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Generalised Hausdorff measure of sets of Dirichlet non-improvable matrices in higher dimensions

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Abstract

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function. A pair $(A, \mathbf{b})$, where A is a real $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$, is said to be ψ-Dirichlet improvable, if the system

$$\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m < \psi(T), \quad \|\mathbf{q}\|^n < T$$

is solvable in $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n$ for all sufficiently large T where $\|\cdot\|$ denotes the supremum norm. For ψ-Dirichlet non-improvable sets, Kleinbock–Wadleigh (2019) proved the Lebesgue measure criterion whereas Kim–Kim (2022) established the Hausdorff measure results. In this paper we obtain the generalised Hausdorff f-measure version of Kim–Kim (2022) results for ψ-Dirichlet non-improvable sets.

1 Introduction

To begin with, we recall the higher dimensional general form of Dirichlet's Theorem (1842). Let m, n be positive integers and let X^{mn} denotes the space of real $m \times n$ matrices.

Theorem 1.1 (Dirichlet's Theorem) *Given any $A \in X^{mn}$ and $T > 1$, there exist $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that*

$$\|A\mathbf{q} - \mathbf{p}\|^m \leq \frac{1}{T} \text{ and } \|\mathbf{q}\|^n < T. \tag{1.1}$$

Here $\|\cdot\|$ denotes the supremum norm in $\mathbb{R}^i$, $i \in \mathbb{N}$. Theorem 1.1 guarantees a nontrivial integer solution for all T. The standard application of (1.1) is the following corollary, guaranteeing that such a system is solvable for an unbounded set of T.

Corollary 1.2 *For any $A \in X^{mn}$ there exist infinitely many integer vectors $\mathbf{q} \in \mathbb{Z}^n$ such that*

$$\|A\mathbf{q} - \mathbf{p}\|^m \leq \frac{1}{\|\mathbf{q}\|^n} \text{ for some } \mathbf{p} \in \mathbb{Z}^m. \tag{1.2}$$

The two statements above give rise to two possible ways to pose Diophantine approximation problems sometimes referred to as uniform vs asymptotic approximation results:

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that is, looking for solvability of inequalities for all large enough T vs. for some arbitrarily large T. The rate of approximation given in above two statements works for all real matrices $A \in X^{mn}$, which serves as the beginning of the *metric theory of Diophantine approximation,* a field concerned with understanding sets of $A \in X^{mn}$ satisfying similar conclusions but with the right hand sides replaced by faster decaying functions of T and $\|\mathbf{q}\|^n$ respectively. Those sets are well studied in the asymptotic setup (1.2) long ago.

Indeed, for a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a matrix $A \in X^{mn}$ is said to be ψ-approximable if the inequality [1]

$$\|A\mathbf{q} - \mathbf{p}\|^m < \psi(\|\mathbf{q}\|^n) \text{ for some } \mathbf{p} \in \mathbb{Z}^m \tag{1.3}$$

is satisfied for infinitely many integer vectors $\mathbf{q} \in \mathbb{Z}^n$. As the set of ψ-approximable matrices is translation invariant under integer vectors, we can restrict attention to mn-dimensional unit cube $[0,1]^{mn}$. Then the set of ψ-approximable matrices in $[0,1]^{mn}$ will be denoted by $W_{m,n}(\psi)$.

The following result gives the size of the set $W_{m,n}(\psi)$ in terms of Lebesgue measure.

Theorem 1.3 (Khintchine–Groshev Theorem, [11]) *Given a non-increasing ψ, the set $W_{m,n}(\psi)$ has zero (respectively full) Lebesgue measure if and only if the series $\sum_k \psi(k)$ converges (respectively, diverges).*

Let us now briefly describe what is known in the setting of (1.1). For a non-increasing function $\psi : [T_0, \infty) \to \mathbb{R}_+$ with $T_0 > 1$ fixed, consider the set $D_{m,n}(\psi)$ of ψ-Dirichlet improvable matrices consisting of $A \in X^{mn}$ such that the system

$$\|A\mathbf{q} - \mathbf{p}\|^m \leq \psi(T) \text{ and } \|\mathbf{q}\|^n < T$$

has a nontrivial integer solution for all large enough T. Elements of the complementary set, $D_{m,n}(\psi)^c$, will be referred as ψ-Dirichlet non-improvable matrices.

With the notation $\psi_a(x) := x^{-a}$, (1.1) implies that $D_{1,1}(\psi_1) = \mathbb{R}$, and that for any m, n every matrix is ψ_1-Dirichlet improvable. It was observed in [8] that for $\min(m,n) = 1$ and in [17] for the general case, that the Lebesgue measure of $D_{m,n}(c\psi_1)$ of the set $c\psi_1$-Dirichlet improvable matrices is zero for any $c < 1$,.

The theory of inhomogeneous Diophantine approximation starts by replacing the values of a system of linear forms $A\mathbf{q}$ by those of a system of affine forms $\mathbf{q} \mapsto A\mathbf{q} + \mathbf{b}$ where $A \in X^{mn}$ and $\mathbf{b} \in \mathbb{R}^m$. Following [15], for a non-increasing function $\psi : [T_0, \infty) \to \mathbb{R}_+$ a pair $(A, \mathbf{b}) \in X^{mn} \times \mathbb{R}^m$ is called ψ-Dirichlet improvable if for all T large enough, one can find nonzero integer vectors $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

$$\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m < \psi(T) \quad \text{and} \quad \|\mathbf{q}\|^n < T. \tag{1.4}$$

Let $\widehat{D}_{m,n}(\psi)$ denote the set of ψ-Dirichlet improvable pairs in the unit cube $[0,1]^{mn+m}$. If the inhomogeneous vector $\mathbf{b} \in \mathbb{R}^m$ is fixed then let $\widehat{D}^{\mathbf{b}}_{m,n}(\psi)$ be the set of all $A \in X^{mn}$ such that (1.4) holds i.e. for a fixed $\mathbf{b} \in \mathbb{R}^m$ we have $\widehat{D}^{\mathbf{b}}_{m,n}(\psi) = \{A \in X^{mn} : (A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)\}$.

[1]Here we use the definition as in [14,16], whereas in Sect. 4 we will consider slightly different definition such as in [4] where instead of (1.3) the inequality $\|A\mathbf{q} - \mathbf{b}\| < \psi(\|\mathbf{q}\|)$ is used.

The Lebesgue measure criterion for the set $\widehat{D}_{m,n}(\psi)$ i.e. doubly metric case has been proved by Kleinbock–Wadleigh [16] by reducing the problem to the shrinking target problem on the space of grids in $\mathbb{R}^{m+n}$. The proof of their theorem is based on a correspondence between Diophantine approximation and homogenous dynamics.

Theorem 1.4 (Kleinbock–Wadleigh, [16]) *Given a non-increasing ψ, the set $\widehat{D}_{m,n}(\psi)$ has zero (respectively full) Lebesgue measure if and only if the series $\sum_j \frac{1}{j^2\psi(j)}$ diverges (respectively converges).*

Recently (2022), Kim–Kim [12] established the Hausdorff measure analogue of Theorem 1.4.

Theorem 1.5 (Kim–Kim, [12]) *Let ψ be non-increasing with $\lim_{T\to\infty}\psi(T) = 0$ and $0 \le s \le mn + m$. Then*

$$\mathcal{H}^s(\widehat{D}_{m,n}(\psi)^c) = \begin{cases} 0 & \text{if} \quad \sum\limits_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn+m-s} < \infty; \\ \mathcal{H}^s([0,1]^{mn+m}) & \text{if} \quad \sum\limits_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn+m-s} = \infty. \end{cases}$$

In the same article Kim–Kim also provided the Hausdorff measure criterion for the singly metric case.

Theorem 1.6 (Kim–Kim, [12]) *Let ψ be non-increasing with $\lim_{T\to\infty}\psi(T) = 0$. Then for any $0 \le s \le mn$*

$$\mathcal{H}^s(\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c) = \begin{cases} 0 & \text{if} \quad \sum\limits_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn-s} < \infty; \\ \mathcal{H}^s([0,1]^{mn}) & \text{if} \quad \sum\limits_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn-s} = \infty, \end{cases}$$

for every $\mathbf{b} \in \mathbb{R}^m \setminus \mathbb{Z}^m$.

