Some existence results for the Toda system on closed surfaces

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ABSTRACT. Given a compact closed surface Σ, we consider the *generalized Toda* system of equations on Σ: $-\Delta u_i = \sum_{j=1}^{2} \rho_j a_{ij} \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} dV_g} - 1 \right)$ for $i = 1, 2$, where ρ_1, ρ_2 are real parameters and h_1, h_2 are smooth positive functions. Exploiting the variational structure of the problem and using a new minimax scheme we prove existence of solutions for generic values of ρ_1 and for $\rho_2 < 4\pi$.

Key Words: Toda System, Variational Methods, Minimax Schemes

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1 Introduction

The following system, defined on a domain $\Omega \subseteq \mathbb{R}^2$,

$$(1) \qquad -\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j}, \qquad i = 1, \dots, N,$$

where $A = (a_{ij})_{ij}$ is the *Cartan matrix* of $SU(N+1)$,

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix},$$

is known as the *Toda system*, and it arises in the study of non-abelian Chern-Simons theory, see for example [16] or [31].

In this paper we consider a generalized version of (1) on a closed surface Σ (which from now on we assume with total volume 1), namely

$$(2) \qquad -\Delta u_i = \sum_{j=1}^{N} \rho_j a_{ij} \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} dV_g} - 1 \right), \qquad i = 1, \dots, N,$$

where h_i are smooth and positive functions on the surface Σ. We specialize here to the case $N = 2$, so the system becomes

$$(3) \qquad \begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right); \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right), \end{cases} \qquad \text{on } \Sigma.$$

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Problem (3) is variational, and solutions can be found as critical points of a functional $J_\rho : H^1(\Sigma)\times H^1(\Sigma)$, $\rho = (\rho_1, \rho_2)$ defined as

$$J_\rho(u_1, u_2) = \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j dV_g \right] + \sum_{i=1}^{2} \rho_i \int_\Sigma u_i dV_g - \sum_{i=1}^{2} \rho_i \log \int_\Sigma h_i e^{u_i} dV_g.$$

Here a^{ij} are the entries of the inverse matrix A^{-1}.

The structure of the functional J_ρ strongly depends on the values of ρ_1 and ρ_2. For example, the condition $\rho_i \leq 4\pi$ for both $i = 1, 2$ has been proven in [18] to be necessary and sufficient for J_ρ to be bounded from below, see Theorem 2.1 (we refer also to [27] and [28]). In particular, for ρ_1 and ρ_2 strictly less than 4π, J_ρ becomes coercive (once we factor out the constants, since J_ρ is invariant under the transformation $u_i \mapsto u_i + c_i$, $c_i \in \mathbb{R}$) and solutions of (3) can be found as global minima.

The case in which one of the ρ_i's becomes equal to 4π (or both of them) is more subtle since the functional is still bounded from below but not coercive anymore. In [17] and [22] some conditions for existence are given in this case, and the proofs involve a delicate analysis of the limit behavior of the solutions when the ρ_i's converge to 4π from below.

On the other hand, when some of the ρ_i's is bigger than 4π, J_ρ is unbounded from below and solutions should be found as saddle points. In [23], [25] and [26] some existence results are given and it is proved that if $h_i \equiv 1$ and if some additional assumptions are satisfied, then $(0, 0)$ is a local minimizer for J_ρ, so the functional has a mountain pass structure and some corresponding critical points. Furthermore in [17] a very refined blow-up behavior of solutions is given (below we report Theorem 2.4 as a consequence of this analysis) and existence is proved if Σ has positive genus and if ρ_1, ρ_2 satisfy either **(i)** $\rho_1 < 4\pi$, $\rho_2 \in (4\pi, 8\pi)$ (and viceversa), or **(ii)** $\rho_1, \rho_2 \in (4\pi, 8\pi)$.

Our goal here is to give a general existence result when one of the coefficients ρ_i can be arbitrarily large. We have indeed the following theorem.

Theorem 1.1 *Suppose m is a positive integer, and let $h_1, h_2 : \Sigma \to \mathbb{R}$ be smooth positive functions. Then for $\rho_1 \in (4\pi m, 4\pi(m + 1))$ and for $\rho_2 < 4\pi$ problem (3) is solvable.*

Remark 1.2 *The solution (u_1, u_2) found in Theorem 1.1 is non-constant provided the following generic conditions hold: either $\rho_2 \neq 0$ and h_1 or h_2 are non-constant, or if $\rho_2 = 0$ and h_1 is non constant. In the latter case the system decouples and $u_2 = -\frac{1}{2}u_1$.*

Remark 1.3 *By Proposition 2.5 below, if we assume also that $\int_\Sigma u_i dV_g = 0$ for $i = 1, 2$, the solutions of (3) stay bounded in $C^l(\Sigma)$ for any integer l.*

To give an idea of the proof of Theorem 1.1 we recall first the analogy of (3) with some nonlinear scalar equations. First of all we should mention that (2) for $N = 1$

$$-\Delta u = 2\rho \left(\frac{he^u}{\int_\Sigma he^u dV_g} - 1 \right) \qquad \text{on } \Sigma \tag{4}$$

arises in the study of mean field limit of point vortices of Euler flows, spherical Onsager vortex theory and condensates in some Chern-Simons-Higgs models, see for example the papers [3], [4], [7], [8], [9], [10], [12], [20], [30] and the references therein.

We also mention the similarity of the scalar equation (4) with the geometric equations

$$-\Delta_g u + K_g = K_{\tilde{g}} e^{2u} \quad \text{on } \Sigma; \qquad P_g u + 2Q_g = 2Q_{\tilde{g}} e^{4u} \quad \text{on } M. \tag{5}$$

Here K_g is the Gauss curvature of Σ, Δ_g the Laplace-Beltrami operator, $\tilde{g} = e^{2u}g$ a conformal metric and $K_{\tilde{g}}$ the Gauss curvature of $\tilde{g}$. The second equation in (5) is the transformation law of the *Q-curvature* on a four-dimensional manifold M under a similar conformal change of metric, and P_g is the *Paneitz operator* associated to (M,g), see for example [15], [24] and the references therein.

We recall next the ideas used in [15] to find conformal metrics of constant Q-curvature. For the reader's convenience we transpose the discussion to equation (4), for which analogous considerations hold. Actually the method in [15] has been used in [13] to study (4) as well, in order to obtain existence results on surfaces of arbitrary genus.

Equation (4) also has variational structure and is the Euler equation of the functional

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 dV_g - 2\rho \int_\Sigma u dV_g - 2\rho \log \int_\Sigma h e^u dV_g; \qquad\qquad u \in H^1(\Sigma)$$

which, as before, is bounded from below if and only if $\rho \leq 4\pi$ by the Moser-Trudinger inequality, see (6). For $\rho > 4\pi$, instead of using degree theory, as in [9] and [10] one can indeed employ directly a minimax scheme based on improvements of (6). In fact, if the integral of e^u is distributed into ℓ different distinct regions, then (naively) the coefficient in (6) reduces by a factor ℓ. For a precise statement see Proposition 2.2 below. As a consequence one has that if $\rho \in (4k\pi, 4(k+1)\pi))$ and if $I_\rho(u_l) \to -\infty$ along a sequence u_l, then e^{u_l} has to concentrate near at most k points in Σ. For such a result we refer to Lemma 2.4 in [15] or in [13]. Assuming that $\int_\Sigma e^{u_l} dV_g = 1$, then we have that $e^{u_l} \rightharpoonup \sum_{i=1}^k t_i \delta_{x_i}$ for some non-negative coefficients t_i such that $\sum_{i=1}^k t_i = 1$. This family of formal convex combinations of Dirac deltas is known as the set of *formal barycenters* of Σ, see Section 2, and we denote it by Σ_k. We notice that for $k = 1$ the set Σ_1 is simply homeomorphic to Σ but for larger k the t_i's do not have any bound from below or the x_i's could collapse onto each-other, so the set could be degenerate near some of its points. In fact, Σ_k is a *stratified manifold*, namely union of sets of different dimensions. Nevertheless, since $e^{u_l} \rightharpoonup \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k$, with some work it is possible to build a continuous and non-trivial map Π_k from sublevels $\{I_\rho \leq -L\}$ (with L large) into Σ_k. By non-triviality we mean that this map is homotopically non-trivial, and indeed for any $L > 0$ there exists a map $\varphi : \Sigma_k \to \{I_\rho \leq -L\}$ (see (26) for the explicit formula, and Proposition 4.1 in [13] for the evaluation of I_ρ) such that $\Pi_k \circ \varphi$ is homotopic to the identity on Σ_k, which is non-contractible. This allows then to define a minimax scheme using maps from the topological cone over Σ_k with values into $H^1(\Sigma)$ (see e.g. [13], Section 5) which coincide with φ on Σ_k (the boundary of the cone).

