepsilon(1 + |y|^C) e^{-|y|} \leq C\varepsilon^{-C} e^{-\frac{1}{C\varepsilon}}; \qquad \frac{\mu_0}{4\varepsilon} \leq |y| \leq \frac{\mu_0}{2\varepsilon}.$$

By last two bounds, formula (15), and the trace Sobolev embedding we find that

$$\left| \int_{\partial\Omega_\varepsilon} \frac{\partial z_{\varepsilon,X}}{\partial\tilde{\nu}} v d\sigma \right| \leq C\varepsilon \|v\|. \tag{19}$$

Furthermore, from (14) and the fact that U solves the equation in (9) we obtain

$$\left| -\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z_{\varepsilon,X}^p \right| \quad \leq \quad C\varepsilon^2 \left(|y'| |\nabla U| + |y'|^2 |\nabla^2 U| \right),$$

for $|y| \leq \left(\frac{1}{4\varepsilon \overline{C} \sup_X \|A_X\|} \right) \frac{1}{\overline{C}}$, and

$$\left| -\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z_{\varepsilon,X}^p \right| \quad \leq \quad C(1 + |y'|^{\overline{C}}) e^{-|y'|} \leq C\varepsilon^{-C} e^{-\frac{1}{C\varepsilon}},$$

for $\left(\frac{1}{4\varepsilon\overline{C}\sup_X \|A_X\|}\right)^{\frac{1}{\overline{C}}} \leq |y| \leq \frac{\mu_0}{2\varepsilon}$. Hence from the last two formulas we deduce that

$$\left|-\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z^p_{\varepsilon,X}\right| \leq C\varepsilon(1+|y|^C)e^{-|y|}; \qquad |y| \leq \left(\frac{1}{4\varepsilon\overline{C}\sup_X \|A_X\|}\right)^{\frac{1}{\overline{C}}},$$

which from Hölder's inequality implies

$$\left|\int_{\Omega_\varepsilon} \left(-\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z^p_{\varepsilon,X}\right) v dy\right| \leq C\varepsilon\|v\|. \tag{20}$$

From (19) and (20) we finally get the conclusion. □

With the aim of applying Proposition 1, we next expand $J_\varepsilon(z_{\varepsilon,X})$ up to the first order in ε.

Lemma 2.2. *As $\varepsilon \to 0$, the following formula holds uniformly on $\partial\Omega_\varepsilon$*

$$J_\varepsilon(z_{\varepsilon,X}) \quad = \quad C_0 - C_1\varepsilon H(X) + O(\varepsilon^2),$$

where

$$C_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N_+} U^{p+1}, \quad C_1 = \left(\int_0^\infty r^n U_r^2 dr\right)\int_{S^n_+} y_N|y'|^2 d\sigma.$$

Proof. Since z is supported in $B_{\frac{\mu_0}{2\varepsilon}}(X)$, we can still use the above coordinates y, so we can write that

$$J_\varepsilon(z_{\varepsilon,X}) = \tfrac{1}{2}\int_{\mathbb{R}^N_+} \left(|\nabla_g z_{\varepsilon,X}|^2 + z^2_{\varepsilon,X}\right) dy - \tfrac{1}{p+1}\int_{\mathbb{R}^N_+} z^{p+1}_{\varepsilon,X} dy.$$

An integration by parts yields

$$J_\varepsilon(z_{\varepsilon,X}) = \frac{1}{2}\int_{\partial\mathbb{R}^N_+} z_{\varepsilon,X}\frac{\partial z_{\varepsilon,X}}{\partial\tilde{\nu}} d\sigma + \frac{1}{2}\int_{\mathbb{R}^N_+} z_{\varepsilon,X}\left(-\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X}\right) dy$$
$$- \frac{1}{p+1}\int_{\mathbb{R}^N_+} |z_{\varepsilon,X}|^{p+1} dy.$$

Using formulas (16) and (14) we obtain

$$\frac{1}{2}\int_{\mathbb{R}^N_+} z_{\varepsilon,X}\left(-\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X}\right) dy - \frac{1}{p+1}\int_{\mathbb{R}^N_+} |z_{\varepsilon,X}|^{p+1} dy =$$
$$= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N_+} U^{p+1} dy + \frac{\varepsilon}{2}\int_{\partial\mathbb{R}^N_+} U\langle A_X y', \nabla_{y'} U\rangle d\sigma$$
$$+\varepsilon\int_{\mathbb{R}^N_+} U\langle A_X y', \nabla_{y'}\partial_{y_N} U\rangle dy + \frac{\varepsilon}{2} tr A_X \int_{\mathbb{R}^N_+} U\partial_{y_N} U dy + O(\varepsilon^2).$$

Also, from (11) we obtain that

$$\frac{1}{2}\int_{\partial\mathbb{R}^N_+} z_{\varepsilon,X}\frac{\partial z_{\varepsilon,X}}{\partial\tilde{\nu}} d\sigma = \frac{\varepsilon}{2}\int_{\partial\mathbb{R}^N_+} U\langle A_X y', \nabla_{y'} U\rangle dy + O(\varepsilon^2).$$

Collecting the above formulas we find

$$\begin{aligned} J_\varepsilon(z) \quad &= \quad \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N_+} U^{p+1} dy + \frac{\varepsilon}{2}\int_{\partial\mathbb{R}^N_+} U\langle A_X y', \nabla_{y'} U\rangle d\sigma \\ &+ \quad \varepsilon\int_{\mathbb{R}^N_+} U\langle A_X y', \nabla_{y'}\partial_{y_N} U\rangle dy + \frac{\varepsilon}{2} tr A_X \int_{\mathbb{R}^N_+} U\partial_{y_N} U d\sigma + O(\varepsilon^2). \end{aligned}$$