Naturally one can ask about the generalization of Theorems 1.5 and 1.6 in terms of f-dimensional Hausdorff measure. Recall that a natural generalization of the s-dimensional Hausdorff measure $\mathcal{H}^s$ is the f-dimensional Hausdorff measure $\mathcal{H}^f$ where f is a dimension function, that is an increasing, continuous function $f:\mathbb{R}_+ \to \mathbb{R}_+$ such that $f(r) \to 0$ as $r \to 0$.

In this article we extend the results of Kim–Kim [12] by establishing the zero-full law for the sets $\widehat{D}_{m,n}(\psi)$ and $\widehat{D}^{\mathbf{b}}_{m,n}(\psi)$ in terms of generalised f-dimensional Hausdorff measure. We obtain the following main results.

Theorem 1.7 *Let ψ be non-increasing and f be a dimension function with*

$$f(xy) \asymp x^s f(y) \quad \forall \quad y^{\alpha} \le x \le y^{\frac{1}{\alpha}} \tag{1.5}$$

where $mn + m - n < s < mn + m$ and $\alpha > 1$ is some absolute constant independent of x and y and suppose that

$$f'(x) = a(x)\frac{f(x)}{x} \tag{1.6}$$

such that $a(x) \to s$ *as* $x \to 0$*. Further, let*

$$(q^{-\frac{m}{n}})^{\alpha} \leq \psi(q) \leq (q^{-\frac{m}{n}})^{\frac{1}{\alpha}}. \tag{1.7}$$

Then

$$\mathcal{H}^f(\widehat{D}_{m,n}(\psi)^c) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2} \left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn+m} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right) < \infty; \\ \mathcal{H}^f([0,1]^{mn+m}) & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2} \left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn+m} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right) = \infty. \end{cases}$$

For the singly metric case we have the following result.

Theorem 1.8 *Let* ψ *be non-increasing and* f *be a dimension function such that* $r^{-mn}f(r) \to \infty$ *as* $r \to 0$*. Suppose that* (1.5)–(1.7) *holds and* $mn - n < s < mn$*. Then*

$$\mathcal{H}^f(\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2} \left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right) < \infty; \\ \mathcal{H}^f([0,1]^{mn}) & \text{if } \sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2} \left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right) = \infty \end{cases}$$

for every $\mathbf{b} \in \mathbb{R}^m \setminus \mathbb{Z}^m$*.*

We remark that the conditions (1.5) and (1.6) are satisfied in a wide variety of cases, for example $f(x) = x^s \log^t(x)$ for some $s > 0$ and $t \in \mathbb{R}$. Indeed, (1.5) follows since $f(xy) = (xy)^s \log^t(xy) \asymp x^s y^s \log^t(y) = x^s f(y)$, and (1.6) follows since

$$\frac{xf'(x)}{f(x)} = x\frac{d}{dx}[s\log(x) + t\log\log(x)] = x\left(\frac{s}{x} + \frac{t}{x\log(x)}\right) \to s \text{ as } x \to 0.$$

2 Preliminaries and auxiliary results

2.1 Hausdorff measure and dimension

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a dimension function i.e. an increasing continuous function such that $f(r) \to 0$ as $r \to 0$ and let $\mathcal{V}$ be an arbitrary subset of $\mathbb{R}^n$. For $\rho > 0$, a ρ-cover for a set $\mathcal{V}$ is defined as a countable collection $\{U_i\}_{i\geq 1}$ of sets in $\mathbb{R}^n$ with diameters $0 < \mathrm{diam}(U_i) \leq \rho$ such that $\mathcal{V} \subseteq \bigcup_{i=1}^{\infty} U_i$. Then for each $\rho > 0$ define

$$\mathcal{H}^f_\rho(\mathcal{V}) = \inf\left\{\sum_{i=1}^{\infty} f\big(\mathrm{diam}(U_i)\big) : \{U_i\} \text{ is a } \rho - \text{cover of } \mathcal{V}\right\}.$$

Note that $\mathcal{H}^f_\rho(\mathcal{V})$ is non-decreasing as ρ decreases and therefore approaches a limit as $\rho \to 0$. Accordingly, the f-dimensional Hausdorff measure of $\mathcal{V}$ is defined as

$$\mathcal{H}^f(\mathcal{V}) := \lim_{\rho \to 0} \mathcal{H}^f_\rho(\mathcal{V}).$$

This limit could be zero or infinity, or take a finite positive value.

If $f(r) = r^s$ where $s > 0$, then $\mathcal{H}^f$ is the s-dimensional Hausdorff measure and is represented by $\mathcal{H}^s$. It can be easily verified that Hausdorff measure is monotonic, that is, if E is contained in F then $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$, countably sub-additive, and satisfies $\mathcal{H}^s(\emptyset) = 0$.

The following property

$$\mathcal{H}^s(\mathcal{V}) < \infty \implies \mathcal{H}^{s'}(\mathcal{V}) = 0 \quad \text{if } s' > s,$$

implies that there is a unique real point s at which the Hasudorff s-measure drops from infinity to zero (unless $\mathcal{V}$ is finite so that $\mathcal{H}^s(\mathcal{V})$ is never infinite). The value taken by s at this discontinuity is referred to as the *Hausdorff dimension* of a set $\mathcal{V}$ and is defined as

$$\dim_{\mathrm{H}} \mathcal{V} := \inf\{s > 0 : \mathcal{H}^s(\mathcal{V}) = 0\}.$$

For establishing the convergent part of Theorems 1.7 and 1.8 we will apply the following Hausdorff measure version of the famous Borel–Cantelli lemma [6, Lemma 3.10]:

Lemma 2.1 *Let $\{B_i\}_{i\geq 1}$ be a sequence of measurable sets in $\mathbb{R}^n$ and suppose that for some dimension function f, $\sum_i f(\mathrm{diam}(B_i)) < \infty$. Then $\mathcal{H}^f(\limsup_{i\to\infty} B_i) = 0$.*

We will use the following principle known as Mass Distribution Principle [10, §4.1] for the divergent part of Theorem 1.7.

Lemma 2.2 *Let μ be a probability measure supported on a subset $\mathcal{V}$ of $\mathbb{R}^k$. Suppose there are positive constants $c > 0$ and $\varepsilon > 0$ such that*

$$\mu(U) \leq cf(\mathrm{diam}(U))$$

for all sets U with $\mathrm{diam}(U) \leq \varepsilon$. Then $\mathcal{H}^f(\mathcal{V}) \geq \mu(\mathcal{V})/c$.

Theorem 2.3 ([1, Theorem 2]) *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be any approximating function and let $mn > 1$. Let f and $g : r \to g(r) := r^{-m(n-1)}f(r)$ be dimension functions such that $r \mapsto r^{-mn}f(r)$ is monotonic. Then*

$$\mathcal{H}^f(W_{m,n}(\psi)) = \begin{cases} 0 & \text{if} \quad \sum\limits_{q=1}^{\infty} q^{m+n-1} g\left(\frac{\widehat{\psi}(q)}{q}\right) < \infty; \\ \mathcal{H}^f([0,1]^{mn}) & \text{if} \quad \sum\limits_{q=1}^{\infty} q^{m+n-1} g\left(\frac{\widehat{\psi}(q)}{q}\right) = \infty, \end{cases}$$

where $\widehat{\psi}(q) = \psi(q^n)^{\frac{1}{m}}$.

2.2 Ubiquitous systems

To prove the divergent parts of Theorem 1.8 we will use the ubiquity technique developed by Beresnevich, Dickinson, and Velani, see [5, §12.1]. The idea and concept of ubiquity was originally formulated by Dodson, Rynne, and Vickers in [9] and coincided in part with the concept of 'regular systems' of Baker and Schmidt [2]. Both have proven to be extremely useful in obtaining lower bounds for the Hausdorff dimension of limsup sets. The ubiquity framework in [5] provides a general and abstract approach for establishing the Lebesgue and Hausdorff measure of a large class of limsup sets.

