



# From additive to transport noise in 2D fluid dynamics

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*This article is dedicated to István Gyöngy on the occasion of his 70th birthday.*

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## Abstract

Additive noise in Partial Differential equations, in particular those of fluid mechanics, has relatively natural motivations. The aim of this work is showing that suitable multiscale arguments lead rigorously, from a model of fluid with additive noise, to transport type noise. The arguments apply both to small-scale random perturbations of the fluid acting on a large-scale passive scalar and to the action of the former on the large scales of the fluid itself. Our approach consists in studying the (stochastic) characteristics associated to small-scale random perturbations of the fluid, here modelled by stochastic 2D Euler equations with additive noise, and their convergence in the infinite scale separation limit.

## 1 Introduction

Let  $T > 0$  be fixed. In this work we are concerned with convergence of characteristics associated with stochastic Euler equations in vorticity form on the two-dimensional torus  $\mathbb{T}^2 := \mathbb{R}^2 / (2\pi\mathbb{Z}^2)$ :

$$d\xi_t^\epsilon + (v_t^\epsilon + u_t^\epsilon) \cdot \nabla \xi_t^\epsilon dt = -\epsilon^{-1} \xi_t^\epsilon dt + \epsilon^{-1} \sum_{k \in \mathbb{N}} \varsigma_k dW_t^k, \quad t \in [0, T], \quad (1.1)$$

where  $\xi^\epsilon$  is the zero-mean unknown vorticity field,  $u^\epsilon$  is the velocity field reconstructed from  $\xi^\epsilon$  via the Biot–Savart kernel:  $u_t^\epsilon = -\nabla^\perp (-\Delta)^{-1} \xi_t^\epsilon$ ,  $v^\epsilon$  is a divergence-free external field with suitable regularity,  $\varsigma_k : \mathbb{T}^2 \rightarrow \mathbb{R}$  with zero average for every

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$k \in \mathbb{N}$ ,  $(W^k)_{k \in \mathbb{N}}$  is a family of i.i.d. Wiener processes defined on a filtered probability space  $(\Omega, \mathcal{F}_t, \mathbb{P})$ , and  $\epsilon \ll 1$  is a scaling parameter.

Equation (1.1) above aim to represent the small-scale component of a two-dimensional incompressible fluid [1], with the additive noise and damping on the right-hand-side modelling the influence on the fluid of a possibly irregular boundary or topography. The choice of the parameter  $\epsilon^{-1}$  in front of both noise and damping is appropriate when looking at the system with respect to the point of view of a large-scale observer, see [21] and Sect. 1.2 for details. In view of this, it makes sense to couple (1.1) with a large-scale scalar dynamics:

$$d\Xi_t^\epsilon + (v_t^\epsilon + u_t^\epsilon) \cdot \nabla \Xi_t^\epsilon dt = \nu \Delta \Xi_t^\epsilon dt + q_t^\epsilon dt, \quad t \in [0, T], \tag{1.2}$$

either passive (in which case the external field  $v^\epsilon$  should be interpreted as given a priori) or active (in which case the external field  $v^\epsilon$  could depend on the large-scale dynamics itself, as for instance in the vorticity formulation of 2D Navier-Stokes equations, where  $v_t^\epsilon = -\nabla^\perp (-\Delta)^{-1} \Xi_t^\epsilon$ ).

In (1.2) above,  $\nu \geq 0$  is a fixed parameter that represents molecular diffusivity (passive dynamics) or viscosity (active dynamics), and  $q^\epsilon$  is a given source term with suitable integrability.

Let  $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$  be an auxiliary probability space and let  $w$  be a standard  $\mathbb{R}^2$ -valued Wiener process defined on  $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ . The (stochastic) characteristics  $\phi^\epsilon$  associated with problem (1.1)–(1.2) are given by the family of maps  $\phi_t^\epsilon : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  satisfying

$$\phi_t^\epsilon(x) = x + \int_0^t v_s^\epsilon(\phi_s^\epsilon(x)) ds + \int_0^t u_s^\epsilon(\phi_s^\epsilon(x)) ds + \sqrt{2\nu} w_t, \tag{1.3}$$

where  $t \in [0, T]$ ,  $x \in \mathbb{T}^2$ . Since  $v^\epsilon$  and  $u^\epsilon$  are divergence-free and have sufficient regularity, the characteristics  $\phi^\epsilon$  defined above constitute a *stochastic flow of measure-preserving homeomorphisms*, in the sense of Definition 2.3 below. The interest in studying the solution of (1.3) is motivated by the following representation formula for the solution of (1.2):

$$\Xi_t^\epsilon = \tilde{\mathbb{E}} \left[ \Xi_0 \circ (\phi_t^\epsilon)^{-1} + \int_0^t q_s^\epsilon \circ \phi_s^\epsilon \circ (\phi_t^\epsilon)^{-1} ds \right], \tag{1.4}$$

where  $\tilde{\mathbb{E}}$  is the expectation on  $\tilde{\Omega}$  with respect to  $\tilde{\mathbb{P}}$  and we have tacitly assumed that the initial condition  $\Xi_t^\epsilon|_{t=0} = \Xi_0$  is independent of  $\epsilon$ . See Definition 2.8 for more details on the notion of solution adopted in the present paper.

The main purpose of this work—cfr. Theorem 2.12—is to investigate conditions allowing to prove convergence in a suitable sense, as  $\epsilon \rightarrow 0$ , of  $\phi^\epsilon$  towards the solution of:

$$\phi_t(x) = x + \int_0^t v_s(\phi_s(x)) ds + \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s(x)) \circ dW_s^k + \sqrt{2\nu} w_t, \tag{1.5}$$

where  $\sigma_k = -\nabla^\perp(-\Delta)^{-1}\zeta_k$  and  $v^\epsilon \rightarrow v$  in a certain sense.

The notion of convergence  $\phi^\epsilon \rightarrow \phi$  contained in Theorem 2.12 permits to prove a notion of weak convergence of the large-scale observable  $\Xi^\epsilon$  given by (1.4) towards

$$\Xi_t = \tilde{\mathbb{E}} \left[ \Xi_0 \circ (\phi_t)^{-1} + \int_0^t q_s \circ \phi_s \circ (\phi_t)^{-1} ds \right], \tag{1.6}$$

that solves the large-scale dynamics with *transport noise*:

$$d\Xi_t + v_t \cdot \nabla \Xi_t dt + \sum_{k \in \mathbb{N}} \sigma_k \cdot \nabla \Xi_t \circ dW_t^k = v \Delta \Xi_t dt + q_t dt, \tag{1.7}$$

where  $v$  is independent of  $\Xi$  for passive dynamics, while it could depend of  $\Xi$  itself for active dynamics, and  $q^\epsilon \rightarrow q$  in a sense to be specified later. The precise meaning of weak convergence is made rigorous in Theorem 2.13 below.

We think that these results could contribute to a proper interpretation of transport noise in SPDEs, at least for the two classes considered here. Several papers considered transport noise so far, either in passive scalars ([11, 15, 23, 24, 32, 33, 40]), passive vector fields ([18, 19, 31, 44]) and fluid mechanics equations themselves ([3, 4, 6, 7, 9, 10, 12–14, 16, 20, 22, 30, 36, 37, 43]). In terms of consequences of transport noise, among the aforementioned works are proved several results concerning well-posedness, enhanced dissipation and mixing properties of fluid dynamics equations perturbed by transport noise, thus being a good starting point towards a rigorous understanding of turbulence in fluids. However, unlike the case of additive noise, that is widely accepted as a source of randomness, transport noise needs a more careful justification. There have been previous attempts by others to derive transport noise by, effectively, perturbing deterministic balance laws [36] and variational principles [30]. In the present paper, we add to the picture an argument based on Wong–Zakai approximation results, largely investigated both in and outside the realm of fluid dynamics ([2, 25, 26, 28, 29, 41, 42]). More specifically, we show that the presence of additive noise at small scales, modelling the influence on the fluid of a possibly irregular boundary or topography, produces transport noise at large scales in a suitable scaling limit. The limit noise is meant in the Stratonovich sense, as usually is the case in Wong–Zakai approximation results.

We point out that, at the moment, we are not able to properly justify our modelling assumptions, but heuristic arguments supporting them are presented in Sect. 1.2; still, would the modelling assumptions be confirmed in successive studies or experiments, this partial picture could be a first step in the understanding of turbulence in fluids.

Let us explain what is added to these works by the present paper. Concerning the passive dynamics, several Wong–Zakai type results of convergence to the white noise transport in Stratonovich form have been proved before (see also [38] for a dissipation enhancement result due to the presence of a Stratonovich-to-Itô corrector), but this seems to be the first work where the velocity field approximating the white noise one is the solution of a nonlinear fluid mechanics equation. Concerning the active dynamics, the results contained in this paper extend and make more precise our previous work [21]: (i) some details in the proof of [21, Proposition 4.1], which after publication

appeared not precise, are fixed here in Theorem 2.12; (ii) more importantly, the term  $u_t^\epsilon \cdot \nabla \xi_t^\epsilon$  was absent in [21], which therefore should be interpreted more along the research lines of model reduction, inspired by Majda et al. [34], instead of multiscale analysis of the full problem.

### 1.1 Examples

Throughout the paper we keep ourselves in a setting as general as possible, in order to comprehend, in our abstract results, the greatest number of particular cases. However, our work has been motivated by two main examples:

- *Advection-diffusion equation* Consider the following system, describing the evolution of the concentration  $\rho^\epsilon$  of a passive scalar advected by the Euler flow and subject to the influence of an external source  $q^\epsilon$ :

$$\begin{cases} d\rho_t^\epsilon + (v_t + u_t^\epsilon) \cdot \nabla \rho_t^\epsilon dt = \nu \Delta \rho_t^\epsilon dt + q_t^\epsilon dt, \\ d\xi_t^\epsilon + (v_t + u_t^\epsilon) \cdot \nabla \xi_t^\epsilon dt = -\epsilon^{-1} \xi_t^\epsilon dt + \epsilon^{-1} \sum_{k \in \mathbb{N}} \varsigma_k dW_t^k, \\ u_t^\epsilon = -\nabla^\perp (-\Delta)^{-1} \xi_t^\epsilon. \end{cases}$$

We have taken  $\nu \geq 0$  and  $v^\epsilon = v$ , independent of  $\epsilon$ , since the passive scalar does not affect the external field. In this setting,  $\rho^\epsilon$  converges towards the solution of the limiting advection-diffusion equation with transport noise:

$$d\rho_t + v_t \cdot \nabla \rho_t dt + \sum_{k \in \mathbb{N}} \sigma_k \cdot \nabla \rho_t \circ dW_t^k = \nu \Delta \rho_t dt + q_t dt.$$

- *Navier–Stokes and Euler equations* Consider the following system, describing the coupling between large-scale Navier–Stokes ( $\nu > 0$ ) or Euler ( $\nu = 0$ ) equations and small-scale stochastic Euler equations:

$$\begin{cases} d\Xi_t^\epsilon + (v_t^\epsilon + u_t^\epsilon) \cdot \nabla \Xi_t^\epsilon dt = \nu \Delta \Xi_t^\epsilon dt + q_t^\epsilon dt, \\ d\xi_t^\epsilon + (v_t^\epsilon + u_t^\epsilon) \cdot \nabla \xi_t^\epsilon dt = -\epsilon^{-1} \xi_t^\epsilon dt + \epsilon^{-1} \sum_{k \in \mathbb{N}} \varsigma_k dW_t^k, \\ v_t^\epsilon = -\nabla^\perp (-\Delta)^{-1} \Xi_t^\epsilon, \\ u_t^\epsilon = -\nabla^\perp (-\Delta)^{-1} \xi_t^\epsilon. \end{cases}$$

We take  $q^\epsilon$  and  $\Xi_0$  with zero spatial average, so that  $\Xi^\epsilon$  is zero mean, too. Notice that in this case the field  $v^\epsilon$  is generated by  $\Xi^\epsilon$  itself through the Biot–Savart law  $v_t^\epsilon = -\nabla^\perp (-\Delta)^{-1} \Xi_t^\epsilon$ , in particular  $v^\epsilon$  is random. On the other hand, the external source  $q^\epsilon$  can be thought as given a priori and deterministic. In this setting,  $\Xi^\epsilon$  converges towards the solution of the limiting Navier–Stokes or Euler equations with transport noise:

$$\begin{cases} d\Xi_t + v_t \cdot \nabla \Xi_t dt + \sum_{k \in \mathbb{N}} \sigma_k \cdot \nabla \Xi_t \circ dW_t^k = \nu \Delta \Xi_t dt + q_t dt, \\ v_t = -\nabla^\perp (-\Delta)^{-1} \Xi_t. \end{cases}$$

It is worth of mention that, also in the limit, the velocity field  $v$  is still generated by  $\Xi$  through the Biot–Savart law  $v_t = -\nabla^\perp(-\Delta)^{-1}\Xi_t$ .

## 1.2 Motivations

As already mentioned in the Introduction, (1.1) aims to represent the small-scale component of a two-dimensional incompressible fluid, looked at by a large-scale observer. At large scales the fluid shows a turbulent behaviour, and its statistical properties are well-described by solutions of stochastic equations, although the underlying continuum mechanics equations that govern the evolution of the fluid are deterministic.

We refer to Flandoli and Pappaletta [21, Chapter 2] and reference therein for a complete discussion about the equations under investigation in this paper and the interest for their asymptotical behaviour as  $\epsilon \rightarrow 0$ .

### 1.2.1 On the additive noise and damping

Additive noise in SPDEs is so common that apparently we do not need a justification for introducing it, as we have done in Eq. (1.1) above. However, a short discussion may help to convince ourselves that it is very natural, and moreover to understand that also the damping term is needed.

Our opinion is that an additive noise is a good compromise to keep into account the vortices produced by obstacles and irregularities at the boundary or internal obstacles, which are not explicitly described in the mathematical formulation, often based on the torus geometry or a domain with smooth boundary. Such obstacles introduce vortices, eddies, that could be idealized and described as a jump Markov process  $W_N(t)$  in the Hilbert space  $H$  of  $L^2(\mathbb{T}^2)$  vorticity fields on the torus; the fluid equation perturbed by the creation of these new vortices takes a priori the form

$$\partial_t \xi_t + (v_t + u_t) \cdot \nabla \xi_t = \partial_t W_N(t)$$

where  $\partial_t W_N(t)$  is a sum of delta Dirac in time, with the effect that  $\xi_t$  jumps at those times, namely (if  $t_i$  denotes one of such times)  $\xi_{t_i^+}$  is equal to  $\xi_{t_i^-}$  plus the created vortex. We have indexed  $W_N(t)$  by  $N$  to anticipate that we consider a regime with frequent creation of vortices of small amplitude.

Scaling the parameters of  $W_N(t)$  in the right way, under suitable assumptions of zero average of  $W_N(t)$  and integrability,  $W_N(t)$  converges in law to a Brownian motion  $W(t)$  in  $H$  with a suitable covariance. This is our motivation for the equation with additive noise

$$d\xi_t + (v_t + u_t) \cdot \nabla \xi_t dt = dW(t).$$

However, as it is easily seen by Itô formula, such additive noise introduces systematically energy, fact that is not acceptable from the physical viewpoint: the vortices created by obstacles do not increase the energy (at most, some energy is lost in thermal dissipation at the boundary). Therefore some sort of compensation is needed. The

simplest is to think that the forces which are responsible for the creation of vortices by the obstacles are somewhat similar to a friction. Thus we introduce a friction term to maintain equilibrium:

$$d\xi_t + (v_t + u_t) \cdot \nabla \xi_t dt = -\lambda \xi_t dt + dW(t).$$

This is the origin of the fluid model. The particular scaling attributed above to the terms  $-\lambda \xi_t dt$  and  $dW(t)$  is related to a different argument, which is explained in the next paragraph.

### 1.2.2 On the parameter $\epsilon^{-1}$

An important feature of (1.1) is the presence of the scaling parameter  $\epsilon^{-1}$  in front of both noise and damping, in contrast to the widely-studied diffusive scaling given by coefficients  $\epsilon^{-1}$  in front of the damping and  $\epsilon^{-1/2}$  in front of the noise. Let us recall briefly where this scaling comes from, referring to Flandoli and Pappalettera [21] for further details.

We suppose to have a small time scale  $\mathcal{T}_S$ , at which we observe the vorticity field  $\xi$ . At this scale, the small scales evolve according to deterministic equations, and the typical intensity and turnover time of  $\xi$  are of order one.

Let us now take an intermediate point of view on the system, say human-scale,  $\mathcal{T}_M := \epsilon^{-1} \mathcal{T}_S$ . At this scale, fluctuations of  $\xi = \xi^\epsilon$  look random and could be well modeled by stochastic equations (1.1), with the crucial difference of a coefficient  $\epsilon^{-1/2}$  in front of the noise rather than  $\epsilon^{-1}$ .