Having sketched this argument for the scalar equation (4), we can now describe our approach to study system (3). First of all we prove a compactness result under the assumptions of Theorem 1.1, see Proposition 2.5. This result exploits the blow-up analysis in [17] when ρ_2 stays positive and away from zero. On the other hand, for $\rho_2 \in (-\infty, \delta]$ with δ positive and small, we use an argument inspired by Brezis and Merle, [6], combined with a compactness result in [21], see Theorem 2.3.

Next, a main ingredient in our proof is again an improved version of the Moser-Trudinger inequality for systems, which was given in [18], see Theorem 2.1. In Proposition 3.1 we see that, in analogy with the scalar case, if e^{u_1} is distributed among disjoint sets, then the Moser-Trudinger inequality improves and the bigger is the spreading, the better the improvement is. The argument relies both on Theorem 2.1 and Proposition 2.2. The way we use them is the following. Assuming e^{u_1} *spread* into ℓ sets $S_1, \dots, S_\ell$, we can find another ℓ-tuple $\tilde{S}_1, \dots, \tilde{S}_\ell \subseteq \Sigma$ such that each of these sets contain a fixed portion of the integral of e^{u_1}, and such that $\tilde{S}_1$ contains also a fixed portion of the integral of e^{u_2}, see Lemma 3.2. Then, by a localization argument through some cutoff functions $g_1, \dots, g_\ell$, we use the Moser-Trudinger inequality for systems near $\tilde{S}_1$, and the improved scalar inequality near $\tilde{S}_2, \dots, \tilde{S}_\ell$. In this step we employ some interpolation inequalities and some cutoffs in the Fourier modes of u_1, u_2 to deal with some lower order terms.

From the improved inequality we derive the following consequence. If $\rho_1 \in (4\pi m, 4\pi(m+1))$, if $\rho_2 < 4\pi$ and if $J_\rho(u_{1,l}, u_{2,l}) \to -\infty$ along a sequence $(u_{1,l}, u_{2,l})$, then $e^{u_{1,l}}$ has to concentrate near at most m points of Σ. Therefore, as for the scalar equation, we can map $e^{u_{1,l}}$ onto Σ_m for l large. Precisely, for $L \gg 1$ we can define a continuous projection $\Psi : \{J_\rho \leq -L\} \to \Sigma_m$ which is homotopically non-trivial. Indeed, recalling that Σ_m is non-contractible (see Lemma 2.6), there exists a map Φ such that $\Psi \circ \Phi$ is homotopic to the identity and such that $J_\rho(\Phi(\Sigma_m))$ can become arbitrarily large negative, so that Ψ is well-defined on its image.

Some comments on the construction of the map Φ are in order. If we want to obtain low values of J_ρ on a couple (u_1, u_2), since e^{u_1} has necessarily to concentrate near at most m points of Σ, a natural choice of the test functions (u_1, u_2) is $\left(\varphi_{\lambda,\sigma}, -\frac{1}{2}\varphi_{\lambda,\sigma}\right)$, where σ is any element of Σ_m, and where $\varphi_{\lambda,\sigma}$ is given in (26). In fact, as λ tends to infinity $e^{\varphi_{\lambda,\sigma}}$ converges to σ in the weak sense of distributions, while the choice of u_2 is done in such a way to obtain the best possible cancelation in the quadratic part of the functional, see Remark 4.3. We notice that this kind of function (for the case $m = 1$ only), was used in [18] to prove unboundedness of J_ρ from below if some of the ρ_i's is bigger than 4π. Letting σ varying in Σ_m, we get a full embedding of Σ_m into low sublevels of J_ρ through the map Φ.

At this point we are in position to run a minimax scheme similar to that described above, based on the topological cone over Σ_m. The scheme yields a Palais-Smale sequence for J_ρ, but since we cannot ensure convergence directly, following Struwe ([29]) we introduce the auxiliary functional $J_{t\rho}$ ($t\rho = (t\rho_1, t\rho_2)$) where t belongs to a small neighborhood of 1. Running the same scheme on the functional $J_{t\rho}$, via some monotonicity argument, yields existence of critical points for almost every value of t, and in particular along a sequence $t_k \to 1$. To conclude, it is sufficient to apply the compactness result in Proposition 2.5.

The plan of the paper is the following. In Section 2 we collect some preliminary results regarding the Moser-Trudinger inequality, the barycentric sets Σ_k and the proof of Proposition 2.5. In Section 3 we give an improved version of the inequality for systems, and we apply it to characterize the low sublevels of J_ρ in terms of the concentration of the function e^{u_1}, see Corollary 3.5. Then in Section 4 we introduce the topological argument to study (3). We first define the global projection Ψ onto Σ_m (where m is the integer in Theorem 1.1) and then we define also the map $\Phi : \Sigma_m \to H^1(\Sigma) \times H^1(\Sigma)$, proving that $\Psi \circ \Phi$ is homotopic to the identity on Σ_m. Finally we run the minimax scheme based on the topological cones over Σ_m.

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2 Notation and preliminaries

In this section we collect some useful preliminary facts. For $x, y \in \Sigma$ we denote by $d(x, y)$ the metric distance between x and y on Σ. In the same way, we denote by $d(S_1, S_2)$ the distance between two sets $S_1, S_2 \subseteq \Sigma$, namely

$$d(S_1, S_2) = \inf \{d(x, y) \, : \, x \in S_1, y \in S_2\} .$$

Recalling that we are assuming $Vol_g(\Sigma) := \int_\Sigma 1 dV_g = 1$, given a function $u \in L^1(\Sigma)$, we denote its average (or integral) as

$$\overline{u} = \int_\Sigma u dV_g.$$

Below, by C we denote large constants which are allowed to vary among different formulas or even within lines. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc.. Also constants with subscripts are allowed to vary.

We now recall some Moser-Trudinger type inequalities and compactness results. The functional under interest is the following

$$J_\rho(u_1, u_2) = \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j dV_g\right] + \sum_{i=1}^{2} \rho_i \int_\Sigma u_i dV_g - \sum_{i=1}^{2} \rho_i \log \int_\Sigma h_i e^{u_i} dV_g,$$

which for large values of ρ_1 and ρ_2 will be in general unbounded from below. In fact, there is a precise criterion for J_ρ to have this boundedness, which has been proved by Jost and Wang.

Theorem 2.1 *([18]) For $\rho = (\rho_1, \rho_2)$ the functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma)$ is bounded from below if and only if both ρ_1 and ρ_2 satisfy the inequality $\rho_i \leq 4\pi$.*

Concerning the scalar Moser-Trudinger inequality

$$\log \int_\Sigma e^{(u-\overline{u})} dV_g \leq C + \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 dV_g, \tag{6}$$

we have the following improvement which occurs if the integral of $e^{(u-\overline{u})}$ is distributed among different sets of positive mutual distance.

Proposition 2.2 *Let $S_1, \dots, S_\ell$ be subsets of Σ satisfying $dist(S_i, S_j) \geq \delta_0$ for $i \neq j$, and let $\gamma_0 \in \left(0, \frac{1}{\ell}\right)$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0)$ such that*

$$\log \int_\Sigma e^{(u-\overline{u})} dV_g \leq C + \frac{1}{16\ell\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla u|^2 dV_g$$

for all the functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{S_i} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \gamma_0; \qquad i \in \{1, \dots, \ell\}.$$

For the proof in the case $\ell = 2$ see [11]. We also refer the reader to [15].

We now recall the following compactness results from [21] and [17].

Theorem 2.3 *([21]) Let $(u_k)_k$ be a sequence of solutions of the equations*

$$-\Delta u_k = \lambda_k \left(\frac{V_k e^{u_k}}{\int_\Sigma V_k e^{u_k} dV_g} - W_k \right),$$

where $(V_k)_k$ and $(W_k)_k$ satisfy

$$\int_\Sigma W_k dV_g = 1; \qquad \|W_k\|_{C^1(\Sigma)} \leq C; \qquad |\log V_k| \leq C; \qquad \|\nabla V_k\|_{L^\infty(\Sigma)} \leq C,$$

and where $\lambda_k \to \lambda_0 > 0$, $\lambda_0 \neq 8k\pi$ for $k = 1, 2, \dots$. Then, under the additional constraint $\int_\Sigma u_k dV_g = 1$, $(u_k)_k$ stays uniformly bounded in $L^\infty(\Sigma)$.

Theorem 2.4 *([17]) Let m_1, m_2 be two non-negative integers, and suppose Λ_1, Λ_2 are two compact sets of the intervals $(4\pi m_1, 4\pi(m_1+1))$ and $(4\pi m_2, 4\pi(m_2+1))$ respectively. Then if $\rho_1 \in \Lambda_1$ and $\rho_2 \in \Lambda_2$ and if we impose $\int_\Sigma u_i dV_g = 0$, $i = 1, 2$, the solutions of (3) stay uniformly bounded in $L^\infty(\Sigma)$ (actually in every $C^l(\Sigma)$ with $l \in \mathbb{N}$).*

This theorem, as stated in [17], requires m_1 and m_2 to be positive. However it is clear from the blow-up analysis there that one can allow also zero values of m_1 or of m_2. Combining Theorem 2.3 and Theorem 2.4 we obtain another compactness result which includes all the possibilities of Theorem 1.1.