Consider the mn-dimensional unit cube $[0,1]^{mn}$ with the supremum norm $\|\cdot\|$. Let $\mathcal{R} = \{R_\kappa \subseteq [0,1]^{mn} : \kappa \in J\}$ be a family of subsets, referred to as resonant sets R_κ of $[0,1]^{mn}$ indexed by an infinite, countable set J. Let $\beta : J \to \mathbb{R}_+ : \kappa \mapsto \beta_\kappa$ be a positive function on J i.e. the function β attaches the weight β_κ to the set R_κ. Next assume that the number of terms κ in J with β_κ bounded above is always finite. Following the ideas from [5, §12.1] and [12] let us assume that the family $\mathcal{R}$ of resonant sets R_κ consists of $(m-1)n$-dimensional, rational hyperplanes and define the following notations. For a set $S \subseteq [0,1]^{mn}$, let

$$\Delta(S, r) := \{V \in [0,1]^{mn} : \mathrm{dist}(V, S) < r\},$$

where $\mathrm{dist}(V, S) := \inf\{\|V - Y\| : Y \in S\}$. Fix a decreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ let

$$\Lambda(\Psi) = \{V \in [0,1]^{mn} : V \in \Delta(R_\kappa, \Psi(\beta_\kappa)) \text{ for i.m. } \kappa \in J\} \tag{2.1}$$

The set $\Lambda(\Psi)$ is a lim sup set; it consists of elements of $[0,1]^{mn}$ which lie in infinitely many of the thickenings $\Delta(R_\kappa, \Psi(\beta_\kappa))$. It is natural to call Ψ the approximating function as it governs the 'rate' at which the elements of $[0,1]^{mn}$ must be approximated by resonant sets in order to lie in $\Lambda(\Psi)$. Let us rewrite the set $\Lambda(\Psi)$ in a way which brings its lim sup nature to the forefront.

For $N \in \mathbb{N}$, let

$$\Delta(\Psi, N) := \bigcup_{\kappa \in J : 2^{N-1} < \beta_\kappa \leq 2^N} \Delta(R_\kappa, \Psi(\beta_\kappa)).$$

Thus $\Lambda(\Psi)$ is the set consisting elements of $[0,1]^{mn}$ which lie in infinitely many $\Delta(\Psi, N)$, that is,

$$\Lambda(\Psi) := \limsup_{N \to \infty} \Delta(\Psi, N) \tag{2.2}$$

Next let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\rho(t) \to 0$ as $t \to \infty$ and let

$$\Delta(\rho, N) := \bigcup_{\kappa \in J : 2^{N-1} < \beta_\kappa \leq 2^N} \Delta(R_\kappa, \rho(\beta_\kappa)). \tag{2.3}$$

Definition 2.4 Let B be an arbitrary ball in $[0,1]^{mn}$. Suppose there exist a function ρ and an absolute constant $\kappa > 0$ such that

$$|B \cap \Delta(\rho, N)| \geq \kappa |B| \text{ for } N \geq N_0(B), \tag{2.4}$$

where $|\cdot|$ denotes the Lebesgue measure on $[0,1]^{mn}$. Then the pair $(\mathcal{R}, \beta)$ is said to be a 'local ubiquitous system' relative to ρ and the function ρ will be referred to as the 'ubiquitous function'.

A function h is said to be 2-regular if there exists a strictly positive constant $\lambda < 1$ such that for N sufficiently large

$$h(2^{N+1}) \leq \lambda h(2^N).$$

The next theorem is a simplified version of Theorem 1 and Theorem 2 from [5]. To state the result we define notions similar to those in [5]. Note that with notions in [5], we have $\Omega := [0,1]^{mn}$, the Lebesgue measure on $[0,1]^{mn}$ is of type (M2) with $\delta = mn$ and $\gamma = (m-1)n$ and the local ubiquitous system $(\mathcal{R}, \beta)$ satisfies the intersection conditions with $\gamma = (m-1)n$ (see [5, section 12.1]). Given that the Lebesgue measure is comparable with $\mathcal{H}^\delta-$ a simple consequence of (M2), we have the following combined version of Theorem 1 and Theorem 2 from [5].

Theorem 2.5 *Suppose that $(\mathcal{R}, \beta)$ is a local ubiquitous system relative to ρ and that Ψ is an approximating function. Let f be a dimension function such that $r^{-nm}f(r)$ is monotonic, $r^{-nm}f(r) \to \infty$ as $r \to 0$ and $r^{-n(m-1)}f(r)$ is increasing. Furthermore, suppose that ρ is 2-regular and*

$$\sum_{n=1}^{\infty} \frac{(\Psi(2^N))^{-n(m-1)} f(\Psi(2^N))}{\rho(2^N)^n} = \infty. \tag{2.5}$$

Then

$$\mathcal{H}^f(\Lambda(\Psi)) = \mathcal{H}^f([0,1]^{mn}). \tag{2.6}$$

Proof With $\delta = mn$, and $\gamma = (m-1)n$ the function g in [5, Theorem 2] becomes $g(r) := f(\Psi(r))\Psi(r)^{-\gamma}\rho(r)^{\gamma-\delta} = f(\Psi(r))\Psi(r)^{-(m-1)n}\rho(r)^{-n}$. Also ρ is 2-regular, thus from [5, Theorem 2] it follows that

$$\mathcal{H}^f(\Lambda(\Psi)) = \infty \quad \text{if} \quad \sum_{n=1}^{\infty} g(2^N) = \infty,$$

which is same as the divergent sum condition in (2.5).

Note that as the dimension function $r^{-nm}f(r) \to \infty$ as $r \to 0$ then $H^f(\Omega) = \infty$ and Theorem 2.5 leads to the same conclusion as Theorem 2 in [5]. □

2.3 Dirichlet improvability and homogenous dynamics

In one dimensional settings, continued fraction expansions have been useful in characterising ψ-Dirichlet improvable numbers [15]. However this machinery is not applicable in higher dimensions. For general dimensions, building on ideas from [7] (also see [13]), a dynamical approach was proposed in [15], reformulating the homogenous approximation problem as a shrinking target problem and a similar approach was used in [16] to solve an analogous inhomogeneous problem. Following the ideas from [12,16] we will use the standard argument usually known as the 'Dani correspondence' which serves as a connection between Diophantine approximation and homogenous dynamics. In order to describe how Dirichlet-improvability is related to dynamics we will start by recalling the dynamics on space of grids. To describe this dynamical interpretation, let us fix some notation.

Fix $d = m + n$. Let

$$G_d = SL_d(\mathbb{R}) \text{ and } \widehat{G}_d = ASL_d(\mathbb{R}) = G_d \ltimes \mathbb{R}^d$$

and put

$$\Gamma_d = SL_d(\mathbb{Z}) \text{ and } \widehat{\Gamma}_d = ASL_d(\mathbb{Z}) = \Gamma_d \ltimes \mathbb{Z}^d.$$

Denote by $\widehat{Y}_d$ the space of affine shifts of unimodular lattices in $\mathbb{R}^d$ (i.e. space of unimodular grids). Clearly, $\widehat{Y}_d$ is canonically identified with $\widehat{G}_d/\widehat{\Gamma}_d$ via

$$< g, \mathbf{w} > \widehat{\Gamma}_d \in \widehat{G}_d/\widehat{\Gamma}_d \longleftrightarrow g\mathbb{Z}^d + \mathbf{w} \in \widehat{Y}_d$$

where $< g, \mathbf{w} >$ is an element of $\widehat{G}_d$ such that $g \in G_d$ and $\mathbf{w} \in \mathbb{R}^d$. Similarly, $Y_d := G_d/\Gamma_d$ is identified with the space of unimodular lattices in $\mathbb{R}^d$ (i.e. the space of unimodular grids containing zero vector). Note that Γ_d (respectively, $\widehat{\Gamma}_d$) is a lattice in G_d (respectively, $\widehat{G}_d$).

Denote by m_{Y_d} the Haar probability measure on Y_d. For any $t \in \mathbb{R}$, the flow of interest a_t is given by the diagonal matrix

$$a_t := \mathrm{diag}(e^{t/m}, \cdots, e^{t/m}, e^{-t/n}, \cdots, e^{-t/n}).$$

Let

$$u_A := \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \in G_d,$$

$$u_{A,\mathbf{b}} := \left\langle \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \right\rangle \in \widehat{G}_d$$

for $A \in X^{mn}$ and $(A, \mathbf{b}) \in X^{mn} \times \mathbb{R}^m$. Let us also denote by

$$\Lambda_A := u_A \mathbb{Z}^d \in Y_d \text{ and } \Lambda_{A,\mathbf{b}} := u_{A,\mathbf{b}} \mathbb{Z}^d \in \widehat{Y}_d,$$

where $u_{A,\mathbf{b}}\mathbb{Z}^d = \left\{ \begin{pmatrix} A\mathbf{q} + \mathbf{b} - \mathbf{p} \\ \mathbf{q} \end{pmatrix} : \mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n \right\}$.