Only when we look at the system with respect to a large time scale  $\mathcal{T}_L := \epsilon^{-1} \mathcal{T}_M$  the scaling of (1.1) appears. As a result of the theory here developed, under this point of view the small scale fluctuations behave as a white noise of multiplicative type.

We remark that, in our arguments, spatial scaling is less important than temporal scaling. As it emerges from computations performed in [21, Subsection 2.3], the spatial scaling only affects the spatial covariance of the noise in (1.1). For the sake of concreteness, suppose that  $W_M(\tilde{t}, \tilde{x})$  is the noise perturbing  $\xi^\epsilon$  at intermediate scales and  $W_L(t, x)$  is the noise perturbing  $\xi^\epsilon$  at large scales, with mesoscopic variables  $\tilde{t}, \tilde{x}$  related to macroscopic variables  $t, x$  by the formulas  $\tilde{t} = \epsilon^{-1} t, \tilde{x} = \epsilon_X^{-1} x$ . Then it holds the equality in law

$$W_M(\tilde{t}, \tilde{x}) = \epsilon^{1/2} W_L\left(t, \epsilon_X^{-1} x\right).$$

Moreover, assuming that the elements producing the noise (topography, boundaries *et cetera*) are actually large-scale, we can suppose that the covariance of  $W_M$  is slowly-varying with respects to  $\tilde{x}$ , or equivalently

$$W_L\left(t, \epsilon_X^{-1} x\right) = \sum_{k \in \mathbb{N}} \varrho_k(x) W_t^k,$$

with  $\varrho_k$  and  $W^k$  as in (1.1).

### 1.3 Structure of the paper

In Sect. 2 we introduce some notation and recall classical results that will be frequently used in the remainder of the paper. This section contains, among others: main properties of the Biot–Savart kernel on the torus  $-\nabla^\perp(-\Delta)^{-1}$ ; a useful Gronwall-type lemma for ODEs with log-Lipschitz drift; notions of solution and well-posedness results for stochastic Euler equations (1.1), equations of characteristics (1.3) and (1.5), and large-scale dynamics (1.2) and (1.7). Also, here we introduce our main working assumptions (A1)–(A7), and in the last part of this section we state our two main results, concerning convergence of characteristics (Theorem 2.12) and subsequent convergence of large-scale dynamics (Theorem 2.13).

In the first part of Sect. 3, we define a linearized version of (1.1), where we neglect the nonlinear term. This approach is similar to that of [21], and the key idea is that, although the solution  $\theta^\epsilon$  of linearized equation is not close to the actual solution  $\xi^\epsilon$  of (1.1), the characteristics generated by  $\theta^\epsilon$  are close to the characteristics generated by  $\xi^\epsilon$ , in particular they have the same limit as  $\epsilon \rightarrow 0$ .

In the same section we present two main technical results, needed in the proof of Theorem 2.12. The first of those results is Proposition 3.1, which ensures that the linear part  $\theta^\epsilon$  of the small-scale dynamics behaves as a Stratonovich white-in-time noise as  $\epsilon \rightarrow 0$ , at least in a distributional sense. The second result Proposition 3.2, instead, aims to rigorously prove the closeness of the characteristics generated by  $\theta^\epsilon$  and  $\xi^\epsilon$ , and it is one of the main novelties of this paper with respect to [21].

The proof of Theorem 2.12 is contained in Sect. 4, and it is based on a Gronwall-type lemma and Itô Formula applied to a smooth approximation  $g_\delta(x)$  of the absolute value  $|x|$ ,  $x \in \mathbb{R}^2$ . The proof of Theorem 2.13 can be found in Sect. 5, and it relies on representation formulas (1.4) and (1.6) and a measure-theoretic argument.

Finally, in Sect. 6 we discuss how our main motivational examples—cfr. Sect. 1.1—fit our abstract setting. In particular, the non-trivial one is the coupled system given by deterministic Navier–Stokes equations at large scales plus stochastic Euler equations at small scales; we identify an additional but very natural condition (A8) on the limit external source  $q$  that allows to verify assumptions (A1)–(A7) for the system under consideration.

## 2 Notations, preliminaries and main results

In this section we collect definitions, notations and classical results needed in the paper. Also, we introduce our main working assumptions (A1)–(A7), and state our main results.

### 2.1 Properties of the Biot–Savart kernel

Here we briefly recall some useful properties of the Biot–Savart kernel  $K$ . We refer to Marchioro and Pulvirenti [35] and Brzeźniak et al. [6] for details and proofs.

First of all, the Biot–Savart kernel  $K$  is defined as  $K = -\nabla^\perp G = (\partial_2 G, -\partial_1 G)$ , where  $G$  is the Green function of the Laplace operator on the torus  $\mathbb{T}^2$  with zero mean.

For  $p \in (1, \infty)$  and  $\xi \in L^p(\mathbb{T}^2)$  with zero-mean, the convolution with  $K$  represents the Biot–Savart operator:

$$K * \xi = -\nabla^\perp(-\Delta)^{-1}\xi,$$

that to every zero-mean  $\xi \in L^p(\mathbb{T}^2)$  associates the unique zero-mean, divergence-free velocity vector field  $u \in W^{1,p}(\mathbb{T}^2, \mathbb{R}^2)$  such that  $\text{curl } u = \xi$ . Moreover, for every  $p \in (1, \infty)$  there exist constants  $c, C$  such that for every zero-mean  $\xi \in L^p(\mathbb{T}^2)$

$$c\|\xi\|_{L^p(\mathbb{T}^2)} \leq \|K * \xi\|_{W^{1,p}(\mathbb{T}^2, \mathbb{R}^2)} \leq C\|\xi\|_{L^p(\mathbb{T}^2)}.$$

Also, recall that since  $K \in L^1(\mathbb{T}^2, \mathbb{R}^2)$  the convolution  $K * \xi$  is well-defined for every  $\xi \in L^p(\mathbb{T}^2)$ ,  $p \in [1, \infty]$  and the following estimate holds:

$$\|K * \xi\|_{L^p(\mathbb{T}^2, \mathbb{R}^2)} \leq \|K\|_{L^1(\mathbb{T}^2, \mathbb{R}^2)} \|\xi\|_{L^p(\mathbb{T}^2)}. \tag{2.1}$$

Let  $r \geq 0$ . Denote  $\gamma : [0, \infty) \rightarrow \mathbb{R}$  the concave function:

$$\gamma(r) = r(1 - \log r)\mathbf{1}_{\{0 < r < 1/e\}} + (r + 1/e)\mathbf{1}_{\{r \geq 1/e\}}.$$

The following two lemmas are proved in [35, Lemma 3.1] and [6, Section 2].

**Lemma 2.1** *There exists a constant  $C$  such that:*

$$\int_{\mathbb{T}^2} |K(x - y) - K(x' - y)| dy \leq C\gamma(|x - x'|)$$

for every  $x, x' \in \mathbb{T}^2$ .

**Lemma 2.2** *Let  $T > 0, \lambda > 0, a_0 \in [0, \exp(1 - 2e^{\lambda T})]$  be constants. Let  $a : [0, T] \rightarrow \mathbb{R}$  be such that for every  $t \in [0, T]$ :*

$$a_t \leq a_0 + \lambda \int_0^t \gamma(a_s) ds.$$

Then for every  $t \in [0, T]$  the following estimate holds:

$$a_t \leq ea_0^{\exp(-\lambda t)}.$$

**2.2 Stochastic flows of measure-preserving homeomorphisms**

As a convention, in the following we say that  $\mathcal{N} \subset \Omega$  (respectively  $\tilde{\mathcal{N}} \subset \tilde{\Omega}$ ) is negligible if it is measurable and  $\mathbb{P}(\mathcal{N}) = 0$  (respectively  $\tilde{\mathbb{P}}(\tilde{\mathcal{N}}) = 0$ ), without explicit

mention of the reference probability measure. Unless otherwise specified, we will always denote with  $\mathcal{N}$  negligible sets in  $\Omega$ , and with  $\tilde{\mathcal{N}}$  negligible sets in  $\tilde{\Omega}$ .

Let us begin this paragraph with the following fundamental definition.

**Definition 2.3** A measurable map  $\phi : \Omega \times \tilde{\Omega} \times [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is a *stochastic flow of measure-preserving homeomorphisms* provided there exist negligible sets  $\mathcal{N} \subset \Omega$  and  $\tilde{\mathcal{N}} \subset \tilde{\Omega}$  such that:

- for every  $\omega \in \mathcal{N}^c$ ,  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$  and  $t \in [0, T]$ , the map  $\phi(\omega, \tilde{\omega}, t, \cdot) : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is a homeomorphism of the torus and

$$\int_{\mathbb{T}^2} f(x)dx = \int_{\mathbb{T}^2} f(\phi(\omega, \tilde{\omega}, t, y))dy$$

for every  $f \in L^1(\mathbb{T}^2)$ ;

- for every  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$  and  $x \in \mathbb{T}^2$ , the stochastic process  $\phi(\cdot, \tilde{\omega}, \cdot, x) : \Omega \times [0, T] \rightarrow \mathbb{T}^2$  is progressively measurable with respect to the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$ .

In some circumstances it can be useful to have the following:

**Definition 2.4** A stochastic flow of measure-preserving homeomorphisms  $\phi$  is called *inviscid* if there exist negligible sets  $\mathcal{N} \subset \Omega$  and  $\tilde{\mathcal{N}} \subset \tilde{\Omega}$ , and a measurable map  $\psi : \Omega \times [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$  such that for every  $\omega \in \mathcal{N}^c$ ,  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$ ,  $t \in [0, T]$  and  $x \in \mathbb{T}^2$

$$\phi(\omega, \tilde{\omega}, t, x) = \psi(\omega, t, x).$$

With a little abuse of notation, hereafter we identify an inviscid stochastic flow of measure-preserving homeomorphisms  $\phi$  with its  $\tilde{\omega}$ -independent representative  $\psi$ .

Let us now clarify the meaning of (1.3), (1.5).

A measurable map  $\phi^\epsilon : \Omega \times \tilde{\Omega} \times [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is a solution of (1.3) if there exist negligible sets  $\mathcal{N} \subset \Omega$  and  $\tilde{\mathcal{N}} \subset \tilde{\Omega}$  such that for every  $\omega \in \mathcal{N}^c$ ,  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$ ,  $t \in [0, T]$  and  $x \in \mathbb{T}^2$ :

$$\begin{aligned} \phi^\epsilon(\omega, \tilde{\omega}, t, x) &= x + \int_0^t v^\epsilon(\omega, s, \phi^\epsilon(\omega, \tilde{\omega}, s, x))ds \\ &\quad + \int_0^t u^\epsilon(\omega, s, \phi^\epsilon(\omega, \tilde{\omega}, s, x))ds + \sqrt{2\nu}w(\tilde{\omega}, t), \end{aligned}$$

where the previous identity can be interpreted as an equation on  $\mathbb{T}^2$  since one can check  $\phi^\epsilon(\omega, \tilde{\omega}, t, x + 2\pi\mathbf{e}) = \phi^\epsilon(\omega, \tilde{\omega}, t, x) + 2\pi\mathbf{e}$  for  $\mathbf{e} = (1, 0)$  and  $\mathbf{e} = (0, 1)$ .

Similarly, a measurable map  $\phi : \Omega \times \tilde{\Omega} \times [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is a solution of (1.5) if there exist negligible sets  $\mathcal{N} \subset \Omega$  and  $\tilde{\mathcal{N}} \subset \tilde{\Omega}$  such that for every  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$  and  $x \in \mathbb{T}^2$ , the stochastic process  $\phi(\cdot, \tilde{\omega}, \cdot, x) : \Omega \times [0, T] \rightarrow \mathbb{T}^2$  is progressively measurable with respect to the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$ , and for every  $\omega \in \mathcal{N}^c$ ,  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$ ,  $t \in [0, T]$  and  $x \in \mathbb{T}^2$ :

$$\begin{aligned} \phi(\omega, \tilde{\omega}, t, x) &= x + \int_0^t v(\omega, s, \phi(\omega, \tilde{\omega}, s, x)) ds \\ &+ \sum_{k \in \mathbb{N}} \left( \int_0^t \sigma_k(\phi(\cdot, \tilde{\omega}, s, x)) \circ dW_s^k \right) (\omega) + \sqrt{2}vw(\tilde{\omega}, t). \end{aligned}$$

Notice that progressive measurability of the process  $\phi(\cdot, \tilde{\omega}, \cdot, x) : \Omega \times [0, T] \rightarrow \mathbb{T}^2$  is necessary to make sense of the Stratonovich stochastic integral appearing in the equation above.

### 2.3 Notions of solution and some well-posedness results

The aim of the present subsection is twofold. On the one hand, we provide a suitable notion of solution for (1.1)–(1.2), in some sense highlighting the minimal requirements on the solutions to prove our results. On the other hand, we show the existence of solutions in the general case, as well as uniqueness in the case of the large-scale process being a passive scalar.

In the following, we say that a field  $v^\epsilon$  is *compatible* with the large-scale process  $\Xi^\epsilon$  if: either  $\Xi^\epsilon$  is a passive scalar, or:  $\Xi^\epsilon$  is an active scalar and  $v^\epsilon$  is reconstructed from the latter by the Biot–Savart law. We adopt a similar terminology for the limiting quantities  $v, \Xi$ . In this subsection we make assumptions directly on the fields  $v^\epsilon, v$ ; we shall see in Sect. 6 that, even for active scalars, fields compatible with large-scale processes satisfy our assumptions.

#### 2.3.1 Well-posedness of small-scale dynamics and characteristics

First we make the following assumptions on the external fields:

- (A1)  $v^\epsilon, v : \Omega \times [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$  and for every  $t \in [0, T]$  the maps  $v^\epsilon, v|_{\Omega \times [0, t]} : \Omega \times [0, t] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$  are  $\mathcal{F}_t \otimes \mathcal{B}_{[0, t]} \otimes \mathcal{B}_{\mathbb{T}^2}$  measurable, where  $\mathcal{B}$  denotes the Borel sigma-field;
- (A2) there exist a constant  $C$  and a negligible set  $\mathcal{N} \subset \Omega$  such that, for every  $\omega \in \mathcal{N}^c$ ,  $\epsilon > 0$  and  $t \in [0, T]$ :  $\operatorname{div} v^\epsilon(\omega, t, \cdot) = \operatorname{div} v(\omega, t, \cdot) = 0$ , and

$$\begin{aligned} |v^\epsilon(\omega, t, x)| &\leq C, \quad |v^\epsilon(\omega, t, x) - v^\epsilon(\omega, t, y)| \leq C\gamma(|x - y|), \\ |v(\omega, t, x)| &\leq C, \quad |v(\omega, t, x) - v(\omega, t, y)| \leq C\gamma(|x - y|), \end{aligned}$$

for every  $x, y \in \mathbb{T}^2$ .

Also, we make the following assumption on the coefficients  $(\varsigma_k)_{k \in \mathbb{N}}$ :

- (A3) there exists  $\ell \geq 1$  such that  $\varsigma_k \in W^{\ell, \infty}(\mathbb{T}^2)$  with zero-mean for every  $k \in \mathbb{N}$ , and moreover

$$\sum_{k \in \mathbb{N}} \|\varsigma_k\|_{W^{\ell, \infty}(\mathbb{T}^2)} < \infty.$$

Similarly to what has been done in the Introduction, given a stochastic flow of measure-preserving homeomorphisms  $\phi$  we will use  $\phi_t(x)$  as a notational shortcut for  $\phi(\omega, \tilde{\omega}, t, x)$ , thus making implicit the dependence of the randomness variables  $\omega, \tilde{\omega}$ . The same convention may be used for the fields  $v, u$ , *et cetera*.

The next result can be proved repeating the arguments contained in [6, 21].

**Proposition 2.5** *Assume (A1)–(A3). Then:*

- for every  $\epsilon > 0$  there exist a unique Lagrangian solution  $\xi^\epsilon$  of (1.1), namely there exists a unique stochastic process  $\xi^\epsilon : \Omega \times [0, T] \rightarrow L^\infty(\mathbb{T}^2)$  weakly progressively measurable with respect to  $(\mathcal{F}_t)_{t \in [0, T]}$  such that the equation

$$\psi_t^\epsilon(x) = x + \int_0^t v_s^\epsilon(\psi_s^\epsilon(x)) ds + \int_0^t u_s^\epsilon(\psi_s^\epsilon(x)) ds,$$

with  $u^\epsilon = K * \xi^\epsilon$ , admits a unique inviscid stochastic flow of measure-preserving homeomorphisms  $\psi^\epsilon$  as a solution, and moreover

$$\xi_t^\epsilon(\psi_t^\epsilon(x)) = \epsilon^{-1} \sum_{k \in \mathbb{N}} \int_0^t e^{-\epsilon^{-1}(t-s)} \mathcal{G}_k(\psi_s^\epsilon(x)) dW_s^k; \tag{2.2}$$

- for every  $\epsilon > 0$  there exists a unique stochastic flow of measure-preserving homeomorphisms  $\phi^\epsilon$  solution of (1.3), with  $u^\epsilon = K * \xi^\epsilon$ ;
- there exists a unique stochastic flow of measure-preserving homeomorphisms  $\phi$  solution of (1.5).