Proposition 2.5 *Suppose h_1, h_2 are smooth positive functions on Σ, and consider the sequence of solutions of the system*

$$\begin{cases} -\Delta u_{1,k} = 2\rho_{1,k} \left(\frac{h_1 e^{u_{1,k}}}{\int_\Sigma h_1 e^{u_{1,k}} dV_g} - 1 \right) - \rho_{2,k} \left(\frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} - 1 \right); \\ -\Delta u_{2,k} = 2\rho_{2,k} \left(\frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} - 1 \right) - \rho_{1,k} \left(\frac{h_1 e^{u_{1,k}}}{\int_\Sigma h_1 e^{u_{1,k}} dV_g} - 1 \right), \end{cases} \quad \text{on } \Sigma. \tag{7}$$

Suppose $(\rho_{1,k})_k$ lie in a compact set K_1 of $\cup_{i=1}^\infty (4i\pi, 4(i+1)\pi)$, and that $(\rho_{2,k})_k$ lie in a compact set K_2 of $(-\infty, 4\pi)$. Then, if $\int_\Sigma u_{i,k} dV_g = 0$ for $i = 1, 2$ and for $k \in \mathbb{N}$, the functions $(u_{1,k}, u_{2,k})$ of (7) stay uniformly bounded in $L^\infty(\Sigma) \times L^\infty(\Sigma)$.

PROOF. First of all we claim that the following property holds true: for any $p > 1$ there exists $\overline{\rho} > 0$ (depending on p, K_1, K_2, h_1 and h_2) such that for $\rho_{2,k} \leq \overline{\rho}$ the solutions of $(e^{u_{2,k}})_k$ stay uniformly bounded in $L^p(\Sigma)$.

The proof of this claim follows an argument in [6]: using the Green's representation formula and the fact that $\rho_1 > 0$ we find (recall that $\int_\Sigma u_{2,k} dV_g = 0$)

$$u_{2,k}(x) \leq C + \int_\Sigma G(x,y) \left(2\rho_{2,k} \frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} \right) dV_g(y),$$

where $G(x,y)$ is the Green's function of $-\Delta$ on Σ. Using the Jensen's inequality we then find

$$e^{p u_{2,k}(x)} \leq C \int_\Sigma \exp(2p\rho_{2,k} G(x,y)) \frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} dV_g(y).$$

Recalling that $G(x,y) \simeq \frac{1}{2\pi} \log \left(\frac{1}{d(x,y)} \right)$ and using also the Fubini theorem we get

$$\int_\Sigma e^{p u_{2,k}} dV_g \leq C \sup_{x \in \Sigma} \int_\Sigma \frac{1}{d(x,y)^{\frac{p\rho_{2,k}}{\pi}}} dV_g(y).$$

Now it is sufficient to take $\overline{\rho} = \frac{\pi}{2p}$ in order to obtain the claim.

For proving the proposition, in the case $\rho_{2,k} \geq \overline{\rho}$ we simply use Theorem 2.4, while for $\rho_{2,k} \leq \overline{\rho}$ we employ the above claim. In fact, from uniform L^p bounds on $e^{u_{2,k}}$ and from elliptic regularity theory, we obtain uniform $W^{2,p}$ bounds on the sequence $(v_k)_k$, where v_k is defined as the unique (we can assume that every v_k has zero average) solution of

$$-\Delta v_k = -\rho_{2,k} \left(\frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} - 1 \right).$$

Taking p sufficiently large, by the Sobolev embedding, we also obtain uniform $C^{1,\alpha}$ bounds on $(v_k)_k$ (and hence on $(e^{v_k})_k$). Now we write $u_{1,k} = w_{1,k} + v_k$, so that $w_{1,k}$ satisfies

$$-\Delta w_{1,k} = 2\rho_{1,k} \left(\frac{h_1 e^{v_k} e^{w_{1,k}}}{\int_\Sigma h_1 e^{v_k} e^{w_{1,k}} dV_g} - 1 \right).$$

Moreover, since we are assuming $\int_\Sigma u_{1,k} dV_g = 0$ and since $\int_\Sigma v_k dV_g = 0$ as well, we have that also $\int_\Sigma w_{1,k} dV_g = 0$. Hence, applying Theorem 2.3 with $u_k = w_{1,k}$, $\lambda_k = 2\rho_{1,k}$, $V_k = h_1 e^{v_k}$ and $W_k \equiv 1$, we obtain uniform bounds on $w_{1,k}$ in $L^\infty(\Sigma)$. Since $(v_k)_k$ stays uniformly bounded in $L^\infty(\Sigma)$, we also get uniform bounds on $u_{1,k}$ in $L^\infty(\Sigma)$. Then, from the second equation in (7) we also achieve uniform bounds on $u_{2,k}$ in $W^{2,p}(\Sigma)$ (and hence in $L^\infty(\Sigma)$ taking p large enough). This concludes the proof. ■

At this point some notation is in order. For $k \in \mathbb{N}$, we let Σ_k denote the family of formal sums

$$\Sigma_k = \sum_{i=1}^k t_i \delta_{x_i}; \qquad t_i \geq 0, \quad \sum_{i=1}^k t_i = 1, \quad x_i \in \Sigma, \tag{8}$$

where δ_x stands for the Dirac delta at the point $x \in \Sigma$. We endow this set with the weak topology of distributions. This is known in literature as the *formal set of barycenters* of Σ (of order k), see [1], [2], [5]. Although this is not in general a smooth manifold (except for $k = 1$), it is a *stratified set*, namely union of cells of different dimensions. The maximal dimension is $3k - 1$, when all the points x_i are distinct and all the t_i's belong to the open interval $(0,1)$.

Next we recall the following result from the last references (see also Lemma 3.7 in [15]), which is necessary in order to carry out the topological argument below.

Lemma 2.6 *(well-known) For any $k \geq 1$ one has $H_{3k-1}(\Sigma_k; \mathbb{Z}_2) \neq 0$. As a consequence Σ_k is non-contractible.*

If $\varphi \in C^1(\Sigma)$ and if $\sigma \in \Sigma_k$, we denote the action of σ on φ as

$$\langle \sigma, \varphi \rangle = \sum_{i=1}^{k} t_i \varphi(x_i), \qquad \sigma = \sum_{i=1}^{k} t_i \delta_{x_i}.$$

Moreover, if f is a non-negative L^1 function on Σ with $\int_\Sigma f dV_g = 1$, we can define a distance of f from Σ_k in the following way

$$dist(f, \Sigma_k) = \inf_{\sigma \in \Sigma_k} \sup \left\{ \left| \int_\Sigma f \varphi dV_g - \langle \sigma, \varphi \rangle \right| \mid \|\varphi\|_{C^1(\Sigma)} = 1 \right\}. \tag{9}$$

We also let

$$\mathcal{D}_{\varepsilon,k} = \left\{ f \in L^1(\Sigma) \; : \; f \geq 0, \|f\|_{L^1(\Sigma)} = 1, dist(f, \Sigma_k) < \varepsilon \right\}.$$

From a straightforward adaptation of the arguments of Proposition 3.1 in [15], we obtain the following result.

Proposition 2.7 *Let k be a positive integer, and for $\varepsilon > 0$ let $\mathcal{D}_{\varepsilon,k}$ be as above. Then there exists $\varepsilon_k > 0$, depending on k and Σ such that, for $\varepsilon \leq \varepsilon_k$ there exists a continuous map $\psi : \mathcal{D}_{\varepsilon,k} \to \Sigma_k$.*

Now we introduce some more notation. For any positive integer m, we let K_{Σ_m} denote the topological cone over Σ_m

$$K_{\Sigma_m} = (\Sigma_m \times [0,1]) /_\sim, \tag{10}$$

where the equivalence relation means that the set $\Sigma_m \times \{1\}$ is collapsed to a single point.

3 An improved Moser-Trudinger inequality with applications

In this section we present an improvement of the Moser-Trudinger type inequality for the Toda system given in [18]. The condition to get this improvement is that the integral of the function e^{u_1} is distributed among different sets with positive mutual distance. Our proof relies heavily on the main result in [18], and is combined with some arguments in [11] and [15]. As an application, see Corollary 3.5, we derive a characterization of the sublevels $\{J_\rho \leq -L\}$, for $L > 0$ large, in terms of the concentration of e^{u_1}.

3.1 The improved inequality

In this subsection we analyze the Moser-Trudinger inequality, depending on the *distribution* of the function e^{u_1}. A consequence of this inequality is that it allows to give an upper bound (depending on ρ_1) for the number of concentration points of e^{u_1}.