A further integration by parts shows that the terms of order ε are given by

$$\begin{array}{rl} & \frac{1}{4}\int_{\partial\mathbb{R}^N_+}\langle A_X y', \nabla_{y'}U^2\rangle d\sigma + \int_{\mathbb{R}^N_+} U\langle A_X y', \nabla_{y'}\partial_{y_N}U\rangle dy + \frac{1}{4} tr A_X \int_{\mathbb{R}^N_+}\partial_{y_N}U^2 dy \\ = & -\frac{1}{2} tr A_X \int_{\partial\mathbb{R}^N_+} U^2 d\sigma - \int_{\partial\mathbb{R}^N_+} U\langle A_X y', \nabla_{y'}U\rangle d\sigma - \int_{\mathbb{R}^N_+}\partial_{y_N}U\langle A_X y', \nabla_{y'}U\rangle dy \\ = & -\int_{\mathbb{R}^N_+}\partial_{y_N}U\langle A_X y', \nabla_{y'}U\rangle dy. \end{array}$$

Since U is radial, we have that

$$\partial_{y_N}U = \frac{y_N}{|y|}U_r; \qquad \nabla_{y'}U = \frac{y'}{|y|},$$

and therefore

$$\int_{\mathbb{R}^N_+}\partial_{y_N}U\langle A_X(y'), \nabla_{y'}U\rangle d\sigma = -\int_{\mathbb{R}^N_+}\frac{y_N\langle A_X(y'), y'\rangle}{|y|^2}dy.$$

Expressing the integral in radial coordinates, we obtain the conclusion. □

The latter result allows to expand the finite-dimensional functional in (6). In fact, from the regularity of J_ε and from the fact that by Lemma 2.1 and by Proposition 1 $\|\omega_\varepsilon(z)\| = O(\varepsilon)$, we have that

$$\begin{aligned} \mathbf{I}_\varepsilon(z) = I_\varepsilon(z+\omega_\varepsilon(z)) &= J_\varepsilon(z_{\varepsilon,X}) + \|\nabla J_\varepsilon(z_{\varepsilon,X})\|\,\|\omega_\varepsilon(z_{\varepsilon,X})\| \\ &+ O(\|\omega_\varepsilon(z_{\varepsilon,X})\|^2) = J_\varepsilon(z_{\varepsilon,X}) + O(\varepsilon^2). \end{aligned}$$

As a consequence we obtain the following:

Proposition 3. *Let Z_ε be as in (17) and let $I_\varepsilon = J_\varepsilon$. Let $\mathbf{I}_\varepsilon(z)$ be as in (6). Then*

$$\mathbf{I}_\varepsilon(z) = C_0 - C_1\varepsilon H(X) + O(\varepsilon^2); \qquad z \in Z_\varepsilon,$$

with C_0, C_1 as in Lemma 2.2.

A similar result holds for the expansion of the derivatives of $\mathbf{I}_\varepsilon$ in terms of the gradient of the mean curvature of Ω. Using a direct maximization (resp., minimization) argument, or a local degree computation one finds the following result.

Theorem 2.3. *Let $p < \frac{N+2}{N-2}$ and suppose P is a strict local minimum (resp., maximum) or a non-degenerate critical point for the mean curvature H of $\partial\Omega$. Then there exist spike-layers u_ε of (P_ε) concentrating at P for $\varepsilon \to 0$.*

As discussed in the introduction, the papers [49], [50] studied the limiting behaviour of solutions with minimal energy. Once it is proven that, after a proper translation and dilation in ε the limiting profile is at the boundary and converges to the radial solution U, it is intuitive from the above proposition that minimality in energy corresponds to maximality of boundary mean curvature. Therefore, from the second part of Proposition 2 one can then show also the following result.

Theorem 2.4. *([50]) Let $p < \frac{N+2}{N-2}$. Then solutions of (P_ε) with minimal energy form, as $\varepsilon \to 0$, spike-layers concentrating at boundary points of Ω with maximal mean curvature.*

As again discussed in the introduction, a variant of the above finite-dimensional reduction allows to find solutions with multiple boundary peaks, concentrating at suitable stationary points of the mean curvature.

2.3. **Concentration at points for the Dirichlet problem.** We consider next the singularly-perturbed Dirichlet problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{D_ε}$$

Our goal is to apply again the abstract method in Subsection 2.1 starting with approximate solutions that are dilations (by a factor ε) of the radial soliton U, and centered at interior points Q of the domain.

We need though to achieve boundary conditions, so these approximate solutions need to be suitably adjusted near the boundary, which is possible to the exponential decay of U. However a generic cut-off function will not be precise enough, and it is useful to consider a *projection operator* which associates to each $u \in H^1(\Omega)$ its closest element (w.r.t. the Sobolev distance) in $H^1_0(\Omega)$. This amounts to subtracting to such a function u the solution of

$$\begin{cases} -\varepsilon^2 \Delta \varphi + \varphi = 0 & \text{in } \Omega; \\ \text{trace}(\varphi) = \text{trace}(u) & \text{on } \partial\Omega. \end{cases}$$

We choose $u = U\left(\frac{x-Q}{\varepsilon}\right)$ for $Q \in \Omega$, and we will need to determine some asymptotic behaviour of φ as $\varepsilon \to 0$. By (2), the trace of u behaves like $e^{-\frac{|x-Q|}{\varepsilon}}$.

It is convenient now to make a change of variables: setting $\psi = -\varepsilon \log \varphi$, one finds that it satisfies

$$\begin{cases} \varepsilon \Delta \psi - |\nabla \psi|^2 + 1 = 0 & \text{in } \Omega; \\ \psi(x) = -\varepsilon \log U\left(\frac{x-Q}{\varepsilon}\right). \end{cases} \tag{21}$$

By the asymptotic behaviour of U at infinity, one has that

$$-\varepsilon \log U\left(\frac{x-Q}{\varepsilon}\right) \to |x-Q| \qquad \text{as } \varepsilon \to 0.$$

Using a barrier argument it was shown in [52] that the above functions ψ are uniformly Lipschitz as $\varepsilon \to 0$. Moreover, it is possible to prove that that their limit, guaranteed by Ascoli's theorem, is a Lipschitz function that can explicitly characterized as follows.

Proposition 4. *([52]) Let $\psi = \psi_\varepsilon$ be the solutions to the above boundary value problem. Then, as $\varepsilon \to 0$, ψ_ε converge uniformly in Ω to a Lipschitz function ψ_0 which is defined as*

$$\psi_0(x) = \inf_{P \in \partial\Omega} \left(|P-Q| + L(P,x) \right); \qquad x \in \Omega.$$

Here $L(x,y)$ stands for the infimum of the numbers T such that there exists $\zeta(s) \in C^{0,1}([0,T]; \bar{\Omega})$ with $\zeta(0) = x$, $\zeta(T) = y$ and $|\frac{d\zeta}{ds}| \leq 1$ a.e. on $[0,T]$.