Following [16], define $\Delta : \widehat{Y}_d \to [-\infty, +\infty)$ by

$$\Delta(\Lambda) := \log \inf_{\mathbf{v} \in \Lambda} \|\mathbf{v}\|.$$

Lemma 2.6 ([14]) *Let $\psi : [T_0, \infty) \to \mathbb{R}_+$ be a continuous, non-increasing function where $T_0 \in \mathbb{R}_+$ and m, n be positive integers. Then there exists a continuous function*

$$z = z_\psi : [t_0, \infty) \to \mathbb{R},$$

where $t_0 := \frac{m}{m+n} \log T_0 - \frac{n}{m+n} \log \psi(T_0)$, such that

(i) *the function $t \mapsto t + nz(t)$ is strictly increasing and unbounded;*
(ii) *the function $t \mapsto t - mz(t)$ is non-decreasing;*
(iii) *$\psi(e^{t+nz(t)}) = e^{-t+mz(t)}$ for all $t \geq t_0$.*

Note that, properties (*i*) and (*ii*) of Lemma 2.6 imply that any $z = z_\psi$ does not oscillate too wildly. Namely, $z(s) - \frac{1}{m} \leq z(u) \leq z(s) + \frac{1}{n}$ whenever $s \leq u \leq s + 1$.

The following lemma, which rephrases ψ-Dirichlet improvable properties of $(A, \mathbf{b}) \in X^{mn} \times \mathbb{R}^m$ as the statement about the orbit of $\Lambda_{A,\mathbf{b}}$ in the dynamical space $(\widehat{Y}_d, a_t)$, is the general version of the correspondence between the improvability of the inhomogeneous Dirichlet theorem and dynamics on $\widehat{Y}_d$.

Lemma 2.7 ([16]) *Let $z = z_\psi$ be the function associated to ψ by Lemma 2.6. Then $(A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)$ if and only if $\Delta(a_t \Lambda_{A,\mathbf{b}}) < z_\psi(t)$ for all sufficiently large t.*

This equivalence is usually called the Dani Correspondence. In view of this interpretation a pair fails to be ψ-Dirichlet improvable if and only if the associated grid visits the target $\Delta^{-1}([z_\psi(t), \infty))$ at unbounded times t under the flow a_t. Note that from the above lemma in the definitions $\widehat{D}_{m,n}(\psi)^c = \limsup_{t\to\infty} \left\{(A, \mathbf{b}) : \Delta(a_t \Lambda_{A,\mathbf{b}}) \geq z_\psi(t)\right\}$ and $\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c = \limsup_{t\to\infty} \left\{A : \Delta(a_t \Lambda_{A,\mathbf{b}}) \geq z_\psi(t)\right\}$, the limsup is taken for real values $t \in \mathbb{R}$. However to prove the convergent part, we need to use Hausdorff–Cantelli lemma (Lemma 2.1), therefore we will consider limsup sets taken for $t \in \mathbb{N}$. Thus we will use the following definitions: there exists a non-zero positive constant C_0 such that

$$\widehat{D}_{m,n}(\psi)^c \subseteq \limsup_{t\to\infty, t\in\mathbb{N}} \left\{(A, \mathbf{b}) : \Delta(a_t \Lambda_{A,\mathbf{b}}) \geq z_\psi(t) - C_0\right\}, \tag{2.7}$$

$$\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c \subseteq \limsup_{t\to\infty, t\in\mathbb{N}} \left\{A : \Delta(a_t\Lambda_{A,\mathbf{b}}) \geq z_\psi(t) - C_0\right\}. \tag{2.8}$$

The validity of these definitions can be observed by the fact that z_ψ does not oscillate wildly by [16, Remark 3.3] and Δ is uniformly continuous on the set $\Delta^{-1}([z,\infty))$ for any $z \in \mathbb{R}$, ([16, Lemma 2.1]).

3 Proof of Theorems 1.7 and 1.8: the convergent case

Lemma 3.1 *Let $\psi : [T_0,\infty) \to \mathbb{R}_+$ be a non-increasing function, and let $z = z_\psi$ be the function associated to ψ by Lemma 2.6. Let f be a dimension function satisfying (1.5) and (1.6) where $nm - n < s \leq nm$. Also suppose that (1.7) holds. Then we have*

$$\sum_{q=\lceil T_0\rceil}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right) < \infty$$
$$\iff \sum_{t=\lceil t_0\rceil}^{\infty} e^{-(m+n)z(t)}e^{(m+n)t}f(e^{-\frac{(m+n)t}{mn}}) < \infty.$$

Proof The proof of this lemma uses ideas introduced in [14, Lemma 8.3] and [16]. Using the monotonicity of ψ and [16, Remark 3.3], let us replace the sums with integrals

$$\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right)dx \text{ and } \int_{t_0}^{\infty} e^{-(m+n)z(t)}e^{(m+n)t}f(e^{-\frac{(m+n)t}{mn}})dt.$$

Define

$$P := -\log\circ\psi\circ\exp : [T_0,\infty) \to \mathbb{R} \text{ and } \lambda(t) := t + nz(t).$$

Since $\psi(e^\lambda) = e^{-P(\lambda)}$, letting $\log x = \lambda$ we have

$$\begin{aligned}\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right)dx &= \int_{\log T_0}^{\infty} \frac{1}{\psi(e^\lambda)e^{2\lambda}}\left(\frac{e^{\lambda m}}{\psi(e^\lambda)^n}\right) f\left(\frac{\psi(e^\lambda)^{\frac{1}{m}}}{e^{\frac{\lambda}{n}}}\right)e^\lambda d\lambda\\ &= \int_{\log T_0}^{\infty} e^{(m-1)\lambda}e^{(1+n)P(\lambda)}f(e^{\frac{-P(\lambda)}{m}}e^{\frac{-\lambda}{n}})d\lambda. \end{aligned}\tag{3.1}$$

Using $P(\lambda(t)) = t - mz(t)$, we have

$$\begin{aligned}&\int_{t_0}^{\infty} e^{-(m+n)z(t)}e^{(m+n)t}f(e^{-\frac{(m+n)t}{mn}})dt\\ &= \int_{t_0}^{\infty} e^{(m-1)\lambda}e^{(1+n)P(\lambda)}f(e^{\frac{-P(\lambda)}{m}}e^{\frac{-\lambda}{n}})d\left[\frac{m}{m+n}\lambda + \frac{n}{m+n}P(\lambda)\right]\\ &= \frac{m}{m+n}\int_{\log T_0}^{\infty} e^{(m-1)\lambda}e^{(1+n)P(\lambda)}f(e^{\frac{-P(\lambda)}{m}}e^{\frac{-\lambda}{n}})d\lambda\\ &\quad + \frac{n}{m+n}\int_{\log T_0}^{\infty} e^{(m-1)\lambda}e^{(1+n)P(\lambda)}f(e^{\frac{-P(\lambda)}{m}}e^{\frac{-\lambda}{n}})d(P(\lambda)).\end{aligned}\tag{3.2}$$

The term in the last line can be expressed by

$$\frac{n}{m+n}\int_{\log T_0}^{\infty} e^{(m-1)\lambda}e^{(1+n)P(\lambda)}f(e^{\frac{-P(\lambda)}{m}}e^{\frac{-\lambda}{n}})d(P(\lambda))$$

$$
\begin{aligned}
&\asymp \frac{n}{m+n}\int_{\log T_0}^{\infty} e^{(m-1)\lambda} f(e^{\frac{-\lambda}{n}}) e^{(1+n)P(\lambda)} e^{-s\frac{P(\lambda)}{m}}\, d(P(\lambda)),\\
&= \frac{n}{m+n}\left(1+\frac{mn-s}{m}\right)^{-1}\int_{\log T_0}^{\infty} e^{(m-1)\lambda} f(e^{\frac{-\lambda}{n}})\, d(e^{((1+n)-\frac{s}{m})P(\lambda)}),
\end{aligned}
\tag{3.3}
$$

the second last equation follows from (1.5) and (1.7). Since by using (1.7) and the fact that $\psi(e^{\lambda}) = e^{-P(\lambda)}$ we obtain the condition

$$
(e^{\frac{-\lambda}{n}})^{\alpha} \le e^{\frac{-P(\lambda)}{m}} \le (e^{\frac{-\lambda}{n}})^{\frac{1}{\alpha}},
$$

therefore by using (1.5) we can write

$$
f(e^{\frac{-P(\lambda)}{m}} e^{\frac{-\lambda}{n}}) \asymp e^{-s\frac{P(\lambda)}{m}} f(e^{\frac{-\lambda}{n}}).
$$