Proposition 3.1 *Let $\delta_0 > 0$, $\ell \in \mathbb{N}$, and let $S_1, \dots, S_\ell$ be subsets of Σ satisfying $dist(S_i, S_j) \geq \delta_0$ for $i \neq j$. Let $\gamma_0 \in \left(0, \frac{1}{\ell}\right)$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0, \ell, \Sigma)$ such that*

$$\ell \log \int_\Sigma e^{(u_1 - \overline{u}_1)} dV_g + \log \int_\Sigma e^{(u_2 - \overline{u}_2)} dV_g \leq C + \frac{1}{4\pi - \tilde{\varepsilon}} \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j dV_g \right]$$

provided the function u_1 satisfies the relations

$$\frac{\int_{S_i} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \gamma_0, \qquad i \in \{1, \dots, \ell\}. \tag{11}$$

Before proving the proposition, we state a preliminary lemma, which will be proved later on.

Lemma 3.2 *Under the assumptions of Proposition 3.1, there exist numbers $\tilde{\gamma}_0, \tilde{\delta}_0 > 0$, depending only on $\gamma_0, \delta_0, \Sigma$, and ℓ sets $\tilde{S}_1, \dots, \tilde{S}_\ell$ such that $d(\tilde{S}_i, \tilde{S}_j) \geq \tilde{\delta}_0$ for $i \neq j$ and such that*

$$\frac{\int_{\tilde{S}_1} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \tilde{\gamma}_0, \quad \frac{\int_{\tilde{S}_1} e^{u_2} dV_g}{\int_\Sigma e^{u_2} dV_g} \geq \tilde{\gamma}_0; \qquad \frac{\int_{\tilde{S}_i} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \tilde{\gamma}_0, \quad i \in \{2, \dots, \ell\}. \tag{12}$$

Proof of Proposition 3.1. We modify the argument in [11] and [15]. Let $\tilde{S}_1, \dots, \tilde{S}_\ell$ be given by Lemma 3.2. Assuming without loss of generality that $\overline{u}_1 = \overline{u}_2 = 0$, we can find ℓ functions $g_1, \dots, g_\ell$ satisfying the properties

$$\begin{cases} g_i(x) \in [0,1], & \text{for every } x \in \Sigma; \\ g_i(x) = 1, & \text{for every } x \in \tilde{S}_i, i = 1, \dots, \ell; \\ supp(g_i) \cap supp(g_j) = \emptyset, & \text{for } i \neq j; \\ \|g_i\|_{C^2(\Sigma)} \leq C_{\tilde{\delta}_0}, \end{cases} \tag{13}$$

where $C_{\tilde{\delta}_0}$ is a positive constant depending only on $\tilde{\delta}_0$. We decompose the functions u_1 and u_2 in the following way

$$u_1 = \hat{u}_1 + \tilde{u}_1; \qquad u_2 = \hat{u}_2 + \tilde{u}_2, \qquad \hat{u}_1, \hat{u}_2 \in L^\infty(\Sigma). \tag{14}$$

The explicit decomposition (via some truncation in the Fourier modes) will be chosen later on. Using (12), for any $b = 2, \dots, \ell$ we can write that

$$\begin{aligned} \ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &= \log \left[\left(\int_\Sigma e^{u_1} dV_g \int_\Sigma e^{u_2} dV_g \right) \left(\int_\Sigma e^{u_1} dV_g \right)^{\ell-1} \right] \\ &\leq \log \left[\left(\int_{\tilde{S}_1} e^{u_1} dV_g \int_{\tilde{S}_1} e^{u_2} dV_g \right) \left(\int_{\tilde{S}_b} e^{u_1} dV_g \right)^{\ell-1} \right] - (\ell+1) \log \tilde{\gamma}_0 \\ &\leq \log \left[\left(\int_\Sigma e^{g_1 u_1} dV_g \int_\Sigma e^{g_1 u_2} dV_g \right) \left(\int_\Sigma e^{g_b u_1} dV_g \right)^{\ell-1} \right] \\ &- (\ell+1) \log \tilde{\gamma}_0, \end{aligned}$$

where C is independent of u_1 and u_2.

Now, using the fact that $\hat{u}_1$ and $\hat{u}_2$ belong to $L^\infty(\Sigma)$, we also write

$$\begin{aligned} \ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &\leq \log \left[\left(\int_\Sigma e^{g_1 \tilde{u}_1} dV_g \int_\Sigma e^{g_1 \tilde{u}_2} dV_g \right) \left(\int_\Sigma e^{g_b \tilde{u}_1} dV_g \right)^{\ell-1} \right] \\ &- (\ell+1) \log \tilde{\gamma}_0 + \ell (\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}). \end{aligned}$$

Therefore we get

$$\begin{aligned} \ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &\leq \log \int_\Sigma e^{g_1 \tilde{u}_1} dV_g + \log \int_\Sigma e^{g_1 \tilde{u}_2} dV_g + (\ell-1) \int_\Sigma e^{g_b \tilde{u}_2} dV_g \\ &- (\ell+1) \log \tilde{\gamma}_0 + \ell (\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}). \end{aligned} \tag{15}$$

At this point we can use Theorem 2.1 with parameters $(4\pi, 4\pi)$, applied to the couple $(g_1 \tilde{u}_1, g_1 \tilde{u}_2)$, and the standard Moser-Trudinger inequality (6) applied to $g_b \tilde{u}_1$ and we get the following estimates

$$\begin{aligned} \log \int_\Sigma e^{g_1 \tilde{u}_1} dV_g + \log \int_\Sigma e^{g_1 \tilde{u}_2} dV_g &\leq \frac{1}{4\pi} \left[\frac{1}{2} \sum_{i,j=1}^2 \int_\Sigma a^{ij} \nabla(g_1 \tilde{u}_i) \cdot \nabla(g_1 \tilde{u}_j) dV_g \right] \\ &+ \left(\overline{g_1 \tilde{u}_1} + \overline{g_1 \tilde{u}_2} \right) + C; \end{aligned} \tag{16}$$

$$(\ell-1)\int_\Sigma e^{g_b\tilde{u}_1}dV_g \leq \frac{(\ell-1)}{16\pi}\int_\Sigma |\nabla(g_b\tilde{u}_1)|^2 dV_g + (\ell-1)\overline{g_b\tilde{u}_1} + (\ell-1)C.$$

Now we notice that for $N=2$ one has

$$a^{ij} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix},$$

and that

$$\frac{1}{2}\sum_{i,j=1}^{2} a^{ij}\xi_i \cdot \xi_j = \frac{1}{4}|\xi_1|^2 + \frac{1}{12}|\xi_1 + 2\xi_2|^2. \tag{17}$$

This implies

$$\frac{1}{2}\sum_{i,j} a^{ij}\xi_i \cdot \xi_j \geq \frac{1}{4}|\xi_1|^2 \qquad \text{for every couple } (\xi_1,\xi_2) \in T_x\Sigma \times T_x\Sigma. \tag{18}$$

Applying this inequality to $(\nabla(g_b\tilde{u}_1), \nabla(g_b\tilde{u}_2))$ and integrating one finds

$$\frac{(\ell-1)}{16\pi}\int_\Sigma |\nabla(g_b\tilde{u}_1)|^2 dV_g \leq \frac{(\ell-1)}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla(g_b\tilde{u}_i)\cdot\nabla(g_b\tilde{u}_j)dV_g\right]. \tag{19}$$

Putting together (15)-(19) we then obtain

$$\begin{aligned} \ell\log\int_\Sigma e^{u_1}dV_g + \log\int_\Sigma e^{u_2}dV_g \quad &\leq \quad \frac{1}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla(g_1\tilde{u}_i)\cdot\nabla(g_1\tilde{u}_j)dV_g\right] \\ &+ \quad \frac{(\ell-1)}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla(g_b\tilde{u}_i)\cdot\nabla(g_b\tilde{u}_j)dV_g\right] \\ &+ \quad \left(\overline{g_1\tilde{u}_1} + \overline{g_1\tilde{u}_2}\right) + (\ell-1)\overline{g_b\tilde{u}_1} + \ell C - (\ell+1)\log\tilde{\gamma}_0 \\ &+ \quad \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}). \end{aligned} \tag{20}$$

Now we notice that, by interpolation, for any $\varepsilon > 0$ there exists $C_{\varepsilon,\tilde{\delta}_0}$ (depending only on ε and $\tilde{\delta}_0$) such that for $a = 1,\dots,\ell$

$$\begin{aligned} \left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla(g_a\tilde{u}_i)\cdot\nabla(g_a\tilde{u}_j)dV_g\right] \quad &\leq \quad \left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma g_a^2 a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_j dV_g\right] \\ &+ \quad \varepsilon\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_j dV_g\right] + C_{\varepsilon,\tilde{\delta}_0}\int_\Sigma(\tilde{u}_1^2+\tilde{u}_2^2)dV_g. \end{aligned} \tag{21}$$

Inserting this inequality into (20) we get

$$\begin{aligned} \ell\log\int_\Sigma e^{u_1}dV_g + \log\int_\Sigma e^{u_2}dV_g \quad &\leq \quad \frac{1}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma g_1^2 a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_j dV_g\right] \\ &+ \quad \frac{(\ell-1)}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma g_b^2 a^{ij}\nabla u_i\cdot\nabla\tilde{u}_j dV_g\right] \\ &+ \quad \frac{\ell}{4\pi}\varepsilon\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_j dV_g\right] + \ell C_{\varepsilon,\tilde{\delta}_0}\int_\Sigma(\tilde{u}_1^2+\tilde{u}_2^2)dV_g \\ &+ \quad \left(\overline{g_1\tilde{u}_1} + \overline{g_1\tilde{u}_2}\right) + (\ell-1)\overline{g_b\tilde{u}_1} + \ell C - (\ell+1)\log\tilde{\gamma}_0 \\ &+ \quad \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}), \end{aligned}$$

for $b = 2, \dots, \ell$.