Taking straight curves from Q to its closest point to the boundary, one obtains the following result.

Corollary 1. *If ψ_0 is as above, one has that*

$$\psi_0(Q) = 2d(Q, \partial\Omega).$$

The above results can be used to generate good approximate solutions. We first scale the boundary as in the previous subsection, and consider the equivalent problem

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon; \end{cases} \qquad \Omega_\varepsilon = \frac{1}{\varepsilon}\Omega. \tag{22}$$

For $Q \in \Omega_\varepsilon$, define

$$u^D_{Q,\varepsilon} = U(x - Q) - \psi_{\varepsilon,Q}(\varepsilon x).$$

By construction the above function $u^D_{Q,\varepsilon}$ satisfies the Dirichlet boundary conditions on Ω_ε. We will next give an idea of the fact that $u^D_{Q,\varepsilon}$ is a good approximate solution for the Dirichlet problem in the following sense. Consider the Euler-Lagrange energy for (22)

$$\hat{I}_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(|\nabla u|^2 + u^2\right) dy - \frac{1}{p+1}\int_{\Omega_\varepsilon} u^{p+1} dy.$$

We have then the following result.

Lemma 2.5. *Suppose $u^D_{Q,\varepsilon}$ is as before, and that Q belongs to the ε-dilation of a fixed compact set of Ω. Then one has*

$$\|\nabla \hat{I}_\varepsilon(u^D_{Q,\varepsilon})\| \leq C e^{-\min\{2,p\} d(Q,\partial\Omega_\varepsilon)}; \qquad u \in H^1_0(\Omega_\varepsilon)$$

Proof. We only give a sketch of the proof, referring to papers mentioned below for full details. Consider any test function $v \in H^1_0(\Omega_\varepsilon)$: then integrating by parts and using the fact that $u^D_{Q,\varepsilon}$ satisfies

$$-\Delta u^D_{Q,\varepsilon} + u^D_{Q,\varepsilon} = U^p_Q,$$

we have that

$$\begin{aligned} \nabla \hat{I}_\varepsilon(u^D_{Q,\varepsilon})[v] &= \int_{\Omega_\varepsilon} \left(\nabla u^D_{Q,\varepsilon} \cdot \nabla v + u^D_{Q,\varepsilon} v\right) dx - \int_{\Omega_\varepsilon} (u^D_{Q,\varepsilon})^p v dx \\ &= \int_{\Omega_\varepsilon} \left(-\Delta u^D_{Q,\varepsilon} + u^D_{Q,\varepsilon} - (u^D_{Q,\varepsilon})^p v\right) v\, dx \\ &= \int_{\Omega_\varepsilon} \Big(U^p_Q - (U_Q - \psi_{\varepsilon,Q}(\varepsilon x))^p\Big)\, v\, dx. \end{aligned} \tag{23}$$

By construction, it turns out that $|\psi_{\varepsilon,Q}(\varepsilon x)| \leq C U_Q$, hence from a Taylor expansion one has

$$(U_Q - \psi_{\varepsilon,Q}(\varepsilon x))^p = U^p_Q - p U^{p-1}_Q \psi_{\varepsilon,Q}(\varepsilon x) + O(U^{p-2}_Q \psi_{\varepsilon,Q}(\varepsilon x)^2). \tag{24}$$

Therefore from the last two formulas it follows that

$$\nabla \hat{I}_\varepsilon(u^D_{Q,\varepsilon})[v] = \int_{\Omega_\varepsilon} \Big(p U^{p-1}_Q \psi_{\varepsilon,Q}(\varepsilon x) + O(U^{p-2}_Q \psi_{\varepsilon,Q}(\varepsilon x)^2)\Big)\, v\, dx$$

Using then Hölder's inequality and the decay properties of U_Q and $\psi_{\varepsilon,Q}$, the conclusion follows. □

We have then the following energy expansion (where we neglect the power-like terms in (2)).

Proposition 5. *Suppose that Q belongs to the ε-dilation of a fixed compact set of Ω. The following asymptotic expansion holds:*

$$\hat{I}_\varepsilon(u^D_{Q,\varepsilon}) = C_2 + C_3 e^{-2d(Q,\partial\Omega_\varepsilon)} + l.o.t..$$

for some $C_2, C_3 \in \mathbb{R}, C_2, C_3 > 0$.

Proof. We again give a sketch of the argument, referring to [32] for full details. Integrating again by parts we write that

$$\begin{aligned}\hat{I}_\varepsilon(u^D_{Q,\varepsilon}) &= \frac{1}{2}\int_{\Omega_\varepsilon}\left(|\nabla u^D_{Q,\varepsilon}|^2 + (u^D_{Q,\varepsilon})^2\right)dx - \frac{1}{p+1}\int_{\Omega_\varepsilon}(u^D_{Q,\varepsilon})^{p+1}dx \\ &= \frac{1}{2}\int_{\Omega_\varepsilon}\left(-\Delta u^D_{Q,\varepsilon} + u^D_{Q,\varepsilon} - (u^D_{Q,\varepsilon})^p\right)u^D_{Q,\varepsilon}dx \\ &+ \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon}(u^D_{Q,\varepsilon})^{p+1}dx.\end{aligned} \tag{25}$$

Using the equation satisfied by $u^D_{Q,\varepsilon}$ we then get

$$\begin{aligned}\hat{I}_\varepsilon(u^D_{Q,\varepsilon}) &= \frac{1}{2}\int_{\Omega_\varepsilon}\left(U^p_Q - (U_Q - \psi_{\varepsilon,Q}(\varepsilon x))^p\right)(U_Q - \psi_{\varepsilon,Q}(\varepsilon x))\,dx \\ &+ \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon}(U_Q - \psi_{\varepsilon,Q}(\varepsilon x))^{p+1}dx.\end{aligned} \tag{26}$$