Next we will use integration by parts to evaluate the integral in (3.3).

$$
\begin{aligned}
&\int_{\log T_0}^{\infty} e^{(m-1)\lambda} f(e^{\frac{-\lambda}{n}})\, d(e^{((1+n)-\frac{s}{m})P(\lambda)})\\
&= -\int_{\log T_0}^{\infty} [(m-1)e^{(m-1)\lambda} f(e^{-\frac{\lambda}{n}}) - \frac{1}{n} e^{(m-1)\lambda} e^{\frac{-\lambda}{n}} f'(e^{\frac{-\lambda}{n}})] e^{((1+n)-\frac{s}{m})P(\lambda)} d\lambda\\
&\quad + e^{(m-1)\lambda} f(e^{-\frac{\lambda}{n}}) e^{((1+n)-\frac{s}{m})P(\lambda)}\Big|_{\log T_0}^{\infty}\\
&= \int_{\log T_0}^{\infty}\left((1-m)e^{(m-1)\lambda} f(e^{-\frac{\lambda}{n}}) e^{((1+n)-\frac{s}{m})P(\lambda)} + \frac{1}{n} e^{(m-1)\lambda} f'(e^{-\frac{\lambda}{n}}) e^{-\frac{\lambda}{n}} e^{((1+n)-\frac{s}{m})P(\lambda)}\right) d\lambda\\
&\quad + \lim_{\lambda\to\infty} e^{(m-1)\lambda} f(e^{-\frac{\lambda}{n}}) e^{((1+n)-\frac{s}{m})P(\lambda)} - T_0^{m-1} f(T_0^{-\frac{1}{n}}) \psi(T_0)^{-((1+n)-\frac{s}{m})},\\
&\text{by (1.6), we have } f'(e^{-\frac{\lambda}{n}}) = a(e^{-\frac{\lambda}{n}}) \frac{f(e^{-\frac{\lambda}{n}})}{e^{-\frac{\lambda}{n}}},\\
&= \int_{\log T_0}^{\infty}\left(1-m+\frac{1}{n} a(e^{-\frac{\lambda}{n}})\right) e^{(m-1)\lambda} f(e^{-\frac{\lambda}{n}}) e^{((1+n)-\frac{s}{m})P(\lambda)} d\lambda\\
&\quad + \lim_{\lambda\to\infty} e^{(m-1)\lambda} f(e^{-\frac{\lambda}{n}}) e^{((1+n)-\frac{s}{m})P(\lambda)} - T_0^{m-1} f(T_0^{-\frac{1}{n}}) \psi(T_0)^{-((1+n)-\frac{s}{m})}\\
&\asymp \int_{\log T_0}^{\infty} (1-m+\frac{1}{n} a(e^{-\frac{\lambda}{n}})) e^{(m-1)\lambda} e^{(1+n)P(\lambda)} f(e^{\frac{-P(\lambda)}{m}} e^{\frac{-\lambda}{n}}) d\lambda && (3.4)\\
&\quad + \lim_{\lambda\to\infty} e^{(m-1)\lambda} e^{(1+n)P(\lambda)} f(e^{\frac{-P(\lambda)}{m}} e^{\frac{-\lambda}{n}}) - T_0^{m-1} f(T_0^{-\frac{1}{n}}) \psi(T_0)^{-((1+n)-\frac{s}{m})}. && (3.5)
\end{aligned}
$$

Note that as $\lambda \to \infty$, $e^{-\frac{\lambda}{n}} \to 0$ thus by assumption $a(e^{-\frac{\lambda}{n}}) \to s$ and therefore

$$
\left(1 - \frac{mn - a(e^{-\frac{\lambda}{n}})}{n}\right) \to \left(1 - \frac{mn-s}{n}\right),
$$

which is finite and positive for T_0 large enough (since $s > nm - n$). Observe that

$$
\lim_{\lambda\to\infty} e^{(m-1)\lambda} e^{(1+n)P(\lambda)} f(e^{\frac{-P(\lambda)}{m}} e^{\frac{-\lambda}{n}}) = 0
$$

if the integral

$$
\int_{\log T_0}^{\infty} e^{(m-1)\lambda} e^{(1+n)P(\lambda)} f(e^{\frac{-P(\lambda)}{m}} e^{\frac{-\lambda}{n}}) d\lambda
$$

converges. Thus the convergence of

$$
\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx \text{ or } \int_{t_0}^{\infty} e^{-(m+n)z(t)} e^{(m+n)t} f(e^{-\frac{(m+n)t}{mn}}) dt
$$

implies the convergence of other since all summands are positive except the finite value $-T_0^{m-1}f(T_0^{-\frac{1}{n}})\psi(T_0)^{-((1+n)-\frac{s}{m})}$. □

In order to apply the Hausdorff–Cantelli lemma (Lemma 2.1) we need a sequence of coverings for the sets $\widehat{D}_{m,n}(\psi)^c$ and $\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c$. Recall that we are considering the supremum norm $\|\cdot\|$ on $[0,1]^{mn}$ and let $\lambda_j(\Lambda)$ denote the j-th successive minimum of a lattice $\Lambda \subseteq \mathbb{R}^d$ i.e. the infimum of λ such that the ball $B_\lambda^{\mathbb{R}^d}(0)$ contains j independent vectors of Λ. Then:

Proposition 3.2 (Kim–Kim, [12, Proposition 3.6]) *Let C_0 be the same constant as in (2.7) and (2.8). For $t \in \mathbb{N}$, let $Z_t := \{A \in [0,1]^{mn} : \log(d\lambda_d(a_t\Lambda_A)) \geq z_\psi(t) - C_0\}$. Then Z_t can be covered with $Ke^{(m+n)(t-z_\psi(t))}$ balls in $X^{mn} = M_{m,n}(\mathbb{R})$ of radius $\frac{1}{2}e^{-(\frac{1}{m}+\frac{1}{n})t}$ for a constant $K > 0$ not depending on t.*

We are now in a position to prove the following statement.

Proposition 3.3 *Let $mn - n < s \leq mn$. If $\sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right) < \infty$, then $\mathcal{H}^f(\limsup_{t\to\infty} Z_t) = 0$ and $\mathcal{H}^{f+m}(\limsup_{t\to\infty} Z_t \times [0,1]^m) = 0$. (Note that $\mathcal{H}^{f+m}$ represents the Hausdorff measure of a set when we take $(f+m)(r) = r^m f(r)$).*

Proof By Lemma 3.1, the assumption $\sum_{q=1}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right) < \infty$ is equivalent to

$$\sum_{t=1}^{\infty} e^{-(m+n)(z(t)-t)} f(e^{-(\frac{1}{m}+\frac{1}{n})t}) < \infty. \tag{3.6}$$

For each $t \in \mathbb{N}$, let $D_{t,1}, D_{t,2}, \cdots, D_{t,p_t}$ be the balls of radius $\frac{1}{2}e^{-(\frac{1}{m}+\frac{1}{n})t}$ covering Z_t as in Proposition 3.2. Note that p_t, the number of the balls, is not greater than $Ke^{(m+n)(t-z_\psi(t))}$ by Proposition 3.2. By applying Lemma 2.1 to the sequence of balls $\{D_{t,j}\}_{t\in\mathbb{N},1\leq j\leq p_t}$, we have $\mathcal{H}^f(\limsup_{t\to\infty} Z_t) \leq \mathcal{H}^f(\limsup_{t\to\infty} D_{t_j}) = 0$.

We prove the second statement by a similar argument. Proposition 3.2 implies that $Z_t \times [0,1]^m$ can be covered with $Ke^{\frac{m+n}{n}t}e^{(m+n)(t-z_\psi(t))}$ balls of radius $\frac{1}{2}e^{-(\frac{1}{m}+\frac{1}{n})t}$. Applying Lemma 2.1 again, we have $\mathcal{H}^{f+m}(\limsup_{t\to\infty} Z_t \times [0,1]^m) = 0$. □

The convergence parts of Theorems 1.7 and 1.8 follow from this proposition. We will adapt a similar method as in [12].