We now choose $b \in \{2, \dots, \ell\}$ such that

$$\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma g_b^2 a^{ij}\nabla u_i \cdot \nabla \tilde{u}_j dV_g \leq \frac{1}{\ell-1}\frac{1}{2}\sum_{i,j=1}^{2}\int_{\cup_{s=1+1}^{\ell} supp(g_s)} a^{ij}\nabla u_i \cdot \nabla \tilde{u}_j dV_g.$$

Since the $g_i's$ have disjoint supports, see (13), the last formula yields

$$\begin{aligned}\ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &\leq \frac{1}{4\pi}(1+\ell\varepsilon)\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla \tilde{u}_i \cdot \nabla \tilde{u}_j dV_g\right] \\ &+ \ell C_{\varepsilon,\tilde{\delta}_0}\int_\Sigma (\tilde{u}_1^2 + \tilde{u}_2^2) dV_g + \left(\overline{g_1\tilde{u}_1} + \overline{g_1\tilde{u}_2}\right) + (\ell-1)\overline{g_b\tilde{u}_1} \\ &+ \ell C - (\ell+1)\log\tilde{\gamma}_0 + \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).\end{aligned}$$

By elementary estimates we find

$$\begin{aligned}\ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &\leq \frac{1}{4\pi}(1+\ell\varepsilon)\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla \tilde{u}_i \cdot \nabla \tilde{u}_j dV_g\right] \\ &+ C_{\varepsilon,\tilde{\delta}_0,\ell}\int_\Sigma (\tilde{u}_1^2 + \tilde{u}_2^2) dV_g \\ &+ C_{\varepsilon,\tilde{\delta}_0,\ell,\tilde{\gamma}_0} + \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).\end{aligned}$$

Now comes the choice of $\hat{u}_1$ and $\hat{u}_2$, see (14). We choose $\tilde{C}_{\varepsilon,\tilde{\delta}_0,\ell}$ to be so large that the following property holds

$$C_{\varepsilon,\tilde{\delta}_0,\ell}\int_\Sigma (v_1^2 + v_2^2) dV_g < \frac{\varepsilon}{2}\int_\Sigma a^{ij}\nabla v_i \cdot \nabla v_j dV_g, \qquad \forall v_1, v_2 \in V_{\varepsilon,\tilde{\delta}_0,\ell} \oplus V_{\varepsilon,\tilde{\delta}_0,\ell},$$

where $V_{\varepsilon,\tilde{\delta}_0,\ell}$ denotes the span of the eigenfunctions of the Laplacian on Σ corresponding to eigenvalues bigger than $\tilde{C}_{\varepsilon,\tilde{\delta}_0,\ell}$.

Then we set

$$\hat{u}_i = P_{V_{\varepsilon,\tilde{\delta}_0,\ell}} u_i; \qquad \tilde{u}_i = P_{V^\perp_{\varepsilon,\tilde{\delta}_0,\ell}} u_i,$$

where $P_{V_{\varepsilon,\tilde{\delta}_0,\ell}}$ (resp. $P_{V^\perp_{\varepsilon,\tilde{\delta}_0,\ell}}$) stands for the orthogonal projection onto $V_{\varepsilon,\tilde{\delta}_0,\ell}$ (resp. $V^\perp_{\varepsilon,\tilde{\delta}_0,\ell}$). Since $\overline{u}_i = 0$, the H^1-norm and the L^∞-norm on $V_{\varepsilon,\tilde{\delta}_0,\ell}$ are equivalent (with a proportionality factor which depends on $\varepsilon, \tilde{\delta}_0$ and ℓ), hence by our choice of u_1 and u_2 there holds

$$\|\hat{u}_i\|^2_{L^\infty(\Sigma)} \leq \hat{C}_{\varepsilon,\tilde{\delta}_0,\ell}\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla u_i \cdot \nabla u_j dV_g; \qquad C_{\varepsilon,\tilde{\delta}_0,\ell}\int_\Sigma (\tilde{u}_1^2 + \tilde{u}_2^2) dV_g < \frac{\varepsilon}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla v \cdot \nabla v_j dV_g.$$

Hence the last formulas imply

$$\begin{aligned}\ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &\leq \frac{1}{4\pi}(1+3\ell\varepsilon)\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla \tilde{u}_i \cdot \nabla \tilde{u}_j dV_g\right] \\ &+ \hat{C}_{\varepsilon,\tilde{\delta}_0,\ell,\tilde{\gamma}_0}.\end{aligned}$$

This concludes the proof. ■

Proof of Lemma 3.2. First of all we fix a number $r_0 < \frac{\delta_0}{80}$. Then we cover Σ with a finite union of metric balls $(B_{r_0}(x_l))_l$. The number of these balls can be bounded by an integer N_{r_0} which depends only on r_0 (and Σ).

Next we cover the closure $\overline{S}_i$ of every set S_i by a finite number of these balls, and we choose a point $y_i \in \cup_l \{x_l\}$ such that

$$\int_{B_{r_0}(y_i)} e^{u_1} dV_g = \max \left\{ \int_{B_{r_0}(x_l)} e^{u_1} dV_g \ : \ B_{r_0}(x_l) \cap \overline{S}_i \neq \emptyset \right\}.$$

We also choose $y \in \cup_l \{x_l\}$ such that

$$\int_{B_{r_0}(y)} e^{u_2} dV_g = \max_l \int_{B_{r_0}(x_l)} e^{u_2} dV_g.$$

Since the total number of balls is bounded by N_{r_0} and since by our assumption the integral of e^{u_1} over S_i is greater or equal than γ_0, it follows that

$$\frac{\int_{B_{r_0}(y_i)} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \frac{\gamma_0}{N_{r_0}}; \qquad \frac{\int_{B_{r_0}(y)} e^{u_2} dV_g}{\int_\Sigma e^{u_2} dV_g} \geq \frac{1}{N_{r_0}}. \tag{22}$$

By the properties of the sets S_i, we have that

$$B_{20 r_0}(y_i) \cap B_{r_0}(y_j) \text{ for } i \neq j; \qquad card \{ y_s \ : \ B_{r_0}(y_s) \cap B_{20 r_0}(y) \neq \emptyset \} \leq 1.$$

In other words, if we fix y_i, the ball $B_{20r_0}(y_i)$ intersects no other of the balls $B_{r_0}(y_j)$ except $B_{r_0}(y_i)$, and given y, $B_{20r_0}(y)$ intersects at most one of the balls $B_{r_0}(y_i)$.

Now, by a relabeling of the points y_i, we can assume that one of the following two possibilities occurs

(a) $B_{20r_0}(y) \cap B_{r_0}(y_1) \neq \emptyset$ (and hence $B_{20r_0}(y) \cap B_{r_0}(y_i) = \emptyset$ for $i > 1$)

(b) $B_{20r_0}(y) \cap B_{r_0}(y_i) = \emptyset$ for every $i = 1, \dots, \ell$.

In case **(a)** we define the sets $\tilde{S}_i$ as

$$\tilde{S}_i = \begin{cases} B_{10r_0}(y_1) \cup B_{10r_0}(y) & \text{for } i = 1; \\ B_{10r_0}(y_i), & \text{for } i = 2 \dots \ell, \end{cases}$$

while in case **(b)** we define

$$\tilde{S}_i = B_{30r_0}(y_i), \qquad \text{for } i = 1, \dots, \ell.$$

We also set $\tilde{\gamma}_0 = \frac{\gamma_0}{N_{r_0}}$ and $\tilde{\delta}_0 = 5r_0$. We notice that $\tilde{\gamma}_0$ and $\tilde{\delta}_0$ depend only on γ_0, δ_0 and Σ, as claimed, and that the sets $\tilde{S}_i$ satisfy the required conditions. This concludes the proof of the lemma. ■

3.2 Application to the study of J_ρ

In this subsection we apply the improved inequality in order to understand the structure of the sublevels of J_ρ. Our main result here is Corollary 3.5.

In the next lemma we show a criterion which implies the situation described by (11). The result is proven in [15] Lemma 2.3, but we repeat here the argument for the reader's convenience.