**Remark 2.6** If  $v = 0$ , then both  $\phi^\epsilon$  and  $\phi$  are inviscid stochastic flows of measure-preserving homeomorphisms, and actually  $\phi^\epsilon = \psi^\epsilon$ . The terminology is thus justified, since  $v = 0$  corresponds to null diffusivity/viscosity in the equations for the large-scale dynamics (1.2) and (1.7).

**Remark 2.7** Formula (2.2) above corresponds to the solution of (1.1) with initial condition  $\xi_0^\epsilon = 0$ , that we assume throughout this paper for the sake of simplicity. More general initial conditions, as those considered in [21], can be taken into account by simply modifying (2.2) into

$$\xi_t^\epsilon(\psi_t^\epsilon(x)) = e^{-\epsilon^{-1}t} \xi_0^\epsilon(x) + \epsilon^{-1} \sum_{k \in \mathbb{N}} \int_0^t e^{-\epsilon^{-1}(t-s)} \mathcal{G}_k(\psi_s^\epsilon(x)) dW_s^k.$$

### 2.3.2 Notion of solution to the large-scale dynamics

By previous Proposition 2.5, under assumption (A1)–(A3) we can use the Euler flow to represent the solutions of (1.2) and (1.7). To be more precise, our notion of solution is given exactly by those processes  $\Xi^\epsilon, \Xi$  for which (1.4) and (1.6) hold true, and it is inspired by the notion of generalized solution in [5, Definition 2.2].

**Definition 2.8** Assume (A1)–(A3),  $q^\epsilon, q \in L^1([0, T], L^\infty(\mathbb{T}^2))$  for every  $\epsilon > 0$  and  $\Xi_0 \in L^\infty(\mathbb{T}^2)$ . Then:

- for every  $\epsilon > 0$ , a measurable map  $\Xi^\epsilon : \Omega \times [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$  is called *generalized solution* to (1.2) if it is compatible with  $v^\epsilon$  and for every  $t \in [0, T]$  it holds

$$\Xi_t^\epsilon = \tilde{\mathbb{E}} \left[ \Xi_0 \circ (\phi_t^\epsilon)^{-1} + \int_0^t q_s^\epsilon \circ \phi_s^\epsilon \circ (\phi_t^\epsilon)^{-1} ds \right],$$

as an equality in  $L^\infty(\Omega \times \mathbb{T}^2)$ , where  $\phi^\epsilon$  is the unique stochastic flow of measure-preserving homeomorphisms solution of (1.3);

- a measurable map  $\Xi : \Omega \times [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$  is called *generalized solution* to (1.7) if it is compatible with  $v$  and for every  $t \in [0, T]$  it holds

$$\Xi_t = \tilde{\mathbb{E}} \left[ \Xi_0 \circ (\phi_t)^{-1} + \int_0^t q_s \circ \phi_s \circ (\phi_t)^{-1} ds \right],$$

as an equality in  $L^\infty(\Omega \times \mathbb{T}^2)$ , where  $\phi$  is the unique stochastic flow of measure-preserving homeomorphisms solution of (1.5).

Notice that this notion of solution immediately implies existence and uniqueness in the case of passive large-scale dynamics: we can state that in the following.

**Proposition 2.9** *Under the same assumptions as above, suppose  $\Xi^\epsilon$  (resp.  $\Xi$ ) are passive scalars. Then there exists a unique generalized solution to (1.2) (resp. (1.7)).*

**Proof** Indeed, for passive scalars the compatibility condition is void, and  $\Xi^\epsilon$  (resp.  $\Xi$ ) depends only on the initial datum  $\Xi_0$ , the external sources  $q^\epsilon$  (resp.  $q$ ), and the characteristics  $\phi^\epsilon$  (resp.  $\phi$ ), the latter existing and being unique by Proposition 2.5.  $\square$

For active dynamics the previous picture is not correct, since the compatibility condition between the external field and the large-scale variable is not encoded in the representation formula itself. However, we will not investigate in this paper well-posedness for this notion of solution in full generality. For active scalars, we limit ourselves to show existence of generalized solutions, see Proposition 2.11 below.

Also, it is worth of mention that every sufficiently smooth generalized solution of (1.2) or (1.7) is also a classical solution, as can be proved following the lines of [8, Theorem 2.2 and Proposition 2.7]. On the other hand, our notion of generalized solution is weaker than the notion of  $L^\infty$ -weak solution contained in [6], that we recall now:

**Definition 2.10** Assume (A1)–(A3),  $q^\epsilon, q \in L^1([0, T], L^\infty(\mathbb{T}^2))$  for every  $\epsilon > 0$  and  $\Xi_0 \in L^\infty(\mathbb{T}^2)$ . For  $f, g : \mathbb{T}^2 \rightarrow \mathbb{R}$ , denote  $\langle f, g \rangle := \int_{\mathbb{T}^2} f(x)g(x)dx$ . Then:

- for every  $\epsilon > 0$ , a stochastic process  $\Xi^\epsilon : \Omega \times [0, T] \rightarrow L^\infty(\mathbb{T}^2)$  is called a  *$L^\infty$ -weak solution* of (1.2) if it is weakly progressively measurable with respect to  $(\mathcal{F}_t)_{t \in [0, T]}$ , it is compatible with  $v^\epsilon$  and for every smooth test function  $f \in C^\infty(\mathbb{T}^2)$  it holds  $\mathbb{P}$ -a.s. for every  $t \in [0, T]$ :

$$\begin{aligned} \langle \Xi_t^\epsilon, f \rangle - \langle \Xi_0^\epsilon, f \rangle &= \int_0^t \langle \Xi_s^\epsilon, (v_s^\epsilon + u_s^\epsilon) \cdot \nabla f \rangle ds \\ &\quad + \int_0^t \langle \Xi_s^\epsilon, v \Delta f \rangle ds + \int_0^t \langle q_s^\epsilon, f \rangle ds; \end{aligned}$$

- a stochastic process  $\Xi : \Omega \times [0, T] \rightarrow L^\infty(\mathbb{T}^2)$  is called a  $L^\infty$ -weak solution of (1.7) if it is weakly progressively measurable with respect to  $(\mathcal{F}_t)_{t \in [0, T]}$ , it is compatible with  $v$  and for every smooth test function  $f \in C^\infty(\mathbb{T}^2)$  it holds  $\mathbb{P}$ -a.s. for every  $t \in [0, T]$ :

$$\begin{aligned} \langle \Xi_t, f \rangle - \langle \Xi_0, f \rangle &= \int_0^t \langle \Xi_s, v_s \cdot \nabla f \rangle ds + \sum_{k \in \mathbb{N}} \int_0^t \langle \Xi_s, \sigma_k \cdot \nabla f \rangle \circ dW_s^k \\ &\quad + \int_0^t \langle \Xi_s, v \Delta f \rangle ds + \int_0^t \langle q_s, f \rangle ds. \end{aligned}$$

In [6] well-posedness of  $L^\infty$ -weak solution to stochastic Euler Equations is shown. With minor modifications in the argument one can prove existence of  $L^\infty$ -weak solutions to (1.2) and (1.7) in the general case. For active scalars, those provide generalized solutions in the sense of Definition 2.8, that is the content of the following:

**Proposition 2.11** *Assume (A1)–(A3),  $q^\epsilon, q \in L^1([0, T], L^\infty(\mathbb{T}^2))$  for every  $\epsilon > 0$  and  $\Xi_0 \in L^\infty(\mathbb{T}^2)$ . Then every  $L^\infty$ -weak solution to (1.2) is also a generalized solution to (1.2), and every  $L^\infty$ -weak solution to (1.7) is also a generalized solution to (1.7).*

**Proof** The strategy of the proof is similar to Brzeźniak et al. [6, Proposition 5.3] and [17, Theorem 20], and consists in taking the convolution of a  $L^\infty$ -weak solution with a smooth mollifier  $\vartheta_\delta = \delta^{-2} \vartheta(\delta \cdot)$ ,  $\delta > 0$ , and then taking the limit for  $\delta \rightarrow 0$ .

Let  $\Xi^\epsilon$  be a  $L^\infty$ -weak solution of (1.2) and  $\Xi$  be a  $L^\infty$ -weak solution of (1.7), in the sense of the previous definition. Using  $f = \vartheta_\delta(y - \cdot)$  as a test function,  $y \in \mathbb{T}^2$ , and denoting  $\Xi_\delta^\epsilon := \vartheta_\delta * \Xi^\epsilon$ ,  $\Xi_\delta := \vartheta_\delta * \Xi$  we get (omitting the parameter  $\omega$ )

$$\begin{aligned} \Xi_\delta^\epsilon(t, y) - \Xi_\delta^\epsilon(0, y) &= \int_0^t \int_{\mathbb{T}^2} \Xi^\epsilon(s, x) (v^\epsilon(s, x) + u^\epsilon(s, x)) \cdot \nabla_x \vartheta_\delta(y - x) dx ds \\ &\quad + v \int_0^t \int_{\mathbb{T}^2} \Xi^\epsilon(s, x) \Delta_x \vartheta_\delta(y - x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{T}^2} q^\epsilon(s, x) \vartheta_\delta(y - x) dx ds, \end{aligned}$$

and

$$\begin{aligned} \Xi_\delta(t, y) - \Xi_\delta(0, y) &= \int_0^t \int_{\mathbb{T}^2} \Xi(s, x)v(s, x) \cdot \nabla_x \vartheta_\delta(y - x) dx ds \\ &\quad + \sum_{k \in \mathbb{N}} \int_0^t \int_{\mathbb{T}^2} \Xi(s, x)\sigma_k(x) \cdot \nabla_x \vartheta_\delta(y - x) dx \circ dW_s^k \\ &\quad + \nu \int_0^t \int_{\mathbb{T}^2} \Xi(s, x)\Delta_x \vartheta_\delta(y - x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{T}^2} q(s, x)\vartheta_\delta(y - x) dx ds. \end{aligned}$$

Since  $\Xi_\delta^\epsilon, \Xi_\delta$  are smooth functions in the variable  $y$ , we can write the equivalent expressions in differential notation

$$\begin{aligned} d\Xi_\delta^\epsilon(t, y) + \nabla \Xi_\delta^\epsilon(t, y) \cdot (v^\epsilon(t, y) + u^\epsilon(t, y)) dt \\ = \int_{\mathbb{T}^2} \Xi^\epsilon(t, x)(v^\epsilon(t, x) + u^\epsilon(t, x)) \cdot \nabla_x \vartheta_\delta(y - x) dx dt \\ + \nu \int_{\mathbb{T}^2} \Xi^\epsilon(t, x)\Delta_x \vartheta_\delta(y - x) dx dt + \int_{\mathbb{T}^2} q^\epsilon(t, x)\vartheta_\delta(y - x) dx dt \\ + \nabla \Xi_\delta^\epsilon(t, y) \cdot (v^\epsilon(t, y) + u^\epsilon(t, y)) dt, \end{aligned}$$

and

$$\begin{aligned} d\Xi_\delta(t, y) + \nabla \Xi_\delta(t, y) \cdot v(t, y) dt + \sum_{k \in \mathbb{N}} \nabla \Xi_\delta(t, y) \cdot \sigma_k(y) \circ dW_t^k \\ = \int_{\mathbb{T}^2} \Xi(t, x)v(t, x) \cdot \nabla_x \vartheta_\delta(y - x) dx dt \\ + \sum_{k \in \mathbb{N}} \int_{\mathbb{T}^2} \Xi(t, x)\sigma_k(x) \cdot \nabla_x \vartheta_\delta(y - x) dx \circ dW_t^k \\ + \nu \int_{\mathbb{T}^2} \Xi(t, x)\Delta_x \vartheta_\delta(y - x) dx dt + \int_{\mathbb{T}^2} q(t, x)\vartheta_\delta(y - x) dx dt \\ + \nabla \Xi_\delta(t, y) \cdot v(t, y) dt + \sum_{k \in \mathbb{N}} \nabla \Xi_\delta(t, y) \cdot \sigma_k(y) \circ dW_t^k. \end{aligned}$$

Notice that the following formulas for the gradient of the convolution hold true:  $\nabla \Xi_\delta^\epsilon(t, y) = - \int_{\mathbb{T}^2} \Xi^\epsilon(t, x)\nabla_x \vartheta_\delta(y - x)$ , and  $\nabla \Xi_\delta(t, y) = - \int_{\mathbb{T}^2} \Xi(t, x)\nabla_x \vartheta_\delta(y - x)$ ; also,  $\Delta_x \vartheta_\delta(y - x) = \Delta_y \vartheta_\delta(y - x)$ . Substituting into the previous expressions, we get

$$\begin{aligned}
 & d\Xi_\delta^\epsilon(t, y) + \nabla \Xi_\delta^\epsilon(t, y) \cdot (v^\epsilon(t, y) + u^\epsilon(t, y))dt \\
 &= [-\vartheta_\delta * (\nabla \Xi_t^\epsilon \cdot (v_t^\epsilon + u_t^\epsilon)) + (v_t^\epsilon + u_t^\epsilon) \cdot (\vartheta_\delta * \nabla \Xi_t^\epsilon)](y)dt \\
 &\quad + \nu \Delta \Xi_\delta^\epsilon(t, y)dt + q_\delta^\epsilon(t, y)dt \\
 &= R_\delta [v_t^\epsilon + u_t^\epsilon, \Xi_t^\epsilon](y)dt + \nu \Delta \Xi_\delta^\epsilon(t, y)dt + q_\delta^\epsilon(t, y)dt,
 \end{aligned}$$

and

$$\begin{aligned}
 & d\Xi_\delta(t, y) + \nabla \Xi_\delta(t, y) \cdot v(t, y)dt + \sum_{k \in \mathbb{N}} \nabla \Xi_\delta(t, y) \cdot \sigma_k(y) \circ dW_t^k \\
 &= [-\vartheta_\delta * (\nabla \Xi_t \cdot v_t) + v_t \cdot (\vartheta_\delta * \nabla \Xi_t)](y)dt \\
 &\quad + \sum_{k \in \mathbb{N}} [-\vartheta_\delta * (\nabla \Xi_t \cdot \sigma_k) + \sigma_k \cdot (\vartheta_\delta * \nabla \Xi_t)](y) \circ dW_t^k \\
 &\quad + \nu \Delta \Xi_\delta(t, y)dt + q_\delta(t, y)dt \\
 &= R_\delta [v_t, \Xi_t](y)dt + \sum_{k \in \mathbb{N}} R_\delta [\sigma_k, \Xi_t](y) \circ dW_t^k \\
 &\quad + \nu \Delta \Xi_\delta(t, y)dt + q_\delta(t, y)dt,
 \end{aligned}$$

where we have defined  $q_\delta^\epsilon := \vartheta_\delta * q^\epsilon$ ,  $q_\delta := \vartheta_\delta * q$  and the commutator

$$R_\delta [v, \Xi] := -\vartheta_\delta * (\nabla \Xi \cdot v) + v \cdot (\vartheta_\delta * \nabla \Xi).$$

We have obtained differential equations for the spatially smooth processes  $\Xi_\delta^\epsilon$  and  $\Xi_\delta$ . Applying the backwards Itô Formula to the processes  $s \mapsto \Xi_\delta^\epsilon(s, \phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y)))$  and  $s \mapsto \Xi_\delta(s, \phi_s((\phi_t)^{-1}(y)))$ , for fixed  $t \in [0, T]$ , and taking the expectation with respect to  $\mathbb{P}$ , we obtain that the process  $\Xi_\delta^\epsilon$  is given by

$$\begin{aligned}
 \Xi_\delta^\epsilon(t, y) &= \tilde{\mathbb{E}} \left[ \Xi_\delta^\epsilon(0, (\phi_t^\epsilon)^{-1}(y)) + \int_0^t q_\delta^\epsilon(s, \phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y)))ds \right] \\
 &\quad + \tilde{\mathbb{E}} \left[ \int_0^t R_\delta [v_s^\epsilon + u_s^\epsilon, \Xi_s^\epsilon](\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y)))ds \right], \tag{2.3}
 \end{aligned}$$

whereas the process  $\Xi_\delta$  is given by

$$\begin{aligned}
 \Xi_\delta(t, y) &= \tilde{\mathbb{E}} \left[ \Xi_\delta(0, (\phi_t)^{-1}(y)) + \int_0^t q_\delta(s, \phi_s((\phi_t)^{-1}(y)))ds \right] \\
 &\quad + \tilde{\mathbb{E}} \left[ \int_0^t R_\delta [v_s, \Xi_s](\phi_s((\phi_t)^{-1}(y)))ds \right] \\
 &\quad + \sum_{k \in \mathbb{N}} \tilde{\mathbb{E}} \left[ \int_0^t R_\delta [\sigma_k, \Xi_s](\phi_s((\phi_t)^{-1}(y))) \circ dW_s^k \right] \tag{2.4}
 \end{aligned}$$

Let us focus on (2.3). By well-known properties of mollifiers, for every fixed  $\omega \in \Omega$  and  $t \in [0, T]$ , the right-hand side  $\Xi_\delta^\epsilon(\omega, t, \cdot) \rightarrow \Xi^\epsilon(\omega, t, \cdot)$  in  $L^1(\mathbb{T}^2)$  as  $\delta \rightarrow 0$ .