Proof We first prove the singly metric case i.e., the convergent part of Theorem 1.8. We claim that $\log(d\lambda_d(a_t\Lambda_A)) \geq \Delta(a_t\Lambda_{A,\mathbf{b}})$ for every $\mathbf{b} \in \mathbb{R}^m$. Let $v_1, \ldots, v_d$ be linearly independent vectors satisfying $\|v_i\| \leq \Lambda_d(a_t\Lambda_A)$ for $1 \leq i \leq d$. The shortest vector of $a_t\Lambda_{A,\mathbf{b}}$ can be written as a form of $\sum_1^d \alpha_i v_i$ for some $-1 \leq \alpha_i \leq 1$, so the length of the shortest vector is less than $\sum_1^d \|v_i\|$. Thus, $\Delta(a_t\Lambda_{A,\mathbf{b}}) \leq \log\sum_i^d \|v_i\| \leq \log(d\lambda_d(a_t\Lambda_A))$. This implies $\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c \subseteq \limsup_{t\to\infty}\{A \in [0,1]^{mn} : \Delta(a_t\Lambda_{A,\mathbf{b}}) \geq z_\psi(t) - C_0\} \subseteq \limsup_{t\to\infty} Z_t$ by Lemma 2.7 and Proposition 3.3, thus we obtain $\mathcal{H}^f(\hat{D}^{\mathbf{b}}_{m,n}(\psi)^c) \leq \mathcal{H}^f(\limsup_{t\to\infty} Z_t) = 0$.

Similarly for the doubly metric case, together with the second statement of Proposition 3.3, $\widehat{D}_{m,n}(\psi)^c \subseteq \limsup_{t\to\infty}\{(A,\mathbf{b}) \in [0,1]^{mn+m} : \Delta(a_t\Lambda_{A,\mathbf{b}}) \geq z_\psi(t) - C_0\} \subseteq \limsup_{t\to\infty} Z_t \times [0,1]^m$ provides the proof of the convergent part of Theorem 1.7. □

4 Proof of Theorems 1.7 and 1.8: the divergent case

Recall that $d = m + n$ and assume that $\psi : [T_0, \infty) \to \mathbb{R}_+$ is a decreasing function satisfying $\lim_{T\to\infty} \psi(T) = 0$. Denote by $\| \cdot \|_{\mathbb{Z}}$ and $|.|_{\mathbb{Z}}$ the distance to the nearest integer vector and number, respectively. Define the function $\widetilde{\psi} : [S_0, \infty) \to \mathbb{R}_+$ by

$$\widetilde{\psi}(S) = (\psi^{-1}(S^{-m}))^{\frac{-1}{n}}$$

where $S_0 = \psi(T_0)^{\frac{-1}{m}}$. The next lemma associates ψ-Dirichlet non-improvability with $\widetilde{\psi}$-approximability via a transference lemma as follows.

Lemma 4.1 [12, Lemma 4.2] *Given $(A, \mathbf{b}) \in X^{mn} \times \mathbb{R}^m$, if the system*

$$\|A^t\mathbf{x}\|_{\mathbb{Z}} < d^{-1}|\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}}\widetilde{\psi}(S) \text{ and } \|\mathbf{x}\| < d^{-1}|\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}}S$$

has a nontrivial solution $\mathbf{x} \in \mathbb{Z}^m$ for an unbounded set of $S \geq S_0$, then $(A, \mathbf{b}) \in \widehat{D}_{m,n}(\psi)^c$.

Following [12] we adopt some notations. Let $W_{S,\varepsilon}$ be the set of $A \in [0, 1]^{mn}$ such that there exists $\mathbf{x}_{A,S} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ satisfying

$$\|A^t\mathbf{x}_{A,S}\|_{\mathbb{Z}} < d^{-1}\varepsilon\widetilde{\psi}(S) \text{ and } \|\mathbf{x}_{A,S}\| < d^{-1}\varepsilon S$$

and let

$$\widehat{W}_{S,\varepsilon} := \{(A, \mathbf{b}) \in [0, 1]^{mn+m} : A \in W_{S,\varepsilon} \text{ and } |\mathbf{b}\cdot\mathbf{x}_{A,S}|_{\mathbb{Z}} > \varepsilon\}.$$

For fixed $\mathbf{b} \in \mathbb{R}^m$, consider the set $W_{\mathbf{b},S,\varepsilon}$ of matrices $A \in [0, 1]^{mn}$ such that there exists $\mathbf{x} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ satisfying

- $|\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}} > \varepsilon$
- $\|A^t\mathbf{x}\|_{\mathbb{Z}} < d^{-1}\varepsilon\widetilde{\psi}(S)$ and $\|\mathbf{x}\| < d^{-1}\varepsilon S$.

Let $W_{\mathbf{b},\varepsilon} := \limsup_{S\to\infty} W_{\mathbf{b},S,\varepsilon}$. Note that $A \in W_{S,\varepsilon}$ if and only if

$$\|A^t\mathbf{x}_{A,S}\|_{\mathbb{Z}} < \Psi_\varepsilon(U) \text{ and } \|\mathbf{x}_{A,S}\| < U \text{ for some } \mathbf{x}_{A,S},$$

where

$$\Psi_\varepsilon(U) := d^{-1}\varepsilon\widetilde{\psi}(d\varepsilon^{-1}U), \quad U = d^{-1}\varepsilon S. \tag{4.1}$$

By Lemma 4.1 $\limsup_{S\to\infty} \widehat{W}_{S,\varepsilon} \subseteq \widehat{D}_{m,n}(\psi)^c$ and $W_{\mathbf{b},\varepsilon} \subseteq \widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c$.

Further $\limsup_{S\to\infty} W_{S,\varepsilon} = \{A \in [0, 1]^{mn} : A^t \in W_{n,m}(\Psi_\varepsilon)\}$ is the set of matrices whose transposes are Ψ_ε-approximable. From here onwards we use a slightly different definition of Ψ_ε-approximability; recall from footnote 1 where the inequality $\|A^t\mathbf{x}\|_{\mathbb{Z}} < \Psi_\varepsilon(\|\mathbf{x}\|)$ is used instead of (1.3). Then, $W_{\mathbf{b},\varepsilon}$ can be considered as the set of matrices whose transposes are Ψ_ε-approximable with solutions restricted on the set $\{\mathbf{x} \in \mathbb{Z}^m : |\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}} > \varepsilon\}$.

4.1 Mass distributions on Ψ_ε-approximable matrices

In this subsection we prove the divergent part of Theorem 1.7 using mass distributions on Ψ_ε-approximable matrices following [1].

Lemma 4.2 *For each $mn - n < s \leq mn$ and $0 < \varepsilon < 1/2$, let $U_0 = d^{-1}\varepsilon S_0$ and f be a dimension function satisfying (1.5) and (1.6). Suppose that (1.7) holds. Then*

$$\sum_{q=\lceil T_0 \rceil}^{\infty} \frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right) < \infty$$
$$\iff \sum_{h=\lceil U_0 \rceil}^{\infty} h^{m+n-1}\left(\frac{\Psi_\varepsilon(h)}{h}\right)^{-n(m-1)} f\left(\frac{\Psi_\varepsilon(h)}{h}\right) < \infty.$$

Proof Similar to Lemma 3.1, we may replace the sums with integrals

$$\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx \text{ and } \int_{U_0}^{\infty} h^{m+n-1}\left(\frac{\Psi_\varepsilon(h)}{h}\right)^{-n(m-1)} f\left(\frac{\Psi_\varepsilon(h)}{h}\right) dh,$$

respectively.