Lemma 3.3 *Let $f \in L^1(\Sigma)$ be a non-negative function with $\|f\|_{L^1(\Sigma)} = 1$. We also fix an integer ℓ and suppose that the following property holds true. There exist $\varepsilon > 0$ and $r > 0$ such that*

$$\int_{\cup_{i=1}^{\ell} B_r(p_i)} f dV_g \leq 1 - \varepsilon \qquad \textit{for all the } \ell\textit{-tuples } p_1, \dots, p_\ell \in \Sigma.$$

Then there exist $\overline{\varepsilon} > 0$ and $\overline{r} > 0$, depending only on ε, r, ℓ and Σ (and not on f), and $\ell + 1$ points $\overline{p}_1, \dots, \overline{p}_{\ell+1} \in \Sigma$ (which depend on f) satisfying

$$\int_{B_{\overline{r}}(\overline{p}_1)} f dV_g > \overline{\varepsilon}, \ \dots, \ \int_{B_{\overline{r}}(\overline{p}_{\ell+1})} f dV_g > \overline{\varepsilon}; \qquad B_{2\overline{r}}(\overline{p}_i) \cap B_{2\overline{r}}(\overline{p}_j) = \emptyset \textit{ for } i \neq j.$$

Proof. Suppose by contradiction that for every $\overline{\varepsilon}, \overline{r} > 0$ and for any $\ell+1$ points $p_1, \dots, p_{\ell+1} \in \Sigma$ there holds

$$(23) \qquad \int_{B_{\overline{r}}(p_1)} f dV_g \geq \overline{\varepsilon}, \; \dots, \; \int_{B_{\overline{r}}(p_{\ell+1})} f dV_g \geq \overline{\varepsilon} \qquad \Rightarrow \qquad B_{2\overline{r}}(p_i) \cap B_{2\overline{r}}(p_j) \neq \emptyset \text{ for some } i \neq j.$$

We let $\overline{r} = \frac{r}{8}$, where r is given in the statement. We can find $h \in \mathbb{N}$ and h points $x_1, \dots, x_h \in \Sigma$ such that Σ is covered by $\cup_{i=1}^h B_{\overline{r}}(x_i)$. If ε is as above, we also set $\overline{\varepsilon} = \frac{\varepsilon}{2h}$. We point out that the choice of $\overline{r}$ and $\overline{\varepsilon}$ depends on r, ε and Σ only, as required.

Let $\{\tilde{x}_1, \dots, \tilde{x}_j\} \subseteq \{x_1, \dots, x_h\}$ be the points for which $\int_{B_{\overline{r}}(\tilde{x}_i)} f dV_g \geq \overline{\varepsilon}$. We define $\tilde{x}_{j_1} = \tilde{x}_1$, and let A_1 denote the set

$$A_1 = \{\cup_i B_{\overline{r}}(\tilde{x}_i) \; : \; B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_1}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_1}).$$

If there exists $\tilde{x}_{j_2}$ such that $B_{2\overline{r}}(\tilde{x}_{j_2}) \cap B_{2\overline{r}}(\tilde{x}_{j_1}) = \emptyset$, we define

$$A_2 = \{\cup_i B_{\overline{r}}(\tilde{x}_i) \; : \; B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_2}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_2}).$$

Proceeding in this way, we define recursively some points $\tilde{x}_{j_3}, \tilde{x}_{j_4}, \dots, \tilde{x}_{j_s}$ satisfying

$$B_{2\overline{r}}(\tilde{x}_{j_s}) \cap B_{2\overline{r}}(\tilde{x}_{j_a}) = \emptyset \; \forall 1 \leq a < s;$$

and some sets $A_3, \dots, A_s$ by

$$A_s = \{\cup_i B_{\overline{r}}(\tilde{x}_i) \; : \; B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_s}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_s}).$$

By (23), the process cannot go further than $\tilde{x}_{j_\ell}$, and hence $s \leq \ell$. Using the definition of $\overline{r}$ we obtain

$$(24) \qquad \cup_{i=1}^j B_{\overline{r}(\tilde{x}_i)} \subseteq \cup_{i=1}^s A_i \subseteq \cup_{i=1}^s B_{4\overline{r}}(\tilde{x}_{j_i}) \subseteq \cup_{i=1}^s B_r(\tilde{x}_{j_i}).$$

Then by our choice of h, $\overline{\varepsilon}$, $\{\tilde{x}_1, \dots, \tilde{x}_j\}$ and by (24) there holds

$$\int_{\Sigma \setminus \cup_{i=1}^s B_r(\tilde{x}_{j_i})} f dV_g \leq \int_{\Sigma \setminus \cup_{i=1}^j B_{\overline{r}(\tilde{x}_i)}} f dV_g < (h-j)\overline{\varepsilon} \leq \frac{\varepsilon}{2}.$$

Finally, if we chose $\overline{p}_i = \tilde{x}_{j_i}$ for $i = 1, \dots, s$ and $\overline{p}_i = \tilde{x}_{j_s}$ for $i = s+1, \dots, \ell$, we get a contradiction to the assumptions. ■

Next we characterize the functions in $H^1(\Sigma) \times H^1(\Sigma)$ for which the value of J_ρ is large negative.

Lemma 3.4 *Suppose $\rho_1 \in (4\pi m, 4\pi(m+1))$ and that $\rho_2 < 4\pi$. Then for any $\varepsilon > 0$ and any $r > 0$ there exists a large positive $L = L(\varepsilon, r)$ such that for every $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ with $J_\rho(u) \leq -L$ and with $\int_\Sigma e^{u_i} dV_g = 1$, $i = 1, 2$, there exists m points $p_{1,u_1}, \dots, p_{m,u_1} \in \Sigma$ such that*

$$(25) \qquad \int_{\Sigma \setminus \cup_{i=1}^m B_r(p_{i,u_1})} e^{u_1} dV_g < \varepsilon.$$

Proof. Suppose by contradiction that the statement is not true. Then we can apply Lemma 3.3 with $\ell = m+1$ and $f = e^{u_1}$ to obtain $\hat{\delta}_0$, $\hat{\gamma}_0$ and sets $\hat{S}_1, \dots \hat{S}_{m+1}$ such that

$$d(\hat{S}_i, \hat{S}_j) \geq \hat{\delta}_0, \qquad i \neq j;$$

$$\int_{\hat{S}_i} e^{u_1} dV_g > \hat{\gamma}_0 \int_\Sigma e^{u_1} dV_g, \qquad i = 1, \dots, m+1.$$

Now we notice that, by the normalization of the u_i's and the Jensen's inequality, there holds $\int_\Sigma u_i dV_g \leq 0$ for $i = 1, 2$, and that two cases may occur

(a) $\rho_2 \leq 0$;

(b) $\rho_2 > 0$.

In case **(a)** we have that $-\rho_2 \int_\Sigma u_2 dV_g \geq 0$. Using also inequality (18) we find

$$J_\rho(u_1,u_2) \geq \frac{1}{4}\int_\Sigma |\nabla u_1|^2 dV_g + \rho_1 \int_\Sigma u_1 dV_g - C.$$

Now it is sufficient to use Proposition 2.2 with $\ell = m+1$, $\delta_0 = \hat{\delta}_0$, $\gamma_0 = \hat{\gamma}_0$, $S_j = \hat{S}_j$, $j = 1, \dots, m+1$ and $\tilde{\varepsilon} \in (0, 16\pi(m+1) - 4\rho_1)$, to get

$$\begin{aligned} J_\rho(u_1,u_2) &\geq \frac{1}{4}\int_\Sigma |\nabla u_1|^2 dV_g - \frac{\rho_1}{16\pi(m+1)-\tilde{\varepsilon}}\int_\Sigma |\nabla u_1|^2 dV_g - C \\ &\geq \frac{16\pi(m+1) - 4\rho_1 - \tilde{\varepsilon}}{4\left[16\pi(m+1)-\tilde{\varepsilon}\right]}\int_\Sigma |\nabla u_1|^2 dV_g - \tilde{C}, \end{aligned}$$

where $\tilde{C}$ is independent of (u_1,u_2). This contradicts the fact that $J_\rho(u) \leq -L$ if L is large enough.

In case **(b)** we use Proposition 3.1 with $\delta_0 = \hat{\delta}_0$, $\gamma_0 = \hat{\gamma}_0$, $\ell = m+1$, $S_j = \hat{S}_j$ and $\tilde{\varepsilon}$ such that $(4\pi - \tilde{\varepsilon})(m+1) > \rho_1$ and such that $4\pi - \tilde{\varepsilon} > \rho_2$ (recall that ρ_1 is strictly less than $4\pi(m+1)$ and that $\rho_2 < 4\pi$), to deduce

$$\begin{aligned} J_\rho(u_1,u_2) &\geq (4\pi - \tilde{\varepsilon})\left[-(m+1)\overline{u}_1 - \overline{u}_2\right] + \rho_1\overline{u}_1 + \rho_2\overline{u}_2 \\ &= (\rho_1 - (m+1)(4\pi - \tilde{\varepsilon}))\,\overline{u}_1 + (\rho_2 - 4\pi + \tilde{\varepsilon})\,\overline{u}_2 - C \geq -C, \end{aligned}$$

by the Jensen inequality, where, again, $\tilde{C}$ is independent of (u_1,u_2). In this way we arrive to a contradiction as before. This concludes the proof. ■

As a consequence of Lemma 3.4 we have the following result, regarding the distance of the functions e^{u_1} (suitably normalized) from Σ_m, see (9).