Concerning the left-hand side, a commutator lemma [17, Lemma 17] yields for every fixed  $\epsilon > 0$

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{T}^2} \left| \tilde{\mathbb{E}} \left[ \int_0^t R_\delta [v_s^\epsilon + u_s^\epsilon, \Xi_s^\epsilon] (\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y))) ds \right] \right| dy = 0,$$

and by well-known properties of mollifiers and Lebesgue dominated convergence Theorem we can prove the convergence

$$\begin{aligned} & \tilde{\mathbb{E}} \left[ \Xi_\delta^\epsilon(0, (\phi_t^\epsilon)^{-1}) + \int_0^t q_\delta^\epsilon(s, \phi_s^\epsilon((\phi_t^\epsilon)^{-1})) ds \right] \\ & + \tilde{\mathbb{E}} \left[ \int_0^t R_\delta [v_s^\epsilon + u_s^\epsilon, \Xi_s^\epsilon] (\phi_s^\epsilon((\phi_t^\epsilon)^{-1})) ds \right] \\ & \rightarrow \tilde{\mathbb{E}} \left[ \Xi^\epsilon(0, (\phi_t^\epsilon)^{-1}) + \int_0^t q^\epsilon(s, \phi_s^\epsilon((\phi_t^\epsilon)^{-1})) ds \right] \end{aligned}$$

in  $L^1(\mathbb{T}^2)$  as  $\delta \rightarrow 0$ , for almost every  $\omega \in \Omega$  and  $t \in [0, T]$ . Therefore, by (2.4) we have and the uniqueness of the  $L^1(\mathbb{T}^2)$  limit, for almost every  $\omega \in \Omega$ ,  $t \in [0, T]$  and  $y \in \mathbb{T}^2$ :

$$\Xi^\epsilon(t, y) = \tilde{\mathbb{E}} \left[ \Xi^\epsilon(0, (\phi_t^\epsilon)^{-1}(y)) + \int_0^t q^\epsilon(s, \phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y))) ds \right],$$

that is exactly the desired representation formula (1.4). The argument for (2.4) is similar, with only a little complication due to the stochastic integral, and we leave it to the reader. □

As a final remark, since we have seen that the notion of generalized solution is weaker than the notion of  $L^\infty$ -weak solution, our results are indeed very general: they can be applied at least to every  $L^\infty$ -weak solution.

### 2.4 Statement of main results

We remind the reader that, for externally given  $v^\epsilon$  and  $v$  satisfying (A1)–(A3), there exist unique solutions of the characteristic equations and the large-scale dynamics, assuming the latter is passive (cfr. Proposition 2.9). Strictly speaking, the results in this section are formulated for passive scalars; however, we shall see in Sect. 6 that, a posteriori, even in the active case, fields generated by large-scale processes satisfy all the needed assumptions. Therefore the following theorems hold true in the more general case, simply looking at an active large-scale process as a passive scalar compatible with the external fields it generates.

#### 2.4.1 Convergence of characteristics

Denote  $|x - y|$  the geodesic distance on the flat two dimensional torus between points  $x, y \in \mathbb{T}^2$ . To keep the notation simple, we define the following quantity associated with a measurable map  $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ :

$$\|\phi\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)} := \int_{\mathbb{T}^2} |\phi(x)| dx.$$

Notice that  $\|\cdot\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)}$  is not a norm on the space of measurable maps  $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ , in particular it is not positively homogeneous. However,  $\|\cdot\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)}$  induces a distance on the space  $C(\mathbb{T}^2, \mathbb{T}^2)$  of continuous maps  $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ . Similarly, we define  $\|\cdot\|_{L^\infty(\mathbb{T}^2, \mathbb{T}^2)}$  as

$$\|\phi\|_{L^\infty(\mathbb{T}^2, \mathbb{T}^2)} := \operatorname{ess\,sup}_{x \in \mathbb{T}^2} |\phi(x)|.$$

In order to prove convergence of characteristics  $\phi^\epsilon \rightarrow \phi$ , it is clear that one needs some sort of control for the difference  $v^\epsilon - v$ . Therefore, we assume:

(A4) there exist a constant  $C$  and a negligible set  $\mathcal{N} \subset \Omega$  such that for every  $\omega \in \mathcal{N}^c$ ,  $\epsilon > 0$  and  $t \in [0, T]$ :

$$\begin{aligned} \|v^\epsilon(\omega, t, \cdot) - v(\omega, t, \cdot)\|_{L^1(\mathbb{T}^2, \mathbb{R}^2)} &\leq C\gamma \left( \tilde{\mathbb{E}} [\|\phi_t^\epsilon - \phi_t\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)}] \right) \\ &\quad + C \int_0^t \gamma \left( \tilde{\mathbb{E}} [\|\phi_s^\epsilon - \phi_s\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)}] \right) ds + c_\epsilon, \end{aligned}$$

where  $c_\epsilon \in \mathbb{R}$  is infinitesimal as  $\epsilon \rightarrow 0$ ,  $\phi_t^\epsilon = \phi^\epsilon(\omega, \tilde{\omega}, t, \cdot)$  is the unique solution of (1.3), and  $\phi_t = \phi(\omega, \tilde{\omega}, t, \cdot)$  is the unique solution of (1.5).

A little less clear, at this point, is our next assumption on the coefficients  $(\zeta_k)_{k \in \mathbb{N}}$ :

(A5) for every  $x \in \mathbb{T}^2$  it holds

$$\sum_{k \in \mathbb{N}} ((K * \zeta_k) \cdot \nabla \zeta_k)(x) = 0.$$

The motivations for assuming (A5) will become evident during the proof of Proposition 3.2 in Sect. 3.

We are ready to state our first main result:

**Theorem 2.12** *Assume (A1)–(A5). Let  $\hat{\mathbb{E}}[\cdot] := \mathbb{E}[\tilde{\mathbb{E}}[\cdot]]$  denote the expectation on  $\hat{\Omega} := \Omega \times \tilde{\Omega}$  with respect to the probability measure  $\hat{\mathbb{P}} := \mathbb{P} \otimes \tilde{\mathbb{P}}$ . Then*

$$\sup_{t \in [0, T]} \hat{\mathbb{E}} [\|\phi_t^\epsilon - \phi_t\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)}] \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

### 2.4.2 Convergence of large-scale dynamics

Let  $q^\epsilon, q : [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$  be such that:

(A6) there exists a constant  $C$  such that for every  $\epsilon > 0$  it holds  $q^\epsilon, q \in L^1([0, T], L^\infty(\mathbb{T}^2))$  and

$$\int_0^T \|q_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} ds \leq C, \quad \int_0^T \|q_s\|_{L^\infty(\mathbb{T}^2)} ds \leq C;$$

(A7)  $q^\epsilon - q$  converges to zero in  $L^1([0, T], L^\infty(\mathbb{T}^2))$ .

Our second main result is the following.

**Theorem 2.13** *Assume (A1)–(A7) and  $\Xi_0 \in L^\infty(\mathbb{T}^2)$ . Then the solution  $\Xi^\epsilon$  of (1.2) converges towards the solution  $\Xi$  of (1.7) in the following sense: for every  $f \in L^1(\mathbb{T}^2)$*

$$\mathbb{E} \left[ \left| \int_{\mathbb{T}^2} \Xi_t^\epsilon(x) f(x) dx - \int_{\mathbb{T}^2} \Xi_t(x) f(x) dx \right| \right] \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,$$

for every fixed  $t \in [0, T]$  and in  $L^p([0, T])$  for every finite  $p$ . Moreover, if  $q \in L^1([0, T], Lip(\mathbb{T}^2))$  then the previous convergence holds uniformly in  $t \in [0, T]$  and  $f \in Lip(\mathbb{T}^2)$  with Lipschitz constant  $[f]_{Lip(\mathbb{T}^2)} \leq 1$  and  $\|f\|_{L^\infty(\mathbb{T}^2)} \leq 1$ .

### 3 Technical results

In this section and after in the paper, the symbol  $\lesssim$  will indicate inequality up to a unimportant multiplicative constant  $C$  not depending of  $\epsilon$ .

#### 3.1 Linearized dynamics

For  $\epsilon > 0$ , denote  $\theta^\epsilon$  the solution of the linear problem

$$d\theta_t^\epsilon = -\epsilon^{-1}\theta_t^\epsilon dt + \epsilon^{-1} \sum_{k \in \mathbb{N}} \varsigma_k dW_t^k,$$

with initial condition  $\theta^\epsilon|_{t=0} = 0$ . The process  $\theta^\epsilon$  is explicitly given by the formula  $\theta_t^\epsilon = \sum_{k \in \mathbb{N}} \varsigma_k \eta_t^{\epsilon,k}$ , where

$$\eta_t^{\epsilon,k} := \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} dW_s^k, \quad k \in \mathbb{N},$$

is the so called Ornstein–Uhlenbeck process with null initial condition. By Jia and Zhao [27, Theorem 2.2], for every fixed  $p \geq 1$  it holds uniformly in  $k \in \mathbb{N}$

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |\eta_t^{\epsilon,k}|^p \right] \lesssim \epsilon^{-p/2} \log^{p/2}(1 + \epsilon^{-1}), \tag{3.1}$$

and therefore by assumption (A3)

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|\theta_t^\epsilon\|_{W^{1, \infty}(\mathbb{T}^2)}^p \right] \lesssim \epsilon^{-p/2} \log^{p/2}(1 + \epsilon^{-1}). \tag{3.2}$$

The difference  $\zeta^\epsilon := \xi^\epsilon - \theta^\epsilon$  between the small-scale vorticity  $\xi^\epsilon$  and  $\theta^\epsilon$  solves the equation

$$d\zeta_t^\epsilon + (v_t^\epsilon + u_t^\epsilon) \cdot \nabla \zeta_t^\epsilon dt = -\epsilon^{-1} \zeta_t^\epsilon dt - (v_t^\epsilon + u_t^\epsilon) \cdot \nabla \theta_t^\epsilon dt$$

with initial condition  $\zeta_0^\epsilon = 0$ , whose solution satisfies

$$\zeta_t^\epsilon(\psi_t^\epsilon(x)) = - \int_0^t e^{-\epsilon^{-1}(t-s)} ((v_s^\epsilon + u_s^\epsilon) \cdot \nabla \theta_s^\epsilon)(\psi_s^\epsilon(x)) ds. \tag{3.3}$$

In the following, for  $t \in [0, T]$  and  $x \in \mathbb{T}^2$  we denote  $z_t^\epsilon(x) = (K * \zeta_t^\epsilon)(x)$ .

### 3.2 Main technical results

We are going to prove two main technical results, needed for the proof of Theorem 2.12. Since our strategy consists in replicating the proof of Flandoli and Pappaletta [21, Proposition 4.1], the first result we need is the following:

**Proposition 3.1** *Assume (A1)–(A3). Then the following inequality holds:*

$$\begin{aligned} & \hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \left\| \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(\cdot)) \eta_s^{\epsilon, k} ds - \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(\cdot)) \circ dW_s^k \right\|_{L^1(\mathbb{T}^2, \mathbb{R}^2)} \right] \\ & \lesssim \epsilon^{1/42} \log^{47/42}(1 + \epsilon). \end{aligned}$$

In [21, Section 4] a similar estimate was proven along the way, using a considerable amount of auxiliary lemmas and computations. In view of this, here we refrain from going again into full detail, and the proof of Proposition 3.1 will only be sketched.

On the other hand, the nonlinear term in (1.1) produces a new term in the equation of characteristics, that was absent in [21]. Although the final results is not affected by this new term, it is not trivial to actually prove so. We need the following.

**Proposition 3.2** *Assume (A1)–(A5). Then:*

$$\hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \left\| \int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \right] \lesssim \epsilon^{1/12} \log^{11/12}(1 + \epsilon^{-1}).$$

This constitutes the main novelty with respect to Flandoli and Pappaletta [21]. The proof of Proposition 3.2 relies strongly on assumption (A5) and the following Itô Formulas, yielding for every fixed  $t \in [0, T]$  and  $k, h \in \mathbb{N}$ :

$$\begin{aligned} \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} &= -\epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \\ &\quad + \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} dW_s^h + \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,h} dW_s^k \\ &\quad + \delta_{k,h} \frac{\epsilon^{-2}}{2} \int_0^t e^{-\epsilon^{-1}(t-s)} ds, \\ \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} &= -2\epsilon^{-1} \int_0^t \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \\ &\quad + \epsilon^{-1} \int_0^t \eta_s^{\epsilon,k} dW_s^h + \epsilon^{-1} \int_0^t \eta_s^{\epsilon,h} dW_s^k + \delta_{k,h} \frac{\epsilon^{-2}t}{2}, \end{aligned}$$

with  $\delta_{k,h}$  being the Kronecker delta function, allowing to control the time integral of quadratics  $\eta_s^{\epsilon,k} \eta_s^{\epsilon,h}$ . In the formula above we have used  $\eta_0^{\epsilon,k} = \eta_0^{\epsilon,h} = 0$ , although the computations could be performed also for more general initial conditions.

### 3.3 Proof of Proposition 3.1

In this paragraph we recall the argument contained in [21]. Roughly speaking, Proposition 3.1 is a sort of Wong–Zakai result for the Ornstein–Uhlenbeck process  $\eta^{\epsilon,k}$  converging to a white-in-time noise, that is the formal time derivative of the Wiener process  $W^k$ . We need to exploit a discretization of (1.3) to show the closeness, in a certain sense to be specified, between the Stratonovich-to-Itô corrector  $c : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ , given by:

$$c(x) = \frac{1}{2} \sum_{k \in \mathbb{N}} \nabla \sigma_k(x) \cdot \sigma_k(x), \quad x \in \mathbb{T}^2,$$

coming from the stochastic integral, and the iterated time integral of the Ornstein–Uhlenbeck process.