Fro the first term we can use formula (24), together with the analogous expansion

$$(U_Q - \psi_{\varepsilon,Q}(\varepsilon x))^{(p+1)} = U^p_Q - (p+1)U^p_Q\psi_{\varepsilon,Q}(\varepsilon x) + O(U^{p-1}_Q\psi_{\varepsilon,Q}(\varepsilon x)^2) \tag{27}$$

to write that

$$\begin{aligned}&\hat{I}_\varepsilon(u^D_{Q,\varepsilon}) \\ &= \frac{1}{2}\int_{\Omega_\varepsilon}\left(pU^{p-1}_Q\psi_{\varepsilon,Q}(\varepsilon x) + O(U^{p-2}_Q\psi_{\varepsilon,Q}(\varepsilon x)^2)\right)(U_Q - \psi_{\varepsilon,Q}(\varepsilon x))\,dx \\ &+ \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon}(U^{p+1}_Q - (p+1)U^p_Q\psi_{\varepsilon,Q}(\varepsilon x) + O(U^{p-1}_Q\psi_{\varepsilon,Q}(\varepsilon x)^2))dx.\end{aligned} \tag{28}$$

Collecting all terms, from the decay of U_Q and $\psi_{\varepsilon,Q}$ one finds that

$$\hat{I}_\varepsilon(u^D_{Q,\varepsilon}) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon}U^{p+1}_Q dx + \frac{1}{2}\int_{\Omega_\varepsilon}U^p_Q\psi_{\varepsilon,Q}(\varepsilon x)dx + l.o.t.$$

For the first term, from the exponential decay of U one has

$$\int_{\Omega_\varepsilon}U^{p+1}_Q dx = \int_{\mathbb{R}^N}U^{p+1}dx - \int_{\mathbb{R}^N\setminus\Omega_\varepsilon}U^{p+1}_Q dx = C_0 + O(e^{-(p+1)d(Q,\partial\Omega_\varepsilon)})$$

For the second term instead, from the decay of U and Corollary XX one has that

$$\int_{\Omega_\varepsilon}U^p_Q\psi_{\varepsilon,Q}(\varepsilon x)dx \simeq \frac{1}{2}\psi_{\varepsilon,Q}(\varepsilon Q)\int_{\mathbb{R}^N}U^p dx + l.o.t. = C_1 e^{-2d(Q,\partial\Omega_\varepsilon)} + l.o.t..$$

This concludes the proof. □

Similarly to Proposition 3, we obtain the following expansion.

Proposition 6. *Fix a compact set K in Ω, define $Z_\varepsilon = \left\{u^D_{Q,\varepsilon} : Q \in \frac{1}{\varepsilon}K\right\}$, and let $I_\varepsilon = \hat{I}_\varepsilon$. Let $\mathbf{I}_\varepsilon(z)$ be as in (6). Then, if $z = u^D_{Q,\varepsilon}$ one has that*

$$\mathbf{I}_\varepsilon(z) = C_2 + C_3 e^{-2d(Q,\partial\Omega_\varepsilon)} + l.o.t.; \qquad z \in Z_\varepsilon,$$

for some positive constants C_2, C_3*.*

Using this proposition and the above abstract arguments, it is possible to prove results of the following type.

Theorem 2.6. *([32]) Let* $p < \frac{N+2}{N-2}$ *and let* $V \subset \Omega$ *be an open set with compact closure in* Ω *and let* d *denote the distance function from* $\partial\Omega$*. Suppose that*

$$deg(d, V, 0) \neq 0.$$

Then as $\varepsilon \to 0$ *problem* (D_ε) *admits spike-layer solutions concentrating at some point in* V*.*

As for Theorem 2.4, the following result for the Dirichlet problem was proved, regarding solutions with minimal energy.

Theorem 2.7. *([52]) Let* $p < \frac{N+2}{N-2}$*. Then solutions of* (D_ε) *with minimal energy form, as* $\varepsilon \to 0$*, spike-layers concentrating at interior points of* Ω *with maximal distance from the boundary.*

Expansions similar to the ones discussed in this subsection were used to construct interior spikes for (P_ε) as well, and solutions with multiple spike-layers, even of mixed interior and boundary types. We refer to the introduction for more precise references.

3. Concentration at spheres in symmetric domains.

Here we consider again problem (P_ε) for the unit ball $\Omega = B_1 = \{x \in \mathbb{R}^N \ : \ |x| < 1\}$, $N \geq 2$, showing the existence of radial solutions concentrating near the boundary, but with the profile of interior one-dimensional spike-layers. The phenomenon is peculiar of the higher-dimensional case and is due to a balancing effect between the *volume energy* of radial spike-layers, which would tend to shrink their radius, and an *attractive force* due to the imposed boundary condition: there are indeed no such solutions in one dimension.

It is convenient to scale the domain by a factor $\frac{1}{\varepsilon}$, i.e. to consider

$$\begin{cases} -\Delta u + u = u^p, & \text{in } B_{\frac{1}{\varepsilon}}, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_{\frac{1}{\varepsilon}}, & u > 0. \end{cases} \tag{29}$$

and to use the functional I_ε defined in (4).

We next construct a family of approximate solutions to (29), imposing approximate Neumann boundary conditions. Given $r_0 < \frac{1}{2}$, let $\phi_\varepsilon(r)$ be a smooth cutoff function satisfying

$$\phi_\varepsilon(r) = \begin{cases} 0 & \text{for } r \in \left[0, \frac{r_0}{8\varepsilon}\right]; \\ 1 & \text{for } r \in \left[\frac{r_0}{4\varepsilon}, \frac{1}{\varepsilon}\right]; \\ |\phi_\varepsilon'(r)| \leq C\varepsilon & \text{for } r \in \left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]; \\ |\phi_\varepsilon''(r)| \leq C\varepsilon^2 & \text{for } r \in \left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]. \end{cases} \tag{30}$$

Consider the one-dimensional solution $\overline{U}$ to

$$-\overline{U}'' + \overline{U} = \overline{U}^p \qquad \text{in } \mathbb{R}. \tag{31}$$