Note that since $\Psi_\varepsilon(h) = d^{-1}\varepsilon\widetilde{\psi}(d\varepsilon^{-1}h)$, if we consider the term $\int_{U_0}^{\infty} h^{m+n-1}\left(\frac{\Psi_\varepsilon(h)}{h}\right)^{-n(m-1)} f\left(\frac{\Psi_\varepsilon(h)}{h}\right) dh$, then

$$\int_{U_0}^{\infty} h^{m+n-1}\left(\frac{\Psi_\varepsilon(h)}{h}\right)^{-n(m-1)} f\left(\frac{\Psi_\varepsilon(h)}{h}\right) dh < \infty$$
$$\iff \int_{S_0}^{\infty} y^{m+n-1}\left(\frac{\widetilde{\psi}(q)}{y}\right)^{-n(m-1)} f\left(\frac{\widetilde{\psi}(q)}{y}\right) dy < \infty.$$

Also, since $\widetilde{\psi}(y) = \psi^{-1}(y^{-m})^{-\frac{1}{n}}$, we have

$$\begin{aligned}
&\int_{S_0}^{\infty} y^{m+n-1}\left(\frac{\widetilde{\psi}(y)}{y}\right)^{-n(m-1)} f\left(\frac{\widetilde{\psi}(y)}{y}\right) dy \\
&= \int_{S_0}^{\infty} y^{mn+m-1}(\psi^{-1}(y^{-m}))^{m-1} f\left(\frac{(\psi^{-1}(y^{-m}))^{-\frac{1}{n}}}{y}\right) dy \\
&= \frac{1}{m}\int_{S_0^m}^{\infty} t^n(\psi^{-1}(t^{-1}))^{m-1} f\left(\frac{(\psi^{-1}(t^{-1}))^{-\frac{1}{n}}}{t^{\frac{1}{m}}}\right) dt \\
&= \frac{1}{m}\int_{\psi^{-1}(S_0^{-m})}^{\infty} x^{m-1}(\psi(x)^{-1})^n f\left(\frac{x^{-\frac{1}{n}}}{(\psi(x)^{-1})^{\frac{1}{m}}}\right) d\psi(x)^{-1} \\
&\asymp \frac{1}{m}\left(n - \frac{s}{m} + 1\right)^{-1} \int_{T_0}^{\infty} x^{m-1} f(x^{-\frac{1}{n}}) d(\psi(x)^{-1})^{n-\frac{s}{m}+1},
\end{aligned}$$

where in the second last line we used the change of variables $x = \psi^{-1}(t^{-1})$, $t = \psi(x)^{-1}$ and in the last line we used (1.5) and (1.7). Since it follows from (1.7) that $(x^{-\frac{1}{n}})^{\alpha} \leq (\psi(x)^{-1})^{-\frac{1}{m}} \leq (x^{-\frac{1}{n}})^{\frac{1}{\alpha}}$. Therefore by using (1.5) we can write

$$f((\psi(x)^{-1})^{-\frac{1}{m}} x^{-\frac{1}{n}}) \asymp (\psi(x)^{-1})^{-\frac{s}{m}} f(x^{-\frac{1}{n}}).$$

Using integration by parts

$$\int_{T_0}^{\infty} x^{m-1} f(x^{-\frac{1}{n}}) d(\psi(x)^{-1})^{n-\frac{s}{m}+1}$$

$$
\begin{aligned}
&= \big(\lim_{x\to\infty} x^{m-1}\psi(x)^{-n-1+\frac{s}{m}} f(x^{-\frac{1}{n}}) - T_0^{m-1}\psi(T_0)^{-n-1+\frac{s}{m}} f(T_0^{\frac{-1}{n}})\big) \\
&\quad + \int_{T_0}^{\infty} \left[-(m-1)x^{m-2} f(x^{-\frac{1}{n}}) + \frac{1}{n} x^{m-1} x^{-\frac{1}{n}-1} f'(x^{-\frac{1}{n}})\right] \psi(x)^{-n-1+\frac{s}{m}}\, dx \\
&= \lim_{x\to\infty} x^{m-1}\psi(x)^{-n-1+\frac{s}{m}} f(x^{-\frac{1}{n}}) - T_0^{m-1}\psi(T_0)^{-n-1+\frac{s}{m}} f(T_0^{\frac{-1}{n}}) \\
&\quad + \int_{T_0}^{\infty} \left[-(m-1) + \frac{1}{n} a(x^{-\frac{1}{n}})\right] x^{m-2} f(x^{-\frac{1}{n}}) \psi(x)^{-n-1+\frac{s}{m}}\, dx, \text{ by (1.6)} \\
&\asymp \lim_{x\to\infty} x^{m-1}\psi(x)^{-n-1} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) - T_0^{m-1}\psi(T_0)^{-n-1+\frac{s}{m}} f\left(\frac{\psi(T_0)^{\frac{1}{m}}}{T_0^{\frac{1}{n}}}\right) \\
&\quad + \int_{T_0}^{\infty} \left[\frac{1}{n} a(x^{-\frac{1}{n}}) - (m-1)\right] x^{m-2}\psi(x)^{-n-1} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx.
\end{aligned}
$$

Note that

$$
\int_{T_0}^{\infty} x^{m-2}\psi(x)^{-n-1} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx = \int_{T_0}^{\infty} x^{m-1}\psi(x)^{-n-1} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) d\log x. \quad (4.2)
$$

Thus the convergence of $\int_{T_0}^{\infty} x^{m-2}\psi(x)^{-1-n} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx$ gives that

$$
\lim_{x\to\infty} x^{m-1}\psi(x)^{-n-1} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) < \infty.
$$

Also observe that as $x \to \infty$, $a(x^{\frac{-1}{n}}) \to s$. Therefore

$$
\frac{1}{n} a(x^{\frac{-1}{n}}) - (m-1) \to \frac{s - n(m-1)}{n}
$$

which is finite and positive (since $s > mn - n$). Therefore the convergence of $\int_{T_0}^{\infty} x^{m-2}\psi(x)^{-1-n} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx$ gives the convergence of

$$
\frac{1}{n}\int_{T_0}^{\infty} a(x^{-\frac{1}{n}}) x^{m-2}\psi(x)^{-n-1} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx - (m-1)\int_{T_0}^{\infty} x^{m-2}\psi(x)^{-n-1} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx.
$$

Hence the convergence of

$$
\int_{T_0}^{\infty} \frac{1}{\psi(x)x^2}\left(\frac{x^{\frac{1}{n}}}{\psi(x)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(x)^{\frac{1}{m}}}{x^{\frac{1}{n}}}\right) dx \text{ or } \int_{S_0}^{\infty} y^{m+n-1}\left(\frac{\widetilde{\psi}(y)}{y}\right)^{-n(m-1)} f\left(\frac{\widetilde{\psi}(y)}{y}\right) dy,
$$

implies the convergence of other one since for T_0 large enough all summands in (4.2) are positive except the finite value

$$
-T_0^{m-1}\psi(T_0)^{-n-1+\frac{s}{m}} f\left(\frac{\psi(T_0)^{\frac{1}{m}}}{T_0^{\frac{1}{n}}}\right).
$$

□

Lemma 4.3 ([1, Section 5]) *Assume that* $\sum_{q=1}^{\infty}\frac{1}{\psi(q)q^2}\left(\frac{q^{\frac{1}{n}}}{\psi(q)^{\frac{1}{m}}}\right)^{mn} f\left(\frac{\psi(q)^{\frac{1}{m}}}{q^{\frac{1}{n}}}\right)=\infty$. *Fix* $0<\varepsilon<\frac{1}{2}$. *Then, for any* $\eta>1$ *there exists a probability measure* μ *on* $\limsup W_{S,\varepsilon}$ *satisfying the condition that for an arbitrary ball* D *of sufficiently small radius* $r(D)$ *we have*

$$\mu(D) \ll \frac{f(r(D))}{\eta},$$

where the implied constant does not depend on D *or* η.

Proof Note that $\limsup_{S\to\infty} W_{S,\varepsilon}=\{A\in[0,1]^{mn}: A^t\in W_{n,m}(\Psi_\varepsilon)\}$. By Lemma 4.2

$$\sum_{h=1}^{\infty} h^{m+n-1}\Big(\frac{\Psi_\varepsilon(h)}{h}\Big)^{-n(m-1)} f\Big(\frac{\Psi_\varepsilon(h)}{h}\Big)=\infty,$$

which is the divergent assumption of Theorem 2.3 for $W_{n,m}(\Psi_\varepsilon)$. From the proof of Jarnik's Theorem in [1] and the construction of probability measure in [1, Section 5] we can obtain a probability measure μ on $\limsup_{S\to\infty} W_{S,\varepsilon}$ satisfying the above condition. □

Let us prove the divergent part of Theorem 1.7.