In order to discretize the problem, for every  $\epsilon > 0$  take a mesh  $\delta > 0$  such that  $T/\delta$  is an integer. For any  $n = 0, \dots, T/\delta - 1$  and fixed  $x \in \mathbb{T}^2$ , consider the following decomposition:

$$\begin{aligned} \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \sigma_k(\phi_s^\epsilon(x)) \eta_s^{\epsilon,k} ds &= \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \left( \int_{n\delta}^s \nabla \sigma_k(\phi_r^\epsilon(x)) \cdot v_r^\epsilon(\phi_r^\epsilon(x)) dr \right) \eta_s^{\epsilon,k} ds \\ &\quad + \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \left( \int_{n\delta}^s \nabla \sigma_k(\phi_r^\epsilon(x)) \cdot z_r^\epsilon(\phi_r^\epsilon(x)) dr \right) \eta_s^{\epsilon,k} ds \\ &\quad + \sum_{k,h \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \left( \int_{n\delta}^s \nabla \sigma_k(\phi_r^\epsilon(x)) \cdot \sigma_h(\phi_r^\epsilon(x)) \eta_r^{\epsilon,h} dr \right) \eta_s^{\epsilon,k} ds \\ &\quad + \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \left( \int_{n\delta}^s \nabla \sigma_k(\phi_r^\epsilon(x)) \cdot \sqrt{2\nu} dw_r \right) \eta_s^{\epsilon,k} ds \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \sigma_k(\phi_{n\delta}^\epsilon(x)) dW_s^k \\
 & - \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \sigma_k(\phi_{n\delta}^\epsilon(x)) \epsilon d\eta_s^{\epsilon,k} \\
 & = : I_1^\epsilon(n) + I_2^\epsilon(n) + I_3^\epsilon(n) + I_4^\epsilon(n) + I_5^\epsilon(n) + I_6^\epsilon(n),
 \end{aligned}$$

where the terms  $I_2^\epsilon(n)$  and  $I_3^\epsilon(n)$  come from the identity  $u_r^\epsilon(\phi_r^\epsilon(x)) = z_r^\epsilon(\phi_r^\epsilon(x)) + \sum_{h \in \mathbb{N}} \sigma_h(\phi_r^\epsilon(x)) \eta_r^{\epsilon,h}$ , which can be obtained applying the Biot–Savart law to the identity  $\zeta^\epsilon = \xi^\epsilon - \theta^\epsilon$  defining  $\zeta^\epsilon$ . Regarding the Stratonovich integral, we can rewrite:

$$\begin{aligned}
 \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \sigma_k(\phi_s^\epsilon(x)) \circ dW_s^k & = \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} (\sigma_k(\phi_s^\epsilon(x)) - \sigma_k(\phi_{n\delta}^\epsilon(x))) dW_s^k \\
 & + \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \sigma_k(\phi_{n\delta}^\epsilon(x)) dW_s^k \\
 & + \int_{n\delta}^{(n+1)\delta} (c(\phi_s^\epsilon(x)) - c(\phi_{n\delta}^\epsilon(x))) ds \\
 & + \int_{n\delta}^{(n+1)\delta} c(\phi_{n\delta}^\epsilon(x)) ds \\
 & = : J_1^\epsilon(n) + J_2^\epsilon(n) + J_3^\epsilon(n) + J_4^\epsilon(n).
 \end{aligned}$$

The ingredients for the proof of Proposition 3.1 are:

- a good estimate on  $\widehat{\mathbb{E}} \left[ \sup_{t \in [0, T]} |z_t^\epsilon(\phi_t^\epsilon(x))| \right]$  (cfr. Lemma 3.3), needed to control  $I_2^\epsilon(n)$ ;
- a good estimate on  $\widehat{\mathbb{E}} \left[ \sup_{\tau \leq \delta} |\phi_{\tau+n\delta}^\epsilon(x) - \phi_{n\delta}^\epsilon(x)| \right]$  (cfr. Lemma 3.4), needed to approximate  $I_3^\epsilon(n)$  with

$$\sum_{k, h \in \mathbb{N}} \nabla \sigma_k(\phi_{n\delta}(x)) \cdot \sigma_h(\phi_{n\delta}(x)) \int_{n\delta}^{(n+1)\delta} \left( \int_{n\delta}^s \eta_r^{\epsilon,h} dr \right) \eta_s^{\epsilon,k} ds; \tag{3.4}$$

- a better estimate on  $\widehat{\mathbb{E}} \left[ |\phi_{(n+1)\delta}^\epsilon(x) - \phi_{n\delta}^\epsilon(x)| \right]$  (cfr. Lemma 3.5), needed to control  $I_6^\epsilon(n)$  with a discrete integration by parts.

Notice that  $I_5^\epsilon(n) = J_2^\epsilon(n)$ . Also, the expression in (3.4) (which approximates  $I_3^\epsilon(n)$ ) must be compensated by subtracting  $J_4^\epsilon(n)$ .

**Lemma 3.3** *Assume (A1)–(A3). Then for every fixed  $p \geq 1$  it holds*

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|\zeta_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}^p \right] \lesssim \log^p(1 + \epsilon^{-1}).$$

In particular, since  $z_t^\epsilon = K * \zeta_t^\epsilon$  we also have

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|z_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}^p \right] \lesssim \log^p(1 + \epsilon^{-1}).$$

**Proof** We prove in the first place the weaker estimate:

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|\zeta_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}^p \right] \lesssim \epsilon^{-p}. \tag{3.5}$$

Since  $\theta^\epsilon$  satisfies the bound above by (3.2), it suffices to prove it for  $\xi^\epsilon$ . Denote  $M_t^\epsilon(x) = \sum_{k \in \mathbb{N}} \int_0^t \zeta_k(\psi_s^\epsilon(x)) dW_s^k$ . Since for every  $s, t \in [0, T]$

$$\mathbb{E} \left[ \|M_t^\epsilon - M_s^\epsilon\|_{L^\infty(\mathbb{T}^2)}^4 \right] \lesssim \left( \sum_{k \in \mathbb{N}} \|\zeta_k\|_{L^\infty(\mathbb{T}^2)}^2 \right)^2 (t - s)^2,$$

by (A3) and Kolmogorov continuity Theorem the process  $M^\epsilon : \Omega \times [0, T] \rightarrow L^\infty(\mathbb{T}^2)$  has a modification  $\tilde{M}^\epsilon$  that is  $\alpha$ -Hölder continuous for every  $\alpha < 1/4$ , with  $\alpha$ -Hölder constant  $K_{\epsilon, \alpha}$  bounded in  $L^p(\Omega)$  for every  $p < \infty$  uniformly in  $\epsilon$ . Since  $M^\epsilon$  has continuous trajectories,  $M_t^\epsilon = \tilde{M}_t^\epsilon$  a.s. as random variables in  $L^\infty(\mathbb{T}^2)$  and

$$\begin{aligned} \xi_t^\epsilon(\psi_t^\epsilon(x)) &= \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} dM_s^\epsilon(x) \\ &= \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} d(M_s^\epsilon(x) - M_t^\epsilon(x)) \\ &= \epsilon^{-1} \left[ e^{-\epsilon^{-1}(t-s)} (M_s^\epsilon(x) - M_t^\epsilon(x)) \right]_{s=0}^{s=t} \\ &\quad - \epsilon^{-2} \int_0^t e^{-\epsilon^{-1}(t-s)} (M_s^\epsilon(x) - M_t^\epsilon(x)) ds. \end{aligned}$$

Clearly  $\|\xi_t^\epsilon\|_{L^\infty(\mathbb{T}^2)} = \|\xi_t^\epsilon \circ \psi_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}$ , and therefore

$$\|\xi_t^\epsilon\|_{L^\infty(\mathbb{T}^2)} \leq \epsilon^{-1} e^{-\epsilon^{-1}t} \|M_t^\epsilon\|_{L^\infty(\mathbb{T}^2)} + \epsilon^{-1} K_{\epsilon, \alpha},$$

and (3.5) follows.

Recalling (3.3), the following inequality holds

$$\|\zeta_t^\epsilon\|_{L^\infty(\mathbb{T}^2)} \leq \int_0^t e^{-\epsilon^{-1}(t-s)} \|(v_s^\epsilon + u_s^\epsilon) \cdot \nabla \theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} ds. \tag{3.6}$$

Using assumption (A2) and  $u_s^\epsilon = K * \zeta_s^\epsilon + K * \theta_s^\epsilon$  we get

$$\|(v_s^\epsilon + u_s^\epsilon) \cdot \nabla \theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} \lesssim (1 + \|\zeta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} + \|\theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)}) \|\nabla \theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)},$$

that can be plugged back into (3.6) to produce the recursive estimate

$$\begin{aligned} \|\zeta_t^\epsilon\|_{L^\infty(\mathbb{T}^2)} &\lesssim \int_0^t e^{-\epsilon^{-1}(t-s)} (1 + \|\theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)}) \|\nabla\theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} ds \\ &\quad + \int_0^t e^{-\epsilon^{-1}(t-s)} \|\zeta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} \|\nabla\theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} ds \\ &\lesssim \epsilon \left( \sup_{s \in [0, T]} \|\theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} + \sup_{s \in [0, T]} \|\zeta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} \right) \sup_{s \in [0, T]} \|\nabla\theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)}. \end{aligned}$$

By Hölder inequality and (3.5) we deduce from the previous inequality

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|\zeta_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}^p \right] \lesssim \log^p(1 + \epsilon^{-1}) + \epsilon^{-p/2} \log^{p/2}(1 + \epsilon^{-1}),$$

improving the bound (3.5) itself. Iterating the same argument one more time we obtain the desired estimate. □

**Lemma 3.4** *Assume (A1)–(A3). Then for every fixed  $p \geq 1$  and  $\alpha \in (0, 1/2)$*

$$\hat{\mathbb{E}} \left[ \sup_{\substack{t+\tau \leq T \\ \tau \leq \delta}} \|\phi_{t+\tau}^\epsilon - \phi_t^\epsilon\|_{L^\infty(\mathbb{T}^2, \mathbb{T}^2)}^p \right] \lesssim \delta^p \epsilon^{-p/2} \log^{p/2}(1 + \epsilon^{-1}) + \delta^{p\alpha}.$$

**Proof** The increment  $\phi_{t+\tau}^\epsilon(x) - \phi_t^\epsilon(x)$  can be written as

$$\begin{aligned} \phi_{t+\tau}^\epsilon(x) - \phi_t^\epsilon(x) &= \int_t^{t+\tau} v_s^\epsilon(\phi_s^\epsilon(x)) ds + \sum_{k \in \mathbb{N}} \int_t^{t+\tau} \sigma_k(\phi_s^\epsilon(x)) \eta_s^{\epsilon, k} ds \\ &\quad + \int_t^{t+\tau} z_s(\phi_s^\epsilon(x)) ds + \sqrt{2\nu}(w_{t+\tau} - w_t), \end{aligned}$$

therefore, by assumption (A2) we have

$$\begin{aligned} \sup_{t+\tau \leq T} \|\phi_{t+\tau}^\epsilon - \phi_t^\epsilon\|_{L^\infty(\mathbb{T}^2, \mathbb{T}^2)} &\lesssim \tau + \tau \sum_{k \in \mathbb{N}} \|\sigma_k\|_{L^\infty(\mathbb{T}^2)} \sup_{s \in [0, T]} |\eta_s^{\epsilon, k}| \\ &\quad + \tau \sup_{s \in [0, T]} \|\zeta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} + K_\alpha \tau^\alpha, \end{aligned}$$

where  $K_\alpha$  denotes the  $\alpha$ -Hölder constant of  $w$ . The thesis follows easily by (A3), (3.1) and Lemma 3.3. □

**Lemma 3.5** *Assume (A1)–(A3). Then for every fixed  $p \geq 1$  we have, uniformly in  $n = 0, \dots, T/\delta - 1$ :*

$$\begin{aligned} \hat{\mathbb{E}} \left[ \|\phi_{(n+1)\delta}^\epsilon - \phi_{n\delta}^\epsilon\|_{L^\infty(\mathbb{T}^2, \mathbb{T}^2)}^p \right] &\lesssim \delta^{2p} \epsilon^{-p} \log^p(1 + \epsilon^{-1}) \\ &\quad + \delta^{p(1+\alpha)} \epsilon^{-p/2} \log^{p/2}(1 + \epsilon^{-1}) \\ &\quad + \delta^{p/2} + \epsilon^{p/2} \log^{p/2}(1 + \epsilon^{-1}). \end{aligned}$$

**Proof** The increment  $\phi_{(n+1)\delta}^\epsilon(x) - \phi_{n\delta}^\epsilon(x)$  can be written as

$$\begin{aligned} \phi_{(n+1)\delta}^\epsilon(x) - \phi_{n\delta}^\epsilon(x) &= \int_{n\delta}^{(n+1)\delta} v_s^\epsilon(\phi_s^\epsilon(x)) ds \\ &\quad + \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} (\sigma_k(\phi_s^\epsilon(x)) - \sigma_k(\phi_{n\delta}^\epsilon(x))) \eta_s^{\epsilon,k} ds \\ &\quad + \sum_{k \in \mathbb{N}} \int_{n\delta}^{(n+1)\delta} \sigma_k(\phi_{n\delta}^\epsilon(x)) \eta_s^{\epsilon,k} ds \\ &\quad + \int_{n\delta}^{(n+1)\delta} z_s^\epsilon(\phi_s^\epsilon(x)) ds + \sqrt{2v}(w_{(n+1)\delta} - w_{n\delta}). \end{aligned}$$

The first, fourth and fifth term are easy. The second one is bounded in  $L^\infty(\mathbb{T}^2, \mathbb{T}^2)$  uniformly in  $n$  by

$$\int_0^\delta \sum_{k \in \mathbb{N}} \|\nabla \sigma_k\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^4)} \sup_{t+s \leq T} \|\phi_{t+s}^\epsilon - \phi_t^\epsilon\|_{L^\infty(\mathbb{T}^2, \mathbb{T}^2)} \sup_{s \in [0, T]} |\eta_s^{\epsilon,k}| ds,$$

and by (A3) and Hölder inequality with exponent  $q > 1$

$$\begin{aligned} &\hat{\mathbb{E}} \left[ \left( \int_0^\delta \sum_{k \in \mathbb{N}} \|\nabla \sigma_k\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^4)} \sup_{t+s \leq T} \|\phi_{t+s}^\epsilon - \phi_t^\epsilon\|_{L^\infty(\mathbb{T}^2, \mathbb{T}^2)} \sup_{s \in [0, T]} |\eta_s^{\epsilon,k}| ds \right)^p \right] \\ &\leq \delta^{p-1} \left( \sum_{k \in \mathbb{N}} \|\nabla \sigma_k\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^4)} \right)^{p-1} \int_0^\delta \sum_{k \in \mathbb{N}} \|\nabla \sigma_k\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^4)} \\ &\quad \times \hat{\mathbb{E}} \left[ \sup_{t+s \leq T} \|\phi_{t+s}^\epsilon - \phi_t^\epsilon\|_{L^\infty(\mathbb{T}^2, \mathbb{T}^2)}^{pq} \right]^{1/q} \hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\eta_s^{\epsilon,k}|^{pq'} \right]^{1/q'} ds \\ &\lesssim \delta^{p-1} \int_0^\delta \left( s^p \epsilon^{-p} \log^p(1 + \epsilon^{-1}) ds + s^{p\alpha} \epsilon^{-p/2} \log^{p/2}(1 + \epsilon^{-1}) \right) ds \\ &\lesssim \delta^{2p} \epsilon^{-p} \log^p(1 + \epsilon^{-1}) + \delta^{p(1+\alpha)} \epsilon^{-p/2} \log^{p/2}(1 + \epsilon^{-1}). \end{aligned}$$

The third term is bounded in  $L^\infty(\mathbb{T}^2, \mathbb{R}^2)$  by

$$\begin{aligned} \sum_{k \in \mathbb{N}} \|\sigma_k\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \left| \int_{n\delta}^{(n+1)\delta} \eta_s^{\epsilon, k} ds \right| &= \sum_{k \in \mathbb{N}} \|\sigma_k\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \left| W_{(n+1)\delta}^k - W_{n\delta}^k \right| \\ &+ \sum_{k \in \mathbb{N}} \|\sigma_k\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \epsilon \left| \eta_{(n+1)\delta}^{\epsilon, k} - \eta_{n\delta}^{\epsilon, k} \right|, \end{aligned}$$

from which we deduce as usual

$$\hat{\mathbb{E}} \left[ \left( \sum_{k \in \mathbb{N}} \|\sigma_k\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \left| \int_{n\delta}^{(n+1)\delta} \eta_s^{\epsilon, k} ds \right| \right)^p \right] \lesssim \delta^{p/2} + \epsilon^{p/2} \log^{p/2}(1 + \epsilon^{-1}).$$

Putting all together, the thesis follows. □

**Proof of Proposition 3.1** For any given  $t \in [0, T]$ , let  $\lfloor t \rfloor = :m\delta$  be the largest multiple of  $\delta$  strictly smaller than  $t$ . We can therefore decompose

$$\begin{aligned} \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(x)) \eta_s^{\epsilon, k} ds &= \sum_{k \in \mathbb{N}} \int_0^{m\delta} \sigma_k(\phi_s^\epsilon(x)) \eta_s^{\epsilon, k} ds + \sum_{k \in \mathbb{N}} \int_{m\delta}^t \sigma_k(\phi_s^\epsilon(x)) \eta_s^{\epsilon, k} ds \\ &= \sum_{j=1}^6 \sum_{n=0}^{m-1} I_j^\epsilon(n) + \sum_{k \in \mathbb{N}} \int_{m\delta}^t \sigma_k(\phi_s^\epsilon(x)) \eta_s^{\epsilon, k} ds, \end{aligned}$$

and in a similar fashion

$$\begin{aligned} \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(x)) \circ dW_s^k &= \sum_{k \in \mathbb{N}} \int_0^{m\delta} \sigma_k(\phi_s^\epsilon(x)) \circ dW_s^k + \sum_{k \in \mathbb{N}} \int_{m\delta}^t \sigma_k(\phi_s^\epsilon(x)) \circ dW_s^k \\ &= \sum_{j=1}^4 \sum_{n=0}^{m-1} J_j^\epsilon(n) + \sum_{k \in \mathbb{N}} \int_{m\delta}^t \sigma_k(\phi_s^\epsilon(x)) \circ dW_s^k. \end{aligned}$$

By (3.1), the following estimate holds true

$$\hat{\mathbb{E}} \left[ \sup_{\substack{m=0, \dots, T/\delta-1 \\ t \leq \delta}} \left\| \sum_{k \in \mathbb{N}} \int_{m\delta}^t \sigma_k(\phi_s^\epsilon(\cdot)) \eta_s^{\epsilon, k} ds \right\|_{L^1(\mathbb{T}^2, \mathbb{R}^2)} \right] \lesssim \delta \epsilon^{-1/2} \log^{1/2}(1 + \epsilon^{-1}).$$