Let $\overline{\alpha} = \lim_{t\to+\infty} e^t \overline{U}(t)$, recalling (2), and let $z_\rho(r) = \overline{U}(r-\rho)$: define then

$$z_\rho^{\mathcal{N}} = \phi_\varepsilon\left(z_\rho + v_\rho\right) := \phi_\varepsilon(\cdot)\left(z_\rho(\cdot) + \overline{\alpha}e^{-\left(\frac{1}{\varepsilon}-\rho\right)}e^{-\left(\frac{1}{\varepsilon}-\cdot\right)}\right); \qquad \rho \geq \frac{3}{4\varepsilon}. \tag{32}$$

For the normal derivative, we have the following estimate

$$\begin{aligned}(z_\rho^{\mathcal{N}})'\left(\frac{1}{\varepsilon}\right) &= z_\rho'\left(\frac{1}{\varepsilon}\right) - \overline{\alpha} e^{-\left(\frac{1}{\varepsilon}-\rho\right)} \\ &= z_\rho\left(\frac{1}{\varepsilon}\right)\left(\frac{z_\rho'}{z_\rho}\left(\frac{1}{\varepsilon}\right) - \frac{\overline{\alpha} e^{-\left(\frac{1}{\varepsilon}-\rho\right)}}{z_\rho\left(\frac{1}{\varepsilon}\right)}\right) = o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right).\end{aligned} \tag{33}$$

The term v_ρ in the definition of $z_\rho^{\mathcal{N}}$ can be heuristically viewed as a *virtual spike* outside Ω, which has the effect of *attracting* the interior spike to the boundary.

We have next the following result concerning approximate solutions.

Lemma 3.1. *Then there exists $C > 0$ such that, testing on radial functions*

$$\|\nabla I_\varepsilon(z_\rho^{\mathcal{N}})\| \leq C\varepsilon^{\frac{1-N}{2}}\left(\varepsilon + o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right)\right) \text{ for every } z_\rho^{\mathcal{N}} \text{ as in (32).}$$

Proof. As $z_\rho = \overline{U}(\cdot - \rho)$ and v_ρ satisfy $-z_\rho'' + z_\rho = z_\rho^p$ and $-v_\rho'' + v_\rho = 0$, for arbitrary radial functions $u \in H_r^1(B_{\frac{1}{\varepsilon}})$ there holds

$$\begin{aligned} I_\varepsilon'(z^{\mathcal{N}})[u] &= \int_0^{\frac{1}{\varepsilon}} \left(-(z_\rho^{\mathcal{N}})'' - \frac{N-1}{r}(z_\rho^{\mathcal{N}})' + V(\varepsilon r) z_\rho^{\mathcal{N}} - (z_\rho^{\mathcal{N}})^p\right) u r^{N-1} dr \\ &+ \varepsilon^{1-N}(z_\rho^{\mathcal{N}})'(1/\varepsilon) u(1/\varepsilon) \\ &= \varepsilon^{1-N}(z_\rho^{\mathcal{N}})'(1/\varepsilon) u(1/\varepsilon) - (N-1)\int_0^{\frac{1}{\varepsilon}} \frac{1}{r}(z_\rho^{\mathcal{N}})' u r^{N-1} dr \\ &- \int_0^{\frac{1}{\varepsilon}} \left(2\phi_\varepsilon'(z_\rho^{\mathcal{N}})' + \phi_\varepsilon''(z_\rho^{\mathcal{N}})\right) u r^{N-1} dr - \int_0^{\frac{1}{\varepsilon}} \left((z_\rho^{\mathcal{N}})^p - \phi_\varepsilon z_\rho^p\right) u r^{N-1} dr. \end{aligned}$$

For brevity, we might omit next the index ρ in z_ρ and v_ρ and for simplicity we will write

$$\int (\cdot) := \int_0^{\frac{1}{\varepsilon}} (\cdot) r^{N-1} dr. \tag{34}$$

From Strauss' Lemma, see [56], and (33) we obtain that

$$\varepsilon^{1-N}|(z^{\mathcal{N}})'(1/\varepsilon) u(1/\varepsilon)| = \varepsilon^{\frac{1-N}{2}} o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right)\|u\|. \tag{35}$$

It is easy to check that $\|(z^{\mathcal{N}})'\| \leq C\varepsilon^{\frac{1-N}{2}}$ and moreover, since $z^{\mathcal{N}}$ is supported in $\left\{r \geq \frac{r_0}{8\varepsilon}\right\}$, one also has

$$\left|\int \frac{1}{r}(z^{\mathcal{N}})' u\right| \leq C\varepsilon \|(z^{\mathcal{N}})'\|\|u\| \leq C\varepsilon^{\frac{3-N}{2}}\|u\|. \tag{36}$$

By the exponential decays of $z = z_\rho$ and $v = v_\rho$, the fact that $\phi_\varepsilon', \phi_\varepsilon''$ are supported in $\left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]$ and from the condition $\rho \geq \frac{3}{4\varepsilon}$, one finds

$$\left|\int \phi_\varepsilon'(z+v)' u\right| \leq C\varepsilon^{1+\frac{1-N}{2}} e^{-\frac{r_0}{4\varepsilon}}\|u\|; \qquad \left|\int \phi_\varepsilon''(z+v) u\right| \leq C\varepsilon^{2+\frac{1-N}{2}} e^{-\frac{r_0}{4\varepsilon}}\|u\|. \tag{37}$$

Let us consider now $\int \left((z^{\mathcal{N}})^p - \phi_\varepsilon z^p\right) u$, noticing that

$$(z^{\mathcal{N}})^p - \phi_\varepsilon z^p = \phi_\varepsilon^p\left((z+v)^p - z^p\right) + \phi_\varepsilon^p\left(\phi_\varepsilon^p z^p - \phi_\varepsilon z^p\right).$$

Since z is uniformly bounded in L^∞ we find that

$$\left|(z+v)^p - z^p - p z^{p-1} v\right| \leq C \max\{|v|^2, |v|^p\}.$$

As a consequence

$$\left|\int \left[(z+v)^p - z^p\right] u\right| \leq p \left|\int z^{p-1} v|u|\right| + C \left|\int |u| \max\{|v|^2, |v|^p\}\right|.$$