Corollary 3.5 *Let $\overline{\varepsilon}$ be a (small) arbitrary positive number, and let $\rho_1 \in (4\pi m, 4\pi(m+1))$, $\rho_2 < 4\pi$. Then there exists $L > 0$ such that, if $J_\rho(u_1,u_2) \leq -L$ and if $\int_\Sigma e^{u_1} dV_g = 1$, we have $dist(e^{u_1}, \Sigma_m) < \overline{\varepsilon}$.*

Proof. We consider ε and r small and positive (to be fixed later), and we let L be the corresponding constant given by Lemma 3.4. We let $p_1, \dots, p_m$ denote the corresponding points. Now we define $\sigma \in \Sigma_m$ by

$$\sigma = \sum_{j=1}^m t_j \delta_{p_j}; \qquad \text{where} \qquad t_j = \int_{A_{r,j}} e^{u_1} dV_g, \quad A_{r,j} := B_r(p_j) \setminus \cup_{k=1}^{j-1} B_r(p_k).$$

Notice that all the sets $A_{r,j}$'s are disjoint by construction. Now, given $\varphi \in C^1(\Sigma)$ with $\|\varphi\|_{C^1(\Sigma)} = 1$, (using also (25)) we have that $\cup_{j=1}^m B_r(p_j) = \cup_{j=1}^m A_{r,j}$ and that

$$\left| \int_{\Sigma \setminus \cup_{j=1}^m B_r(p_j)} e^{u_1} \varphi dV_g \right| < \varepsilon; \qquad \left| \int_{A_{r,j}} \varphi e^{u_1} dV_g - t_j \varphi(p_j) \right| \leq C_\Sigma r \|\varphi\|_{C^1(\Sigma)} \leq C_\Sigma r.$$

By (9) then it follows that

$$dist(e^{u_1}, \Sigma_m) \leq \sup \left\{ \left| \int_\Sigma e^{u_1} \varphi dV_g - \langle \sigma, \varphi \rangle \right| \;|\; \|\varphi\|_{C^1(\Sigma)} = 1 \right\} \leq \varepsilon + m C_\Sigma r.$$

Now it is sufficient to choose ε and r such that $\varepsilon + mC_\Sigma r < \overline{\varepsilon}$. This concludes the proof. ■

4 The minimax argument

In this section we perform the topological construction to be used in order to produce solutions of (3). First of all, Corollary 3.5 allows to construct a projection Ψ from suitable sublevels of J_ρ onto Σ_m. Next, the main idea is to use for the minimax some maps from the cone K_m over Σ_m, see (10), into $H^1(\Sigma) \times H^1(\Sigma)$. We impose that these maps at the boundary all coincide with a given function Φ, which is defined in the next subsection.

The map Φ is chosen so that (see Proposition 4.2) $\Psi \circ \Phi$ is homotopic to the identity on Σ_m, and so that the functional J_ρ on the image is very large negative. Considering then the image of K_m with respect to the above maps (with fixed boundary datum), in Proposition 2.7 we will verify that the maximal value of J_ρ on the image will be strictly greater than the maximum on the boundary. By standard arguments (considering a pseudo-gradient flow for J_ρ), we can conclude that the functional possesses a Palais-Smale sequence at some level α_ρ.

At this point, in order to recover boundedness of the Palais-Smale sequences, we employ crucially a method due to Struwe. We introduce a modified functional $J_{t\rho}$ and we prove a sort of monotonicity of $\alpha_{t\rho}$ with respect to t. This allows to prove existence of solutions of (3) with ρ replaced by $t_k\rho$ where $t_k \to 1$ as $k \to \infty$. Finally we apply the compactness result in Proposition 2.5 to achieve existence for $t = 1$ as well.

4.1 Construction of the maps Ψ and Φ

Proposition 4.1 *Suppose m is a positive integer, and suppose that $\rho_1 \in (4\pi m, 4\pi(m+1))$, and that $\rho_2 < 4\pi$. Then there exists a large $L > 0$ and a continuous projection Ψ from $\{J_\rho \leq -L\} \cap \left\{\int_\Sigma e^{u_1} dV_g = 1\right\}$ (with the natural topology of $H^1(\Sigma) \times H^1(\Sigma)$) onto Σ_m which is homotopically non-trivial.*

PROOF. We fix ε_m so small that Proposition 2.7 applies with $k = m$. Then we apply Corollary 3.5 with $\bar{\varepsilon} = \varepsilon_m$. We let L be the corresponding large number, so that if $J_\rho(u) \leq -L$, then $dist(e^{u_1}, \Sigma_m) < \varepsilon_m$. Hence for these ranges of u_1 and u_2, since the map $u \mapsto e^u$ is continuous from $H^1(\Sigma)$ into $L^1(\Sigma)$, the projections Π_m from $H^1(\Sigma)$ onto Σ_m is well defined and continuous. The non-triviality of this map is a consequence of Proposition 4.2 **(ii)**, which proof is given below. ■

The next step consists in mapping Σ_m into arbitrarily negative sublevels of J_ρ. In order to do this, we need some preliminary notation. Given $\sigma \in \Sigma_m$, $\sigma = \sum_{i=1}^m t_i \delta_{x_i}$ ($\sum_{i=1}^m t_i = 1$) and $\lambda > 0$, we define the function $\varphi_{\lambda,\sigma} : \Sigma \to \mathbb{R}$ by

$$\varphi_{\lambda,\sigma}(y) = \log \sum_{i=1}^m t_i \left(\frac{\lambda}{1 + \lambda^2 d_i^2(y)} \right)^2, \tag{26}$$

where we have set

$$d_i(y) = d(y, x_i), \qquad\qquad x_i, y \in \Sigma.$$

We point out that, since the distance from a fixed point of Σ is a Lipschitz function, $\varphi_{\lambda,\sigma}(y)$ is also Lipschitz in y, and hence it belongs to $H^1(\Sigma)$.

Proposition 4.2 *Suppose m is a positive integer, that $\rho_1 \in (4\pi m, 4\pi(m+1))$, and that $\rho_2 < 4\pi$. For $\lambda > 0$ and for $\sigma \in \Sigma_m$, we define $\Phi : \Sigma_m \to H^1(\Sigma) \times H^1(\Sigma)$ as*

$$(\Phi(\sigma))(\cdot) = (\Phi(\sigma)_1(\cdot), \Phi(\sigma)_2(\cdot)) := \left(\varphi_{\lambda,\sigma}(\cdot), -\frac{1}{2}\varphi_{\lambda,\sigma}(\cdot) \right), \tag{27}$$

where $\varphi_{\lambda,\sigma}$ is given in (26). Then for L sufficiently large there exists $\lambda > 0$ such that

(i) *$J_\rho(\Phi(\sigma)) \leq -L$ uniformly in $\sigma \in \Sigma_m$;*

(ii) *$\Psi \circ \Phi$ is homotopic to the identity on Σ_m,* *where Ψ is given by Proposition 4.1, and where we assume L to be so large that Ψ is well defined on $\{J_\rho \leq -L\}$.*

Proof. The main ideas follow the strategy in [15], but for the reader's convenience we present here a simplified argument (for the H^2 setting in [15] it was necessary to introduce a cutoff function on the distances d_i which made the computations more involved).

The proof of **(i)** relies on showing the following two pointwise estimates on the gradient of $\varphi_{\lambda,\sigma}$

$$|\nabla \varphi_{\lambda,\sigma}(y)| \leq C\lambda; \qquad \text{for every } y \in \Sigma, \tag{28}$$

where C is a constant independent of σ and λ, and

$$|\nabla \varphi_{\lambda,\sigma}(y)| \leq \frac{4}{d_{min}(y)} \qquad \text{where} \qquad d_{min}(y) = \min_{i=1,\dots,m} d(y, x_i). \tag{29}$$

For proving (28) we notice that the following inequality holds

$$\frac{\lambda^2 d(y, x_i)}{1 + \lambda^2 d^2(y, x_i)} \leq C\lambda, \qquad i = 1, \dots, m, \tag{30}$$

where C is a fixed constant (independent of λ and x_i). Moreover we have

$$\nabla \varphi_{\lambda,\sigma}(y) = -2\lambda^2 \frac{\sum_i t_i (1 + \lambda^2 d_i^2(y))^{-3} \nabla_y (d_i^2(y))}{\sum_j t_j (1 + \lambda^2 d_j^2(y))^{-2}}. \tag{31}$$

Using the fact that $|\nabla_y (d_i^2(y))| \leq 2d_i(y)$ and inserting (30) into (31) we obtain immediately (28). Similarly we find

$$\begin{aligned} |\nabla \varphi_{\lambda,\sigma}(y)| &\leq 4\lambda^2 \frac{\sum_i t_i (1 + \lambda^2 d_i^2(y))^{-3} d_i(y)}{\sum_j t_j (1 + \lambda^2 d_j^2(y))^{-2}} \leq 4\lambda^2 \frac{\sum_i t_i (1 + \lambda^2 d_i^2(y))^{-2} \frac{d_i(y)}{\lambda^2 d_i^2(y)}}{\sum_j t_j (1 + \lambda^2 d_j^2(y))^{-2}} \\ &\leq 4 \frac{\sum_i t_i (1 + \lambda^2 d_i^2(y))^{-2} \frac{1}{d_{min}(y)}}{\sum_j t_j (1 + \lambda^2 d_j^2(y))^{-2}} \leq \frac{4}{d_{min}(y)}, \end{aligned}$$

which is (29).