Also, by (A3) and Kolmogorov continuity Theorem, for every fixed  $\alpha \in (0, 1/2)$  we have

$$\hat{\mathbb{E}} \left[ \sup_{\substack{m=0, \dots, T/\delta-1 \\ t \leq \delta}} \left\| \sum_{k \in \mathbb{N}} \int_{m\delta}^t \sigma_k(\phi_s^\epsilon(\cdot)) \circ dW_s^k \right\|_{L^1(\mathbb{T}^2, \mathbb{R}^2)} \right] \lesssim \delta^\alpha.$$

Finally, by calculations similar to those performed in Lemma 4.6 and Lemma 4.7 of [21], for every fixed  $\alpha \in (0, 1/2)$

$$\begin{aligned} & \hat{\mathbb{E}} \left[ \sup_{m=0, \dots, T/\delta-1} \left\| \sum_{j=1}^6 \sum_{n=0}^{m-1} I_j^\epsilon(n) - \sum_{j=1}^4 \sum_{n=0}^{m-1} J_j^\epsilon(n) \right\|_{L^1(\mathbb{T}^2, \mathbb{R}^2)} \right] \\ & \lesssim \delta \epsilon^{-1/2} \log^{3/2}(1 + \epsilon^{-1}) + \delta^{\alpha-1} \epsilon^{1/2} \log(1 + \epsilon^{-1}) \\ & \delta^2 \epsilon^{-3/2} \log^{3/2}(1 + \epsilon^{-1}) + \delta^{1+\alpha} \epsilon^{-1} \log(1 + \epsilon^{-1}) + \delta^\alpha. \end{aligned}$$

We conclude the proof fixing  $\alpha$  close to  $1/2$  so that  $(1 + \alpha)^{-1} < 3/4 < (2 - 2\alpha)^{-1}$ , for instance  $\alpha = 3/8$ , and optimizing over  $\delta$ : for  $\delta = \epsilon^{16/21} \log^{-4/21}(1 + \epsilon^{-1})$ , it follows the desired inequality

$$\begin{aligned} & \hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \left\| \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(\cdot)) \eta_s^{\epsilon, k} ds - \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(\cdot)) \circ dW_s^k \right\|_{L^1(\mathbb{T}^2, \mathbb{R}^2)} \right] \\ & \lesssim \epsilon^{1/42} \log^{47/42}(1 + \epsilon^{-1}). \end{aligned}$$

□

### 3.4 Proof of Proposition 3.2

Recall the content of Proposition 3.2: we need to prove, under assumptions (A1)–(A5)

$$\hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \left\| \int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \right] \lesssim \epsilon^{1/12} \log^{11/12}(1 + \epsilon^{-1}).$$

Comparing the desired inequality with Lemma 3.3, one realizes that time integration of the process  $z_s^\epsilon(\phi_s^\epsilon(x))$  allows a better control due to cancellation of opposite-sign oscillations, even if the latter may become of large magnitude for  $\epsilon$  going to zero.

Concerning the strategy of the proof, in the first place we prove the following:

**Lemma 3.6** *For every fixed  $t \in [0, T]$  it holds*

$$\hat{\mathbb{E}} \left[ \left\| \int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \right] \lesssim \epsilon^{1/6} \log^{5/6}(1 + \epsilon^{-1}).$$

Having at hands the previous result, the proof of Proposition 3.2 goes as follows: for some parameter  $\delta = T/m > 0, m \in \mathbb{N}$  to be chosen, write

$$\begin{aligned} \sup_{t \in [0, T]} \left\| \int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} &\leq \sup_{n=0, \dots, m-1} \left\| \int_0^{n\delta} z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \\ &\quad + \sup_{\substack{n=0, \dots, m-1 \\ t \leq \delta}} \left\| \int_{n\delta}^{n\delta+t} z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \\ &\leq \sum_{n=0}^{m-1} \left\| \int_0^{n\delta} z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \\ &\quad + \delta \sup_{s \in [0, T]} \|z_s^\epsilon(\phi_s^\epsilon(\cdot))\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)}. \end{aligned}$$

Hence, by Lemmas 3.3 and 3.6

$$\begin{aligned} \widehat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \left\| \int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \right] &\leq \sum_{n=0}^{m-1} \widehat{\mathbb{E}} \left[ \left\| \int_0^{n\delta} z_s^\epsilon(\phi_s^\epsilon(\cdot)) ds \right\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \right] \\ &\quad + \delta \widehat{\mathbb{E}} \left[ \sup_{s \in [0, T]} \|z_s^\epsilon(\phi_s^\epsilon(\cdot))\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \right] \\ &\lesssim \delta^{-1} \epsilon^{1/6} \log^{5/6}(1 + \epsilon^{-1}) + \delta \log(1 + \epsilon^{-1}), \end{aligned}$$

and the thesis follows by optimizing the choice of  $\delta$ .

**Proof of Lemma 3.6** We will work with fixed  $x \in \mathbb{T}^2$ . The reader can easily check that all the inequalities present in the proof hold uniformly in  $x$ . Recall  $z_t^\epsilon = K * \zeta_t^\epsilon$ , and for  $\psi_{t,s}^\epsilon(x) := \psi_s^\epsilon((\psi_t^\epsilon)^{-1}(x))$  the formula

$$\begin{aligned} \zeta_t^\epsilon(x) &= - \int_0^t e^{-\epsilon^{-1}(t-s)} ((v_s^\epsilon + K * \zeta_s^\epsilon) \cdot \nabla \theta_s^\epsilon)(\psi_{t,s}^\epsilon(x)) ds \\ &\quad - \int_0^t e^{-\epsilon^{-1}(t-s)} ((K * \theta_s^\epsilon) \cdot \nabla \theta_s^\epsilon)(\psi_{t,s}^\epsilon(x)) ds. \end{aligned}$$

For notational simplicity let  $\Theta_s^\epsilon := (K * \theta_s^\epsilon) \cdot \nabla \theta_s^\epsilon$ , and rewrite

$$\begin{aligned} \zeta_t^\epsilon(x) &= - \int_0^t e^{-\epsilon^{-1}(t-s)} ((v_s^\epsilon + K * \zeta_s^\epsilon) \cdot \nabla \theta_s^\epsilon)(\psi_{t,s}^\epsilon(x)) ds \\ &\quad - \int_0^t e^{-\epsilon^{-1}(t-s)} (\Theta_s^\epsilon(\psi_{t,s}^\epsilon(x)) - \Theta_s^\epsilon(x)) ds \\ &\quad - \int_0^t e^{-\epsilon^{-1}(t-s)} \Theta_s^\epsilon(x) ds \\ &= : \zeta_t^{\epsilon,1}(x) + \zeta_t^{\epsilon,2}(x) + \zeta_t^{\epsilon,3}(x). \end{aligned}$$

Let us focus on the terms  $\zeta^{\epsilon,j}$ ,  $j = 1, 2, 3$  separately. Concerning  $\zeta^{\epsilon,1}$ ,

$$\begin{aligned} \|\zeta_t^{\epsilon,1}\|_{L^\infty(\mathbb{T}^2)} &\lesssim \int_0^t e^{-\epsilon^{-1}(t-s)} ds \left( 1 + \sup_{s \in [0,T]} \|\zeta_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} \right) \\ &\quad \times \sup_{s \in [0,T]} \|\nabla \theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)}, \end{aligned}$$

and thus the following holds by assumption (A2) and Lemma 3.3

$$\sup_{t \in [0,T]} \widehat{\mathbb{E}} \left[ \|\zeta_t^{\epsilon,1}\|_{L^\infty(\mathbb{T}^2)} \right] \lesssim \epsilon^{1/2} \log^{3/2}(1 + \epsilon^{-1}). \tag{3.7}$$

Moving to  $\zeta^{\epsilon,2}$ , notice that  $|\psi_{t,s}^\epsilon(x) - x| = |\psi_{t,s}^\epsilon(x) - \psi_{t,t}^\epsilon(x)|$ , and letting  $y = (\psi_t^\epsilon)^{-1}(x)$  we have

$$\begin{aligned} |\psi_{t,s}^\epsilon(x) - \psi_{t,t}^\epsilon(x)| &= |\psi_s^\epsilon(y) - \psi_t^\epsilon(y)| \\ &\leq \int_s^t |v_r^\epsilon(\psi_r^\epsilon(y))| dr + \int_s^t |u_r^\epsilon(\psi_r^\epsilon(y))| dr \\ &\lesssim |t - s| \left( 1 + \sup_{r \in [0,T]} \|\zeta_r^\epsilon\|_{L^\infty(\mathbb{T}^2)} + \sup_{r \in [0,T]} \|\theta_r^\epsilon\|_{L^\infty(\mathbb{T}^2)} \right), \end{aligned}$$

therefore

$$\begin{aligned} \|\zeta_t^{\epsilon,2}\|_{L^\infty(\mathbb{T}^2)} &\lesssim \int_0^t e^{-\epsilon^{-1}(t-s)} |t - s| ds \sup_{s \in [0,T]} \|\nabla \Theta_s^\epsilon\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \\ &\quad \times \left( 1 + \sup_{r \in [0,T]} \|\zeta_r^\epsilon\|_{L^\infty(\mathbb{T}^2)} + \sup_{r \in [0,T]} \|\theta_r^\epsilon\|_{L^\infty(\mathbb{T}^2)} \right), \end{aligned}$$

that implies

$$\sup_{t \in [0,T]} \widehat{\mathbb{E}} \left[ \|\zeta_t^{\epsilon,2}\|_{L^\infty(\mathbb{T}^2)} \right] \lesssim \epsilon^{1/2} \log^{3/2}(1 + \epsilon^{-1}). \tag{3.8}$$

Finally, let us consider the term  $\zeta^{\epsilon,3}$ , which requires a preliminary manipulation. Since  $\theta_s^\epsilon(x) = \sum_{k \in \mathbb{N}} \sigma_k(x) \eta_s^{\epsilon,k}$ , we can rewrite for every  $x \in \mathbb{T}^2$

$$\Theta_s^\epsilon(x) = \sum_{k,h \in \mathbb{N}} (\sigma_k \cdot \nabla \zeta_h)(x) \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} = : \sum_{k,h \in \mathbb{N}} \Theta_{k,h}(x) \eta_s^{\epsilon,k} \eta_s^{\epsilon,h},$$

where we have used  $\sigma_k = K * \zeta_k$  and  $\Theta_{k,h} := \sigma_k \cdot \nabla \zeta_h$ . Also, rewrite:

$$\begin{aligned} \zeta_t^{\epsilon,3}(x) &= - \int_0^t e^{-\epsilon^{-1}(t-s)} \Theta_s^\epsilon(x) ds \\ &= - \sum_{k,h \in \mathbb{N}} \Theta_{k,h}(x) \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds. \end{aligned}$$

By Itô Formula, for every fixed  $t$  and  $k, h \in \mathbb{N}$  it holds

$$\begin{aligned} \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} &= -\epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \\ &\quad + \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} dW_s^h + \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,h} dW_s^k \\ &\quad + \frac{\epsilon^{-2}}{2} \delta_{k,h} \int_0^t e^{-\epsilon^{-1}(t-s)} ds, \end{aligned}$$

with  $\delta_{k,h}$  being the Kronecker delta function:  $\delta_{k,h} = 1$  if  $k = h$  and  $\delta_{k,h} = 0$  if  $k \neq h$ . Otherwise said:

$$\begin{aligned} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds &= -\epsilon \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} \tag{3.9} \\ &\quad + \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} dW_s^h + \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,h} dW_s^k \\ &\quad + \frac{1 - e^{\epsilon^{-1}t}}{2} \delta_{k,h}. \end{aligned}$$

By (3.9) and assumption (A5), for every  $x \in \mathbb{T}^2$  we have

$$\begin{aligned} \zeta_t^{\epsilon,3}(x) &= \sum_{k,h \in \mathbb{N}} \Theta_{k,h}^\epsilon(x) \epsilon \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} \\ &\quad - \sum_{k,h \in \mathbb{N}} \Theta_{k,h}^\epsilon(x) \left( \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} dW_s^h + \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,h} dW_s^k \right), \end{aligned}$$

and therefore we can rewrite

$$\begin{aligned} \int_0^t (K * \zeta_s^{\epsilon,3})(\phi_s^\epsilon(x)) ds &= \sum_{k,h \in \mathbb{N}} \int_0^t (K * \Theta_{k,h}^\epsilon)(\phi_s^\epsilon(x)) \epsilon \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \\ &\quad - \sum_{k,h \in \mathbb{N}} \int_0^t (K * \Theta_{k,h})(\phi_s^\epsilon(x)) \left( \int_0^s e^{-\epsilon^{-1}(s-r)} \eta_r^{\epsilon,k} dW_r^h \right. \\ &\quad \left. + \int_0^s e^{-\epsilon^{-1}(s-r)} \eta_r^{\epsilon,h} dW_r^k \right) ds. \end{aligned}$$

$$\begin{aligned}
 & \widehat{\mathbb{E}} \left[ \left| \int_0^t (K * \Theta_{k,h}^\epsilon)(\phi_s^\epsilon(x)) \int_0^s e^{-\epsilon^{-1}(s-r)} \eta_r^{\epsilon,k} dW_r^h ds \right| \right] \\
 &= \widehat{\mathbb{E}} \left[ \left| \int_0^t \left( \int_r^t (K * \Theta_{k,h}^\epsilon)(\phi_s^\epsilon(x)) e^{-\epsilon^{-1}(s-r)} ds \right) \eta_r^{\epsilon,k} dW_r^h \right| \right] \\
 &\lesssim \widehat{\mathbb{E}} \left[ \left| \int_0^t \left( \int_r^t (K * \Theta_{k,h}^\epsilon)(\phi_s^\epsilon(x)) e^{-\epsilon^{-1}(s-r)} ds \right) \eta_r^{\epsilon,k} dW_r^h \right|^2 \right]^{1/2} \\
 &\lesssim \widehat{\mathbb{E}} \left[ \int_0^t \left( \int_r^t (K * \Theta_{k,h}^\epsilon)(\phi_s^\epsilon(x)) e^{-\epsilon^{-1}(s-r)} ds \right)^2 |\eta_r^{\epsilon,k}|^2 dr \right]^{1/2} \\
 &\lesssim \epsilon^{1/2} \log^{1/2}(1 + \epsilon^{-1}).
 \end{aligned}$$

The last non-trivial term is manipulated as follows. Let  $\delta = t/m > 0$ ,  $m \in \mathbb{N}$  to be suitably chosen. We have

$$\begin{aligned}
 & \sum_{k,h \in \mathbb{N}} \int_0^t (K * \Theta_{k,h})(\phi_s^\epsilon(x)) \epsilon \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \tag{3.10} \\
 &= \sum_{k,h \in \mathbb{N}} \sum_{n=0}^{m-1} \int_{n\delta}^{(n+1)\delta} ((K * \Theta_{k,h})(\phi_s^\epsilon(x)) - (K * \Theta_{k,h})(\phi_{n\delta}^\epsilon(x))) \epsilon \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \\
 &+ \sum_{k,h \in \mathbb{N}} \sum_{n=0}^{m-1} (K * \Theta_{k,h})(\phi_{n\delta}^\epsilon(x)) \int_{n\delta}^{(n+1)\delta} \epsilon \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds.
 \end{aligned}$$

Recalling (1.3), for every  $\alpha \in (0, 1/2)$  it holds

$$\begin{aligned}
 |\phi_t^\epsilon(x) - \phi_s^\epsilon(x)| &\leq \int_s^t |v_r^\epsilon(\phi_r^\epsilon(x))| dr + \int_s^t |u_r^\epsilon(\phi_r^\epsilon(x))| dr + \sqrt{2v}(w_t - w_s) \\
 &\lesssim |t - s| \left( 1 + \sup_{r \in [0,T]} \|\zeta_r^\epsilon\|_{L^\infty(\mathbb{T}^2)} + \sup_{r \in [0,T]} \|\theta_r^\epsilon\|_{L^\infty(\mathbb{T}^2)} \right) + |t - s|^\alpha,
 \end{aligned}$$

which implies

$$\begin{aligned}
 & \widehat{\mathbb{E}} \left[ \left| \sum_{k,h \in \mathbb{N}} \sum_{n=0}^{m-1} \int_{n\delta}^{(n+1)\delta} ((K * \Theta_{k,h})(\phi_s^\epsilon(x)) - (K * \Theta_{k,h})(\phi_{n\delta}^\epsilon(x))) \epsilon \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \right| \right] \tag{3.11} \\
 &\lesssim \delta \epsilon^{-1/2} \log^{3/2}(1 + \epsilon^{-1}) + \delta^\alpha \log(1 + \epsilon^{-1}).
 \end{aligned}$$