From Hölder's inequality we obtain

$$\left|\int |v|^{2\wedge p}|u|\right| \leq C e^{-(2\wedge p)\left(\frac{1}{\varepsilon}-\rho\right)} \int e^{-(2\wedge p)\left(\frac{1}{\varepsilon}-r\right)}|u| \leq C e^{-(2\wedge p)\left(\frac{1}{\varepsilon}-\rho\right)} \varepsilon^{\frac{1-N}{2}} \|u\|,$$

and we notice that also $\left|\int z^{p-1} v|u|\right| \leq \left(\int z^{2(p-1)} v^2\right)^{\frac{1}{2}} \|u\|$. For the latter integral we consider separately the sets $r \leq \frac{\rho+\varepsilon^{-1}}{2}$ and $r \geq \frac{\rho+\varepsilon^{-1}}{2}$. For $r \leq \frac{\rho+\varepsilon^{-1}}{2}$, v satifies $|v| \leq e^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)}$ and therefore

$$\begin{aligned} \left(\int_{r\leq\frac{\rho+\varepsilon}{2}} z^{2(p-1)} v^2 r^{N-1} dr\right)^{\frac{1}{2}} &\leq C e^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)} \left(\int_{r\leq\frac{\rho+\varepsilon}{2}} z^{2(p-1)} r^{N-1} dr\right)^{\frac{1}{2}} \\ &\leq C e^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)} \varepsilon^{\frac{1-N}{2}}. \end{aligned}$$

If instead $r \geq \frac{\rho+\varepsilon^{-1}}{2}$, z satisfies $|z(r)| \leq e^{-\frac{1}{2}\left(\frac{1}{\varepsilon}-\rho\right)}$ so one finds

$$\begin{aligned} \left(\int_{r\geq\frac{\rho+\varepsilon}{2}} z^{2(p-1)} v^2 r^{N-1} dr\right)^{\frac{1}{2}} &\leq C e^{-\frac{p-1}{2}\left(\frac{1}{\varepsilon}-\rho\right)} \left(\int_0^{\frac{1}{\varepsilon}} |v|^2 r^{N-1} dr\right)^{\frac{1}{2}} \\ &\leq C e^{-\left(1+\frac{p-1}{2}\right)\left(\frac{1}{\varepsilon}-\rho\right)} \varepsilon^{\frac{1-N}{2}}. \end{aligned}$$

Notice that also

$$\left|\int \left(\phi_\varepsilon^p z^p - \phi_\varepsilon z^p\right) u\right| \leq C \left(\int \left(\phi_\varepsilon^p - \phi_\varepsilon\right)^2 \overline{U}^{2p}\right)^{\frac{1}{2}} \|u\| \leq C e^{-\frac{p r_0}{2\varepsilon}} \varepsilon^{\frac{1-N}{2}} \|u\|.$$

All the above inequalities yield

$$\left|\int \left((z^{\mathcal{N}})^p - \phi_\varepsilon \overline{u}^p\right) u\right| \leq C \varepsilon^{\frac{1-N}{2}} \left(e^{-\left(\frac{3\wedge(p+1)}{2}\right)\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-\frac{p r_0}{4\varepsilon}}\right) \|u\|. \tag{38}$$

Therefore (35)-(38) guarantee that

$$\|I_\varepsilon'(z_\rho^{\mathcal{N}})\| \leq C \varepsilon^{\frac{1-N}{2}} \left(\varepsilon + o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right) + e^{-\frac{r_0}{4\varepsilon}}\right),$$

concluding the proof. □

Even though the norm of the gradient in Lemma 3.1 is not small in ε, it is small relatively to that of the $z_\rho^{\mathcal{N}}$'s. It is possible then to perform a contraction argument as in the previous sections, working in the set of radial functions

$$\widetilde{\mathcal{C}}_\varepsilon = \left\{ w \in H_r^1(B_{\frac{1}{\varepsilon}}) \ : \ \|w\|_{H_r^1(B_{\frac{1}{\varepsilon}})} \leq \gamma\varepsilon \|z_\rho^{\mathcal{N}}\|_{H_r^1(B_{\frac{1}{\varepsilon}})}, \ |w(r)| \leq \gamma\varepsilon \text{ for } r > 0 \right\}.$$

One can then prove the following result (see [2]) for complete details.

Proposition 7. *For ε small there exists $\mu > 0$ such that for $\rho \in \left[\frac{r_0}{\varepsilon}, \frac{1}{\varepsilon} - \mu\right]$, there exists a function $w^{\mathcal{N}} = w^{\mathcal{N}}(z_{\rho,\varepsilon}) \in \widetilde{\mathcal{C}}_\varepsilon$ with the following property. Set*

$$\mathbf{I}_\varepsilon(\rho) = I_\varepsilon(z_\rho^{\mathcal{N}} + w_{\rho,\varepsilon}^{\mathcal{N}}) :$$

if ρ_ε is stationary point of $\mathbf{I}_\varepsilon$, then $\widetilde{u}_\varepsilon = z_{\rho_\varepsilon}^{\mathcal{N}} + w_{\rho_\varepsilon,\varepsilon}^{\mathcal{N}}$ is a critical point of I_ε.

The reduced functional $\mathbf{I}_\varepsilon$ can then be expanded as follows.

Proposition 8. *Let $z_\rho^{\mathcal{N}}$ be defined in* (32)*, and set*

$$\alpha = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) \int_{\mathbb{R}} \overline{U}^{p+1}; \qquad\qquad \beta = \tfrac{1}{2}\overline{\alpha} \int_{\mathbb{R}} \overline{U}^{p} e^{r}. \tag{39}$$

Then for all $\rho \in \left[\frac{3}{4\varepsilon}, \frac{1}{\varepsilon}\right]$ one has

$$\mathbf{I}_\varepsilon(\rho) \quad = \quad \varepsilon^{1-N}(\varepsilon\rho)^{N-1}\left[\alpha - \beta e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right] + O(\varepsilon^{2-N}) + \varepsilon^{1-N} o\left(e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right).$$