Proof Assume that $mn+m-n<s<mn+m$ and fix $0<\varepsilon<\frac{1}{2}$. For any fixed $\eta>1$, let μ be a probability measure on $\limsup_{S\to\infty} W_{S,\varepsilon}$ as in Lemma 4.3 with $f(r(D))$ replaced by $r(D)^{-m}f(r(D))$.

Here we remark that since $f(r)$ satisfies (1.5) and (1.6) it is not hard to check that the new function $f^*(r):=\frac{f(r)}{r^m}$ satisfies conditions (1.5) and (1.6) with s replaced by $s-m$. Indeed, (1.5) (with s replaced by $s-m$) follows since $f^*(xy)=\frac{f(xy)}{(xy)^m}\asymp\frac{x^s f(y)}{x^m y^m}=x^{s-m}f^*(y)$, and (1.6) (with s replaced by $s-m$) follows since

$$\frac{rf^{*'}(r)}{f^*(r)}=\frac{r}{f^*(r)}[r^{-m}f'(r)-mr^{-m-1}f(r)]=\left[r\frac{f'(r)}{f(r)}-m\right]\to(s-m)\text{ as } r\to 0.$$

Now consider the product measure $\nu=\mu\times m_{\mathbb{R}^m}$, where $m_{\mathbb{R}^m}$ is the canonical Lebesgue measure on $\mathbb{R}^m$ and let π_1 and π_2 be the natural projections from $\mathbb{R}^{mn+m}$ to $\mathbb{R}^{mn}$ and $\mathbb{R}^m$, respectively.

For any fixed integer $N\geq 1$, let $V_{S,\varepsilon}=W_{S,\varepsilon}\setminus\bigcup_{k=N}^{S-1}W_{k,\varepsilon}$ and $\widehat{V}_{S,\varepsilon}=\{(A,\mathbf{b})\in\widehat{W}_{S,\varepsilon}: A\in V_{S,\varepsilon}\}$. Then $\nu(\bigcup_{S\geq N}\widehat{W}_{S,\varepsilon})=\nu(\bigcup_{S\geq N}\widehat{V}_{S,\varepsilon})\geq 1-2\varepsilon$, see [12, p.21].

Since $N\geq 1$ is arbitrary, we have $\nu(\limsup_{S\to\infty}\widehat{W}_{S,\varepsilon})\geq 1-2\epsilon$. For an arbitrary ball $B\subseteq\mathbb{R}^{mn+m}$ of sufficiently small radius $r(B)$, we have

$$\nu(B)\leq\mu(\pi_1(B))\times m_{\mathbb{R}^m}(\pi_2(B))\ll\frac{f(r(B))}{\eta},$$

where the implied constant does not depend on B or η. By using the Mass Distribution Principle i.e. Lemma 2.2 and the Transference Lemma i.e. Lemma 4.1, we have

$$\mathcal{H}^f(\widehat{D}_{m,n}(\psi)^c)\geq\mathcal{H}^f(\limsup_{S\to\infty}\widehat{W}_{S,\varepsilon})\gg(1-2\varepsilon)\eta,$$

and by letting $\eta\to\infty$ we obtain the desired result. □

4.2 Local ubiquity for $W_{\mathbf{b},\varepsilon}$

We will use the idea of local ubiquity for $W_{\mathbf{b},\varepsilon}$ to prove the divergent part of Theorem 1.8. Following [12] we define

$$\varepsilon(\mathbf{b}) = \min_{1\le j\le m,\ |b_j|_{\mathbb{Z}}>0} \frac{|b_j|_{\mathbb{Z}}}{4}, \tag{4.3}$$

for $\mathbf{b} = (b_1, \cdots, b_m) \in \mathbb{R}^m \setminus \mathbb{Z}^m$. Note that $\varepsilon(\mathbf{b}) > 0$ is due to the fact that $\mathbf{b} \in \mathbb{R}^m \setminus \mathbb{Z}^m$.

The following lemma is used when we count the number of integral vectors $\mathbf{z} \in \mathbb{Z}^m$ such that

$$|\mathbf{b}\cdot\mathbf{z}|_{\mathbb{Z}} \le \varepsilon(\mathbf{b}). \tag{4.4}$$

Lemma 4.4 ([12, Lemma 4.4]) *For $\mathbf{b} = (b_1, \cdots, b_m) \in \mathbb{R}^m \setminus \mathbb{Z}^m$, let $\varepsilon(\mathbf{b})$ be as in (4.3) and $1 \le i \le m$ be an index such that $\varepsilon(\mathbf{b}) = \frac{|b_i|_{\mathbb{Z}}}{4}$. Then, for any $\mathbf{x} \in \mathbb{Z}^m$, at most one of $\mathbf{x}$ and $\mathbf{x} + \mathbf{e}_i$ satisfies (4.4) where $\mathbf{e}_i$ denotes the vector with a 1 in the ith coordinate and 0's elsewhere.*

For a fixed $\mathbf{b} \in \mathbb{R}^m \setminus \mathbb{Z}^m$, let $\varepsilon_0 := \varepsilon(\mathbf{b})$, $\Psi_0 := \Psi_{\varepsilon_0}$ and $\Psi(h) = \frac{\Psi_0(h)}{h}$. With notions in the Subsection 2.2, which are defined for the ubiquitous system construction, let

$$J := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^m \times \mathbb{Z}^n : \|\mathbf{y}\| \le m\|\mathbf{x}\| \text{ and } |\mathbf{b}\cdot\mathbf{x}|_{\mathbb{Z}} > \varepsilon_0\} \text{ and} \tag{4.5}$$

$$\text{for } \kappa := (\mathbf{x}, \mathbf{y}) \in J \text{ denote } \beta_\kappa := \|\mathbf{x}\| \text{ and } R_\kappa := \{A \in [0, 1]^{mn} : A^t\mathbf{x} = \mathbf{y}\}. \tag{4.6}$$

Note that $W_{\mathbf{b},\varepsilon_0} \subset \Lambda(\Psi)$ and the family $\mathcal{R}$ of resonant sets R_κ consists of $(m-1)n$-dimensional, rational affine subspaces.

By Lemma 4.2, now we assume that the divergence part of Theorem 1.8 is satisfied. Then we can find a strictly increasing sequence of positive integers $\{h_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{h_{i-1}<h\le h_i}^{\infty} h^{m+n-1}\left(\frac{\Psi_0(h)}{h}\right)^{-n(m-1)} f\left(\frac{\Psi_0(h)}{h}\right) > 1 \tag{4.7}$$

and $h_i > 2h_{i-1}$. Put $\omega(h) := i^{\frac{1}{n}}$ if $h_{i-1} < h \le h_i$. Then

$$\sum_{h=1}^{\infty} h^{m+n-1}\left(\frac{\Psi_0(h)}{h}\right)^{-n(m-1)} f\left(\frac{\Psi_0(h)}{h}\right)\omega(h)^{-n} = \infty.$$

For a constant $c > 0$, define the ubiquitous function $\rho_c : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\rho_c(h) = \begin{cases} ch^{-\frac{1+n}{n}} & \text{if} \quad m = 1; \\ ch^{-\frac{m+n}{n}}\omega(h) & \text{if} \quad m \ge 2. \end{cases} \tag{4.8}$$

Clearly the ubiquitous function is 2-regular.

Theorem 4.5 *([12, Theorem 4.5]) The pair $(\mathcal{R}, \beta)$ is a locally ubiquitous system relative to $\rho = \rho_c$ for some constant $c > 0$.*

The divergent part of Theorem 1.8.

Assume that $(m-1)n < s \leq mn$ and $r^{-nm}f(r) \to \infty$ as $r \to 0$. It follows from Theorem 2.5 and Theorem 4.5 that

$$\mathcal{H}^f(\widehat{D}^{\mathbf{b}}_{m,n}(\psi)^c) \geq \mathcal{H}^f(W_{\mathbf{b},\varepsilon_0}) = \mathcal{H}^f([0,1]^{mn}).$$

Similar as in [12] here we have used the fact that the divergence and convergence of the sums

$$\sum_{N=1}^{\infty} 2^{\kappa N}\mathcal{F}(2^N) \text{ and } \sum_{h=1}^{\infty} h^{\kappa-1}\mathcal{F}(h)$$

coincide for any monotonic function $\mathcal{F} : \mathbb{Z}_+ \to \mathbb{Z}_+$ and $\kappa \in \mathbb{R}$. This completes the proof of the divergent part of Theorem 1.8.

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