Also, we can apply Itô Formula again to find an alternative representation for the time integral of the quadratics  $\eta_s^{\epsilon,k} \eta_s^{\epsilon,h}$ , similar to (3.9). Indeed,

$$\begin{aligned} \eta_{(n+1)\delta}^{\epsilon,k} \eta_{(n+1)\delta}^{\epsilon,h} - \eta_{n\delta}^{\epsilon,k} \eta_{n\delta}^{\epsilon,h} &= -2\epsilon^{-1} \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} dt \\ &\quad + \epsilon^{-1} \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k} dW_t^h + \epsilon^{-1} \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,h} dW_t^k \\ &\quad + \frac{\epsilon^{-2}\delta}{2} \delta_{k,h}, \end{aligned}$$

and rearranging the terms we obtain

$$\begin{aligned} \int_{n\delta}^{(n+1)\delta} \epsilon \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} dt &= \frac{\epsilon^2}{2} \left( \eta_{n\delta}^{\epsilon,k} \eta_{n\delta}^{\epsilon,h} - \eta_{(n+1)\delta}^{\epsilon,k} \eta_{(n+1)\delta}^{\epsilon,h} \right) \\ &\quad + \frac{\epsilon}{2} \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k} dW_t^h + \frac{\epsilon}{2} \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,h} dW_t^k + \frac{\delta}{4} \delta_{k,h}. \end{aligned} \tag{3.12}$$

Finally, making use of (3.12) above and assumption (A5) we can rewrite

$$\begin{aligned} &\sum_{k,h \in \mathbb{N}} \sum_{n=0}^{m-1} (K * \Theta_{k,h})(\phi_{n\delta}^\epsilon(x)) \int_{n\delta}^{(n+1)\delta} \epsilon \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \\ &= \sum_{k,h \in \mathbb{N}} \sum_{n=0}^{m-1} (K * \Theta_{k,h})(\phi_{n\delta}^\epsilon(x)) \frac{\epsilon^2}{2} \left( \eta_{n\delta}^{\epsilon,k} \eta_{n\delta}^{\epsilon,h} - \eta_{(n+1)\delta}^{\epsilon,k} \eta_{(n+1)\delta}^{\epsilon,h} \right) \\ &\quad + \sum_{k,h \in \mathbb{N}} \sum_{n=0}^{m-1} (K * \Theta_{k,h})(\phi_{n\delta}^\epsilon(x)) \left( \frac{\epsilon}{2} \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k} dW_t^h + \frac{\epsilon}{2} \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,h} dW_t^k \right). \end{aligned}$$

We have

$$\begin{aligned} &\widehat{\mathbb{E}} \left[ \left| \sum_{k,h \in \mathbb{N}} \sum_{n=0}^{m-1} (K * \Theta_{k,h})(\phi_{n\delta}^\epsilon(x)) \frac{\epsilon^2}{2} \left( \eta_{n\delta}^{\epsilon,k} \eta_{n\delta}^{\epsilon,h} - \eta_{(n+1)\delta}^{\epsilon,k} \eta_{(n+1)\delta}^{\epsilon,h} \right) \right| \right] \\ &\lesssim \delta^{-1} \epsilon \log(1 + \epsilon^{-1}), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\widehat{\mathbb{E}} \left[ \left| \sum_{k,h \in \mathbb{N}} \sum_{n=0}^{m-1} (K * \Theta_{k,h})(\phi_{n\delta}^\epsilon(x)) \frac{\epsilon}{2} \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k} dW_t^h \right| \right] \\ &\lesssim \sum_{n=0}^{m-1} \epsilon \widehat{\mathbb{E}} \left[ \left| \int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k} dW_t^h \right|^2 \right]^{1/2} \lesssim \delta^{-1/2} \epsilon^{1/2} \log^{1/2}(1 + \epsilon^{-1}). \end{aligned} \tag{3.14}$$

It only remains to choose  $\delta$  in a suitable way, so that all the terms (3.11) and (3.13) are infinitesimal in the limit  $\epsilon \rightarrow 0$ . Taking for instance  $\alpha = 1/3$  and optimizing over  $\delta$  gives

$$\hat{\mathbb{E}} \left[ \left| \int_0^t (K * \zeta_s^{\epsilon,3})(\phi_s^\epsilon(x)) ds \right| \right] \lesssim \epsilon^{1/6} \log^{5/6}(1 + \epsilon^{-1}). \tag{3.15}$$

Considering (3.7), (3.8) and (3.15), we finally get the desired estimate: the proof is complete.  $\square$

### 4 Convergence of characteristics

In this section we prove our first major result Theorem 2.12.

We take the opportunity to point out a mistake in [21, Lemma 3.8], where BDG inequality was applied incorrectly. The present proof also corrects this previous mistake, and it is based on Itô Formula for a smooth approximation  $g_\delta(x)$  of the absolute value  $|x|$ .

**Proof of Theorem 2.12** The strategy of the proof is very similar to that of [21, Proposition 4.1]. Indeed, the difference  $\phi^\epsilon - \phi$  solves  $\hat{\mathbb{P}}$ -a.s. for every  $t \in [0, T]$  and  $x \in \mathbb{T}^2$ :

$$\begin{aligned} \phi_t^\epsilon(x) - \phi_t(x) &= \int_0^t v_s^\epsilon(\phi_s^\epsilon(x)) ds - \int_0^t v_s(\phi_s(x)) ds \\ &+ \int_0^t v_s(\phi_s^\epsilon(x)) ds - \int_0^t v_s(\phi_s(x)) ds \\ &+ \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(x)) \eta_s^{\epsilon,k} ds - \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(x)) \circ dW_s^k \\ &+ \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(x)) \circ dW_s^k - \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s(x)) \circ dW_s^k \\ &+ \int_0^t z_s^\epsilon(\phi_s^\epsilon(x)) ds. \end{aligned}$$

For  $\delta > 0$ , introduce the smooth function  $g_\delta : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $g_\delta(x) := (|x|^2 + \delta)^{1/2}$ . It holds  $\partial_{x_j} g_\delta(x) = x_j g_\delta(x)^{-1}$  and  $\partial_{x_j} \partial_{x_i} g_\delta(x) = g_\delta(x)^{-1} (\delta_{i,j} - x_i x_j g_\delta(x)^{-2})$  for every  $x \in \mathbb{R}^2$  and  $j = 1, 2$ , and moreover  $|x| \leq g_\delta(x) \leq |x| + \delta^{1/2}$ .

Denote

$$R_t^\epsilon(x) := \int_0^t z_s^\epsilon(\phi_s^\epsilon(x)) ds + \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(x)) \eta_s^{\epsilon,k} ds - \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s^\epsilon(x)) \circ dW_s^k,$$

and

$$Z_t^\epsilon(x) := \phi_t^\epsilon(x) - \phi_t(x) - R_t^\epsilon(x),$$

both seen as functions on the whole plane  $\mathbb{R}^2$ . Applying Itô Formula to  $g_\delta(Z_t^\epsilon(x))$  yields:

$$\begin{aligned} dg_\delta(Z_t^\epsilon(x)) &= Z_t^\epsilon(x)g_\delta(Z_t^\epsilon(x))^{-1} \cdot (v_t^\epsilon(\phi_t^\epsilon(x)) - v_t(\phi_t(x))) dt \\ &\quad + Z_t^\epsilon(x)g_\delta(Z_t^\epsilon(x))^{-1} \cdot (v_t(\phi_t^\epsilon(x)) - v_t(\phi_t(x))) dt \\ &\quad + \sum_{k \in \mathbb{N}} Z_t^\epsilon(x)g_\delta(Z_t^\epsilon(x))^{-1} \cdot (\sigma_k(\phi_t^\epsilon(x)) - \sigma_k(\phi_t(x))) dW_t^k \\ &\quad + Z_t^\epsilon(x)g_\delta(Z_t^\epsilon(x))^{-1} \cdot (c(\phi_t^\epsilon(x)) - c(\phi_t(x))) dt \\ &\quad + \sum_{k \in \mathbb{N}} \sum_{i,j=1}^2 g_\delta(Z_t^\epsilon(x))^{-1} (\delta_{i,j} - (Z_t^\epsilon(x))^i (Z_t^\epsilon(x))^j g_\delta(Z_t^\epsilon(x))^{-2}) \\ &\quad \times (\sigma_k(\phi_t^\epsilon(x)) - \sigma_k(\phi_t(x)))^i (\sigma_k(\phi_t^\epsilon(x)) - \sigma_k(\phi_t(x)))^j dt, \end{aligned}$$

and therefore

$$\begin{aligned} \hat{\mathbb{E}} [|\phi_t^\epsilon(x) - \phi_t^\delta(x)|] &\leq \hat{\mathbb{E}} [ |Z_t^\epsilon(x)| ] + \hat{\mathbb{E}} [ |R_t^\epsilon(x)| ] \leq \hat{\mathbb{E}} [ g_\delta(Z_t^\epsilon(x)) ] + \hat{\mathbb{E}} [ |R_t^\epsilon(x)| ] \\ &\lesssim \delta^{1/2} + \hat{\mathbb{E}} [ |R_t^\epsilon(x)| ] + \hat{\mathbb{E}} \left[ \int_0^t |v_s^\epsilon(\phi_s^\epsilon(x)) - v_s(\phi_s^\delta(x))| ds \right] \\ &\quad + \hat{\mathbb{E}} \left[ \int_0^t |v_s(\phi_s^\epsilon(x)) - v_s(\phi_s^\delta(x))| ds \right] \\ &\quad + \hat{\mathbb{E}} \left[ \int_0^t |\phi_s^\epsilon(x) - \phi_s^\delta(x)| ds \right] + \delta^{-1/2} \hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} |R_t^\epsilon(x)| \right], \end{aligned}$$

where in the last line we have used  $g_\delta(Z_s^\epsilon(x))^{-1} \leq \delta^{-1/2}$  and  $|\phi_s^\epsilon(x) - \phi_s^\delta(x)| \lesssim |Z_s^\epsilon(x)| + |R_s^\epsilon(x)|$ .

Taking the integral over  $x \in \mathbb{T}^2$  and using assumptions (A2), (A4), concavity of the function  $\gamma$ , Jensen inequality, Propositions 3.1 and 3.2 we get

$$\begin{aligned} \hat{\mathbb{E}} [ \|\phi_t^\epsilon - \phi_t^\delta\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)} ] &\lesssim \delta^{1/2} + \delta^{-1/2} \epsilon^{1/42} \log^{47/42}(1 + \epsilon^{-1}) + c_\epsilon \\ &\quad + \int_0^t \gamma \left( \hat{\mathbb{E}} [ \|\phi_s^\epsilon - \phi_s^\delta\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)} ] \right) ds \end{aligned}$$

uniformly in  $t \in [0, T]$  and  $\delta > 0$ . Taking  $\delta = \epsilon^{1/42} \log^{47/42}(1 + \epsilon^{-1})$  we deduce the desired result by Lemma 2.2. □

### 5 Convergence of large-scale dynamics

Recall the representation formulas for the solutions of (1.2) and (1.7)

$$\begin{aligned} \Xi_t^\epsilon &= \tilde{\mathbb{E}} \left[ \Xi_0 \circ (\phi_t^\epsilon)^{-1} + \int_0^t q_s^\epsilon \circ \phi_s^\epsilon \circ (\phi_t^\epsilon)^{-1} ds \right], \\ \Xi_t &= \tilde{\mathbb{E}} \left[ \Xi_0 \circ (\phi_t)^{-1} + \int_0^t q_s \circ \phi_s \circ (\phi_t)^{-1} ds \right], \end{aligned}$$

with  $\phi^\epsilon$  and  $\phi$  solving respectively (1.3) and (1.5).

As made clear by the following proof, these representation formulas are the key ingredient needed to show Theorem 2.13, thus justifying our Definition 2.8 in terms of these identities.

**Proof of Theorem 2.13** Let  $f \in L^1(\mathbb{T}^2)$  and  $t \in [0, T]$ . We have

$$\begin{aligned} & \left| \int_{\mathbb{T}^2} \Xi_t^\epsilon(x) f(x) dx - \int_{\mathbb{T}^2} \Xi_t(x) f(x) dx \right| \\ & \leq \left| \int_{\mathbb{T}^2} \tilde{\mathbb{E}} \left[ \Xi_0((\phi_t^\epsilon)^{-1}(x)) \right] f(x) dx - \int_{\mathbb{T}^2} \tilde{\mathbb{E}} \left[ \Xi_0((\phi_t)^{-1}(x)) \right] f(x) dx \right| \\ & \quad + \left| \int_{\mathbb{T}^2} \tilde{\mathbb{E}} \left[ \int_0^t q_s^\epsilon(\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(x))) ds \right] f(x) dx - \int_{\mathbb{T}^2} \tilde{\mathbb{E}} \left[ \int_0^t q_s(\phi_s((\phi_t)^{-1}(x))) ds \right] f(x) dx \right| \\ & = \left| \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \Xi_0((\phi_t^\epsilon)^{-1}(x)) f(x) dx - \int_{\mathbb{T}^2} \Xi_0((\phi_t)^{-1}(x)) f(x) dx \right] \right| \\ & \quad + \left| \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \int_0^t q_s^\epsilon(\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(x))) ds f(x) dx - \int_{\mathbb{T}^2} \int_0^t q_s(\phi_s((\phi_t)^{-1}(x))) ds f(x) dx \right] \right| \\ & = \left| \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \Xi_0(y) f(\phi_t^\epsilon(y)) dy - \int_{\mathbb{T}^2} \Xi_0(y) f(\phi_t(y)) dy \right] \right| \\ & \quad + \left| \tilde{\mathbb{E}} \left[ \int_0^t \int_{\mathbb{T}^2} q_s^\epsilon(\phi_s^\epsilon(y)) f(\phi_t^\epsilon(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s(y)) f(\phi_t(y)) dy ds \right] \right|. \end{aligned}$$

Taking expectation with respect to  $\mathbb{P}$ , the first summand is bounded by

$$\begin{aligned} & \mathbb{E} \left[ \left| \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \Xi_0(y) f(\phi_t^\epsilon(y)) dy - \int_{\mathbb{T}^2} \Xi_0(y) f(\phi_t(y)) dy \right] \right| \right] \\ & \leq \|\Xi_0\|_{L^\infty(\mathbb{T}^2)} \hat{\mathbb{E}} \left[ \int_{\mathbb{T}^2} |f(\phi_t^\epsilon(y)) - f(\phi_t(y))| dy \right]. \end{aligned} \tag{5.1}$$

As for the second term, we can rewrite

$$\begin{aligned} & \int_0^t \int_{\mathbb{T}^2} q_s^\epsilon(\phi_s^\epsilon(y)) f(\phi_t^\epsilon(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s(y)) f(\phi_t(y)) dy ds \\ & = \int_0^t \int_{\mathbb{T}^2} q_s^\epsilon(\phi_s^\epsilon(y)) f(\phi_t^\epsilon(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s^\epsilon(\phi_s^\epsilon(y)) f(\phi_t(y)) dy ds \\ & \quad + \int_0^t \int_{\mathbb{T}^2} q_s^\epsilon(\phi_s^\epsilon(y)) f(\phi_t(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s^\epsilon(y)) f(\phi_t(y)) dy ds \\ & \quad + \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s^\epsilon(y)) f(\phi_t(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s(y)) f(\phi_t(y)) dy ds, \end{aligned}$$

with estimates

$$\begin{aligned} & \hat{\mathbb{E}} \left[ \left| \int_0^t \int_{\mathbb{T}^2} q_s^\epsilon(\phi_s^\epsilon(y)) f(\phi_t^\epsilon(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s^\epsilon(\phi_s^\epsilon(y)) f(\phi_t(y)) dy ds \right| \right] \\ & \leq \int_0^t \|q_s^\epsilon\|_{L^\infty(\mathbb{T}^2)} ds \hat{\mathbb{E}} \left[ \int_{\mathbb{T}^2} |f(\phi_t^\epsilon(y)) - f(\phi_t(y))| dy \right]; \end{aligned} \tag{5.2}$$

$$\begin{aligned} & \hat{\mathbb{E}} \left[ \left| \int_0^t \int_{\mathbb{T}^2} q_s^\epsilon(\phi_s^\epsilon(y)) f(\phi_t(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s^\epsilon(y)) f(\phi_t(y)) dy ds \right| \right] \\ & \leq \int_0^t \|q_s^\epsilon - q_s\|_{L^\infty(\mathbb{T}^2)} ds \|f\|_{L^1(\mathbb{T}^2)}; \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} & \hat{\mathbb{E}} \left[ \left| \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s^\epsilon(y)) f(\phi_t(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s(y)) f(\phi_t(y)) dy ds \right| \right] \\ & \leq \hat{\mathbb{E}} \left[ \int_0^t \int_{\mathbb{T}^2} |q_s(\phi_s^\epsilon(y)) - q_s(\phi_s(y))| |f(\phi_t(y))| dy ds \right] \\ & = : \hat{\mathbb{E}} \left[ \int_0^t \int_{\mathbb{T}^2} |q_s(\phi_s^\epsilon(y)) - q_s(\phi_s(y))| d\mu(y) ds \right], \end{aligned} \tag{5.4}$$

where  $d\mu(y) := |f(\phi_t(y))| dy$  is a random Radon measure on  $\mathbb{T}^2$ .