Proof. It will be sufficient to estimate $I_\varepsilon(z_\rho^{\mathcal{N}})$ since the contribution of $w_{\rho,\varepsilon}^{\mathcal{N}}$ will be negligible, as for the previous cases. Integrating by parts we obtain

$$\begin{aligned}
I_\varepsilon(z^{\mathcal{N}}) =& \tfrac{1}{2}\int \left(|(z^{\mathcal{N}})'|^2 + (z^{\mathcal{N}})^2\right) - \tfrac{1}{p+1}\int |z^{\mathcal{N}}|^{p+1} \\
&= \tfrac{1}{2}\int \left(-(z^{\mathcal{N}})'' - \tfrac{N-1}{r}(z^{\mathcal{N}})' + z^{\mathcal{N}}\right) z^{\mathcal{N}} \\
&+ \tfrac{1}{2}\varepsilon^{1-N} z^{\mathcal{N}}(1/\varepsilon)(z^{\mathcal{N}})'(1/\varepsilon) - \tfrac{1}{p+1}\int |z^{\mathcal{N}}|^{p+1} \\
&= \tfrac{1}{2}\varepsilon^{1-N} z^{\mathcal{N}}(1/\varepsilon)(z^{\mathcal{N}})'(1/\varepsilon) + \tfrac{1}{2}\int \phi_\varepsilon z^p z^{\mathcal{N}} - \tfrac{1}{p+1}\int |z^{\mathcal{N}}|^{p+1} \\
&- \tfrac{N-1}{2}\int \frac{(z^{\mathcal{N}})' z^{\mathcal{N}}}{r} - \int \phi_\varepsilon' z^{\mathcal{N}}(z+v)' - \tfrac{1}{2}\int \phi_\varepsilon'' z^{\mathcal{N}}(z+v).
\end{aligned} \tag{40}$$

We next estimate each term separately. By (33) we get

$$\varepsilon^{1-N}\left|z^{\mathcal{N}}(1/\varepsilon)(z^{\mathcal{N}})'(1/\varepsilon)\right| = \varepsilon^{1-N} o\left(e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right). \tag{41}$$

To control the second and the third terms in the r.h.s. of (40), we can write

$$\begin{aligned}
\tfrac{1}{2}\int \phi_\varepsilon z^p z^{\mathcal{N}} - \tfrac{1}{p+1}\int |z^{\mathcal{N}}|^{p+1} &= \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right)\int \phi_\varepsilon^{p+1} z^{p+1} \\
&+ \tfrac{1}{2}\int \left(\phi_\varepsilon^2 - \phi_\varepsilon^{p+1}\right) z^p(z+v) - \tfrac{1}{2}\int \phi_\varepsilon^{p+1} z^p v \\
&- \tfrac{1}{p+1}\int \phi_\varepsilon^{p+1}\left(|z+v|^{p+1} - z^{p+1} - (p+1)z^p v\right).
\end{aligned} \tag{42}$$

There holds

$$\begin{aligned}
&\left|\int \phi_\varepsilon^{p+1} z^{p+1} - \rho^{N-1}\int_{\mathbb{R}} \overline{U}^{p+1} dr\right| \\
&\le \rho^{N-1}\int_{r\ge 1/\varepsilon} \overline{U}^{p+1}(r-\rho)dr + \int (1-\phi_\varepsilon^{p+1}) z^{p+1} \\
&+ \left|\int_0^{\frac{1}{\varepsilon}} (r^{N-1} - \rho^{N-1})\overline{U}^{p+1}(r-\rho)dr\right|.
\end{aligned}$$

Taylor expanding $r^{N-1} - \rho^{N-1}$ and using $r \le C(r_0)\rho$ (by $\rho \ge r_0/\varepsilon$), we find

$$\begin{aligned}
&\left|\int_0^{\frac{1}{\varepsilon}} (r^{N-1} - \rho^{N-1})\overline{U}^{p+1}(r-\rho)dr\right| \\
&\le C(n, r_0)\rho^{N-2}\int_0^{\frac{1}{\varepsilon}} |r-\rho|\overline{U}^{p+1}(r-\rho)dr \le C\rho^{N-2}.
\end{aligned}$$

By the exponential decay of $\overline{U}$, we obtain

$$\begin{aligned}\rho^{N-1}\int_{r\geq 1/\varepsilon}\overline{U}^{p+1}(r-\rho)dr &\leq C\varepsilon^{1-N}\left(e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)}+e^{-\frac{(p+1)r_0}{4\varepsilon}}\right);\\ \int_0^{\frac{1}{\varepsilon}} r^{N-1}(1-\phi_\varepsilon^{p+1})\overline{U}^{p+1} &\leq C\varepsilon^{1-N}e^{-\frac{(p+1)r_0}{4\varepsilon}}.\end{aligned}$$

From the last three formulas we get

$$\left|\int \phi_\varepsilon^{p+1}z^{p+1}-\rho^{N-1}\int_{\mathbb{R}}\overline{U}^{p+1}dr\right|\leq C\varepsilon^{1-N}\left(e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)}+\varepsilon\right). \tag{43}$$

The term $\int \phi_\varepsilon^{p+1}\left(|z+v|^{p+1}-z^{p+1}-(p+1)z^pv\right)$ in (42) can be estimated in the following way: from

$$\left||z+v|^{p+1}-z^{p+1}-(p+1)z^pv-p(p+1)z^{p-1}v^2\right|\leq C\max\{|v|^3,|v|^{p+1}\},$$

we get

$$\int\left||z+v|^{p+1}-z^{p+1}-(p+1)z^pv\right|\leq C\int z^{p-1}v^2+C\int\max\{|v|^3,|v|^{p+1}\}.$$

The first term in the r.h.s. can be controlled considering separately the sets $\{r\leq\frac{\rho+\varepsilon^{-1}}{2}\}$ and $\{r\geq\frac{\rho+\varepsilon^{-1}}{2}\}$, as before, while for the second it is sufficient to use the explicit expression of v. We then get

$$\begin{aligned}&\left|\int \phi_\varepsilon^{p+1}\left(|z+v|^{p+1}-z^{p+1}-(p+1)z^pv\right)\right|\\ &\leq C\varepsilon^{1-N}\left(e^{-3\left(\frac{1}{\varepsilon}-\rho\right)}+e^{-\frac{(p+3)}{2}\left(\frac{1}{\varepsilon}-\rho\right)}+e^{-(3\wedge(p+1))\left(\frac{1}{\varepsilon}-\rho\right)}\right).\end{aligned} \tag{44}$$