Now, using (28), (29) and the fact that $\nabla\Phi(\sigma)_2 = -\frac{1}{2}\nabla\Phi(\sigma)_1$, one easily finds that

$$\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} (\nabla\Phi(\sigma)_i) \cdot (\nabla\Phi(\sigma)_j) dV_g \leq C + 4 \int_{\Sigma \setminus \cup_i B_{\frac{1}{\lambda}}(x_i)} \frac{1}{d_{min}^2(y)} dV_g(y).$$

Reasoning as in [15] one can show that

$$\int_{\Sigma \setminus \cup_i B_{\frac{1}{\lambda}}(x_i)} \frac{1}{d_{min}^2(y)} dV_g(y) \leq 8\pi m (1 + o_\lambda(1)) \log \lambda, \qquad (o_\lambda(1) \to 0 \text{ as } \lambda \to +\infty),$$

and that

$$\int_\Sigma \varphi_{\lambda,\sigma} dV_g = -2(1 + o_\lambda(1)) \log \lambda; \quad \log \int_\Sigma e^{\varphi_{\lambda,\sigma}} dV_g = O(1); \quad \log \int_\Sigma e^{-\frac{1}{2}\varphi_{\lambda,\sigma}} dV_g = (1 + o_\lambda(1)) \log \lambda.$$

Using the last four inequalities one then obtains

$$J_\rho(\Phi(\sigma)) \leq (8m\pi - 2\rho_1 + o_\lambda(1)) \log \lambda + C,$$

where C is independent of λ and σ. Since we are assuming that ρ_1 is bigger than $4m\pi$, we achieve **(i)**.

To prove **(ii)** it is sufficient to consider the family of maps $T_\lambda : \Sigma_m \to \Sigma_m$ defined by

$$T_\lambda(\sigma) = \Psi(\Phi_\lambda(\sigma)), \qquad \sigma \in \Sigma_m.$$

We recall that when λ is sufficiently large this composition is well defined. Therefore, since $\frac{e^{\varphi_{\lambda,\sigma}}}{\int_\Sigma e^{\varphi_{\lambda,\sigma}} dV_g} \rightharpoonup \sigma$ in the weak sense of distributions, letting $\lambda \to \infty$ we obtain an homotopy between $\Psi \circ \Phi$ and Id_{Σ_m}. This concludes the proof. ■

Remark 4.3 *We point out that, fixing $p \in \Sigma$ and $\xi_1 \in T_p\Sigma$, the choice of ξ_2 which minimizes the quadratic form $\sum_{i,j} a^{ij}\xi_1 \cdot \xi_j$ is $\xi_2 = -\frac{1}{2}\xi_1$, see also* (17)*. This motivates the coefficient $-\frac{1}{2}$ in the second component of Φ.*

4.2 The minimax scheme: proof of Theorem 1.1

In this section we prove Theorem 1.1 employing a minimax scheme based on the cone over Σ_m, see Lemma 4.4. As anticipated in the introduction, we then define a modified functional $J_{t\rho_1,t\rho_2}$ for which we can prove existence of solutions in a dense set of the values of t. Following an idea of Struwe, this is done proving the a.e. differentiability of the map $t \mapsto \alpha_{t\rho}$, where $\alpha_{t\rho}$ is the minimax value for the functional $J_{t\rho_1,t\rho_2}$ given by the scheme.

Let K_m be the topological cone over Σ_m, see (10). First, let L be so large that Proposition 4.1 applies with $\frac{L}{4}$, and choose then Φ such that Proposition 4.2 applies for L. Fixing L and Φ, we define the class of maps

$$\Pi_\Phi = \left\{\pi : K_m \to H^1_*(\Sigma) \times H^1_*(\Sigma) \; : \; \pi \text{ is continuous and } \pi|_{\Sigma_m(=\partial K_m)} = \Phi\right\}, \tag{32}$$

where

$$H^1_* = \left\{ u \in H^1(\Sigma) \; : \; \int_\Sigma e^u dV_g = 1 \right\}.$$

Then we have the following properties.

Lemma 4.4 *The set Π_Φ is non-empty and moreover, letting*

$$\alpha_\rho = \inf_{\pi \in \Pi_\Phi} \sup_{m \in K_m} J_{\rho_1,\rho_2}(\pi(m)), \qquad \textit{there holds} \qquad \alpha_\rho > -\frac{L}{2}.$$

PROOF. To prove that $\Pi_\Phi \neq \emptyset$, we just notice that the following map

$$\overline{\pi}(\sigma,t) = t\Phi(\sigma) - \log\left(\int_\Sigma e^{t\Phi(\sigma)} dV_g\right); \qquad \sigma \in \Sigma_m, t \in [0,1] \quad ((\sigma,t) \in K_m) \tag{33}$$

belongs to Π_Φ. Assuming by contradiction that $\alpha_\rho \leq -\frac{L}{2}$, there would exist a map $\pi \in \Pi_\Phi$ with $\sup_{\tilde\sigma \in K_m} II(\pi(\tilde\sigma)) \leq -\frac{3}{8}L$. Then, since Proposition 4.1 applies with $\frac{L}{4}$, writing $\tilde\sigma = (\sigma,t)$, with $\sigma \in \Sigma_m$, the map

$$t \mapsto \Psi \circ \pi(\cdot, t)$$

would be an homotopy in Σ_m between $\Psi \circ \Phi$ and a constant map. But this is impossible since Σ_m is non-contractible (see Lemma 2.6) and since $\Psi \circ \Phi$ is homotopic to the identity on Σ_m, by Proposition 4.2. Therefore we deduce $\alpha_\rho > -\frac{L}{2}$. ■

PROOF OF THEOREM 1.1 We introduce a variant of the above minimax scheme, following [29] and [12]. For t close to 1, we consider the functional

$$\begin{aligned} J_{t\rho_1,t\rho_2}(u) &= \frac{1}{2}\sum_{i,j}\int_\Sigma a^{ij}\nabla u_i \cdot \nabla u_j dV_g + t\rho_1 \int_\Sigma u_1 dV_g + t\rho_2 \int_\Sigma u_2 dV_g \\ &- t\rho_1 \log \int_\Sigma h_1 e^{u_1} dV_g - t\rho_2 \log \int_\Sigma h_2 e^{u_2} dV_g. \end{aligned}$$

Repeating the estimates of the previous sections, one easily checks that the above minimax scheme applies uniformly for $t \in [1-t_0, 1+t_0]$ with t_0 sufficiently small. More precisely, given $L > 0$ as before, for t_0 sufficiently small we have

$$\sup_{\pi \in \Pi_\Phi} \sup_{m \in \partial K_m} J_{t\rho_1, t\rho_2}(\pi(m)) < -2L; \quad \alpha_{t\rho} := \inf_{\pi \in \Pi_\Phi} \sup_{m \in K_m} J_{t\rho_1, t\rho_2}(\pi(m)) > -\frac{L}{2}; \tag{34}$$
$$\text{for every } t \in [1-t_0, 1+t_0],$$

where Π_Φ is defined in (32).

Next we notice that for $t' \geq t$ there holds

$$\frac{J_{t\rho_1, t\rho_2}(u)}{t} - \frac{J_{t'\rho_1, t'\rho_2}(u)}{t'} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t'}\right) \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j dV_g \geq 0, \qquad u \in H^1(\Sigma) \times H^1(\Sigma).$$

Therefore it follows easily that also

$$\frac{\alpha_{t\rho}}{t} - \frac{\alpha_{t'\rho}}{t'} \geq 0,$$

namely the function $t \mapsto \frac{\alpha_{t\rho}}{t}$ is non-increasing, and hence is almost everywhere differentiable. Using Struwe's monotonicity argument, see for example [12], one can see that at the points where $\frac{\alpha_{t\rho}}{t}$ is differentiable $J_{t\rho_1, t\rho_2}$ admits a bounded Palais-Smale sequence at level $\alpha_{t\rho}$, which converges to a critical point of $J_{t\rho_1, t\rho_2}$. Therefore, since the points with differentiability fill densely the interval $[1-t_0, 1+t_0]$, there exists $t_k \to 1$ such that the following system has a solution $(u_{1,k}, u_{2,k})$

$$-\Delta u_{i,k} = \sum_{j=1}^N t_k \rho_j a_{ij} \left(\frac{h_j e^{u_{j,k}}}{\int_\Sigma h_j e^{u_{j,k}} dV_g} - 1 \right), \qquad i = 1, 2. \tag{35}$$

Now it is sufficient to apply Proposition 2.5 to obtain a limit (u_1, u_2) which is a solution of (3). This concludes the proof. ■

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