By assumptions (A6) and (A7), the terms (5.1), (5.2) and (5.3) go to zero as  $\epsilon \rightarrow 0$ , using the same reasoning of Flandoli and Pappaletta [21, Theorem 5.1]. Therefore, here we restrict ourselves to only consider the remaining term (5.4).

Let us argue *per absurdum*. Suppose by contradiction that there exists a subsequence  $\epsilon_k \rightarrow 0$  such that

$$\hat{\mathbb{E}} \left[ \int_0^t \int_{\mathbb{T}^2} |q_s(\phi_s^{\epsilon_k}(y)) - q_s(\phi_s(y))| d\mu(y) ds \right] \geq C \tag{5.5}$$

for some  $C > 0$  and for every  $\epsilon_k$ .

Let  $\mathcal{N}$  and  $\tilde{\mathcal{N}}$  be negligible sets such that  $\phi_t$  is measure preserving for every  $\omega \in \mathcal{N}^c$  and  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$ .

Take  $\delta > 0$ . By Lusin Theorem [39, Theorem 2.23] there exists a measurable set  $C_\delta \subset [0, t] \times \mathbb{T}^2$  with  $\mathcal{L}_{[0,t]} \otimes \mathcal{L}_{\mathbb{T}^2}([0, t] \times \mathbb{T}^2 \setminus C_\delta) < \delta$  and a continuous function  $Q_\delta \in C([0, t] \times \mathbb{T}^2)$  that coincides with  $q$  on  $C_\delta$ . Therefore

$$\begin{aligned} \int_0^t \int_{\mathbb{T}^2} |q_s(\phi_s^{\epsilon_k}(y)) - q_s(\phi_s(y))| d\mu(y) ds &= \int_{C_\delta} |q_s(\phi_s^{\epsilon_k}(y)) - q_s(\phi_s(y))| d\mu(y) ds \\ &+ \int_{[0,t] \times \mathbb{T}^2 \setminus C_\delta} |q_s(\phi_s^{\epsilon_k}(y)) - q_s(\phi_s(y))| d\mu(y) ds \\ &\leq \int_{[0,t] \times \mathbb{T}^2} |Q_\delta(s, \phi_s^{\epsilon_k}(y)) - Q_\delta(s, \phi_s(y))| d\mu(y) ds \\ &+ 2 \int_{[0,t] \times \mathbb{T}^2 \setminus C_\delta} \|q_s\|_{L^\infty(\mathbb{T}^2)} d\mu(y) ds. \end{aligned}$$

Let us consider the second term first. Recalling  $d\mu(y) = |f(\phi_t(y))|dy$ , we have

$$\begin{aligned} \int_{[0,t] \times \mathbb{T}^2 \setminus C_\delta} \|q_s\|_{L^\infty(\mathbb{T}^2)} d\mu(y) ds &= \int_{[0,t] \times \mathbb{T}^2 \setminus C_\delta} \|q_s\|_{L^\infty(\mathbb{T}^2)} |f(\phi_t(y))| dy ds \\ &= \int_{\phi_t^{-1}(C_\delta^c)} \|q_s\|_{L^\infty(\mathbb{T}^2)} |f(y)| dy ds, \end{aligned}$$

with  $\phi_t^{-1}(C_\delta^c) := \{(s, y) : (s, \phi_t(y)) \in C_\delta^c\}$ . Since  $\phi_t$  is measure preserving for every  $\omega \in \mathcal{N}^c$  and  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$ , it is easy to check

$$\mathcal{L}_{[0,t]} \otimes \mathcal{L}_{\mathbb{T}^2}(\phi_t^{-1}(C_\delta^c)) = \mathcal{L}_{[0,t]} \otimes \mathcal{L}_{\mathbb{T}^2}(C_\delta^c) < \delta$$

$\hat{\mathbb{P}}$ -almost surely, and since  $\|q\|_{L^\infty(\mathbb{T}^2)}|f| \in L^1([0, t] \times \mathbb{T}^2)$ , absolute continuity of Lebesgue integral gives the existence of  $\delta > 0$  such that for every  $\omega \in \mathcal{N}^c$  and  $\tilde{\omega} \in \tilde{\mathcal{N}}^c$

$$\int_{[0,t] \times \mathbb{T}^2 \setminus C_\delta} \|q_s\|_{L^\infty(\mathbb{T}^2)} d\mu(y) ds < C/3.$$

We fix such a  $\delta$  hereafter. For the first term we argue as follows: since we have proved

$$\sup_{t \in [0, T]} \hat{\mathbb{E}} \left[ \|\phi_t^{\epsilon_k} - \phi_t\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)} \right] \rightarrow 0$$

as  $\epsilon_k \rightarrow 0$ , then for every fixed  $s \in [0, T]$  there exists a subsequence (that we still denote  $\epsilon_k$ ) such that the maps

$$\begin{aligned} \Phi_s^{\epsilon_k} : \hat{\Omega} \times \mathbb{T}^2 &\rightarrow [0, T] \times \mathbb{T}^2, \\ \Phi_s^{\epsilon_k}(\hat{\omega}, y) &= (s, \phi^\epsilon(\hat{\omega}, s, y)) \end{aligned}$$

converge  $\hat{\mathbb{P}} \otimes \mathcal{L}_{\mathbb{T}^2}$ -almost everywhere to  $\Phi_s$  given by  $\Phi_s(\hat{\omega}, y) = (s, \phi(\hat{\omega}, s, y))$ . By almost sure continuity in time of  $\Phi_s^{\epsilon_k}$  and  $\Phi_s$ , it is possible to extract a common subsequence  $\epsilon_k \rightarrow 0$  such that  $\Phi_s^{\epsilon_k}$  converges  $\hat{\mathbb{P}} \otimes \mathcal{L}_{\mathbb{T}^2}$ -almost everywhere to  $\Phi_s$  simultaneously for all  $s \in [0, T]$ .

Therefore, since  $Q_\delta$  is continuous on  $[0, t] \times \mathbb{T}^2$ , also  $Q_\delta(\Phi^{\epsilon_k})$  converges  $\hat{\mathbb{P}} \otimes \mathcal{L}_{[0,t]} \otimes \mathcal{L}_{\mathbb{T}^2}$ -almost everywhere to  $Q_\delta(\Phi)$ , and since  $\mu$  is absolutely continuous with respect to  $\mathcal{L}_{\mathbb{T}^2}$  for almost every  $\hat{\omega} \in \hat{\Omega}$ , the convergence is actually  $\hat{\mathbb{P}} \otimes \mathcal{L}_{[0,t]} \otimes \mu_{\hat{\omega}}$ -almost everywhere; moreover,  $Q_\delta(\Phi^{\epsilon_k})$  is dominated by the constant  $\sup_{s \in [0,t], y \in \mathbb{T}^2} |Q_\delta(s, y)|$ , and Lebesgue dominated convergence yields convergence in  $L^1(\hat{\Omega} \times [0, T] \times \mathbb{T}^2, \hat{\mathbb{P}} \otimes \mathcal{L}_{[0,t]} \otimes \mu_{\hat{\omega}})$ , that is

$$\hat{\mathbb{E}} \left[ \int_{[0,t] \times \mathbb{T}^2} |Q_\delta(s, \phi_s^{\epsilon_k}(y)) - Q_\delta(s, \phi_s(y))| d\mu(y) ds \right] \rightarrow 0,$$

as  $\epsilon_k \rightarrow 0$ . This contradicts (5.5), and therefore we have proved: for every  $f \in L^1(\mathbb{T}^2)$

$$\mathbb{E} \left[ \left| \int_{\mathbb{T}^2} \Xi_t^\epsilon(x) f(x) dx - \int_{\mathbb{T}^2} \Xi_t(x) f(x) dx \right| \right] \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,$$

for every fixed  $t \in [0, T]$ . Since  $\|\Xi_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}$  is bounded uniformly in  $\epsilon > 0$  and  $t \in [0, T]$ , pointwise convergence implies convergence in  $L^p([0, T])$  for every finite  $p$  by Lebesgue dominated convergence Theorem.

Finally, if  $q \in L^1([0, T], Lip(\mathbb{T}^2))$  and  $f \in Lip(\mathbb{T}^2)$  with  $[f]_{Lip(\mathbb{T}^2)} \leq 1$ , we have

$$\begin{aligned} \hat{\mathbb{E}} \left[ \int_{\mathbb{T}^2} |f(\phi_t^\epsilon(y)) - f(\phi_t(y))| dy \right] &\leq \hat{\mathbb{E}} \left[ \int_{\mathbb{T}^2} |\phi_t^\epsilon(y) - \phi_t(y)| dy \right] \\ &\leq \sup_{t \in [0, T]} \hat{\mathbb{E}} [\|\phi_t^\epsilon - \phi_t\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)}], \end{aligned}$$

controlling (5.1) and (5.2) uniformly in  $f$ ; also, since  $\|f\|_{L^\infty(\mathbb{T}^2)} \leq 1$  it holds

$$\begin{aligned} &\hat{\mathbb{E}} \left[ \int_0^t \int_{\mathbb{T}^2} |q_s(\phi_s^\epsilon(y)) - q_s(\phi_s(y))| |f(\phi_t(y))| dy ds \right] \\ &\leq \hat{\mathbb{E}} \left[ \int_0^t \int_{\mathbb{T}^2} \|q_s\|_{Lip(\mathbb{T}^2)} |\phi_s^\epsilon(y) - \phi_s(y)| dy ds \right] \\ &\leq \int_0^t \|q_s\|_{Lip(\mathbb{T}^2)} ds \sup_{s \in [0, T]} \hat{\mathbb{E}} [\|\phi_s^\epsilon - \phi_s\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)}], \end{aligned}$$

allowing to bound (5.4) in a simpler way. Putting all together, we have proved the desired convergence uniformly in  $t \in [0, T]$  and  $f \in Lip(\mathbb{T}^2)$  with  $[f]_{Lip(\mathbb{T}^2)} \leq 1$ ,  $\|f\|_{L^\infty(\mathbb{T}^2)} \leq 1$ . The proof is complete.  $\square$

### 6 Examples

In this final section, we discuss how assumptions (A1)-(A7) are fulfilled by our main motivational examples, namely advection-diffusion or Navier–Stokes equations at large scales coupled with stochastic Euler equations at small scales - cfr. subsection 1.1 for details.

First of all, notice that in the case of passive scalars, like in the advection-diffusion equations, there is nothing to actually prove since all the subjects of assumptions (A1)-(A7) are given *a priori*. On the other hand, in the Navier–Stokes system the fields  $v^\epsilon, v$  are given by  $v^\epsilon = K * \Xi^\epsilon, v = K * \Xi$ , and therefore (A1), (A2) and (A4) need to be checked. The verification of (A4) needs an additional requirement on the external source  $q$ : assume

(A8) there exists a constant  $C$  such that for almost every  $t \in [0, T]$  and almost every  $x, y \in \mathbb{T}^2$

$$|q(t, x) - q(t, y)| \leq C\gamma(|x - y|).$$

**Proposition 6.1** *Let  $v \geq 0, \Xi_0 \in L^\infty(\mathbb{T}^2)$  with zero spatial average and consider the Navier–Stokes ( $v > 0$ ) or Euler ( $v = 0$ ) system*

$$\begin{cases} d\Xi_t^\epsilon + (v_t^\epsilon + u_t^\epsilon) \cdot \nabla \Xi_t^\epsilon dt = \nu \Delta \Xi_t^\epsilon dt + q_t^\epsilon dt, \\ d\xi_t^\epsilon + (v_t^\epsilon + u_t^\epsilon) \cdot \nabla \xi_t^\epsilon dt = -\epsilon^{-1} \xi_t^\epsilon dt + \epsilon^{-1} \sum_{k \in \mathbb{N}} \zeta_k dW_t^k, \\ v_t^\epsilon = -\nabla^\perp (-\Delta)^{-1} \Xi_t^\epsilon, \\ u_t^\epsilon = -\nabla^\perp (-\Delta)^{-1} \xi_t^\epsilon, \end{cases}$$

and the limiting large-scale dynamics

$$\begin{cases} d\Xi_t + v_t \cdot \nabla \Xi_t dt + \sum_{k \in \mathbb{N}} \sigma_k \cdot \nabla \Xi_t \circ dW_t^k = \nu \Delta \Xi_t dt + q_t dt, \\ v_t = -\nabla^\perp (-\Delta)^{-1} \Xi_t. \end{cases}$$

Assume (A3), (A5)–(A8) and take  $q_t^\epsilon, q_t$  with zero spatial average for almost every  $t \in [0, T]$ . Then the velocity fields  $v^\epsilon, v$  satisfy (A1), (A2) and (A4).

**Proof** Concerning (A1), measurability can be deduced by  $v^\epsilon = K * \Xi^\epsilon, v = K * \Xi$ , representation formulas (1.4) and (1.6), and the fact that  $\phi^\epsilon, \phi$  are stochastic flows of measure-preserving homeomorphisms. Assumption (A2) is given by  $v^\epsilon = K * \Xi^\epsilon, v = K * \Xi$ , (2.1) and Lemma 2.1.

Finally, let us then verify (A4). Recall

$$\begin{aligned} v_t^\epsilon(x) &= \int_{\mathbb{T}^2} K(x - y) \Xi_t^\epsilon(y) dy \\ &= \int_{\mathbb{T}^2} K(x - y) \tilde{\mathbb{E}} \left[ \Xi_0((\phi_t^\epsilon)^{-1}(y)) + \int_0^t q_s^\epsilon(\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y))) ds \right] dy \\ &= \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} K(x - \phi_t^\epsilon(y)) \Xi_0(y) dy \right] + \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} K(x - \phi_t^\epsilon(y)) \int_0^t q_s^\epsilon(\phi_s^\epsilon(y)) ds dy \right], \end{aligned}$$

and

$$\begin{aligned} v_t(x) &= \int_{\mathbb{T}^2} K(x - y) \Xi_t(y) dy \\ &= \int_{\mathbb{T}^2} K(x - y) \tilde{\mathbb{E}} \left[ \Xi_0((\phi_t)^{-1}(y)) + \int_0^t q_s(\phi_s((\phi_t)^{-1}(y))) ds \right] dy \\ &= \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} K(x - \phi_t(y)) \Xi_0(y) dy \right] + \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} K(x - \phi_t(y)) \int_0^t q_s(\phi_s(y)) ds dy \right]. \end{aligned}$$

We have

$$\begin{aligned}
 \int_{\mathbb{T}^2} |v_t^\epsilon(x) - v_t(x)| dx &\leq \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} |K(x - \phi_t^\epsilon(y)) - K(x - \phi_t(y))| |\Xi_0(y)| dy dx \right] \\
 &\quad + \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \left| \int_{\mathbb{T}^2} K(x - \phi_t^\epsilon(y)) \int_0^t q_s^\epsilon(\phi_s^\epsilon(y)) ds dy \right. \right. \\
 &\quad \left. \left. - \int_{\mathbb{T}^2} K(x - \phi_t(y)) \int_0^t q_s(\phi_s(y)) ds dy \right| dx \right] \\
 &\leq \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} |K(x - \phi_t^\epsilon(y)) - K(x - \phi_t(y))| |\Xi_0(y)| dy dx \right] \\
 &\quad + \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} |K(x - \phi_t^\epsilon(y)) - K(x - \phi_t(y))| \left| \int_0^t q_s^\epsilon(\phi_s^\epsilon(y)) ds \right| dy dx \right] \\
 &\quad + \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} |K(x - \phi_t(y))| \int_0^t |q_s^\epsilon(\phi_s^\epsilon(y)) - q_s(\phi_s^\epsilon(y))| ds | dy dx \right] \\
 &\quad + \tilde{\mathbb{E}} \left[ \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} |K(x - \phi_t(y))| \int_0^t |q_s(\phi_s^\epsilon(y)) - q_s(\phi_s(y))| ds | dy dx \right] \\
 &\lesssim \gamma \left( \tilde{\mathbb{E}} \left[ \|\phi_t^\epsilon - \phi_t\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)} \right] \right) + \int_0^t \|q_s^\epsilon - q_s\|_{L^\infty(\mathbb{T}^2)} ds \\
 &\quad + \int_0^t \gamma \left( \tilde{\mathbb{E}} \left[ \|\phi_s^\epsilon - \phi_s\|_{L^1(\mathbb{T}^2, \mathbb{T}^2)} \right] \right) ds,
 \end{aligned}$$

that is the desired estimate, since  $\int_0^t \|q_s^\epsilon - q_s\|_{L^\infty(\mathbb{T}^2)} ds \rightarrow 0$  as  $\epsilon \rightarrow 0$  by assumption (A7).  $\square$

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