The term $\int\phi_\varepsilon^{p+1}z^pv$ in (42) is of order $\varepsilon^{1-N}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}$. We also have

$$\begin{aligned}\int\phi_\varepsilon^{p+1}z^pv&=\overline{\alpha}\rho^{N-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\int_{\mathbb{R}}\overline{U}^pe^rdr\\ &-\overline{\alpha}\rho^{N-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\int_{r\geq 1/\varepsilon}\overline{U}^p(r-\rho)e^{(r-\rho)}dr\\ &+\int_0^{\frac{1}{\varepsilon}}(r^{N-1}-\rho^{N-1})z^pv+\int(\phi_\varepsilon^{p+1}-1)z^pv.\end{aligned}$$

As before, we find

$$\begin{aligned}\rho^{N-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\int_{r\geq 1/\varepsilon}\overline{U}^p(r-\rho)e^{(r-\rho)}dr &\leq C\varepsilon^{1-N}e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)};\\ \left|\int_0^{\frac{1}{\varepsilon}}(r^{N-1}-\rho^{N-1})z^pvdr\right| &\leq C\varepsilon^{2-N}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)};\\ \int(1-\phi_\varepsilon^{p+1})z^pv &\leq C\varepsilon^{1-N}e^{-\frac{(p+1)r_0}{4\varepsilon}}.\end{aligned}$$

The last formulas imply

$$\begin{aligned}
&\int \phi_\varepsilon^{p+1} z^p v \\
&= \overline{\alpha}\rho^{N-1} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \int_{\mathbb{R}} \overline{U}^p e^r dr + \varepsilon^{1-N} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} O\left(\varepsilon + e^{-(p-1)\left(\frac{1}{\varepsilon}-\rho\right)}\right) \\
&= \overline{\alpha}\varepsilon^{1-N} (\varepsilon\rho)^{N-1} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} \int_{\mathbb{R}} \overline{U}^p e^r dr + \varepsilon^{1-N} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)} O\left(\varepsilon + e^{-(p-1)\left(\frac{1}{\varepsilon}-\rho\right)}\right),
\end{aligned} \tag{45}$$

for ε small. The fourth term in (40) can be controlled similalry to (36), and yields

$$\left| \int \frac{(z^{\mathcal{N}})' z^{\mathcal{N}}}{r} \right| \leq C\varepsilon^{2-N}. \tag{46}$$

The fifth and the sixth terms in (40) can be controlled by

$$\left| \int \phi_\varepsilon' z^{\mathcal{N}} (z+v)' \right| \leq C\varepsilon^{2-N} e^{-\frac{r_0}{2\varepsilon}} \qquad \left| \int \phi_\varepsilon'' z^{\mathcal{N}} (z+v) \right| \leq C\varepsilon^{3-N} e^{-\frac{r_0}{2\varepsilon}}, \tag{47}$$

concluding the proof. □

Choosing some special values and using the above expansion, it is possible to show that Hence it follows that the reduced functional $\mathbf{I}_\varepsilon$ possesses a maximum point in a suitable interval $(\rho_{1,\varepsilon}, \rho_{2,\varepsilon})$, where both values approach $\frac{1}{\varepsilon}$ at a logarithmic rate in ε. From the first part of Proposition 7 one then finds the following result.

Theorem 3.2. [4] *Given* $n \geq 2$ *and* $p > 1$, *there exists a family of radial solutions* u_ε *of* (P_ε) *concentrating at* $|x| = r_\varepsilon$, *where* r_ε *is a local maximum point of* u_ε *satisfying* $1 - r_\varepsilon \sim \varepsilon |\log \varepsilon|$.

The same proof, with minor modifications, also applies when Ω is an annulus: in this case there are still solutions concentrating near the exterior boundary. However when Dirichlet conditions are imposed the boundary has a *repelling effect* on radial spike-layers, so concentration occurs at inner boundaries of annuli. One has indeed the following result.

Theorem 3.3. *([4]) Let* $\Omega \subseteq \mathbb{R}^N$ *be the annulus* $\{a < |x| < 1\}$, *with* $a \in (0,1)$. *Then there exists a family of radial solutions* u_ε *of* (D_ε) *concentrating near* $|x| = a$. *More precisely,* u_ε *possesses a local maximum point* $a < r_\varepsilon < 1$ *for which* $r_\varepsilon - a \sim \varepsilon |\log \varepsilon|$.

As for the construction of multiple peaks mentioned at the end of the previous section, it is possible to construct via a finite-dimensional analysis solutions with multiple spherical layers that approach parts boundary of balls or of annuli, depending on the boundary conditions one imposes, see [44].

The above results hold more in general for the problems

$$\begin{cases} -\varepsilon^2 \Delta u + V(|x|) u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, & u > 0 \text{ in } \Omega; \end{cases} \qquad \begin{cases} -\varepsilon^2 \Delta u + V(|x|) u = u^p & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, & u > 0 \text{ in } \Omega, \end{cases} \tag{48}$$

or for the above equations in the whole Euclidean space. Here one assumes V to be positive, bounded in C^2 norm and bounded away from zero. In this case, the location of an interior concentration set is determined by the critical points of the *auxiliary function* $M(r) = r^{N-1}V^{\theta}(r)$ (see also [12]). We also mention [7], [10] for similar results obtained with different techniques and [8], [9] for problems with reduced symmetries. For general potentials (without symmetry restrictions), see [19], [38] and [58], especially for what concerns a conjecture in [3].

Concerning concentration at the boundary, it occurs for the Neumann problem provided $M'(1) > 0$ or $M'(a) < 0$: for the Dirichlet problem, opposite inequalities are needed.

In [3], where the equation appearing in (48) was studied in the whole $\mathbb{R}^N$ was studied, it was also shown that, as $\varepsilon \to 0$, there is bifurcation of non-radial solutions from the radial one. This is related to the divergence of the Morse index of such solutions within Sobolev spaces of general (non-radial) functions, as discussed at the end of the introduction.

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